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# **A First Course in Quantitative Economics with Python**

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This lecture series provides an introduction to quantitative economics using Python.

The lectures were designed and written by Thomas J. Sargent and John Stachurski, with extensive help from the rest of the QuantEcon team.

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# **Part I**

# **Introduction**



## ABOUT THESE LECTURES

### 1.1 About

This lecture series introduces quantitative economics using elementary mathematics and statistics plus computer code written in [Python](#).

The lectures emphasize simulation and visualization through code as a way to convey ideas, rather than focusing on mathematical details.

Although the presentation is quite novel, the ideas are rather foundational.

We emphasize the deep and fundamental importance of economic theory, as well as the value of analyzing data and understanding stylized facts.

The lectures can be used for university courses, self-study, reading groups or workshops.

Researchers and policy professionals might also find some parts of the series valuable for their work.

We hope the lectures will be of interest to students of economics who want to learn both economics and computing, as well as students from fields such as computer science and engineering who are curious about economics.

### 1.2 Level

The lecture series is aimed at undergraduate students.

The level of the lectures varies from truly introductory (suitable for first year undergraduates or even high school students) to more intermediate.

The more intermediate lectures require comfort with linear algebra and some mathematical maturity (e.g., calmly reading theorems and trying to understand their meaning).

In general, easier lectures occur earlier in the lecture series and harder lectures occur later.

We assume that readers have covered the easier parts of the QuantEcon lecture series [on Python programming](#).

In particular, readers should be familiar with basic Python syntax including Python functions. Knowledge of classes and Matplotlib will be beneficial but not essential.

## 1.3 Credits

In building this lecture series, we had invaluable assistance from research assistants at QuantEcon, as well as our QuantEcon colleagues. Without their help this series would not have been possible.

In particular, we sincerely thank and give credit to

- [Aakash Gupta](#)
- [Shu Hu](#)
- [Jiacheng Li](#)
- [Smit Lunagariya](#)
- [Matthew McKay](#)
- [Maanasee Sharma](#)
- [Humphrey Yang](#)

We also thank Noritaka Kudoh for encouraging us to start this project and providing thoughtful suggestions.

## **Part II**

# **Economic Data**



## ECONOMIC GROWTH EVIDENCE

### 2.1 Overview

In this lecture we use Python, Pandas, and Matplotlib to download, organize, and visualize historical data on GDP growth. In addition to learning how to deploy these tools more generally, we'll use them to describe facts about economic growth experiences across many countries over several centuries.

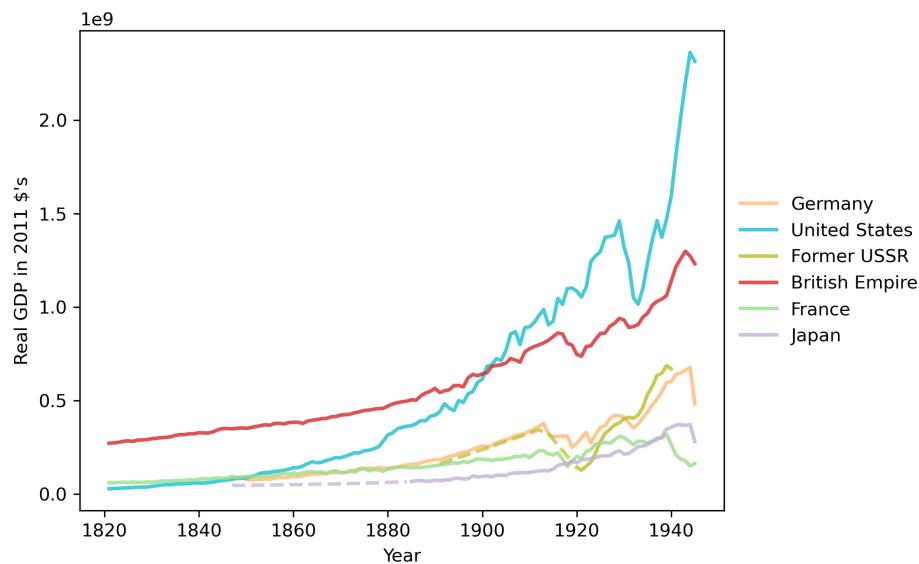
Such “growth facts” are interesting for a variety of reasons.

Explaining growth facts is a principal purpose of both “development economics” and “economic history”.

And growth facts are important inputs into historians’ studies of geopolitical forces and dynamics.

Thus, Adam Tooze’s account of the geopolitical precedents and antecedents of World War I begins by describing how Gross National Products of European Great Powers had evolved during the 70 years preceding 1914 (see chapter 1 of [Too14]).

Using the very same data that Tooze used to construct his figure, here is our version of his chapter 1 figure.



(This is just a copy of our figure Fig. 2.6. We describe how we constructed it later in this lecture.)

Chapter 1 of [Too14] used his graph to show how US GDP started the 19th century way behind the GDP of the British Empire.

By the end of the nineteenth century, US GDP had caught up with GDP of the British Empire, and how during the first half of the 20th century, US GDP surpassed that of the British Empire.

For Adam Tooze, that fact was a key geopolitical underpinning for the “American century”.

Looking at this graph and how it set the geopolitical stage for “the American (20th) century” naturally tempts one to want a counterpart to his graph for 2014 or later.

(An impatient reader seeking a hint at the answer might now want to jump ahead and look at figure [Fig. 2.7.](#))

As we’ll see, reasoning by analogy, this graph perhaps set the stage for an “XXX (21st) century”, where you are free to fill in your guess for country XXX.

As we gather data to construct those two graphs, we’ll also study growth experiences for a number of countries for time horizons extending as far back as possible.

These graphs will portray how the “Industrial Revolution” began in Britain in the late 18th century, then migrated to one country after another.

In a nutshell, this lecture records growth trajectories of various countries over long time periods.

While some countries have experienced long term rapid growth across that has lasted a hundred years, others have not.

Since populations differ across countries and vary within a country over time, it will be interesting to describe both total GDP and GDP per capita as it evolves within a country.

First let’s import the packages needed to explore what the data says about long run growth

```
import pandas as pd
import os
import matplotlib as mpl
import matplotlib.pyplot as plt
import matplotlib.cm as cm
import numpy as np
from collections import namedtuple
from matplotlib.lines import Line2D
```

## 2.2 Setting up

A project initiated by [Angus Maddison](#) has collected many historical time series related to economic growth, some dating back to the first century.

The data can be downloaded from the [Maddison Historical Statistics webpage](#) by clicking on the “Latest Maddison Project Release”.

For convenience, here is a copy of the 2020 data in Excel format.

Let’s read it into a pandas dataframe:

```
data = pd.read_excel("datasets/mpd2020.xlsx", sheet_name='Full data')
data
```

	countrycode	country	year	gdppc	pop
0	AFG	Afghanistan	1820	NaN	3280.00000
1	AFG	Afghanistan	1870	NaN	4207.00000
2	AFG	Afghanistan	1913	NaN	5730.00000
3	AFG	Afghanistan	1950	1156.0000	8150.00000
4	AFG	Afghanistan	1951	1170.0000	8284.00000

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```
...      ...      ...      ...      ...      ...
21677    ZWE    Zimbabwe  2014  1594.0000  13313.99205
21678    ZWE    Zimbabwe  2015  1560.0000  13479.13812
21679    ZWE    Zimbabwe  2016  1534.0000  13664.79457
21680    ZWE    Zimbabwe  2017  1582.3662  13870.26413
21681    ZWE    Zimbabwe  2018  1611.4052  14096.61179

[21682 rows x 5 columns]
```

We can see that this dataset contains GDP per capita (gdppc) and population (pop) for many countries and years.

Let's look at how many and which countries are available in this dataset

```
len(data.country.unique())
```

169

We can now explore some of the 169 countries that are available.

Let's loop over each country to understand which years are available for each country

```
cntry_years = []
for cntry in data.country.unique():
    cy_data = data[data.country == cntry][ 'year' ]
    ymin, ymax = cy_data.min(), cy_data.max()
    cntry_years.append((cntry, ymin, ymax))
cntry_years = pd.DataFrame(cntry_years, columns=[ 'country', 'Min Year', 'Max Year' ]).
    ↪set_index('country')
cntry_years
```

country	Min Year	Max Year
Afghanistan	1820	2018
Angola	1950	2018
Albania	1	2018
United Arab Emirates	1950	2018
Argentina	1800	2018
...	...	...
Yemen	1820	2018
Former Yugoslavia	1	2018
South Africa	1	2018
Zambia	1950	2018
Zimbabwe	1950	2018

[169 rows x 2 columns]

Let's now reshape the original data into some convenient variables to enable quicker access to countries time series data.

We can build a useful mapping between country codes and country names in this dataset

```
code_to_name = data[ [ 'countrycode', 'country' ] ].drop_duplicates().reset_
    ↪index(drop=True).set_index([ 'countrycode' ])
```

Then we can quickly focus on GDP per capita (gdp)

```
data
```

	countrycode	country	year	gdppc	pop
0	AFG	Afghanistan	1820	NaN	3280.00000
1	AFG	Afghanistan	1870	NaN	4207.00000
2	AFG	Afghanistan	1913	NaN	5730.00000
3	AFG	Afghanistan	1950	1156.0000	8150.00000
4	AFG	Afghanistan	1951	1170.0000	8284.00000
...	...	...	...	...	...
21677	ZWE	Zimbabwe	2014	1594.0000	13313.99205
21678	ZWE	Zimbabwe	2015	1560.0000	13479.13812
21679	ZWE	Zimbabwe	2016	1534.0000	13664.79457
21680	ZWE	Zimbabwe	2017	1582.3662	13870.26413
21681	ZWE	Zimbabwe	2018	1611.4052	14096.61179

[21682 rows x 5 columns]

```
gdppc = data.set_index(['countrycode', 'year'])['gdppc']
gdppc = gdppc.unstack('countrycode')
```

```
gdppc
```

countrycode	AFG	AGO	ALB	ARE	ARG	\
year						
1	NaN	NaN	NaN	NaN	NaN	
730	NaN	NaN	NaN	NaN	NaN	
1000	NaN	NaN	NaN	NaN	NaN	
1090	NaN	NaN	NaN	NaN	NaN	
1120	NaN	NaN	NaN	NaN	NaN	
...	...	...	...	...	...	
2014	2022.0000	8673.0000	9808.0000	72601.0000	19183.0000	
2015	1928.0000	8689.0000	10032.0000	74746.0000	19502.0000	
2016	1929.0000	8453.0000	10342.0000	75876.0000	18875.0000	
2017	2014.7453	8146.4354	10702.1201	76643.4984	19200.9061	
2018	1934.5550	7771.4418	11104.1665	76397.8181	18556.3831	

countrycode	ARM	AUS	AUT	AZE	BDI	...	\
year							
1	NaN	NaN	NaN	NaN	NaN	NaN	
730	NaN	NaN	NaN	NaN	NaN	NaN	
1000	NaN	NaN	NaN	NaN	NaN	NaN	
1090	NaN	NaN	NaN	NaN	NaN	NaN	
1120	NaN	NaN	NaN	NaN	NaN	NaN	
...	...	...	...	...	...	...	
2014	9735.0000	47867.0000	41338.0000	17439.0000	748.0000	...	
2015	10042.0000	48357.0000	41294.0000	17460.0000	694.0000	...	
2016	10080.0000	48845.0000	41445.0000	16645.0000	665.0000	...	
2017	10859.3783	49265.6135	42177.3706	16522.3072	671.3169	...	
2018	11454.4251	49830.7993	42988.0709	16628.0553	651.3589	...	

countrycode	URY	USA	UZB	VEN	VNM	\
year						
1	NaN	NaN	NaN	NaN	NaN	
730	NaN	NaN	NaN	NaN	NaN	

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1000	NaN	NaN	NaN	NaN	NaN
1090	NaN	NaN	NaN	NaN	NaN
1120	NaN	NaN	NaN	NaN	NaN
...	...	...	...	...	...
2014	19160.0000	51664.0000	9085.0000	20317.0000	5455.0000
2015	19244.0000	52591.0000	9720.0000	18802.0000	5763.0000
2016	19468.0000	53015.0000	10381.0000	15219.0000	6062.0000
2017	19918.1361	54007.7698	10743.8666	12879.1350	6422.0865
2018	20185.8360	55334.7394	11220.3702	10709.9506	6814.1423
countrycode	YEM	YUG	ZAF	ZMB	ZWE
year					
1	NaN	NaN	NaN	NaN	NaN
730	NaN	NaN	NaN	NaN	NaN
1000	NaN	NaN	NaN	NaN	NaN
1090	NaN	NaN	NaN	NaN	NaN
1120	NaN	NaN	NaN	NaN	NaN
...	...	...	...	...	...
2014	4054.0000	14627.0000	12242.0000	3478.0000	1594.0000
2015	2844.0000	14971.0000	12246.0000	3478.0000	1560.0000
2016	2506.0000	15416.0000	12139.0000	3479.0000	1534.0000
2017	2321.9239	15960.8432	12189.3579	3497.5818	1582.3662
2018	2284.8899	16558.3123	12165.7948	3534.0337	1611.4052

[772 rows x 169 columns]

We create a color mapping between country codes and colors for consistency

## 2.3 GDP plots

Looking at the United Kingdom we can first confirm we are using the correct country code

```
fig, ax = plt.subplots(dpi=300)
cntry = 'GBR'
_ = gdppc[cntry].plot(
    ax = fig.gca(),
    ylabel = 'International \$\'s',
    xlabel = 'Year',
    linestyle='--',
    color=color_mapping['GBR'])
```

**Note:** International Dollars are a hypothetical unit of currency that has the same purchasing power parity that the U.S. Dollar has in the United States at any given time. They are also known as Geary–Khamis dollars (GK Dollars).

We can see that the data is non-continuous for longer periods in the early 250 years of this millennium, so we could choose to interpolate to get a continuous line plot.

Here we use dashed lines to indicate interpolated trends

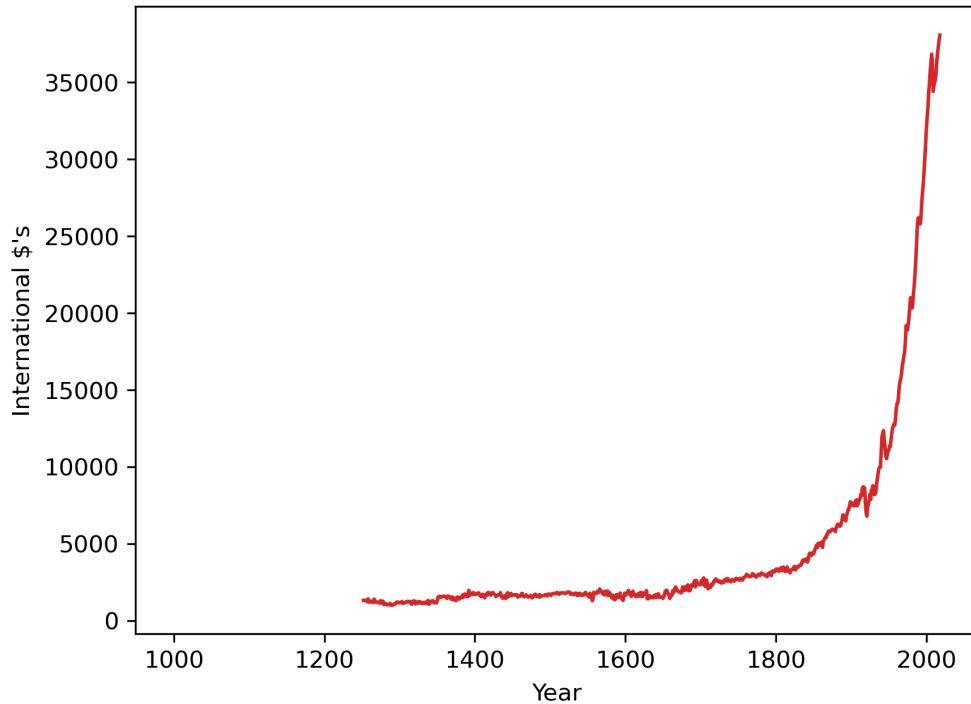


Fig. 2.1: GDP per Capita (GBR)

```
fig, ax = plt.subplots(dpi=300)
cntry = 'GBR'
ax.plot(gdppc[cntry].interpolate(),
         linestyle='--',
         lw=2,
         color=color_mapping[cntry])

ax.plot(gdppc[cntry],
         linestyle='-' ,
         lw=2,
         color=color_mapping[cntry])
ax.set_ylabel('International $\'s')
ax.set_xlabel('Year')
plt.show()
```

We can now put this into a function to generate plots for a list of countries

```
def draw_interp_plots(series, ylabel, xlabel, color_mapping, code_to_name, lw,_
                      logscale, ax):

    for i, c in enumerate(cntry):
        # Get the interpolated data
        df_interpolated = series[c].interpolate(limit_area='inside')
        interpolated_data = df_interpolated[series[c].isnull()]

        # Plot the interpolated data with dashed lines
```

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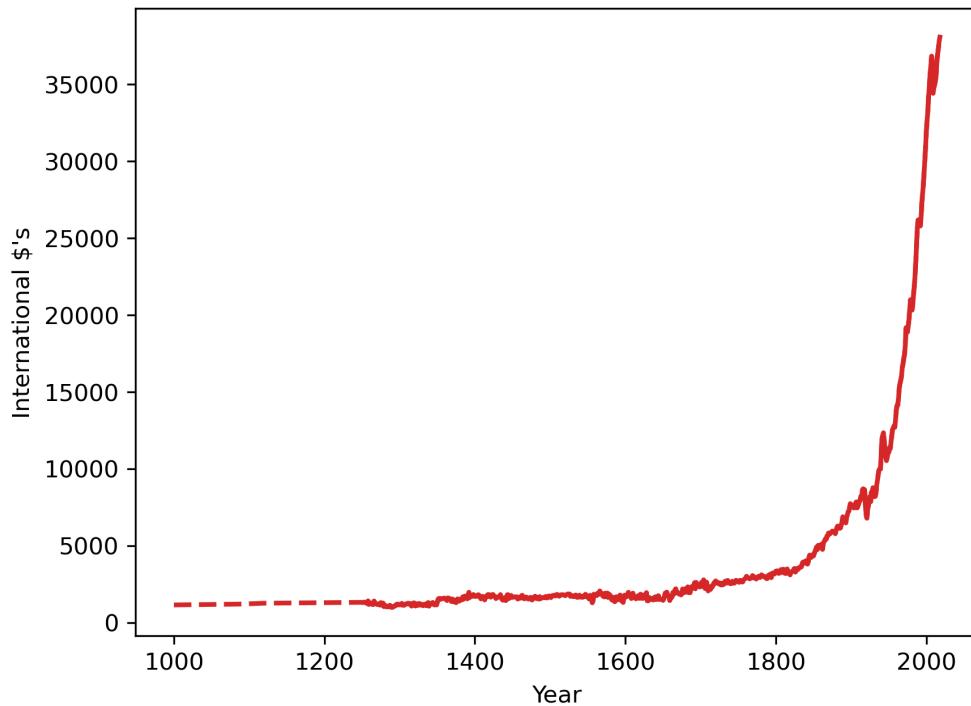


Fig. 2.2: GDP per Capita (GBR)

(continued from previous page)

```

ax.plot(interpolated_data,
        linestyle='--',
        lw=lw,
        alpha=0.7,
        color=color_mapping[c])

# Plot the non-interpolated data with solid lines
ax.plot(series[c],
        linestyle='--',
        lw=lw,
        color=color_mapping[c],
        alpha=0.8,
        label=code_to_name.loc[c]['country'])

if logscale == True:
    ax.set_yscale('log')

# Draw the legend outside the plot
ax.legend(loc='center left', bbox_to_anchor=(1, 0.5), frameon=False)
ax.set_ylabel(ylabel)
ax.set_xlabel(xlabel)

return ax

```

As you can see from this chart, economic growth started in earnest in the 18th century and continued for the next two hundred years.

How does this compare with other countries' growth trajectories?

Let's look at the United States (USA), United Kingdom (GBR), and China (CHN)

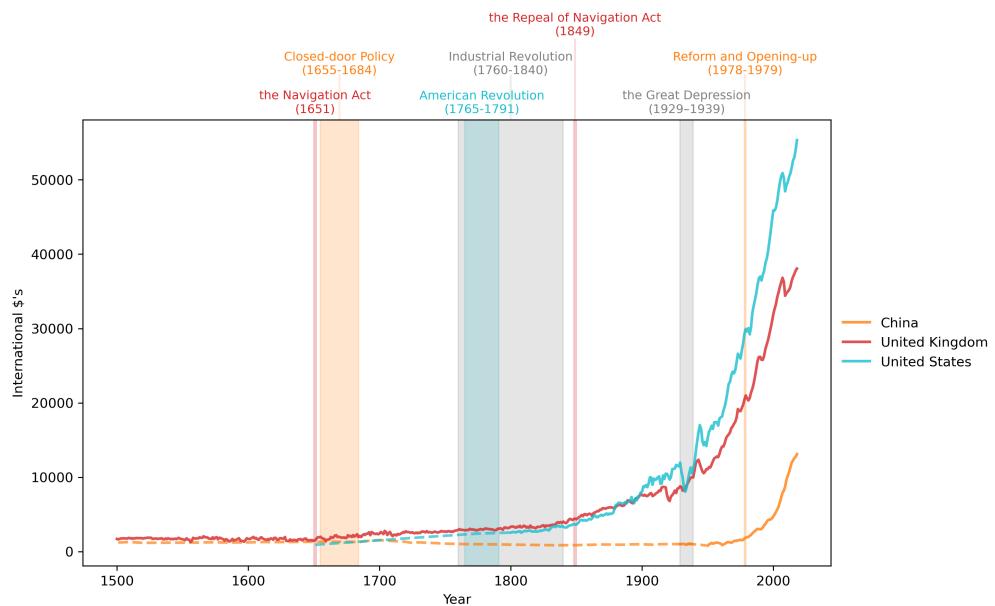


Fig. 2.3: GDP per Capita, 1500- (China, UK, USA)

The preceding graph of per capita GDP strikingly reveals how the spread of the industrial revolution has over time gradually lifted the living standards of substantial groups of people

- most of the growth happened in the past 150 years after the industrial revolution.
- per capita GDP in the US and UK rose and diverged from that of China from 1820 to 1940.
- the gap has closed rapidly after 1950 and especially after the late 1970s.
- these outcomes reflect complicated combinations of technological and economic-policy factors that students of economic growth try to understand and quantify.

It is fascinating to see China's GDP per capita levels from 1500 through to the 1970s.

Notice the long period of declining GDP per capital levels from the 1700s until the early 20th century.

Thus, the graph indicates

- a long economic downturn and stagnation after the Closed-door Policy by the Qing government.
- China's very different experience than the UK's after the onset of the industrial revolution in the UK.
- how the Self-Strengthening Movement seemed mostly to help China to grow.
- how stunning have been the growth achievements of modern Chinese economic policies by the PRC that culminated with its late 1970s reform and liberalization.

We can also look at the United States (USA) and United Kingdom (GBR) in more detail

In the following graph, please watch for

- impact of trade policy (Navigation Act).
- productivity changes brought by the industrial revolution.

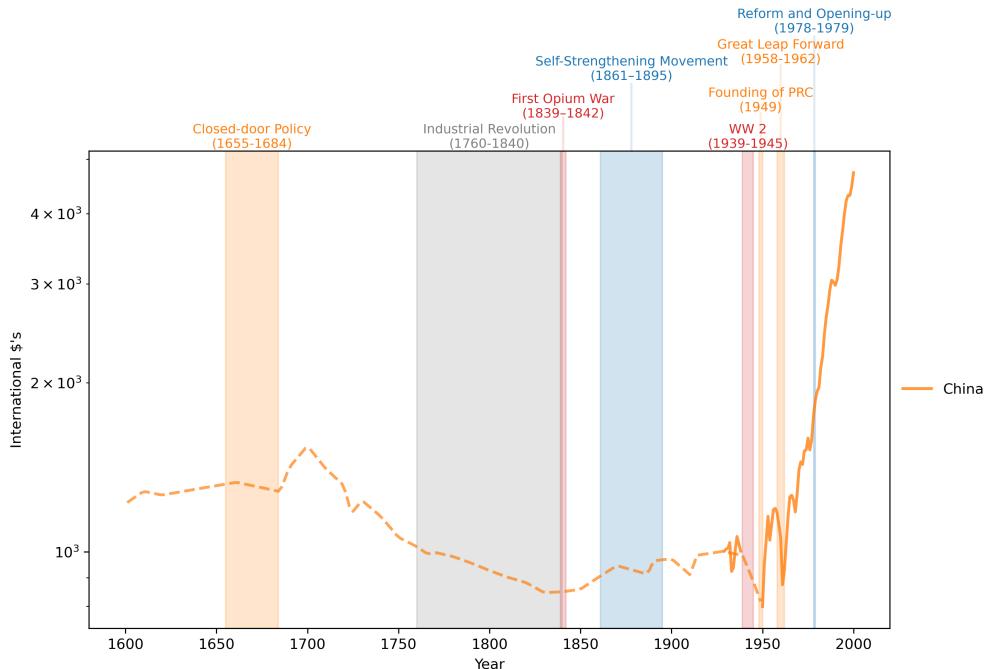


Fig. 2.4: GDP per Capita, 1500-2000 (China)

- how the US gradually approaches and then surpasses the UK, setting the stage for the “American Century”.
- the often unanticipated consequences of wars.
- interruptions and scars left by *business cycle* recessions and depressions.

## 2.4 The industrialized world

Now we'll construct some graphs of interest to geopolitical historians like Adam Tooze.

We'll focus on total Gross Domestic Product (GDP) (as a proxy for “national geopolitical-military power”) rather than focusing on GDP per capita (as a proxy for living standards).

```
data = pd.read_excel("datasets/mpd2020.xlsx", sheet_name='Full data')
data.set_index(['countrycode', 'year'], inplace=True)
data['gdp'] = data['gdppc'] * data['pop']
gdp = data['gdp'].unstack('countrycode')
```

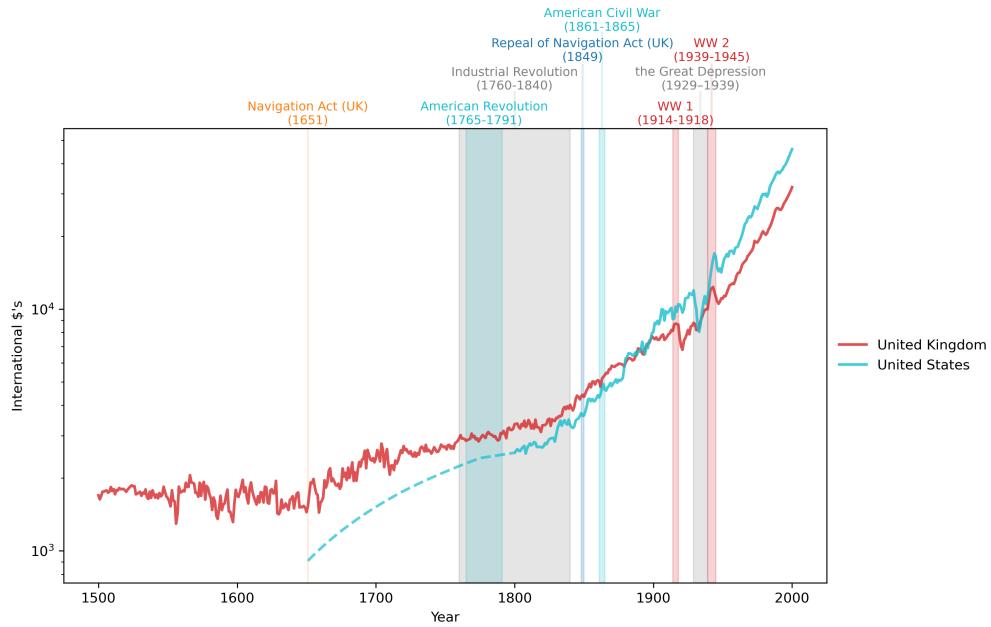


Fig. 2.5: GDP per Capita, 1500-2000 (UK and US)

#### 2.4.1 Early industrialization (1820 to 1940)

We first visualize the trend of China, the Former Soviet Union, Japan, the UK and the US.

The most notable trend is the rise of the US, surpassing the UK in the 1860s and China in the 1880s.

The growth continued until the large dip in the 1930s when the Great Depression hit.

Meanwhile, Russia experienced significant setbacks during World War I and recovered significantly after the February Revolution.

```
fig, ax = plt.subplots(dpi=300)
ax = fig.gca()
cntry = ['CHN', 'SUN', 'JPN', 'GBR', 'USA']
start_year, end_year = (1820, 1945)
ax = draw_interp_plots(gdp[cntry].loc[start_year:end_year],
    'International $\'s', 'Year',
    color_mapping, code_to_name, 2, False, ax)
```

## 2.5 Constructing a plot similar to Tooke's

In this section we describe how we have constructed a version of the striking figure from chapter 1 of [Too14] that we discussed at the start of this lecture.

Let's first define a collection of countries that consist of the British Empire (BEM) so we can replicate that series in Tooke's chart.

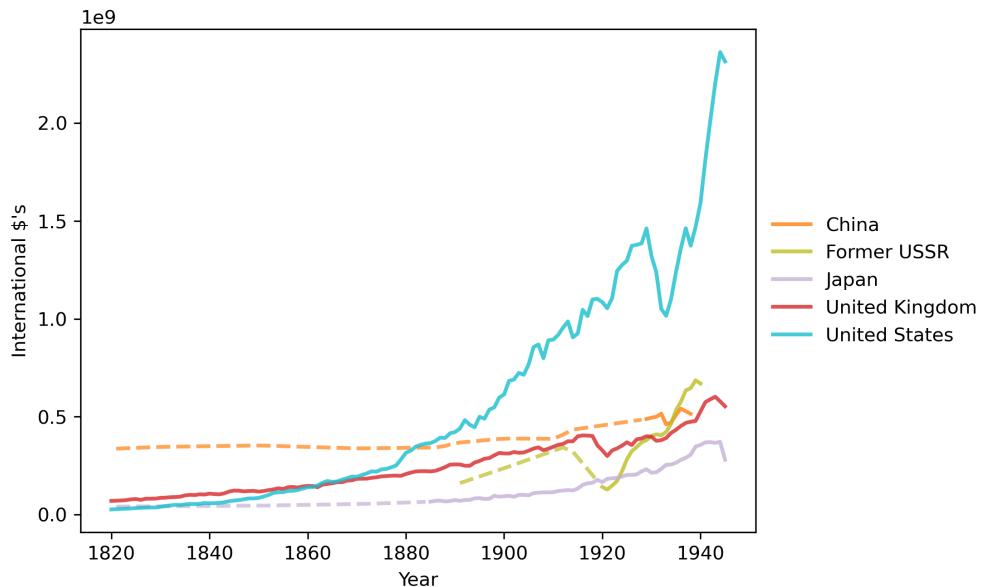


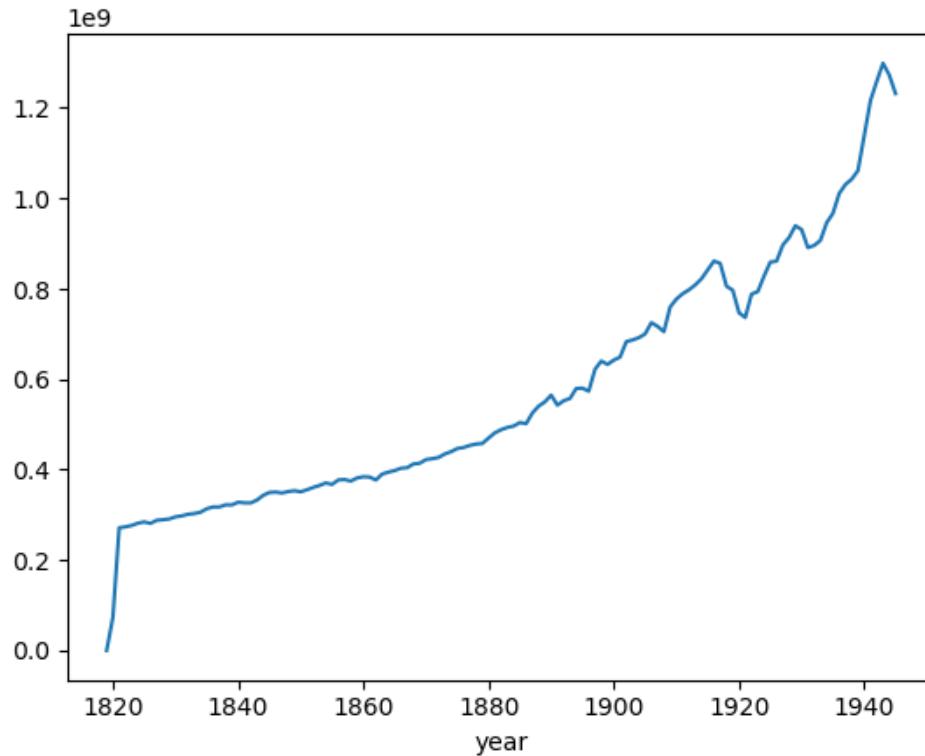
Fig. 2.6: GDP in the early industrialization era

```
BEM = ['GBR', 'IND', 'AUS', 'NZL', 'CAN', 'ZAF']
gdp['BEM'] = gdp[BEM].loc[start_year-1:end_year].interpolate(method='index').
    sum(axis=1) # Interpolate incomplete time-series
```

Let's take a look at the aggregation that represents the British Empire.

```
gdp['BEM'].plot() # The first year is np.nan due to interpolation
```

```
<Axes: xlabel='year'>
```



```
code_to_name
```

countrycode	country
AFG	Afghanistan
AGO	Angola
ALB	Albania
ARE	United Arab Emirates
ARG	Argentina
...	...
YEM	Yemen
YUG	Former Yugoslavia
ZAF	South Africa
ZMB	Zambia
ZWE	Zimbabwe

[169 rows x 1 columns]

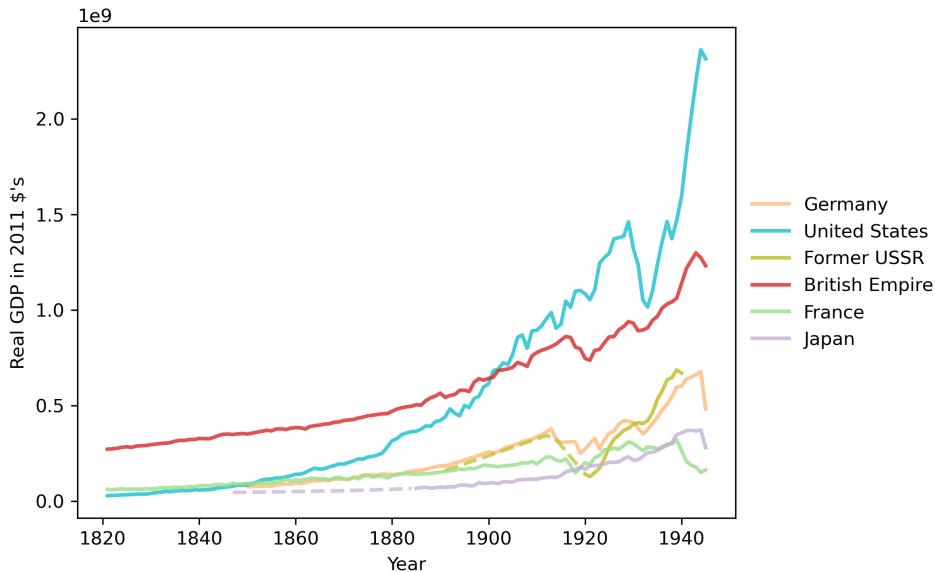
Now let's assemble our series and get ready to plot them.

```
# Define colour mapping and name for BEM
color_mapping['BEM'] = color_mapping['GBR'] # Set the color to be the same as Great Britain
# Add British Empire to code_to_name
bem = pd.DataFrame(["British Empire"], index=["BEM"], columns=['country'])
bem.index.name = 'countrycode'
code_to_name = pd.concat([code_to_name, bem])
```

```

fig, ax = plt.subplots(dpi=300)
ax = fig.gca()
cntry = ['DEU', 'USA', 'SUN', 'BEM', 'FRA', 'JPN']
start_year, end_year = (1821, 1945)
ax = draw_interp_plots(gdp[cntry].loc[start_year:end_year],
    'Real GDP in 2011 \$\'s', 'Year',
    color_mapping, code_to_name, 2, False, ax)
plt.savefig("./_static/lecture_specific/long_run_growth/tooze_ch1_graph.png", dpi=300,
    bbox_inches='tight')
plt.show()

```



At the start of this lecture, we noted how US GDP came from “nowhere” at the start of the 19th century to rival and then overtake the GDP of the British Empire by the end of the 19th century, setting the geopolitical stage for the “American (twentieth) century”.

Let’s move forward in time and start roughly where Tooze’s graph stopped after World War II.

In the spirit of Tooze’s chapter 1 analysis, doing this will provide some information about geopolitical realities today.

### 2.5.1 The modern era (1950 to 2020)

The following graph displays how quickly China has grown, especially since the late 1970s.

```

fig, ax = plt.subplots(dpi=300)
ax = fig.gca()
cntry = ['CHN', 'SUN', 'JPN', 'GBR', 'USA']
start_year, end_year = (1950, 2020)
ax = draw_interp_plots(gdp[cntry].loc[start_year:end_year],
    'International \$\'s', 'Year',
    color_mapping, code_to_name, 2, False, ax)

```

It is tempting to compare this graph with figure Fig. 2.6 that showed the US overtaking the UK near the start of the “American Century”, a version of the graph featured in chapter 1 of [Too14].

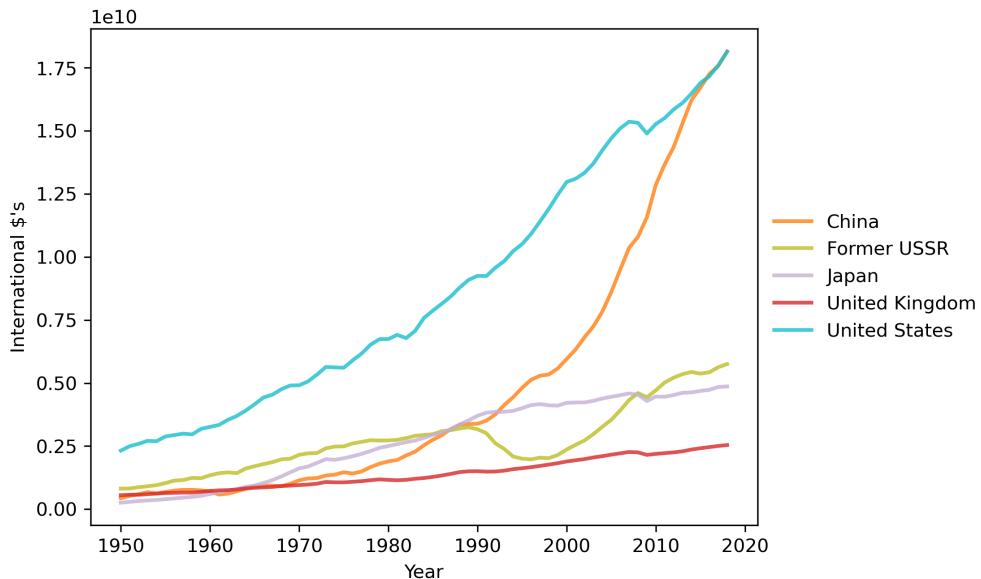


Fig. 2.7: GDP in the modern era

## 2.6 Regional analysis

We often want to study historical experiences of countries outside the club of “World Powers”.

Fortunately, the [Maddison Historical Statistics](#) dataset also includes regional aggregations

```
data = pd.read_excel("datasets/mpd2020.xlsx", sheet_name='Regional data', header=(0,1,
                                                               2), index_col=0)
data.columns = data.columns.droplevel(level=2)
```

We can save the raw data in a more convenient format to build a single table of regional GDP per capita

```
regionalgdppc = data['gdppc_2011'].copy()
regionalgdppc.index = pd.to_datetime(regionalgdppc.index, format='%Y')
```

Let's interpolate based on time to fill in any gaps in the dataset for the purpose of plotting

```
regionalgdppc.interpolate(method='time', inplace=True)
```

and record a dataset of world GDP per capita

```
worldgdppc = regionalgdppc['World GDP pc']
```

```
fig = plt.figure(dpi=300)
ax = fig.gca()
ax = worldgdppc.plot(
    ax = ax,
    xlabel='Year',
    ylabel='2011 US$',
)
```

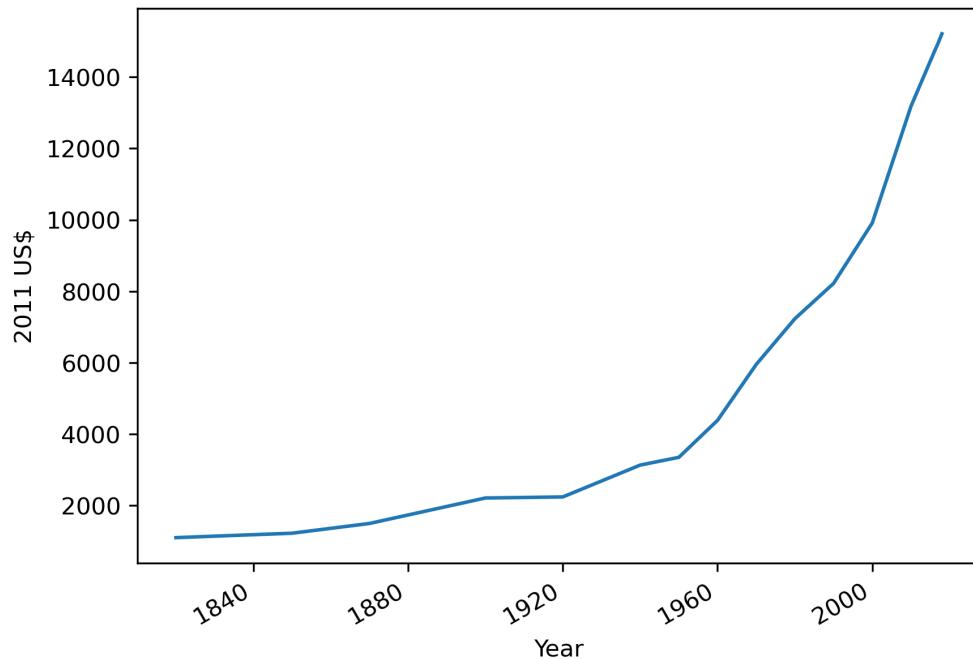


Fig. 2.8: World GDP per capita

Looking more closely, let's compare the time series for Western Offshoots and Sub-Saharan Africa and more broadly at a number of different regions around the world.

Again we see the divergence of the West from the rest of the world after the industrial revolution and the convergence of the world after the 1950s

```
fig = plt.figure(dpi=300)
ax = fig.gca()
line_styles = [':', '--', ':', '-.', '.', 'o', '-', '--', '-']
ax = regionalgdppc.plot(ax = ax, style=line_styles)
ax.set_yscale('log')
plt.legend(loc='lower center',
ncol=3, bbox_to_anchor=[0.5, -0.4])
plt.show()
```

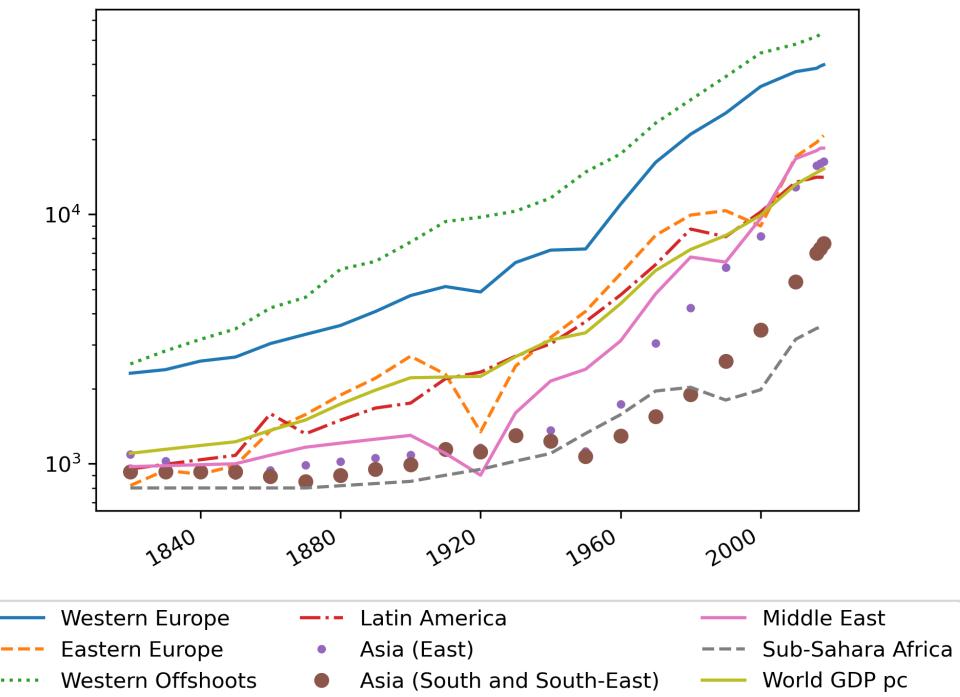


Fig. 2.9: Regional GDP per capita

## BUSINESS CYCLES

### 3.1 Overview

In this lecture we review some empirical aspects of business cycles.

Business cycles are fluctuations in economic activity over time.

These include expansions (also called booms) and contractions (also called recessions).

For our study, we will use economic indicators from the [World Bank](#) and [FRED](#).

In addition to the packages already installed by Anaconda, this lecture requires

```
!pip install wbgapi
!pip install pandas-datareader
```

We use the following imports

```
import matplotlib.pyplot as plt
import pandas as pd
import numpy as np
import scipy.stats as st
import datetime
import wbgapi as wb
import pandas_datareader.data as web
```

Here's some minor code to help with colors in our plots.

### 3.2 Data acquisition

We will use the World Bank's data API `wbgapi` and `pandas_datareader` to retrieve data.

We can use `wb.series.info` with the argument `q` to query available data from the [World Bank](#).

For example, let's retrieve the GDP growth data ID to query GDP growth data.

```
wb.series.info(q='GDP growth')
```

id	value
NY.GDP.MKTP.KD.ZG	GDP growth (annual %)
	1 elements

Now we use this series ID to obtain the data.

```
gdp_growth = wb.data.DataFrame('NY.GDP.MKTP.KD.ZG',
                               ['USA', 'ARG', 'GBR', 'GRC', 'JPN'],
                               labels=True)
gdp_growth
```

		Country	YR1960	YR1961	YR1962	YR1963	YR1964	\	
economy									
JPN	Japan	NaN	12.043536	8.908973	8.473642	11.676708			
GRC	Greece	NaN	13.203841	0.364811	11.844868	9.409677			
GBR	United Kingdom	NaN	2.677119	1.102910	4.874384	5.533659			
ARG	Argentina	NaN	5.427843	-0.852022	-5.308197	10.130298			
USA	United States	NaN	2.300000	6.100000	4.400000	5.800000			
		YR1965	YR1966	YR1967	YR1968	...	YR2013	YR2014	\
economy									
JPN	5.819708	10.638562	11.082142	12.882468	...	2.005100	0.296206		
GRC	10.768011	6.494502	5.669485	7.203719	...	-2.515997	0.475696		
GBR	2.142177	1.573100	2.786475	5.441083	...	1.819863	3.199703		
ARG	10.569433	-0.659726	3.191997	4.822501	...	2.405324	-2.512615		
USA	6.400000	6.500000	2.500000	4.800000	...	1.841875	2.287776		
		YR2015	YR2016	YR2017	YR2018	YR2019	YR2020	\	
economy									
JPN	1.560627	0.753827	1.675332	0.643391	-0.402169	-4.278604			
GRC	-0.196088	-0.487173	1.092149	1.668429	1.884342	-9.004044			
GBR	2.393103	2.165206	2.443570	1.705021	1.604309	-11.030858			
ARG	2.731160	-2.080328	2.818503	-2.617396	-2.000861	-9.943235			
USA	2.706370	1.667472	2.241921	2.945385	2.294439	-2.767803			
		YR2021	YR2022						
economy									
JPN	2.142487	1.028625							
GRC	8.434426	5.913708							
GBR	7.597471	4.101621							
ARG	10.398249	5.243044							
USA	5.945485	2.061593							

[5 rows x 64 columns]

We can look at the series' metadata to learn more about the series (click to expand).

```
wb.series.metadata.get('NY.GDP.MKTP.KD.ZG')
```

### 3.3 GDP growth rate

First we look at GDP growth.

Let's source our data from the World Bank and clean it.

```
# Use the series ID retrieved before
gdp_growth = wb.data.DataFrame('NY.GDP.MKTP.KD.ZG',
                               ['USA', 'ARG', 'GBR', 'GRC', 'JPN'],
```

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```

    labels=True)
gdp_growth = gdp_growth.set_index('Country')
gdp_growth.columns = gdp_growth.columns.str.replace('YR', '').astype(int)

```

Here's a first look at the data

```
gdp_growth
```

	1960	1961	1962	1963	1964	1965	\
Country							
Japan	NaN	12.043536	8.908973	8.473642	11.676708	5.819708	
Greece	NaN	13.203841	0.364811	11.844868	9.409677	10.768011	
United Kingdom	NaN	2.677119	1.102910	4.874384	5.533659	2.142177	
Argentina	NaN	5.427843	-0.852022	-5.308197	10.130298	10.569433	
United States	NaN	2.300000	6.100000	4.400000	5.800000	6.400000	
	1966	1967	1968	1969	...	2013	\
Country							
Japan	10.638562	11.082142	12.882468	12.477895	...	2.005100	
Greece	6.494502	5.669485	7.203719	11.563667	...	-2.515997	
United Kingdom	1.573100	2.786475	5.441083	1.924097	...	1.819863	
Argentina	-0.659726	3.191997	4.822501	9.679526	...	2.405324	
United States	6.500000	2.500000	4.800000	3.100000	...	1.841875	
	2014	2015	2016	2017	2018	2019	\
Country							
Japan	0.296206	1.560627	0.753827	1.675332	0.643391	-0.402169	
Greece	0.475696	-0.196088	-0.487173	1.092149	1.668429	1.884342	
United Kingdom	3.199703	2.393103	2.165206	2.443570	1.705021	1.604309	
Argentina	-2.512615	2.731160	-2.080328	2.818503	-2.617396	-2.000861	
United States	2.287776	2.706370	1.667472	2.241921	2.945385	2.294439	
	2020	2021	2022				
Country							
Japan	-4.278604	2.142487	1.028625				
Greece	-9.004044	8.434426	5.913708				
United Kingdom	-11.030858	7.597471	4.101621				
Argentina	-9.943235	10.398249	5.243044				
United States	-2.767803	5.945485	2.061593				

[5 rows x 63 columns]

We write a function to generate plots for individual countries taking into account the recessions.

Let's start with the United States.

```

fig, ax = plt.subplots()

country = 'United States'
ylabel = 'GDP growth rate (%)'
plot_series(gdp_growth, country,
            ylabel, 0.1, ax,
            g_params, b_params, t_params)
plt.show()

```

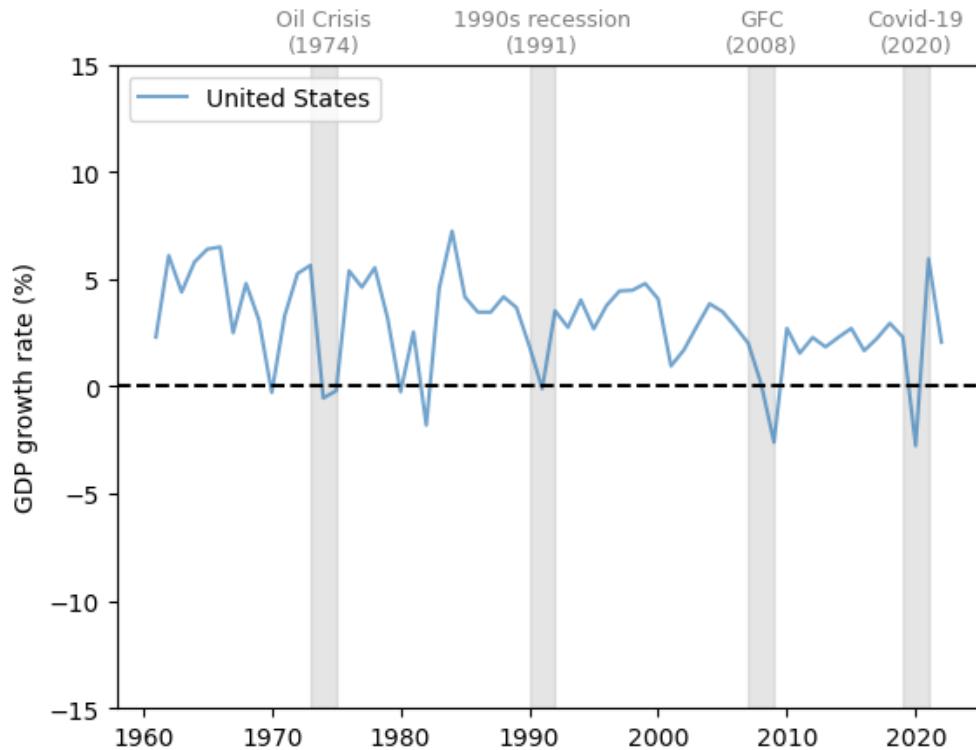


Fig. 3.1: United States (GDP growth rate %)

GDP growth is positive on average and trending slightly downward over time.

We also see fluctuations over GDP growth over time, some of which are quite large.

Let's look at a few more countries to get a basis for comparison.

The United Kingdom (UK) has a similar pattern to the US, with a slow decline in the growth rate and significant fluctuations.

Notice the very large dip during the Covid-19 pandemic.

```
fig, ax = plt.subplots()

country = 'United Kingdom'
plot_series(gdp_growth, country,
            ylabel, 0.1, ax,
            g_params, b_params, t_params)
plt.show()
```

Now let's consider Japan, which experienced rapid growth in the 1960s and 1970s, followed by slowed expansion in the past two decades.

Major dips in the growth rate coincided with the Oil Crisis of the 1970s, the Global Financial Crisis (GFC) and the Covid-19 pandemic.

```
fig, ax = plt.subplots()
```

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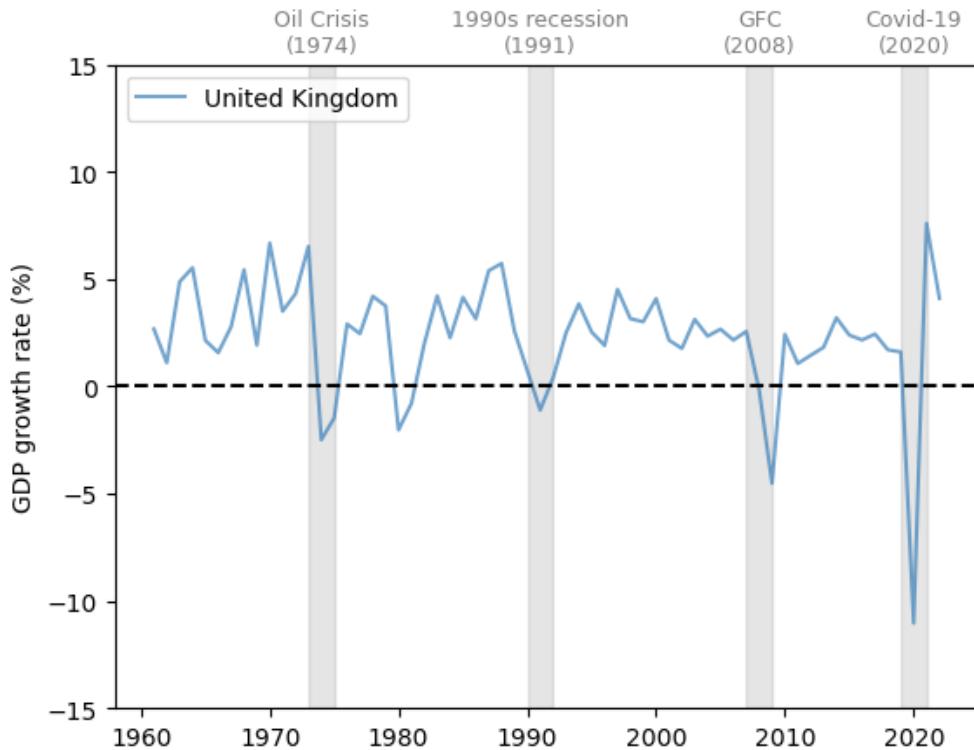


Fig. 3.2: United Kingdom (GDP growth rate %)

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```
country = 'Japan'
plot_series(gdp_growth, country,
            ylabel, 0.1, ax,
            g_params, b_params, t_params)
plt.show()
```

Now let's study Greece.

```
fig, ax = plt.subplots()

country = 'Greece'
plot_series(gdp_growth, country,
            ylabel, 0.1, ax,
            g_params, b_params, t_params)
plt.show()
```

Greece experienced a very large drop in GDP growth around 2010-2011, during the peak of the Greek debt crisis.

Next let's consider Argentina.

```
fig, ax = plt.subplots()
```

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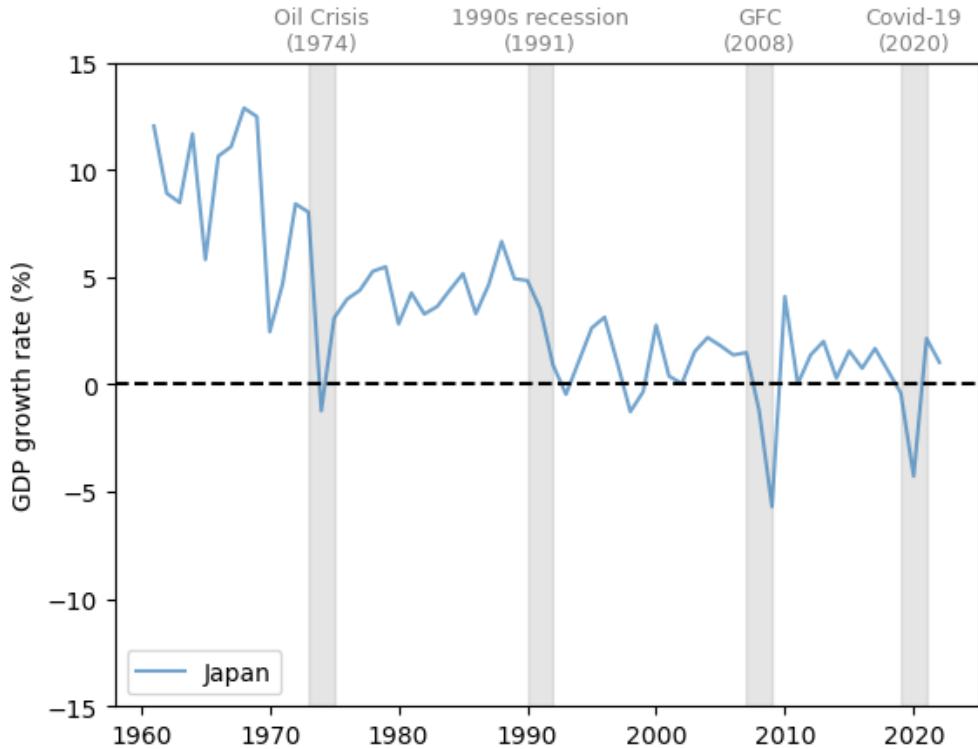


Fig. 3.3: Japan (GDP growth rate %)

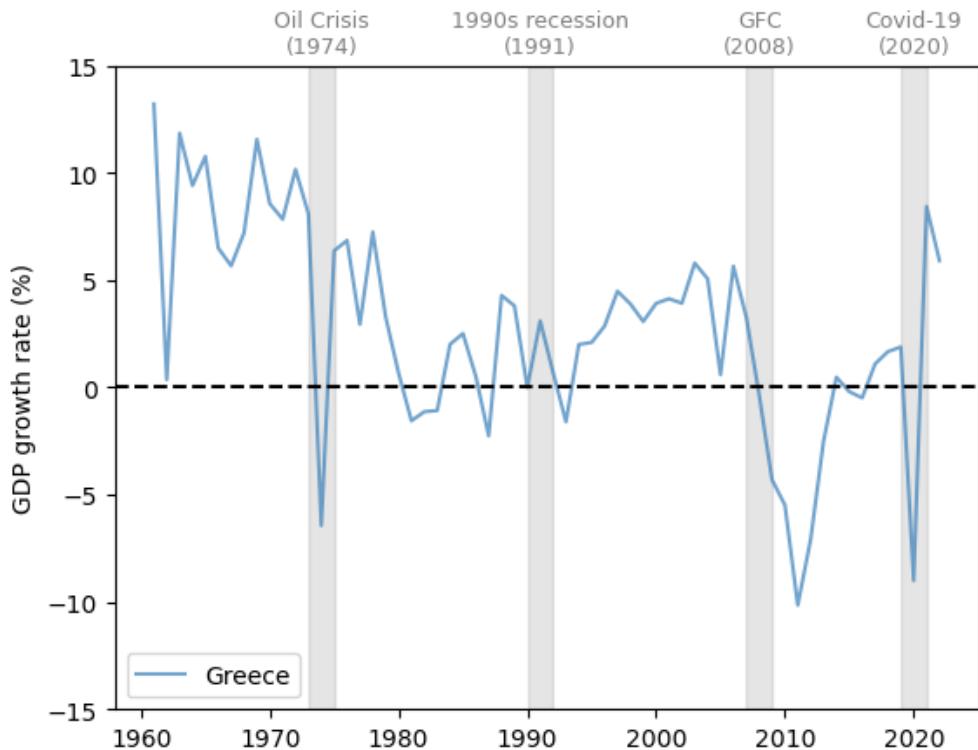


Fig. 3.4: Greece (GDP growth rate %)

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```
country = 'Argentina'
plot_series(gdp_growth, country,
            ylabel, 0.1, ax,
            g_params, b_params, t_params)
plt.show()
```

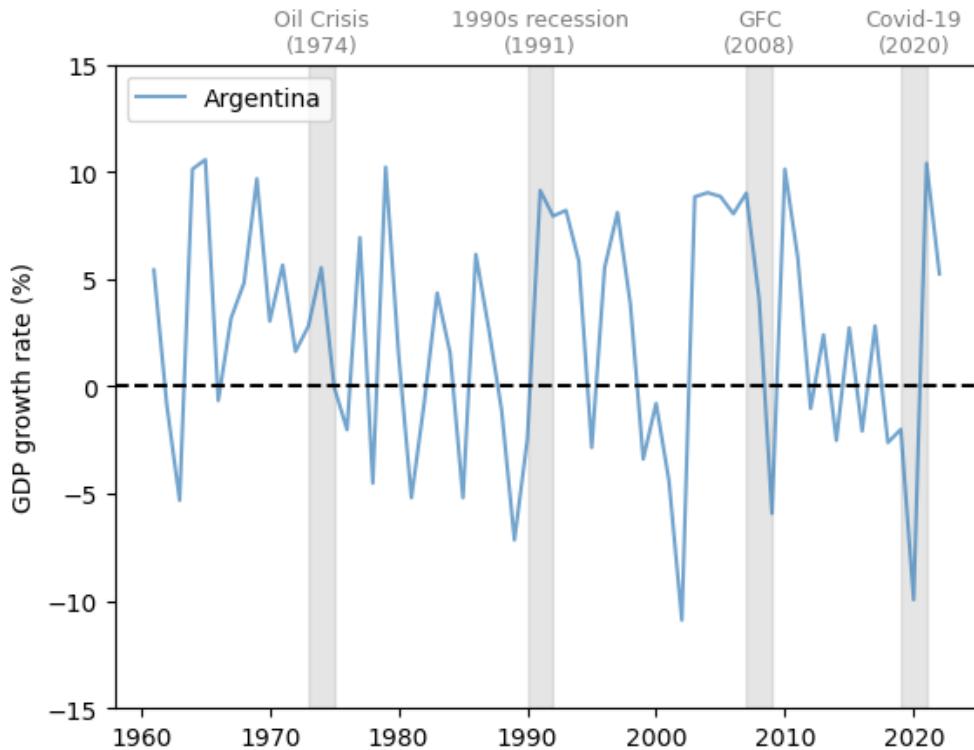


Fig. 3.5: Argentina (GDP growth rate %)

Notice that Argentina has experienced far more volatile cycles than the economies examined above.

At the same time, Argentina's growth rate did not fall during the two developed economy recessions in the 1970s and 1990s.

## 3.4 Unemployment

Another important measure of business cycles is the unemployment rate.

We study unemployment using rate data from FRED spanning from 1929-1942 to 1948-2022, combined unemployment rate data over 1942-1948 estimated by the [Census Bureau](#).

Let's plot the unemployment rate in the US from 1929 to 2022 with recessions defined by the NBER.

The plot shows that

- expansions and contractions of the labor market have been highly correlated with recessions.

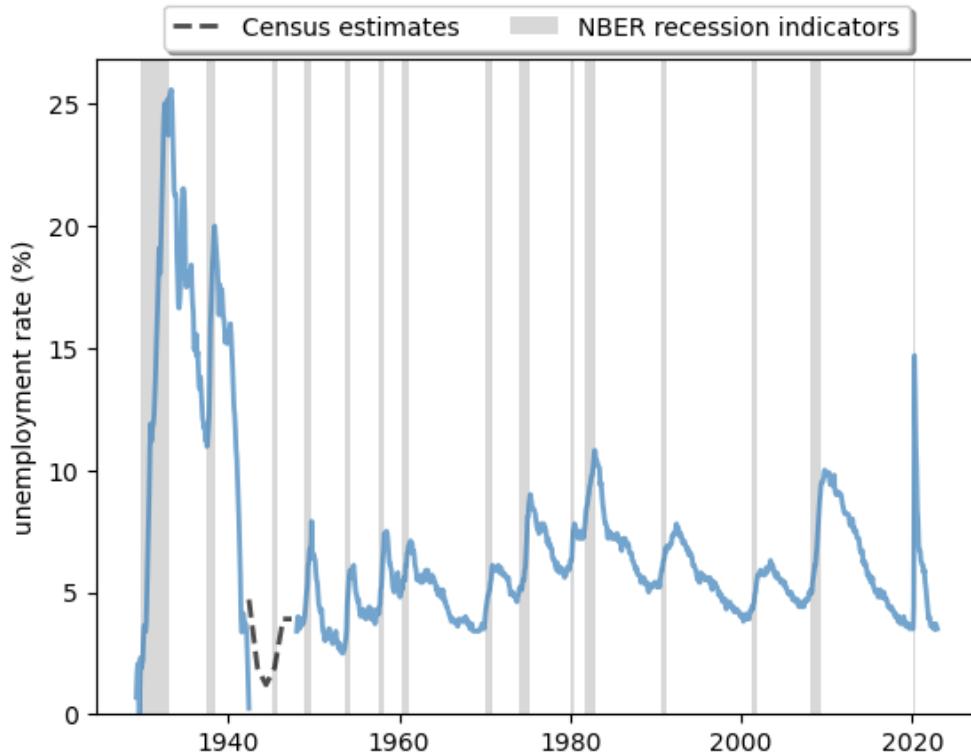


Fig. 3.6: Long-run unemployment rate, US (%)

- cycles are, in general, asymmetric: sharp rises in unemployment are followed by slow recoveries.
- It also shows us how unique labor market conditions were in the US during the post-pandemic recovery. The labor market recovered at an unprecedented rate after the shock in 2020-2021.

## 3.5 Synchronization

In our *previous discussion*, we found that developed economies have had relatively synchronized periods of recession.

At the same time, this synchronization did not appear in Argentina until the 2000s.

Let's examine this trend further.

With slight modifications, we can use our previous function to draw a plot that includes multiple countries.

Here we compare the GDP growth rate of developed economies and developing economies.

We use the United Kingdom, United States, Germany, and Japan as examples of developed economies.

We choose Brazil, China, Argentina, and Mexico as representative developing economies.

The comparison of GDP growth rates above suggests that business cycles are becoming more synchronized in 21st-century recessions.

However, emerging and less developed economies often experience more volatile changes throughout the economic cycles.

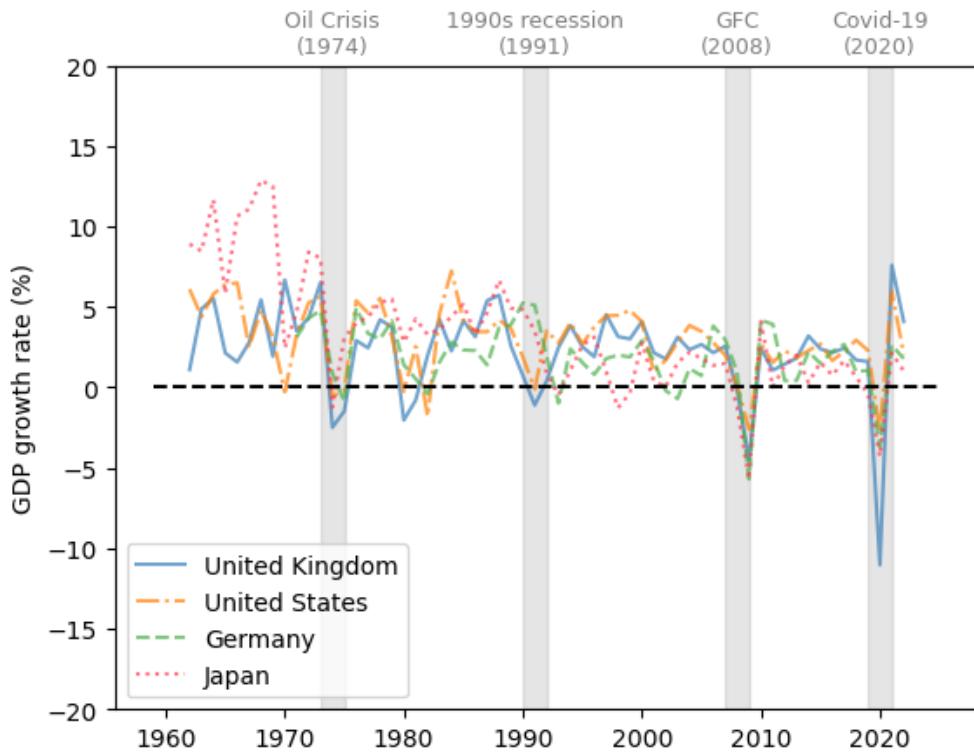


Fig. 3.7: Developed economies (GDP growth rate %)

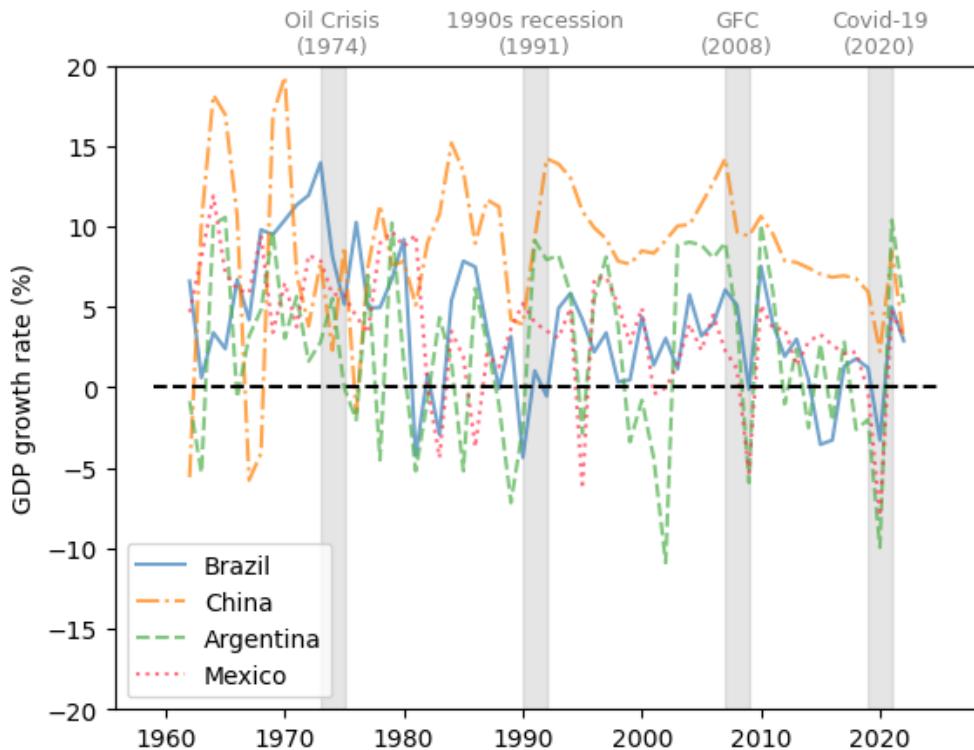


Fig. 3.8: Developing economies (GDP growth rate %)

Despite the synchronization in GDP growth, the experience of individual countries during the recession often differs.

We use the unemployment rate and the recovery of labor market conditions as another example.

Here we compare the unemployment rate of the United States, the United Kingdom, Japan, and France.

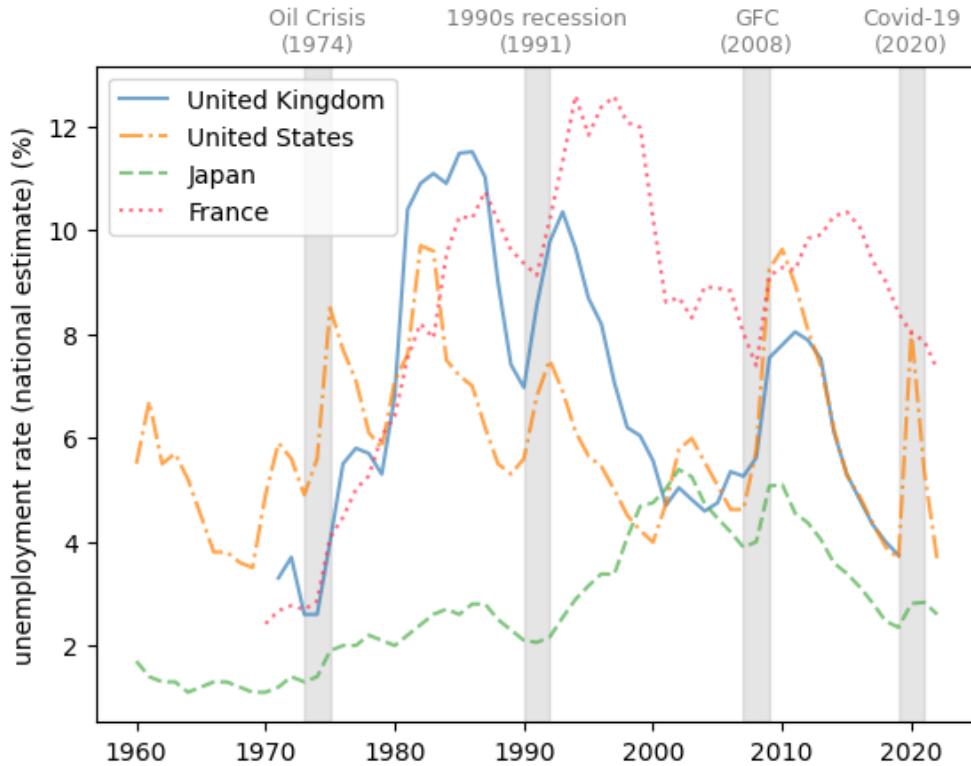


Fig. 3.9: Developed economies (unemployment rate %)

We see that France, with its strong labor unions, typically experiences relatively slow labor market recoveries after negative shocks.

We also notice that Japan has a history of very low and stable unemployment rates.

## 3.6 Leading indicators and correlated factors

Examining leading indicators and correlated factors helps policymakers to understand the causes and results of business cycles.

We will discuss potential leading indicators and correlated factors from three perspectives: consumption, production, and credit level.

### 3.6.1 Consumption

Consumption depends on consumers' confidence towards their income and the overall performance of the economy in the future.

One widely cited indicator for consumer confidence is the [consumer sentiment index](#) published by the University of Michigan.

Here we plot the University of Michigan Consumer Sentiment Index and year-on-year [core consumer price index \(CPI\)](#) change from 1978-2022 in the US.

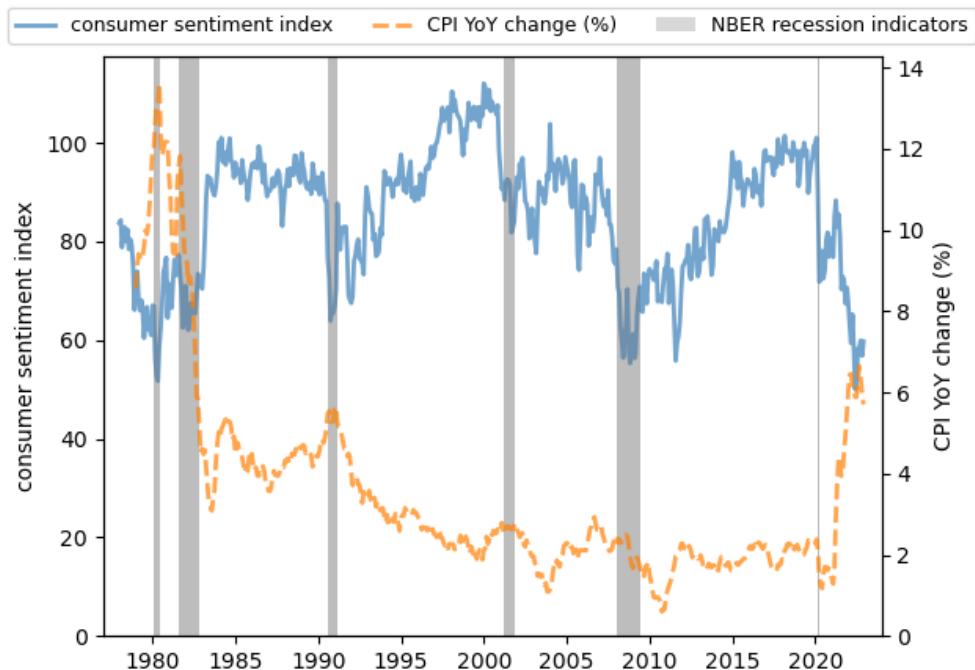


Fig. 3.10: Consumer sentiment index and YoY CPI change, US

We see that

- consumer sentiment often remains high during expansions and drops before recessions.
- there is a clear negative correlation between consumer sentiment and the CPI.

When the price of consumer commodities rises, consumer confidence diminishes.

This trend is more significant during [stagflation](#).

### 3.6.2 Production

Real industrial output is highly correlated with recessions in the economy.

However, it is not a leading indicator, as the peak of contraction in production is delayed relative to consumer confidence and inflation.

We plot the real industrial output change from the previous year from 1919 to 2022 in the US to show this trend.

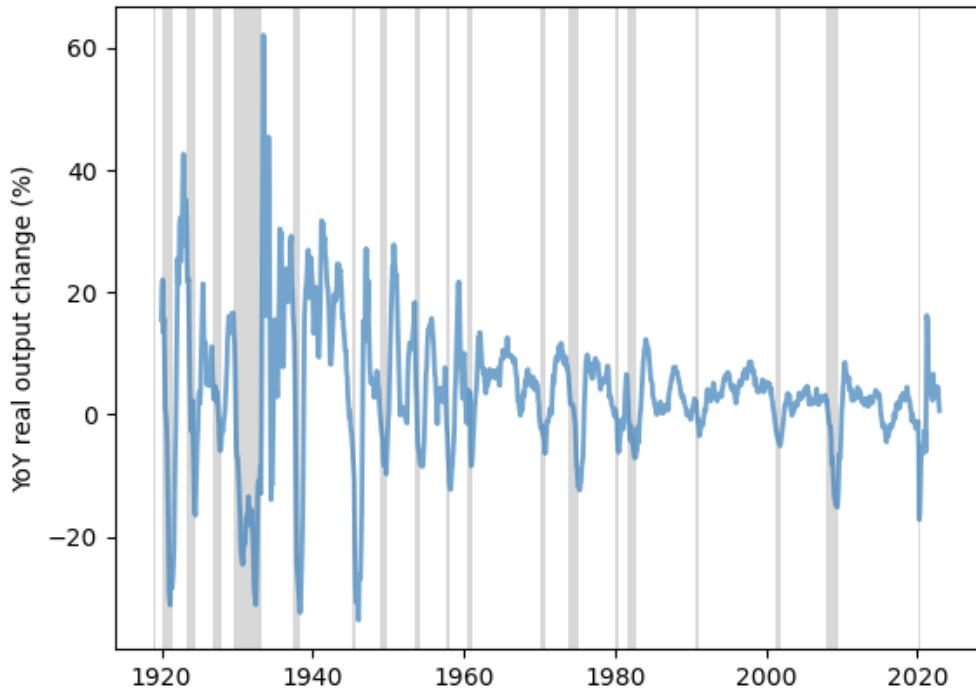


Fig. 3.11: YoY real output change, US (%)

We observe the delayed contraction in the plot across recessions.

### 3.6.3 Credit level

Credit contractions often occur during recessions, as lenders become more cautious and borrowers become more hesitant to take on additional debt.

This is due to factors such as a decrease in overall economic activity and gloomy expectations for the future.

One example is domestic credit to the private sector by banks in the UK.

The following graph shows the domestic credit to the private sector as a percentage of GDP by banks from 1970 to 2022 in the UK.

Note that the credit rises during economic expansions and stagnates or even contracts after recessions.

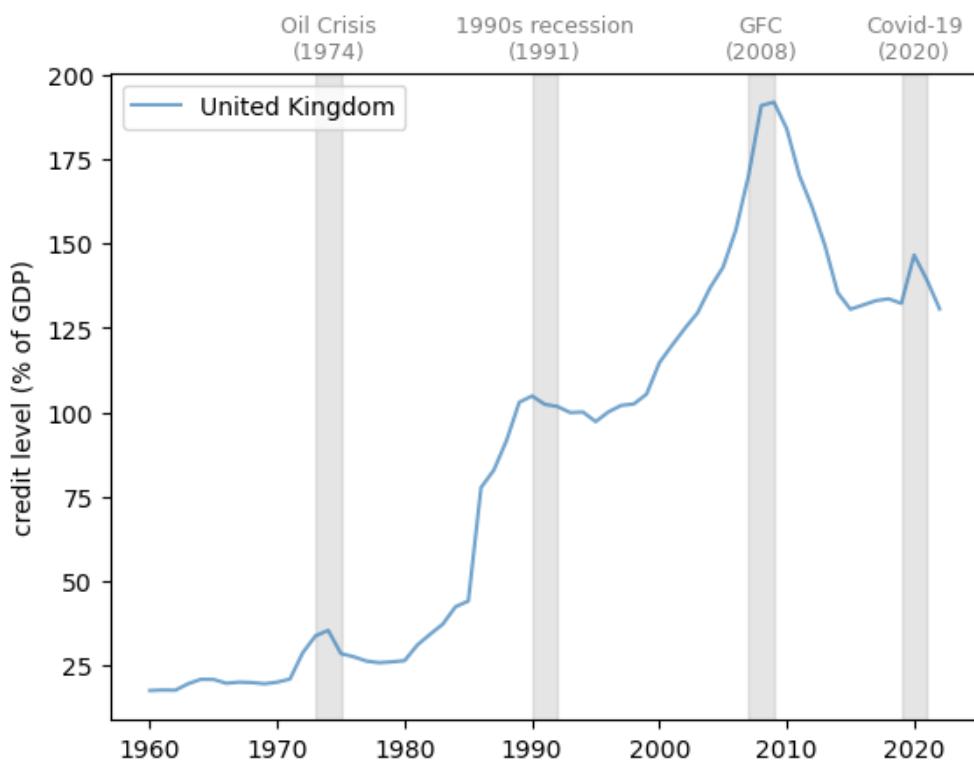


Fig. 3.12: Domestic credit to private sector by banks (% of GDP)



# INCOME AND WEALTH INEQUALITY

## 4.1 Overview

In this section we

- provide motivation for the techniques deployed in the lecture and
- import code libraries needed for our work.

### 4.1.1 Some history

Many historians argue that inequality played a key role in the fall of the Roman Republic.

After defeating Carthage and invading Spain, money flowed into Rome and greatly enriched those in power.

Meanwhile, ordinary citizens were taken from their farms to fight for long periods, diminishing their wealth.

The resulting growth in inequality caused political turmoil that shook the foundations of the republic.

Eventually, the Roman Republic gave way to a series of dictatorships, starting with Octavian (Augustus) in 27 BCE.

This history is fascinating in its own right, and we can see some parallels with certain countries in the modern world.

Many recent political debates revolve around inequality.

Many economic policies, from taxation to the welfare state, are aimed at addressing inequality.

### 4.1.2 Measurement

One problem with these debates is that inequality is often poorly defined.

Moreover, debates on inequality are often tied to political beliefs.

This is dangerous for economists because allowing political beliefs to shape our findings reduces objectivity.

To bring a truly scientific perspective to the topic of inequality we must start with careful definitions.

In this lecture we discuss standard measures of inequality used in economic research.

For each of these measures, we will look at both simulated and real data.

We will install the following libraries.

```
! pip install --upgrade quantecon interpolation
```

And we use the following imports.

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import quantecon as qe
import random as rd
from interpolation import interp
```

```
/usr/share/miniconda3/envs/quantecon/lib/python3.11/site-packages/numba/core/
decorators.py:262: NumbaDeprecationWarning: numba.generated_jit is deprecated.
Please see the documentation at: https://numba.readthedocs.io/en/stable/
reference/deprecation.html#deprecation-of-generated-jit for more information and
advice on a suitable replacement.
warnings.warn(msg, NumbaDeprecationWarning)
```

## 4.2 The Lorenz curve

One popular measure of inequality is the Lorenz curve.

In this section we define the Lorenz curve and examine its properties.

### 4.2.1 Definition

The Lorenz curve takes a sample  $w_1, \dots, w_n$  and produces a curve  $L$ .

We suppose that the sample  $w_1, \dots, w_n$  has been sorted from smallest to largest.

To aid our interpretation, suppose that we are measuring wealth

- $w_1$  is the wealth of the poorest member of the population and
- $w_n$  is the wealth of the richest member of the population.

The curve  $L$  is just a function  $y = L(x)$  that we can plot and interpret.

To create it we first generate data points  $(x_i, y_i)$  according to

$$x_i = \frac{i}{n}, \quad y_i = \frac{\sum_{j \leq i} w_j}{\sum_{j \leq n} w_j}, \quad i = 1, \dots, n$$

Now the Lorenz curve  $L$  is formed from these data points using interpolation.

(If we use a line plot in Matplotlib, the interpolation will be done for us.)

The meaning of the statement  $y = L(x)$  is that the lowest  $(100 \times x)\%$  of people have  $(100 \times y)\%$  of all wealth.

- if  $x = 0.5$  and  $y = 0.1$ , then the bottom 50% of the population owns 10% of the wealth.

In the discussion above we focused on wealth but the same ideas apply to income, consumption, etc.

## 4.2.2 Lorenz curves of simulated data

Let's look at some examples and try to build understanding.

In the next figure, we generate  $n = 2000$  draws from a lognormal distribution and treat these draws as our population.

The straight line ( $x = L(x)$  for all  $x$ ) corresponds to perfect equality.

The lognormal draws produce a less equal distribution.

For example, if we imagine these draws as being observations of wealth across a sample of households, then the dashed lines show that the bottom 80% of households own just over 40% of total wealth.

```
n = 2000
sample = np.exp(np.random.randn(n))

fig, ax = plt.subplots()

f_vals, l_vals = qe.lorenz_curve(sample)
ax.plot(f_vals, l_vals, label='lognormal sample', lw=2)
ax.plot(f_vals, f_vals, label='equality', lw=2)

ax.legend(fontsize=12)

ax.vlines([0.8], [0.0], [0.43], alpha=0.5, colors='k', ls='--')
ax.hlines([0.43], [0], [0.8], alpha=0.5, colors='k', ls='--')

ax.set_xlim((0, 1))
ax.set_ylim((0, 1))

plt.show()
```

## 4.2.3 Lorenz curves for US data

Next let's look at the real data, focusing on income and wealth in the US in 2016.

The following code block imports a subset of the dataset SCF\_plus, which is derived from the Survey of Consumer Finances (SCF).

```
url = 'https://media.githubusercontent.com/media/QuantEcon/high_dim_data/main/SCF_'
      ↪plus/SCF_plus_mini.csv'
df = pd.read_csv(url)
df = df.dropna()
df_income_wealth = df
```

```
df_income_wealth.head()
```

	year	n_wealth	t_income	l_income	weights	nw_groups	ti_groups
0	1950	266933.75	55483.027	0.0	0.998732	50–90%	50–90%
1	1950	87434.46	55483.027	0.0	0.998732	50–90%	50–90%
2	1950	795034.94	55483.027	0.0	0.998732	Top 10%	50–90%
3	1950	94531.78	55483.027	0.0	0.998732	50–90%	50–90%
4	1950	166081.03	55483.027	0.0	0.998732	50–90%	50–90%

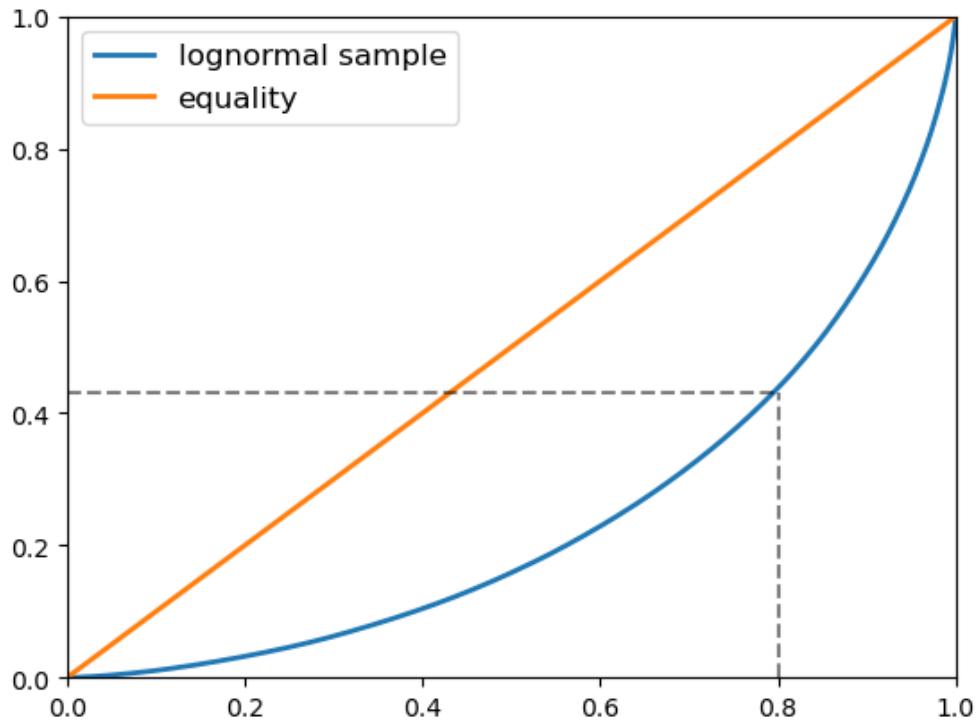


Fig. 4.1: Lorenz curve of simulated data

The following code block uses data stored in dataframe `df_income_wealth` to generate the Lorenz curves.

(The code is somewhat complex because we need to adjust the data according to population weights supplied by the SCF.)

Now we plot Lorenz curves for net wealth, total income and labor income in the US in 2016.

```
fig, ax = plt.subplots()

ax.plot(f_vals_nw[-1], l_vals_nw[-1], label=f'net wealth')
ax.plot(f_vals_ti[-1], l_vals_ti[-1], label=f'total income')
ax.plot(f_vals_li[-1], l_vals_li[-1], label=f'labor income')
ax.plot(f_vals_nw[-1], f_vals_nw[-1], label=f'equality')

ax.legend(fontsize=12)
plt.show()
```

Here all the income and wealth measures are pre-tax.

Total income is the sum of households' all income sources, including labor income but excluding capital gains.

One key finding from this figure is that wealth inequality is significantly more extreme than income inequality.

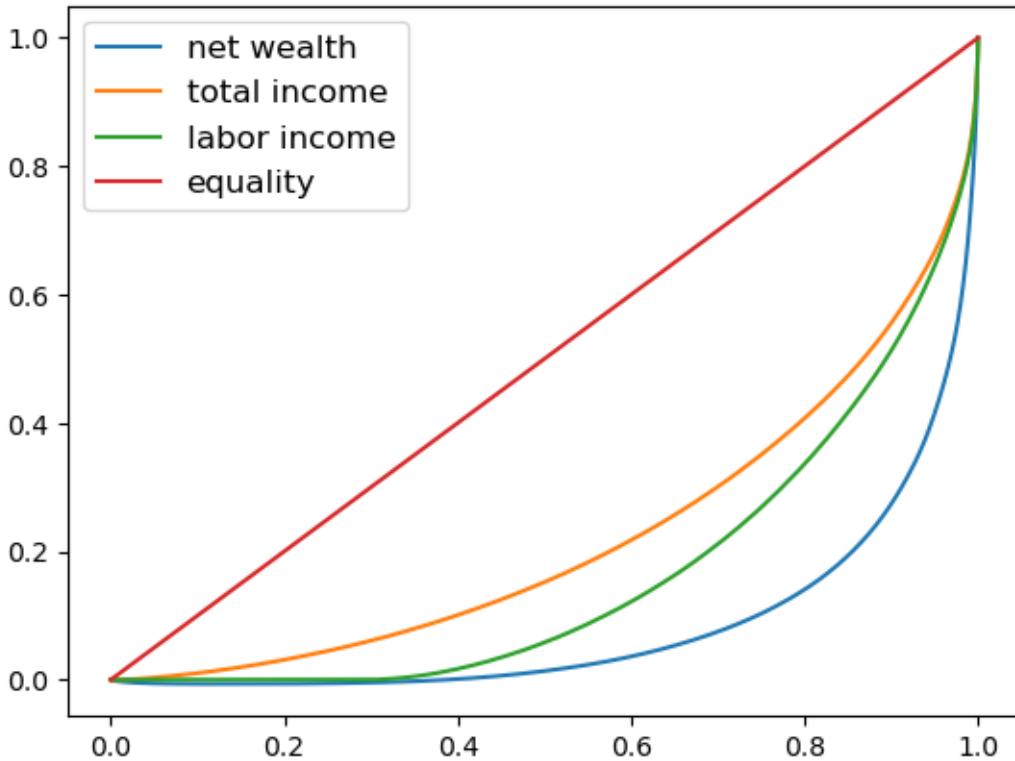


Fig. 4.2: 2016 US Lorenz curves

## 4.3 The Gini coefficient

The Lorenz curve is a useful visual representation of inequality in a distribution.

Another popular measure of income and wealth inequality is the Gini coefficient.

The Gini coefficient is just a number, rather than a curve.

In this section we discuss the Gini coefficient and its relationship to the Lorenz curve.

### 4.3.1 Definition

As before, suppose that the sample  $w_1, \dots, w_n$  has been sorted from smallest to largest.

The Gini coefficient is defined for the sample above as

$$G := \frac{\sum_{i=1}^n \sum_{j=1}^n |w_j - w_i|}{2n \sum_{i=1}^n w_i}. \quad (4.1)$$

The Gini coefficient is closely related to the Lorenz curve.

In fact, it can be shown that its value is twice the area between the line of equality and the Lorenz curve (e.g., the shaded area in the following Figure below).

The idea is that  $G = 0$  indicates complete equality, while  $G = 1$  indicates complete inequality.

```

fig, ax = plt.subplots()

f_vals, l_vals = qe.lorenz_curve(sample)
ax.plot(f_vals, l_vals, label='lognormal sample', lw=2)
ax.plot(f_vals, f_vals, label='equality', lw=2)

ax.legend(fontsize=12)

ax.vlines([0.8], [0.0], [0.43], alpha=0.5, colors='k', ls='--')
ax.hlines([0.43], [0], [0.8], alpha=0.5, colors='k', ls='--')

ax.fill_between(f_vals, l_vals, f_vals, alpha=0.06)

ax.set_xlim((0, 1))
ax.set_ylim((0, 1))

ax.text(0.04, 0.5, r'$G = 2 \times \text{shaded area}$', fontsize=12)

plt.show()

```

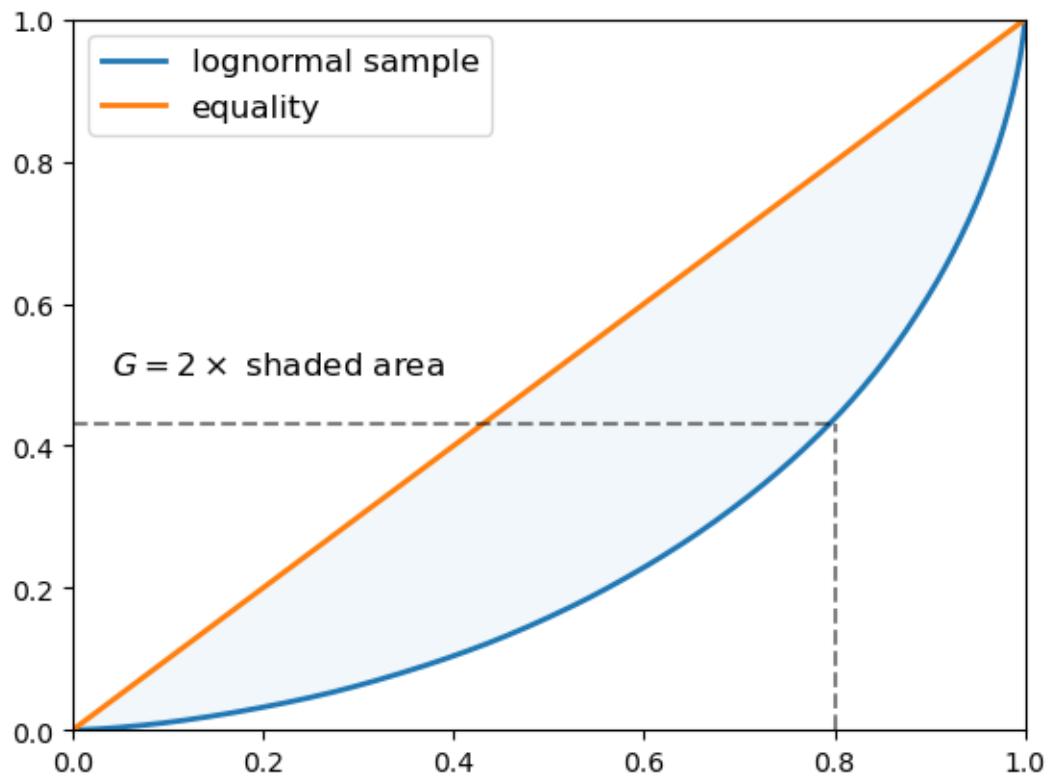


Fig. 4.3: Shaded Lorenz curve of simulated data

### 4.3.2 Gini coefficient dynamics of simulated data

Let's examine the Gini coefficient in some simulations.

The following code computes the Gini coefficients for five different populations.

Each of these populations is generated by drawing from a lognormal distribution with parameters  $\mu$  (mean) and  $\sigma$  (standard deviation).

To create the five populations, we vary  $\sigma$  over a grid of length 5 between 0.2 and 4.

In each case we set  $\mu = -\sigma^2/2$ .

This implies that the mean of the distribution does not change with  $\sigma$ .

(You can check this by looking up the expression for the mean of a lognormal distribution.)

```
k = 5
sigma_vals = np.linspace(0.2, 4, k)
n = 2_000

ginis = []

for sigma in sigma_vals:
    mu = -sigma**2 / 2
    y = np.exp(mu + sigma * np.random.randn(n))
    ginis.append(qe.gini_coefficient(y))
```

```
def plot_inequality_measures(x, y, legend, xlabel, ylabel):
    fig, ax = plt.subplots()
    ax.plot(x, y, marker='o', label=legend)

    ax.set_xlabel(xlabel, fontsize=12)
    ax.set_ylabel(ylabel, fontsize=12)

    ax.legend(fontsize=12)
    plt.show()
```

```
plot_inequality_measures(sigma_vals,
                        ginis,
                        'simulated',
                        '$\sigma$',
                        'gini coefficients')
```

The plots show that inequality rises with  $\sigma$ , according to the Gini coefficient.

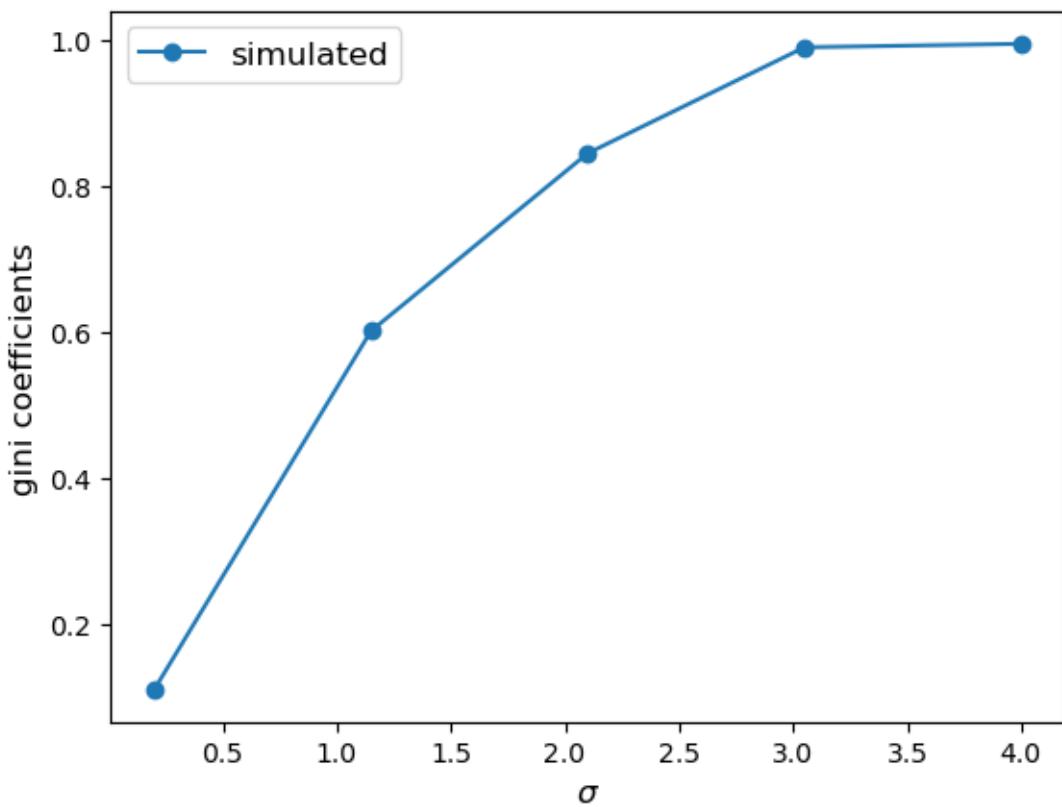


Fig. 4.4: Gini coefficients of simulated data

### 4.3.3 Gini coefficient dynamics for US data

Now let's look at Gini coefficients for US data derived from the SCF.

The following code creates a list called `Ginis`.

It stores data of Gini coefficients generated from the dataframe `df_income_wealth` and method `gini_coefficient`, from `QuantEcon` library.

```
ginis_nw, ginis_ti, ginis_li = Ginis
```

Let's plot the Gini coefficients for net wealth, labor income and total income.

```
# use an average to replace an outlier in labor income gini
ginis_li_new = ginis_li
ginis_li_new[5] = (ginis_li[4] + ginis_li[6]) / 2
```

```
xlabel = "year"
ylabel = "gini coefficient"

fig, ax = plt.subplots()

ax.plot(years, ginis_nw, marker='o')

ax.set_xlabel(xlabel, fontsize=12)
ax.set_ylabel(ylabel, fontsize=12)

plt.show()
```

```
xlabel = "year"
ylabel = "gini coefficient"

fig, ax = plt.subplots()

ax.plot(years, ginis_li_new, marker='o', label="labor income")
ax.plot(years, ginis_ti, marker='o', label="total income")

ax.set_xlabel(xlabel, fontsize=12)
ax.set_ylabel(ylabel, fontsize=12)

ax.legend(fontsize=12)
plt.show()
```

We see that, by this measure, inequality in wealth and income has risen substantially since 1980.

The wealth time series exhibits a strong U-shape.

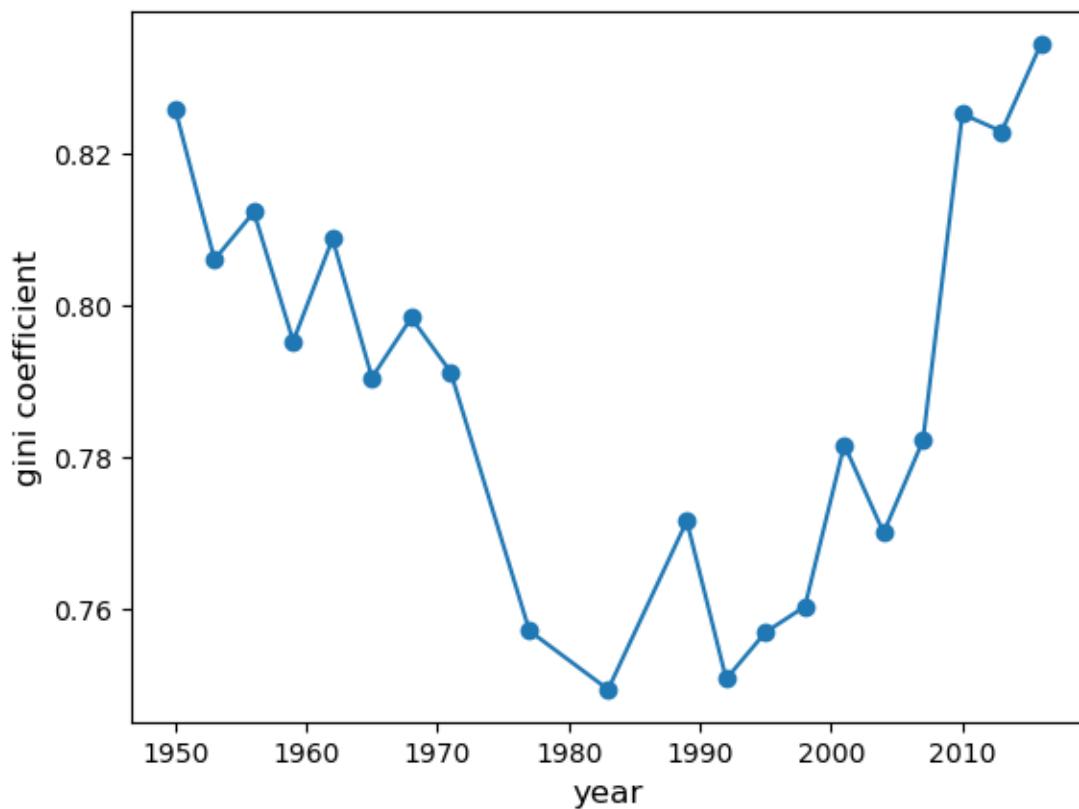


Fig. 4.5: Gini coefficients of US net wealth



Fig. 4.6: Gini coefficients of US income

## 4.4 Top shares

Another popular measure of inequality is the top shares.

Measuring specific shares is less complex than the Lorenz curve or the Gini coefficient.

In this section we show how to compute top shares.

### 4.4.1 Definition

As before, suppose that the sample  $w_1, \dots, w_n$  has been sorted from smallest to largest.

Given the Lorenz curve  $y = L(x)$  defined above, the top  $100 \times p\%$  share is defined as

$$T(p) = 1 - L(1 - p) \approx \frac{\sum_{j \geq i} w_j}{\sum_{j \leq n} w_j}, \quad i = \lfloor n(1 - p) \rfloor \quad (4.2)$$

Here  $\lfloor \cdot \rfloor$  is the floor function, which rounds any number down to the integer less than or equal to that number.

The following code uses the data from dataframe `df_income_wealth` to generate another dataframe `df_topshares`.

`df_topshares` stores the top 10 percent shares for the total income, the labor income and net wealth from 1950 to 2016 in US.

Then let's plot the top shares.

```
xlabel = "year"
ylabel = "top $10\%$ share"

fig, ax = plt.subplots()

ax.plot(years, df_topshares["topshare_l_income"],
        marker='o', label="labor income")
ax.plot(years, df_topshares["topshare_n_wealth"],
        marker='o', label="net wealth")
ax.plot(years, df_topshares["topshare_t_income"],
        marker='o', label="total income")

ax.set_xlabel(xlabel, fontsize=12)
ax.set_ylabel(ylabel, fontsize=12)

ax.legend(fontsize=12)
plt.show()
```

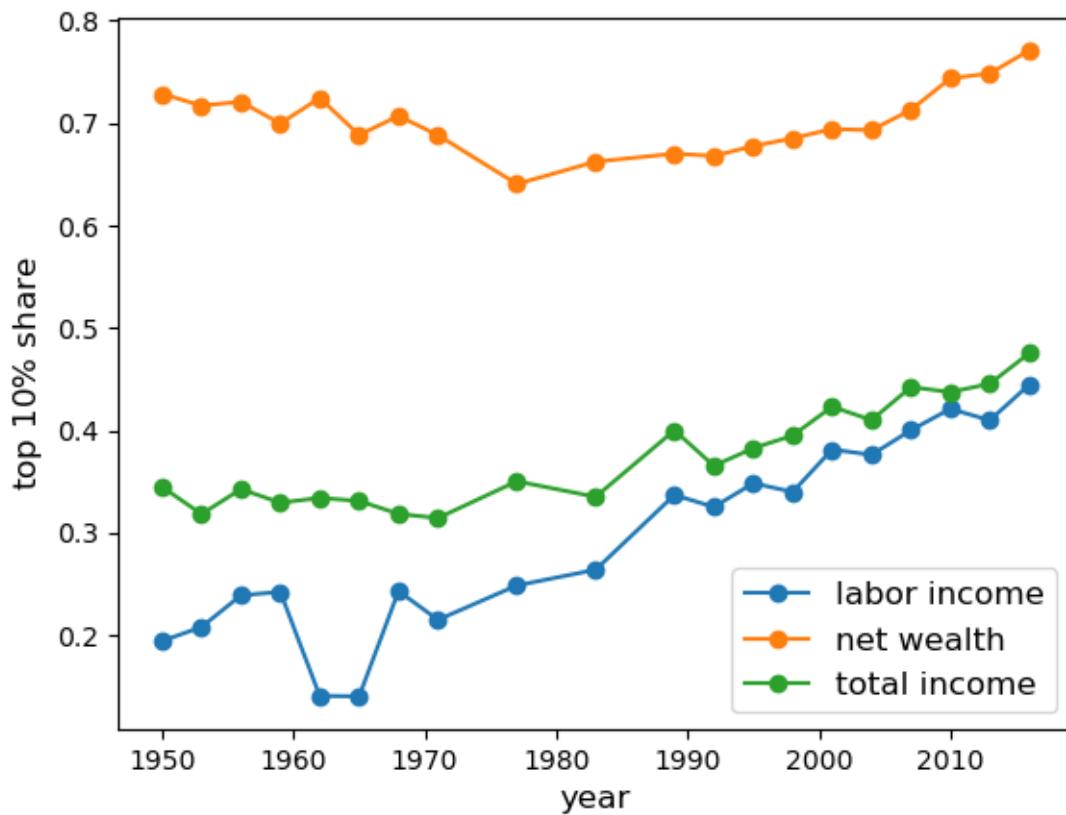


Fig. 4.7: US top shares

## 4.5 Exercises

---

### Exercise 4.5.1

Using simulation, compute the top 10 percent shares for the collection of lognormal distributions associated with the random variables  $w_\sigma = \exp(\mu + \sigma Z)$ , where  $Z \sim N(0, 1)$  and  $\sigma$  varies over a finite grid between 0.2 and 4.

As  $\sigma$  increases, so does the variance of  $w_\sigma$ .

To focus on volatility, adjust  $\mu$  at each step to maintain the equality  $\mu = -\sigma^2/2$ .

For each  $\sigma$ , generate 2,000 independent draws of  $w_\sigma$  and calculate the Lorenz curve and Gini coefficient.

Confirm that higher variance generates more dispersion in the sample, and hence greater inequality.

---



---

### Solution to Exercise 4.5.1

Here is one solution:

```
def calculate_top_share(s, p=0.1):

    s = np.sort(s)
    n = len(s)
    index = int(n * (1 - p))
    return s[index:].sum() / s.sum()

k = 5
sigma_vals = np.linspace(0.2, 4, k)
n = 2_000

topshares = []
ginis = []
f_vals = []
l_vals = []

for sigma in sigma_vals:
    mu = -sigma ** 2 / 2
    y = np.exp(mu + sigma * np.random.randn(n))
    f_val, l_val = qe._inequality.lorenz_curve(y)
    f_vals.append(f_val)
    l_vals.append(l_val)
    ginis.append(qe._inequality.gini_coefficient(y))
    topshares.append(calculate_top_share(y))

plot_inequality_measures(sigma_vals,
                        topshares,
                        "simulated data",
                        "\$\sigma\$",
                        "top 10% share")

plot_inequality_measures(sigma_vals,
                        ginis,
```

(continues on next page)

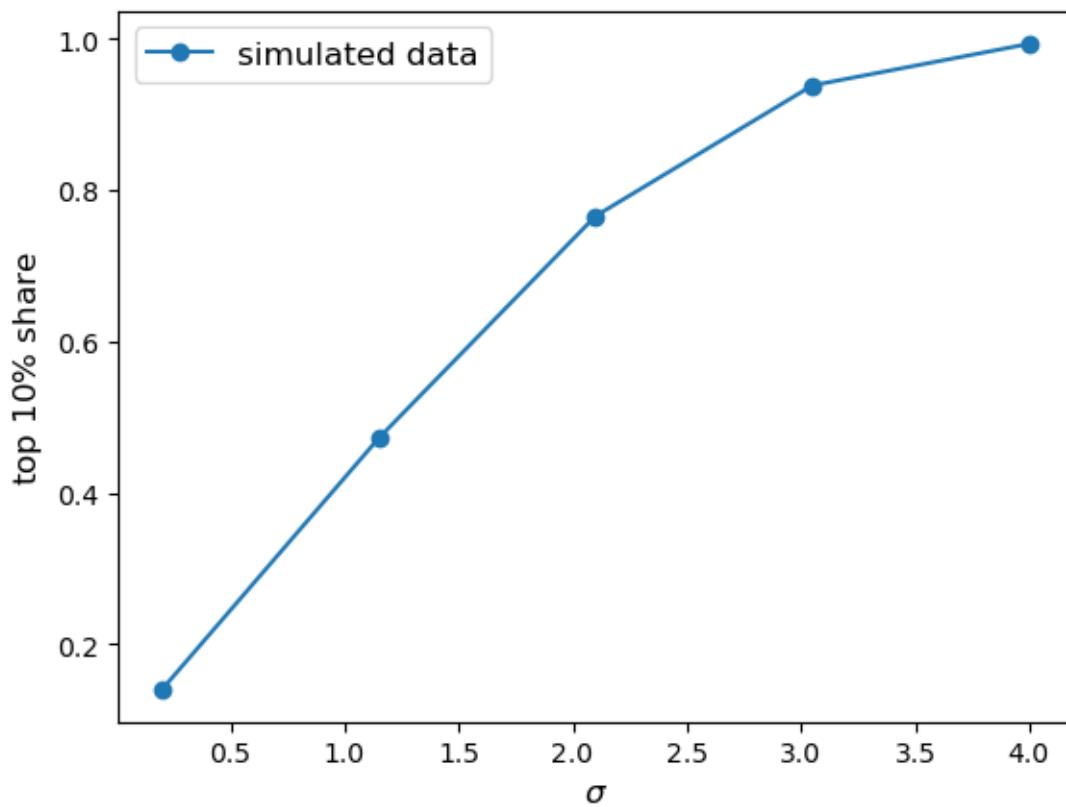


Fig. 4.8: Top shares of simulated data

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```

"simulated data",
"\$\\sigma\$",
"gini coefficient")

```

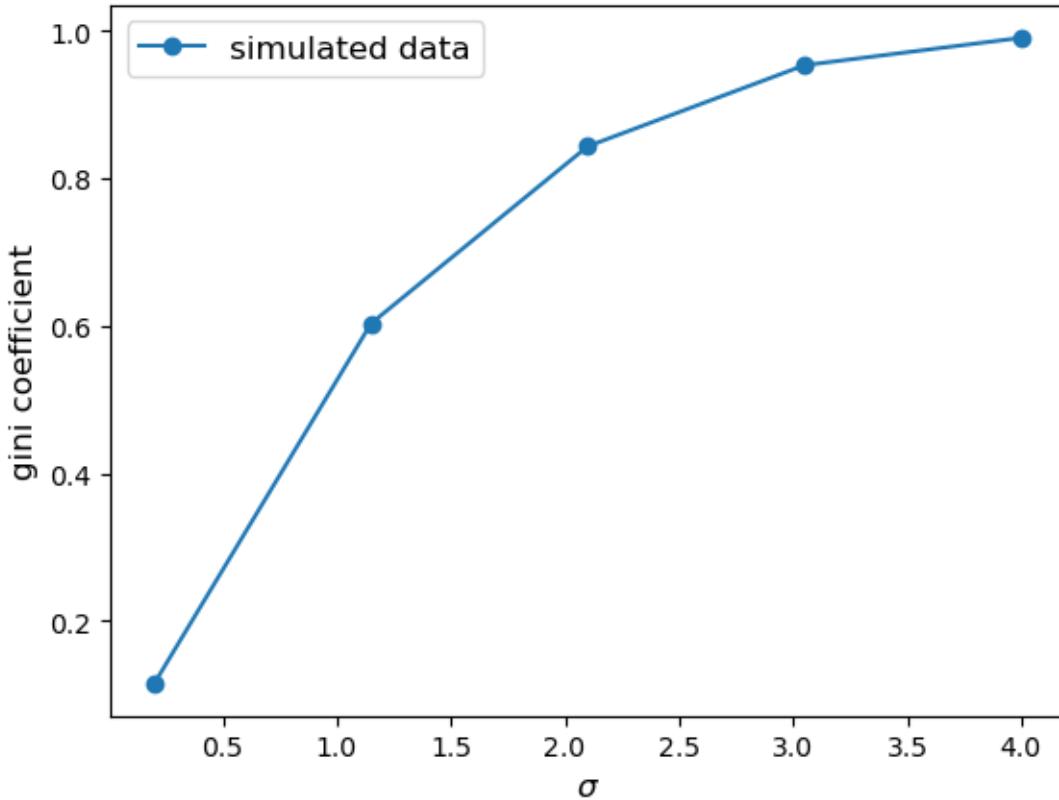


Fig. 4.9: Gini coefficients of simulated data

```

fig, ax = plt.subplots()
ax.plot([0,1],[0,1], label=f"equality")
for i in range(len(f_vals)):
    ax.plot(f_vals[i], l_vals[i], label=f"\$\\sigma\$ = {\sigma_vals[i]}")
plt.legend()
plt.show()

```

---

#### Exercise 4.5.2

According to the definition of the top shares (4.2) we can also calculate the top percentile shares using the Lorenz curve. Compute the top shares of US net wealth using the corresponding Lorenz curves data: `f_vals_nw`, `l_vals_nw` and linear interpolation.

Plot the top shares generated from Lorenz curve and the top shares approximated from data together.

---

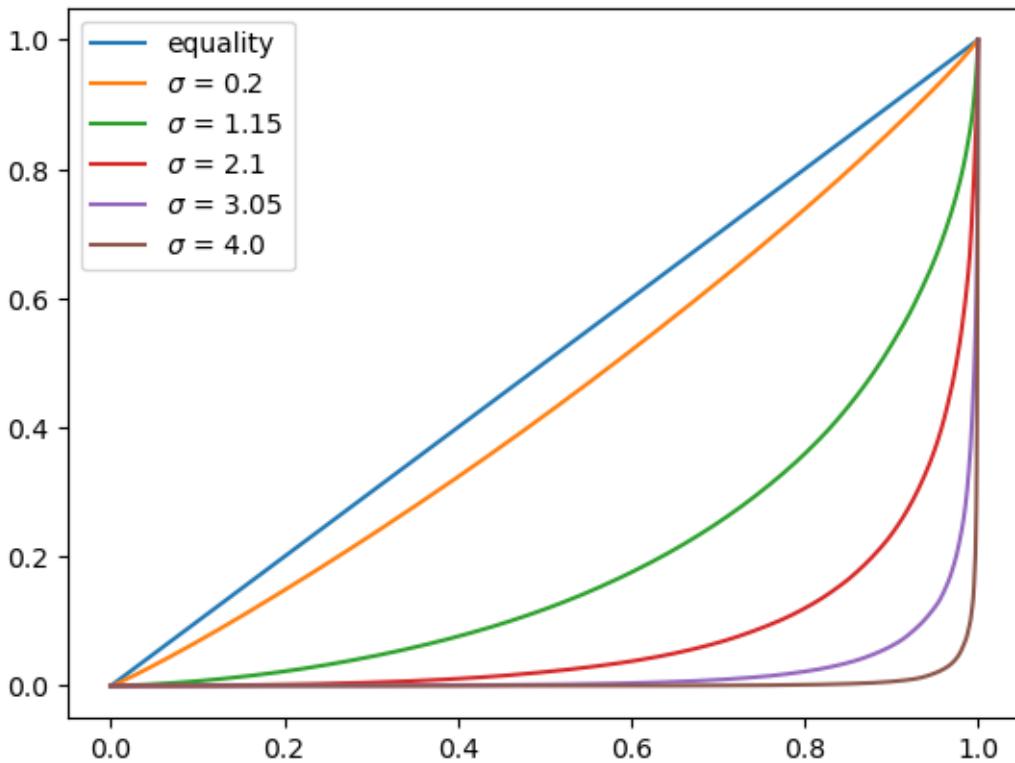


Fig. 4.10: Lorenz curves for simulated data

---

### Solution to Exercise 4.5.2

Here is one solution:

```
def lorenz2top(f_val, l_val, p=0.1):
    t = lambda x: interp(f_val, l_val, x)
    return 1 - t(1 - p)
```

```
top_shares_nw = []
for f_val, l_val in zip(f_vals_nw, l_vals_nw):
    top_shares_nw.append(lorenz2top(f_val, l_val))
```

```
xlabel = "year"
ylabel = "top $10\%$ share"

fig, ax = plt.subplots()

ax.plot(years, df_topshares["topshare_n_wealth"], marker='o', \
        label="net wealth-approx")
ax.plot(years, top_shares_nw, marker='o', label="net wealth-lorenz")

ax.set_xlabel(xlabel, fontsize=12)
ax.set_ylabel(ylabel, fontsize=12)
```

(continues on next page)

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```
ax.legend(fontsize=12)
plt.show()
```

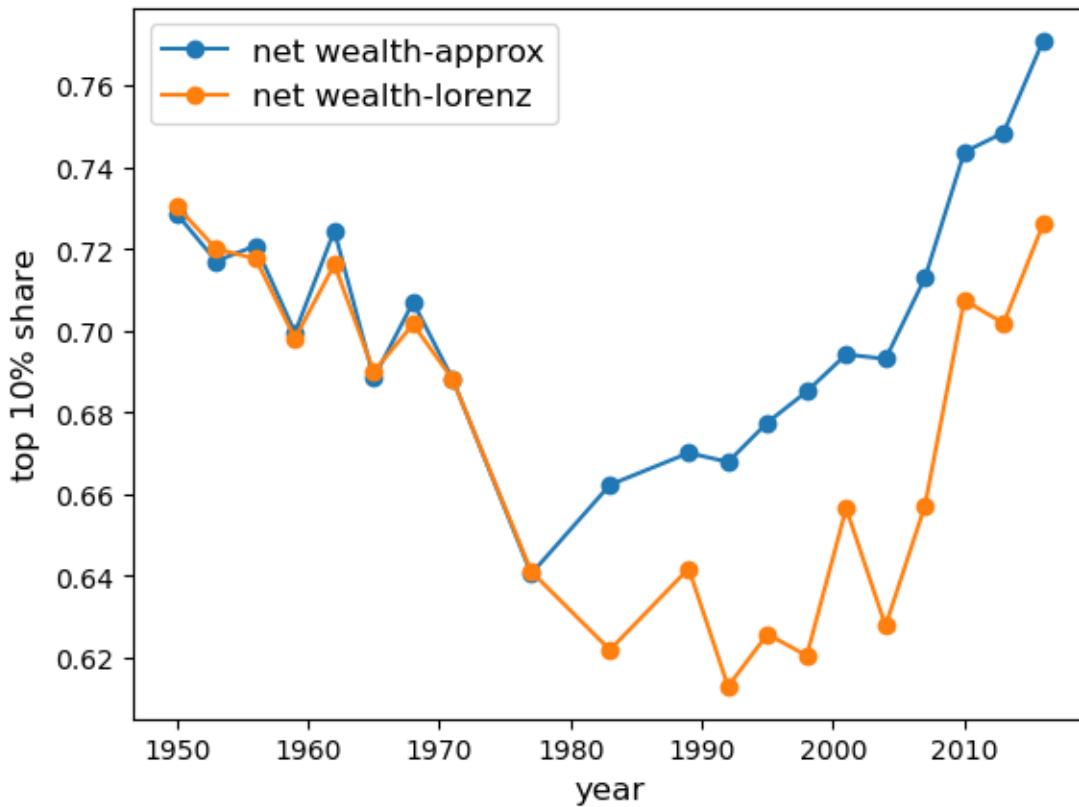


Fig. 4.11: US top shares: approximation vs Lorenz

## **Part III**

# **Essential Tools**



## **LINEAR EQUATIONS AND MATRIX ALGEBRA**

### **5.1 Overview**

Many problems in economics and finance require solving linear equations.

In this lecture we discuss linear equations and their applications.

To illustrate the importance of linear equations, we begin with a two good model of supply and demand.

The two good case is so simple that solutions can be calculated by hand.

But often we need to consider markets containing many goods.

In the multiple goods case we face large systems of linear equations, with many equations and unknowns.

To handle such systems we need two things:

- matrix algebra (and the knowledge of how to use it) plus
- computer code to apply matrix algebra to the problems of interest.

This lecture covers these steps.

We will use the following packages:

```
import numpy as np
import matplotlib.pyplot as plt
```

### **5.2 A two good example**

In this section we discuss a simple two good example and solve it by

1. pencil and paper
2. matrix algebra

The second method is more general, as we will see.

### 5.2.1 Pencil and paper methods

Suppose that we have two related goods, such as

- propane and ethanol, and
- rice and wheat, etc.

To keep things simple, we label them as good 0 and good 1.

The demand for each good depends on the price of both goods:

$$\begin{aligned} q_0^d &= 100 - 10p_0 - 5p_1 \\ q_1^d &= 50 - p_0 - 10p_1 \end{aligned} \tag{5.1}$$

(We are assuming demand decreases when the price of either good goes up, but other cases are also possible.)

Let's suppose that supply is given by

$$\begin{aligned} q_0^s &= 10p_0 + 5p_1 \\ q_1^s &= 5p_0 + 10p_1 \end{aligned} \tag{5.2}$$

Equilibrium holds when supply equals demand ( $q_0^s = q_0^d$  and  $q_1^s = q_1^d$ ).

This yields the linear system

$$\begin{aligned} 100 - 10p_0 - 5p_1 &= 10p_0 + 5p_1 \\ 50 - p_0 - 10p_1 &= 5p_0 + 10p_1 \end{aligned} \tag{5.3}$$

We can solve this with pencil and paper to get

$$p_0 = 4.41 \quad \text{and} \quad p_1 = 1.18.$$

Inserting these results into either (5.1) or (5.2) yields the equilibrium quantities

$$q_0 = 50 \quad \text{and} \quad q_1 = 33.82.$$

### 5.2.2 Looking forward

Pencil and paper methods are easy in the two good case.

But what if there are many goods?

For such problems we need matrix algebra.

Before solving problems with matrix algebra, let's first recall the basics of vectors and matrices, in both theory and computation.

## 5.3 Vectors

A **vector** of length  $n$  is just a sequence (or array, or tuple) of  $n$  numbers, which we write as  $x = (x_1, \dots, x_n)$  or  $x = [x_1, \dots, x_n]$ .

We can write these sequences either horizontally or vertically.

But when we use matrix operations, our default assumption is that vectors are column vectors.

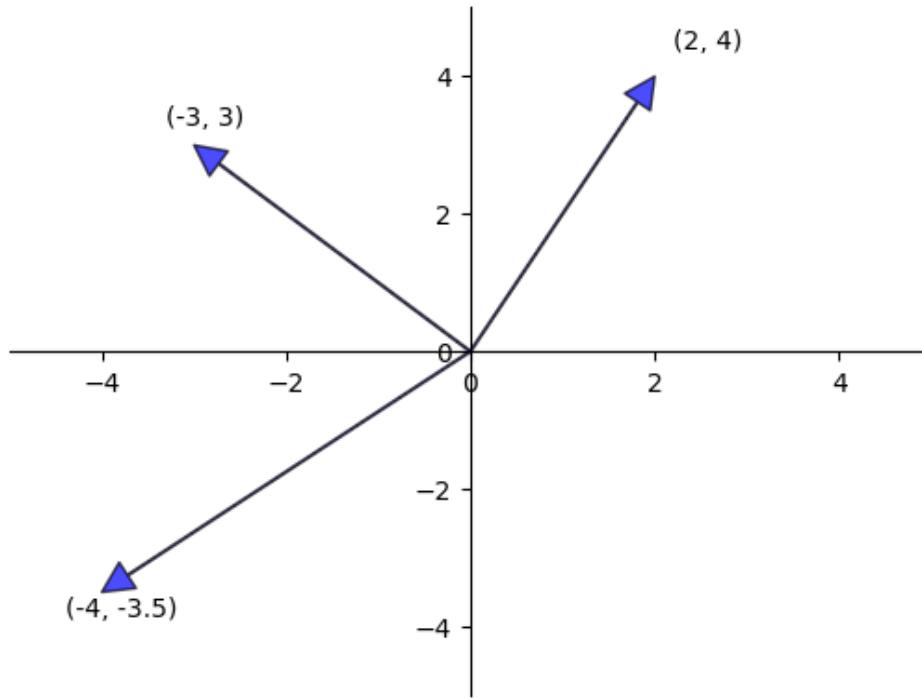
The set of all  $n$ -vectors is denoted by  $\mathbb{R}^n$ .

For example,

- $\mathbb{R}^2$  is the plane — the set of pairs  $(x_1, x_2)$ .
- $\mathbb{R}^3$  is 3 dimensional space — the set of vectors  $(x_1, x_2, x_3)$ .

Often vectors are represented visually as arrows from the origin to the point.

Here's a visualization.



### 5.3.1 Vector operations

Sometimes we want to modify vectors.

The two most common operators on vectors are addition and scalar multiplication, which we now describe.

When we add two vectors, we add them element-by-element.

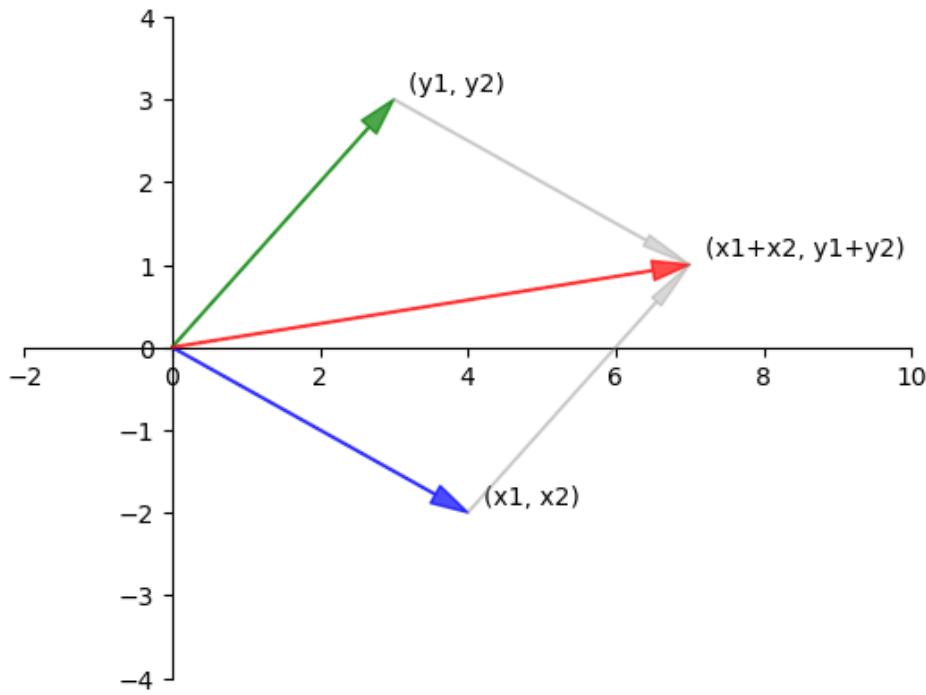
For example,

$$\begin{bmatrix} 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 & + & 3 \\ -2 & + & 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}.$$

In general,

$$x + y = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

We can visualise vector addition in  $\mathbb{R}^2$  as follows.



Scalar multiplication is an operation that multiplies a vector  $x$  with a scalar elementwise.

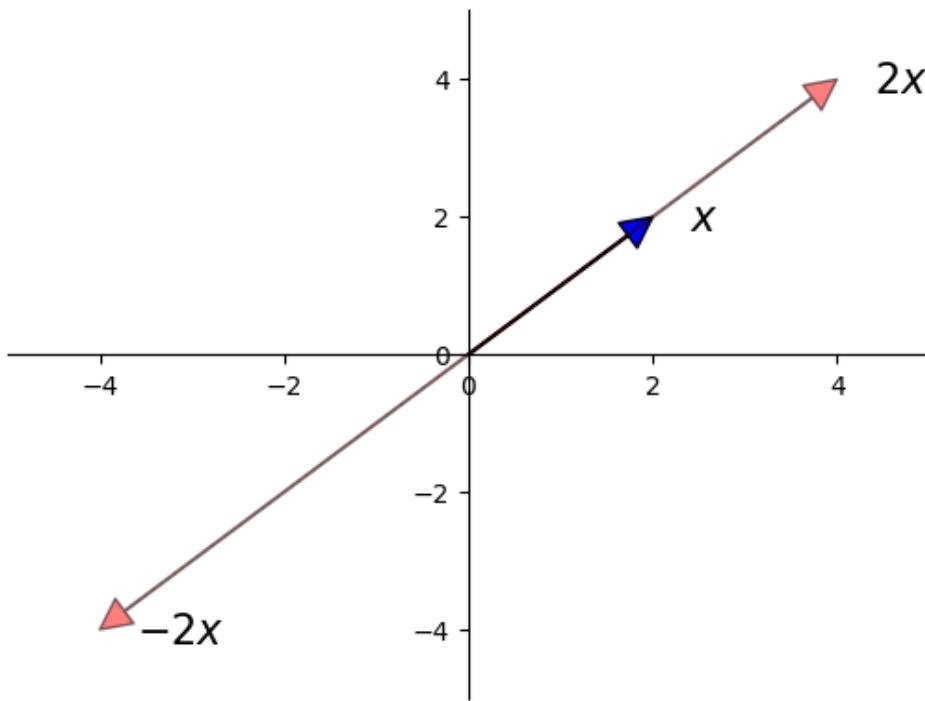
For example,

$$-2 \begin{bmatrix} 3 \\ -7 \end{bmatrix} = \begin{bmatrix} -2 & \times & 3 \\ -2 & \times & -7 \end{bmatrix} = \begin{bmatrix} -6 \\ 14 \end{bmatrix}.$$

More generally, it takes a number  $\gamma$  and a vector  $x$  and produces

$$\gamma x := \begin{bmatrix} \gamma x_1 \\ \gamma x_2 \\ \vdots \\ \gamma x_n \end{bmatrix}.$$

Scalar multiplication is illustrated in the next figure.



In Python, a vector can be represented as a list or tuple, such as `x = [2, 4, 6]` or `x = (2, 4, 6)`.

However, it is more common to represent vectors with NumPy arrays.

One advantage of NumPy arrays is that scalar multiplication and addition have very natural syntax.

```
x = np.ones(3)          # Vector of three ones
y = np.array((2, 4, 6)) # Converts tuple (2, 4, 6) into a NumPy array
x + y                  # Add (element-by-element)
```

```
array([3., 5., 7.])
```

```
4 * x                  # Scalar multiply
```

```
array([4., 4., 4.])
```

### 5.3.2 Inner product and norm

The **inner product** of vectors  $x, y \in \mathbb{R}^n$  is defined as

$$x^\top y = [x_1 \ x_2 \ \dots \ x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n := \sum_{i=1}^n x_i y_i.$$

The **norm** of a vector  $x$  represents its “length” (i.e., its distance from the zero vector) and is defined as

$$\|x\| := \sqrt{x^\top x} := \left( \sum_{i=1}^n x_i^2 \right)^{1/2}.$$

The expression  $\|x - y\|$  can be thought of as the “distance” between  $x$  and  $y$ .

The inner product and norm can be computed as follows

```
np.sum(x*y)      # Inner product of x and y
```

```
12.0
```

```
x @ y          # Another way to compute the inner product
```

```
12.0
```

```
np.sqrt(np.sum(x**2))  # Norm of x, method one
```

```
1.7320508075688772
```

```
np.linalg.norm(x)      # Norm of x, method two
```

```
1.7320508075688772
```

## 5.4 Matrix operations

When we discussed linear price systems, we mentioned using matrix algebra.

Matrix algebra is similar to algebra for numbers.

Let's review some details.

### 5.4.1 Addition and scalar multiplication

Just as was the case for vectors, we can add, subtract and scalar multiply matrices.

Scalar multiplication and addition are generalizations of the vector case:

Here is an example of scalar multiplication

$$3 \begin{bmatrix} 2 & -13 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 6 & -39 \\ 0 & 15 \end{bmatrix}.$$

In general for a number  $\gamma$  and any matrix  $A$ ,

$$\gamma A = \gamma \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} \end{bmatrix} := \begin{bmatrix} \gamma a_{11} & \cdots & \gamma a_{1k} \\ \vdots & \ddots & \vdots \\ \gamma a_{n1} & \cdots & \gamma a_{nk} \end{bmatrix}.$$

Consider this example of matrix addition,

$$\begin{bmatrix} 1 & 5 \\ 7 & 3 \end{bmatrix} + \begin{bmatrix} 12 & -1 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 13 & 4 \\ 7 & 12 \end{bmatrix}.$$

In general,

$$A + B = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nk} \end{bmatrix} := \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1k} + b_{1k} \\ \vdots & \ddots & \vdots \\ a_{n1} + b_{n1} & \cdots & a_{nk} + b_{nk} \end{bmatrix}.$$

In the latter case, the matrices must have the same shape in order for the definition to make sense.

## 5.4.2 Matrix multiplication

We also have a convention for *multiplying* two matrices.

The rule for matrix multiplication generalizes the idea of inner products discussed above.

If  $A$  and  $B$  are two matrices, then their product  $AB$  is formed by taking as its  $i, j$ -th element the inner product of the  $i$ -th row of  $A$  and the  $j$ -th column of  $B$ .

If  $A$  is  $n \times k$  and  $B$  is  $j \times m$ , then to multiply  $A$  and  $B$  we require  $k = j$ , and the resulting matrix  $AB$  is  $n \times m$ .

Here's an example of a  $2 \times 2$  matrix multiplied by a  $2 \times 1$  vector.

$$Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$$

As an important special case, consider multiplying  $n \times k$  matrix  $A$  and  $k \times 1$  column vector  $x$ .

According to the preceding rule, this gives us an  $n \times 1$  column vector.

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ik} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{bmatrix}_{n \times k} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}_{k \times 1} := \begin{bmatrix} a_{11}x_1 + a_{22}x_2 + \cdots + a_{1k}x_k \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ik}x_k \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nk}x_k \end{bmatrix}_{n \times 1} \quad (5.4)$$

Here is a simple illustration of multiplication of two matrices.

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} := \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

There are many tutorials to help you further visualize this operation, such as

- [this one](#), or
- the discussion on the [Wikipedia page](#).

---

**Note:** Unlike number products,  $AB$  and  $BA$  are not generally the same thing.

---

One important special case is the [identity matrix](#), which has ones on the principal diagonal and zero elsewhere:

$$I = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

It is a useful exercise to check the following:

- if  $A$  is  $n \times k$  and  $I$  is the  $k \times k$  identity matrix, then  $AI = A$ , and
- if  $I$  is the  $n \times n$  identity matrix, then  $IA = A$ .

### 5.4.3 Matrices in NumPy

NumPy arrays are also used as matrices, and have fast, efficient functions and methods for all the standard matrix operations.

You can create them manually from tuples of tuples (or lists of lists) as follows

```
A = ((1, 2),  
     (3, 4))  
  
type(A)
```

```
tuple
```

```
A = np.array(A)  
  
type(A)
```

```
numpy.ndarray
```

```
A.shape
```

```
(2, 2)
```

The `shape` attribute is a tuple giving the number of rows and columns — see [here](#) for more discussion.

To get the transpose of `A`, use `A.transpose()` or, more simply, `A.T`.

There are many convenient functions for creating common matrices (matrices of zeros, ones, etc.) — see [here](#).

Since operations are performed elementwise by default, scalar multiplication and addition have very natural syntax.

```
A = np.identity(3)      # 3 x 3 identity matrix  
B = np.ones((3, 3))    # 3 x 3 matrix of ones  
2 * A
```

```
array([[2., 0., 0.],  
       [0., 2., 0.],  
       [0., 0., 2.]])
```

```
A + B
```

```
array([[2., 1., 1.],  
       [1., 2., 1.],  
       [1., 1., 2.]])
```

To multiply matrices we use the `@` symbol.

---

**Note:** In particular, `A @ B` is matrix multiplication, whereas `A * B` is element-by-element multiplication.

---

### 5.4.4 Two good model in matrix form

We can now revisit the two good model and solve (5.3) numerically via matrix algebra.

This involves some extra steps but the method is widely applicable — as we will see when we include more goods.

First we rewrite (5.1) as

$$q^d = Dp + h \quad \text{where} \quad q^d = \begin{bmatrix} q_0^d \\ q_1^d \end{bmatrix} \quad D = \begin{bmatrix} -10 & -5 \\ -1 & -10 \end{bmatrix} \quad \text{and} \quad h = \begin{bmatrix} 100 \\ 50 \end{bmatrix}. \quad (5.5)$$

Recall that  $p \in \mathbb{R}^2$  is the price of two goods.

(Please check that  $q^d = Dp + h$  represents the same equations as (5.1).)

We rewrite (5.2) as

$$q^s = Cp \quad \text{where} \quad q^s = \begin{bmatrix} q_0^s \\ q_1^s \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix}. \quad (5.6)$$

Now equality of supply and demand can be expressed as  $q^s = q^d$ , or

$$Cp = Dp + h.$$

We can rearrange the terms to get

$$(C - D)p = h.$$

If all of the terms were numbers, we could solve for prices as  $p = h/(C - D)$ .

Matrix algebra allows us to do something similar: we can solve for equilibrium prices using the inverse of  $C - D$ :

$$p = (C - D)^{-1}h. \quad (5.7)$$

Before we implement the solution let us consider a more general setting.

### 5.4.5 More goods

It is natural to think about demand systems with more goods.

For example, even within energy commodities there are many different goods, including crude oil, gasoline, coal, natural gas, ethanol, and uranium.

The prices of these goods are related, so it makes sense to study them together.

Pencil and paper methods become very time consuming with large systems.

But fortunately the matrix methods described above are essentially unchanged.

In general, we can write the demand equation as  $q^d = Dp + h$ , where

- $q^d$  is an  $n \times 1$  vector of demand quantities for  $n$  different goods.
- $D$  is an  $n \times n$  “coefficient” matrix.
- $h$  is an  $n \times 1$  vector of constant values.

Similarly, we can write the supply equation as  $q^s = Cp + e$ , where

- $q^s$  is an  $n \times 1$  vector of supply quantities for the same goods.
- $C$  is an  $n \times n$  “coefficient” matrix.

- $e$  is an  $n \times 1$  vector of constant values.

To find an equilibrium, we solve  $Dp + h = Cp + e$ , or

$$(D - C)p = e - h. \quad (5.8)$$

Then the price vector of the  $n$  different goods is

$$p = (D - C)^{-1}(e - h).$$

### 5.4.6 General linear systems

A more general version of the problem described above looks as follows.

$$\begin{array}{lclclcl} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n = b_1 \\ \vdots & & \vdots & & & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n = b_n \end{array} \quad (5.9)$$

The objective here is to solve for the “unknowns”  $x_1, \dots, x_n$ .

We take as given the coefficients  $a_{11}, \dots, a_{nn}$  and constants  $b_1, \dots, b_n$ .

Notice that we are treating a setting where the number of unknowns equals the number of equations.

This is the case where we are most likely to find a well-defined solution.

(The other cases are referred to as [overdetermined](#) and [underdetermined](#) systems of equations — we defer discussion of these cases until later lectures.)

In matrix form, the system (5.9) becomes

$$Ax = b \quad \text{where} \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}. \quad (5.10)$$

For example, (5.8) has this form with

$$A = D - C, \quad b = e - h \quad \text{and} \quad x = p.$$

When considering problems such as (5.10), we need to ask at least some of the following questions

- Does a solution actually exist?
- If a solution exists, how should we compute it?

## 5.5 Solving systems of equations

Recall again the system of equations (5.9), which we write here again as

$$Ax = b. \quad (5.11)$$

The problem we face is to find a vector  $x \in \mathbb{R}^n$  that solves (5.11), taking  $b$  and  $A$  as given.

We may not always find a unique vector  $x$  that solves (5.11).

We illustrate two such cases below.

### 5.5.1 No solution

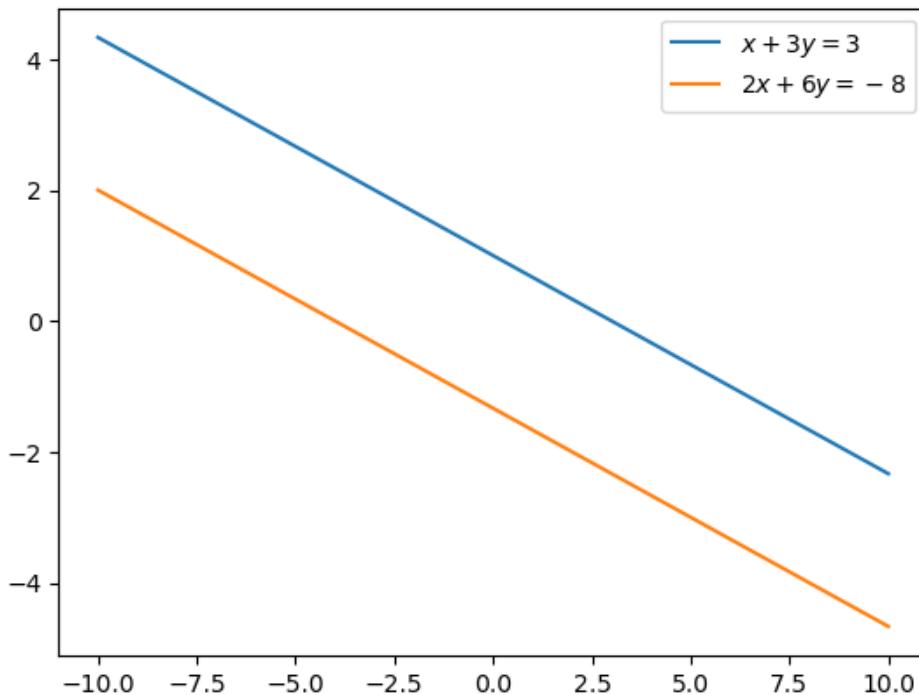
Consider the system of equations given by,

$$\begin{aligned} x + 3y &= 3 \\ 2x + 6y &= -8. \end{aligned}$$

It can be verified manually that this system has no possible solution.

To illustrate why this situation arises let's plot the two lines.

```
fig, ax = plt.subplots()
x = np.linspace(-10, 10)
plt.plot(x, (3-x)/3, label=f'$x + 3y = 3$')
plt.plot(x, (-8-2*x)/6, label=f'$2x + 6y = -8$')
plt.legend()
plt.show()
```



Clearly, these are parallel lines and hence we will never find a point  $x \in \mathbb{R}^2$  such that these lines intersect.

Thus, this system has no possible solution.

We can rewrite this system in matrix form as

$$Ax = b \quad \text{where} \quad A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ -8 \end{bmatrix}. \quad (5.12)$$

It can be noted that the  $2^{nd}$  row of matrix  $A = (2, 6)$  is just a scalar multiple of the  $1^{st}$  row of matrix  $A = (1, 3)$ .

The rows of matrix  $A$  in this case are called **linearly dependent**.

---

**Note:** Advanced readers can find a detailed explanation of linear dependence and independence [here](#).

But these details are not needed in what follows.

---

## 5.5.2 Many solutions

Now consider,

$$\begin{aligned} x - 2y &= -4 \\ -2x + 4y &= 8. \end{aligned}$$

Any vector  $v = (x, y)$  such that  $x = 2y - 4$  will solve the above system.

Since we can find infinite such vectors this system has infinitely many solutions.

This is because the rows of the corresponding matrix

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}. \quad (5.13)$$

are linearly dependent — can you see why?

We now impose conditions on  $A$  in (5.11) that rule out these problems.

## 5.5.3 Nonsingular matrices

To every square matrix we can assign a unique number called the [determinant](#).

For  $2 \times 2$  matrices, the determinant is given by,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

If the determinant of  $A$  is not zero, then we say that  $A$  is *nonsingular*.

A square matrix  $A$  is nonsingular if and only if the rows and columns of  $A$  are linearly independent.

A more detailed explanation of matrix inverse can be found [here](#).

You can check yourself that the in (5.12) and (5.13) with linearly dependent rows are singular matrices.

This gives us a useful one-number summary of whether or not a square matrix can be inverted.

In particular, a square matrix  $A$  has a nonzero determinant, if and only if it possesses an *inverse matrix*  $A^{-1}$ , with the property that  $AA^{-1} = A^{-1}A = I$ .

As a consequence, if we pre-multiply both sides of  $Ax = b$  by  $A^{-1}$ , we get

$$x = A^{-1}b. \quad (5.14)$$

This is the solution to  $Ax = b$  — the solution we are looking for.

## 5.5.4 Linear equations with NumPy

In the two good example we obtained the matrix equation,

$$p = (C - D)^{-1}h.$$

where  $C$ ,  $D$  and  $h$  are given by (5.5) and (5.6).

This equation is analogous to (5.14) with  $A = (C - D)^{-1}$ ,  $b = h$ , and  $x = p$ .

We can now solve for equilibrium prices with NumPy's `linalg` submodule.

All of these routines are Python front ends to time-tested and highly optimized FORTRAN code.

```
C = ((10, 5),      # Matrix C
     (5, 10))
```

Now we change this to a NumPy array.

```
C = np.array(C)
```

```
D = ((-10, -5),    # Matrix D
      (-1, -10))
D = np.array(D)
```

```
h = np.array((100, 50))    # Vector h
h.shape = 2,1              # Transforming h to a column vector
```

```
from numpy.linalg import det, inv
A = C - D
# Check that A is nonsingular (non-zero determinant), and hence invertible
det(A)
```

```
340.0000000000001
```

```
A_inv = inv(A)  # compute the inverse
A_inv
```

```
array([[ 0.05882353, -0.02941176],
       [-0.01764706,  0.05882353]])
```

```
p = A_inv @ h  # equilibrium prices
p
```

```
array([[4.41176471],
       [1.17647059]])
```

```
q = C @ p  # equilibrium quantities
q
```

```
array([[50.          ],
       [33.82352941]])
```

Notice that we get the same solutions as the pencil and paper case.

We can also solve for  $p$  using `solve(A, h)` as follows.

```
from numpy.linalg import solve
p = solve(A, h)  # equilibrium prices
p
```

```
array([[4.41176471],
       [1.17647059]])
```

```
q = C @ p # equilibrium quantities
q
```

```
array([[50.          ],
       [33.82352941]])
```

Observe how we can solve for  $x = A^{-1}y$  by either via `inv(A) @ y`, or using `solve(A, y)`.

The latter method uses a different algorithm that is numerically more stable and hence should be the default option.

## 5.6 Exercises

---

### Exercise 5.6.1

Let's consider a market with 3 commodities - good 0, good 1 and good 2.

The demand for each good depends on the price of the other two goods and is given by:

$$\begin{aligned}q_0^d &= 90 - 15p_0 + 5p_1 + 5p_2 \\q_1^d &= 60 + 5p_0 - 10p_1 + 10p_2 \\q_2^d &= 50 + 5p_0 + 5p_1 - 5p_2\end{aligned}$$

(Here demand decreases when own price increases but increases when prices of other goods increase.)

The supply of each good is given by:

$$\begin{aligned}q_0^s &= -10 + 20p_0 \\q_1^s &= -15 + 15p_1 \\q_2^s &= -5 + 10p_2\end{aligned}$$

Equilibrium holds when supply equals demand, i.e,  $q_0^d = q_0^s$ ,  $q_1^d = q_1^s$  and  $q_2^d = q_2^s$ .

1. Set up the market as a system of linear equations.
  2. Use matrix algebra to solve for equilibrium prices. Do this using both the `numpy.linalg.solve` and `inv(A)` methods. Compare the solutions.
- 

---

### Solution to Exercise 5.6.1

The generated system would be:

$$\begin{aligned}35p_0 - 5p_1 - 5p_2 &= 100 \\-5p_0 + 25p_1 - 10p_2 &= 75 \\-5p_0 - 5p_1 + 15p_2 &= 55\end{aligned}$$

In matrix form we will write this as:

$$Ap = b \quad \text{where} \quad A = \begin{bmatrix} 35 & -5 & -5 \\ -5 & 25 & -10 \\ -5 & -5 & 15 \end{bmatrix}, \quad p = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 100 \\ 75 \\ 55 \end{bmatrix}$$

```

import numpy as np
from numpy.linalg import det

A = np.array([[35, -5, -5], # matrix A
              [-5, 25, -10],
              [-5, -5, 15]])

b = np.array((100, 75, 55)) # column vector b
b.shape = (3, 1)

det(A) # check if A is nonsingular

```

9999.99999999999

```

# Using inverse
from numpy.linalg import det

A_inv = inv(A)

p = A_inv @ b
p

```

array([[4.9625],
 [7.0625],
 [7.675 ]])

```

# Using numpy.linalg.solve
from numpy.linalg import solve
p = solve(A, b)
p

```

array([[4.9625],
 [7.0625],
 [7.675 ]])

The solution is given by:  $p_0 = 4.6925$ ,  $p_1 = 7.0625$  and  $p_2 = 7.675$

### Exercise 5.6.2

Earlier in the lecture we discussed cases where the system of equations given by  $Ax = b$  has no solution.

In this case  $Ax = b$  is called an *inconsistent* system of equations.

When faced with an inconsistent system we try to find the best “approximate” solution.

There are various methods to do this, one such method is the **method of least squares**.

Suppose we have an inconsistent system

$$Ax = b \quad (5.15)$$

where  $A$  is an  $m \times n$  matrix and  $b$  is an  $m \times 1$  column vector.

A **least squares solution** to (5.15) is an  $n \times 1$  column vector  $\hat{x}$  such that, for all other vectors  $x \in \mathbb{R}^n$ , the distance from  $A\hat{x}$  to  $b$  is less than the distance from  $Ax$  to  $b$ .

That is,

$$\|A\hat{x} - b\| \leq \|Ax - b\|$$

It can be shown that, for the system of equations  $Ax = b$ , the least squares solution  $\hat{x}$  is

$$\hat{x} = (A^T A)^{-1} A^T b \quad (5.16)$$

Now consider the general equation of a linear demand curve of a good given by:

$$p = m - nq$$

where  $p$  is the price of the good and  $q$  is the quantity demanded.

Suppose we are trying to *estimate* the values of  $m$  and  $n$ .

We do this by repeatedly observing the price and quantity (for example, each month) and then choosing  $m$  and  $n$  to fit the relationship between  $p$  and  $q$ .

We have the following observations:

Price	Quantity Demanded
1	9
3	7
8	3

Requiring the demand curve  $p = m - nq$  to pass through all these points leads to the following three equations:

$$\begin{aligned} 1 &= m - 9n \\ 3 &= m - 7n \\ 8 &= m - 3n \end{aligned}$$

Thus we obtain a system of equations  $Ax = b$  where  $A = \begin{bmatrix} 1 & -9 \\ 1 & -7 \\ 1 & -3 \end{bmatrix}$ ,  $x = \begin{bmatrix} m \\ n \end{bmatrix}$  and  $b = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}$ .

It can be verified that this system has no solutions.

(The problem is that we have three equations and only two unknowns.)

We will thus try to find the best approximate solution for  $x$ .

1. Use (5.16) and matrix algebra to find the least squares solution  $\hat{x}$ .
  2. Find the least squares solution using `numpy.linalg.lstsq` and compare the results.
- 

---

### Solution to Exercise 5.6.2

```
import numpy as np
from numpy.linalg import inv
```

```
# Using matrix algebra
A = np.array([[1, -9], # matrix A
              [1, -7],
              [1, -3]])
```

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```
A_T = np.transpose(A) # transpose of matrix A
b = np.array((1, 3, 8)) # column vector b
b.shape = (3, 1)

x = inv(A_T @ A) @ A_T @ b
x
```

```
array([[11.46428571],
       [ 1.17857143]])
```

```
# Using numpy.linalg.lstsq
x, res, _, _ = np.linalg.lstsq(A, b, rcond=None)
```

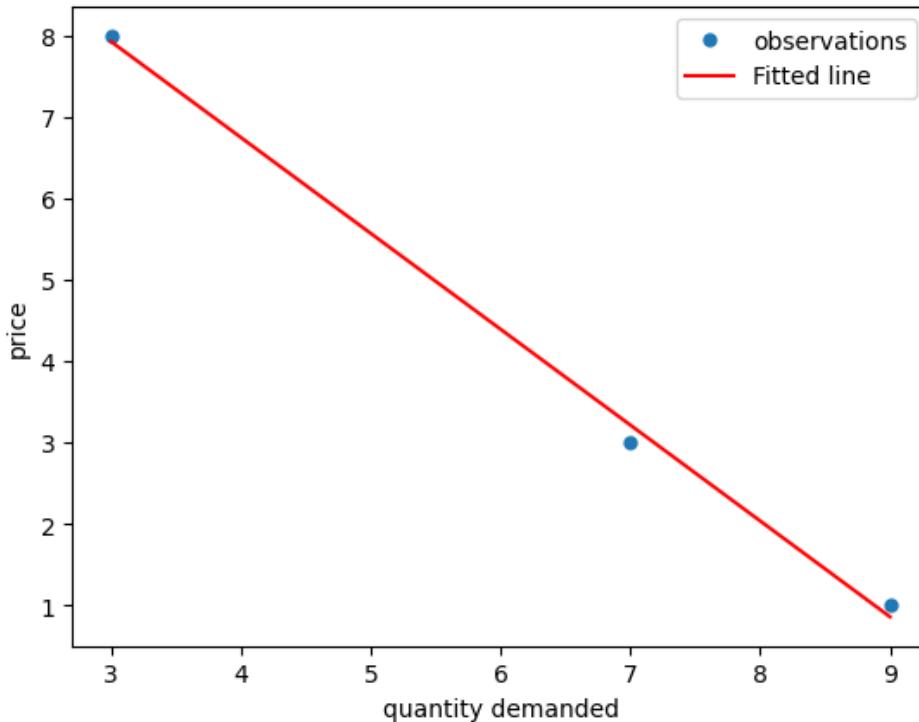
```
hat_x = [[11.46428571]
          [ 1.17857143]]
||Ahat_x - b||^2 = 0.07142857142857081
```

Here is a visualization of how the least squares method approximates the equation of a line connecting a set of points. We can also describe this as “fitting” a line between a set of points.

```
fig, ax = plt.subplots()
p = np.array((1, 3, 8))
q = np.array((9, 7, 3))

a, b = x

ax.plot(q, p, 'o', label='observations', markersize=5)
ax.plot(q, a - b*q, 'r', label='Fitted line')
plt.xlabel('quantity demanded')
plt.ylabel('price')
plt.legend()
plt.show()
```



### 5.6.1 Further reading

The documentation of the `numpy.linalg` submodule can be found [here](#).

More advanced topics in linear algebra can be found [here](#).

## EIGENVALUES AND EIGENVECTORS

### 6.1 Overview

Eigenvalues and eigenvectors are a relatively advanced topic in linear algebra.

At the same time, these concepts are extremely useful for

- economic modeling (especially dynamics!)
- statistics
- some parts of applied mathematics
- machine learning
- and many other fields of science.

In this lecture we explain the basics of eigenvalues and eigenvectors and introduce the Neumann Series Lemma.

We assume in this lecture that students are familiar with matrices and understand *the basics of matrix algebra*.

We will use the following imports:

```
import matplotlib.pyplot as plt
import numpy as np
from numpy.linalg import matrix_power
from matplotlib.lines import Line2D
from matplotlib.patches import FancyArrowPatch
from mpl_toolkits.mplot3d import proj3d
```

### 6.2 Matrices as transformations

Let's start by discussing an important concept concerning matrices.

### 6.2.1 Mapping vectors to vectors

One way to think about a matrix is as a rectangular collection of numbers.

Another way to think about a matrix is as a *map* (i.e., as a function) that transforms vectors to new vectors.

To understand the second point of view, suppose we multiply an  $n \times m$  matrix  $A$  with an  $m \times 1$  column vector  $x$  to obtain an  $n \times 1$  column vector  $y$ :

$$Ax = y$$

If we fix  $A$  and consider different choices of  $x$ , we can understand  $A$  as a map transforming  $x$  to  $Ax$ .

Because  $A$  is  $n \times m$ , it transforms  $m$ -vectors to  $n$ -vectors.

We can write this formally as  $A: \mathbb{R}^m \rightarrow \mathbb{R}^n$ .

You might argue that if  $A$  is a function then we should write  $A(x) = y$  rather than  $Ax = y$  but the second notation is more conventional.

### 6.2.2 Square matrices

Let's restrict our discussion to square matrices.

In the above discussion, this means that  $m = n$  and  $A$  maps  $\mathbb{R}^n$  to itself.

This means  $A$  is an  $n \times n$  matrix that maps (or “transforms”) a vector  $x$  in  $\mathbb{R}^n$  to a new vector  $y = Ax$  also in  $\mathbb{R}^n$ .

Here's one example:

$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Here, the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

transforms the vector  $x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$  to the vector  $y = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ .

Let's visualize this using Python:

```
A = np.array([[2, 1],
             [-1, 1]])
```

```
from math import sqrt

fig, ax = plt.subplots()
# Set the axes through the origin

for spine in ['left', 'bottom']:
    ax.spines[spine].set_position('zero')
for spine in ['right', 'top']:
    ax.spines[spine].set_color('none')

ax.set(xlim=(-2, 6), ylim=(-2, 4), aspect=1)
```

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```

vecs = ((1, 3), (5, 2))
c = ['r', 'black']
for i, v in enumerate(vecs):
    ax.annotate('',
        xy=v, xytext=(0, 0),
        arrowprops=dict(color=c[i],
                        shrink=0,
                        alpha=0.7,
                        width=0.5))

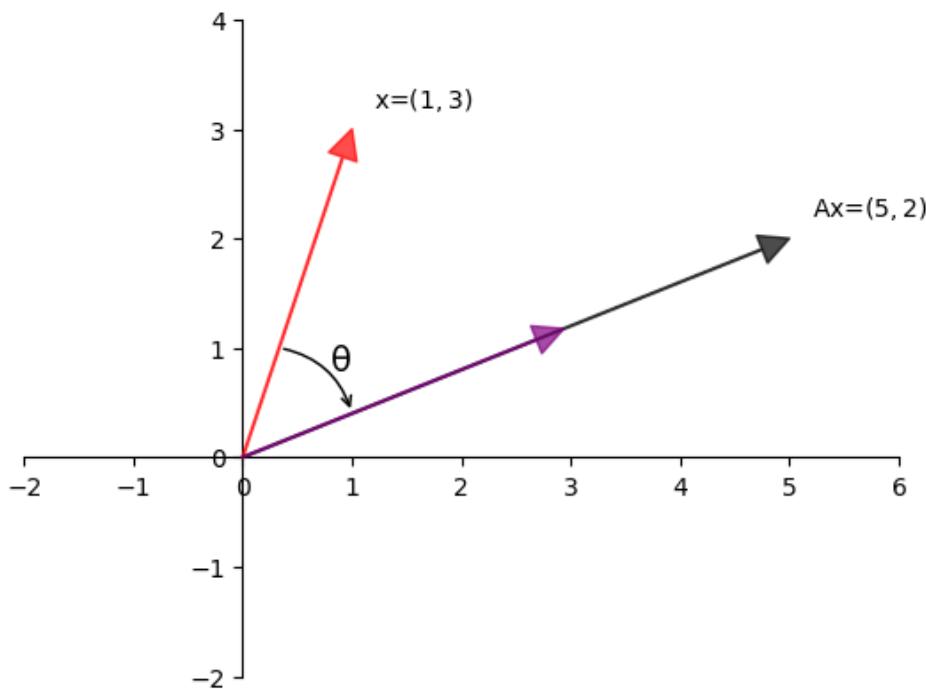
ax.text(0.2 + 1, 0.2 + 3, 'x=$(1,3)$')
ax.text(0.2 + 5, 0.2 + 2, 'Ax=$(5,2)$')

ax.annotate('',
    xy=(sqrt(10/29) * 5, sqrt(10/29) * 2), xytext=(0, 0),
    arrowprops=dict(color='purple',
                    shrink=0,
                    alpha=0.7,
                    width=0.5))

ax.annotate('',
    xy=(1, 2/5), xytext=(1/3, 1),
    arrowprops={'arrowstyle': '->',
                'connectionstyle': 'arc3,rad=-0.3'},
    horizontalalignment='center')
ax.text(0.8, 0.8, f'\theta', fontsize=14)

plt.show()

```



One way to understand this transformation is that  $A$

- first rotates  $x$  by some angle  $\theta$  and
- then scales it by some scalar  $\gamma$  to obtain the image  $y$  of  $x$ .

## 6.3 Types of transformations

Let's examine some standard transformations we can perform with matrices.

Below we visualize transformations by thinking of vectors as points instead of arrows.

We consider how a given matrix transforms

- a grid of points and
- a set of points located on the unit circle in  $\mathbb{R}^2$ .

To build the transformations we will use two functions, called `grid_transform` and `circle_transform`.

Each of these functions visualizes the actions of a given  $2 \times 2$  matrix  $A$ .

### 6.3.1 Scaling

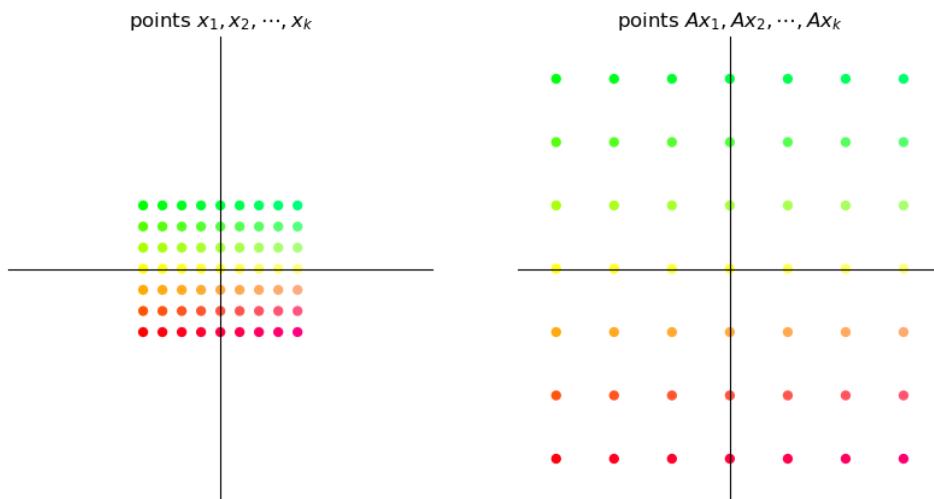
A matrix of the form

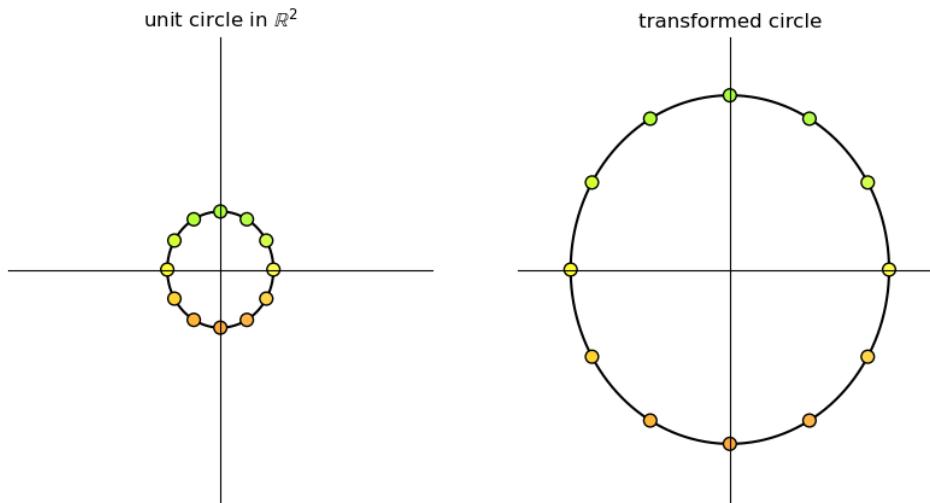
$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

scales vectors across the x-axis by a factor  $\alpha$  and along the y-axis by a factor  $\beta$ .

Here we illustrate a simple example where  $\alpha = \beta = 3$ .

```
A = np.array([[3, 0], # scaling by 3 in both directions
              [0, 3]])
grid_transform(A)
circle_transform(A)
```





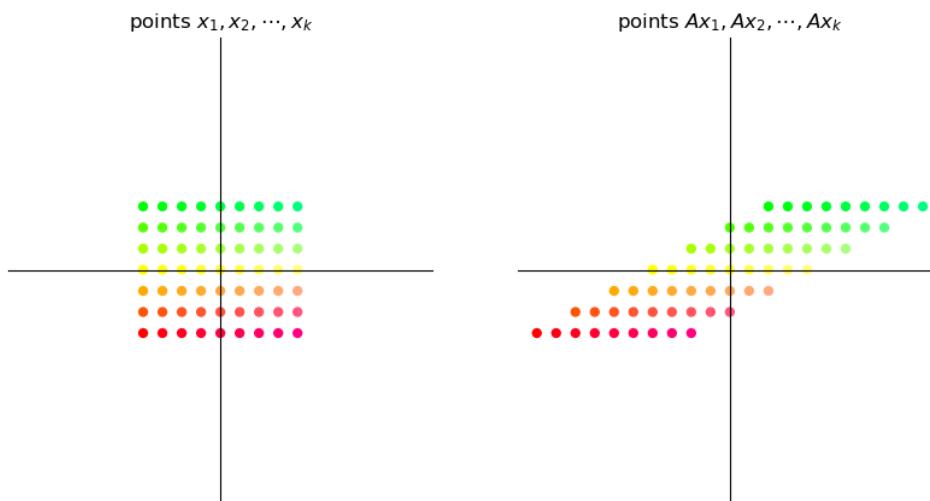
### 6.3.2 Shearing

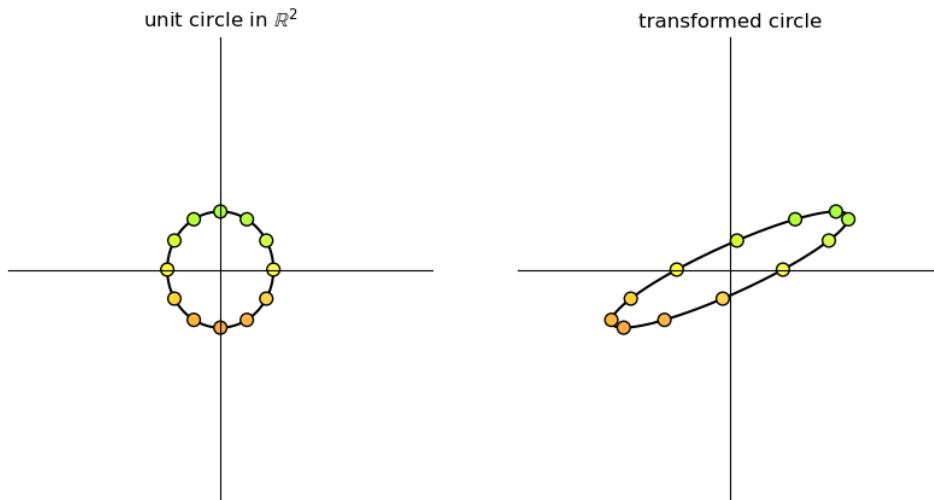
A “shear” matrix of the form

$$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$$

stretches vectors along the x-axis by an amount proportional to the y-coordinate of a point.

```
A = np.array([[1, 2],      # shear along x-axis
              [0, 1]])
grid_transform(A)
circle_transform(A)
```





### 6.3.3 Rotation

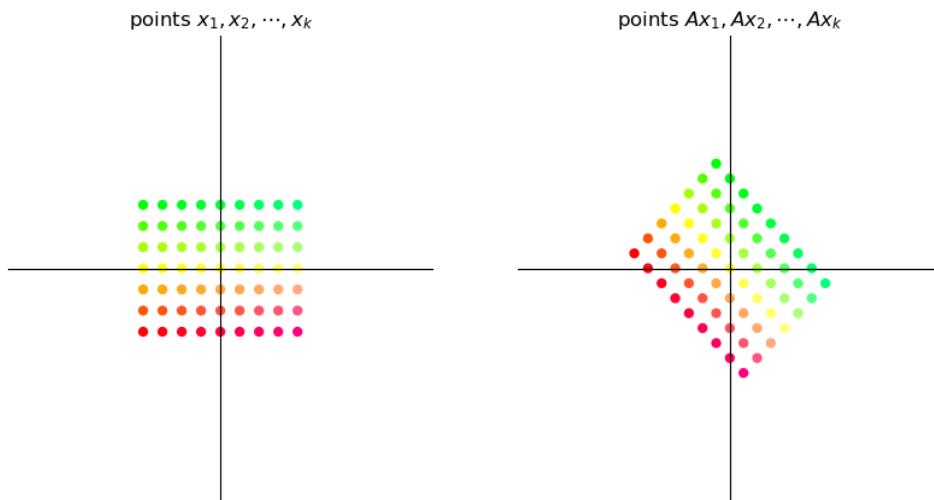
A matrix of the form

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

is called a *rotation matrix*.

This matrix rotates vectors clockwise by an angle  $\theta$ .

```
θ = np.pi/4 # 45 degree clockwise rotation
A = np.array([[np.cos(θ), np.sin(θ)],
              [-np.sin(θ), np.cos(θ)]]))
grid_transform(A)
```



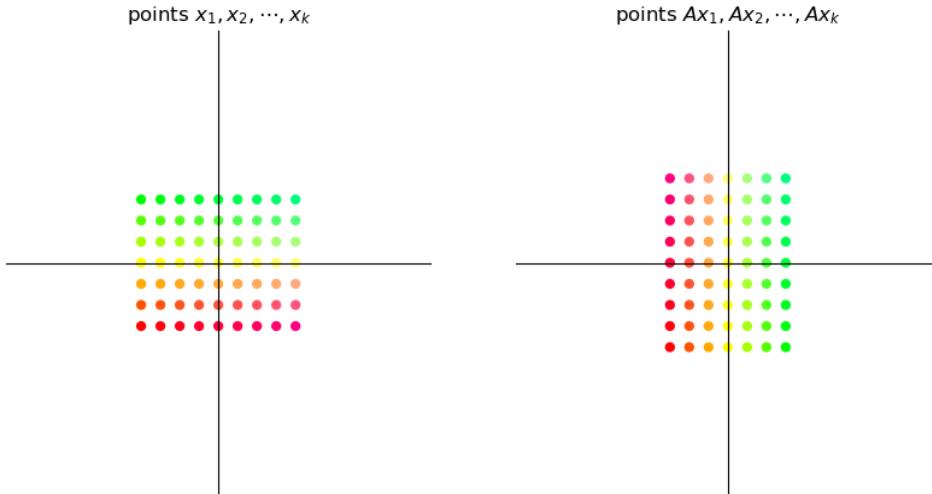
### 6.3.4 Permutation

The permutation matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

interchanges the coordinates of a vector.

```
A = np.column_stack([[0, 1], [1, 0]])
grid_transform(A)
```



More examples of common transition matrices can be found [here](#).

## 6.4 Matrix multiplication as composition

Since matrices act as functions that transform one vector to another, we can apply the concept of function composition to matrices as well.

### 6.4.1 Linear compositions

Consider the two matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

What will the output be when we try to obtain  $ABx$  for some  $2 \times 1$  vector  $x$ ?

$$\begin{array}{c} \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_B \xrightarrow{\quad x \quad} \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}}_{AB} \xrightarrow{\quad 1 \quad} \xrightarrow{\quad y \quad} \begin{bmatrix} 3 \\ -7 \end{bmatrix} \\ \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_B \xrightarrow{\quad x \quad} \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_A \xrightarrow{\quad Bx \quad} \xrightarrow{\quad 7 \quad} \xrightarrow{\quad y \quad} \begin{bmatrix} 3 \\ -7 \end{bmatrix} \end{array}$$

We can observe that applying the transformation  $AB$  on the vector  $x$  is the same as first applying  $B$  on  $x$  and then applying  $A$  on the vector  $Bx$ .

Thus the matrix product  $AB$  is the [composition](#) of the matrix transformations  $A$  and  $B$

This means first apply transformation  $B$  and then transformation  $A$ .

When we matrix multiply an  $n \times m$  matrix  $A$  with an  $m \times k$  matrix  $B$  the obtained matrix product is an  $n \times k$  matrix  $AB$ .

Thus, if  $A$  and  $B$  are transformations such that  $A: \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $B: \mathbb{R}^k \rightarrow \mathbb{R}^m$ , then  $AB$  transforms  $\mathbb{R}^k$  to  $\mathbb{R}^n$ .

Viewing matrix multiplication as composition of maps helps us understand why, under matrix multiplication,  $AB$  is generally not equal to  $BA$ .

(After all, when we compose functions, the order usually matters.)

## 6.4.2 Examples

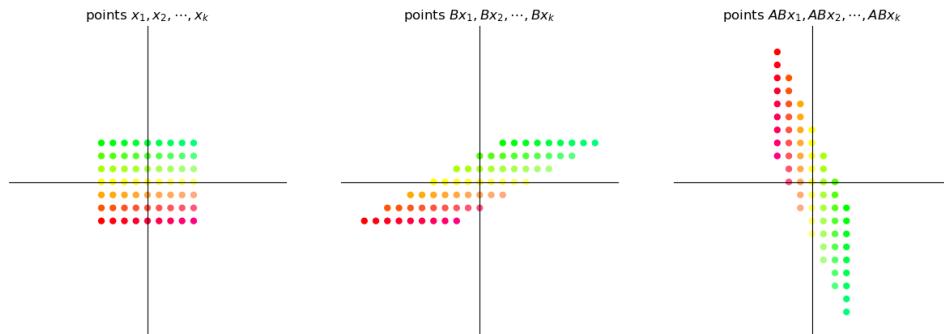
Let  $A$  be the  $90^\circ$  clockwise rotation matrix given by  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  and let  $B$  be a shear matrix along the x-axis given by  $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ .

We will visualize how a grid of points changes when we apply the transformation  $AB$  and then compare it with the transformation  $BA$ .

```
A = np.array([[0, 1],      # 90 degree clockwise rotation
              [-1, 0]])
B = np.array([[1, 2],      # shear along x-axis
              [0, 1]])
```

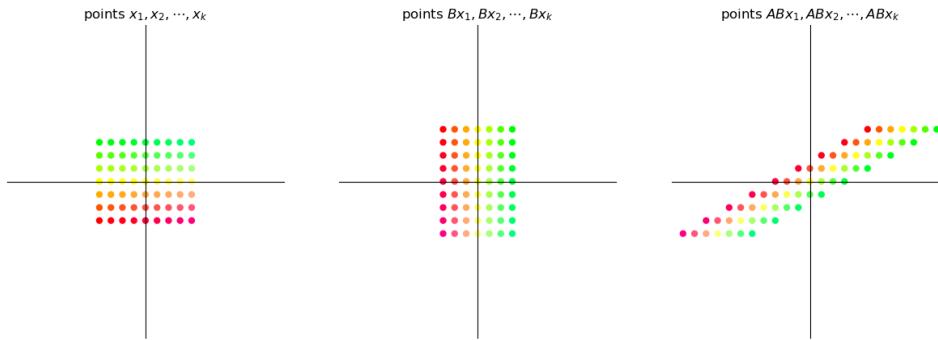
### Shear then rotate

```
grid_composition_transform(A, B)  # transformation AB
```



### Rotate then shear

```
grid_composition_transform(B, A)          # transformation BA
```



It is evident that the transformation  $AB$  is not the same as the transformation  $BA$ .

## 6.5 Iterating on a fixed map

In economics (and especially in dynamic modeling), we are often interested in analyzing behavior where we repeatedly apply a fixed matrix.

For example, given a vector  $v$  and a matrix  $A$ , we are interested in studying the sequence

$$v, \quad Av, \quad AA v = A^2 v, \quad \dots$$

Let's first see examples of a sequence of iterates  $(A^k v)_{k \geq 0}$  under different maps  $A$ .

```
def plot_series(A, v, n):
    B = np.array([[1, -1],
                  [1, 0]])

    fig, ax = plt.subplots()

    ax.set(xlim=(-4, 4), ylim=(-4, 4))
    ax.set_xticks([])
    ax.set_yticks([])
    for spine in ['left', 'bottom']:
        ax.spines[spine].set_position('zero')
    for spine in ['right', 'top']:
        ax.spines[spine].set_color('none')

    theta = np.linspace(0, 2 * np.pi, 150)
    r = 2.5
    x = r * np.cos(theta)
    y = r * np.sin(theta)
    x1 = x.reshape(1, -1)
    y1 = y.reshape(1, -1)
    xy = np.concatenate((x1, y1), axis=0)

    ellipse = B @ xy
    ax.plot(ellipse[0, :], ellipse[1, :], color='black',
```

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```

        linestyle=(0, (5, 10)), linewidth=0.5)

# Initialize holder for trajectories
colors = plt.cm.rainbow(np.linspace(0, 1, 20))

for i in range(n):
    iteration = matrix_power(A, i) @ v
    v1 = iteration[0]
    v2 = iteration[1]
    ax.scatter(v1, v2, color=colors[i])
    if i == 0:
        ax.text(v1+0.25, v2, f'$v$')
    elif i == 1:
        ax.text(v1+0.25, v2, f'$Av$')
    elif 1 < i < 4:
        ax.text(v1+0.25, v2, f'$A^{i}v$')
plt.show()

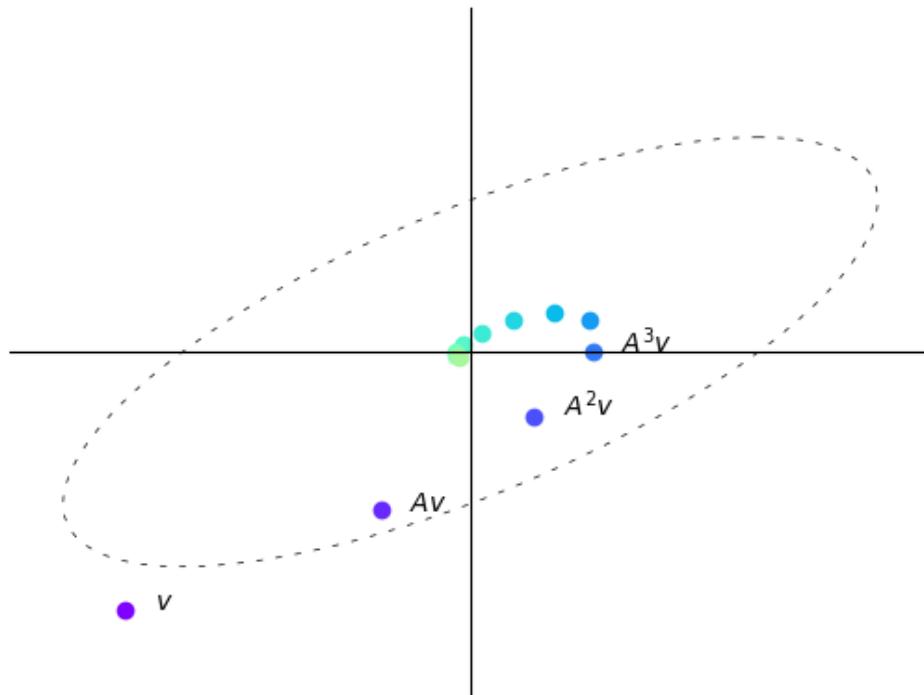
```

```

A = np.array([[sqrt(3) + 1, -2],
              [1, sqrt(3) - 1]])
A = (1/(2*sqrt(2))) * A
v = (-3, -3)
n = 12

plot_series(A, v, n)

```



Here with each iteration the vectors get shorter, i.e., move closer to the origin.

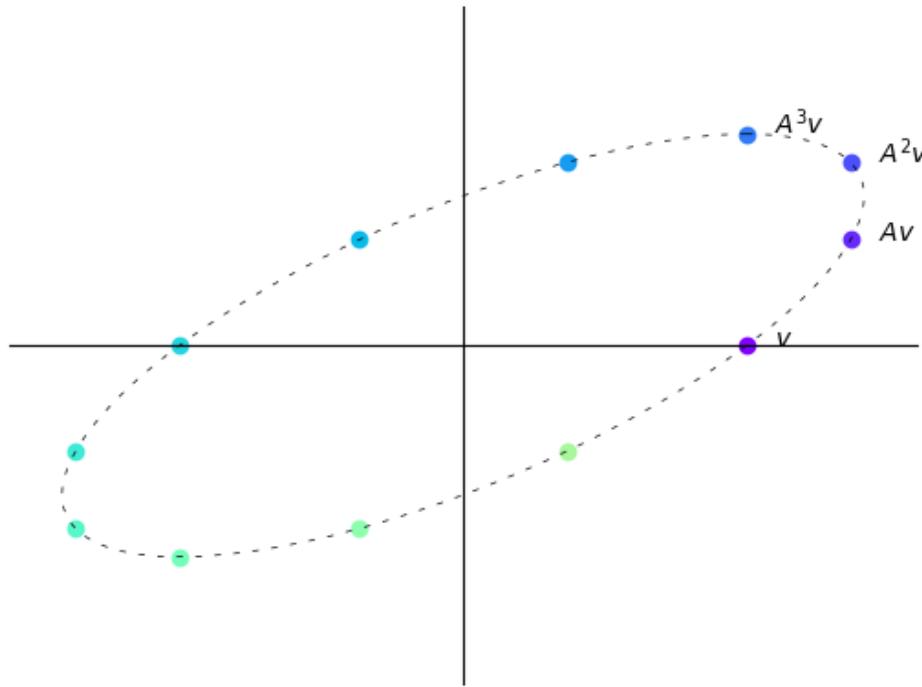
In this case, repeatedly multiplying a vector by  $A$  makes the vector “spiral in”.

```

B = np.array([[sqrt(3) + 1, -2],
              [1, sqrt(3) - 1]])
B = (1/2) * B
v = (2.5, 0)
n = 12

plot_series(B, v, n)

```



Here with each iteration vectors do not tend to get longer or shorter.

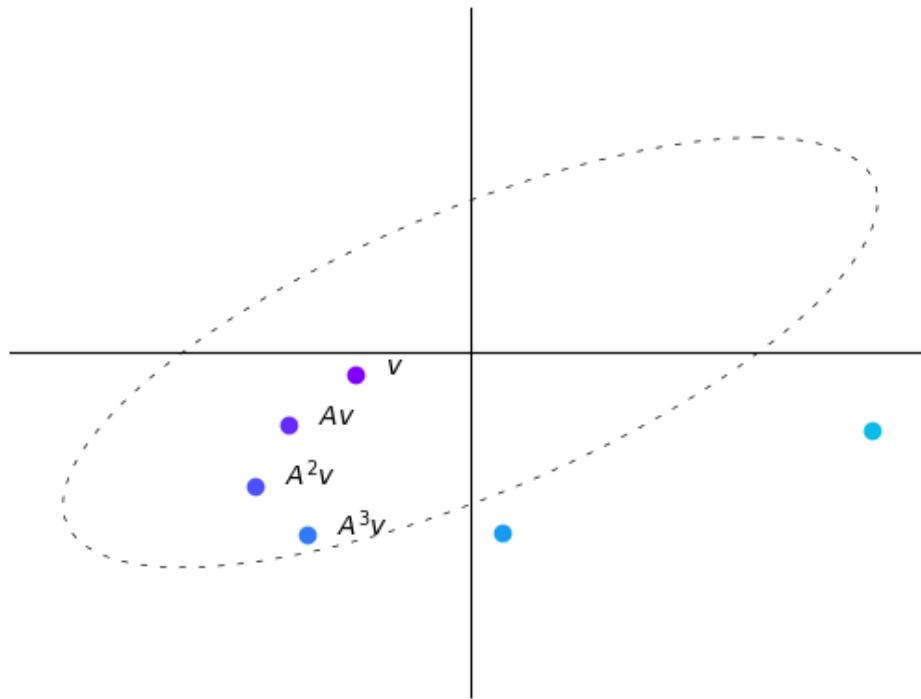
In this case, repeatedly multiplying a vector by  $A$  simply “rotates it around an ellipse”.

```

B = np.array([[sqrt(3) + 1, -2],
              [1, sqrt(3) - 1]])
B = (1/sqrt(2)) * B
v = (-1, -0.25)
n = 6

plot_series(B, v, n)

```



Here with each iteration vectors tend to get longer, i.e., farther from the origin.

In this case, repeatedly multiplying a vector by  $A$  makes the vector “spiral out”.

We thus observe that the sequence  $(A^k v)_{k \geq 0}$  behaves differently depending on the map  $A$  itself.

We now discuss the property of  $A$  that determines this behavior.

## 6.6 Eigenvalues

In this section we introduce the notions of eigenvalues and eigenvectors.

### 6.6.1 Definitions

Let  $A$  be an  $n \times n$  square matrix.

If  $\lambda$  is scalar and  $v$  is a non-zero  $n$ -vector such that

$$Av = \lambda v.$$

Then we say that  $\lambda$  is an *eigenvalue* of  $A$ , and  $v$  is the corresponding *eigenvector*.

Thus, an eigenvector of  $A$  is a nonzero vector  $v$  such that when the map  $A$  is applied,  $v$  is merely scaled.

The next figure shows two eigenvectors (blue arrows) and their images under  $A$  (red arrows).

As expected, the image  $Av$  of each  $v$  is just a scaled version of the original

```
from numpy.linalg import eig
A = [[1, 2],
```

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```

[2, 1]]
A = np.array(A)
evals, evecs = eig(A)
evecs = evecs[:, 0], evecs[:, 1]

fig, ax = plt.subplots(figsize=(10, 8))
# Set the axes through the origin
for spine in ['left', 'bottom']:
    ax.spines[spine].set_position('zero')
for spine in ['right', 'top']:
    ax.spines[spine].set_color('none')
# ax.grid(alpha=0.4)

xmin, xmax = -3, 3
ymin, ymax = -3, 3
ax.set(xlim=(xmin, xmax), ylim=(ymin, ymax))

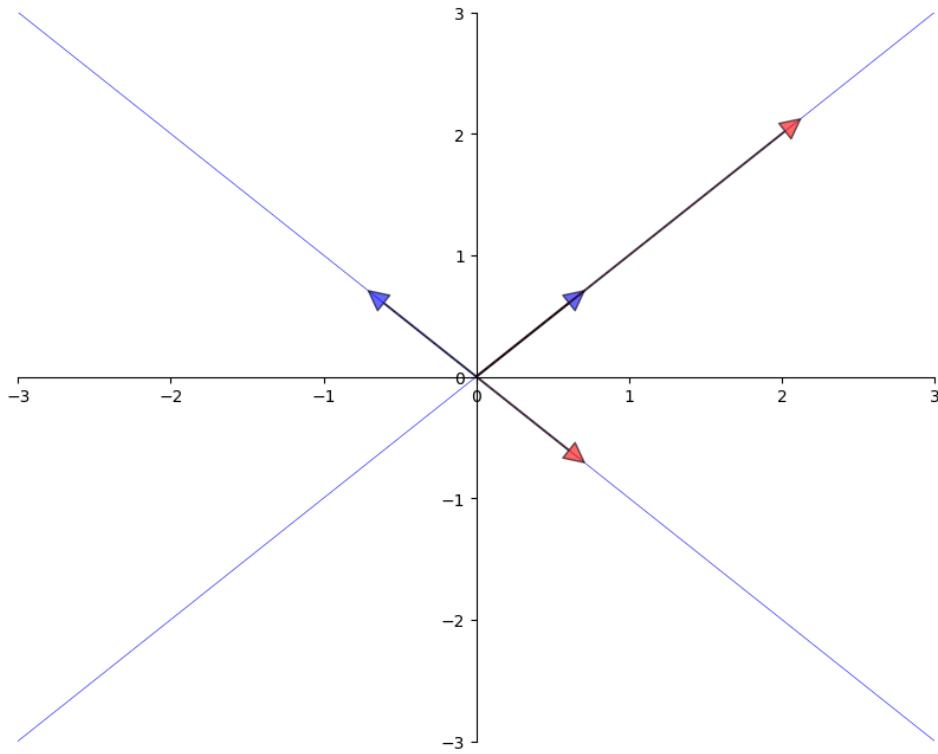
# Plot each eigenvector
for v in evecs:
    ax.annotate('',
        xy=v, xytext=(0, 0),
        arrowprops=dict(facecolor='blue',
                        shrink=0,
                        alpha=0.6,
                        width=0.5))

# Plot the image of each eigenvector
for v in evecs:
    v = A @ v
    ax.annotate('',
        xy=v, xytext=(0, 0),
        arrowprops=dict(facecolor='red',
                        shrink=0,
                        alpha=0.6,
                        width=0.5))

# Plot the lines they run through
x = np.linspace(xmin, xmax, 3)
for v in evecs:
    a = v[1] / v[0]
    ax.plot(x, a * x, 'b-', lw=0.4)

plt.show()

```



## 6.6.2 Complex values

So far our definition of eigenvalues and eigenvectors seems straightforward.

There is one complication we haven't mentioned yet:

When solving  $Av = \lambda v$ ,

- $\lambda$  is allowed to be a complex number and
- $v$  is allowed to be an  $n$ -vector of complex numbers.

We will see some examples below.

## 6.6.3 Some mathematical details

We note some mathematical details for more advanced readers.

(Other readers can skip to the next section.)

The eigenvalue equation is equivalent to  $(A - \lambda I)v = 0$ .

This equation has a nonzero solution  $v$  only when the columns of  $A - \lambda I$  are linearly dependent.

This in turn is equivalent to stating the determinant is zero.

Hence, to find all eigenvalues, we can look for  $\lambda$  such that the determinant of  $A - \lambda I$  is zero.

This problem can be expressed as one of solving for the roots of a polynomial in  $\lambda$  of degree  $n$ .

This in turn implies the existence of  $n$  solutions in the complex plane, although some might be repeated.

## 6.6.4 Facts

Some nice facts about the eigenvalues of a square matrix  $A$  are as follows:

1. the determinant of  $A$  equals the product of the eigenvalues
2. the trace of  $A$  (the sum of the elements on the principal diagonal) equals the sum of the eigenvalues
3. if  $A$  is symmetric, then all of its eigenvalues are real
4. if  $A$  is invertible and  $\lambda_1, \dots, \lambda_n$  are its eigenvalues, then the eigenvalues of  $A^{-1}$  are  $1/\lambda_1, \dots, 1/\lambda_n$ .

A corollary of the last statement is that a matrix is invertible if and only if all its eigenvalues are nonzero.

## 6.6.5 Computation

Using NumPy, we can solve for the eigenvalues and eigenvectors of a matrix as follows

```
from numpy.linalg import eig

A = ((1, 2),
      (2, 1))

A = np.array(A)
evals, evecs = eig(A)
evals # eigenvalues

array([ 3., -1.])

evecs # eigenvectors

array([[ 0.70710678, -0.70710678],
       [ 0.70710678,  0.70710678]])
```

Note that the *columns* of `evecs` are the eigenvectors.

Since any scalar multiple of an eigenvector is an eigenvector with the same eigenvalue (which can be verified), the `eig` routine normalizes the length of each eigenvector to one.

The eigenvectors and eigenvalues of a map  $A$  determine how a vector  $v$  is transformed when we repeatedly multiply by  $A$ .

This is discussed further later.

## 6.7 The Neumann Series Lemma

In this section we present a famous result about series of matrices that has many applications in economics.

### 6.7.1 Scalar series

Here's a fundamental result about series:

If  $a$  is a number and  $|a| < 1$ , then

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} = (1-a)^{-1} \quad (6.1)$$

For a one-dimensional linear equation  $x = ax + b$  where  $x$  is unknown we can thus conclude that the solution  $x^*$  is given by:

$$x^* = \frac{b}{1-a} = \sum_{k=0}^{\infty} a^k b$$

### 6.7.2 Matrix series

A generalization of this idea exists in the matrix setting.

Consider the system of equations  $x = Ax + b$  where  $A$  is an  $n \times n$  square matrix and  $x$  and  $b$  are both column vectors in  $\mathbb{R}^n$ .

Using matrix algebra we can conclude that the solution to this system of equations will be given by:

$$x^* = (I - A)^{-1}b \quad (6.2)$$

What guarantees the existence of a unique vector  $x^*$  that satisfies (6.2)?

The following is a fundamental result in functional analysis that generalizes (6.1) to a multivariate case.

---

#### Theorem 6.7.1 (Neumann Series Lemma)

Let  $A$  be a square matrix and let  $A^k$  be the  $k$ -th power of  $A$ .

Let  $r(A)$  be the **spectral radius** of  $A$ , defined as  $\max_i |\lambda_i|$ , where

- $\{\lambda_i\}_i$  is the set of eigenvalues of  $A$  and
- $|\lambda_i|$  is the modulus of the complex number  $\lambda_i$

Neumann's Theorem states the following: If  $r(A) < 1$ , then  $I - A$  is invertible, and

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$$


---

We can see the Neumann Series Lemma in action in the following example.

```
A = np.array([[0.4, 0.1],
              [0.7, 0.2]])

evals, evecs = eig(A)    # finding eigenvalues and eigenvectors

r = max(abs(lam) for lam in evals)    # compute spectral radius
print(r)
```

```
0.582842712474619
```

The spectral radius  $r(A)$  obtained is less than 1.

Thus, we can apply the Neumann Series Lemma to find  $(I - A)^{-1}$ .

```
I = np.identity(2)  # 2 x 2 identity matrix
B = I - A

B_inverse = np.linalg.inv(B)  # direct inverse method

A_sum = np.zeros((2, 2))  # power series sum of A
A_power = I
for i in range(50):
    A_sum += A_power
    A_power = A_power @ A
```

Let's check equality between the sum and the inverse methods.

```
np.allclose(A_sum, B_inverse)
```

```
True
```

Although we truncate the infinite sum at  $k = 50$ , both methods give us the same result which illustrates the result of the Neumann Series Lemma.

## 6.8 Exercises

---

### Exercise 6.8.1

Power iteration is a method for finding the greatest absolute eigenvalue of a diagonalizable matrix.

The method starts with a random vector  $b_0$  and repeatedly applies the matrix  $A$  to it

$$b_{k+1} = \frac{Ab_k}{\|Ab_k\|}$$

A thorough discussion of the method can be found [here](#).

In this exercise, first implement the power iteration method and use it to find the greatest absolute eigenvalue and its corresponding eigenvector.

Then visualize the convergence.

---



---

### Solution to Exercise 6.8.1

Here is one solution.

We start by looking into the distance between the eigenvector approximation and the true eigenvector.

```

# Define a matrix A
A = np.array([[1, 0, 3],
              [0, 2, 0],
              [3, 0, 1]])

num_iters = 20

# Define a random starting vector b
b = np.random.rand(A.shape[1])

# Get the leading eigenvector of matrix A
eigenvector = np.linalg.eig(A)[1][:, 0]

errors = []
res = []

# Power iteration loop
for i in range(num_iters):
    # Multiply b by A
    b = A @ b
    # Normalize b
    b = b / np.linalg.norm(b)
    # Append b to the list of eigenvector approximations
    res.append(b)
    err = np.linalg.norm(np.array(b) - eigenvector)
    errors.append(err)

greatest_eigenvalue = np.dot(A @ b, b) / np.dot(b, b)
print(f'The approximated greatest absolute eigenvalue is \
      {greatest_eigenvalue:.2f}')
print('The real eigenvalue is', np.linalg.eig(A)[0])

# Plot the eigenvector approximations for each iteration
plt.figure(figsize=(10, 6))
plt.xlabel('iterations')
plt.ylabel('error')
_ = plt.plot(errors)

```

```

The approximated greatest absolute eigenvalue is      4.00
The real eigenvalue is [ 4. -2.  2.]

```

Then we can look at the trajectory of the eigenvector approximation.

```

# Set up the figure and axis for 3D plot
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')

# Plot the eigenvectors
ax.scatter(eigenvector[0],
           eigenvector[1],
           eigenvector[2],
           color='r', s=80)

for i, vec in enumerate(res):
    ax.scatter(vec[0], vec[1], vec[2],

```

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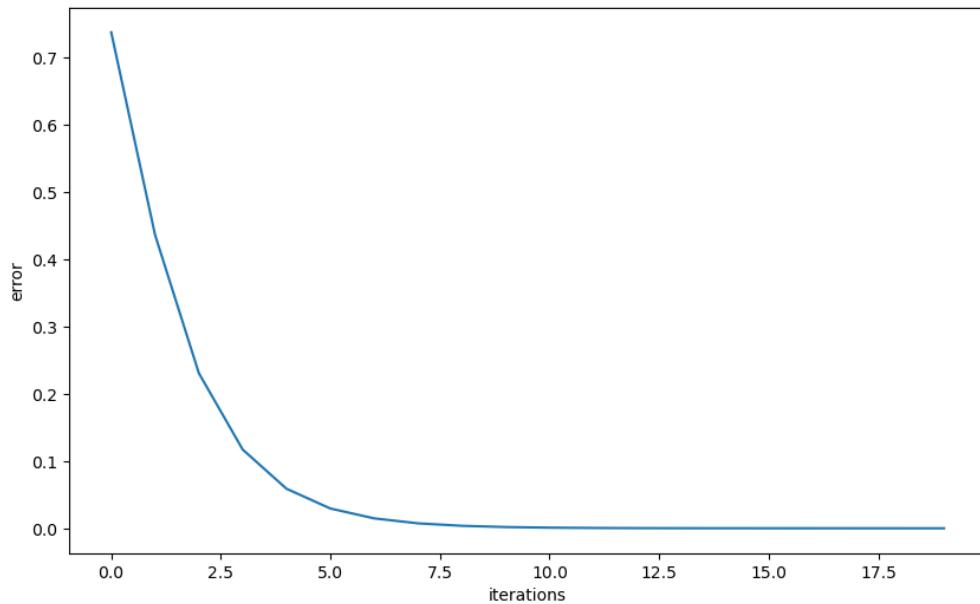


Fig. 6.1: Power iteration

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```

color='b',
alpha=(i+1)/(num_iters+1),
s=80)

ax.set_xlabel('x')
ax.set_ylabel('y')
ax.set_zlabel('z')
ax.tick_params(axis='both', which='major', labelsize=7)

points = [plt.Line2D([0], [0], linestyle='none',
                    c=i, marker='o') for i in ['r', 'b']]
ax.legend(points, ['actual eigenvector',
                   r'approximated eigenvector ($b_k$)'])
ax.set_box_aspect(aspect=None, zoom=0.8)

plt.show()

```

---

### Exercise 6.8.2

We have discussed the trajectory of the vector  $v$  after being transformed by  $A$ .

Consider the matrix  $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$  and the vector  $v = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ .

Try to compute the trajectory of  $v$  after being transformed by  $A$  for  $n = 4$  iterations and plot the result.

---

### Solution to Exercise 6.8.2

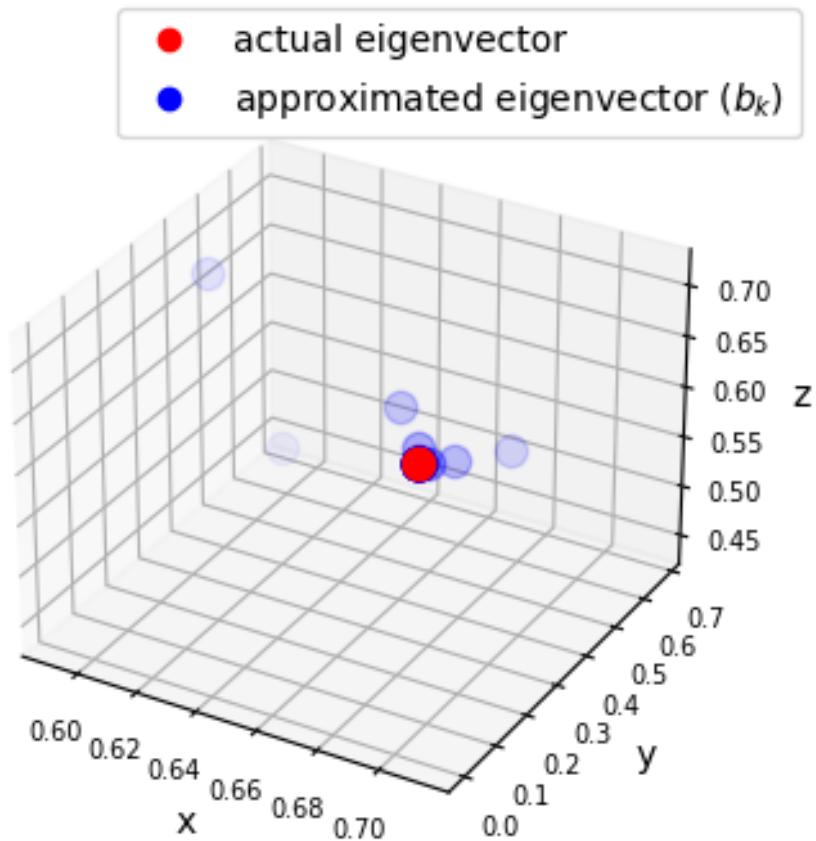


Fig. 6.2: Power iteration trajectory

```

A = np.array([[1, 2],
              [1, 1]])
v = (0.4, -0.4)
n = 11

# Compute eigenvectors and eigenvalues
eigenvalues, eigenvectors = np.linalg.eig(A)

print(f'eigenvalues:\n {eigenvalues}')
print(f'eigenvectors:\n {eigenvectors}')

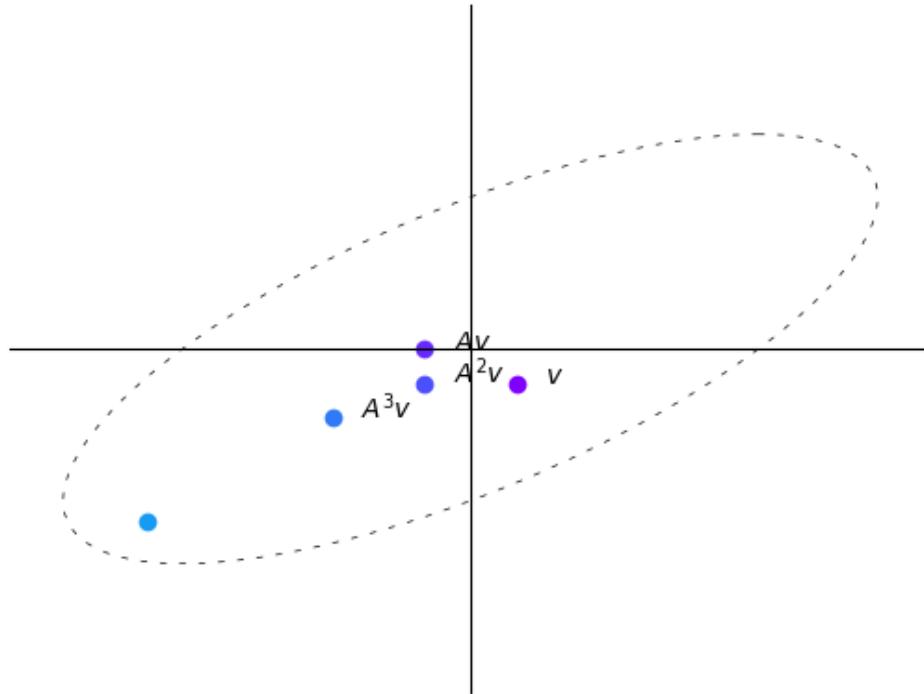
plot_series(A, v, n)

```

```

eigenvalues:
[ 2.41421356 -0.41421356]
eigenvectors:
[[ 0.81649658 -0.81649658]
 [ 0.57735027  0.57735027]]

```



The result seems to converge to the eigenvector of  $A$  with the largest eigenvalue.

Let's use a `vector field` to visualize the transformation brought by  $A$ .

(This is a more advanced topic in linear algebra, please step ahead if you are comfortable with the math.)

```

# Create a grid of points
x, y = np.meshgrid(np.linspace(-5, 5, 15),
                   np.linspace(-5, 5, 20))

# Apply the matrix A to each point in the vector field
vec_field = np.stack([x, y])

```

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```

u, v = np.tensordot(A, vec_field, axes=1)

# Plot the transformed vector field
c = plt.streamplot(x, y, u - x, v - y,
                    density=1, linewidth=None, color="#A23BEC")
c.lines.set_alpha(0.5)
c.arrows.set_alpha(0.5)

# Draw eigenvectors
origin = np.zeros((2, len(eigenvectors)))
parameters = {'color': ['b', 'g'], 'angles': 'xy',
              'scale_units': 'xy', 'scale': 0.1, 'width': 0.01}
plt.quiver(*origin, eigenvectors[0],
            eigenvectors[1], **parameters)
plt.quiver(*origin, -eigenvectors[0],
            -eigenvectors[1], **parameters)

colors = ['b', 'g']
lines = [Line2D([0], [0], color=c, linewidth=3) for c in colors]
labels = ["2.4 eigenspace", "0.4 eigenspace"]
plt.legend(lines, labels, loc='center left',
           bbox_to_anchor=(1, 0.5))

plt.xlabel("x")
plt.ylabel("y")
plt.grid()
plt.gca().set_aspect('equal', adjustable='box')
plt.show()

```

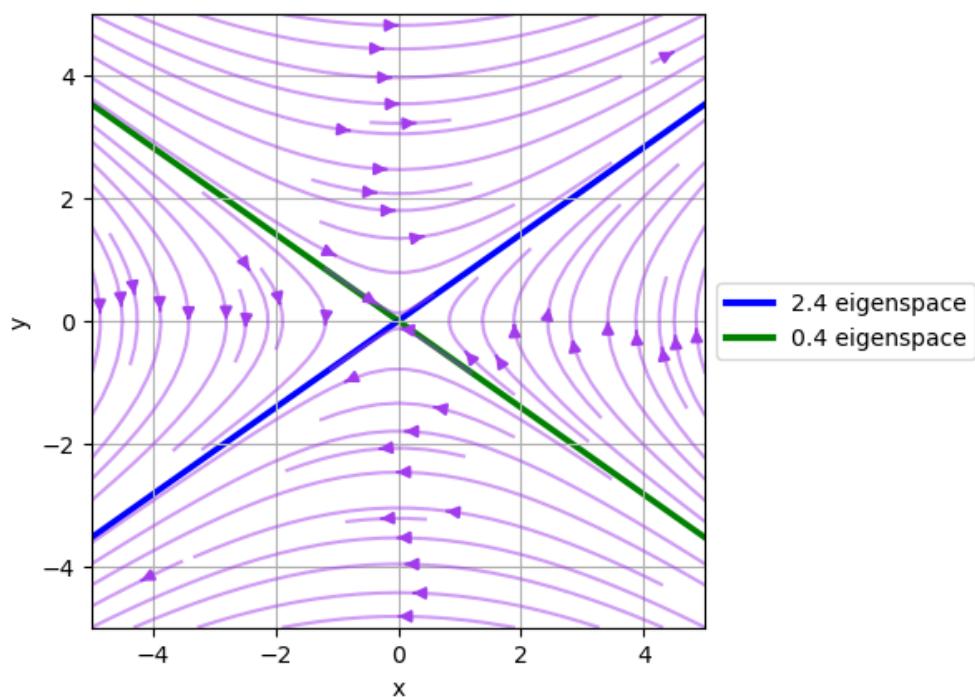


Fig. 6.3: Convergence towards eigenvectors

Note that the vector field converges to the eigenvector of  $A$  with the largest eigenvalue and diverges from the eigenvector of  $A$  with the smallest eigenvalue.

In fact, the eigenvectors are also the directions in which the matrix  $A$  stretches or shrinks the space.

Specifically, the eigenvector with the largest eigenvalue is the direction in which the matrix  $A$  stretches the space the most.

We will see more intriguing examples in the following exercise.

---

### Exercise 6.8.3

*Previously*, we demonstrated the trajectory of the vector  $v$  after being transformed by  $A$  for three different matrices.

Use the visualization in the previous exercise to explain the trajectory of the vector  $v$  after being transformed by  $A$  for the three different matrices.

---

### Solution to Exercise 6.8.3

Here is one solution

```
figure, ax = plt.subplots(1, 3, figsize=(15, 5))
A = np.array([[sqrt(3) + 1, -2],
              [1, sqrt(3) - 1]])
A = (1/(2*sqrt(2))) * A

B = np.array([[sqrt(3) + 1, -2],
              [1, sqrt(3) - 1]])
B = (1/2) * B

C = np.array([[sqrt(3) + 1, -2],
              [1, sqrt(3) - 1]])
C = (1/sqrt(2)) * C

examples = [A, B, C]

for i, example in enumerate(examples):
    M = example

    # Compute right eigenvectors and eigenvalues
    eigenvalues, eigenvectors = np.linalg.eig(M)
    print(f'Example {i+1}:\n')
    print(f'eigenvalues:\n {eigenvalues}\n')
    print(f'eigenvectors:\n {eigenvectors}\n')

    eigenvalues_real = eigenvalues.real
    eigenvectors_real = eigenvectors.real

    # Create a grid of points
    x, y = np.meshgrid(np.linspace(-20, 20, 15),
                       np.linspace(-20, 20, 20))

    # Apply the matrix A to each point in the vector field
    vec_field = np.stack([x, y])
    u, v = np.tensordot(M, vec_field, axes=1)
```

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```

# Plot the transformed vector field
c = ax[i].streamplot(x, y, u - x, v - y, density=1,
                      linewidth=None, color='#A23BEC')
c.lines.set_alpha(0.5)
c.arrows.set_alpha(0.5)

# Draw eigenvectors
parameters = {'color': ['b', 'g'], 'angles': 'xy',
               'scale_units': 'xy', 'scale': 1,
               'width': 0.01, 'alpha': 0.5}
origin = np.zeros((2, len(eigenvectors)))
ax[i].quiver(*origin, eigenvectors_real[0],
              eigenvectors_real[1], **parameters)
ax[i].quiver(*origin,
              - eigenvectors_real[0],
              - eigenvectors_real[1],
              **parameters)

ax[i].set_xlabel("x-axis")
ax[i].set_ylabel("y-axis")
ax[i].grid()
ax[i].set_aspect('equal', adjustable='box')

plt.show()

```

Example 1:

```

eigenvalues:
[0.61237244+0.35355339j 0.61237244-0.35355339j]
eigenvectors:
[[0.81649658+0.j 0.81649658-0.j]
 [0.40824829-0.40824829j 0.40824829+0.40824829j]]

```

Example 2:

```

eigenvalues:
[0.8660254+0.5j 0.8660254-0.5j]
eigenvectors:
[[0.81649658+0.j 0.81649658-0.j]
 [0.40824829-0.40824829j 0.40824829+0.40824829j]]

```

Example 3:

```

eigenvalues:
[1.22474487+0.70710678j 1.22474487-0.70710678j]
eigenvectors:
[[0.81649658+0.j 0.81649658-0.j]
 [0.40824829-0.40824829j 0.40824829+0.40824829j]]

```

The vector fields explain why we observed the trajectories of the vector  $v$  multiplied by  $A$  iteratively before.

The pattern demonstrated here is because we have complex eigenvalues and eigenvectors.

We can plot the complex plane for one of the matrices using `Arrow3D` class retrieved from [stackoverflow](#).

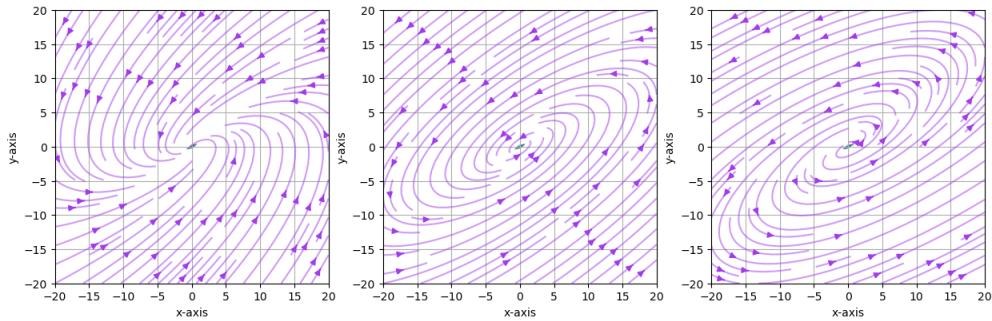


Fig. 6.4: Vector fields of the three matrices

```

class Arrow3D(FancyArrowPatch):
    def __init__(self, xs, ys, zs, *args, **kwargs):
        super().__init__((0, 0), (0, 0), *args, **kwargs)
        self._verts3d = xs, ys, zs

    def do_3d_projection(self):
        xs3d, ys3d, zs3d = self._verts3d
        xs, ys, zs = proj3d.proj_transform(xs3d, ys3d, zs3d,
                                           self.axes.M)
        self.set_positions((0.1*xs[0], 0.1*ys[0]),
                           (0.1*xs[1], 0.1*ys[1]))

    return np.min(zs)

eigenvalues, eigenvectors = np.linalg.eig(A)

# Create meshgrid for vector field
x, y = np.meshgrid(np.linspace(-2, 2, 15),
                   np.linspace(-2, 2, 15))

# Calculate vector field (real and imaginary parts)
u_real = A[0][0] * x + A[0][1] * y
v_real = A[1][0] * x + A[1][1] * y
u_imag = np.zeros_like(x)
v_imag = np.zeros_like(y)

# Create 3D figure
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
vlength = np.linalg.norm(eigenvectors)
ax.quiver(x, y, u_imag, u_real-x, v_real-y, v_imag-u_imag,
           colors='b', alpha=0.3, length=.2,
           arrow_length_ratio=0.01)

arrow_prop_dict = dict(mutation_scale=5,
                      arrowstyle='->', shrinkA=0, shrinkB=0)

# Plot 3D eigenvectors
for c, i in zip(['b', 'g'], [0, 1]):
    a = Arrow3D([0, eigenvectors[0][i].real],
               [0, eigenvectors[1][i].real],

```

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```
[0, eigenvectors[1][i].imag],  
color=c, **arrow_prop_dict)  
ax.add_artist(a)  
  
# Set axis labels and title  
ax.set_xlabel('x')  
ax.set_ylabel('y')  
ax.set_zlabel('Im')  
ax.set_box_aspect(aspect=None, zoom=0.8)  
  
plt.draw()  
plt.show()
```

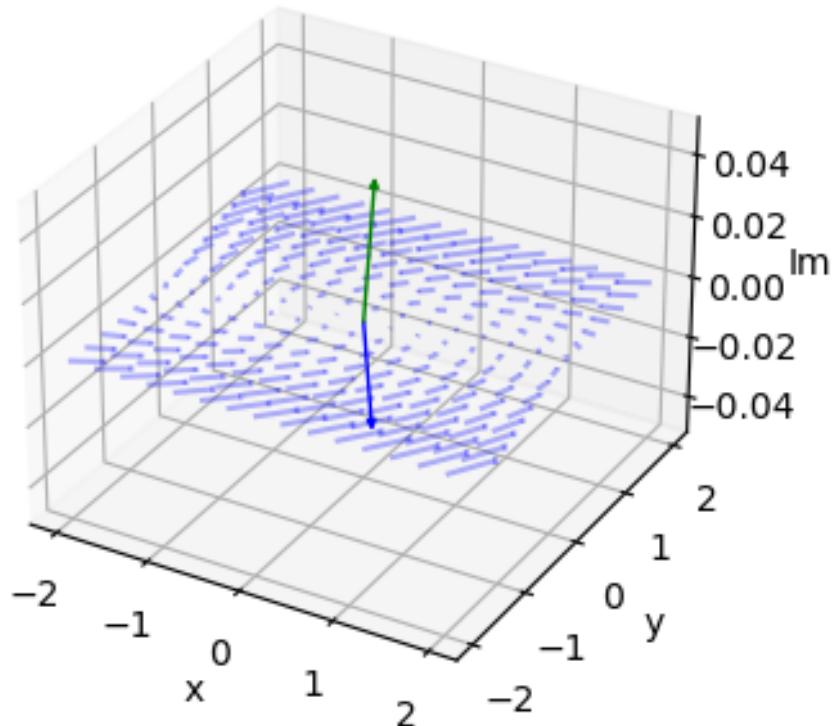


Fig. 6.5: 3D plot of the vector field

## INTRODUCTION TO SUPPLY AND DEMAND

### 7.1 Overview

This lecture is about some models of equilibrium prices and quantities, one of the main topics of elementary microeconomics.

Throughout the lecture, we focus on models with one good and one price.

In a *subsequent lecture* we will investigate settings with many goods.

Key infrastructure concepts that we'll encounter in this lecture are

- inverse demand curves
- inverse supply curves
- consumer surplus
- producer surplus
- social welfare as the sum of consumer and producer surpluses
- relationship between equilibrium quantity and social welfare optimum

Throughout the lectures, we'll assume that inverse demand and supply curves are **affine** functions of quantity.

(“Affine” means “linear plus a constant” and [here](#) is a nice discussion about it.)

We'll also assume affine inverse supply and demand functions when we study models with multiple consumption goods in our *subsequent lecture*.

We do this in order to simplify the exposition and enable us to use just a few tools from linear algebra, namely, matrix multiplication and matrix inversion.

In our exposition we will use the following imports.

```
import numpy as np
import matplotlib.pyplot as plt
```

## 7.2 Supply and demand

We study a market for a single good in which buyers and sellers exchange a quantity  $q$  for a price  $p$ .

Quantity  $q$  and price  $p$  are both scalars.

We assume that inverse demand and supply curves for the good are:

$$p = d_0 - d_1 q, \quad d_0, d_1 > 0$$

$$p = s_0 + s_1 q, \quad s_0, s_1 > 0$$

We call them inverse demand and supply curves because price is on the left side of the equation rather than on the right side as it would be in a direct demand or supply function.

Here is a class that stores parameters for our single good market, as well as implementing the inverse demand and supply curves.

```
class Market:

    def __init__(self,
                 d_0=1.0,          # demand intercept
                 d_1=0.6,          # demand slope
                 s_0=0.1,          # supply intercept
                 s_1=0.4):         # supply slope

        self.d_0, self.d_1 = d_0, d_1
        self.s_0, self.s_1 = s_0, s_1

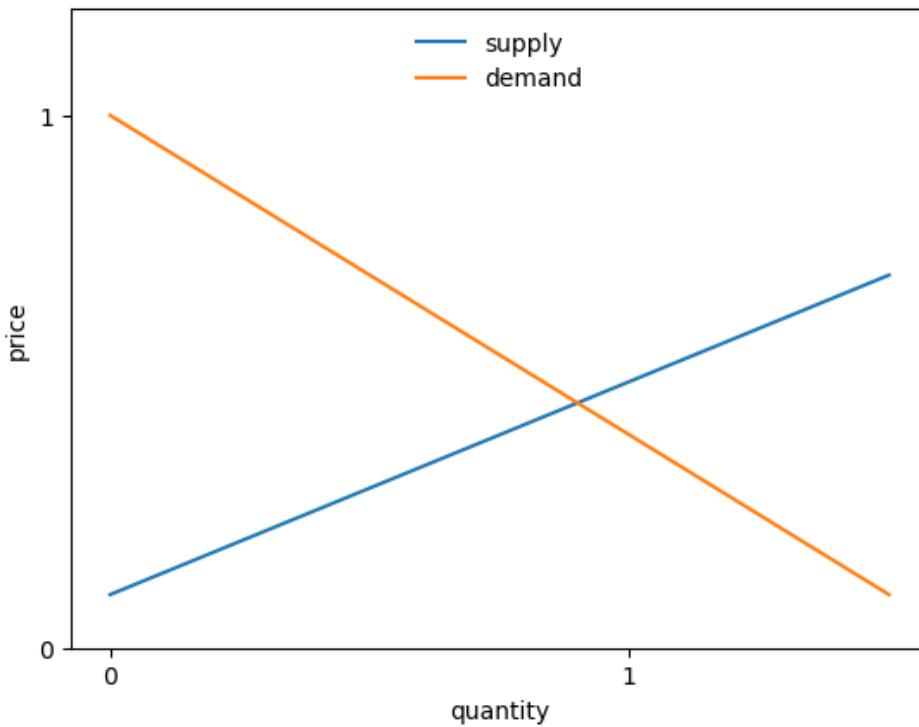
    def inverse_demand(self, q):
        return self.d_0 - self.d_1 * q

    def inverse_supply(self, q):
        return self.s_0 + self.s_1 * q
```

Let's create an instance.

```
market = Market()
```

Here is a plot of these two functions using `market`.



In the above graph, an **equilibrium** price-quantity pair occurs at the intersection of the supply and demand curves.

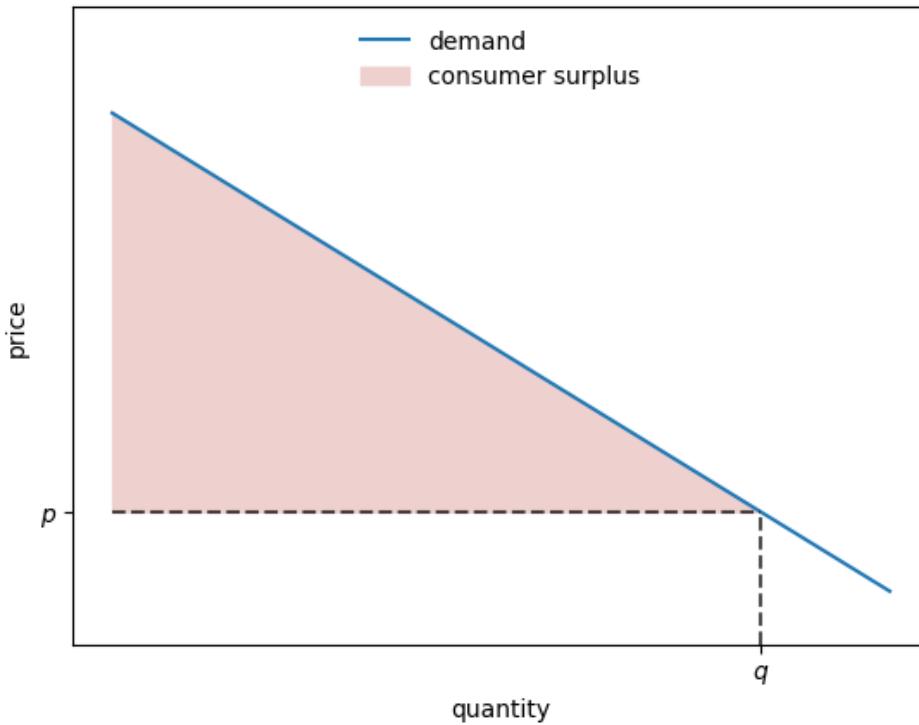
### 7.2.1 Consumer surplus

Let a quantity  $q$  be given and let  $p := d_0 - d_1q$  be the corresponding price on the inverse demand curve.

We define **consumer surplus**  $S_c(q)$  as the area under an inverse demand curve minus  $pq$ :

$$S_c(q) := \int_0^q (d_0 - d_1x)dx - pq \quad (7.1)$$

The next figure illustrates



Consumer surplus provides a measure of total consumer welfare at quantity  $q$ .

The idea is that the inverse demand curve  $d_0 - d_1q$  shows a consumer's willingness to pay for an additional increment of the good at a given quantity  $q$ .

The difference between willingness to pay and the actual price is consumer surplus.

The value  $S_c(q)$  is the “sum” (i.e., integral) of these surpluses when the total quantity purchased is  $q$  and the purchase price is  $p$ .

Evaluating the integral in the definition of consumer surplus (7.1) gives

$$S_c(q) = d_0q - \frac{1}{2}d_1q^2 - pq$$

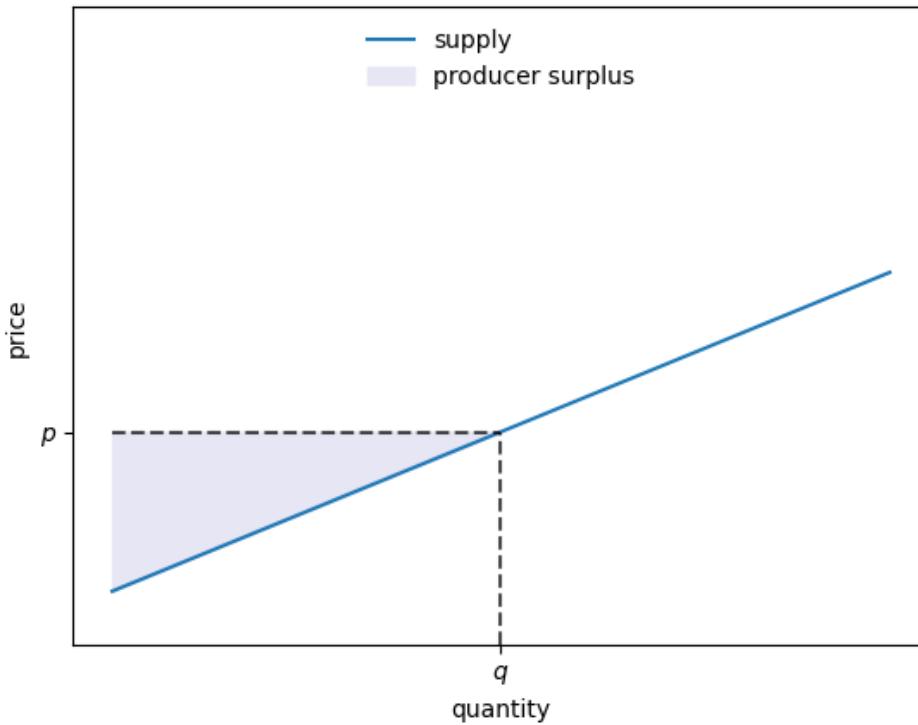
## 7.2.2 Producer surplus

Let a quantity  $q$  be given and let  $p := s_0 + s_1q$  be the corresponding price on the inverse supply curve.

We define **producer surplus** as  $pq$  minus the area under an inverse supply curve

$$S_p(q) := pq - \int_0^q (s_0 + s_1x)dx \quad (7.2)$$

The next figure illustrates



Producer surplus measures total producer welfare at quantity  $q$

The idea is similar to that of consumer surplus.

The inverse supply curve  $s_0 + s_1 q$  shows the price at which producers are prepared to sell, given quantity  $q$ .

The difference between willingness to sell and the actual price is producer surplus.

The value  $S_p(q)$  is the integral of these surpluses.

Evaluating the integral in the definition of producer surplus (7.2) gives

$$S_p(q) = pq - s_0 q - \frac{1}{2} s_1 q^2$$

### 7.2.3 Social welfare

Sometimes economists measure social welfare by a **welfare criterion** that equals consumer surplus plus producer surplus, assuming that consumers and producers pay the same price:

$$W(q) = \int_0^q (d_0 - d_1 x) dx - \int_0^q (s_0 + s_1 x) dx$$

Evaluating the integrals gives

$$W(q) = (d_0 - s_0)q - \frac{1}{2}(d_1 + s_1)q^2$$

Here is a Python function that evaluates this social welfare at a given quantity  $q$  and a fixed set of parameters.

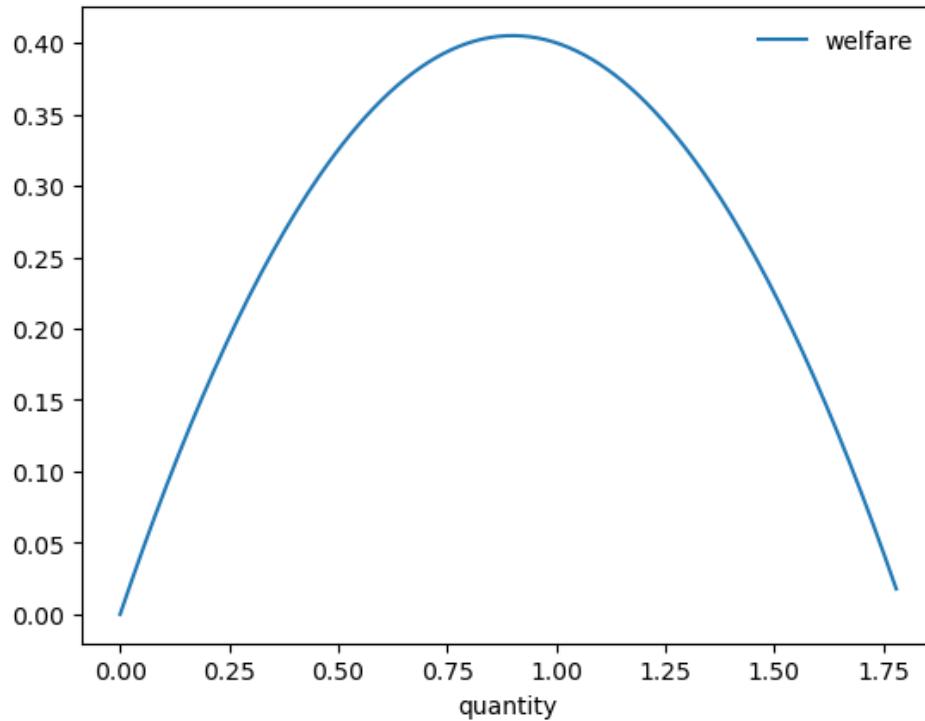
```
def W(q, market):
    # Unpack
    d_0, d_1, s_0, s_1 = market.d_0, market.d_1, market.s_0, market.s_1
```

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```
# Compute and return welfare
return (d_0 - s_0) * q - 0.5 * (d_1 + s_1) * q**2
```

The next figure plots welfare as a function of  $q$ .



Let's now give a social planner the task of maximizing social welfare.

To compute a quantity that maximizes the welfare criterion, we differentiate  $W$  with respect to  $q$  and then set the derivative to zero.

$$\frac{dW(q)}{dq} = d_0 - s_0 - (d_1 + s_1)q = 0$$

Solving for  $q$  yields

$$q = \frac{d_0 - s_0}{s_1 + d_1} \tag{7.3}$$

Let's remember the quantity  $q$  given by equation (7.3) that a social planner would choose to maximize consumer surplus plus producer surplus.

We'll compare it to the quantity that emerges in a competitive equilibrium that equates supply to demand.

## 7.2.4 Competitive equilibrium

Instead of equating quantities supplied and demanded, we can accomplish the same thing by equating demand price to supply price:

$$p = d_0 - d_1 q = s_0 + s_1 q$$

If we solve the equation defined by the second equality in the above line for  $q$ , we obtain

$$q = \frac{d_0 - s_0}{s_1 + d_1} \quad (7.4)$$

This is the competitive equilibrium quantity.

Observe that the equilibrium quantity equals the same  $q$  given by equation (7.3).

The outcome that the quantity determined by equation (7.3) equates supply to demand brings us a **key finding**:

- a competitive equilibrium quantity maximizes our welfare criterion

This is a version of the [first fundamental welfare theorem](#),

It also brings a useful **competitive equilibrium computation strategy**:

- after solving the welfare problem for an optimal quantity, we can read a competitive equilibrium price from either supply price or demand price at the competitive equilibrium quantity

## 7.3 Generalizations

In a [later lecture](#), we'll derive generalizations of the above demand and supply curves from other objects.

Our generalizations will extend the preceding analysis of a market for a single good to the analysis of  $n$  simultaneous markets in  $n$  goods.

In addition

- we'll derive **demand curves** from a consumer problem that maximizes a **utility function** subject to a **budget constraint**.
- we'll derive **supply curves** from the problem of a producer who is price taker and maximizes his profits minus total costs that are described by a **cost function**.

## 7.4 Exercises

Suppose now that the inverse demand and supply curves are modified to take the form

$$p = i_d(q) := d_0 - d_1 q^{0.6}$$

$$p = i_s(q) := s_0 + s_1 q^{1.8}$$

All parameters are positive, as before.

---

### Exercise 7.4.1

Define a new `Market` class that holds the same parameter values as before by changing the `inverse_demand` and `inverse_supply` methods to match these new definitions.

Using the class, plot the inverse demand and supply curves  $i_d$  and  $i_s$

---

### Solution to Exercise 7.4.1

```
class Market:

    def __init__(self,
                 d_0=1.0,          # demand intercept
                 d_1=0.6,          # demand slope
                 s_0=0.1,          # supply intercept
                 s_1=0.4):         # supply slope

        self.d_0, self.d_1 = d_0, d_1
        self.s_0, self.s_1 = s_0, s_1

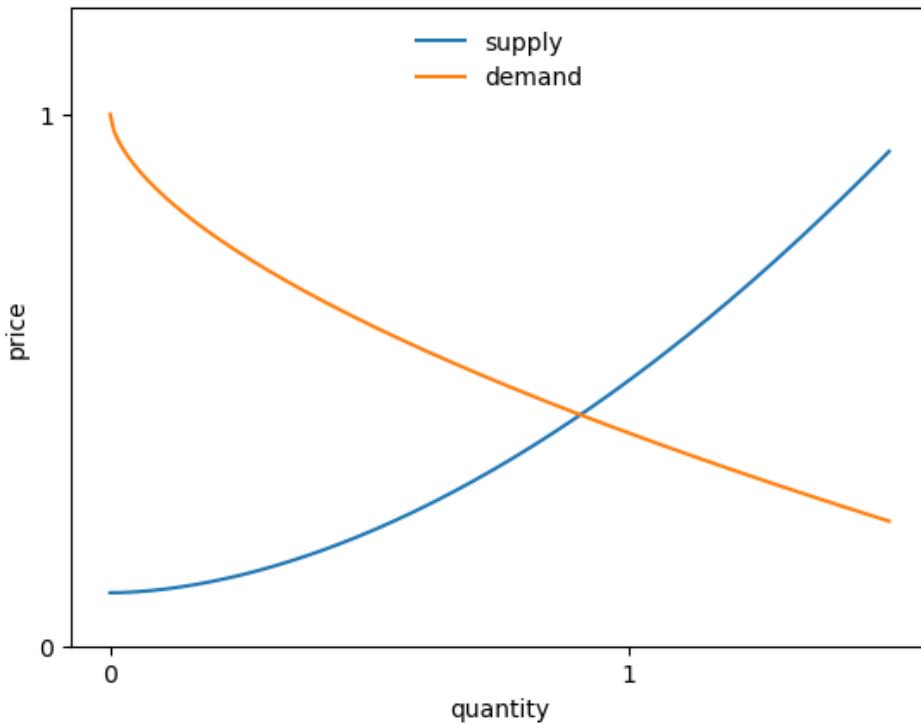
    def inverse_demand(self, q):
        return self.d_0 - self.d_1 * q**0.6

    def inverse_supply(self, q):
        return self.s_0 + self.s_1 * q**1.8
```

Let's create an instance.

```
market = Market()
```

Here is a plot of inverse supply and demand.



---

### Exercise 7.4.2

As before, consumer surplus at  $q$  is the area under the demand curve minus price times quantity:

$$S_c(q) = \int_0^q i_d(x)dx - pq$$

Here  $p$  is set to  $i_d(q)$

Producer surplus is price times quantity minus the area under the inverse supply curve:

$$S_p(q) = pq - \int_0^q i_s(x)dx$$

Here  $p$  is set to  $i_s(q)$ .

Social welfare is the sum of consumer and producer surplus under the assumption that the price is the same for buyers and sellers:

$$W(q) = \int_0^q i_d(x)dx - \int_0^q i_s(x)dx$$

Solve the integrals and write a function to compute this quantity numerically at given  $q$ .

Plot welfare as a function of  $q$ .

---

### Solution to Exercise 7.4.2

Solving the integrals gives

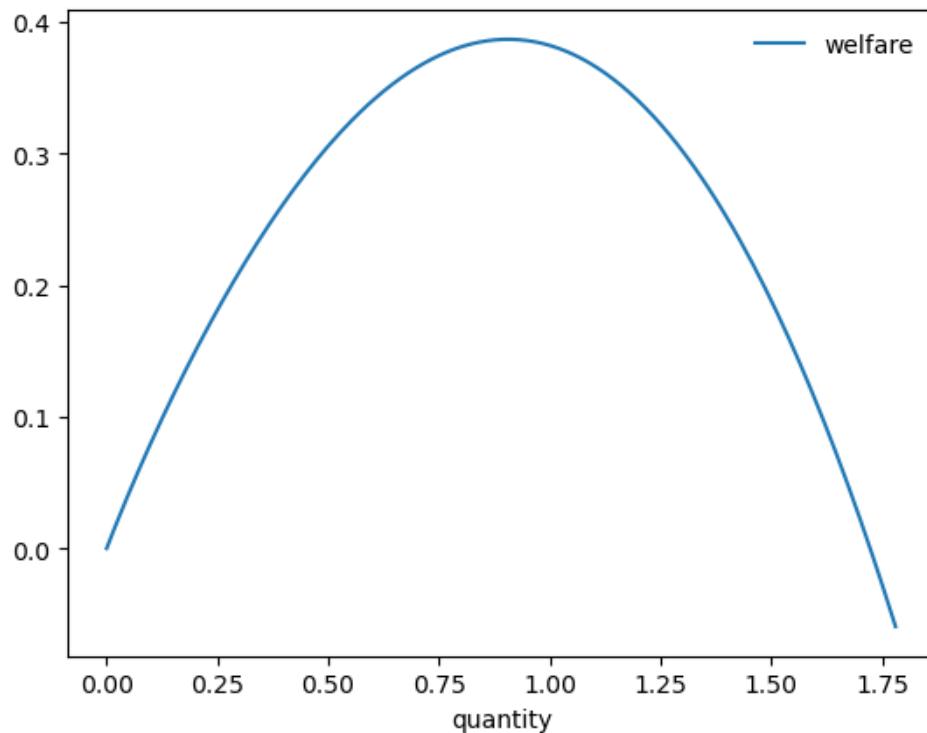
$$W(q) = d_0q - \frac{d_1q^{1.6}}{1.6} - \left( s_0q + \frac{s_1q^{2.8}}{2.8} \right)$$

Here's a Python function that computes this value:

```
def W(q, market):
    # Unpack
    d_0, d_1 = market.d_0, market.d_1
    s_0, s_1 = market.s_0, market.s_1
    # Compute and return welfare
    S_c = d_0 * q - d_1 * q**1.6 / 1.6
    S_p = s_0 * q + s_1 * q**2.8 / 2.8
    return S_c - S_p
```

The next figure plots welfare as a function of  $q$ .

```
fig, ax = plt.subplots()
ax.plot(q_vals, W(q_vals, market), label='welfare')
ax.legend(frameon=False)
ax.set_xlabel('quantity')
plt.show()
```



---

#### Exercise 7.4.3

Due to nonlinearities, the new welfare function is not easy to maximize with pencil and paper.

Maximize it using `scipy.optimize.minimize_scalar` instead.

---

#### Solution to Exercise 7.4.3

```
from scipy.optimize import minimize_scalar

def objective(q):
    return -W(q, market)

result = minimize_scalar(objective, bounds=(0, 10))
print(result.message)
```

Solution found.

```
maximizing_q = result.x
print(f"maximizing_q: {maximizing_q:.5f}")
```

0.90564

---

#### Exercise 7.4.4

Now compute the equilibrium quantity by finding the price that equates supply and demand.

You can do this numerically by finding the root of the excess demand function

$$e_d(q) := i_d(q) - i_s(q)$$

You can use `scipy.optimize.newton` to compute the root.

Initialize `newton` with a starting guess somewhere close to 1.0.

(Similar initial conditions will give the same result.)

You should find that the equilibrium price agrees with the welfare maximizing price, in line with the first fundamental welfare theorem.

---

---

#### Solution to Exercise 7.4.4

```
from scipy.optimize import newton

def excess_demand(q):
    return market.inverse_demand(q) - market.inverse_supply(q)

equilibrium_q = newton(excess_demand, 0.99)
print(f"equilibrium_q: {equilibrium_q:.5f}")
```

0.90564

---



## **Part IV**

# **Linear Dynamics**



## PRESENT VALUES

### 8.1 Overview

This lecture describes the **present value model** that is a starting point of much asset pricing theory.

We'll use the calculations described here in several subsequent lectures.

Our only tool is some elementary linear algebra operations, namely, matrix multiplication and matrix inversion.

Let's dive in.

Let

- $\{d_t\}_{t=0}^T$  be a sequence of dividends or “payouts”
- $\{p_t\}_{t=0}^T$  be a sequence of prices of a claim on the continuation of the asset stream from date  $t$  on, namely,  $\{d_s\}_{s=t}^T$
- $\delta \in (0, 1)$  be a one-period “discount factor”
- $p_{T+1}^*$  be a terminal price of the asset at time  $T + 1$

We assume that the dividend stream  $\{d_t\}_{t=0}^T$  and the terminal price  $p_{T+1}^*$  are both exogenous.

Assume the sequence of asset pricing equations

$$p_t = d_t + \delta p_{t+1}, \quad t = 0, 1, \dots, T \tag{8.1}$$

This is a “cost equals benefits” formula.

It says that the cost of buying the asset today equals the reward for holding it for one period (which is the dividend  $d_t$ ) and then selling it, at  $t + 1$ .

The future value  $p_{t+1}$  is discounted using  $\delta$  to shift it to a present value, so it is comparable with  $d_t$  and  $p_t$ .

We want to solve for the asset price sequence  $\{p_t\}_{t=0}^T$  as a function of  $\{d_t\}_{t=0}^T$  and  $p_{T+1}^*$ .

In this lecture we show how this can be done using matrix algebra.

We will use the following imports

```
import numpy as np
import matplotlib.pyplot as plt
```

## 8.2 Present value calculations

The equations in (8.1.1) can be stacked, as in

$$\begin{aligned}
p_0 &= d_0 + \delta p_1 \\
p_1 &= d_1 + \delta p_2 \\
&\vdots \\
p_{T-1} &= d_{T-1} + \delta p_T \\
p_T &= d_T + \delta p_{T+1}^*
\end{aligned} \tag{8.2}$$

Write the system (8.2) of  $T + 1$  asset pricing equations as the single matrix equation

$$\begin{bmatrix} 1 & -\delta & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\delta & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -\delta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\delta \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{T-1} \\ p_T \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{T-1} \\ d_T \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \delta p_{T+1}^* \end{bmatrix} \tag{8.3}$$

---

### Exercise 8.2.1

Carry out the matrix multiplication in (8.2.2) by hand and confirm that you recover the equations in (8.2.1).

---

In vector-matrix notation, we can write the system (8.2.2) as

$$Ap = d + b \tag{8.4}$$

Here  $A$  is the matrix on the left side of equation (8.3), while

$$p = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_T \end{bmatrix}, \quad d = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_T \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \delta p_{T+1}^* \end{bmatrix}$$

The solution for prices is given by

$$p = A^{-1}(d + b) \tag{8.5}$$

Here is a small example, where the dividend stream is given by

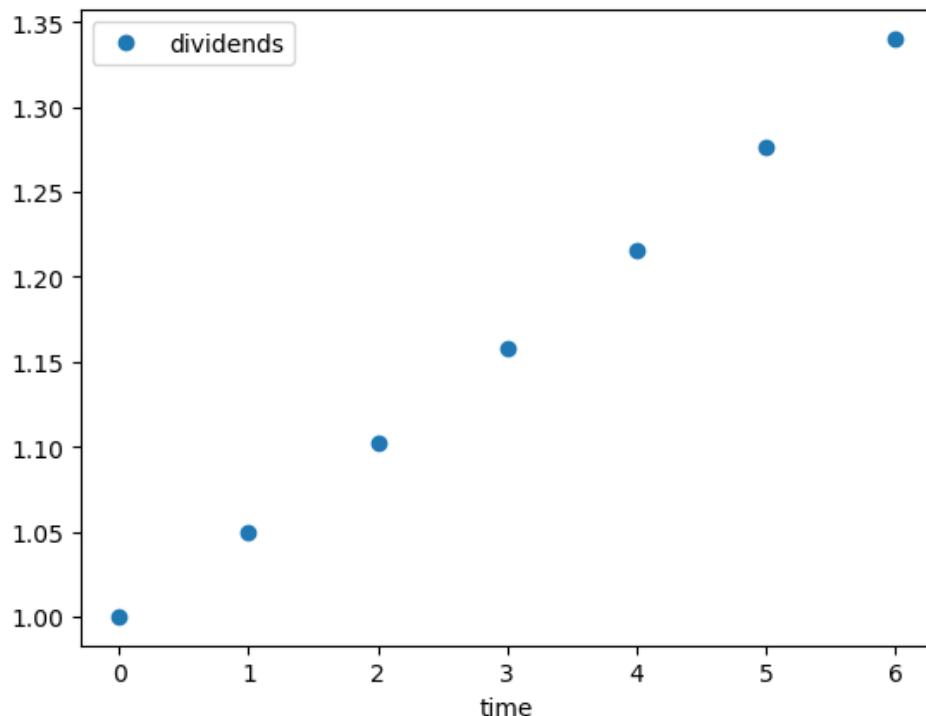
$$d_{t+1} = 1.05d_t, \quad t = 0, 1, \dots, T-1.$$

```

T = 6
current_d = 1.0
d = []
for t in range(T+1):
    d.append(current_d)
    current_d = current_d * 1.05

fig, ax = plt.subplots()
ax.plot(d, 'o', label='dividends')
ax.legend()
ax.set_xlabel('time')
plt.show()

```



We set  $\delta$  and  $p_{T+1}^*$  to

```
δ = 0.99
p_star = 10.0
```

Let's build the matrix  $A$

```
A = np.zeros((T+1, T+1))
for i in range(T+1):
    for j in range(T+1):
        if i == j:
            A[i, j] = 1
            if j < T:
                A[i, j+1] = -δ
```

Let's inspect  $A$

```
A
```

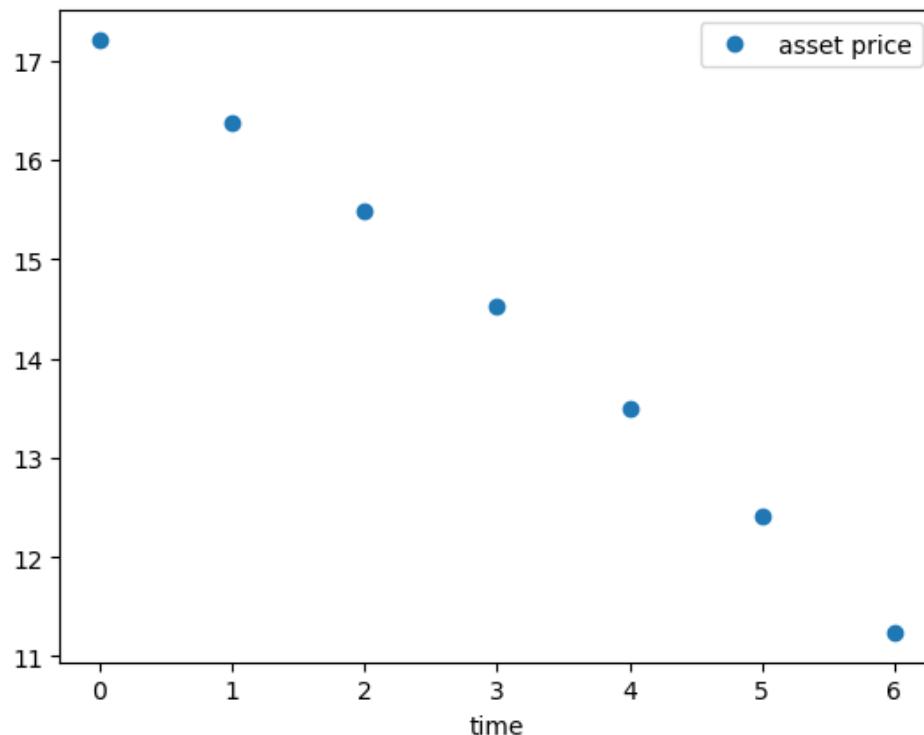
```
array([[ 1. , -0.99,  0. ,  0. ,  0. ,  0. ,  0. ,  0. ],
       [ 0. ,  1. , -0.99,  0. ,  0. ,  0. ,  0. ,  0. ],
       [ 0. ,  0. ,  1. , -0.99,  0. ,  0. ,  0. ,  0. ],
       [ 0. ,  0. ,  0. ,  1. , -0.99,  0. ,  0. ,  0. ],
       [ 0. ,  0. ,  0. ,  0. ,  1. , -0.99,  0. ,  0. ],
       [ 0. ,  0. ,  0. ,  0. ,  0. ,  1. , -0.99,  0. ],
       [ 0. ,  0. ,  0. ,  0. ,  0. ,  0. ,  1. , -0.99],
       [ 0. ,  0. ,  0. ,  0. ,  0. ,  0. ,  0. ,  1. ]])
```

Now let's solve for prices using (8.2.4).

```

b = np.zeros(T+1)
b[-1] = δ * p_star
p = np.linalg.solve(A, d + b)
fig, ax = plt.subplots()
ax.plot(p, 'o', label='asset price')
ax.legend()
ax.set_xlabel('time')
plt.show()

```



We can also consider a cyclically growing dividend sequence, such as

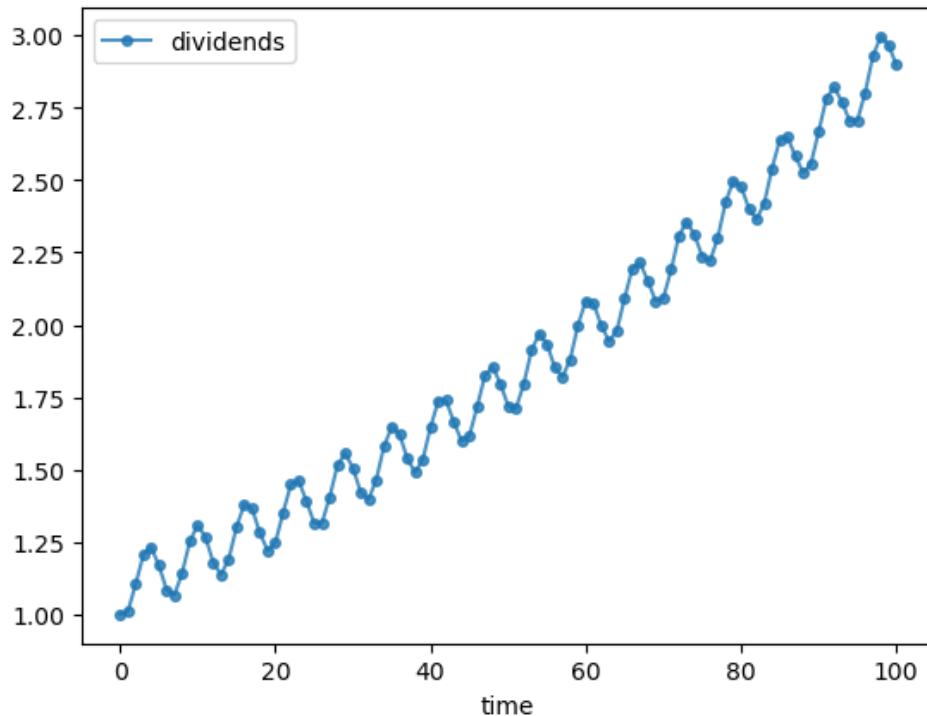
$$d_{t+1} = 1.01d_t + 0.1 \sin t, \quad t = 0, 1, \dots, T-1.$$

```

T = 100
current_d = 1.0
d = []
for t in range(T+1):
    d.append(current_d)
    current_d = current_d * 1.01 + 0.1 * np.sin(t)

fig, ax = plt.subplots()
ax.plot(d, 'o-', ms=4, alpha=0.8, label='dividends')
ax.legend()
ax.set_xlabel('time')
plt.show()

```




---

**Exercise 8.2.2**

Compute the corresponding asset price sequence when  $p_{T+1}^* = 0$  and  $\delta = 0.98$ .

---

**Solution to Exercise 8.2.2**

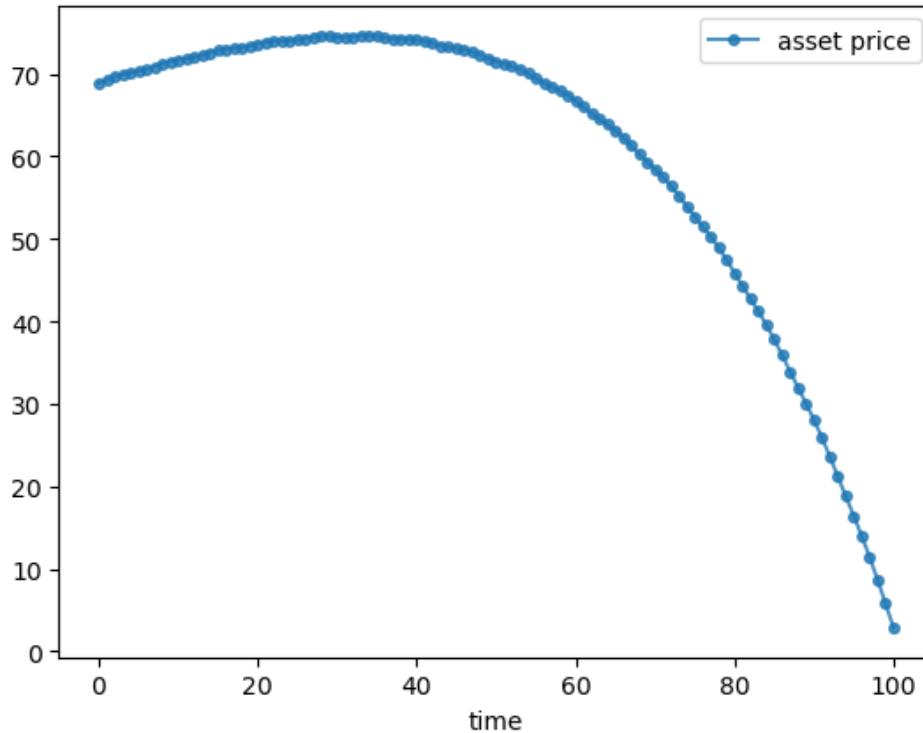
We proceed as above after modifying parameters and  $A$ .

```

delta = 0.98
p_star = 0.0
A = np.zeros((T+1, T+1))
for i in range(T+1):
    for j in range(T+1):
        if i == j:
            A[i, j] = 1
            if j < T:
                A[i, j+1] = -delta

b = np.zeros(T+1)
b[-1] = delta * p_star
p = np.linalg.solve(A, d + b)
fig, ax = plt.subplots()
ax.plot(p, 'o-', ms=4, alpha=0.8, label='asset price')
ax.legend()
ax.set_xlabel('time')
plt.show()

```



The weighted averaging associated with the present value calculation largely eliminates the cycles.

---

### 8.3 Analytical expressions

It can be verified that the inverse of the matrix  $A$  in (8.3) is

$$A^{-1} = \begin{bmatrix} 1 & \delta & \delta^2 & \dots & \delta^{T-1} & \delta^T \\ 0 & 1 & \delta & \dots & \delta^{T-2} & \delta^{T-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \delta \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \quad (8.6)$$

---

#### Exercise 8.3.1

Check this by showing that  $AA^{-1}$  is equal to the identity matrix.

(By the [inverse matrix theorem](#), a matrix  $B$  is the inverse of  $A$  whenever  $AB$  is the identity.)

---

If we use the expression (8.3.1) in (8.2.4) and perform the indicated matrix multiplication, we shall find that

$$p_t = \sum_{s=t}^T \delta^{s-t} d_s + \delta^{T+1-t} p_{T+1}^* \quad (8.7)$$

Pricing formula (8.7) asserts that two components sum to the asset price  $p_t$ :

- a **fundamental component**  $\sum_{s=t}^T \delta^{s-t} d_s$  that equals the discounted present value of prospective dividends

- a **bubble component**  $\delta^{T+1-t} p_{T+1}^*$

The fundamental component is pinned down by the discount factor  $\delta$  and the “fundamentals” of the asset (in this case, the dividends).

The bubble component is the part of the price that is not pinned down by fundamentals.

It is sometimes convenient to rewrite the bubble component as

$$c\delta^{-t}$$

where

$$c \equiv \delta^{T+1} p_{T+1}^*$$

## 8.4 More about bubbles

For a few moments, let's focus on the special case of an asset that will never pay dividends, in which case

$$\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{T-1} \\ d_T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

In this case system (8.1) of our  $T + 1$  asset pricing equations takes the form of the single matrix equation

$$\begin{bmatrix} 1 & -\delta & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\delta & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -\delta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\delta \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{T-1} \\ p_T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \delta p_{T+1}^* \end{bmatrix} \quad (8.8)$$

Evidently, if  $p_{T+1}^* = 0$ , a price vector  $p$  of all entries zero solves this equation and the only the **fundamental** component of our pricing formula (8.7) is present.

But let's activate the **bubble** component by setting

$$p_{T+1}^* = c\delta^{-(T+1)} \quad (8.9)$$

for some positive constant  $c$ .

In this case, it can be verified that when we multiply both sides of (8.8) by the matrix  $A^{-1}$  presented in equation (8.6), we shall find that

$$p_t = c\delta^{-t} \quad (8.10)$$

## 8.5 Gross rate of return

Define the gross rate of return on holding the asset from period  $t$  to period  $t + 1$  as

$$R_t = \frac{p_{t+1}}{p_t} \quad (8.11)$$

Equation (8.10) confirms that an asset whose sole source of value is a bubble that earns a gross rate of return

$$R_t = \delta^{-1} > 1.$$

## 8.6 Exercises

---

### Exercise 8.6.1

Give analytical expressions for the asset price  $p_t$  under the following settings for  $d$  and  $p_{T+1}^*$ :

1.  $p_{T+1}^* = 0, d_t = g^t d_0$  (a modified version of the Gordon growth formula)
  2.  $p_{T+1}^* = g^{T+1} d_0, d_t = g^t d_0$  (the plain vanilla Gordon growth formula)
  3.  $p_{T+1}^* = 0, d_t = 0$  (price of a worthless stock)
  4.  $p_{T+1}^* = c\delta^{-(T+1)}, d_t = 0$  (price of a pure bubble stock)
-

## CONSUMPTION SMOOTHING

### 9.1 Overview

Technically, this lecture is a sequel to this quantecon lecture *present values*, although it might not seem so at first.

It will take a while for a “present value” or asset price explicitly to appear in this lecture, but when it does it will be a key actor.

In this lecture, we’ll study a famous model of the “consumption function” that Milton Friedman [Fri56] and Robert Hall [Hal78]) proposed to fit some empirical data patterns that the simple Keynesian model described in this quantecon lecture *geometric series* had missed.

The key insight of Friedman and Hall was that today’s consumption ought not to depend just on today’s non-financial income: it should also depend on a person’s anticipations of her **future** non-financial incomes at various dates.

In this lecture, we’ll study what is sometimes called the “consumption-smoothing model” using only linear algebra, in particular matrix multiplication and matrix inversion.

Formulas presented in *present value formulas* are at the core of the consumption smoothing model because they are used to define a consumer’s “human wealth”.

As usual, we’ll start with by importing some Python modules.

```
import numpy as np
import matplotlib.pyplot as plt
from collections import namedtuple
```

Our model describes the behavior of a consumer who lives from time  $t = 0, 1, \dots, T$ , receives a stream  $\{y_t\}_{t=0}^T$  of non-financial income and chooses a consumption stream  $\{c_t\}_{t=0}^T$ .

We usually think of the non-financial income stream as coming from the person’s salary from supplying labor.

The model takes that non-financial income stream as an input, regarding it as “exogenous” in the sense of not being determined by the model.

The consumer faces a gross interest rate of  $R > 1$  that is constant over time, at which she is free to borrow or lend, up to some limits that we’ll describe below.

To set up the model, let

- $T \geq 2$  be a positive integer that constitutes a time-horizon
- $y = \{y_t\}_{t=0}^T$  be an exogenous sequence of non-negative non-financial incomes  $y_t$
- $a = \{a_t\}_{t=0}^{T+1}$  be a sequence of financial wealth
- $c = \{c_t\}_{t=0}^T$  be a sequence of non-negative consumption rates
- $R \geq 1$  be a fixed gross one period rate of return on financial assets

- $\beta \in (0, 1)$  be a fixed discount factor
- $a_0$  be a given initial level of financial assets
- $a_{T+1} \geq 0$  be a terminal condition on final assets

While the sequence of financial wealth  $a$  is to be determined by the model, it must satisfy two **boundary conditions** that require it to be equal to  $a_0$  at time 0 and  $a_{T+1}$  at time  $T + 1$ .

The **terminal condition**  $a_{T+1} \geq 0$  requires that the consumer not die leaving debts.

(We'll see that a utility maximizing consumer won't **want** to die leaving positive assets, so she'll arrange her affairs to make  $a_{T+1} = 0$ .)

The consumer faces a sequence of budget constraints that constrains the triple of sequences  $y, c, a$

$$a_{t+1} = R(a_t + y_t - c_t), \quad t = 0, 1, \dots, T \quad (9.1)$$

Notice that there are  $T + 1$  such budget constraints, one for each  $t = 0, 1, \dots, T$ .

Given a sequence  $y$  of non-financial income, there is a big set of **pairs**  $(a, c)$  of (financial wealth, consumption) sequences that satisfy the sequence of budget constraints (9.1).

Our model has the following logical flow.

- start with an exogenous non-financial income sequence  $y$ , an initial financial wealth  $a_0$ , and a candidate consumption path  $c$ .
- use the system of equations (9.1) for  $t = 0, \dots, T$  to compute a path  $a$  of financial wealth
- verify that  $a_{T+1}$  satisfies the terminal wealth constraint  $a_{T+1} \geq 0$ .
  - If it does, declare that the candidate path is **budget feasible**.
  - if the candidate consumption path is not budget feasible, propose a path with less consumption sometimes and start over

Below, we'll describe how to execute these steps using linear algebra – matrix inversion and multiplication.

The above procedure seems like a sensible way to find “budget-feasible” consumption paths  $c$ , i.e., paths that are consistent with the exogenous non-financial income stream  $y$ , the initial financial asset level  $a_0$ , and the terminal asset level  $a_{T+1}$ .

In general, there will be many budget feasible consumption paths  $c$ .

Among all budget-feasible consumption paths, which one **should** the consumer want to choose?

We shall eventually evaluate alternative budget feasible consumption paths  $c$  using the following **welfare criterion**

$$W = \sum_{t=0}^T \beta^t (g_1 c_t - \frac{g_2}{2} c_t^2) \quad (9.2)$$

where  $g_1 > 0, g_2 > 0$ .

The fact that the utility function  $g_1 c_t - \frac{g_2}{2} c_t^2$  has diminishing marginal utility imparts a preference for consumption that is very smooth when  $\beta R \approx 1$ .

Indeed, we shall see that when  $\beta R = 1$  (a condition assumed by Milton Friedman [Fri56] and Robert Hall [Hal78]), this criterion assigns higher welfare to **smoother** consumption paths.

By **smoother** we mean as close as possible to being constant over time.

The preference for smooth consumption paths that is built into the model gives it the name “consumption smoothing model”.

Let's dive in and do some calculations that will help us understand how the model works.

Here we use default parameters  $R = 1.05$ ,  $g_1 = 1$ ,  $g_2 = 1/2$ , and  $T = 65$ .

We create a namedtuple to store these parameters with default values.

```
ConsumptionSmoothing = namedtuple("ConsumptionSmoothing",
                                   ["R", "g1", "g2", "beta_seq", "T"])

def create_cs_model(R=1.05, g1=1, g2=1/2, T=65):
    beta = 1/R
    beta_seq = np.array([beta**i for i in range(T+1)])
    return ConsumptionSmoothing(R=R, g1=g1, g2=g2, beta_seq=beta_seq, T=T)
```

## 9.2 Friedman-Hall consumption-smoothing model

A key object in the model is what Milton Friedman called “human” or “non-financial” wealth at time 0:

$$h_0 \equiv \sum_{t=0}^T R^{-t} y_t = [1 \ R^{-1} \ \dots \ R^{-T}] \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_T \end{bmatrix}$$

Human or non-financial wealth is evidently just the present value at time 0 of the consumer’s non-financial income stream  $y$ .

Notice that formally it very much resembles the asset price that we computed in this quantecon lecture *present values*.

Indeed, this is why Milton Friedman called it “human capital”.

By iterating on equation (9.1) and imposing the terminal condition

$$a_{T+1} = 0,$$

it is possible to convert a sequence of budget constraints into the single intertemporal constraint

$$\sum_{t=0}^T R^{-t} c_t = a_0 + h_0,$$

which says that the present value of the consumption stream equals the sum of financial and non-financial (or human) wealth.

Robert Hall [Hal78] showed that when  $\beta R = 1$ , a condition Milton Friedman had also assumed, it is “optimal” for a consumer to **smooth consumption** by setting

$$c_t = c_0 \quad t = 0, 1, \dots, T$$

(Later we’ll present a “variational argument” that shows that this constant path is indeed optimal when  $\beta R = 1$ .)

In this case, we can use the intertemporal budget constraint to write

$$c_t = c_0 = \left( \sum_{t=0}^T R^{-t} \right)^{-1} (a_0 + h_0), \quad t = 0, 1, \dots, T. \quad (9.3)$$

Equation (9.3) is the consumption-smoothing model in a nutshell.

## 9.3 Mechanics of Consumption smoothing model

As promised, we'll provide step by step instructions on how to use linear algebra, readily implemented in Python, to compute all the objects in play in the consumption-smoothing model.

In the calculations below, we'll set default values of  $R > 1$ , e.g.,  $R = 1.05$ , and  $\beta = R^{-1}$ .

### 9.3.1 Step 1

For some  $(T + 1) \times 1$   $y$  vector, use matrix algebra to compute  $h_0$

$$\sum_{t=0}^T R^{-t} y_t = [1 \quad R^{-1} \quad \dots \quad R^{-T}] \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_T \end{bmatrix}$$

### 9.3.2 Step 2

Compute the optimal level of consumption  $c_0$

$$c_t = c_0 = \left( \frac{1 - R^{-1}}{1 - R^{-(T+1)}} \right) (a_0 + \sum_{t=0}^T R^t y_t), \quad t = 0, 1, \dots, T$$

### 9.3.3 Step 3

In this step, we use the system of equations (9.1) for  $t = 0, \dots, T$  to compute a path  $a$  of financial wealth.

To do this, we translated that system of difference equations into a single matrix equation as follows (we'll say more about the mechanics of using linear algebra to solve such difference equations later in the last part of this lecture):

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -R & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -R & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -R & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -R & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_T \\ a_{T+1} \end{bmatrix} = R \begin{bmatrix} y_0 + a_0 - c_0 \\ y_1 - c_0 \\ y_2 - c_0 \\ \vdots \\ y_{T-1} - c_0 \\ y_T - c_0 \end{bmatrix}$$

Multiply both sides by the inverse of the matrix on the left side to compute

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_T \\ a_{T+1} \end{bmatrix}$$

It should turn out automatically that

$$a_{T+1} = 0.$$

We have built into the our calculations that the consumer leaves life with exactly zero assets, just barely satisfying the terminal condition that  $a_{T+1} \geq 0$ .

Let's verify this with our Python code.

First we implement this model in `compute_optimal`

```

def compute_optimal(model, a0, y_seq):
    R, T = model.R, model.T

    # non-financial wealth
    h0 = model.β_seq @ y_seq      # since β = 1/R

    # c0
    c0 = (1 - 1/R) / (1 - (1/R)**(T+1)) * (a0 + h0)
    c_seq = c0*np.ones(T+1)

    # verify
    A = np.diag(-R*np.ones(T), k=-1) + np.eye(T+1)
    b = y_seq - c_seq
    b[0] = b[0] + a0

    a_seq = np.linalg.inv(A) @ b
    a_seq = np.concatenate([[a0], a_seq])

    return c_seq, a_seq

```

We use an example where the consumer inherits  $a_0 < 0$  (which can be interpreted as a student debt).

The non-financial process  $\{y_t\}_{t=0}^T$  is constant and positive up to  $t = 45$  and then becomes zero afterward.

```

# Financial wealth
a0 = -2      # such as "student debt"

# non-financial Income process
y_seq = np.concatenate([np.ones(46), np.zeros(20)])

cs_model = creat_cs_model()
c_seq, a_seq = compute_optimal(cs_model, a0, y_seq)

print('check a_T+1=0:',
      np.abs(a_seq[-1] - 0) <= 1e-8)

```

```
check a_T+1=0: True
```

The visualization shows the path of non-financial income, consumption, and financial assets.

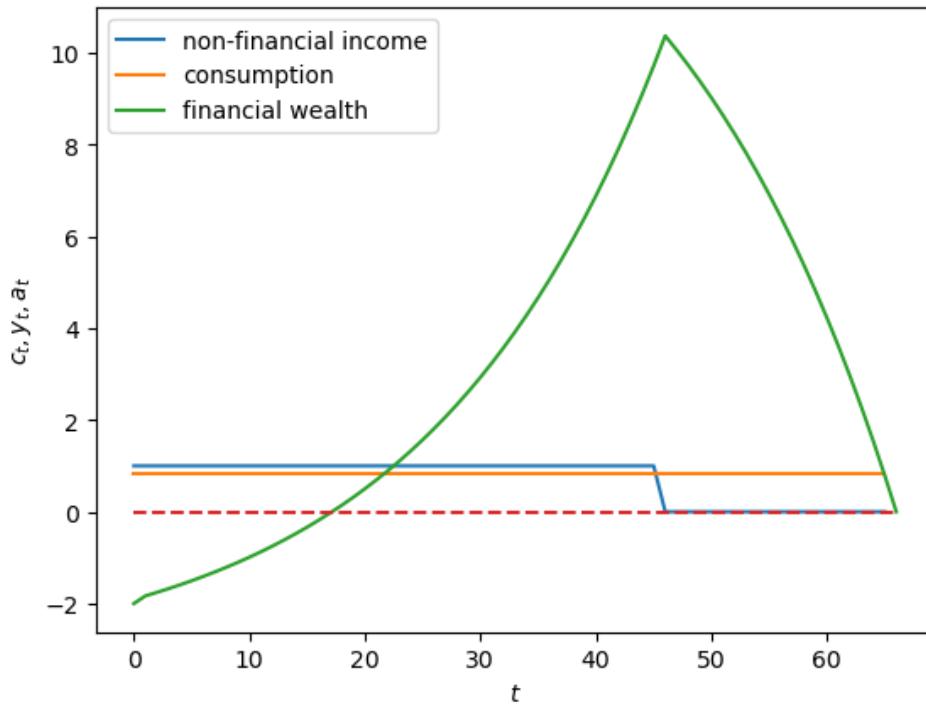
```

# Sequence Length
T = cs_model.T

plt.plot(range(T+1), y_seq, label='non-financial income')
plt.plot(range(T+1), c_seq, label='consumption')
plt.plot(range(T+2), a_seq, label='financial wealth')
plt.plot(range(T+2), np.zeros(T+2), '--')

plt.legend()
plt.xlabel(r'$t$')
plt.ylabel(r'$c_t, y_t, a_t$')
plt.show()

```



Note that  $a_{T+1} = 0$  is satisfied.

We can further evaluate the welfare using the formula (9.2)

```
def welfare(model, c_seq):
    beta_seq, g1, g2 = model.beta_seq, model.g1, model.g2

    u_seq = g1 * c_seq - g2/2 * c_seq**2
    return beta_seq @ u_seq

print('Welfare:', welfare(cs_model, c_seq))
```

Welfare: 13.285050962183433

### 9.3.4 Feasible consumption variations

Earlier, we had promised to present an argument that supports our claim that a constant consumption play  $c_t = c_0$  for all  $t$  is optimal.

Let's do that now.

Although simple and direct, the approach we'll take is actually an example of what is called the “calculus of variations”.

Let's dive in and see what the key idea is.

To explore what types of consumption paths are welfare-improving, we shall create an **admissible consumption path variation sequence**  $\{v_t\}_{t=0}^T$  that satisfies

$$\sum_{t=0}^T R^{-t} v_t = 0$$

This equation says that the **present value** of admissible variations must be zero.

(So once again, we encounter our formula for the present value of an “asset”.)

Here we’ll compute a two-parameter class of admissible variations of the form

$$v_t = \xi_1 \phi^t - \xi_0$$

We say two and not three-parameter class because  $\xi_0$  will be a function of  $(\phi, \xi_1; R)$  that guarantees that the variation is feasible.

Let’s compute that function.

We require

$$\sum_{t=0}^T [\xi_1 \phi^t - \xi_0] = 0$$

which implies that

$$\xi_1 \sum_{t=0}^T \phi_t R^{-t} - \xi_0 \sum_{t=0}^T R^{-t} = 0$$

which implies that

$$\xi_1 \frac{1 - (\phi R^{-1})^{T+1}}{1 - \phi R^{-1}} - \xi_0 \frac{1 - R^{-(T+1)}}{1 - R^{-1}} = 0$$

which implies that

$$\xi_0 = \xi_0(\phi, \xi_1; R) = \xi_1 \left( \frac{1 - R^{-1}}{1 - R^{-(T+1)}} \right) \left( \frac{1 - (\phi R^{-1})^{T+1}}{1 - \phi R^{-1}} \right)$$

This is our formula for  $\xi_0$ .

Evidently, if  $c^o$  is a budget-feasible consumption path, then so is  $c^o + v$ , where  $v$  is a budget-feasible variation.

Given  $R$ , we thus have a two parameter class of budget feasible variations  $v$  that we can use to compute alternative consumption paths, then evaluate their welfare.

Now let’s compute and visualize the variations

```
def compute_variation(model, xi1, phi, a0, y_seq, verbose=1):
    R, T, beta_seq = model.R, model.T, model.beta_seq

    xi0 = xi1 * ((1 - 1/R) / (1 - (1/R)**(T+1))) * ((1 - (phi/R)**(T+1)) / (1 - phi/R))
    v_seq = np.array([(xi1 * phi**t - xi0) for t in range(T+1)])

    if verbose == 1:
        print('check feasible:', np.isclose(beta_seq @ v_seq, 0))      # since beta = 1/R

    c_opt, _ = compute_optimal(model, a0, y_seq)
    cvar_seq = c_opt + v_seq

    return cvar_seq
```

We visualize variations with  $\xi_1 \in \{.01, .05\}$  and  $\phi \in \{.95, 1.02\}$

```

fig, ax = plt.subplots()

ξ1s = [.01, .05]
ϕs = [.95, 1.02]
colors = {.01: 'tab:blue', .05: 'tab:green'}

params = np.array(np.meshgrid(ξ1s, ϕs)).T.reshape(-1, 2)

for i, param in enumerate(params):
    ξ1, ϕ = param
    print(f'variation {i}: ξ1={ξ1}, ϕ={ϕ}')
    cvar_seq = compute_variation(model=cs_model,
                                   ξ1=ξ1, ϕ=ϕ, a0=a0,
                                   y_seq=y_seq)
    print(f'welfare={welfare(cs_model, cvar_seq)}')
    print('-'*64)
    if i % 2 == 0:
        ls = '-.'
    else:
        ls = '-'
    ax.plot(range(T+1), cvar_seq, ls=ls,
            color=colors[ξ1],
            label=fr'$\xi_1 = \{ξ1\}, \phi = \{ϕ\}$')

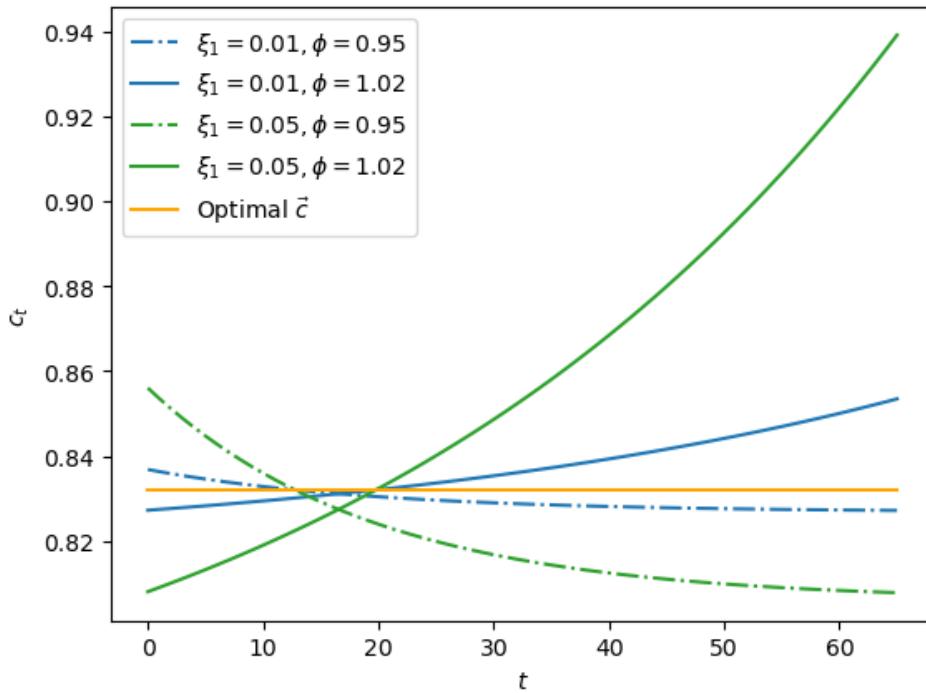
plt.plot(range(T+1), c_seq,
         color='orange', label=r'Optimal $\vec{c}$')

plt.legend()
plt.xlabel(r'$t$')
plt.ylabel(r'$c_t$')
plt.show()

```

```

variation 0: ξ1=0.01, ϕ=0.95
check feasible: True
welfare=13.285009346064834
-----
variation 1: ξ1=0.01, ϕ=1.02
check feasible: True
welfare=13.284911631015438
-----
variation 2: ξ1=0.05, ϕ=0.95
check feasible: True
welfare=13.284010559218512
-----
variation 3: ξ1=0.05, ϕ=1.02
check feasible: True
welfare=13.28156768298361
-----
```



We can even use the Python `np.gradient` command to compute derivatives of welfare with respect to our two parameters.

We are teaching the key idea beneath the **calculus of variations**.

First, we define the welfare with respect to  $\xi_1$  and  $\phi$

```
def welfare_rel(xi1, phi):
    """
    Compute welfare of variation sequence
    for given phi, xi1 with a consumption smoothing model
    """
    cvar_seq = compute_variation(cs_model, xi1=xi1,
                                 phi=phi, a0=a0,
                                 y_seq=y_seq,
                                 verbose=0)
    return welfare(cs_model, cvar_seq)

# Vectorize the function to allow array input
welfare_vec = np.vectorize(welfare_rel)
```

Then we can visualize the relationship between welfare and  $\xi_1$  and compute its derivatives

```
xi1_arr = np.linspace(-0.5, 0.5, 20)

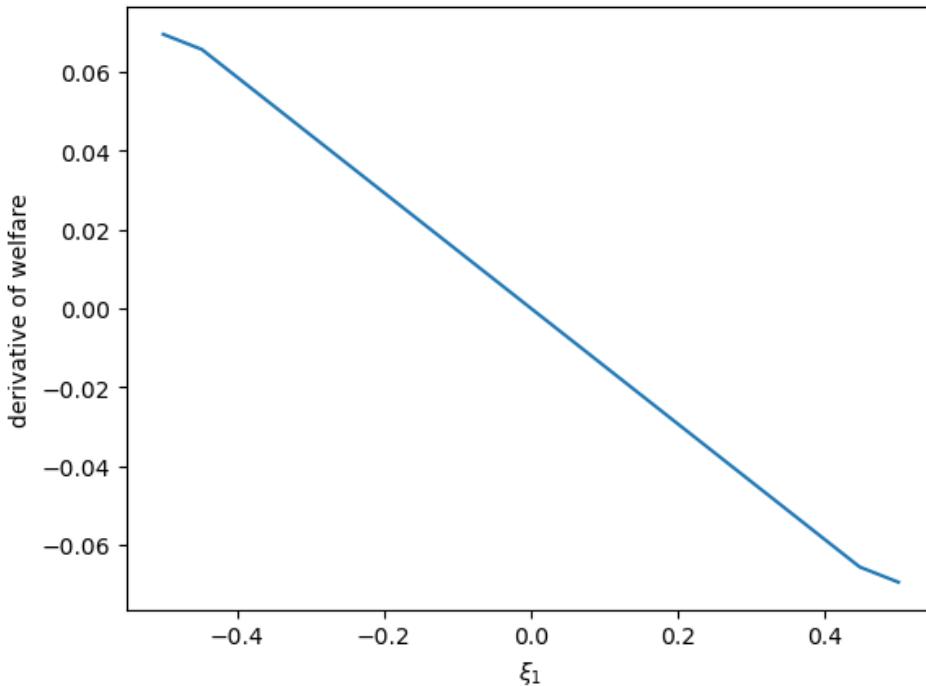
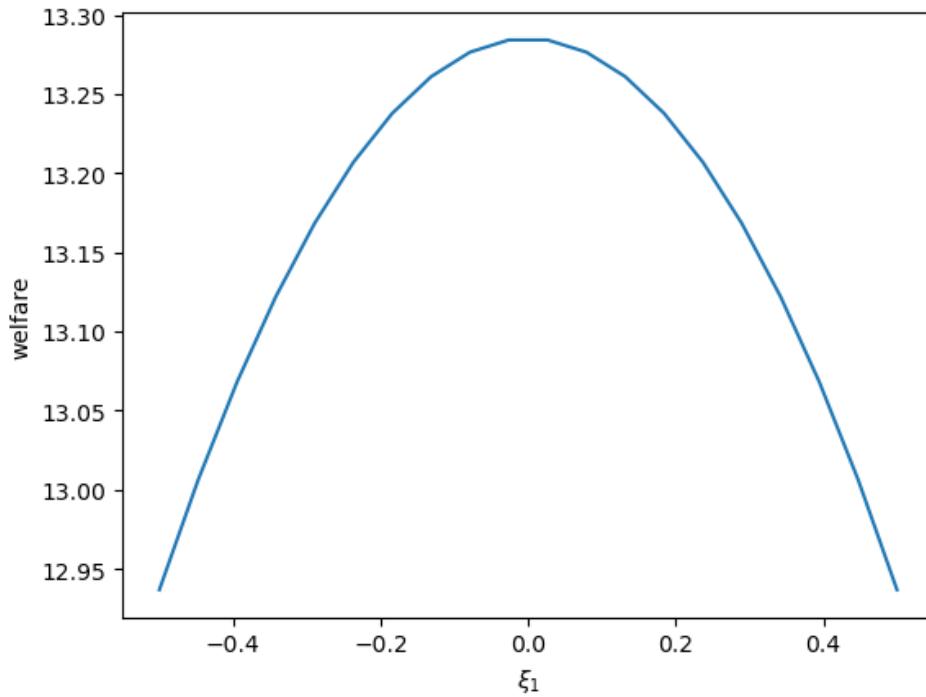
plt.plot(xi1_arr, welfare_vec(xi1_arr, 1.02))
plt.ylabel('welfare')
plt.xlabel(r'$\xi_1$')
plt.show()

welfare_grad = welfare_vec(xi1_arr, 1.02)
welfare_grad = np.gradient(welfare_grad)
```

(continues on next page)

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```
plt.plot(xi1_arr, welfare_grad)
plt.ylabel('derivative of welfare')
plt.xlabel(r'$\xi_1$')
plt.show()
```

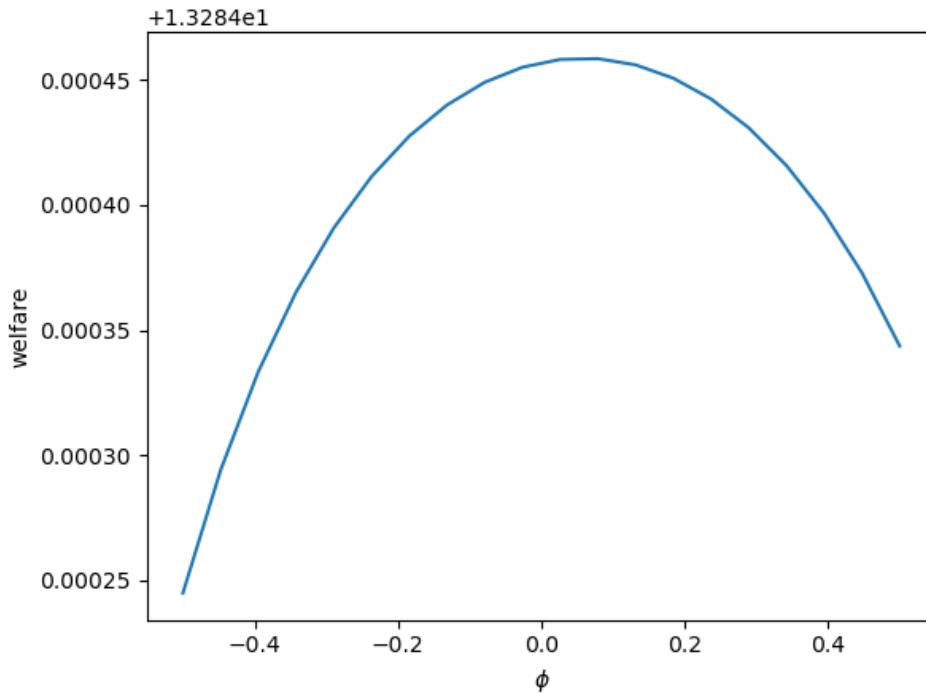


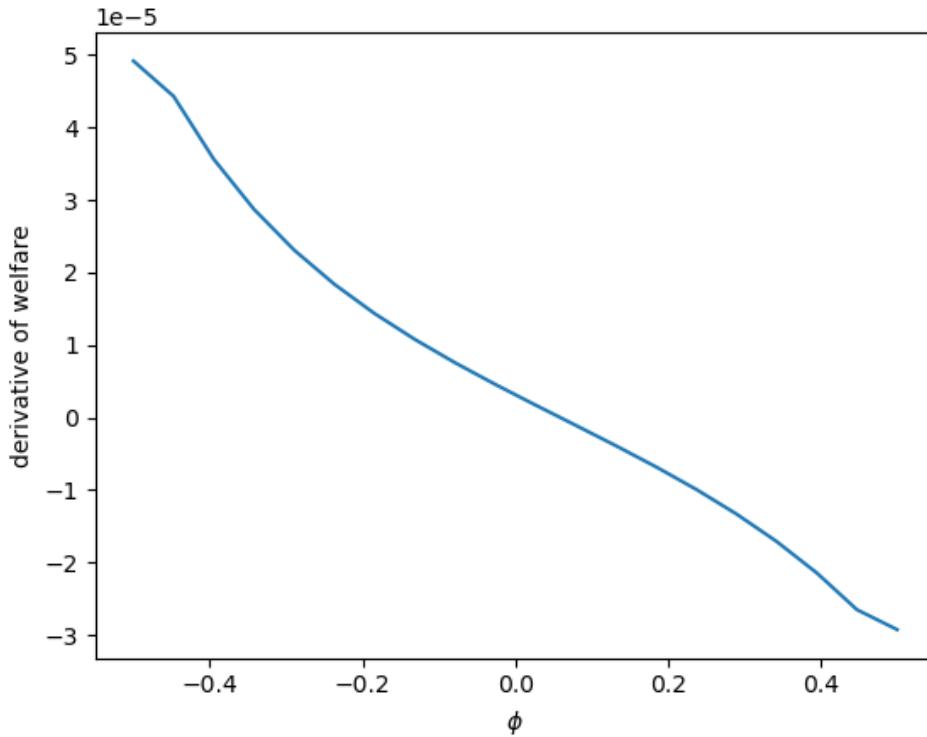
The same can be done on  $\phi$

```
phi_arr = np.linspace(-0.5, 0.5, 20)

plt.plot(xi1_arr, welfare_vec(0.05, phi_arr))
plt.ylabel('welfare')
plt.xlabel(r'$\phi$')
plt.show()

welfare_grad = welfare_vec(0.05, phi_arr)
welfare_grad = np.gradient(welfare_grad)
plt.plot(xi1_arr, welfare_grad)
plt.ylabel('derivative of welfare')
plt.xlabel(r'$\phi$')
plt.show()
```





## 9.4 Wrapping up the consumption-smoothing model

The consumption-smoothing model of Milton Friedman [Fri56] and Robert Hall [Hal78]) is a cornerstone of modern macro that has important ramifications about the size of the Keynesian “fiscal policy multiplier” described briefly in quantecon lecture *geometric series*.

In particular, Milton Friedman and others showed that it **lowered** the fiscal policy multiplier relative to the one implied by the simple Keynesian consumption function presented in *geometric series*.

Friedman and Hall’s work opened the door to a lively literature on the aggregate consumption function and implied fiscal multipliers that remains very active today.

## 9.5 Difference equations with linear algebra

In the preceding sections we have used linear algebra to solve a consumption smoothing model.

The same tools from linear algebra – matrix multiplication and matrix inversion – can be used to study many other dynamic models too.

We’ll conclude this lecture by giving a couple of examples.

In particular, we’ll describe a useful way of representing and “solving” linear difference equations.

To generate some  $y$  vectors, we’ll just write down a linear difference equation with appropriate initial conditions and then use linear algebra to solve it.

### 9.5.1 First-order difference equation

We'll start with a first-order linear difference equation for  $\{y_t\}_{t=0}^T$ :

$$y_t = \lambda y_{t-1}, \quad t = 1, 2, \dots, T$$

where  $y_0$  is a given initial condition.

We can cast this set of  $T$  equations as a single matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_T \end{bmatrix} = \begin{bmatrix} \lambda y_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Multiplying both sides by inverse of the matrix on the left provides the solution

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \lambda & 1 & 0 & \cdots & 0 & 0 \\ \lambda^2 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda^{T-1} & \lambda^{T-2} & \lambda^{T-3} & \cdots & \lambda & 1 \end{bmatrix} \begin{bmatrix} \lambda y_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (9.4)$$

---

#### Exercise 9.5.1

In the (9.4), we multiply the inverse of the matrix  $A$ . In this exercise, please confirm that

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \lambda & 1 & 0 & \cdots & 0 & 0 \\ \lambda^2 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda^{T-1} & \lambda^{T-2} & \lambda^{T-3} & \cdots & \lambda & 1 \end{bmatrix}$$

is the inverse of  $A$  and check that  $AA^{-1} = I$

---

### 9.5.2 Second order difference equation

A second-order linear difference equation for  $\{y_t\}_{t=0}^T$  is

$$y_t = \lambda_1 y_{t-1} + \lambda_2 y_{t-2}, \quad t = 1, 2, \dots, T$$

where now  $y_0$  and  $y_{-1}$  are two given initial equations determined outside the model.

As we did with the first-order difference equation, we can cast this set of  $T$  equations as a single matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\lambda_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -\lambda_2 & -\lambda_1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_2 & -\lambda_1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_T \end{bmatrix} = \begin{bmatrix} \lambda_1 y_0 + \lambda_2 y_{-1} \\ \lambda_2 y_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Multiplying both sides by inverse of the matrix on the left again provides the solution.

---

**Exercise 9.5.2**

As an exercise, we ask you to represent and solve a **third order linear difference equation**. How many initial conditions must you specify?

---

## EQUALIZING DIFFERENCE MODEL

### 10.1 Overview

This lecture presents a model of the college-high-school wage gap in which the “time to build” a college graduate plays a key role.

The model is “incomplete” in the sense that it is just one “condition” in the form of a single equation that would be part of set equations comprising all “equilibrium conditions” of a more fully articulated model.

The condition featured in our model determines a college, high-school wage ratio that equalizes the present values of a high school worker and a college educated worker.

The idea behind this condition is that lifetime earnings have to adjust to make someone indifferent between going to college and not going to college.

(The job of the “other equations” in a more complete model would be to fill in details about what adjusts to bring about this outcome.)

It is just one instance of an “equalizing difference” theory of relative wage rates, a class of theories dating back at least to Adam Smith’s **Wealth of Nations** [Smi10].

For most of this lecture, the only mathematical tools that we’ll use are from linear algebra, in particular, matrix multiplication and matrix inversion.

However, at the very end of the lecture, we’ll use calculus just in case readers want to see how computing partial derivatives could let us present some findings more concisely.

(And doing that will let us show off how good Python is at doing calculus!)

But if you don’t know calculus, our tools from linear algebra are certainly enough.

As usual, we’ll start by importing some Python modules.

```
import numpy as np
import matplotlib.pyplot as plt
```

## 10.2 The indifference condition

The key idea is that the initial college wage premium has to adjust to make a representative worker indifferent between going to college and not going to college.

Let

- $R > 1$  be the gross rate of return on a one-period bond
- $t = 0, 1, 2, \dots, T$  denote the years that a person either works or attends college
- 0 denote the first period after high school that a person can go to work
- $T$  denote the last period that a person works
- $w_t^h$  be the wage at time  $t$  of a high school graduate
- $w_t^c$  be the wage at time  $t$  of a college graduate
- $\gamma_h > 1$  be the (gross) rate of growth of wages of a high school graduate, so that  $w_t^h = w_0^h \gamma_h^t$
- $\gamma_c > 1$  be the (gross) rate of growth of wages of a college graduate, so that  $w_t^c = w_0^c \gamma_c^t$
- $D$  be the upfront monetary costs of going to college

If someone goes to work immediately after high school and works for the  $T + 1$  years  $t = 0, 1, 2, \dots, T$ , she earns present value

$$h_0 = \sum_{t=0}^T R^{-t} w_t^h = w_0^h \left[ \frac{1 - (R^{-1} \gamma_h)^{T+1}}{1 - R^{-1} \gamma_h} \right] \equiv w_0^h A_h$$

where

$$A_h = \left[ \frac{1 - (R^{-1} \gamma_h)^{T+1}}{1 - R^{-1} \gamma_h} \right].$$

The present value  $h_0$  is the “human wealth” at the beginning of time 0 of someone who chooses not to attend college but instead to go to work immediately at the wage of a high school graduate.

If someone goes to college for the four years  $t = 0, 1, 2, 3$  during which she earns 0, but then goes to work immediately after college and works for the  $T - 3$  years  $t = 4, 5, \dots, T$ , she earns present value

$$c_0 = \sum_{t=4}^T R^{-t} w_t^c = w_0^c (R^{-1} \gamma_c)^4 \left[ \frac{1 - (R^{-1} \gamma_c)^{T-3}}{1 - R^{-1} \gamma_c} \right] \equiv w_0^c A_c$$

where

$$A_c = (R^{-1} \gamma_c)^4 \left[ \frac{1 - (R^{-1} \gamma_c)^{T-3}}{1 - R^{-1} \gamma_c} \right]$$

The present value  $c_0$  is the “human wealth” at the beginning of time 0 of someone who chooses to attend college for four years and then start to work at time  $t = 4$  at the wage of a college graduate.

Assume that college tuition plus four years of room and board paid for up front costs  $D$ .

So net of monetary cost of college, the present value of attending college as of the first period after high school is

$$c_0 - D$$

We now formulate a pure **equalizing difference** model of the initial college-high school wage gap  $\phi$  defined by

Let

$$w_0^c = \phi w_0^h$$

We suppose that  $R, \gamma_h, \gamma_c, T$  and also  $w_0^h$  are fixed parameters.

We start by noting that the pure equalizing difference model asserts that the college-high-school wage gap  $\phi$  solves “equalizing” equation that sets the present value not going to college equal to the present value of going to college:

$$h_0 = c_0 - D$$

or

$$w_0^h A_h = \phi w_0^h A_c - D. \quad (10.1)$$

This is the “indifference condition” that is at the heart of the model.

Solving equation (10.1) for the college wage premium  $\phi$  we obtain

$$\phi = \frac{A_h}{A_c} + \frac{D}{w_0^h A_c}. \quad (10.2)$$

In a **free college** special case  $D = 0$  so that the only cost of going to college is the forgone earnings from not working as a high school worker.

In that case,

$$\phi = \frac{A_h}{A_c}.$$

Soon we'll write Python code to compute the gap and plot it as a function of its determinants.

But first we'll describe a possible alternative interpretation of our model.

## 10.3 Reinterpreting the model: workers and entrepreneurs

We can add a parameter and reinterpret variables to get a model of entrepreneurs versus workers.

We now let  $h$  be the present value of a “worker”.

We define the present value of an entrepreneur to be

$$c_0 = \pi \sum_{t=4}^T R^{-t} w_t^c$$

where  $\pi \in (0, 1)$  is the probability that an entrepreneur's “project” succeeds.

For our model of workers and firms, we'll interpret  $D$  as the cost of becoming an entrepreneur.

This cost might include costs of hiring workers, office space, and lawyers.

What we used to call the college, high school wage gap  $\phi$  now becomes the ratio of a successful entrepreneur's earnings to a worker's earnings.

We'll find that as  $\pi$  decreases,  $\phi$  increases.

Now let's write some Python code to compute  $\phi$  and plot it as a function of some of its determinants.

We can have some fun providing some example calculations that tweak various parameters, prominently including  $\gamma_h, \gamma_c, R$ .

```

class equalizing_diff:
    """
    A class of the equalizing difference model
    """

    def __init__(self, R, T, y_h, y_c, w_h0, D=0, pi=None):
        # one switches to the weak model by setting pi
        self.R, self.y_h, self.y_c, self.w_h0, self.D = R, y_h, y_c, w_h0, D
        self.T, self.pi = T, pi

    def compute_gap(self):
        R, y_h, y_c, w_h0, D = self.R, self.y_h, self.y_c, self.w_h0, self.D
        T, pi = self.T, self.pi

        A_h = (1 - (y_h/R)**(T+1)) / (1 - y_h/R)
        A_c = (1 - (y_c/R)**(T-3)) / (1 - y_c/R) * (y_c/R)**4

        # tweaked model
        if pi != None:
            A_c = pi * A_c

        phi = A_h/A_c + D / (w_h0*A_c)
        return phi

```

We can build some functions to help do comparative statics using vectorization instead of loops.

For a given instance of the class, we want to compute  $\phi$  when one parameter changes and others remain unchanged.

Let's do an example.

```

# phi_R
def phi_R(mc, R_new):
    mc_new = equalizing_diff(R_new, mc.T, mc.y_h, mc.y_c, mc.w_h0, mc.D, mc.pi)
    return mc_new.compute_gap()

phi_R = np.vectorize(phi_R)

# phi_yh
def phi_yh(mc, yh_new):
    mc_new = equalizing_diff(mc.R, mc.T, yh_new, mc.y_c, mc.w_h0, mc.D, mc.pi)
    return mc_new.compute_gap()

phi_yh = np.vectorize(phi_yh)

# phi_yc
def phi_yc(mc, yc_new):
    mc_new = equalizing_diff(mc.R, mc.T, mc.y_h, yc_new, mc.w_h0, mc.D, mc.pi)
    return mc_new.compute_gap()

phi_yc = np.vectorize(phi_yc)

# phi_pi
def phi_pi(mc, pi_new):
    mc_new = equalizing_diff(mc.R, mc.T, mc.y_h, mc.y_c, mc.w_h0, mc.D, pi_new)
    return mc_new.compute_gap()

phi_pi = np.vectorize(phi_pi)

```

```
# set benchmark parameters
R = 1.05
T = 40
Y_h, Y_c = 1.01, 1.01
w_h0 = 1
D = 10

# create an instance
ex1 = equalizing_diff(R=R, T=T, Y_h=Y_h, Y_c=Y_c, w_h0=w_h0, D=D)
gap1 = ex1.compute_gap()

print(gap1)
```

1.8041412724969135

Let's not charge for college and recompute  $\phi$ .

The initial college wage premium should go down.

```
# free college
ex2 = equalizing_diff(R, T, Y_h, Y_c, w_h0, D=0)
gap2 = ex2.compute_gap()

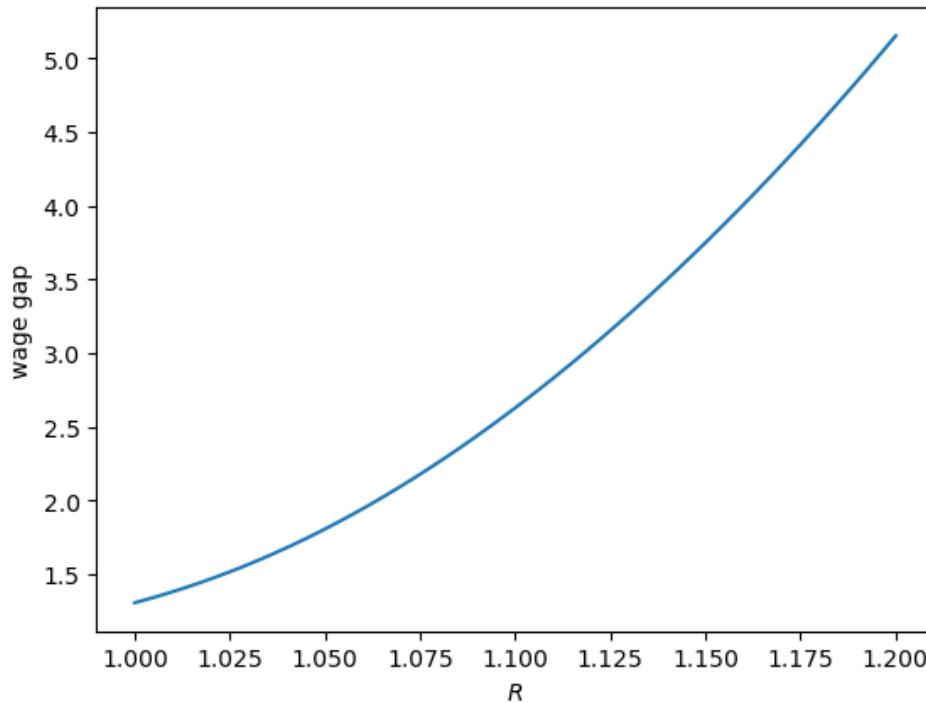
print(gap2)
```

1.2204649517903732

Let us construct some graphs that show us how the initial college-high-school wage ratio  $\phi$  would change if one of its determinants were to change.

Let's start with the gross interest rate  $R$ .

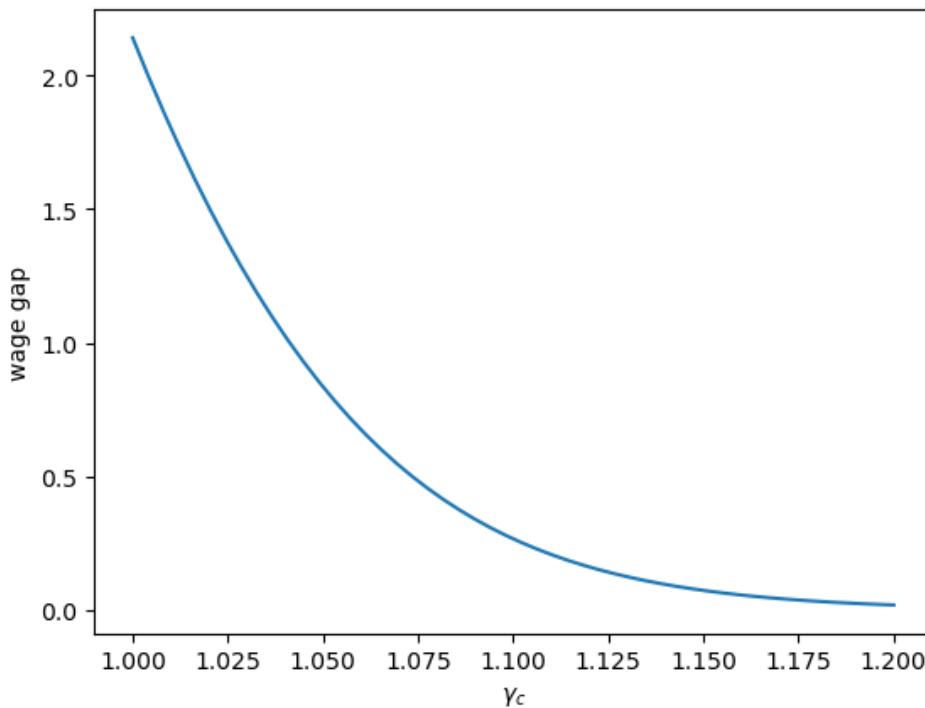
```
R_arr = np.linspace(1, 1.2, 50)
plt.plot(R_arr, phi_R(ex1, R_arr))
plt.xlabel(r'$R$')
plt.ylabel(r'wage gap')
plt.show()
```



Evidently, the initial wage ratio  $\phi$  must rise to compensate a prospective high school student for **waiting** to start receiving income – remember that while she is earning nothing in years  $t = 0, 1, 2, 3$ , the high school worker is earning a salary.

Not let's study what happens to the initial wage ratio  $\phi$  if the rate of growth of college wages rises, holding constant other determinants of  $\phi$ .

```
yc_arr = np.linspace(1, 1.2, 50)
plt.plot(yc_arr, ψ yc(ex1, yc_arr))
plt.xlabel(r'$\gamma_c$')
plt.ylabel('wage gap')
plt.show()
```



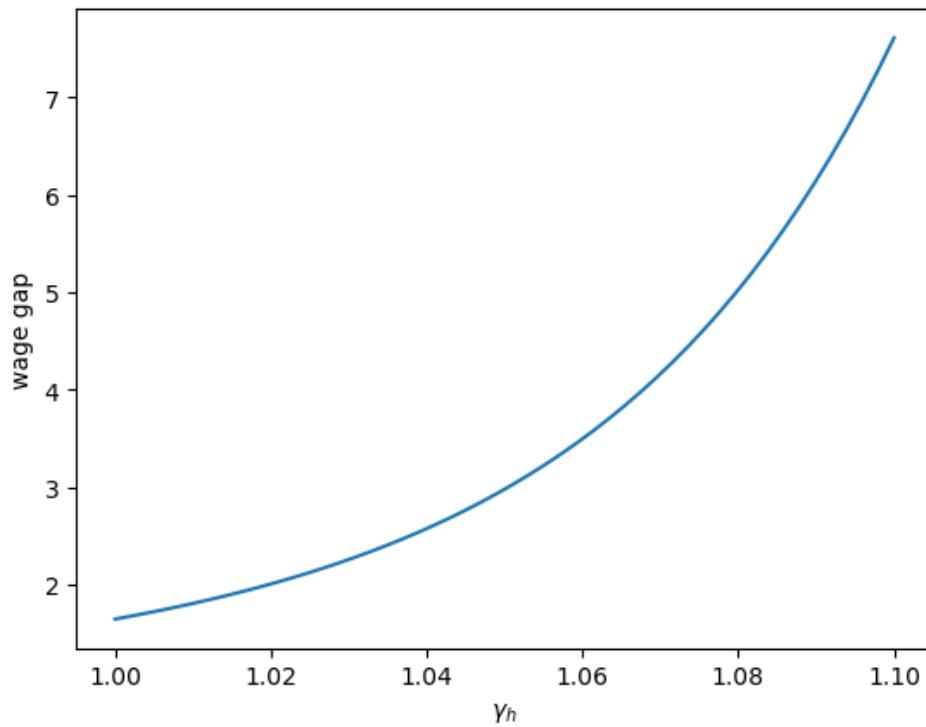
Notice how the initial wage gap falls when the rate of growth  $\gamma_c$  of college wages rises.

It falls to “equalize” the present values of the two types of career, one as a high school worker, the other as a college worker.

Can you guess what happens to the initial wage ratio  $\phi$  when next we vary the rate of growth of high school wages, holding all other determinants of  $\phi$  constant?

The following graph shows what happens.

```
vh_arr = np.linspace(1, 1.1, 50)
plt.plot(vh_arr, ψ_yh(ex1, vh_arr))
plt.xlabel(r'$\gamma_h$')
plt.ylabel('wage gap')
plt.show()
```



## 10.4 Entrepreneur-worker interpretation

Now let's adopt the entrepreneur-worker interpretation of our model.

If the probability that a new business succeeds is .2, let's compute the initial wage premium for successful entrepreneurs.

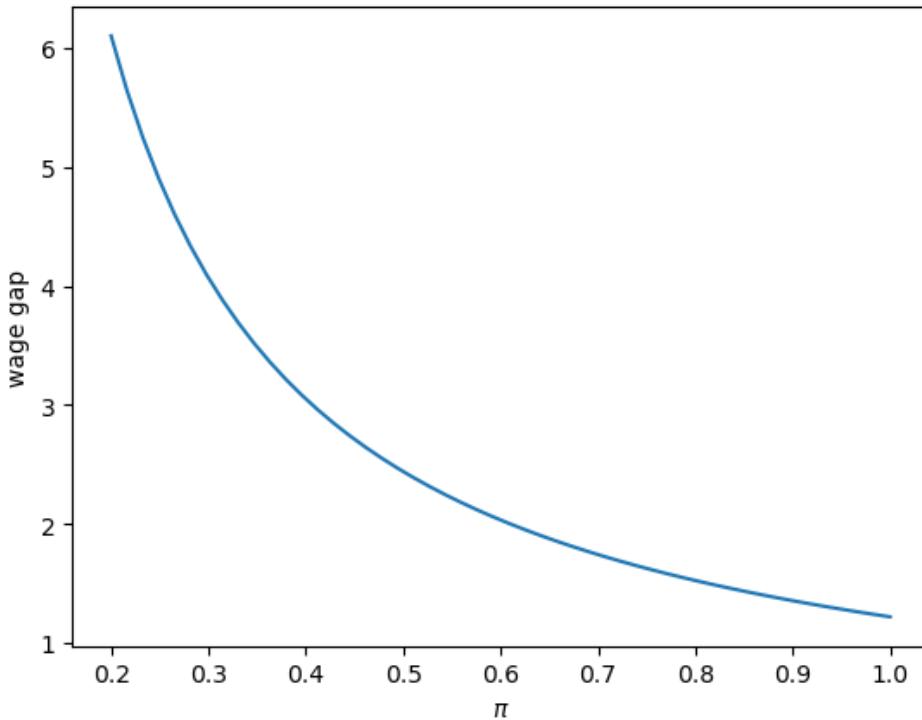
```
# a model of entrepreneur
ex3 = equalizing_diff(R, T, y_h, y_c, w_h0, pi=0.2)
gap3 = ex3.compute_gap()

print(gap3)
```

6.102324758951866

Now let's study how the initial wage premium for successful entrepreneurs depend on the success probability.

```
pi_arr = np.linspace(0.2, 1, 50)
plt.plot(pi_arr, gap(ex3, pi_arr))
plt.ylabel(r'wage gap')
plt.xlabel(r'$\pi$')
plt.show()
```



Does the graph make sense to you?

## 10.5 An application of calculus

So far, we have used only linear algebra and it has been a good enough tool for us to figure out how our model works.

However, someone who knows calculus might ask “Instead of plotting those graphs, why didn’t you just take partial derivatives?”

We’ll briefly do just that, yes, the questioner is correct and that partial derivatives are indeed a good tool for discovering the “comparative statics” properties of our model.

A reader who doesn’t know calculus could read no further and feel confident that applying linear algebra has taught us the main properties of the model.

But for a reader interested in how we can get Python to do all the hard work involved in computing partial derivatives, we’ll say a few things about that now.

We’ll use the Python module ‘sympy’ to compute partial derivatives of  $\phi$  with respect to the parameters that determine it.

Let’s import key functions from sympy.

```
from sympy import Symbol, Lambda, symbols
```

Define symbols

```
y_h, y_c, w_h0, D = symbols('\gamma_h, \gamma_h_c, w_0^h, D', real=True)
R, T = Symbol('R', real=True), Symbol('T', integer=True)
```

Define function  $A_h$

```
A_h = Lambda((y_h, R, T), (1 - (y_h/R)**(T+1)) / (1 - y_h/R))
```

$$\left( (\gamma_h, R, T) \mapsto \frac{1 - \left(\frac{\gamma_h}{R}\right)^{T+1}}{1 - \frac{\gamma_h}{R}} \right)$$

Define function  $A_c$

```
A_c = Lambda((y_c, R, T), (1 - (y_c/R)**(T-3)) / (1 - y_c/R) * (y_c/R)**4)
```

$$\left( (\gamma_{hc}, R, T) \mapsto \frac{\gamma_{hc}^4 \cdot \left(1 - \left(\frac{\gamma_{hc}}{R}\right)^{T-3}\right)}{R^4 \cdot \left(1 - \frac{\gamma_{hc}}{R}\right)} \right)$$

Now, define  $\phi$

```
phi = Lambda((D, y_h, y_c, R, T, w_h0), A_h(y_h, R, T) / A_c(y_c, R, T) + D / (w_h0 * A_c(y_c, R, T)))
```

$\phi$

$$\left( (D, \gamma_h, \gamma_{hc}, R, T, w_h^0) \mapsto \frac{DR^4 \cdot \left(1 - \frac{\gamma_{hc}}{R}\right)}{\gamma_{hc}^4 w_h^0 \cdot \left(1 - \left(\frac{\gamma_{hc}}{R}\right)^{T-3}\right)} + \frac{R^4 \cdot \left(1 - \left(\frac{\gamma_h}{R}\right)^{T+1}\right) \left(1 - \frac{\gamma_{hc}}{R}\right)}{\gamma_{hc}^4 \cdot \left(1 - \frac{\gamma_h}{R}\right) \left(1 - \left(\frac{\gamma_{hc}}{R}\right)^{T-3}\right)} \right)$$

We begin by setting default parameter values.

```
R_value = 1.05
T_value = 40
y_h_value, y_c_value = 1.01, 1.01
w_h0_value = 1
D_value = 10
```

Now let's compute  $\frac{\partial \phi}{\partial D}$  and then evaluate it at the default values

```
phi_D = phi(D, y_h, y_c, R, T, w_h0).diff(D)
phi_D
```

$$\frac{R^4 \cdot \left(1 - \frac{\gamma_{hc}}{R}\right)}{\gamma_{hc}^4 w_h^0 \cdot \left(1 - \left(\frac{\gamma_{hc}}{R}\right)^{T-3}\right)}$$

```
# Numerical value at default parameters
phi_D_func = Lambda((D, y_h, y_c, R, T, w_h0), phi_D)
phi_D_func(D_value, y_h_value, y_c_value, R_value, T_value, w_h0_value)
```

0.058367632070654

Thus, as with our graph above, we find that raising  $R$  increases the initial college wage premium  $\phi$ .

Compute  $\frac{\partial \phi}{\partial T}$  and evaluate it at default parameters

```
phi_T = phi(D, y_h, y_c, R, T, w_h0).diff(T)
phi_T
```

$$\frac{DR^4 \left(\frac{\gamma_{hc}}{R}\right)^{T-3} \cdot \left(1 - \frac{\gamma_{hc}}{R}\right) \log\left(\frac{\gamma_{hc}}{R}\right)}{\gamma_{hc}^4 w_0^h \left(1 - \left(\frac{\gamma_{hc}}{R}\right)^{T-3}\right)^2} - \frac{R^4 \left(\frac{\gamma_h}{R}\right)^{T+1} \cdot \left(1 - \frac{\gamma_{hc}}{R}\right) \log\left(\frac{\gamma_h}{R}\right)}{\gamma_{hc}^4 \cdot \left(1 - \frac{\gamma_h}{R}\right) \left(1 - \left(\frac{\gamma_{hc}}{R}\right)^{T-3}\right)} + \frac{R^4 \left(\frac{\gamma_{hc}}{R}\right)^{T-3} \cdot \left(1 - \left(\frac{\gamma_h}{R}\right)^{T+1}\right) \left(1 - \frac{\gamma_{hc}}{R}\right) \log\left(\frac{\gamma_{hc}}{R}\right)}{\gamma_{hc}^4 \cdot \left(1 - \frac{\gamma_h}{R}\right) \left(1 - \left(\frac{\gamma_{hc}}{R}\right)^{T-3}\right)^2}$$

```
# Numerical value at default parameters
phi_T_func = Lambda((D, y_h, y_c, R, T, w_h0), phi_T)
phi_T_func(D_value, y_h_value, y_c_value, R_value, T_value, w_h0_value)
```

-0.00973478032996598

We find that raising  $T$  decreases the initial college wage premium  $\phi$ .

This is because college graduates now have longer career lengths to “pay off” the time and other costs they paid to go to college

Let's compute  $\frac{\partial \phi}{\partial \gamma_h}$  and evaluate it at default parameters.

```
phi_y_h = phi(D, y_h, y_c, R, T, w_h0).diff(y_h)
phi_y_h
```

$$-\frac{R^4 \left(\frac{\gamma_h}{R}\right)^{T+1} \cdot \left(1 - \frac{\gamma_{hc}}{R}\right) (T+1)}{\gamma_h \gamma_{hc}^4 \cdot \left(1 - \frac{\gamma_h}{R}\right) \left(1 - \left(\frac{\gamma_{hc}}{R}\right)^{T-3}\right)} + \frac{R^3 \cdot \left(1 - \left(\frac{\gamma_h}{R}\right)^{T+1}\right) \left(1 - \frac{\gamma_{hc}}{R}\right)}{\gamma_{hc}^4 \left(1 - \frac{\gamma_h}{R}\right)^2 \cdot \left(1 - \left(\frac{\gamma_{hc}}{R}\right)^{T-3}\right)}$$

```
# Numerical value at default parameters
phi_y_h_func = Lambda((D, y_h, y_c, R, T, w_h0), phi_y_h)
phi_y_h_func(D_value, y_h_value, y_c_value, R_value, T_value, w_h0_value)
```

17.8590485545256

We find that raising  $\gamma_h$  increases the initial college wage premium  $\phi$ , as we did with our graphical analysis earlier

Compute  $\frac{\partial \phi}{\partial \gamma_c}$  and evaluate it numerically at default parameter values

```
phi_y_c = phi(D, y_h, y_c, R, T, w_h0).diff(y_c)
phi_y_c
```

$$\frac{DR^4 \left(\frac{\gamma_{hc}}{R}\right)^{T-3} \cdot (1 - \frac{\gamma_{hc}}{R})(T-3)}{\gamma_{hc}^5 w_0^h \left(1 - \left(\frac{\gamma_{hc}}{R}\right)^{T-3}\right)^2} - \frac{4DR^4 \cdot (1 - \frac{\gamma_{hc}}{R})}{\gamma_{hc}^5 w_0^h \cdot \left(1 - \left(\frac{\gamma_{hc}}{R}\right)^{T-3}\right)} - \frac{DR^3}{\gamma_{hc}^4 w_0^h \cdot \left(1 - \left(\frac{\gamma_{hc}}{R}\right)^{T-3}\right)} + \frac{R^4 \left(\frac{\gamma_h}{R}\right)^{T-3} \cdot \left(1 - \left(\frac{\gamma_h}{R}\right)^{T+1}\right)}{\gamma_{hc}^5 \cdot (1 - \frac{\gamma_h}{R}) \left(1 - \left(\frac{\gamma_h}{R}\right)^{T+1}\right)}$$

```
# Numerical value at default parameters
ϕ_Y_c_func = Lambda((D, γ_h, γ_c, R, T, w_h0), ϕ_Y_c)
ϕ_Y_c_func(D_value, γ_h_value, γ_c_value, R_value, T_value, w_h0_value)
```

-31.6486401973376

We find that raising  $\gamma_c$  decreases the initial college wage premium  $\phi$ , as we did with our graphical analysis earlier

Let's compute  $\frac{\partial \phi}{\partial R}$  and evaluate it numerically at default parameter values

```
ϕ_R = ϕ(D, γ_h, γ_c, R, T, w_h0).diff(R)
ϕ_R
```

$$-\frac{DR^3 \left(\frac{\gamma_{hc}}{R}\right)^{T-3} \cdot (1 - \frac{\gamma_{hc}}{R})(T-3)}{\gamma_{hc}^4 w_0^h \left(1 - \left(\frac{\gamma_{hc}}{R}\right)^{T-3}\right)^2} + \frac{4DR^3 \cdot (1 - \frac{\gamma_{hc}}{R})}{\gamma_{hc}^4 w_0^h \cdot \left(1 - \left(\frac{\gamma_{hc}}{R}\right)^{T-3}\right)} + \frac{DR^2}{\gamma_{hc}^3 w_0^h \cdot \left(1 - \left(\frac{\gamma_{hc}}{R}\right)^{T-3}\right)} + \frac{R^3 \left(\frac{\gamma_h}{R}\right)^{T+1} \cdot \left(1 - \frac{\gamma_h}{R}\right)(T+1)}{\gamma_{hc}^4 \cdot (1 - \frac{\gamma_h}{R}) \left(1 - \left(\frac{\gamma_h}{R}\right)^{T+1}\right)}$$

```
# Numerical value at default parameters
ϕ_R_func = Lambda((D, γ_h, γ_c, R, T, w_h0), ϕ_R)
ϕ_R_func(D_value, γ_h_value, γ_c_value, R_value, T_value, w_h0_value)
```

13.2642738659429

We find that raising the gross interest rate  $R$  increases the initial college wage premium  $\phi$ , as we did with our graphical analysis earlier

## PRICE LEVEL HISTORIES

This lecture offers some scraps of historical evidence about fluctuations in levels of aggregate price indexes.

The rate of growth of the price level is called **inflation** in the popular press and in discussions among central bankers and treasury officials.

The price level is measured in units of domestic currency per units of a representative bundle of consumption goods.

Thus, in the US, the price level at  $t$  is measured in dollars in month  $t$  or year  $t$  per unit of the consumption bundle.

Until the early 20th century, throughout much of the west, although price levels fluctuated from year to year, they didn't have much of a trend.

Thus, they tended to end a century at close to a level at which they started it.

Things were different in the 20th century, as we shall see in this lecture.

This lecture will set the stage for some subsequent lectures about a particular theory that economists use to think about determinants of the price level.

### 11.1 Four Centuries of Price Levels

We begin by displaying some data that originally appeared on page 35 of [SV02].

The data price levels for four “hard currency” countries from 1600 to 1914.

The four countries are

- France
- Spain (Castile)
- United Kingdom
- United States

In the present context, the phrase hard currency means that the countries were on a commodity-money standard: money consisted of gold and silver coins that circulated at values largely determined by the weights of their gold and silver contents.

(Under a gold or silver standard, some money also consisted of “warehouse certificates” that represented paper claims on gold or silver coins. Bank notes issued by the government or private banks can be viewed as examples of such “warehouse certificate”.)

The data we want to study data originally appeared in a graph on page 35 of [SV02].

As usual, we'll start by importing some Python modules.

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import matplotlib.dates as mdates
import datetime
```

```
!pip install xlrd
```

We'll start by bringing these data into Pandas from a spreadsheet.

```
# import data
df_fig5 = pd.read_excel('datasets/longprices.xls', sheet_name='all', header=2, index_
~col=0).iloc[1:]
df_fig5.index = df_fig5.index.astype(int)

df_fig5.head(5)
```

	UK	US	France	Castile
1600	72.455009	NaN	NaN	48.559223
1601	84.609771	NaN	NaN	41.760932
1602	74.349258	NaN	NaN	40.158477
1603	70.718615	NaN	NaN	40.595510
1604	63.773036	NaN	NaN	44.480248

We first plot price levels over the period 1600-1914.

During most years in this time interval, the countries were on a gold or silver standard.

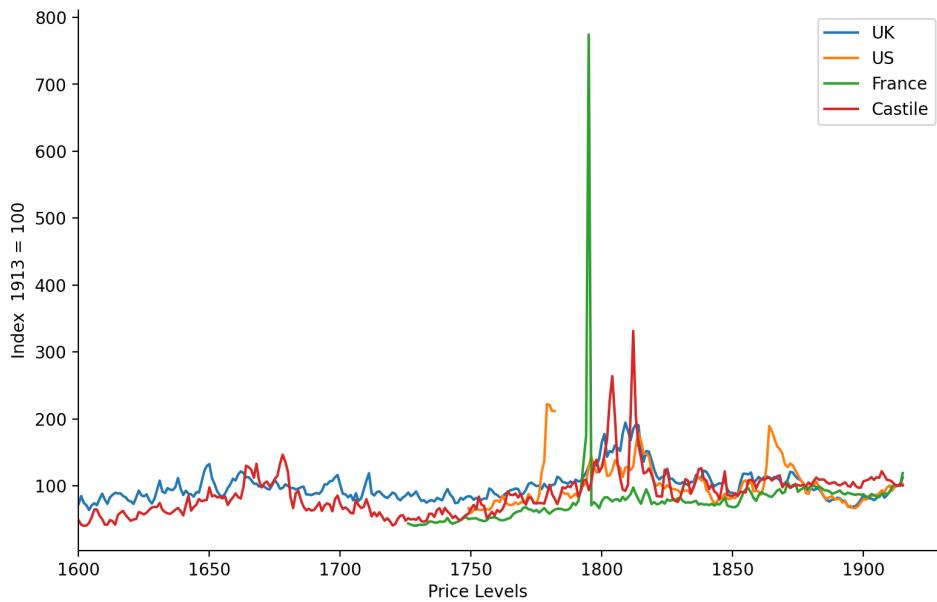
```
df_fig5_bef1914 = df_fig5[df_fig5.index <= 1915]

# create plot
cols = ['UK', 'US', 'France', 'Castile']

fig, ax = plt.subplots(1, 1, figsize=[8, 5], dpi=200)

for col in cols:
    ax.plot(df_fig5_bef1914.index, df_fig5_bef1914[col], label=col)

ax.spines[['right', 'top']].set_visible(False)
ax.legend()
ax.set_ylabel('Index 1913 = 100')
ax.set_xlim(xmin=1600)
plt.tight_layout()
fig.text(.5, .0001, "Price Levels", ha='center')
plt.show()
```



We say “most years” because there were temporary lapses from the gold or silver standard.

By staring at the graph carefully, you might be able to guess when these temporary lapses occurred, because they were also times during which price levels rose markedly from what had been average values during more typical years.

- 1791-1797 in France (the French Revolution)
- 1776-1793 in the US (the US War for Independence from Great Britain)
- 1861-1865 in the US (the US Civil War)

During each of these episodes, the gold/silver standard was temporarily abandoned as a government printed paper money to help it finance war expenditures.

Despite these temporary lapses, a striking thing about the figure is that price levels hovered around roughly constant long-term levels for over three centuries.

Two other features of the figure attracted the attention of leading economists such as Irving Fisher of Yale University and John Maynard Keynes of Cambridge University in the early century.

- There was considerable year-to-year instability of the price levels despite their long term being anchored to the same average level in the long term
- While using valuable gold and silver as coins was a time-tested way to anchor the price level by limiting the supply of money, it cost real resources.
  - that is, society paid a high “opportunity cost” for using gold and silver as coins; gold and silver could instead be used as valuable jewelry and also as an industrial input.

Keynes and Fisher proposed what they suggested would be a socially more efficient way to achieve a price level that would be at least as firmly anchored, and would also exhibit less year-to-year short-term fluctuations.

In particular, they argued that a well-managed central bank could achieve price level stability by

- issuing a **limited supply** of paper currency
- guaranteeing that it would not print money to finance government expenditures

Thus, the waste from using gold and silver as coins prompted John Maynard Keynes to call a commodity standard a “barbarous relic.”

A paper fiat money system dispenses of all reserves behind a currency.

But notice that in doing so, it also eliminates an automatic supply mechanism constraining the price level.

A low-inflation paper fiat money system replaces that automatic mechanism with an enlightened government that commits itself to limiting the quantity of a pure token, no-cost currency.

Now let's see what happened to the price level in our four countries when after 1914 one after another of them left the gold/silver standard.

We'll show a version of the complete graph that originally appeared on page 35 of [SV02].

The graph shows logarithms of price levels our four "hard currency" countries from 1600 to 2000.

Although we didn't have to use logarithms in our earlier graphs that had stopped in 1914 – we use logarithms now because we want also to fit observations after 1914 in the same graph as the earlier observations.

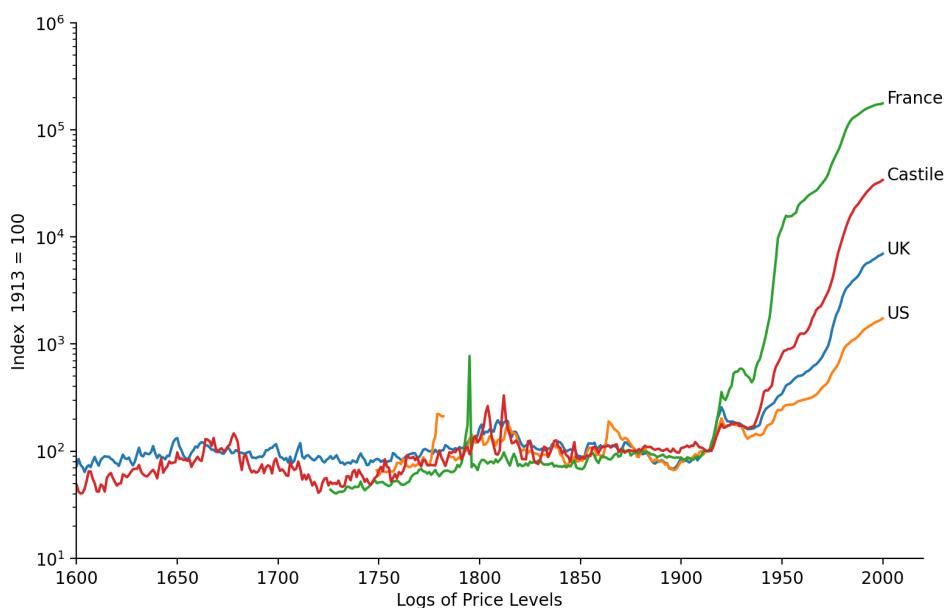
All four of the countries eventually permanently left the gold standard by modifying their monetary and fiscal policies in several ways, starting the outbreak of the Great War in 1914.

```
# create plot
cols = ['UK', 'US', 'France', 'Castile']

fig, ax = plt.subplots(1, 1, figsize=[8, 5], dpi=200)

for col in cols:
    ax.plot(df_fig5.index, df_fig5[col])
    ax.text(x=df_fig5.index[-1]+2, y=df_fig5[col].iloc[-1], s=col)

ax.spines[['right', 'top']].set_visible(False)
ax.set_yscale('log')
ax.set_ylabel('Index 1913 = 100')
ax.set_xlim(xmin=1600)
ax.set_ylim([10, 1e6])
plt.tight_layout()
fig.text(.5, .0001, "Logs of Price Levels", ha='center')
plt.show()
```



The graph shows that achieving a price level system with a well-managed paper money system proved to be more challenging than Irving Fisher and Keynes perhaps imagined.

Actually, earlier economists and statesmen knew about the possibility of fiat money systems long before Keynes and Fisher advocated them in the early 20th century.

It was because earlier proponents of a commodity money system did not trust governments properly to manage a fiat money system that they were willing to pay the resource costs associated with setting up and maintaining a commodity money system.

In light of the high inflation episodes that many countries experienced in the twentieth century after they abandoned commodity monies, it is difficult to criticize them for their preference to stay on the pre-1914 gold/silver standard.

The breadth and length of the inflationary experiences of the twentieth century, the century of paper money, are historically unprecedented.

## 11.2 Ends of Four Big Inflations

In the wake of World War I, which ended in November 1918, monetary and fiscal authorities struggled to achieve price level stability without being on a gold or silver standard.

We present four graphs from “The Ends of Four Big Inflations” from chapter 3 of [Sar13].

The graphs depict logarithms of price levels during the early post World War I years for four countries:

- Figure 3.1, Retail prices Austria, 1921-1924 (page 42)
- Figure 3.2, Wholesale prices Hungary, 1921-1924 (page 43)
- Figure 3.3, Wholesale prices, Poland, 1921-1924 (page 44)
- Figure 3.4, Wholesale prices, Germany, 1919-1924 (page 45)

We have added logarithms of the exchange rates vis a vis the US dollar to each of the four graphs from chapter 3 of [Sar13].

Data underlying our graphs appear in tables in an appendix to chapter 3 of [Sar13]. We have transcribed all of these data into a spreadsheet `chapter_3.xls` that we read into Pandas.

```
# import data
xls = pd.ExcelFile('datasets/chapter_3.xlsx')

# unpack and combine all series
sheet_index = [(2, 3, 4), (9, 10), (14, 15, 16), (21, 18, 19)]
remove_row = [(-2, -2, -2), (-7, -10), (-6, -4, -3), (-19, -3, -6)]

df_list = []

for i in range(4):

    indices, rows = sheet_index[i], remove_row[i]
    sheet_list = [pd.read_excel(xls, 'Table3.' + str(ind), header=1).iloc[:row].
        applymap(process_entry)
        for ind, row in zip(indices, rows)]

    sheet_list = [process_df(df) for df in sheet_list]
    df_list.append(pd.concat(sheet_list, axis=1))

df_Aus, df_Hung, df_Pol, df_Germ = df_list
```

Let's dive in and construct graphs for our four countries.

For each country, we'll plot two graphs.

The first graph plots logarithms of

- price levels
- exchange rates vis a vis US dollars

For each country, the scale on the right side of a graph will pertain to the price level while the scale on the left side of a graph will pertain to the exchange rate.

For each country, the second graph plots a three-month moving average of the inflation rate defined as  $p_t - p_{t-1}$ .

### 11.2.1 Austria

The sources of our data are:

- Table 3.3,  $\exp p$
- Table 3.4, exchange rate with US

```
df_Aus.head(5)
```

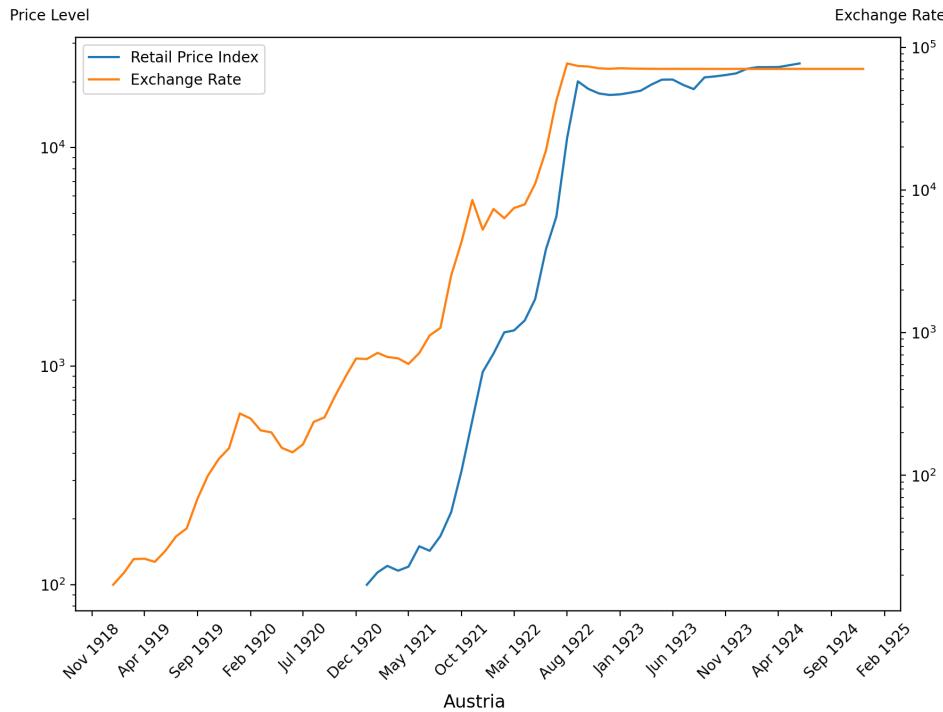
	Total	Note Circulation	Retail price index, 52 commodities	\
1919-01-01		NaN	NaN	
1919-02-01		NaN	NaN	
1919-03-01		4687056.0	NaN	
1919-04-01		5577851.0	NaN	
1919-05-01		5960003.0	NaN	
	Exchange	Rate		
1919-01-01		17.09		
1919-02-01		20.72		
1919-03-01		25.85		
1919-04-01		26.03		
1919-05-01		24.75		

```
p_seq = df_Aus['Retail price index, 52 commodities']
e_seq = df_Aus['Exchange Rate']

lab = ['Retail Price Index', 'Exchange Rate']

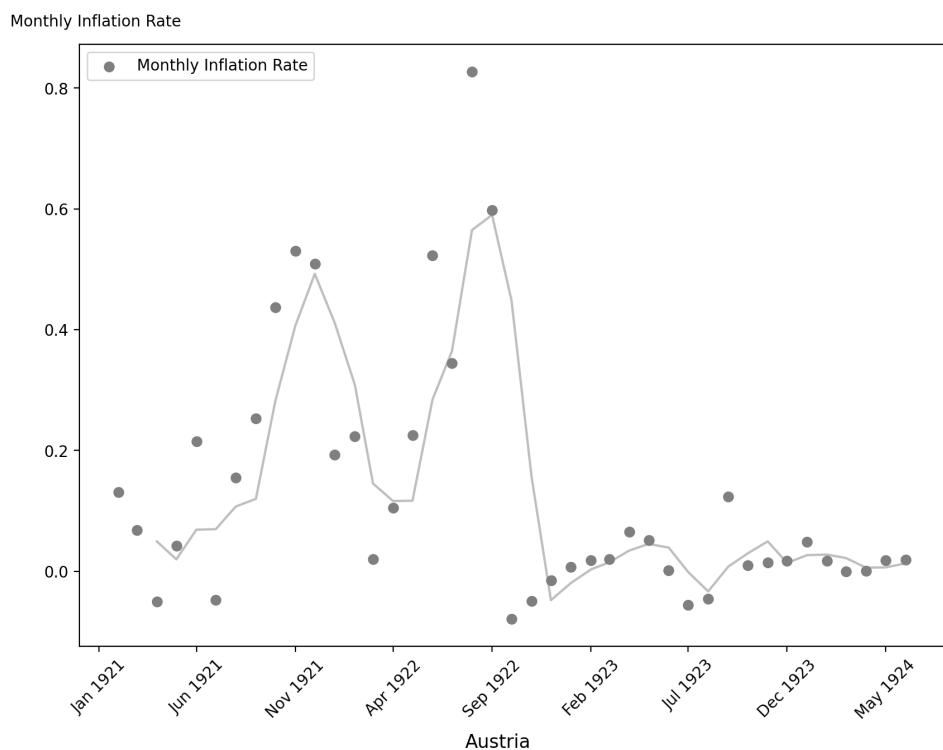
# create plot
fig, ax = plt.subplots(figsize=[10, 7], dpi=200)
_ = create_pe_plot(p_seq, e_seq, df_Aus.index, lab, ax)

# connect disjunct parts
plt.figtext(0.5, -0.02, 'Austria', horizontalalignment='center', fontsize=12)
plt.show()
```



```
fig, ax = plt.subplots(figsize=[10, 7], dpi=200)
_ = create_pr_plot(p_seq, df_Aus.index, ax)

plt.figtext(0.5, -0.02, 'Austria', horizontalalignment='center', fontsize=12)
plt.show()
```



Staring at the above graphs conveys the following impressions to the authors of this lecture at quantecon.

- an episode of “hyperinflation” with rapidly rising log price level and very high monthly inflation rates
- a sudden stop of the hyperinflation as indicated by the abrupt flattening of the log price level and a marked permanent drop in the three-month average of inflation
- a US dollar exchange rate that shadows the price level.

We'll see similar patterns in the next three episodes that we'll study now.

### 11.2.2 Hungary

The source of our data for Hungary is:

- Table 3.10, price level  $\exp p$  and exchange rate

```
df_Hung.head(5)
```

	Gold coin and bullion	Silver coin	Foreign currency and exchange	\
1921-01-01	NaN	NaN	NaN	NaN
1921-02-01	NaN	NaN	NaN	NaN
1921-03-01	NaN	NaN	NaN	NaN
1921-04-01	NaN	NaN	NaN	NaN
1921-05-01	NaN	NaN	NaN	NaN
	Bills discounted	Advances on securities	Advances to treasury	\
1921-01-01	10924.0	195.0	NaN	NaN
1921-02-01	13202.0	162.0	NaN	NaN
1921-03-01	12862.0	160.0	NaN	NaN
1921-04-01	12178.0	110.0	NaN	NaN
1921-05-01	11847.0	111.0	NaN	NaN
	Notes in circulation	Current accounts and deposits	\	
1921-01-01	15206.0	3851.0		
1921-02-01	15571.0	5531.0		
1921-03-01	15650.0	5246.0		
1921-04-01	13114.0	6802.0		
1921-05-01	13686.0	5760.0		
	Hungarian index of prices	Cents per crown in New York		
1921-01-01	NaN	NaN		
1921-02-01	NaN	NaN		
1921-03-01	NaN	NaN		
1921-04-01	NaN	NaN		
1921-05-01	NaN	NaN		

```
m_seq = df_Hung['Notes in circulation']
p_seq = df_Hung['Hungarian index of prices']
e_seq = 1/df_Hung['Cents per crown in New York']
rb_seq = np.log(m_seq) - np.log(p_seq)

lab = ['Hungarian Index of Prices', '1/Cents per Crown in New York']

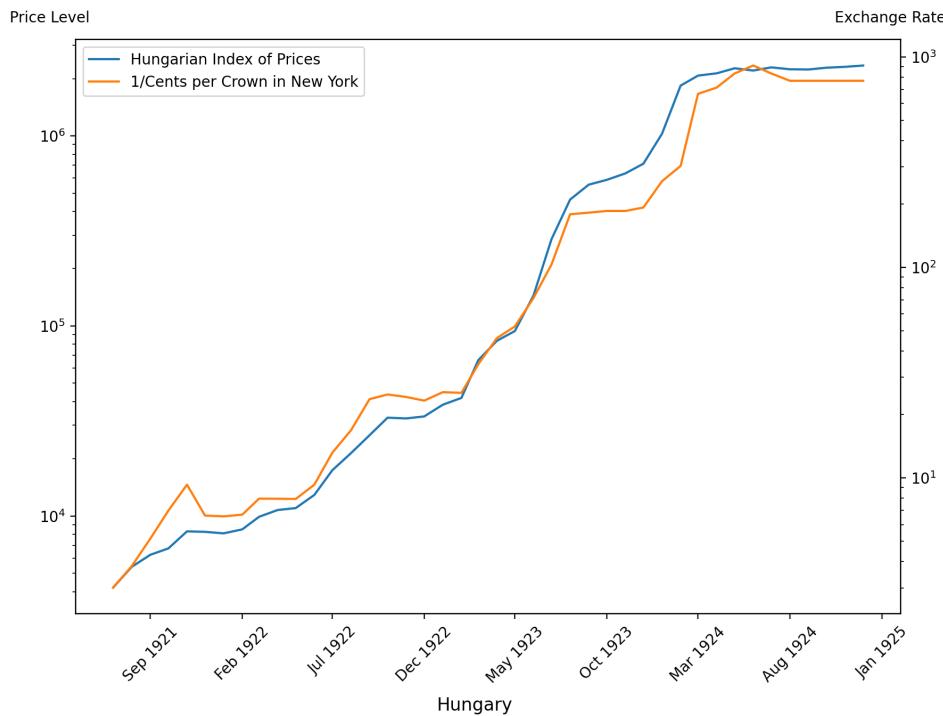
# create plot
fig, ax = plt.subplots(figsize=[10, 7], dpi=200)
```

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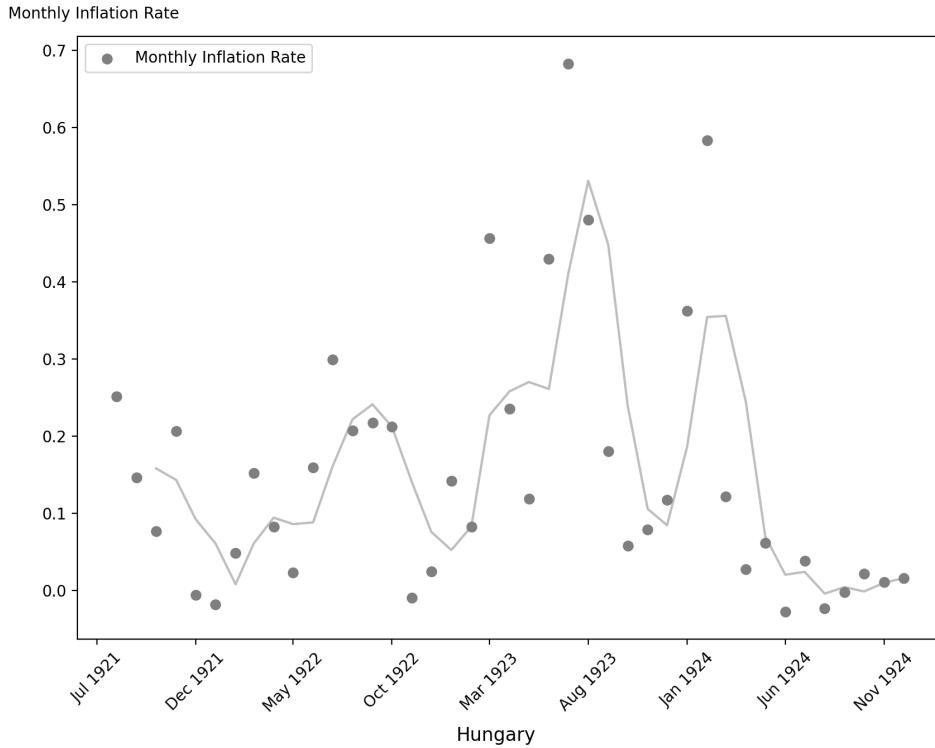
```
_ = create_pe_plot(p_seq, e_seq, df_Hung.index, lab, ax)

plt.figtext(0.5, -0.02, 'Hungary', horizontalalignment='center', fontsize=12)
plt.show()
```



```
fig, ax = plt.subplots(figsize=[10, 7], dpi=200)
_ = create_pr_plot(p_seq, df_Hung.index, ax)

plt.figtext(0.5, -0.02, 'Hungary', horizontalalignment='center', fontsize=12)
plt.show()
```



### 11.2.3 Poland

The sources of our data for Poland are:

- Table 3.15, price level  $\exp p$
- Table 3.15, exchange rate

**Note:** To construct the price level series from the data in the spreadsheet, we instructed Pandas to follow the same procedures implemented in chapter 3 of [Sar13]. We spliced together three series - Wholesale price index, Wholesale Price Index: On paper currency basis, and Wholesale Price Index: On zloty basis. We adjusted the sequence based on the price level ratio at the last period of the available previous series and glued them to construct a single series. We dropped the exchange rate after June 1924, when the zloty was adopted. We did this because we don't have the price measured in zloty. We used the old currency in June to compute the exchange rate adjustment.

```
df_Pol.head(5)
```

	Gold	Silver (including base coin)	Balances with foreign banks	\
1919-01-01	NaN		NaN	NaN
1919-02-01	NaN		NaN	NaN
1919-03-01	3.7		4.2	3.9
1919-04-01	3.7		4.4	9.4
1919-05-01	3.7		8.9	5.8
	Discounts	Advances: Commercial	Advances: Government	\
1919-01-01	5.0	194.7	209.9	
1919-02-01	4.2	196.4	315.0	

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1919-03-01	3.5	189.7	400.0
1919-04-01	2.5	192.8	575.0
1919-05-01	1.8	193.2	925.0
	Note circulation	Gold and Silver (together) \	
1919-01-01	1098.1	NaN	
1919-02-01	1160.0	NaN	
1919-03-01	1223.2	NaN	
1919-04-01	1346.0	NaN	
1919-05-01	1548.3	NaN	
	Wholesale price index \		
1919-01-01		NaN	
1919-02-01		NaN	
1919-03-01		NaN	
1919-04-01		NaN	
1919-05-01		NaN	
	Wholesale Price Index: On paper currency basis \		
1919-01-01		NaN	
1919-02-01		NaN	
1919-03-01		NaN	
1919-04-01		NaN	
1919-05-01		NaN	
	Wholesale Price Index: On zloty basis \		
1919-01-01		NaN	
1919-02-01		NaN	
1919-03-01		NaN	
1919-04-01		NaN	
1919-05-01		NaN	
	Cents per Polish mark (zloty after May 1924)		
1919-01-01		NaN	
1919-02-01		NaN	
1919-03-01		NaN	
1919-04-01		NaN	
1919-05-01		NaN	

```

# splice three price series in different units
p_seq1 = df_Pol['Wholesale price index'].copy()
p_seq2 = df_Pol['Wholesale Price Index: On paper currency basis'].copy()
p_seq3 = df_Pol['Wholesale Price Index: On zloty basis'].copy()

# non-nan part
ch_index_1 = p_seq1[~p_seq1.isna()].index[-1]
ch_index_2 = p_seq2[~p_seq2.isna()].index[-2]

adj_ratio12 = p_seq1[ch_index_1]/p_seq2[ch_index_1]
adj_ratio23 = p_seq2[ch_index_2]/p_seq3[ch_index_2]

# glue three series
p_seq = pd.concat([p_seq1[:ch_index_1],
                   adj_ratio12 * p_seq2[ch_index_1:ch_index_2],
                   adj_ratio23 * p_seq3[ch_index_2:]])

```

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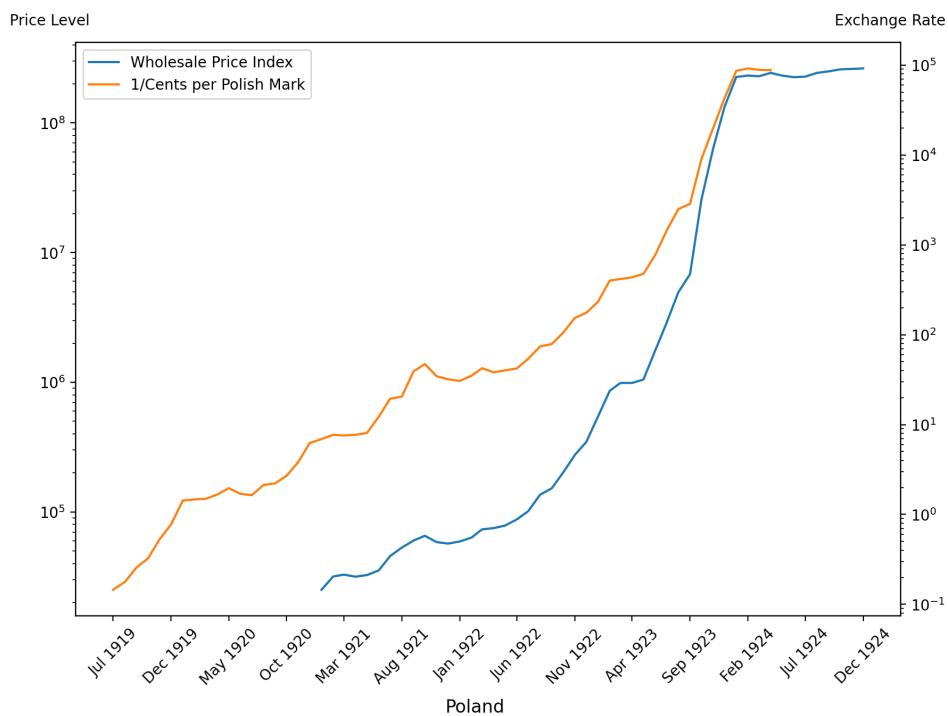
```
p_seq = p_seq[~p_seq.index.duplicated(keep='first')]

# exchange rate
e_seq = 1/df_Pol['Cents per Polish mark (zloty after May 1924)']
e_seq[e_seq.index > '05-01-1924'] = np.nan
```

```
lab = ['Wholesale Price Index', '1/Cents per Polish Mark']

# create plot
fig, ax = plt.subplots(figsize=[10, 7], dpi=200)
ax1 = create_pe_plot(p_seq, e_seq, df_Pol.index, lab, ax)

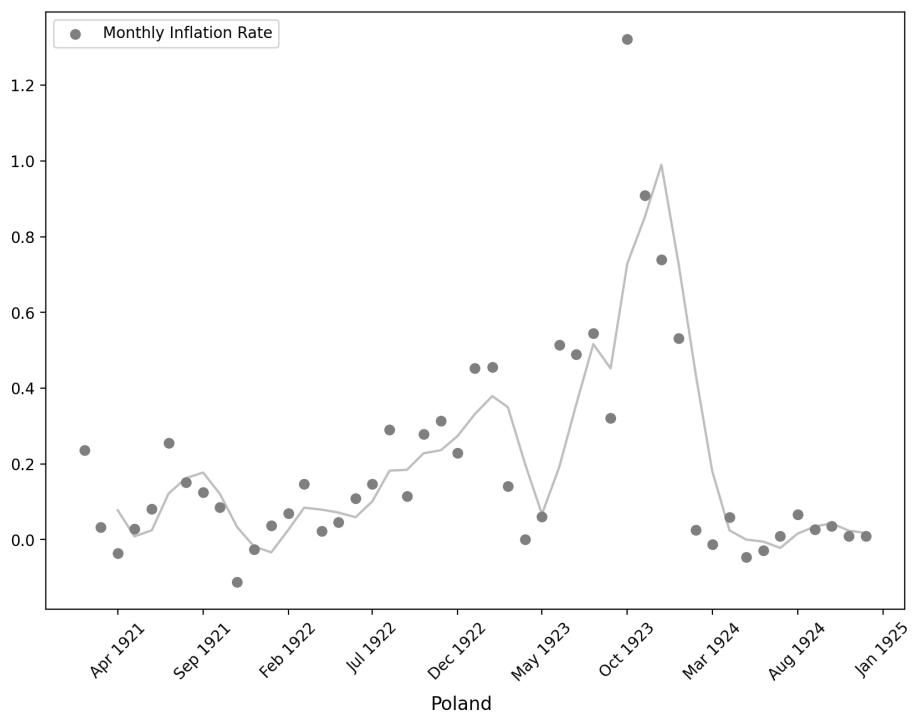
plt.figtext(0.5, -0.02, 'Poland', horizontalalignment='center', fontsize=12)
plt.show()
```



```
fig, ax = plt.subplots(figsize=[10, 7], dpi=200)
_ = create_pr_plot(p_seq, df_Pol.index, ax)

plt.figtext(0.5, -0.02, 'Poland', horizontalalignment='center', fontsize=12)
plt.show()
```

Monthly Inflation Rate



#### 11.2.4 Germany

The sources of our data for Germany are the following tables from chapter 3 of [Sar13]:

- Table 3.18, wholesale price level  $\exp p$
- Table 3.19, exchange rate

```
df_Germ.head(5)
```

	Day	Gold Held Abroad	Gold In vault	Gold Total	\
1919-01-01	NaN	NaN	NaN	NaN	
1919-02-01	NaN	NaN	NaN	NaN	
1919-03-01	NaN	NaN	NaN	NaN	
1919-04-01	NaN	NaN	NaN	NaN	
1919-05-01	NaN	NaN	NaN	NaN	
	Other metallic currency	Total coin and bullion	Foreign exchange		\
1919-01-01		NaN	NaN	NaN	NaN
1919-02-01		NaN	NaN	NaN	NaN
1919-03-01		NaN	NaN	NaN	NaN
1919-04-01		NaN	NaN	NaN	NaN
1919-05-01		NaN	NaN	NaN	NaN
	Treasury and loan bank notes	Notes of other banks		\	
1919-01-01		NaN	NaN		
1919-02-01		NaN	NaN		
1919-03-01		NaN	NaN		
1919-04-01		NaN	NaN		

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1919-05-01		NaN	NaN
	Rentenbank notes	... \	
1919-01-01	NaN	...	
1919-02-01	NaN	...	
1919-03-01	NaN	...	
1919-04-01	NaN	...	
1919-05-01	NaN	...	
	Total discounted treasury and commercial bills	Advances \	
1919-01-01		NaN	NaN
1919-02-01		NaN	NaN
1919-03-01		NaN	NaN
1919-04-01		NaN	NaN
1919-05-01		NaN	NaN
	Securities	Notes in circulation	Public Demand deposits \
1919-01-01	NaN	NaN	NaN
1919-02-01	NaN	NaN	NaN
1919-03-01	NaN	NaN	NaN
1919-04-01	NaN	NaN	NaN
1919-05-01	NaN	NaN	NaN
	Other Demand deposits	Total demand deposits \	
1919-01-01		NaN	NaN
1919-02-01		NaN	NaN
1919-03-01		NaN	NaN
1919-04-01		NaN	NaN
1919-05-01		NaN	NaN
	Demand deposits due to the Rentenbank	\	
1919-01-01		NaN	
1919-02-01		NaN	
1919-03-01		NaN	
1919-04-01		NaN	
1919-05-01		NaN	
	Price index (on basis of marks before July 1924, reichsmarks after)	\	
1919-01-01		262.0	
1919-02-01		270.0	
1919-03-01		274.0	
1919-04-01		286.0	
1919-05-01		297.0	
	Cents per mark		
1919-01-01	NaN		
1919-02-01	NaN		
1919-03-01	NaN		
1919-04-01	NaN		
1919-05-01	NaN		

[5 rows x 22 columns]

```
p_seq = df_Germ['Price index (on basis of marks before July 1924, reichsmarks after)
→'].copy()
```

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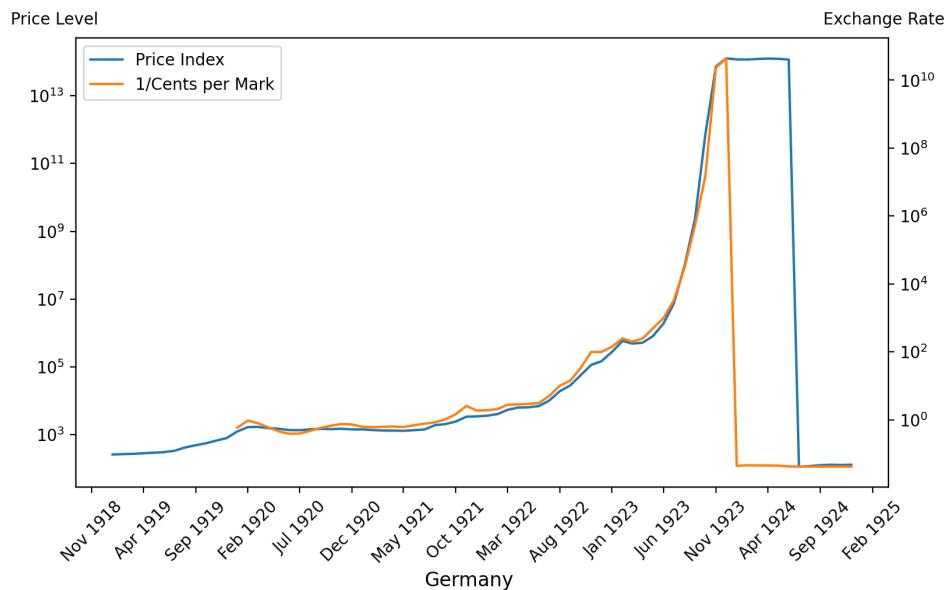
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```
e_seq = 1/df_Germ['Cents per mark']

lab = ['Price Index', '1/Cents per Mark']

# create plot
fig, ax = plt.subplots(figsize=[9,5], dpi=200)
ax1 = create_pe_plot(p_seq, e_seq, df_Germ.index, lab, ax)

plt.figtext(0.5, -0.06, 'Germany', horizontalalignment='center', fontsize=12)
plt.show()
```



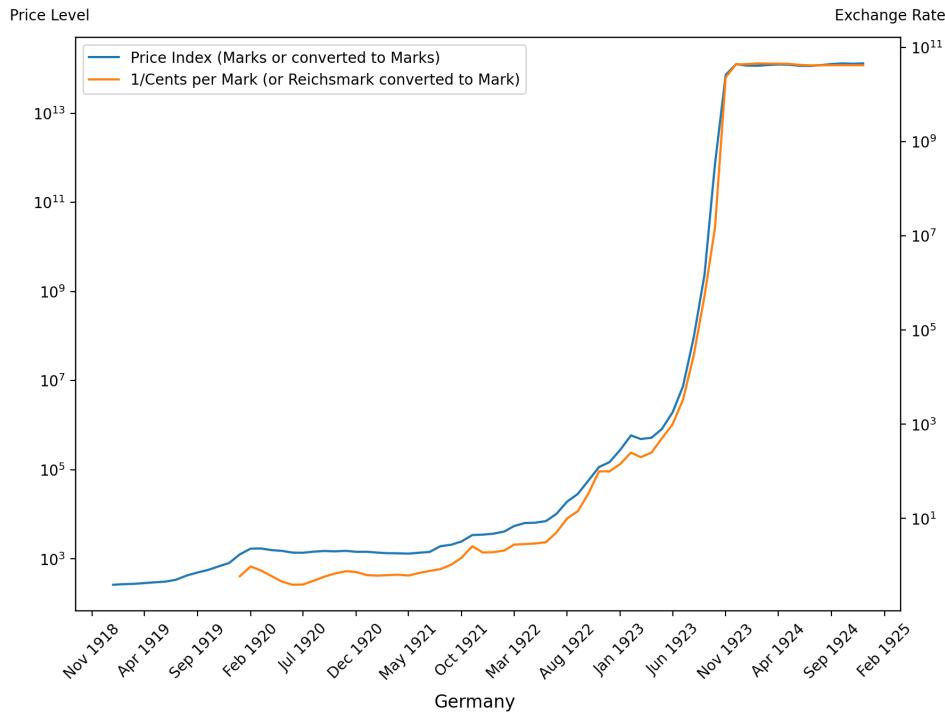
```
p_seq = df_Germ['Price index (on basis of marks before July 1924, reichsmarks after)']
e_seq = 1/df_Germ['Cents per mark'].copy()

# adjust the price level/exchange rate after the currency reform
p_seq[p_seq.index > '06-01-1924'] = p_seq[p_seq.index > '06-01-1924'] * 1e12
e_seq[e_seq.index > '12-01-1923'] = e_seq[e_seq.index > '12-01-1923'] * 1e12

lab = ['Price Index (Marks or converted to Marks)', '1/Cents per Mark (or Reichsmark converted to Mark)']

# create plot
fig, ax = plt.subplots(figsize=[10,7], dpi=200)
ax1 = create_pe_plot(p_seq, e_seq, df_Germ.index, lab, ax)

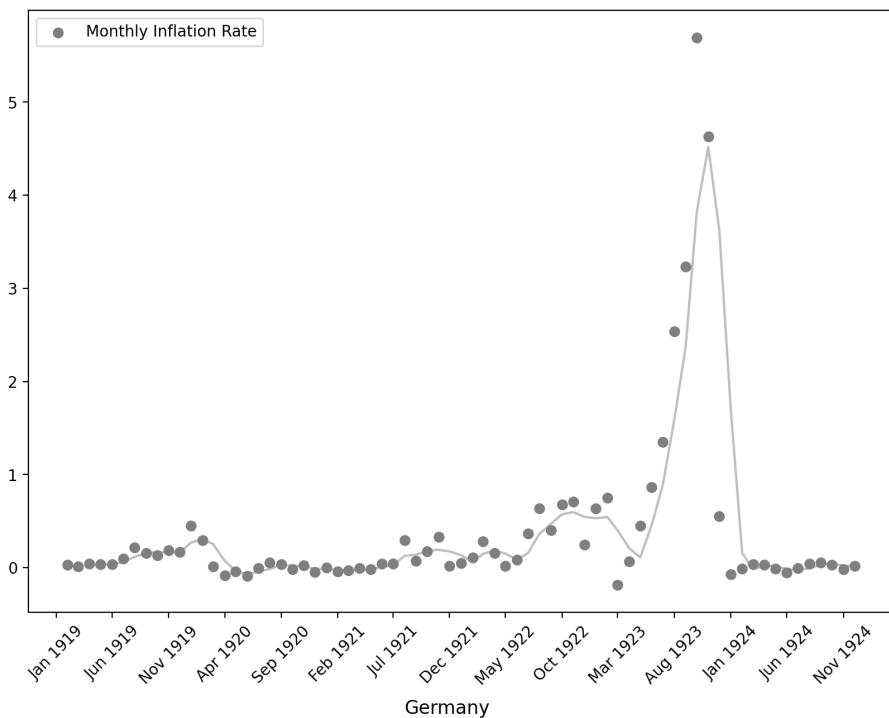
plt.figtext(0.5, -0.02, 'Germany', horizontalalignment='center', fontsize=12)
plt.show()
```



```
fig, ax = plt.subplots(figsize=[10, 7], dpi=200)
_ = create_pr_plot(p_seq, df_Germ.index, ax)

plt.figtext(0.5, -0.02, 'Germany', horizontalalignment='center', fontsize=12)
plt.show()
```

Monthly Inflation Rate



## 11.3 Starting and Stopping Big Inflations

A striking thing about our four graphs is how **quickly** the (log) price levels in Austria, Hungary, Poland, and Germany leveled off after having been rising so quickly.

These “sudden stops” are also revealed by the permanent drops in three-month moving averages of inflation for the four countries.

In addition, the US dollar exchange rates for each of the four countries shadowed their price levels.

- This pattern is an instance of a force modeled in the **purchasing power parity** theory of exchange rates.

Each of these big inflations seemed to have “stopped on a dime”.

Chapter 3 of [SV02] attempts to offer an explanation for this remarkable pattern.

In a nutshell, here is his story.

After World War I, the United States was on the gold standard. The US government stood ready to convert a dollar into a specified amount of gold on demand. To understate things, immediately after the war, Hungary, Austria, Poland, and Germany were not on the gold standard.

In practice, their currencies were largely “fiat” or “unbacked”, meaning that they were not backed by credible government promises to convert them into gold or silver coins on demand. The governments of these countries resorted to the printing of new unbacked money to finance government deficits. (The notes were “backed” mainly by treasury bills that, in those times, could not be expected to be paid off by levying taxes, but only by printing more notes or treasury bills.) This was done on such a scale that it led to a depreciation of the currencies of spectacular proportions. In the end, the German mark stabilized at 1 trillion ( $10^{12}$ ) paper marks to the prewar gold mark, the Polish mark at 1.8 million paper marks to the gold zloty, the Austrian crown at 14,400 paper crowns to the prewar Austro-Hungarian crown, and the Hungarian krone at 14,500 paper crowns to the prewar Austro-Hungarian crown.

Chapter 3 of [SV02] focuses on the deliberate changes in policy that Hungary, Austria, Poland, and Germany made to end their hyperinflations. The hyperinflations were each ended by restoring or virtually restoring convertibility to the dollar or equivalently to gold.

The story told in [SV02] is grounded in a “monetarist theory of the price level” described in *this lecture* and further discussed in *this lecture*.

Those lectures discuss theories about what holders of those rapidly depreciating currencies were thinking about them and how that shaped responses of inflation to government policies.



## A MONETARIST THEORY OF THE PRICE LEVEL

### 12.1 Introduction

We'll use linear algebra first to explain and then do some experiments with a "monetarist theory of the price level".

Sometimes this theory is also called a "fiscal theory of the price level".

Such a of the price level was described by Thomas Sargent and Neil Wallace in chapter 5 of [Sar13], which reprints a 1981 article title "Unpleasant Monetarist Arithmetic".

Sometimes people call it a "monetary" or "monetarist" theory of the price level because fiscal effects on the price level occur through the effects of government fiscal policy decisions on the path of the money supply.

- a government's fiscal policies determine whether its **expenditures** exceed its **tax collections**
- if its expenditures exceeds its tax collections, it can cover the difference by **printing money**
- that leads to effects on the price level as price level path adjusts to equate the supply of money to the demand for money

The theory has been extended, criticized, and applied by John Cochrane in [Coc23].

In another lecture *price level histories*, we described some European hyperinflations that occurred in the wake of World War I.

Elemental forces at work in the fiscal theory of the price level help to understand those episodes.

According to this theory, when the government persistently spends more than it collects in taxes and prints money to finance the shortfall (the "shortfall" is called the "government deficit"), it puts upward pressure on the price level and generates persistent inflation.

The "monetarist or fiscal theory of the price level" asserts that

- to **start** a persistent inflation the government simply persistently runs a money-financed government deficit
- to **stop** a persistent inflation the government simply stops persistently running a money-financed government deficit

Our model is a "rational expectations" (or "perfect foresight") version of a model that Philip Cagan [Cag56] used to study the monetary dynamics of hyperinflations.

While Cagan didn't use that "rational expectations" version of the model, Thomas Sargent [Sar82] did when he studied the Ends of Four Big Inflations in Europe after World War I.

- this lecture *fiscal theory of the price level with adaptive expectations* describes a version of the model that does not impose "rational expectations" but instead uses what Cagan and his teacher Milton Friedman called "adaptive expectations"
  - a reader of both lectures will notice that the algebra is easier and more streamlined in the present rational expectations version of the model

- this can be traced to the following source: the adaptive expectations version of the model has more endogenous variables and more free parameters

Some of our quantitative experiments with our rational expectations version of the model are designed to illustrate how the fiscal theory explains the abrupt end of those big inflations.

In those experiments, we'll encounter an instance of a “velocity dividend” that has sometimes accompanied successful inflation stabilization programs.

To facilitate using linear matrix algebra as our main mathematical tool, we'll use a finite horizon version of the model.

As in the *present values* and *consumption smoothing* lectures, the only linear algebra that we'll be using are matrix multiplication and matrix inversion.

## 12.2 Structure of the model

The model consists of

- a function that expresses the demand for real balances of government printed money as an inverse function of the public's expected rate of inflation
- an exogenous sequence of rates of growth of the money supply. The money supply grows because the government is printing it to finance some of its expenditures
- an equilibrium condition that equates the demand for money to the supply
- a “perfect foresight” assumption that the public's expected rate of inflation equals the actual rate of inflation.

To represent the model formally, let

- $m_t$  be the log of the supply of nominal money balances;
- $\mu_t = m_{t+1} - m_t$  be the net rate of growth of nominal balances;
- $p_t$  be the log of the price level;
- $\pi_t = p_{t+1} - p_t$  be the net rate of inflation between  $t$  and  $t + 1$ ;
- $\pi_t^*$  be the public's expected rate of inflation between  $t$  and  $t + 1$ ;
- $T$  the horizon – i.e., the last period for which the model will determine  $p_t$
- $\pi_{T+1}^*$  the terminal rate of inflation between times  $T$  and  $T + 1$ .

The demand for real balances  $\exp\left(\frac{m_t^d}{p_t}\right)$  is governed by the following version of the Cagan demand function

$$m_t^d - p_t = -\alpha\pi_t^*, \quad \alpha > 0; \quad t = 0, 1, \dots, T. \quad (12.1)$$

This equation asserts that the demand for real balances is inversely related to the public's expected rate of inflation.

People somehow acquire **perfect foresight** by their having solved a forecasting problem.

This lets us set

$$\pi_t^* = \pi_t, \quad (12.2)$$

while equating demand for money to supply lets us set  $m_t^d = m_t$  for all  $t \geq 0$ .

The preceding equations then imply

$$m_t - p_t = -\alpha(p_{t+1} - p_t), \quad \alpha > 0 \quad (12.3)$$

To fill in details about what it means for private agents to have perfect foresight, we subtract equation (12.3) at time  $t$  from the same equation at  $t + 1$  to get

$$\mu_t - \pi_t = -\alpha\pi_{t+1} + \alpha\pi_t,$$

which we rewrite as a forward-looking first-order linear difference equation in  $\pi_s$  with  $\mu_s$  as a “forcing variable”:

$$\pi_t = \frac{\alpha}{1+\alpha}\pi_{t+1} + \frac{1}{1+\alpha}\mu_t, \quad t = 0, 1, \dots, T$$

where  $0 < \frac{\alpha}{1+\alpha} < 1$ .

Setting  $\delta = \frac{\alpha}{1+\alpha}$  let's us represent the preceding equation as

$$\pi_t = \delta\pi_{t+1} + (1-\delta)\mu_t, \quad t = 0, 1, \dots, T$$

Write this system of  $T + 1$  equations as the single matrix equation

$$\begin{bmatrix} 1 & -\delta & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\delta & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -\delta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\delta \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \vdots \\ \pi_{T-1} \\ \pi_T \end{bmatrix} = (1-\delta) \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{T-1} \\ \mu_T \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \delta\pi_{T+1}^* \end{bmatrix} \quad (12.4)$$

By multiplying both sides of equation (12.4) by the inverse of the matrix on the left side, we can calculate

$$\pi \equiv \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \vdots \\ \pi_{T-1} \\ \pi_T \end{bmatrix}$$

It turns out that

$$\pi_t = (1-\delta) \sum_{s=t}^T \delta^{s-t} \mu_s + \delta^{T+1-t} \pi_{T+1}^* \quad (12.5)$$

We can represent the equations

$$m_{t+1} = m_t + \mu_t, \quad t = 0, 1, \dots, T$$

as the matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_T \\ m_{T+1} \end{bmatrix} = \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{T-1} \\ \mu_T \end{bmatrix} + \begin{bmatrix} m_0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad (12.6)$$

Multiplying both sides of equation (12.6) with the inverse of the matrix on the left will give

$$m_t = m_0 + \sum_{s=0}^{t-1} \mu_s, \quad t = 1, \dots, T + 1 \quad (12.7)$$

Equation (12.7) shows that the log of the money supply at  $t$  equals the log of the initial money supply  $m_0$  plus accumulation of rates of money growth between times 0 and  $T$ .

## 12.3 Continuation values

To determine the continuation inflation rate  $\pi_{T+1}^*$  we shall proceed by applying the following infinite-horizon version of equation (12.5) at time  $t = T + 1$ :

$$\pi_t = (1 - \delta) \sum_{s=t}^{\infty} \delta^{s-t} \mu_s, \quad (12.8)$$

and by also assuming the following continuation path for  $\mu_t$  beyond  $T$ :

$$\mu_{t+1} = \gamma^* \mu_t, \quad t \geq T.$$

Plugging the preceding equation into equation (12.8) at  $t = T + 1$  and rearranging we can deduce that

$$\pi_{T+1}^* = \frac{1 - \delta}{1 - \delta \gamma^*} \gamma^* \mu_T \quad (12.9)$$

where we require that  $|\gamma^* \delta| < 1$ .

Let's implement and solve this model.

As usual, we'll start by importing some Python modules.

```
import numpy as np
from collections import namedtuple
import matplotlib.pyplot as plt
```

First, we store parameters in a namedtuple:

```
# Create the rational expectation version of Cagan model in finite time
CaganREE = namedtuple("CaganREE",
                      ["m0", "T", "mu_seq", "a", "delta", "pi_end"])

def create_cagan_model(m0, a, T, mu_seq):
    delta = a / (1 + a)
    pi_end = mu_seq[-1]      # compute terminal expected inflation
    return CaganREE(m0, T, mu_seq, a, delta, pi_end)
```

Here we use the following parameter values:

```
# parameters
T = 80
T1 = 60
a = 5
m0 = 1

mu0 = 0.5
mu_star = 0
```

Now we can solve the model to compute  $\pi_t$ ,  $m_t$  and  $p_t$  for  $t = 1, \dots, T + 1$  using the matrix equation above

```
def solve(model):
    model_params = model.m0, model.T, model.pi_end, model.mu_seq, model.a, model.delta
    m0, T, pi_end, mu_seq, a, delta = model_params
    A1 = np.eye(T+1, T+1) - delta * np.eye(T+1, T+1, k=1)
    A2 = np.eye(T+1, T+1) - np.eye(T+1, T+1, k=-1)
```

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```

b1 = (1-δ) * μ_seq + np.concatenate([np.zeros(T), [δ * π_end]])
b2 = μ_seq + np.concatenate([[m0], np.zeros(T)])

π_seq = np.linalg.inv(A1) @ b1
m_seq = np.linalg.inv(A2) @ b2

π_seq = np.append(π_seq, π_end)
m_seq = np.append(m0, m_seq)

p_seq = m_seq + α * π_seq

return π_seq, m_seq, p_seq

```

### 12.3.1 Some quantitative experiments

In the experiments below, we'll use formula (12.9) as our terminal condition for expected inflation.

In devising these experiments, we'll make assumptions about  $\{\mu_t\}$  that are consistent with formula (12.9).

We describe several such experiments.

In all of them,

$$\mu_t = \mu^*, \quad t \geq T_1$$

so that, in terms of our notation and formula for  $\theta_{T+1}^*$  above,  $\tilde{\gamma} = 1$ .

#### Experiment 1: Foreseen sudden stabilization

In this experiment, we'll study how, when  $\alpha > 0$ , a foreseen inflation stabilization has effects on inflation that proceed it.

We'll study a situation in which the rate of growth of the money supply is  $\mu_0$  from  $t = 0$  to  $t = T_1$  and then permanently falls to  $\mu^*$  at  $t = T_1$ .

Thus, let  $T_1 \in (0, T)$ .

So where  $\mu_0 > \mu^*$ , we assume that

$$\mu_{t+1} = \begin{cases} \mu_0, & t = 0, \dots, T_1 - 1 \\ \mu^*, & t \geq T_1 \end{cases}$$

We'll start by executing a version of our “experiment 1” in which the government implements a **foreseen** sudden permanent reduction in the rate of money creation at time  $T_1$ .

The following code performs the experiment and plots outcomes.

```

def solve_and_plot(m0, α, T, μ_seq):
    model_params = create_cagan_model(m0=m0, α=α, T=T, μ_seq=μ_seq)
    π_seq, m_seq, p_seq = solve(model_params)
    T_seq = range(T + 2)

    fig, ax = plt.subplots(5, figsize=[5, 12], dpi=200)
    ax[0].plot(T_seq[:-1], μ_seq)

```

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```

ax[0].set_ylabel(r'$\mu$')
ax[1].plot(T_seq, pi_seq)
ax[1].set_ylabel(r'$\pi$')

ax[2].plot(T_seq, m_seq - p_seq)
ax[2].set_ylabel(r'$m - p$')

ax[3].plot(T_seq, m_seq)
ax[3].set_ylabel(r'$m$')

ax[4].plot(T_seq, p_seq)
ax[4].set_ylabel(r'$p$')

for i in range(5):
    ax[i].set_xlabel(r'$t$')

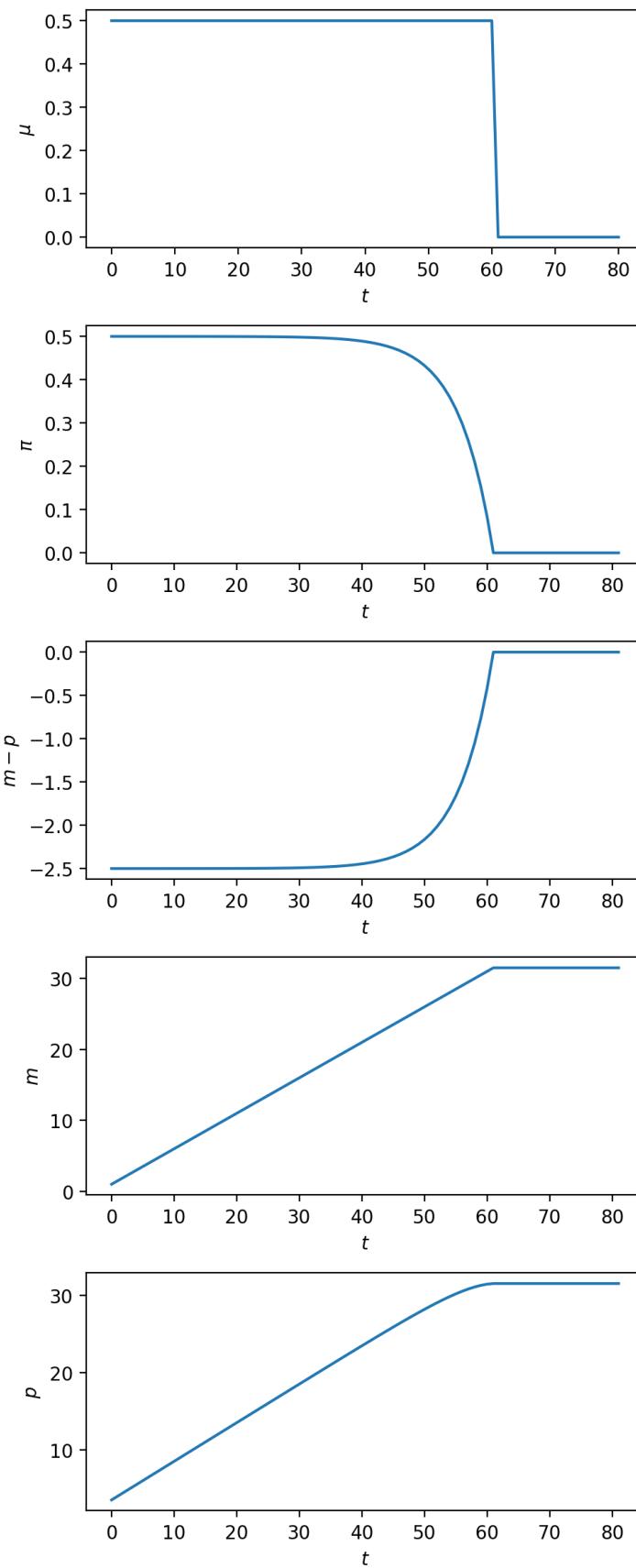
plt.tight_layout()
plt.show()

return pi_seq, m_seq, p_seq

mu_seq_1 = np.append(mu0*np.ones(T1+1), mu_star*np.ones(T-T1))

# solve and plot
pi_seq_1, m_seq_1, p_seq_1 = solve_and_plot(m0=m0, a=a,
                                              T=T, mu_seq=mu_seq_1)

```



The plot of the money growth rate  $\mu_t$  in the top level panel portrays a sudden reduction from .5 to 0 at time  $T_1 = 60$ . This brings about a gradual reduction of the inflation rate  $\pi_t$  that precedes the money supply growth rate reduction at time  $T_1$ .

Notice how the inflation rate declines smoothly (i.e., continuously) to 0 at  $T_1$  – unlike the money growth rate, it does not suddenly “jump” downward at  $T_1$ .

This is because the reduction in  $\mu$  at  $T_1$  has been foreseen from the start.

While the log money supply portrayed in the bottom panel has a kink at  $T_1$ , the log price level does not – it is “smooth” – once again a consequence of the fact that the reduction in  $\mu$  has been foreseen.

To set the stage for our next experiment, we want to study the determinants of the price level a little more.

### 12.3.2 The log price level

We can use equations (12.1) and (12.2) to discover that the log of the price level satisfies

$$p_t = m_t + \alpha\pi_t \quad (12.10)$$

or, by using equation (12.5),

$$p_t = m_t + \alpha \left[ (1 - \delta) \sum_{s=t}^T \delta^{s-t} \mu_s + \delta^{T+1-t} \pi_{T+1}^* \right] \quad (12.11)$$

In our next experiment, we'll study a “surprise” permanent change in the money growth that beforehand was completely unanticipated.

At time  $T_1$  when the “surprise” money growth rate change occurs, to satisfy equation (12.10), the log of real balances jumps *\*upward* as  $\pi_t$  jumps **downward**.

But in order for  $m_t - p_t$  to jump, which variable jumps,  $m_{T_1}$  or  $p_{T_1}$ ?

### 12.3.3 What jumps?

What jumps at  $T_1$ ?

Is it  $p_{T_1}$  or  $m_{T_1}$ ?

If we insist that the money supply  $m_{T_1}$  is locked at its value  $m_{T_1}^1$  inherited from the past, then formula (12.10) implies that the price level jumps downward at time  $T_1$ , to coincide with the downward jump in  $\pi_{T_1}$ .

An alternative assumption about the money supply level is that as part of the “inflation stabilization”, the government resets  $m_{T_1}$  according to

$$m_{T_1}^2 - m_{T_1}^1 = \alpha(\pi^1 - \pi^2) \quad (12.12)$$

By letting money jump according to equation (12.12) the monetary authority prevents the price level from **falling** at the moment that the unanticipated stabilization arrives.

In various research papers about stabilizations of high inflations, the jump in the money supply described by equation (12.12) has been called “the velocity dividend” that a government reaps from implementing a regime change that sustains a permanently lower inflation rate.

### Technical details about whether $p$ or $m$ jumps at $T_1$

We have noted that with a constant expected forward sequence  $\mu_s = \bar{\mu}$  for  $s \geq t$ ,  $\pi_t = \bar{\mu}$ .

A consequence is that at  $T_1$ , either  $m$  or  $p$  must “jump” at  $T_1$ .

We'll study both cases.

$m_{T_1}$  **does not jump.**

$$m_{T_1} = m_{T_1-1} + \mu_0$$

$$\pi_{T_1} = \mu^*$$

$$p_{T_1} = m_{T_1} + \alpha\pi_{T_1}$$

Simply glue the sequences  $t \leq T_1$  and  $t > T_1$ .

$m_{T_1}$  **jumps.**

We reset  $m_{T_1}$  so that  $p_{T_1} = (m_{T_1-1} + \mu_0) + \alpha\mu_0$ , with  $\pi_{T_1} = \mu^*$ .

Then,

$$m_{T_1} = p_{T_1} - \alpha\pi_{T_1} = (m_{T_1-1} + \mu_0) + \alpha(\mu_0 - \mu^*)$$

We then compute for the remaining  $T - T_1$  periods with  $\mu_s = \mu^*$ ,  $\forall s \geq T_1$  and the initial condition  $m_{T_1}$  from above.

We are now technically equipped to discuss our next experiment.

### Experiment 2: an unforeseen sudden stabilization

This experiment deviates a little bit from a pure version of our “perfect foresight” assumption by assuming that a sudden permanent reduction in  $\mu_t$  like that analyzed in experiment 1 is completely unanticipated.

Such a completely unanticipated shock is popularly known as an “MIT shock”.

The mental experiment involves switching at time  $T_1$  from an initial “continuation path” for  $\{\mu_t, \pi_t\}$  to another path that involves a permanently lower inflation rate.

**Initial Path:**  $\mu_t = \mu_0$  for all  $t \geq 0$ . So this path is for  $\{\mu_t\}_{t=0}^{\infty}$ ; the associated path for  $\pi_t$  has  $\pi_t = \mu_0$ .

**Revised Continuation Path** Where  $\mu_0 > \mu^*$ , we construct a continuation path  $\{\mu_s\}_{s=T_1}^{\infty}$  by setting  $\mu_s = \mu^*$  for all  $s \geq T_1$ . The perfect foresight continuation path for  $\pi$  is  $\pi_s = \mu^*$

To capture a “completely unanticipated permanent shock to the  $\{\mu\}$  process at time  $T_1$ , we simply glue the  $\mu_t, \pi_t$  that emerges under path 2 for  $t \geq T_1$  to the  $\mu_t, \pi_t$  path that had emerged under path 1 for  $t = 0, \dots, T_1 - 1$ .

We can do the MIT shock calculations mostly by hand.

Thus, for path 1,  $\pi_t = \mu_0$  for all  $t \in [0, T_1 - 1]$ , while for path 2,  $\mu_s = \mu^*$  for all  $s \geq T_1$ .

We now move on to experiment 2, our “MIT shock”, completely unforeseen sudden stabilization.

We set this up so that the  $\{\mu_t\}$  sequences that describe the sudden stabilization are identical to those for experiment 1, the foreseen sudden stabilization.

The following code does the calculations and plots outcomes.

```

# path 1
μ_seq_2_path1 = μ0 * np.ones(T+1)

mc1 = create_cagan_model(m0=m0, a=a,
                         T=T, μ_seq=μ_seq_2_path1)
π_seq_2_path1, m_seq_2_path1, p_seq_2_path1 = solve(mc1)

# continuation path
μ_seq_2_cont = μ_star * np.ones(T-T1)

mc2 = create_cagan_model(m0=m_seq_2_path1[T1+1],
                         a=a, T=T-1-T1, μ_seq=μ_seq_2_cont)
π_seq_2_cont, m_seq_2_cont1, p_seq_2_cont1 = solve(mc2)

# regime 1 - simply glue π_seq, μ_seq
μ_seq_2 = np.concatenate([μ_seq_2_path1[:T1+1],
                           μ_seq_2_cont])
π_seq_2 = np.concatenate([π_seq_2_path1[:T1+1],
                           π_seq_2_cont])
m_seq_2_regime1 = np.concatenate([m_seq_2_path1[:T1+1],
                                   m_seq_2_cont1])
p_seq_2_regime1 = np.concatenate([p_seq_2_path1[:T1+1],
                                   p_seq_2_cont1])

# regime 2 - reset m_T1
m_T1 = (m_seq_2_path1[T1] + μ0) + a*(μ0 - μ_star)

mc = create_cagan_model(m0=m_T1, a=a,
                         T=T-1-T1, μ_seq=μ_seq_2_cont)
π_seq_2_cont2, m_seq_2_cont2, p_seq_2_cont2 = solve(mc)

m_seq_2_regime2 = np.concatenate([m_seq_2_path1[:T1+1],
                                   m_seq_2_cont2])
p_seq_2_regime2 = np.concatenate([p_seq_2_path1[:T1+1],
                                   p_seq_2_cont2])

T_seq = range(T+2)

# plot both regimes
fig, ax = plt.subplots(5, 1, figsize=[5, 12], dpi=200)

ax[0].plot(T_seq[:-1], μ_seq_2)
ax[0].set_ylabel(r'$\mu$')

ax[1].plot(T_seq, π_seq_2)
ax[1].set_ylabel(r'$\pi$')

ax[2].plot(T_seq, m_seq_2_regime1 - p_seq_2_regime1)
ax[2].set_ylabel(r'$m - p$')

ax[3].plot(T_seq, m_seq_2_regime1,
            label='Smooth $m_{T_1}$')
ax[3].plot(T_seq, m_seq_2_regime2,
            label='Jumpy $m_{T_1}$')
ax[3].set_ylabel(r'$m$')


```

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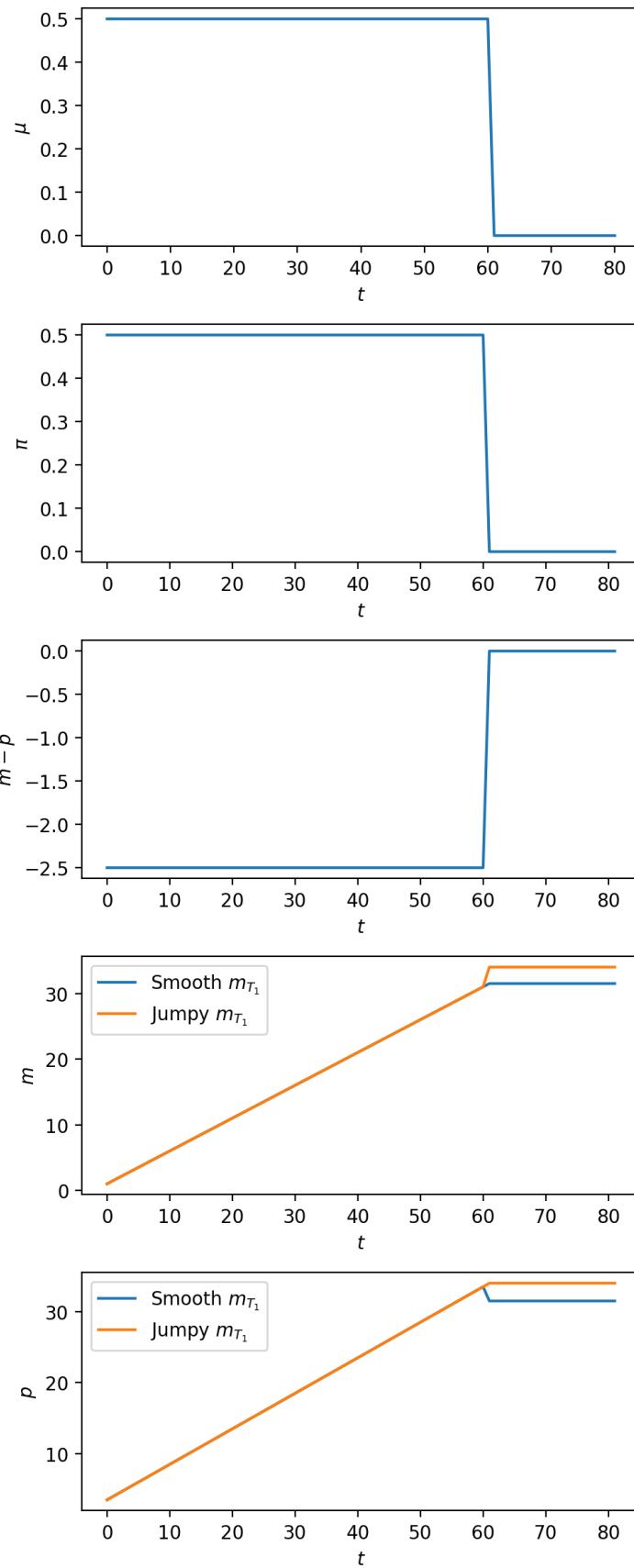
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```
ax[4].plot(T_seq, p_seq_2_regime1,
            label='Smooth $m_{T_1}$')
ax[4].plot(T_seq, p_seq_2_regime2,
            label='Jumpy $m_{T_1}$')
ax[4].set_ylabel(r'$p$')

for i in range(5):
    ax[i].set_xlabel(r'$t$')

for i in [3, 4]:
    ax[i].legend()

plt.tight_layout()
plt.show()
```



We invite you to compare these graphs with corresponding ones for the foreseen stabilization analyzed in experiment 1 above.

Note how the inflation graph in the top middle panel is now identical to the money growth graph in the top left panel, and how now the log of real balances portrayed in the top right panel jumps upward at time  $T_1$ .

The bottom panels plot  $m$  and  $p$  under two possible ways that  $m_{T_1}$  might adjust as required by the upward jump in  $m - p$  at  $T_1$ .

- the orange line lets  $m_{T_1}$  jump upward in order to make sure that the log price level  $p_{T_1}$  does not fall.
- the blue line lets  $p_{T_1}$  fall while stopping the money supply from jumping.

Here is a way to interpret what the government is doing when the orange line policy is in place.

The government prints money to finance expenditure with the “velocity dividend” that it reaps from the increased demand for real balances brought about by the permanent decrease in the rate of growth of the money supply.

The next code generates a multi-panel graph that includes outcomes of both experiments 1 and 2.

That allows us to assess how important it is to understand whether the sudden permanent drop in  $\mu_t$  at  $t = T_1$  is fully unanticipated, as in experiment 1, or completely unanticipated, as in experiment 2.

```
# compare foreseen vs unforeseen shock
fig, ax = plt.subplots(5, figsize=[5, 12], dpi=200)

ax[0].plot(T_seq[:-1], mu_seq_2)
ax[0].set_ylabel(r'$\mu$')

ax[1].plot(T_seq, pi_seq_2,
            label='Unforeseen')
ax[1].plot(T_seq, pi_seq_1,
            label='Foreseen', color='tab:blue')
ax[1].set_ylabel(r'$\pi$')

ax[2].plot(T_seq,
            m_seq_2_regime1 - p_seq_2_regime1,
            label='Unforeseen')
ax[2].plot(T_seq, m_seq_1 - p_seq_1,
            label='Foreseen', color='tab:blue')
ax[2].set_ylabel(r'$m - p$')

ax[3].plot(T_seq, m_seq_2_regime1,
            label=r'Unforeseen (Smooth $m_{T_1}$)')
ax[3].plot(T_seq, m_seq_2_regime2,
            label=r'Unforeseen ($m_{T_1}$ jumps)')
ax[3].plot(T_seq, m_seq_1,
            label='Foreseen shock')
ax[3].set_ylabel(r'$m$')

ax[4].plot(T_seq, p_seq_2_regime1,
            label=r'Unforeseen (Smooth $m_{T_1}$)')
ax[4].plot(T_seq, p_seq_2_regime2,
            label=r'Unforeseen ($m_{T_1}$ jumps)')
ax[4].plot(T_seq, p_seq_1,
            label='Foreseen')
ax[4].set_ylabel(r'$p$')

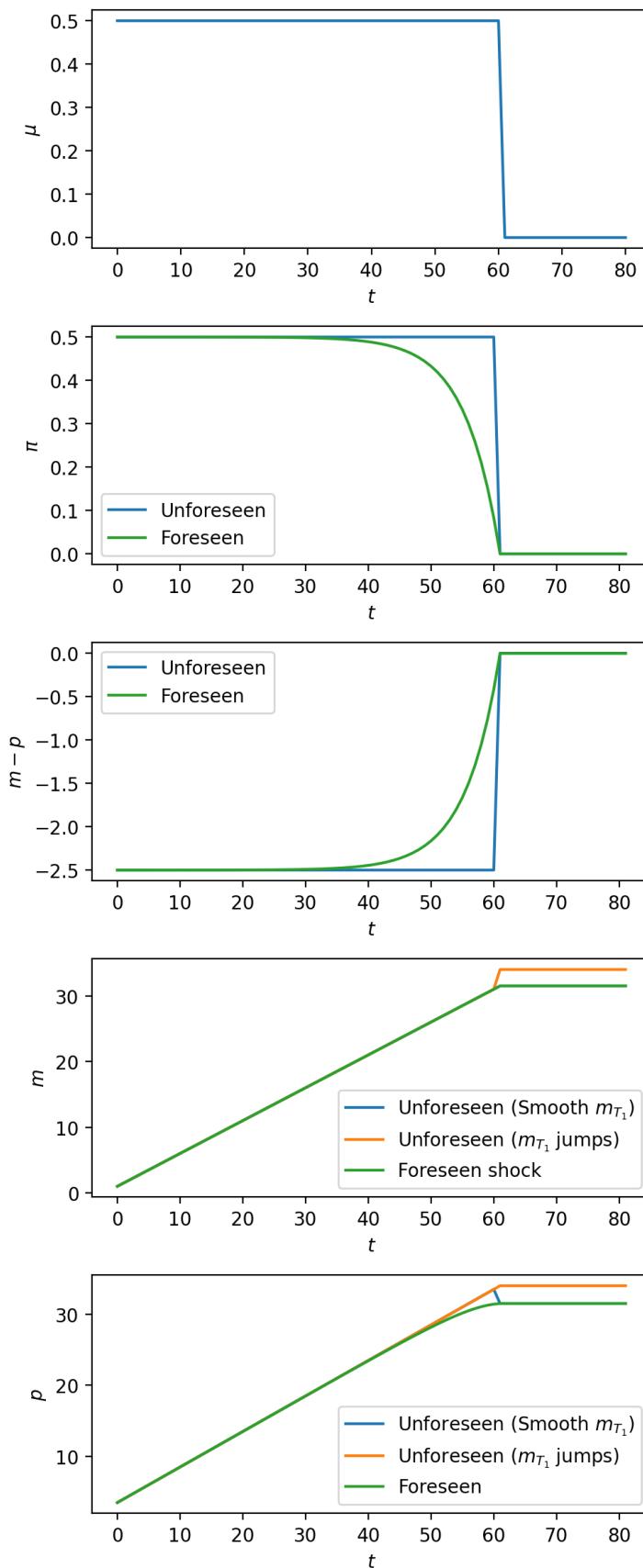
for i in range(5):
    ax[i].set_xlabel(r'$t$')
```

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```
for i in range(1, 5):
    ax[i].legend()

plt.tight_layout()
plt.show()
```



It is instructive to compare the preceding graphs with graphs of log price levels and inflation rates for data from four big inflations described in [this lecture](#).

In particular, in the above graphs, notice how a gradual fall in inflation precedes the “sudden stop” when it has been anticipated long beforehand, but how inflation instead falls abruptly when the permanent drop in money supply growth is unanticipated.

It seems to the author team at quantecon that the drops in inflation near the ends of the four hyperinflations described in [this lecture](#) more closely resemble outcomes from the experiment 2 “unforeseen stabilization”.

(It is fair to say that the preceding informal pattern recognition exercise should be supplemented with a more formal structural statistical analysis.)

### Experiment 3

#### Foreseen gradual stabilization

Instead of a foreseen sudden stabilization of the type studied with experiment 1, it is also interesting to study the consequences of a foreseen gradual stabilization.

Thus, suppose that  $\phi \in (0, 1)$ , that  $\mu_0 > \mu^*$ , and that for  $t = 0, \dots, T - 1$

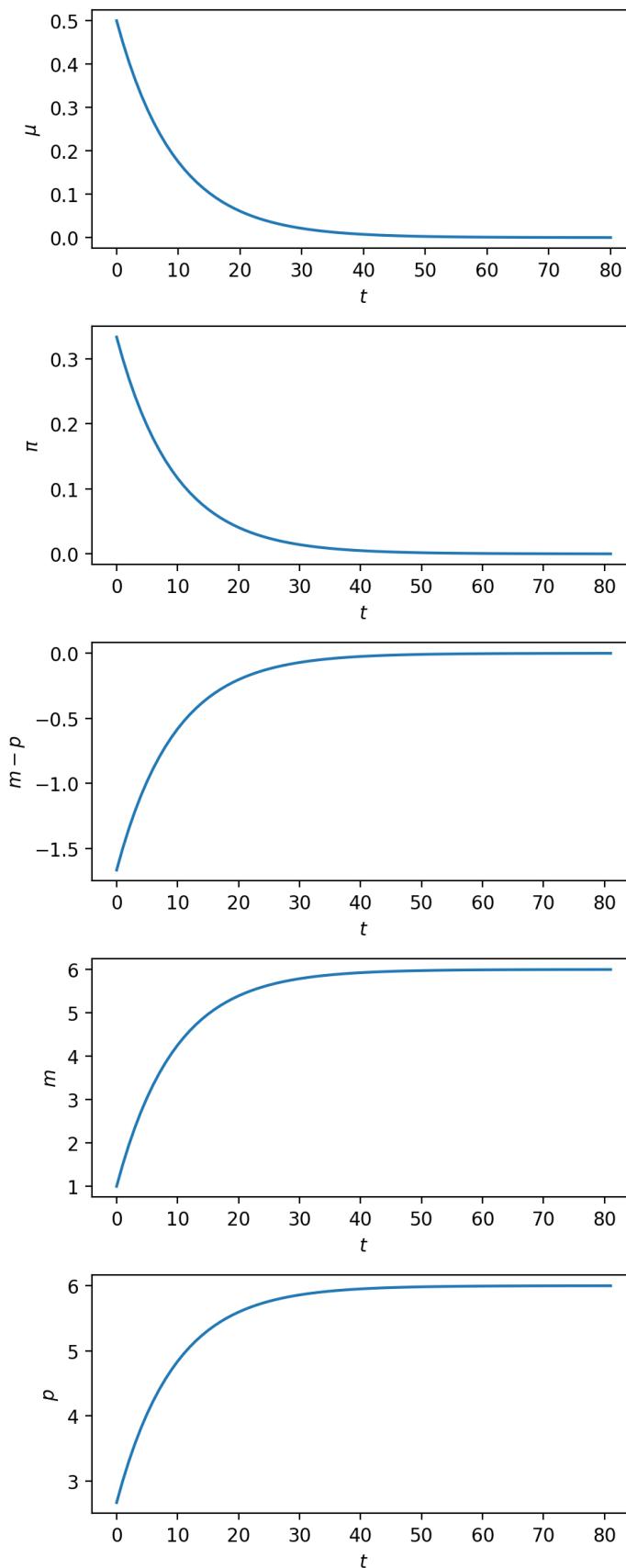
$$\mu_t = \phi^t \mu_0 + (1 - \phi^t) \mu^*.$$

Next we perform an experiment in which there is a perfectly foreseen **gradual** decrease in the rate of growth of the money supply.

The following code does the calculations and plots the results.

```
# parameters
phi = 0.9
mu_seq = np.array([phi**t * mu0 + (1-phi**t)*mu_star for t in range(T)])
mu_seq = np.append(mu_seq, mu_star)

# solve and plot
pi_seq, m_seq, p_seq = solve_and_plot(m0=m0, a=a, T=T, mu_seq=mu_seq)
```



## 12.4 Sequel

This lecture *monetarist theory of the price level with adaptive expectations* describes an “adaptive expectations” version of Cagan’s model.

The dynamics become more complicated and so does the algebra.

Nowadays, the “rational expectations” version of the model is more popular among central bankers and economists advising them.

## A MONETARIST THEORY OF PRICE LEVEL WITH ADAPTIVE EXPECTATIONS

### 13.1 Introduction

This lecture is a sequel or prequel to this lecture *monetarist theory of the price level*.

We'll use linear algebra to do some experiments with an alternative "monetarist" or "fiscal" theory of the price level".

Like the model in this lecture *monetarist theory of the price level*, the model asserts that when a government persistently spends more than it collects in taxes and prints money to finance the shortfall, it puts upward pressure on the price level and generates persistent inflation.

Instead of the "perfect foresight" or "rational expectations" version of the model in this lecture *monetarist theory of the price level*, our model in the present lecture is an "adaptive expectations" version of a model that Philip Cagan [Cag56] used to study the monetary dynamics of hyperinflations.

It combines these components:

- a demand function for real money balances that asserts that the logarithm of the quantity of real balances demanded depends inversely on the public's expected rate of inflation
- an **adaptive expectations** model that describes how the public's anticipated rate of inflation responds to past values of actual inflation
- an equilibrium condition that equates the demand for money to the supply
- an exogenous sequence of rates of growth of the money supply

Our model stays quite close to Cagan's original specification.

As in the *present values* and *consumption smoothing* lectures, the only linear algebra operations that we'll be using are matrix multiplication and matrix inversion.

To facilitate using linear matrix algebra as our principal mathematical tool, we'll use a finite horizon version of the model.

### 13.2 Structure of the model

Let

- $m_t$  be the log of the supply of nominal money balances;
- $\mu_t = m_{t+1} - m_t$  be the net rate of growth of nominal balances;
- $p_t$  be the log of the price level;
- $\pi_t = p_{t+1} - p_t$  be the net rate of inflation between  $t$  and  $t + 1$ ;

- $\pi_t^*$  be the public's expected rate of inflation between  $t$  and  $t + 1$ ;
- $T$  the horizon – i.e., the last period for which the model will determine  $p_t$
- $\pi_0^*$  public's initial expected rate of inflation between time 0 and time 1.

The demand for real balances  $\exp\left(\frac{m_t^d}{p_t}\right)$  is governed by the following version of the Cagan demand function

$$m_t^d - p_t = -\alpha\pi_t^*, \quad \alpha > 0; \quad t = 0, 1, \dots, T. \quad (13.1)$$

This equation asserts that the demand for real balances is inversely related to the public's expected rate of inflation.

Equating the logarithm  $m_t^d$  of the demand for money to the logarithm  $m_t$  of the supply of money in equation (13.1) and solving for the logarithm  $p_t$  of the price level gives

$$p_t = m_t + \alpha\pi_t^* \quad (13.2)$$

Taking the difference between equation (13.2) at time  $t + 1$  and at time  $t$  gives

$$\pi_t = \mu_t + \alpha\pi_{t+1}^* - \alpha\pi_t^* \quad (13.3)$$

We assume that the expected rate of inflation  $\pi_t^*$  is governed by the Friedman-Cagan adaptive expectations scheme

$$\pi_{t+1}^* = \lambda\pi_t^* + (1 - \lambda)\pi_t \quad (13.4)$$

As exogenous inputs into the model, we take initial conditions  $m_0, \pi_0^*$  and a money growth sequence  $\mu = \{\mu_t\}_{t=0}^T$ .

As endogenous outputs of our model we want to find sequences  $\pi = \{\pi_t\}_{t=0}^T, p = \{p_t\}_{t=0}^T$  as functions of the endogenous inputs.

We'll do some mental experiments by studying how the model outputs vary as we vary the model inputs.

### 13.3 Representing key equations with linear algebra

We begin by writing the equation (13.4) adaptive expectations model for  $\pi_t^*$  for  $t = 0, \dots, T$  as

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & 1 \end{bmatrix} \begin{bmatrix} \pi_0^* \\ \pi_1^* \\ \pi_2^* \\ \vdots \\ \pi_{T+1}^* \end{bmatrix} = (1 - \lambda) \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \vdots \\ \pi_T \end{bmatrix} + \begin{bmatrix} \pi_0^* \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Write this equation as

$$A\pi^* = (1 - \lambda)B\pi + \pi_0^* \quad (13.5)$$

where the  $(T + 2) \times (T + 2)$  matrix  $A$ , the  $(T + 2) \times (T + 1)$  matrix  $B$ , and the vectors  $\pi^*, \pi_0, \pi_0^*$  are defined implicitly by aligning these two equations.

Next we write the key equation (13.3) in matrix notation as

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \vdots \\ \pi_T \end{bmatrix} = \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_T \end{bmatrix} + \begin{bmatrix} -\alpha & \alpha & 0 & \cdots & 0 & 0 \\ 0 & -\alpha & \alpha & \cdots & 0 & 0 \\ 0 & 0 & -\alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \alpha & 0 \\ 0 & 0 & 0 & \cdots & -\alpha & \alpha \end{bmatrix} \begin{bmatrix} \pi_0^* \\ \pi_1^* \\ \pi_2^* \\ \vdots \\ \pi_{T+1}^* \end{bmatrix}$$

Represent the previous equation system in terms of vectors and matrices as

$$\pi = \mu + C\pi^* \quad (13.6)$$

where the  $(T + 1) \times (T + 2)$  matrix  $C$  is defined implicitly to align this equation with the preceding equation system.

## 13.4 Harvesting returns from our matrix formulation

We now have all of the ingredients we need to solve for  $\pi$  as a function of  $\mu, \pi_0, \pi_0^*$ .

Combine equations (13.5) and (13.6) to get

$$\begin{aligned} A\pi^* &= (1 - \lambda)B\pi + \pi_0^* \\ &= (1 - \lambda)B[\mu + C\pi^*] + \pi_0^* \end{aligned}$$

which implies that

$$[A - (1 - \lambda)BC]\pi^* = (1 - \lambda)B\mu + \pi_0^*$$

Multiplying both sides of the above equation by the inverse of the matrix on the left side gives

$$\pi^* = [A - (1 - \lambda)BC]^{-1}[(1 - \lambda)B\mu + \pi_0^*] \quad (13.7)$$

Having solved equation (13.7) for  $\pi^*$ , we can use equation (13.6) to solve for  $\pi$ :

$$\pi = \mu + C\pi^*$$

We have thus solved for two of the key endogenous time series determined by our model, namely, the sequence  $\pi^*$  of expected inflation rates and the sequence  $\pi$  of actual inflation rates.

Knowing these, we can then quickly calculate the associated sequence  $p$  of the logarithm of the price level from equation (13.2).

Let's fill in the details for this step.

Since we now know  $\mu$  it is easy to compute  $m$ .

Thus, notice that we can represent the equations

$$m_{t+1} = m_t + \mu_t, \quad t = 0, 1, \dots, T$$

as the matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_T \\ m_{T+1} \end{bmatrix} = \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{T-1} \\ \mu_T \end{bmatrix} + \begin{bmatrix} m_0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad (13.8)$$

Multiplying both sides of equation (13.8) with the inverse of the matrix on the left will give

$$m_t = m_0 + \sum_{s=0}^{t-1} \mu_s, \quad t = 1, \dots, T+1 \quad (13.9)$$

Equation (13.9) shows that the log of the money supply at  $t$  equals the log  $m_0$  of the initial money supply plus accumulation of rates of money growth between times 0 and  $t$ .

We can then compute  $p_t$  for each  $t$  from equation (13.2).

We can write a compact formula for  $p$  as

$$p = m + \alpha\hat{\pi}^*$$

where

$$\hat{\pi}^* = \begin{bmatrix} \pi_0^* \\ \pi_1^* \\ \pi_2^* \\ \vdots \\ \pi_T^* \end{bmatrix},$$

which is just  $\pi^*$  with the last element dropped.

## 13.5 Forecast errors

Our computations will verify that

$$\hat{\pi}^* \neq \pi,$$

so that in general

$$\pi_t^* \neq \pi_t, \quad t = 0, 1, \dots, T \quad (13.10)$$

This outcome is typical in models in which adaptive expectations hypothesis like equation (13.4) appear as a component.

In this lecture *monetarist theory of the price level*, we studied a version of the model that replaces hypothesis (13.4) with a “perfect foresight” or “rational expectations” hypothesis.

But now, let’s dive in and do some computations with the adaptive expectations version of the model.

As usual, we’ll start by importing some Python modules.

```
import numpy as np
from collections import namedtuple
import matplotlib.pyplot as plt

Cagan_Adaptive = namedtuple("Cagan_Adaptive",
                            ["a", "m0", "Epi0", "T", "lambda"])

def create_cagan_model(a, m0, Epi0, T, lambda):
    return Cagan_Adaptive(a, m0, Epi0, T, lambda)
```

Here we define the parameters.

```
# parameters
T = 80
T1 = 60
a = 5
lambda = 0.9    # 0.7
m0 = 1

mu0 = 0.5
mu_star = 0

md = create_cagan_model(a=a, m0=m0, Epi0=mu0, T=T, lambda=lambda)
```

We solve the model and plot variables of interests using the following functions.

```

def solve(model, mu_seq):
    " Solve the Cagan model in finite time. "

    model_params = model.a, model.m0, model.Epi0, model.T, model.lam
    a, m0, Epi0, T, lam = model_params

    A = np.eye(T+2, T+2) - lam*np.eye(T+2, T+2, k=-1)
    B = np.eye(T+2, T+1, k=-1)
    C = -a*np.eye(T+1, T+2) + a*np.eye(T+1, T+2, k=1)
    Epi0_seq = np.append(Epi0, np.zeros(T+1))

    # Epi_seq is of length T+2
    Epi_seq = np.linalg.inv(A - (1-lam)*B @ C) @ ((1-lam) * B @ mu_seq + Epi0_seq)

    # pi_seq is of length T+1
    pi_seq = mu_seq + C @ Epi_seq

    D = np.eye(T+1, T+1) - np.eye(T+1, T+1, k=-1)
    m0_seq = np.append(m0, np.zeros(T))

    # m_seq is of length T+2
    m_seq = np.linalg.inv(D) @ (mu_seq + m0_seq)
    m_seq = np.append(m0, m_seq)

    # p_seq is of length T+2
    p_seq = m_seq + a * Epi_seq

    return pi_seq, Epi_seq, m_seq, p_seq

```

```

def solve_and_plot(model, mu_seq):

    pi_seq, Epi_seq, m_seq, p_seq = solve(model, mu_seq)

    T_seq = range(model.T+2)

    fig, ax = plt.subplots(5, 1, figsize=[5, 12], dpi=200)
    ax[0].plot(T_seq[:-1], mu_seq)
    ax[1].plot(T_seq[:-1], pi_seq, label=r'$\pi_t$')
    ax[1].plot(T_seq, Epi_seq, label=r'$\pi^*_{t+1}$')
    ax[2].plot(T_seq, m_seq - p_seq)
    ax[3].plot(T_seq, m_seq)
    ax[4].plot(T_seq, p_seq)

    y_labs = [r'$\mu$', r'$\pi_t$', r'$m - p$', r'$m$', r'$p$']

    for i in range(5):
        ax[i].set_xlabel(r'$t$')
        ax[i].set_ylabel(y_labs[i])

    ax[1].legend()
    plt.tight_layout()
    plt.show()

    return pi_seq, Epi_seq, m_seq, p_seq

```

## 13.6 Technical condition for stability

In constructing our examples, we shall assume that  $(\lambda, \alpha)$  satisfy

$$\left| \frac{\lambda - \alpha(1 - \lambda)}{1 - \alpha(1 - \lambda)} \right| < 1 \quad (13.11)$$

The source of this condition is the following string of deductions:

$$\begin{aligned} \pi_t &= \mu_t + \alpha\pi_{t+1}^* - \alpha\pi_t^* \\ \pi_{t+1}^* &= \lambda\pi_t^* + (1 - \lambda)\pi_t \\ \pi_t &= \frac{\mu_t}{1 - \alpha(1 - \lambda)} - \frac{\alpha(1 - \lambda)}{1 - \alpha(1 - \lambda)}\pi_t^* \\ \Rightarrow \pi_t^* &= \frac{1}{\alpha(1 - \lambda)}\mu_t - \frac{1 - \alpha(1 - \lambda)}{\alpha(1 - \lambda)}\pi_t \\ \pi_{t+1} &= \frac{\mu_{t+1}}{1 - \alpha(1 - \lambda)} - \frac{\alpha(1 - \lambda)}{1 - \alpha(1 - \lambda)}(\lambda\pi_t^* + (1 - \lambda)\pi_t) \\ &= \frac{\mu_{t+1}}{1 - \alpha(1 - \lambda)} - \frac{\lambda}{1 - \alpha(1 - \lambda)}\mu_t + \frac{\lambda - \alpha(1 - \lambda)}{1 - \alpha(1 - \lambda)}\pi_t \end{aligned}$$

By assuring that the coefficient on  $\pi_t$  is less than one in absolute value, condition (13.11) assures stability of the dynamics of  $\{\pi_t\}$  described by the last line of our string of deductions.

The reader is free to study outcomes in examples that violate condition (13.11).

```
print(np.abs((lambda - alpha * (1 - lambda)) / (1 - alpha * (1 - lambda))))
```

0.8

```
print(lambda - alpha * (1 - lambda))
```

0.4000000000000013

Now we'll turn to some experiments.

### 13.6.1 Experiment 1

We'll study a situation in which the rate of growth of the money supply is  $\mu_0$  from  $t = 0$  to  $t = T_1$  and then permanently falls to  $\mu^*$  at  $t = T_1$ .

Thus, let  $T_1 \in (0, T)$ .

So where  $\mu_0 > \mu^*$ , we assume that

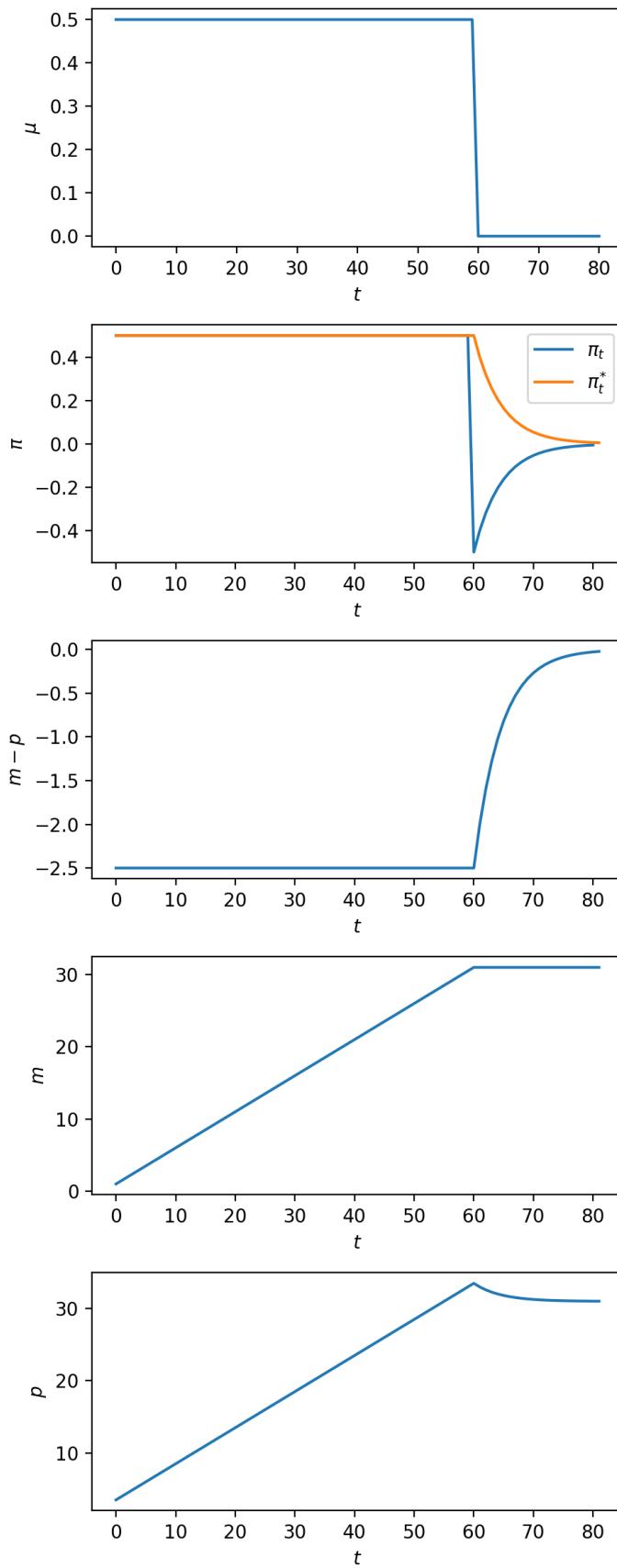
$$\mu_{t+1} = \begin{cases} \mu_0, & t = 0, \dots, T_1 - 1 \\ \mu^*, & t \geq T_1 \end{cases}$$

Notice that we studied exactly this experiment in a rational expectations version of the model in this lecture [monetarist theory of the price level](#).

So by comparing outcomes across the two lectures, we can learn about consequences of assuming adaptive expectations, as we do here, instead of rational expectations as we assumed in that other lecture.

```
μ_seq_1 = np.append(μ0*np.ones(T1), μ_star*np.ones(T+1-T1))

# solve and plot
π_seq_1, Eπ_seq_1, m_seq_1, p_seq_1 = solve_and_plot(md, μ_seq_1)
```



We invite the reader to compare outcomes with those under rational expectations studied in this lecture *monetarist theory of the price level*.

Please note how the actual inflation rate  $\pi_t$  “overshoots” its ultimate steady-state value at the time of the sudden reduction in the rate of growth of the money supply at time  $T_1$ .

We invite you to explain to yourself the source of this overshooting and why it does not occur in the rational expectations version of the model.

### 13.6.2 Experiment 2

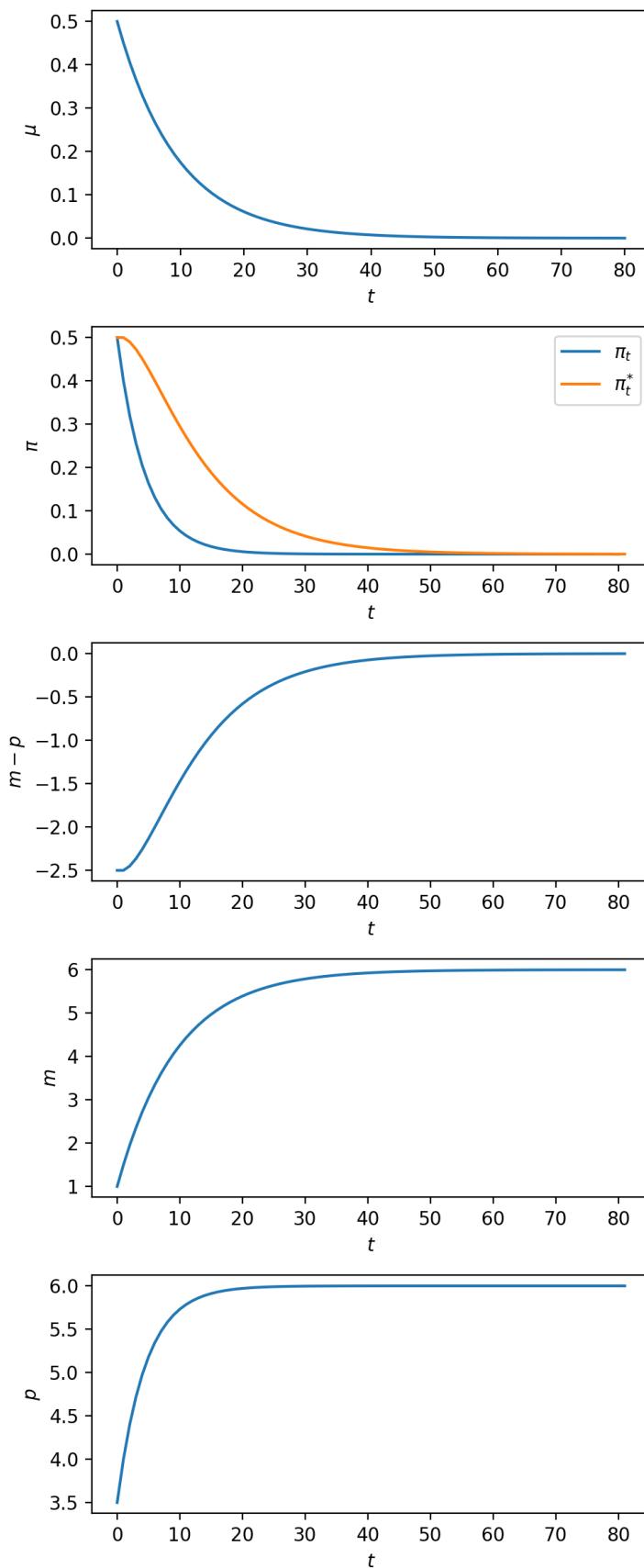
Now we'll do a different experiment, namely, a gradual stabilization in which the rate of growth of the money supply smoothly decline from a high value to a persistently low value.

While price level inflation eventually falls, it falls more slowly than the driving force that ultimately causes it to fall, namely, the falling rate of growth of the money supply.

The sluggish fall in inflation is explained by how anticipated inflation  $\pi_t^*$  persistently exceeds actual inflation  $\pi_t$  during the transition from a high inflation to a low inflation situation.

```
# parameters
phi = 0.9
mu_seq_2 = np.array([phi**t * mu0 + (1-phi**t)*mu_star for t in range(T)])
mu_seq_2 = np.append(mu_seq_2, mu_star)

# solve and plot
pi_seq_2, Epi_seq_2, m_seq_2, p_seq_2 = solve_and_plot(md, mu_seq_2)
```



## GEOMETRIC SERIES FOR ELEMENTARY ECONOMICS

### Contents

- *Geometric Series for Elementary Economics*
  - *Overview*
  - *Key formulas*
  - *Example: The Money Multiplier in Fractional Reserve Banking*
  - *Example: The Keynesian Multiplier*
  - *Example: Interest Rates and Present Values*
  - *Back to the Keynesian multiplier*

### 14.1 Overview

The lecture describes important ideas in economics that use the mathematics of geometric series.

Among these are

- the Keynesian **multiplier**
- the money **multiplier** that prevails in fractional reserve banking systems
- interest rates and present values of streams of payouts from assets

(As we shall see below, the term **multiplier** comes down to meaning **sum of a convergent geometric series**)

These and other applications prove the truth of the wise crack that

“in economics, a little knowledge of geometric series goes a long way “

Below we'll use the following imports:

```
%matplotlib inline
import matplotlib.pyplot as plt
plt.rcParams["figure.figsize"] = (11, 5)  #set default figure size
import numpy as np
import sympy as sym
from sympy import init_printing, latex
from matplotlib import cm
from mpl_toolkits.mplot3d import Axes3D
```

## 14.2 Key formulas

To start, let  $c$  be a real number that lies strictly between  $-1$  and  $1$ .

- We often write this as  $c \in (-1, 1)$ .
- Here  $(-1, 1)$  denotes the collection of all real numbers that are strictly less than  $1$  and strictly greater than  $-1$ .
- The symbol  $\in$  means *in* or *belongs to the set after the symbol*.

We want to evaluate geometric series of two types – infinite and finite.

### 14.2.1 Infinite geometric series

The first type of geometric that interests us is the infinite series

$$1 + c + c^2 + c^3 + \dots$$

Where  $\dots$  means that the series continues without end.

The key formula is

$$1 + c + c^2 + c^3 + \dots = \frac{1}{1 - c} \quad (14.1)$$

To prove key formula (14.1), multiply both sides by  $(1 - c)$  and verify that if  $c \in (-1, 1)$ , then the outcome is the equation  $1 = 1$ .

### 14.2.2 Finite geometric series

The second series that interests us is the finite geometric series

$$1 + c + c^2 + c^3 + \dots + c^T$$

where  $T$  is a positive integer.

The key formula here is

$$1 + c + c^2 + c^3 + \dots + c^T = \frac{1 - c^{T+1}}{1 - c}$$

**Remark:** The above formula works for any value of the scalar  $c$ . We don't have to restrict  $c$  to be in the set  $(-1, 1)$ .

We now move on to describe some famous economic applications of geometric series.

## 14.3 Example: The Money Multiplier in Fractional Reserve Banking

In a fractional reserve banking system, banks hold only a fraction  $r \in (0, 1)$  of cash behind each **deposit receipt** that they issue

- In recent times
  - cash consists of pieces of paper issued by the government and called dollars or pounds or ...
  - a *deposit* is a balance in a checking or savings account that entitles the owner to ask the bank for immediate payment in cash

- When the UK and France and the US were on either a gold or silver standard (before 1914, for example)
  - cash was a gold or silver coin
  - a *deposit receipt* was a *bank note* that the bank promised to convert into gold or silver on demand; (sometimes it was also a checking or savings account balance)

Economists and financiers often define the **supply of money** as an economy-wide sum of **cash** plus **deposits**.

In a **fractional reserve banking system** (one in which the reserve ratio  $r$  satisfies  $0 < r < 1$ ), **banks create money** by issuing deposits *backed* by fractional reserves plus loans that they make to their customers.

A geometric series is a key tool for understanding how banks create money (i.e., deposits) in a fractional reserve system.

The geometric series formula (14.1) is at the heart of the classic model of the money creation process – one that leads us to the celebrated **money multiplier**.

### 14.3.1 A simple model

There is a set of banks named  $i = 0, 1, 2, \dots$

Bank  $i$ 's loans  $L_i$ , deposits  $D_i$ , and reserves  $R_i$  must satisfy the balance sheet equation (because **balance sheets balance**):

$$L_i + R_i = D_i \quad (14.2)$$

The left side of the above equation is the sum of the bank's **assets**, namely, the loans  $L_i$  it has outstanding plus its reserves of cash  $R_i$ .

The right side records bank  $i$ 's liabilities, namely, the deposits  $D_i$  held by its depositors; these are IOU's from the bank to its depositors in the form of either checking accounts or savings accounts (or before 1914, bank notes issued by a bank stating promises to redeem note for gold or silver on demand).

Each bank  $i$  sets its reserves to satisfy the equation

$$R_i = rD_i \quad (14.3)$$

where  $r \in (0, 1)$  is its **reserve-deposit ratio** or **reserve ratio** for short

- the reserve ratio is either set by a government or chosen by banks for precautionary reasons

Next we add a theory stating that bank  $i + 1$ 's deposits depend entirely on loans made by bank  $i$ , namely

$$D_{i+1} = L_i \quad (14.4)$$

Thus, we can think of the banks as being arranged along a line with loans from bank  $i$  being immediately deposited in  $i + 1$

- in this way, the debtors to bank  $i$  become creditors of bank  $i + 1$

Finally, we add an *initial condition* about an exogenous level of bank 0's deposits

$D_0$  is given exogenously

We can think of  $D_0$  as being the amount of cash that a first depositor put into the first bank in the system, bank number  $i = 0$ .

Now we do a little algebra.

Combining equations (14.2) and (14.3) tells us that

$$L_i = (1 - r)D_i \quad (14.5)$$

This states that bank  $i$  loans a fraction  $(1 - r)$  of its deposits and keeps a fraction  $r$  as cash reserves.

Combining equation (14.5) with equation (14.4) tells us that

$$D_{i+1} = (1 - r)D_i \text{ for } i \geq 0$$

which implies that

$$D_i = (1 - r)^i D_0 \text{ for } i \geq 0 \quad (14.6)$$

Equation (14.6) expresses  $D_i$  as the  $i$  th term in the product of  $D_0$  and the geometric series

$$1, (1 - r), (1 - r)^2, \dots$$

Therefore, the sum of all deposits in our banking system  $i = 0, 1, 2, \dots$  is

$$\sum_{i=0}^{\infty} (1 - r)^i D_0 = \frac{D_0}{1 - (1 - r)} = \frac{D_0}{r} \quad (14.7)$$

### 14.3.2 Money multiplier

The **money multiplier** is a number that tells the multiplicative factor by which an exogenous injection of cash into bank 0 leads to an increase in the total deposits in the banking system.

Equation (14.7) asserts that the **money multiplier** is  $\frac{1}{r}$

- An initial deposit of cash of  $D_0$  in bank 0 leads the banking system to create total deposits of  $\frac{D_0}{r}$ .
- The initial deposit  $D_0$  is held as reserves, distributed throughout the banking system according to  $D_0 = \sum_{i=0}^{\infty} R_i$ .

## 14.4 Example: The Keynesian Multiplier

The famous economist John Maynard Keynes and his followers created a simple model intended to determine national income  $y$  in circumstances in which

- there are substantial unemployed resources, in particular **excess supply** of labor and capital
- prices and interest rates fail to adjust to make aggregate **supply equal demand** (e.g., prices and interest rates are frozen)
- national income is entirely determined by aggregate demand

### 14.4.1 Static version

An elementary Keynesian model of national income determination consists of three equations that describe aggregate demand for  $y$  and its components.

The first equation is a national income identity asserting that consumption  $c$  plus investment  $i$  equals national income  $y$ :

$$c + i = y$$

The second equation is a Keynesian consumption function asserting that people consume a fraction  $b \in (0, 1)$  of their income:

$$c = by$$

The fraction  $b \in (0, 1)$  is called the **marginal propensity to consume**.

The fraction  $1 - b \in (0, 1)$  is called the **marginal propensity to save**.

The third equation simply states that investment is exogenous at level  $i$ .

- *exogenous* means *determined outside this model*.

Substituting the second equation into the first gives  $(1 - b)y = i$ .

Solving this equation for  $y$  gives

$$y = \frac{1}{1 - b}i$$

The quantity  $\frac{1}{1-b}$  is called the **investment multiplier** or simply the **multiplier**.

Applying the formula for the sum of an infinite geometric series, we can write the above equation as

$$y = i \sum_{t=0}^{\infty} b^t$$

where  $t$  is a nonnegative integer.

So we arrive at the following equivalent expressions for the multiplier:

$$\frac{1}{1 - b} = \sum_{t=0}^{\infty} b^t$$

The expression  $\sum_{t=0}^{\infty} b^t$  motivates an interpretation of the multiplier as the outcome of a dynamic process that we describe next.

### 14.4.2 Dynamic version

We arrive at a dynamic version by interpreting the nonnegative integer  $t$  as indexing time and changing our specification of the consumption function to take time into account

- we add a one-period lag in how income affects consumption

We let  $c_t$  be consumption at time  $t$  and  $i_t$  be investment at time  $t$ .

We modify our consumption function to assume the form

$$c_t = b y_{t-1}$$

so that  $b$  is the marginal propensity to consume (now) out of last period's income.

We begin with an initial condition stating that

$$y_{-1} = 0$$

We also assume that

$$i_t = i \text{ for all } t \geq 0$$

so that investment is constant over time.

It follows that

$$y_0 = i + c_0 = i + b y_{-1} = i$$

and

$$y_1 = c_1 + i = b y_0 + i = (1 + b)i$$

and

$$y_2 = c_2 + i = b y_1 + i = (1 + b + b^2)i$$

and more generally

$$y_t = b y_{t-1} + i = (1 + b + b^2 + \dots + b^t)i$$

or

$$y_t = \frac{1 - b^{t+1}}{1 - b} i$$

Evidently, as  $t \rightarrow +\infty$ ,

$$y_t \rightarrow \frac{1}{1 - b} i$$

**Remark 1:** The above formula is often applied to assert that an exogenous increase in investment of  $\Delta i$  at time 0 ignites a dynamic process of increases in national income by successive amounts

$$\Delta i, (1 + b)\Delta i, (1 + b + b^2)\Delta i, \dots$$

at times 0, 1, 2, ....

**Remark 2** Let  $g_t$  be an exogenous sequence of government expenditures.

If we generalize the model so that the national income identity becomes

$$c_t + i_t + g_t = y_t$$

then a version of the preceding argument shows that the **government expenditures multiplier** is also  $\frac{1}{1-b}$ , so that a permanent increase in government expenditures ultimately leads to an increase in national income equal to the multiplier times the increase in government expenditures.

## 14.5 Example: Interest Rates and Present Values

We can apply our formula for geometric series to study how interest rates affect values of streams of dollar payments that extend over time.

We work in discrete time and assume that  $t = 0, 1, 2, \dots$  indexes time.

We let  $r \in (0, 1)$  be a one-period **net nominal interest rate**

- if the nominal interest rate is 5 percent, then  $r = .05$

A one-period **gross nominal interest rate**  $R$  is defined as

$$R = 1 + r \in (1, 2)$$

- if  $r = .05$ , then  $R = 1.05$

**Remark:** The gross nominal interest rate  $R$  is an **exchange rate** or **relative price** of dollars at between times  $t$  and  $t + 1$ . The units of  $R$  are dollars at time  $t + 1$  per dollar at time  $t$ .

When people borrow and lend, they trade dollars now for dollars later or dollars later for dollars now.

The price at which these exchanges occur is the gross nominal interest rate.

- If I sell  $x$  dollars to you today, you pay me  $Rx$  dollars tomorrow.
- This means that you borrowed  $x$  dollars for me at a gross interest rate  $R$  and a net interest rate  $r$ .

We assume that the net nominal interest rate  $r$  is fixed over time, so that  $R$  is the gross nominal interest rate at times  $t = 0, 1, 2, \dots$

Two important geometric sequences are

$$1, R, R^2, \dots \quad (14.8)$$

and

$$1, R^{-1}, R^{-2}, \dots \quad (14.9)$$

Sequence (14.8) tells us how dollar values of an investment **accumulate** through time.

Sequence (14.9) tells us how to **discount** future dollars to get their values in terms of today's dollars.

### 14.5.1 Accumulation

Geometric sequence (14.8) tells us how one dollar invested and re-invested in a project with gross one period nominal rate of return accumulates

- here we assume that net interest payments are reinvested in the project
- thus, 1 dollar invested at time 0 pays interest  $r$  dollars after one period, so we have  $r + 1 = R$  dollars at time 1
- at time 1 we reinvest  $1 + r = R$  dollars and receive interest of  $rR$  dollars at time 2 plus the *principal*  $R$  dollars, so we receive  $rR + R = (1 + r)R = R^2$  dollars at the end of period 2
- and so on

Evidently, if we invest  $x$  dollars at time 0 and reinvest the proceeds, then the sequence

$$x, xR, xR^2, \dots$$

tells how our account accumulates at dates  $t = 0, 1, 2, \dots$

### 14.5.2 Discounting

Geometric sequence (14.9) tells us how much future dollars are worth in terms of today's dollars.

Remember that the units of  $R$  are dollars at  $t + 1$  per dollar at  $t$ .

It follows that

- the units of  $R^{-1}$  are dollars at  $t$  per dollar at  $t + 1$
- the units of  $R^{-2}$  are dollars at  $t$  per dollar at  $t + 2$
- and so on; the units of  $R^{-j}$  are dollars at  $t$  per dollar at  $t + j$

So if someone has a claim on  $x$  dollars at time  $t + j$ , it is worth  $xR^{-j}$  dollars at time  $t$  (e.g., today).

### 14.5.3 Application to asset pricing

A **lease** requires a payments stream of  $x_t$  dollars at times  $t = 0, 1, 2, \dots$  where

$$x_t = G^t x_0$$

where  $G = (1 + g)$  and  $g \in (0, 1)$ .

Thus, lease payments increase at  $g$  percent per period.

For a reason soon to be revealed, we assume that  $G < R$ .

The **present value** of the lease is

$$\begin{aligned} p_0 &= x_0 + x_1/R + x_2/(R^2) + \dots \\ &= x_0(1 + GR^{-1} + G^2R^{-2} + \dots) \\ &= x_0 \frac{1}{1 - GR^{-1}} \end{aligned}$$

where the last line uses the formula for an infinite geometric series.

Recall that  $R = 1 + r$  and  $G = 1 + g$  and that  $R > G$  and  $r > g$  and that  $r$  and  $g$  are typically small numbers, e.g., .05 or .03.

Use the Taylor series of  $\frac{1}{1+r}$  about  $r = 0$ , namely,

$$\frac{1}{1+r} = 1 - r + r^2 - r^3 + \dots$$

and the fact that  $r$  is small to approximate  $\frac{1}{1+r} \approx 1 - r$ .

Use this approximation to write  $p_0$  as

$$\begin{aligned} p_0 &= x_0 \frac{1}{1 - GR^{-1}} \\ &= x_0 \frac{1}{1 - (1 + g)(1 - r)} \\ &= x_0 \frac{1}{1 - (1 + g - r - rg)} \\ &\approx x_0 \frac{1}{r - g} \end{aligned}$$

where the last step uses the approximation  $rg \approx 0$ .

The approximation

$$p_0 = \frac{x_0}{r - g}$$

is known as the **Gordon formula** for the present value or current price of an infinite payment stream  $x_0 G^t$  when the nominal one-period interest rate is  $r$  and when  $r > g$ .

We can also extend the asset pricing formula so that it applies to finite leases.

Let the payment stream on the lease now be  $x_t$  for  $t = 1, 2, \dots, T$ , where again

$$x_t = G^t x_0$$

The present value of this lease is:

$$\begin{aligned} p_0 &= x_0 + x_1/R + \dots + x_T/R^T \\ &= x_0(1 + GR^{-1} + \dots + G^T R^{-T}) \\ &= \frac{x_0(1 - G^{T+1} R^{-(T+1)})}{1 - GR^{-1}} \end{aligned}$$

Applying the Taylor series to  $R^{-(T+1)}$  about  $r = 0$  we get:

$$\frac{1}{(1+r)^{T+1}} = 1 - r(T+1) + \frac{1}{2}r^2(T+1)(T+2) + \dots \approx 1 - r(T+1)$$

Similarly, applying the Taylor series to  $G^{T+1}$  about  $g = 0$ :

$$(1+g)^{T+1} = 1 + (T+1)g + \frac{T(T+1)}{2!}g^2 + \frac{(T-1)T(T+1)}{3!}g^3 + \dots \approx 1 + (T+1)g$$

Thus, we get the following approximation:

$$p_0 = \frac{x_0(1 - (1 + (T+1)g)(1 - r(T+1)))}{1 - (1-r)(1+g)}$$

Expanding:

$$\begin{aligned} p_0 &= \frac{x_0(1 - 1 + (T+1)^2rg - r(T+1) + g(T+1))}{1 - 1 + r - g + rg} \\ &= \frac{x_0(T+1)((T+1)rg + r - g)}{r - g + rg} \\ &\approx \frac{x_0(T+1)(r - g)}{r - g} + \frac{x_0rg(T+1)}{r - g} \\ &= x_0(T+1) + \frac{x_0rg(T+1)}{r - g} \end{aligned}$$

We could have also approximated by removing the second term  $rgx_0(T+1)$  when  $T$  is relatively small compared to  $1/(rg)$  to get  $x_0(T+1)$  as in the finite stream approximation.

We will plot the true finite stream present-value and the two approximations, under different values of  $T$ , and  $g$  and  $r$  in Python.

First we plot the true finite stream present-value after computing it below

```
# True present value of a finite lease
def finite_lease_pv_true(T, g, r, x_0):
    G = (1 + g)
    R = (1 + r)
    return (x_0 * (1 - G***(T + 1) * R**(-T - 1))) / (1 - G * R**(-1))

# First approximation for our finite lease

def finite_lease_pv_approx_1(T, g, r, x_0):
    p = x_0 * (T + 1) + x_0 * r * g * (T + 1) / (r - g)
    return p

# Second approximation for our finite lease
def finite_lease_pv_approx_2(T, g, r, x_0):
    return (x_0 * (T + 1))

# Infinite lease
def infinite_lease(g, r, x_0):
    G = (1 + g)
    R = (1 + r)
    return x_0 / (1 - G * R**(-1))
```

Now that we have defined our functions, we can plot some outcomes.

First we study the quality of our approximations

```

def plot_function(axes, x_vals, func, args):
    axes.plot(x_vals, func(*args), label=func.__name__)

T_max = 50

T = np.arange(0, T_max+1)
g = 0.02
r = 0.03
x_0 = 1

our_args = (T, g, r, x_0)
funcs = [finite_lease_pv_true,
         finite_lease_pv_approx_1,
         finite_lease_pv_approx_2]
# the three functions we want to compare

fig, ax = plt.subplots()
for f in funcs:
    plot_function(ax, T, f, our_args)
ax.legend()
ax.set_xlabel('$T$ Periods Ahead')
ax.set_ylabel('Present Value, $p_0$')
plt.show()

```

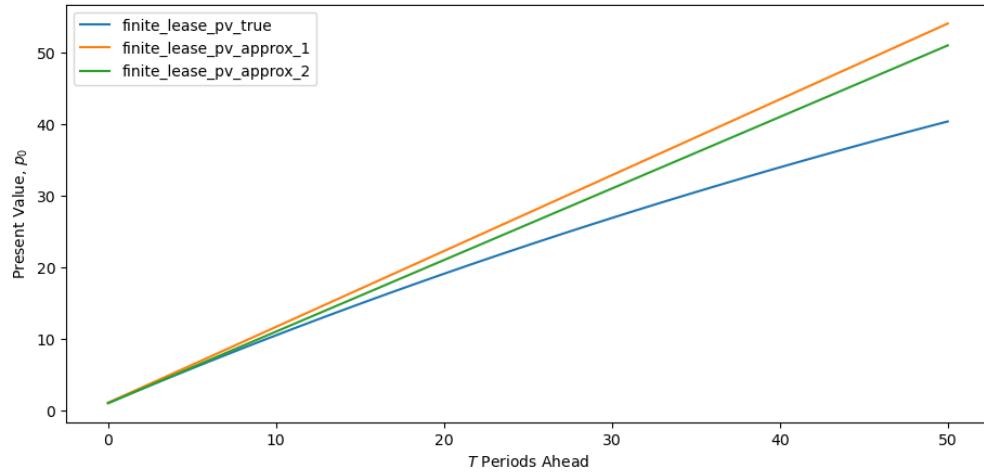


Fig. 14.1: Finite lease present value  $T$  periods ahead

Evidently our approximations perform well for small values of  $T$ .

However, holding  $g$  and  $r$  fixed, our approximations deteriorate as  $T$  increases.

Next we compare the infinite and finite duration lease present values over different lease lengths  $T$ .

```

# Convergence of infinite and finite
T_max = 1000
T = np.arange(0, T_max+1)
fig, ax = plt.subplots()
f_1 = finite_lease_pv_true(T, g, r, x_0)

```

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```

f_2 = np.full(T_max+1, infinite_lease(g, r, x_0))
ax.plot(T, f_1, label='T-period lease PV')
ax.plot(T, f_2, '--', label='Infinite lease PV')
ax.set_xlabel('$T$ Periods Ahead')
ax.set_ylabel('Present Value, $p_0$')
ax.legend()
plt.show()

```

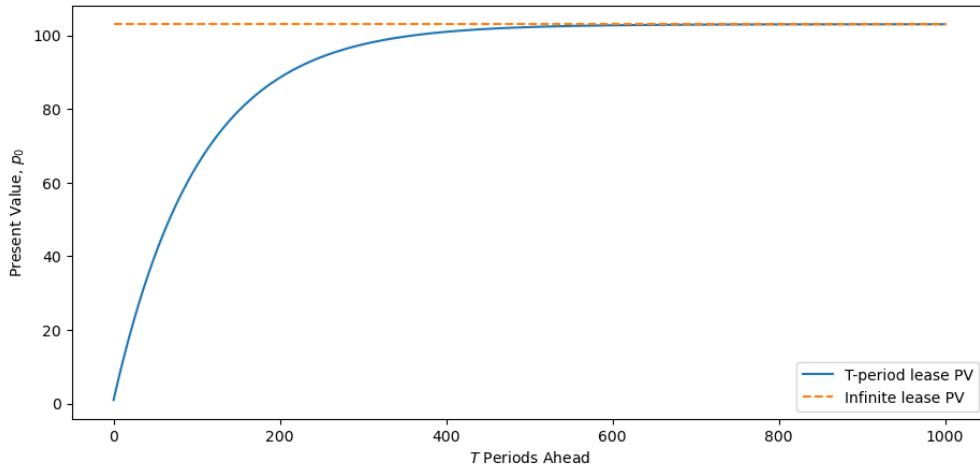


Fig. 14.2: Infinite and finite lease present value  $T$  periods ahead

The graph above shows how as duration  $T \rightarrow +\infty$ , the value of a lease of duration  $T$  approaches the value of a perpetual lease.

Now we consider two different views of what happens as  $r$  and  $g$  covary

```

# First view
# Changing r and g
fig, ax = plt.subplots()
ax.set_ylabel('Present Value, $p_0$')
ax.set_xlabel('$T$ periods ahead')
T_max = 10
T=np.arange(0, T_max+1)

rs, gs = (0.9, 0.5, 0.4001, 0.4), (0.4, 0.4, 0.4, 0.5),
comparisons = ('$g > r$', '$g < r$', '$g \approx r$', '$g = r$')
for r, g, comp in zip(rs, gs, comparisons):
    ax.plot(finite_lease_pv_true(T, g, r, x_0), label=f'r={r} {comp} g={g}' )

ax.legend()
plt.show()

```

This graph gives a big hint for why the condition  $r > g$  is necessary if a lease of length  $T = +\infty$  is to have finite value.

For fans of 3-d graphs the same point comes through in the following graph.

If you aren't enamored of 3-d graphs, feel free to skip the next visualization!

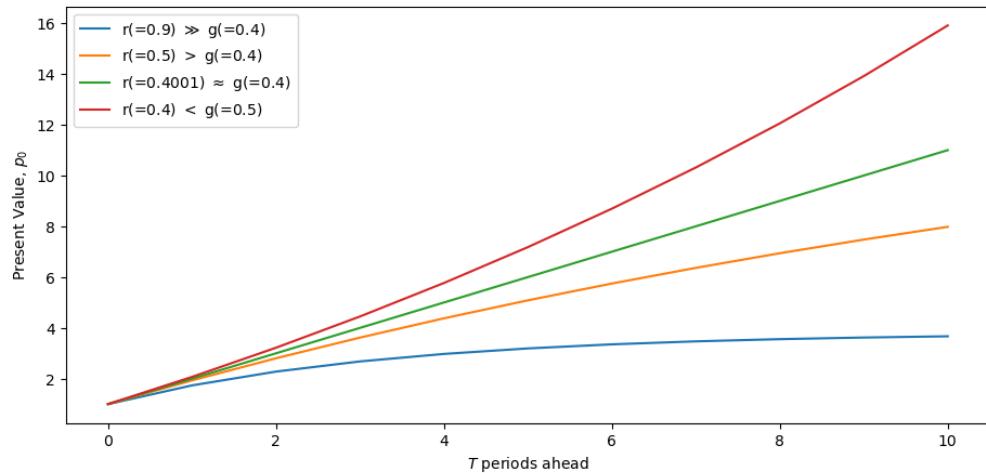


Fig. 14.3: Value of lease of length  $T$

```

# Second view
fig = plt.figure(figsize = [16, 5])
T = 3
ax = plt.subplot(projection='3d')
r = np.arange(0.01, 0.99, 0.005)
g = np.arange(0.011, 0.991, 0.005)

rr, gg = np.meshgrid(r, g)
z = finite_lease_pv_true(T, gg, rr, x_0)

# Removes points where undefined
same = (rr == gg)
z[same] = np.nan
surf = ax.plot_surface(rr, gg, z, cmap=cm.coolwarm,
    antialiased=True, clim=(0, 15))
fig.colorbar(surf, shrink=0.5, aspect=5)
ax.set_xlabel('$r$')
ax.set_ylabel('$g$')
ax.set_zlabel('Present Value, $p_0$')
ax.view_init(20, 8)
plt.show()

```

We can use a little calculus to study how the present value  $p_0$  of a lease varies with  $r$  and  $g$ .

We will use a library called [SymPy](#).

SymPy enables us to do symbolic math calculations including computing derivatives of algebraic equations.

We will illustrate how it works by creating a symbolic expression that represents our present value formula for an infinite lease.

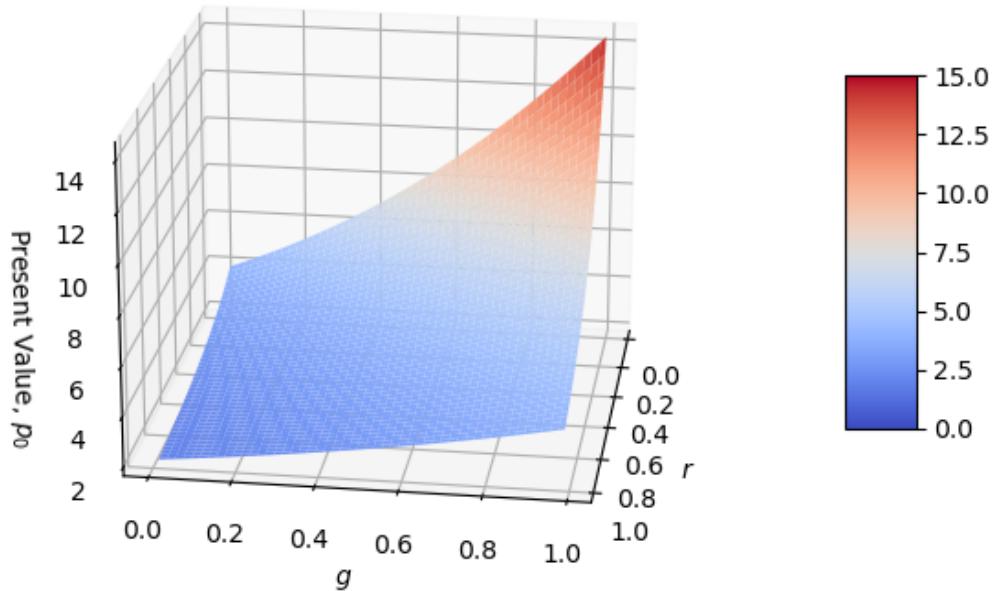
After that, we'll use SymPy to compute derivatives

```

# Creates algebraic symbols that can be used in an algebraic expression
g, r, x0 = sym.symbols('g, r, x0')
G = (1 + g)

```

(continues on next page)

Fig. 14.4: Three period lease PV with varying  $g$  and  $r$ 

(continued from previous page)

```
R = (1 + r)
p0 = x0 / (1 - G * R**(-1))
init_printing(use_latex='mathjax')
print('Our formula is:')
p0
```

Our formula is:

$$\frac{x_0}{-\frac{g+1}{r+1} + 1}$$

```
print('dp0 / dg is:')
dp_dg = sym.diff(p0, g)
dp_dg
```

dp0 / dg is:

$$\frac{x_0}{(r+1) \left(-\frac{g+1}{r+1} + 1\right)^2}$$

```
print('dp0 / dr is:')
dp_dr = sym.diff(p0, r)
dp_dr
```

```
dp0 / dr is:
```

$$-\frac{x_0 (g + 1)}{(r + 1)^2 \left(-\frac{g+1}{r+1} + 1\right)^2}$$

We can see that for  $\frac{\partial p_0}{\partial r} < 0$  as long as  $r > g$ ,  $r > 0$  and  $g > 0$  and  $x_0$  is positive, so  $\frac{\partial p_0}{\partial r}$  will always be negative.

Similarly,  $\frac{\partial p_0}{\partial g} > 0$  as long as  $r > g$ ,  $r > 0$  and  $g > 0$  and  $x_0$  is positive, so  $\frac{\partial p_0}{\partial g}$  will always be positive.

## 14.6 Back to the Keynesian multiplier

We will now go back to the case of the Keynesian multiplier and plot the time path of  $y_t$ , given that consumption is a constant fraction of national income, and investment is fixed.

```
# Function that calculates a path of y
def calculate_y(i, b, g, T, y_init):
    y = np.zeros(T+1)
    y[0] = i + b * y_init + g
    for t in range(1, T+1):
        y[t] = b * y[t-1] + i + g
    return y

# Initial values
i_0 = 0.3
g_0 = 0.3
# 2/3 of income goes towards consumption
b = 2/3
y_init = 0
T = 100

fig, ax = plt.subplots()
ax.set_xlabel('$t$')
ax.set_ylabel('$y_t$')
ax.plot(np.arange(0, T+1), calculate_y(i_0, b, g_0, T, y_init))
# Output predicted by geometric series
ax.hlines(i_0 / (1 - b) + g_0 / (1 - b), xmin=-1, xmax=101, linestyles='--')
plt.show()
```

In this model, income grows over time, until it gradually converges to the infinite geometric series sum of income.

We now examine what will happen if we vary the so-called **marginal propensity to consume**, i.e., the fraction of income that is consumed

```
bs = (1/3, 2/3, 5/6, 0.9)
```

(continues on next page)

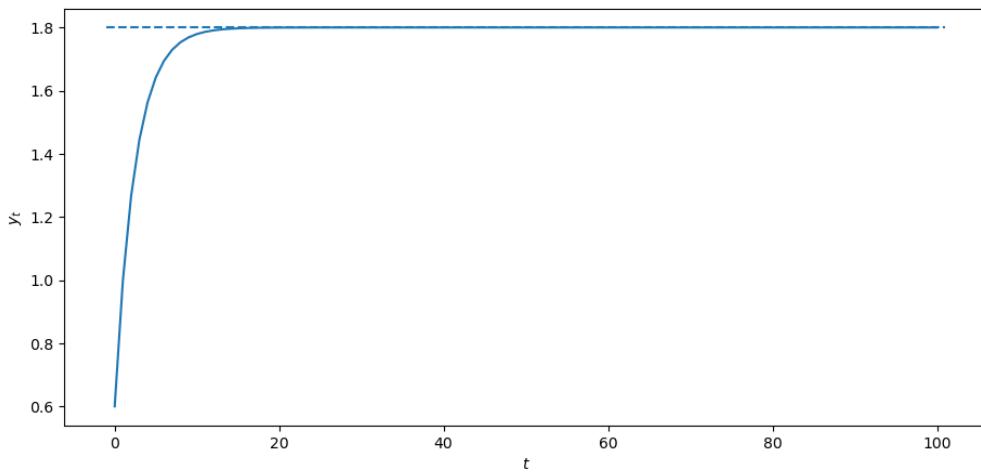


Fig. 14.5: Path of aggregate output over time

(continued from previous page)

```
fig, ax = plt.subplots()
ax.set_ylabel('$y_t$')
ax.set_xlabel('$t$')
x = np.arange(0, T+1)
for b in bs:
    y = calculate_y(i_0, b, g_0, T, y_init)
    ax.plot(x, y, label=r'$b=$' + f'{b:.2f}')
ax.legend()
plt.show()
```

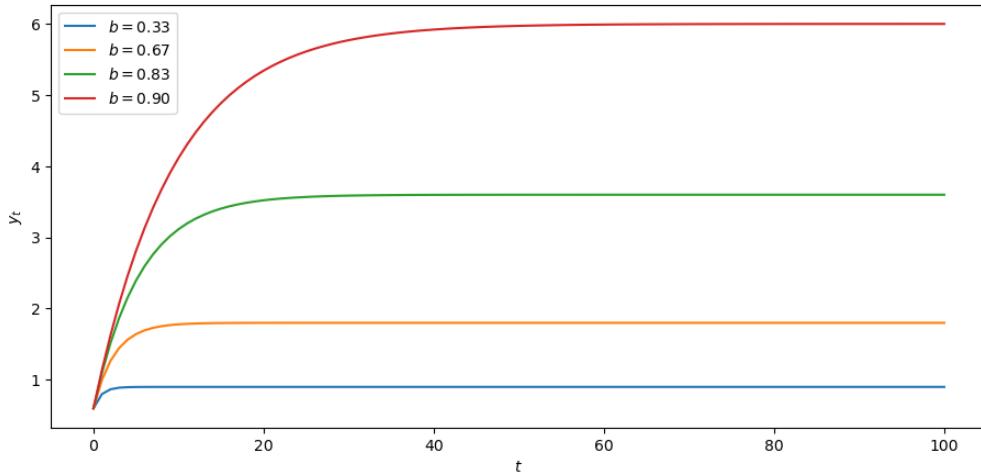


Fig. 14.6: Changing consumption as a fraction of income

Increasing the marginal propensity to consume  $b$  increases the path of output over time.

Now we will compare the effects on output of increases in investment and government spending.

```
fig, (ax1, ax2) = plt.subplots(2, 1, figsize=(6, 10))
fig.subplots_adjust(hspace=0.3)

x = np.arange(0, T+1)
values = [0.3, 0.4]

for i in values:
    y = calculate_y(i, b, g_0, T, y_init)
    ax1.plot(x, y, label=f"i={i}")
for g in values:
    y = calculate_y(i_0, b, g, T, y_init)
    ax2.plot(x, y, label=f"g={g}")

axes = ax1, ax2
param_labels = "Investment", "Government Spending"
for ax, param in zip(axes, param_labels):
    ax.set_title(f'An Increase in {param} on Output')
    ax.legend(loc = "lower right")
    ax.set_ylabel('$y_t$')
    ax.set_xlabel('$t$')
plt.show()
```

Notice here, whether government spending increases from 0.3 to 0.4 or investment increases from 0.3 to 0.4, the shifts in the graphs are identical.

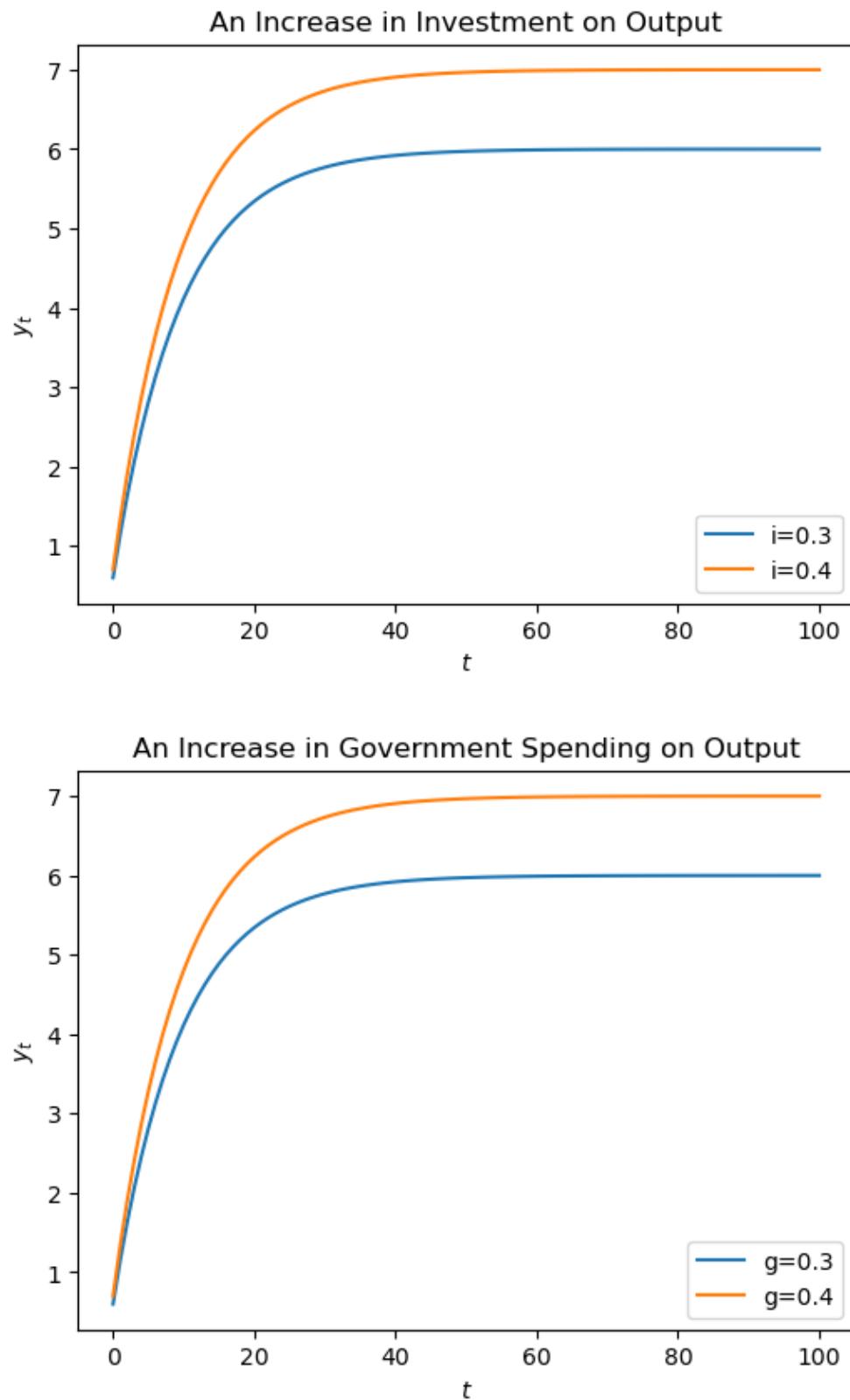


Fig. 14.7: Different increase on output



## **Part V**

# **Probability and Distributions**



## DISTRIBUTIONS AND PROBABILITIES

### Contents

- *Distributions and Probabilities*
  - *Outline*
  - *Common distributions*
  - *Observed distributions*

## 15.1 Outline

In this lecture we give a quick introduction to data and probability distributions using Python

```
!pip install --upgrade yfinance
```

```
import matplotlib.pyplot as plt
import pandas as pd
import numpy as np
import yfinance as yf
import scipy.stats
import seaborn as sns
```

## 15.2 Common distributions

In this section we recall the definitions of some well-known distributions and show how to manipulate them with SciPy.

### 15.2.1 Discrete distributions

Let's start with discrete distributions.

A discrete distribution is defined by a set of numbers  $S = \{x_1, \dots, x_n\}$  and a **probability mass function** (PMF) on  $S$ , which is a function  $p$  from  $S$  to  $[0, 1]$  with the property

$$\sum_{i=1}^n p(x_i) = 1$$

We say that a random variable  $X$  **has distribution**  $p$  if  $X$  takes value  $x_i$  with probability  $p(x_i)$ .

That is,

$$\mathbb{P}\{X = x_i\} = p(x_i) \quad \text{for } i = 1, \dots, n$$

The **mean** or **expected value** of a random variable  $X$  with distribution  $p$  is

$$\mathbb{E}X = \sum_{i=1}^n x_i p(x_i)$$

Expectation is also called the *first moment* of the distribution.

We also refer to this number as the mean of the distribution (represented by)  $p$ .

The **variance** of  $X$  is defined as

$$\mathbb{V}X = \sum_{i=1}^n (x_i - \mathbb{E}X)^2 p(x_i)$$

Variance is also called the *second central moment* of the distribution.

The **cumulative distribution function** (CDF) of  $X$  is defined by

$$F(x) = \mathbb{P}\{X \leq x\} = \sum_{i=1}^n \mathbb{1}\{x_i \leq x\} p(x_i)$$

Here  $\mathbb{1}\{\text{statement}\} = 1$  if “statement” is true and zero otherwise.

Hence the second term takes all  $x_i \leq x$  and sums their probabilities.

### Uniform distribution

One simple example is the **uniform distribution**, where  $p(x_i) = 1/n$  for all  $n$ .

We can import the uniform distribution on  $S = \{1, \dots, n\}$  from SciPy like so:

```
n = 10
u = scipy.stats.randint(1, n+1)
```

Here's the mean and variance

```
u.mean(), u.var()
```

```
(5.5, 8.25)
```

The formula for the mean is  $(n + 1)/2$ , and the formula for the variance is  $(n^2 - 1)/12$ .

Now let's evaluate the PMF

```
u.pmf(1)
```

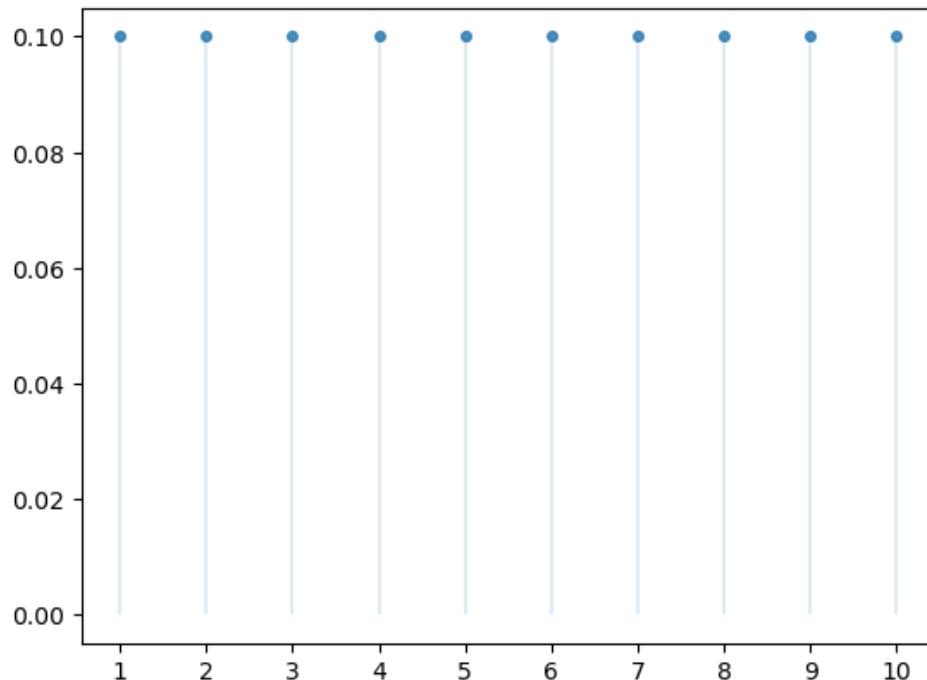
0.1

```
u.pmf(2)
```

0.1

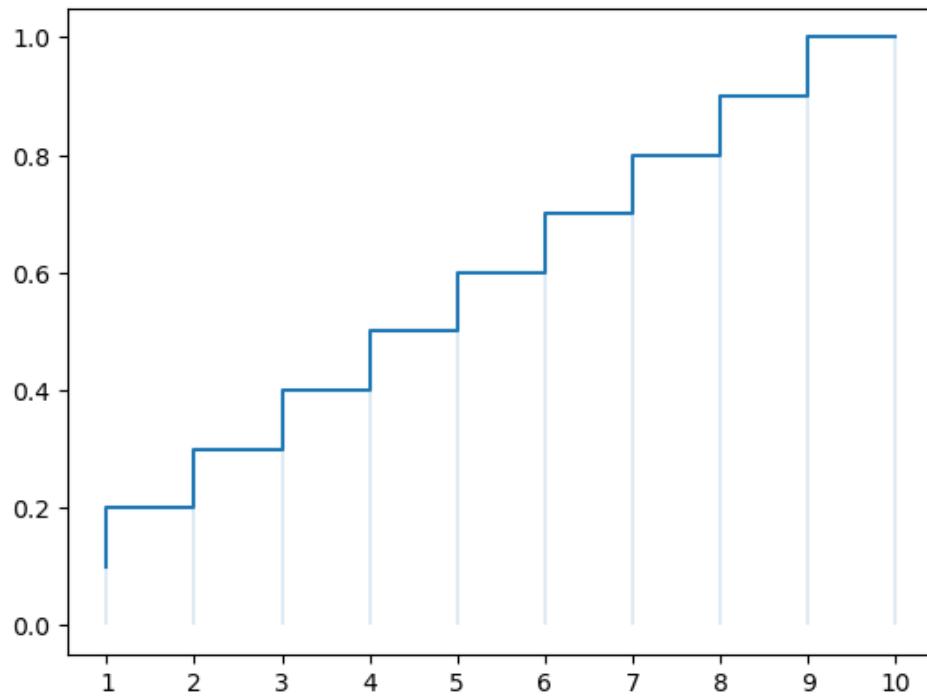
Here's a plot of the probability mass function:

```
fig, ax = plt.subplots()
S = np.arange(1, n+1)
ax.plot(S, u.pmf(S), linestyle=' ', marker='o', alpha=0.8, ms=4)
ax.vlines(S, 0, u.pmf(S), lw=0.2)
ax.set_xticks(S)
plt.show()
```



Here's a plot of the CDF:

```
fig, ax = plt.subplots()
S = np.arange(1, n+1)
ax.step(S, u.cdf(S))
ax.vlines(S, 0, u.cdf(S), lw=0.2)
ax.set_xticks(S)
plt.show()
```



The CDF jumps up by  $p(x_i)$  and  $x_i$ .

---

### Exercise 15.2.1

Calculate the mean and variance for this parameterization (i.e.,  $n = 10$ ) directly from the PMF, using the expressions given above.

Check that your answers agree with `u.mean()` and `u.var()`.

---

## Binomial distribution

Another useful (and more interesting) distribution is the **binomial distribution** on  $S = \{0, \dots, n\}$ , which has PMF

$$p(i) = \binom{n}{i} \theta^i (1 - \theta)^{n-i}$$

Here  $\theta \in [0, 1]$  is a parameter.

The interpretation of  $p(i)$  is: the number of successes in  $n$  independent trials with success probability  $\theta$ .

(If  $\theta = 0.5$ , this is “how many heads in  $n$  flips of a fair coin”)

The mean and variance are

```
n = 10
θ = 0.5
u = scipy.stats.binom(n, θ)
```

```
u.mean(), u.var()
```

(5.0, 2.5)

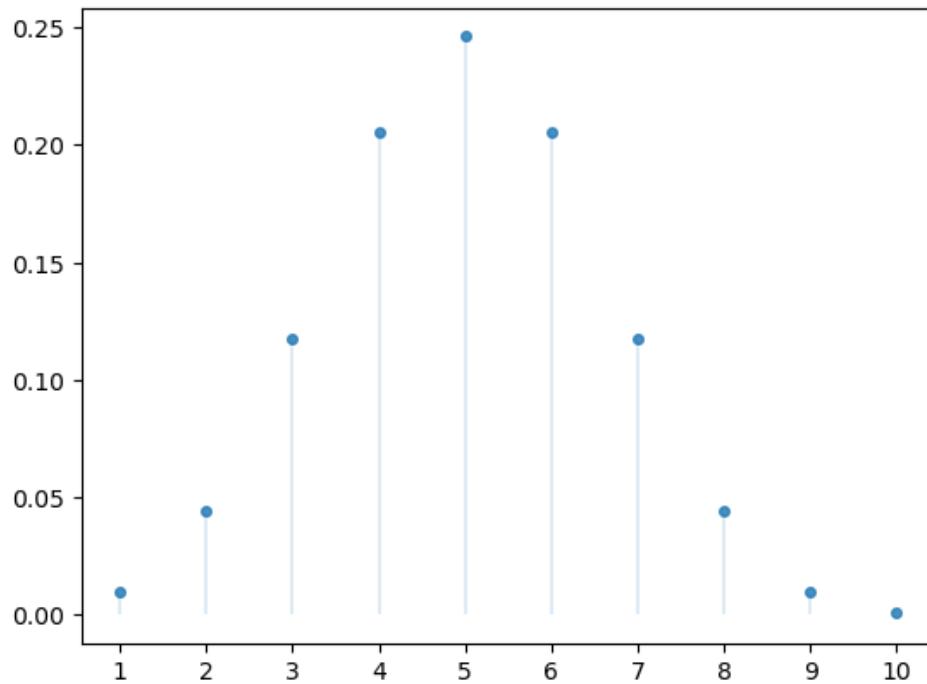
The formula for the mean is  $n\theta$  and the formula for the variance is  $n\theta(1 - \theta)$ .

Here's the PDF

u.pmf(1)

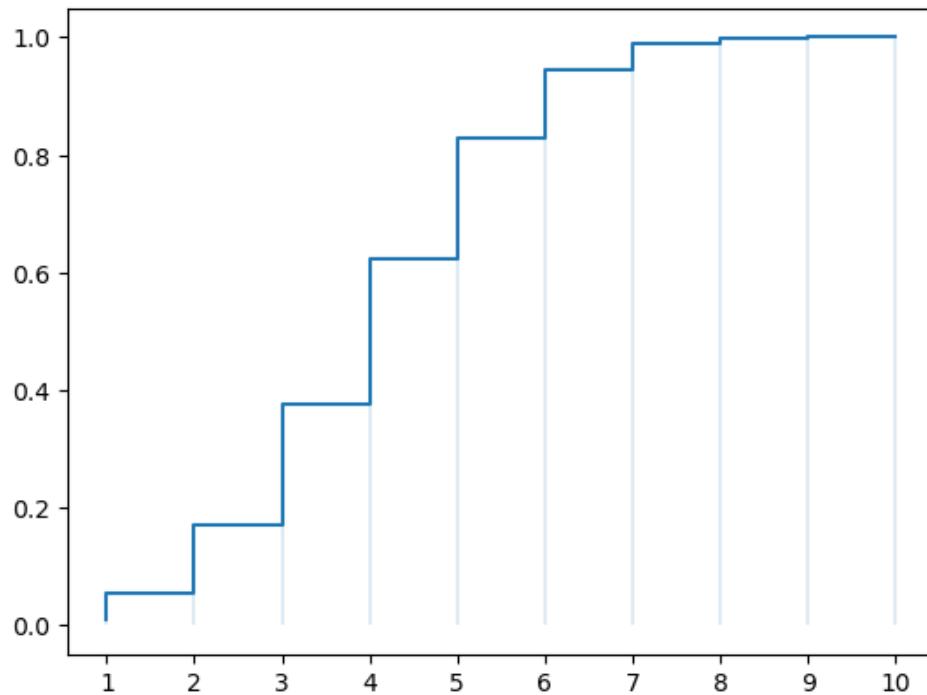
0.009765625000000002

```
fig, ax = plt.subplots()
S = np.arange(1, n+1)
ax.plot(S, u.pmf(S), linestyle=' ', marker='o', alpha=0.8, ms=4)
ax.vlines(S, 0, u.pmf(S), lw=0.2)
ax.set_xticks(S)
plt.show()
```



Here's the CDF

```
fig, ax = plt.subplots()
S = np.arange(1, n+1)
ax.step(S, u.cdf(S))
ax.vlines(S, 0, u.cdf(S), lw=0.2)
ax.set_xticks(S)
plt.show()
```



---

### Exercise 15.2.2

Using `u.pmf`, check that our definition of the CDF given above calculates the same function as `u.cdf`.

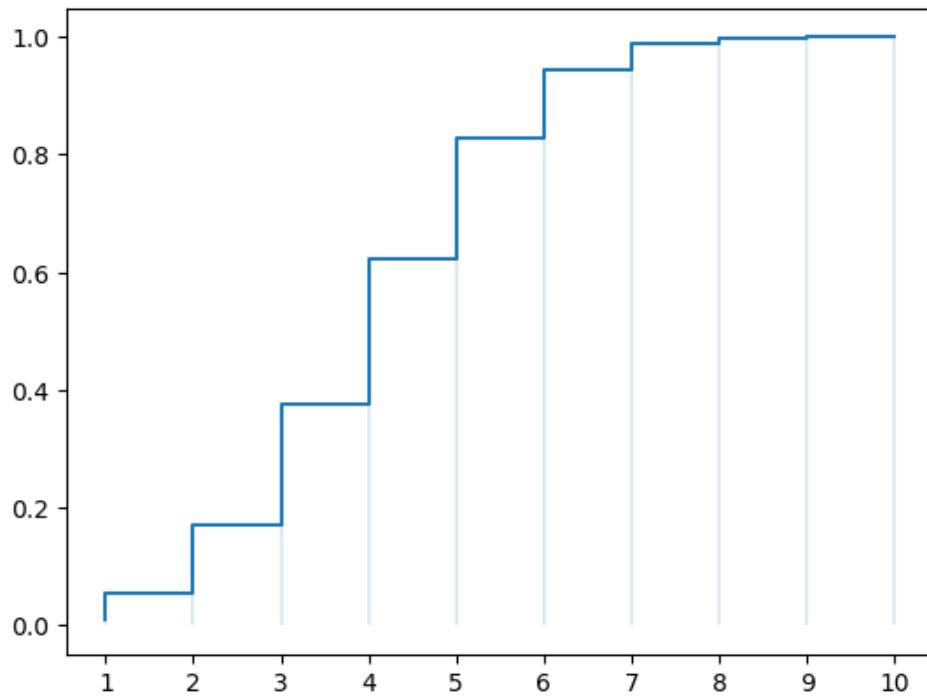
---

---

### Solution to Exercise 15.2.2

Here is one solution

```
fig, ax = plt.subplots()
S = np.arange(1, n+1)
u_sum = np.cumsum(u.pmf(S))
ax.step(S, u_sum)
ax.vlines(S, 0, u_sum, lw=0.2)
ax.set_xticks(S)
plt.show()
```



We can see that the output graph is the same as the one above.

## Poisson distribution

Poisson distribution on  $S = \{0, 1, \dots\}$  with parameter  $\lambda > 0$  has PMF

$$p(i) = \frac{\lambda^i}{i!} e^{-\lambda}$$

The interpretation of  $p(i)$  is: the number of events in a fixed time interval, where the events occur at a constant rate  $\lambda$  and independently of each other.

The mean and variance are

```
λ = 2
u = scipy.stats.poisson(λ)
```

```
u.mean(), u.var()
```

```
(2.0, 2.0)
```

The the expectation of Poisson distribution is  $\lambda$  and the variance is also  $\lambda$ .

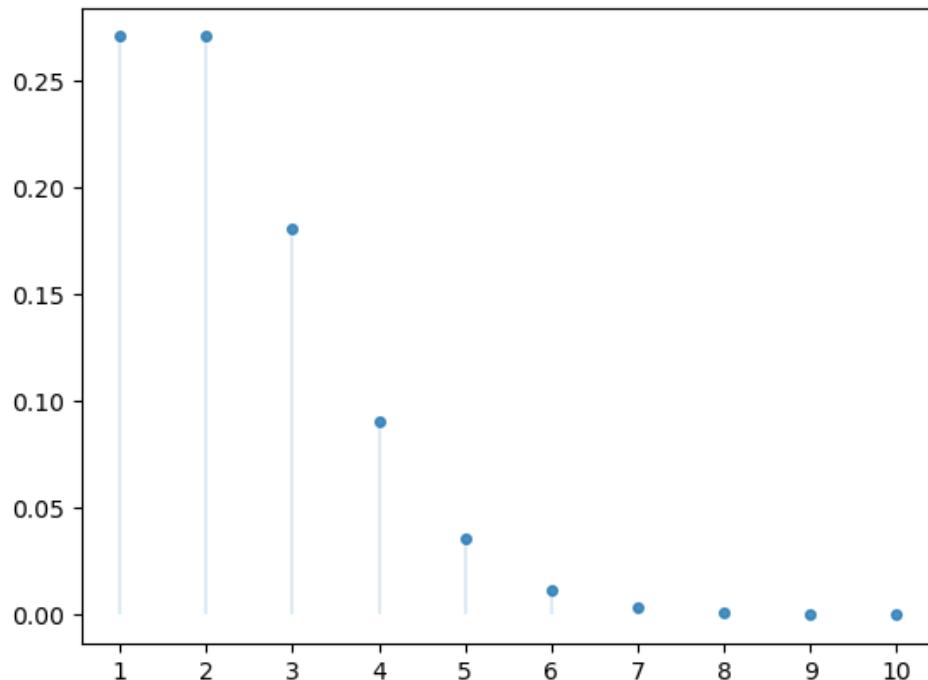
Here's the PMF

```
λ = 2
u = scipy.stats.poisson(λ)
```

```
u.pmf(1)
```

```
0.2706705664732254
```

```
fig, ax = plt.subplots()
S = np.arange(1, n+1)
ax.plot(S, u.pmf(S), linestyle='', marker='o', alpha=0.8, ms=4)
ax.vlines(S, 0, u.pmf(S), lw=0.2)
ax.set_xticks(S)
plt.show()
```



### 15.2.2 Continuous distributions

Continuous distributions are represented by a **density function**, which is a function  $p$  over  $\mathbb{R}$  (the set of all numbers) such that  $p(x) \geq 0$  for all  $x$  and

$$\int_{-\infty}^{\infty} p(x)dx = 1$$

We say that random variable  $X$  has distribution  $p$  if

$$\mathbb{P}\{a < X < b\} = \int_a^b p(x)dx$$

for all  $a \leq b$ .

The definition of the mean and variance of a random variable  $X$  with distribution  $p$  are the same as the discrete case, after replacing the sum with an integral.

For example, the mean of  $X$  is

$$\mathbb{E}X = \int_{-\infty}^{\infty} xp(x)dx$$

The **cumulative distribution function** (CDF) of  $X$  is defined by

$$F(x) = \mathbb{P}\{X \leq x\} = \int_{-\infty}^x p(x)dx$$

## Normal distribution

Perhaps the most famous distribution is the **normal distribution**, which has density

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

This distribution has two parameters,  $\mu$  and  $\sigma$ .

It can be shown that, for this distribution, the mean is  $\mu$  and the variance is  $\sigma^2$ .

We can obtain the moments, PDF, and CDF of the normal density as follows:

```
μ, σ = 0.0, 1.0
u = scipy.stats.norm(μ, σ)
```

```
u.mean(), u.var()
```

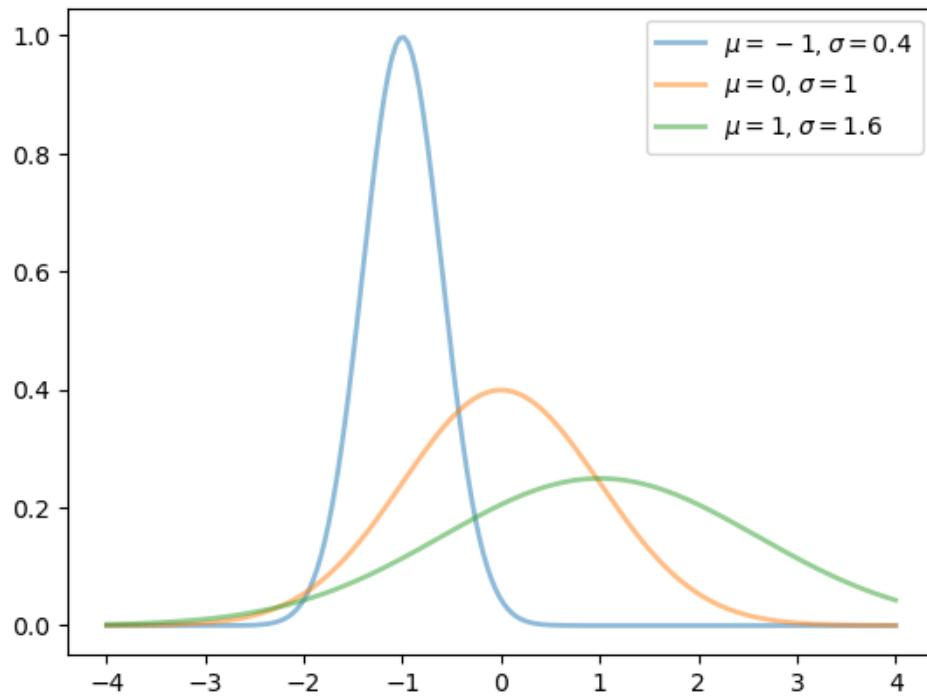
```
(0.0, 1.0)
```

Here's a plot of the density — the famous “bell-shaped curve”:

```
μ_vals = [-1, 0, 1]
σ_vals = [0.4, 1, 1.6]
fig, ax = plt.subplots()
x_grid = np.linspace(-4, 4, 200)

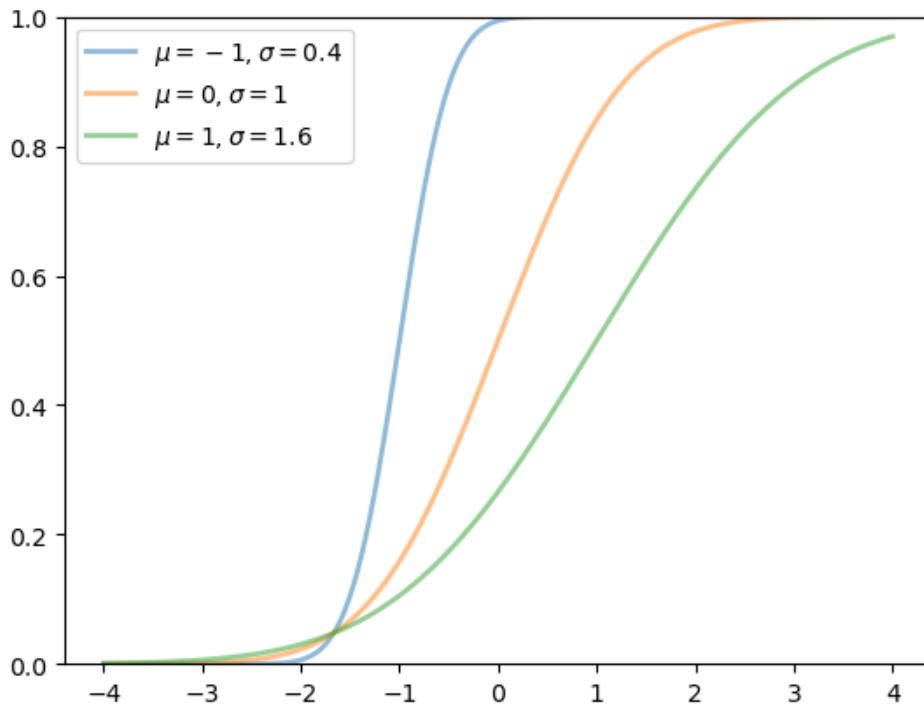
for μ, σ in zip(μ_vals, σ_vals):
    u = scipy.stats.norm(μ, σ)
    ax.plot(x_grid, u.pdf(x_grid),
            alpha=0.5, lw=2,
            label=f'$\mu={μ}$, $\sigma={σ}$')

plt.legend()
plt.show()
```



Here's a plot of the CDF:

```
fig, ax = plt.subplots()
for mu, sigma in zip(mu_vals, sigma_vals):
    u = scipy.stats.norm(mu, sigma)
    ax.plot(x_grid, u.cdf(x_grid),
            alpha=0.5, lw=2,
            label=f'$\mu={mu}$, $\sigma={sigma}$')
ax.set_ylim(0, 1)
plt.legend()
plt.show()
```



### Lognormal distribution

The **lognormal distribution** is a distribution on  $(0, \infty)$  with density

$$p(x) = \frac{1}{\sigma x \sqrt{2\pi}} \exp\left(-\frac{(\log x - \mu)^2}{2\sigma^2}\right)$$

This distribution has two parameters,  $\mu$  and  $\sigma$ .

It can be shown that, for this distribution, the mean is  $\exp(\mu + \sigma^2/2)$  and the variance is  $[\exp(\sigma^2) - 1] \exp(2\mu + \sigma^2)$ .

It has a nice interpretation: if  $X$  is lognormally distributed, then  $\log X$  is normally distributed.

It is often used to model variables that are “multiplicative” in nature, such as income or asset prices.

We can obtain the moments, PDF, and CDF of the normal density as follows:

```
μ, σ = 0.0, 1.0
u = scipy.stats.lognorm(s=σ, scale=np.exp(μ))
```

```
u.mean(), u.var()
```

```
(1.6487212707001282, 4.670774270471604)
```

```
μ_vals = [-1, 0, 1]
σ_vals = [0.25, 0.5, 1]
x_grid = np.linspace(0, 3, 200)

fig, ax = plt.subplots()
```

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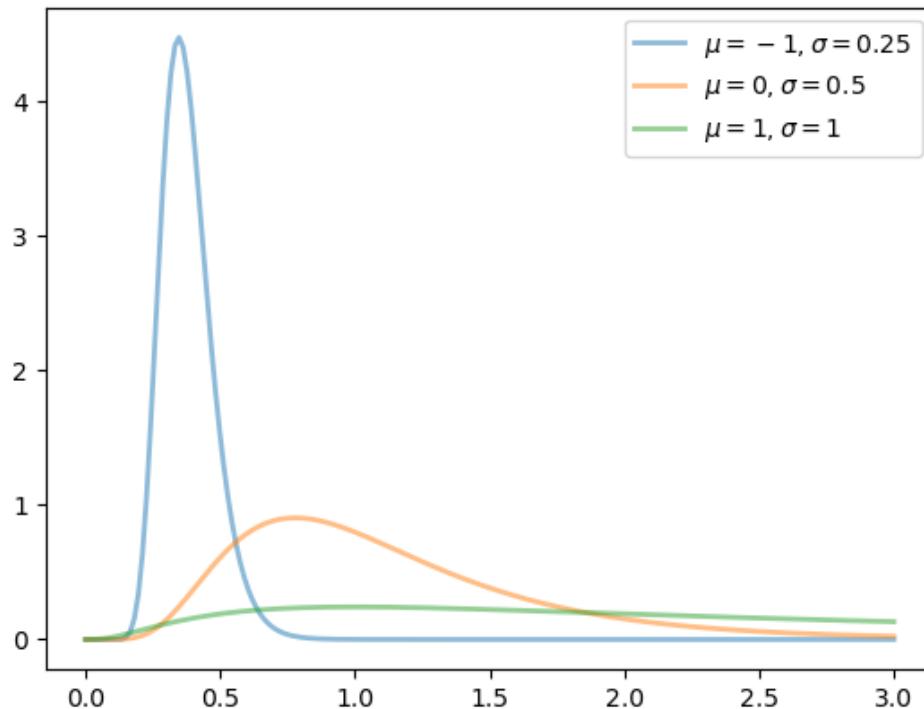
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```

for mu, sigma in zip(mu_vals, sigma_vals):
    u = scipy.stats.lognorm(sigma, scale=np.exp(mu))
    ax.plot(x_grid, u.pdf(x_grid),
             alpha=0.5, lw=2,
             label=f'$\mu={mu}, \sigma={sigma}$')

plt.legend()
plt.show()

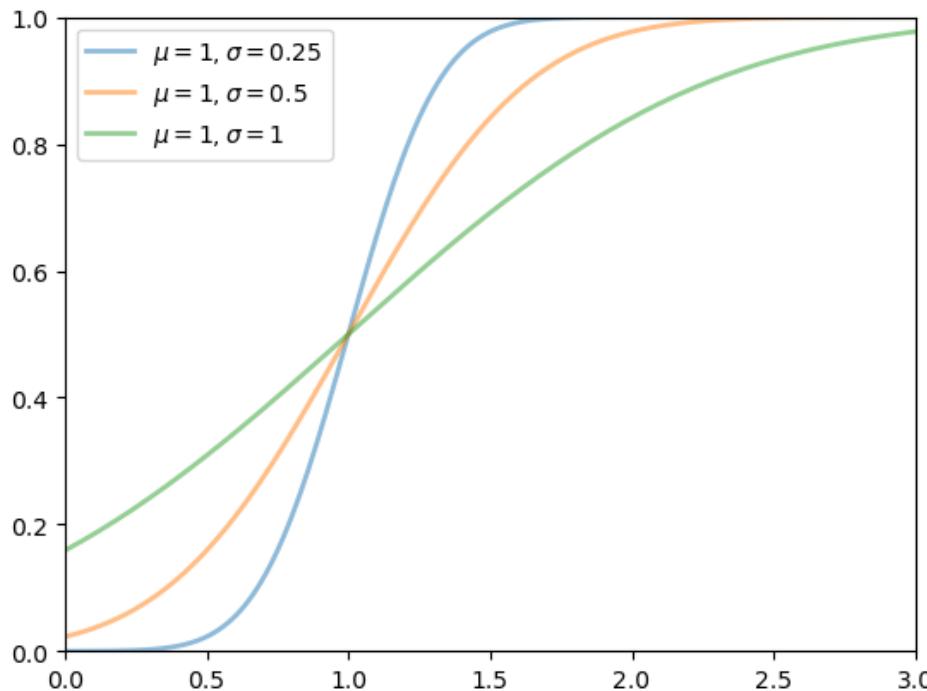
```



```

fig, ax = plt.subplots()
mu = 1
for sigma in sigma_vals:
    u = scipy.stats.norm(mu, sigma)
    ax.plot(x_grid, u.cdf(x_grid),
             alpha=0.5, lw=2,
             label=f'$\mu={mu}, \sigma={sigma}$')
    ax.set_ylim(0, 1)
    ax.set_xlim(0, 3)
plt.legend()
plt.show()

```



## Exponential distribution

The **exponential distribution** is a distribution on  $(0, \infty)$  with density

$$p(x) = \lambda \exp(-\lambda x)$$

This distribution has one parameter,  $\lambda$ .

It is related to the Poisson distribution as it describes the distribution of the length of the time interval between two consecutive events in a Poisson process.

It can be shown that, for this distribution, the mean is  $1/\lambda$  and the variance is  $1/\lambda^2$ .

We can obtain the moments, PDF, and CDF of the normal density as follows:

```
λ = 1.0
u = scipy.stats.expon(scale=1/λ)
```

```
u.mean(), u.var()
```

```
(1.0, 1.0)
```

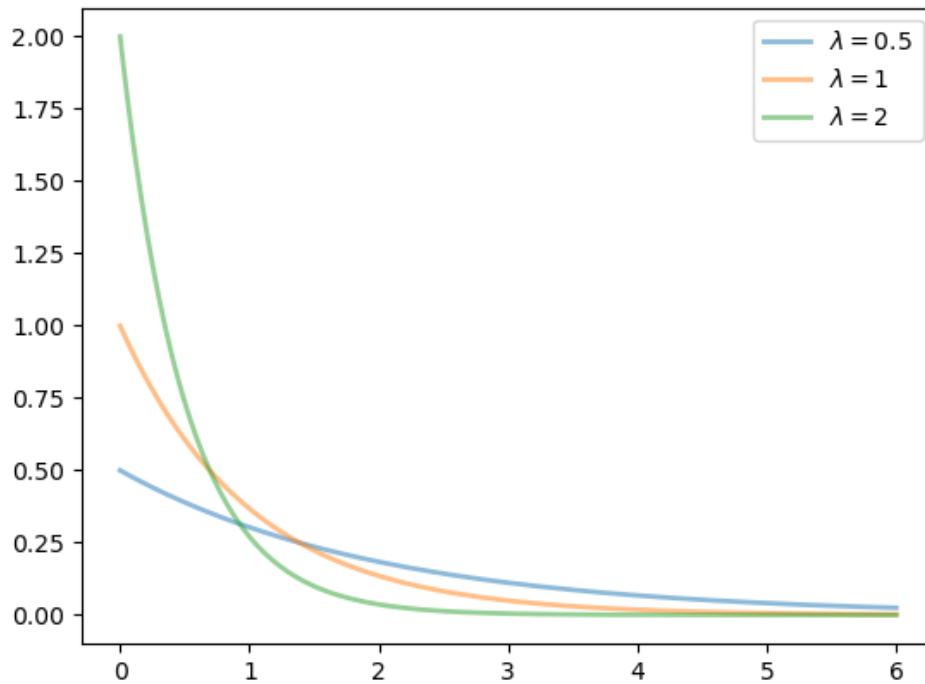
```
fig, ax = plt.subplots()
λ_vals = [0.5, 1, 2]
x_grid = np.linspace(0, 6, 200)

for λ in λ_vals:
    u = scipy.stats.expon(scale=1/λ)
    ax.plot(x_grid, u.pdf(x_grid),
```

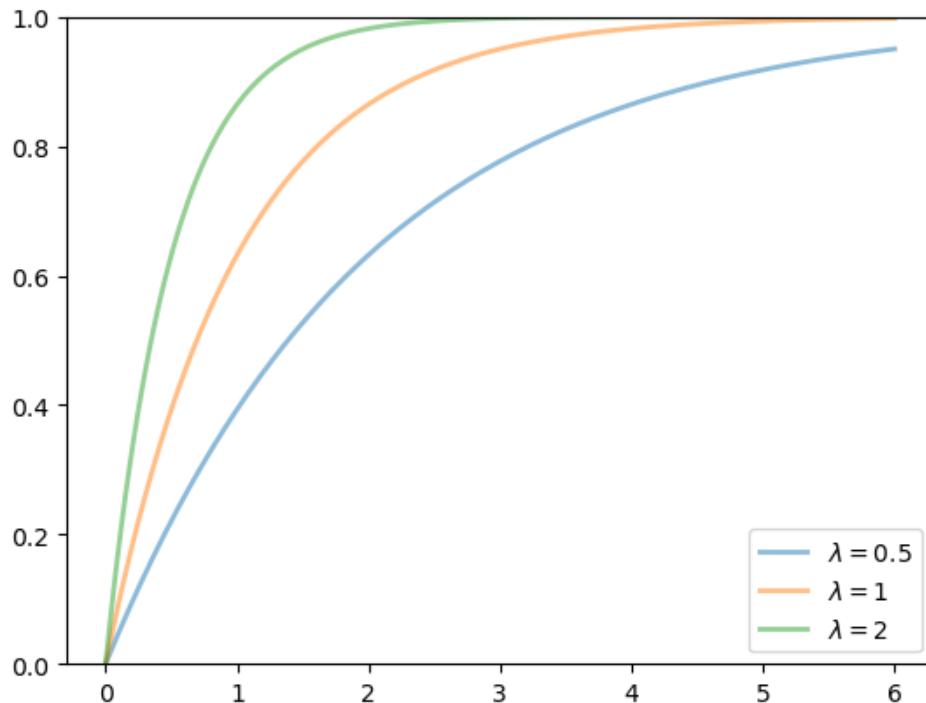
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```
alpha=0.5, lw=2,  
label=f'$\lambda={\lambda}$')  
plt.legend()  
plt.show()
```



```
fig, ax = plt.subplots()  
for lambda in lambda_vals:  
    u = scipy.stats.expon(scale=1/lambda)  
    ax.plot(x_grid, u.cdf(x_grid),  
            alpha=0.5, lw=2,  
            label=f'$\lambda={\lambda}$')  
    ax.set_ylimits(0, 1)  
plt.legend()  
plt.show()
```



## Beta distribution

The **beta distribution** is a distribution on  $(0, 1)$  with density

$$p(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$$

where  $\Gamma$  is the gamma function.

(The role of the gamma function is just to normalize the density, so that it integrates to one.)

This distribution has two parameters,  $\alpha > 0$  and  $\beta > 0$ .

It can be shown that, for this distribution, the mean is  $\alpha/(\alpha + \beta)$  and the variance is  $\alpha\beta/(\alpha + \beta)^2(\alpha + \beta + 1)$ .

We can obtain the moments, PDF, and CDF of the normal density as follows:

```
a, β = 3.0, 1.0
u = scipy.stats.beta(a, β)
```

```
u.mean(), u.var()
```

```
(0.75, 0.0375)
```

```
a_vals = [0.5, 1, 5, 25, 3]
β_vals = [3, 1, 10, 20, 0.5]
x_grid = np.linspace(0, 1, 200)

fig, ax = plt.subplots()
for a, β in zip(a_vals, β_vals):
```

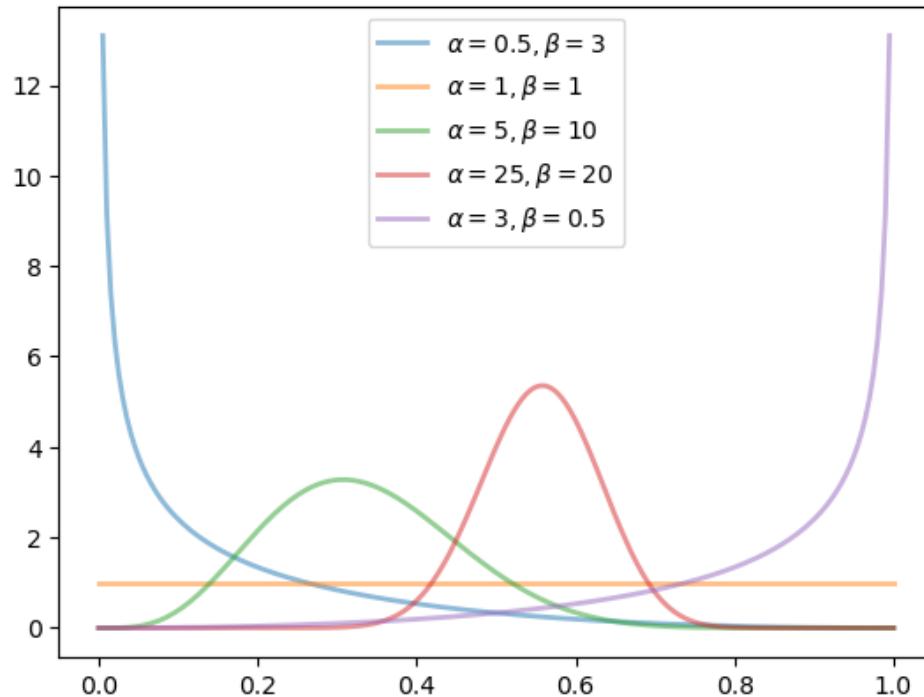
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```

u = scipy.stats.beta(a, b)
ax.plot(x_grid, u.pdf(x_grid),
alpha=0.5, lw=2,
label=fr'$\alpha={a}, \beta={b}$')
plt.legend()
plt.show()

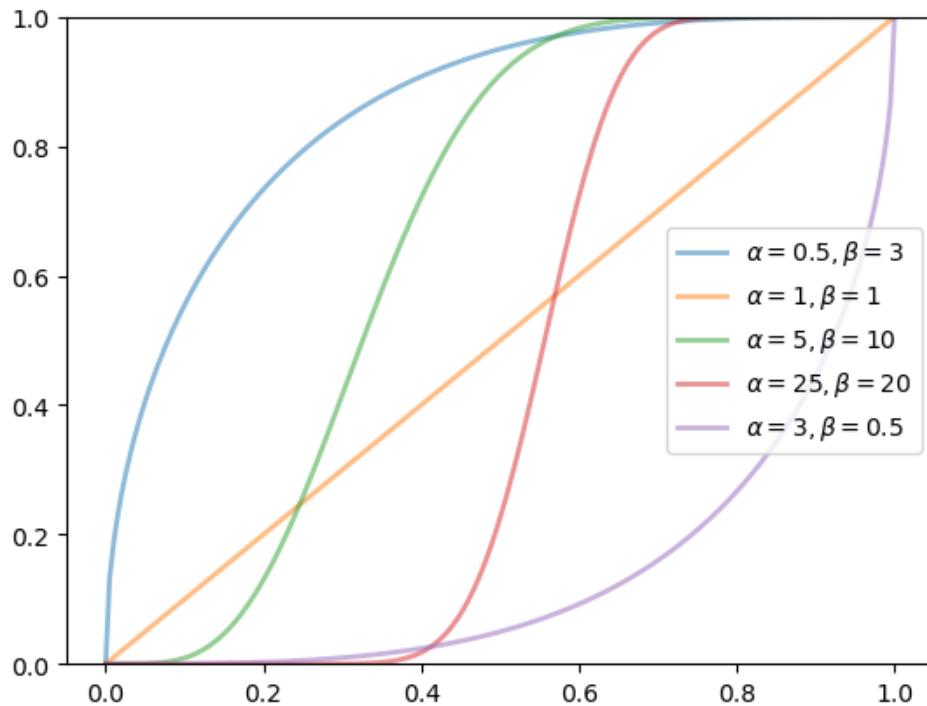
```



```

fig, ax = plt.subplots()
for a, b in zip(a_vals, b_vals):
    u = scipy.stats.beta(a, b)
    ax.plot(x_grid, u.cdf(x_grid),
alpha=0.5, lw=2,
label=fr'$\alpha={a}, \beta={b}$')
    ax.set_ylim(0, 1)
plt.legend()
plt.show()

```



### Gamma distribution

The **gamma distribution** is a distribution on  $(0, \infty)$  with density

$$p(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x)$$

This distribution has two parameters,  $\alpha > 0$  and  $\beta > 0$ .

It can be shown that, for this distribution, the mean is  $\alpha/\beta$  and the variance is  $\alpha/\beta^2$ .

One interpretation is that if  $X$  is gamma distributed and  $\alpha$  is an integer, then  $X$  is the sum of  $\alpha$  independent exponentially distributed random variables with mean  $1/\beta$ .

We can obtain the moments, PDF, and CDF of the normal density as follows:

```
a, β = 3.0, 2.0
u = scipy.stats.gamma(a, scale=1/β)
```

```
u.mean(), u.var()
```

```
(1.5, 0.75)
```

```
a_vals = [1, 3, 5, 10]
β_vals = [3, 5, 3, 3]
x_grid = np.linspace(0, 7, 200)

fig, ax = plt.subplots()
for a, β in zip(a_vals, β_vals):
```

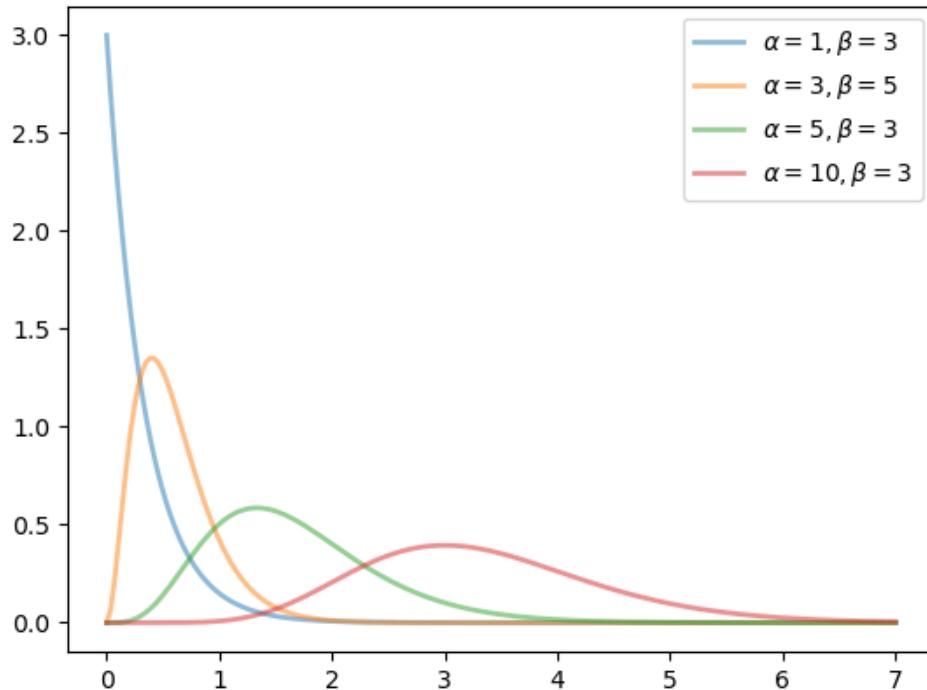
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```

u = scipy.stats.gamma(a, scale=1/β)
ax.plot(x_grid, u.pdf(x_grid),
alpha=0.5, lw=2,
label=fr'$\alpha={a}, \beta={b}$')
plt.legend()
plt.show()

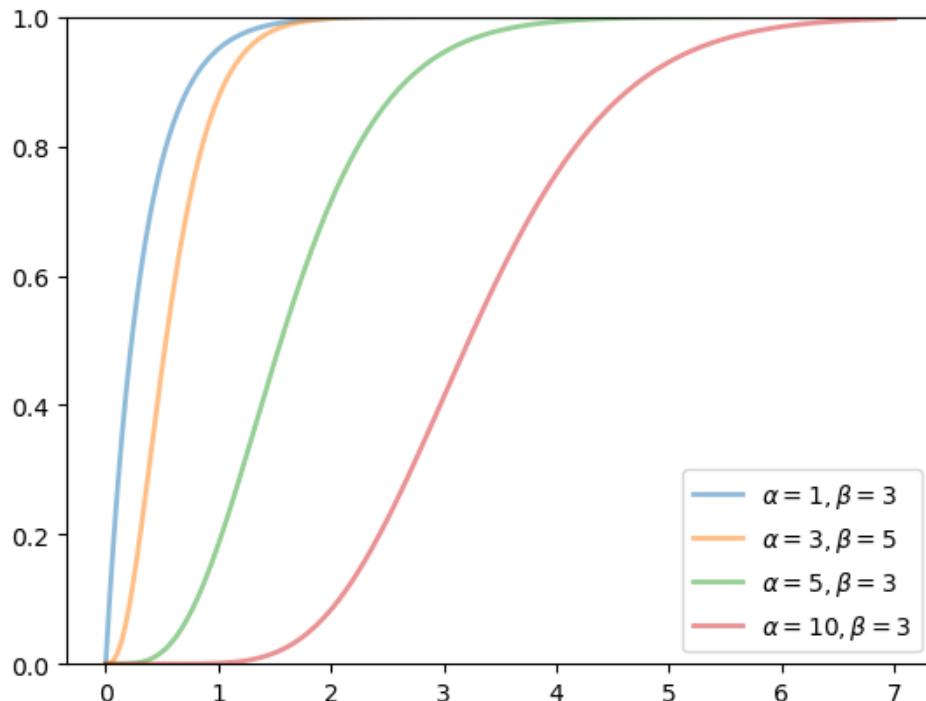
```



```

fig, ax = plt.subplots()
for a, b in zip(a_vals, β_vals):
    u = scipy.stats.gamma(a, scale=1/β)
    ax.plot(x_grid, u.cdf(x_grid),
    alpha=0.5, lw=2,
    label=fr'$\alpha={a}, \beta={b}$')
    ax.set_ylimits(0, 1)
plt.legend()
plt.show()

```



## 15.3 Observed distributions

Sometimes we refer to observed data or measurements as “distributions”.

For example, let’s say we observe the income of 10 people over a year:

```
data = [['Hiroshi', 1200],
        ['Ako', 1210],
        ['Emi', 1400],
        ['Daiki', 990],
        ['Chiyo', 1530],
        ['Taka', 1210],
        ['Katsuhiko', 1240],
        ['Daisuke', 1124],
        ['Yoshi', 1330],
        ['Rie', 1340]]

df = pd.DataFrame(data, columns=['name', 'income'])
df
```

	name	income
0	Hiroshi	1200
1	Ako	1210
2	Emi	1400
3	Daiki	990
4	Chiyo	1530
5	Taka	1210
6	Katsuhiko	1240
7	Daisuke	1124

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8	Yoshi	1330
9	Rie	1340

In this situation, we might refer to the set of their incomes as the “income distribution.”

The terminology is confusing because this set is not a probability distribution — it’s just a collection of numbers.

However, as we will see, there are connections between observed distributions (i.e., sets of numbers like the income distribution above) and probability distributions.

Below we explore some observed distributions.

### 15.3.1 Summary statistics

Suppose we have an observed distribution with values  $\{x_1, \dots, x_n\}$

The **sample mean** of this distribution is defined as

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

The **sample variance** is defined as

$$\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

For the income distribution given above, we can calculate these numbers via

```
x = np.asarray(df['income'])
```

```
x.mean(), x.var()
```

```
(1257.4, 20412.83999999997)
```

---

#### Exercise 15.3.1

Check that the formulas given above produce the same numbers.

---

### 15.3.2 Visualization

Let’s look at different ways that we can visualize one or more observed distributions.

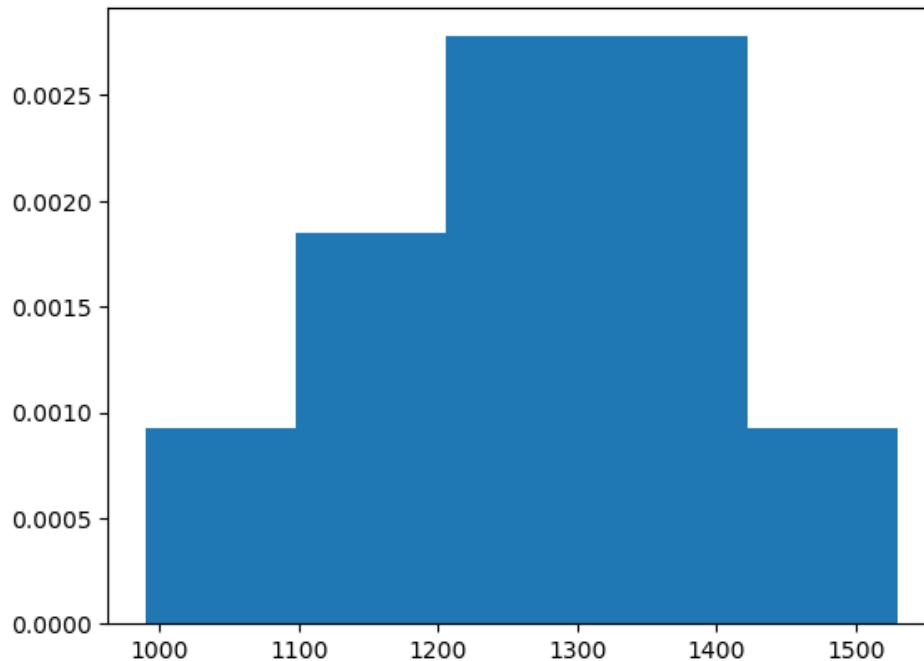
We will cover

- histograms
- kernel density estimates and
- violin plots

## Histograms

We can histogram the income distribution we just constructed as follows

```
x = df['income']
fig, ax = plt.subplots()
ax.hist(x, bins=5, density=True, histtype='bar')
plt.show()
```



Let's look at a distribution from real data.

In particular, we will look at the monthly return on Amazon shares between 2000/1/1 and 2023/1/1.

The monthly return is calculated as the percent change in the share price over each month.

So we will have one observation for each month.

```
df = yf.download('AMZN', '2000-1-1', '2023-1-1', interval='1mo')
prices = df['Adj Close']
data = prices.pct_change() [1:] * 100
data.head()
```

```
[*****100%*****] 1 of 1 completed
```

Date	Return (%)
2000-02-01	6.679568
2000-03-01	-2.722323
2000-04-01	-17.630592
2000-05-01	-12.457531

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```
2000-06-01    -24.838297
Name: Adj Close, dtype: float64
```

The first observation is the monthly return (percent change) over January 2000, which was

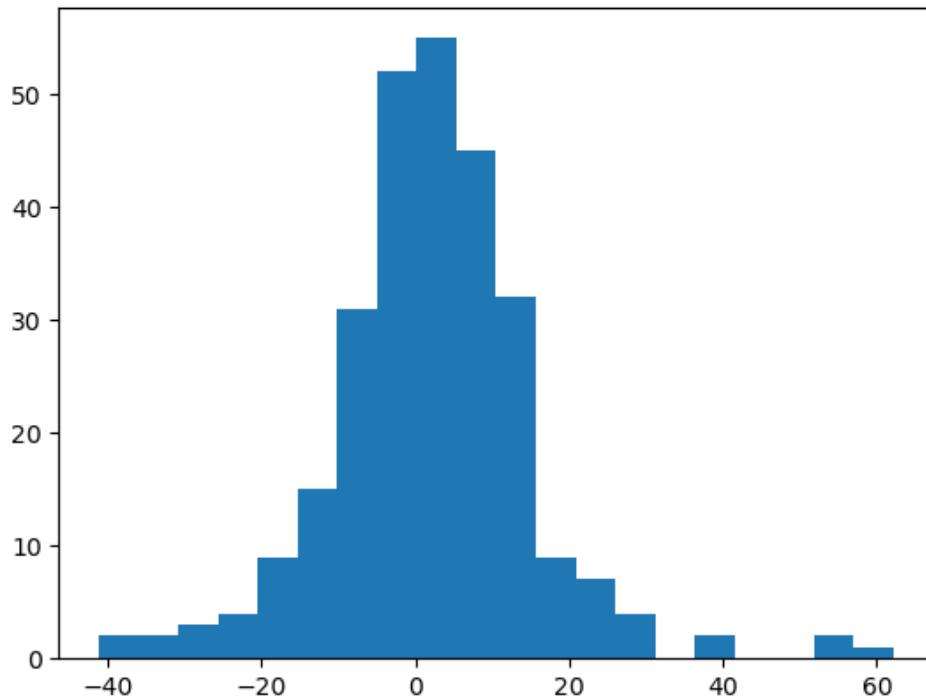
```
data[0]
```

```
6.6795679502808625
```

Let's turn the return observations into an array and histogram it.

```
x_amazon = np.asarray(data)
```

```
fig, ax = plt.subplots()
ax.hist(x_amazon, bins=20)
plt.show()
```

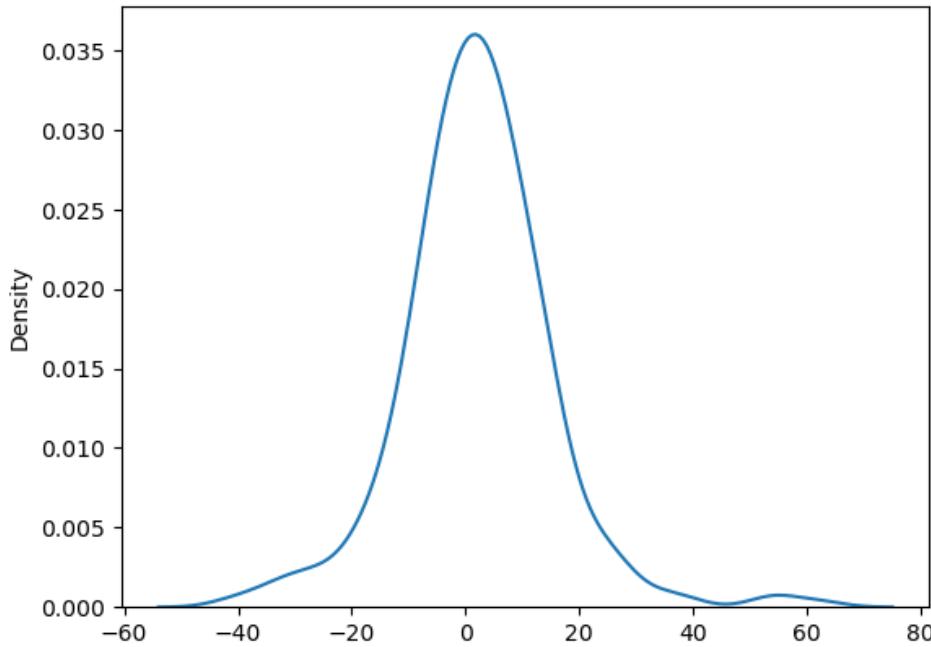


## Kernel density estimates

Kernel density estimate (KDE) is a non-parametric way to estimate and visualize the PDF of a distribution.

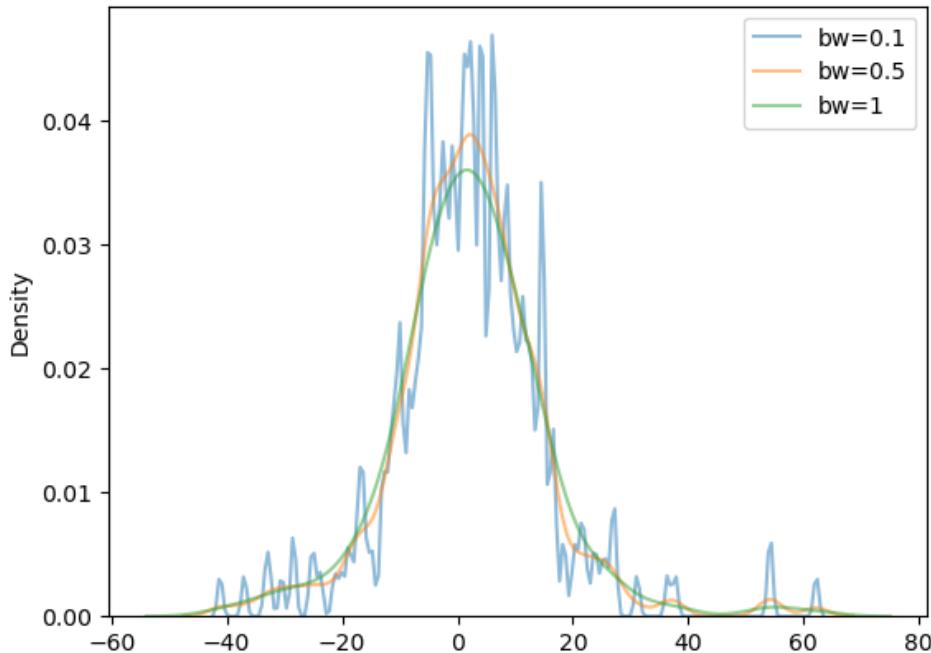
KDE will generate a smooth curve that approximates the PDF.

```
fig, ax = plt.subplots()
sns.kdeplot(x_amazon, ax=ax)
plt.show()
```



The smoothness of the KDE is dependent on how we choose the bandwidth.

```
fig, ax = plt.subplots()
sns.kdeplot(x_amazon, ax=ax, bw_adjust=0.1, alpha=0.5, label="bw=0.1")
sns.kdeplot(x_amazon, ax=ax, bw_adjust=0.5, alpha=0.5, label="bw=0.5")
sns.kdeplot(x_amazon, ax=ax, bw_adjust=1, alpha=0.5, label="bw=1")
plt.legend()
plt.show()
```



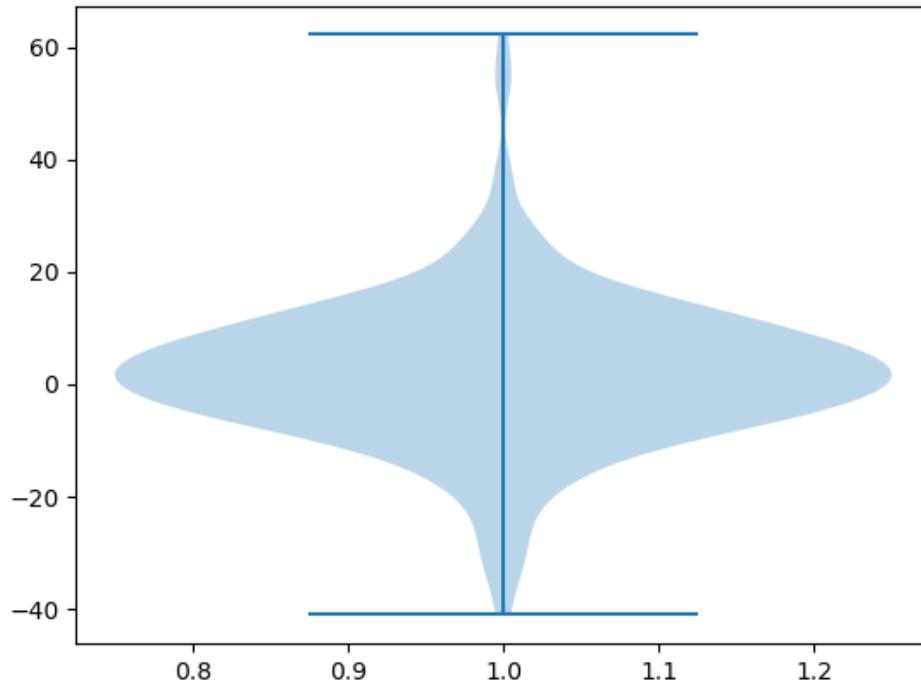
When we use a larger bandwidth, the KDE is smoother.

A suitable bandwidth is not too smooth (underfitting) or too wiggly (overfitting).

### Violin plots

Yet another way to display an observed distribution is via a violin plot.

```
fig, ax = plt.subplots()
ax.violinplot(x_amazon)
plt.show()
```



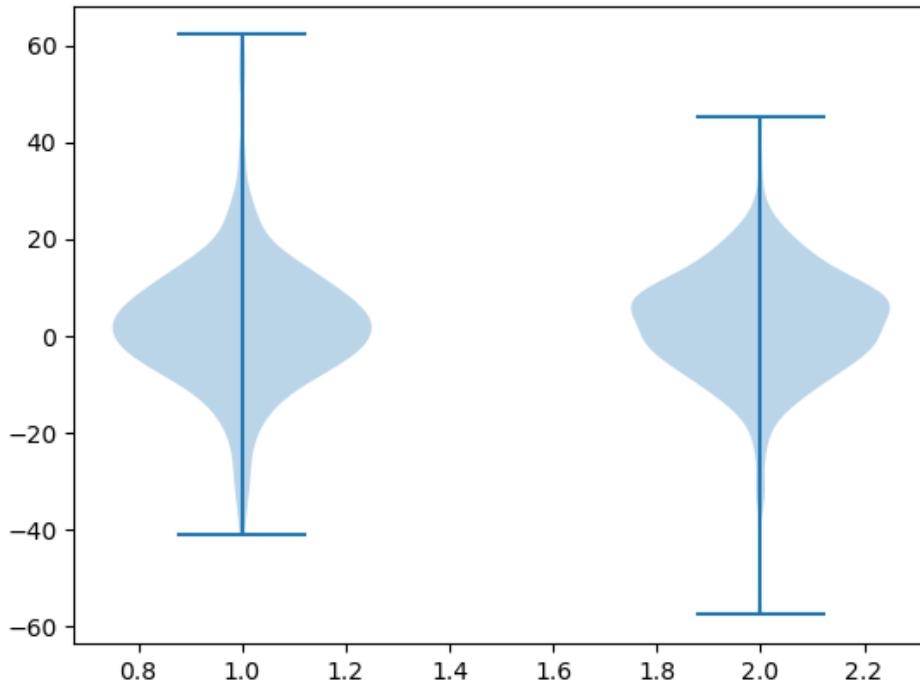
Violin plots are particularly useful when we want to compare different distributions.

For example, let's compare the monthly returns on Amazon shares with the monthly return on Apple shares.

```
df = yf.download('AAPL', '2000-1-1', '2023-1-1', interval='1mo' )
prices = df['Adj Close']
data = prices.pct_change()[1:] * 100
x_apple = np.asarray(data)
```

```
[*****100%*****] 1 of 1 completed
```

```
fig, ax = plt.subplots()
ax.violinplot([x_amazon, x_apple])
plt.show()
```



### 15.3.3 Connection to probability distributions

Let's discuss the connection between observed distributions and probability distributions.

Sometimes it's helpful to imagine that an observed distribution is generated by a particular probability distribution.

For example, we might look at the returns from Amazon above and imagine that they were generated by a normal distribution.

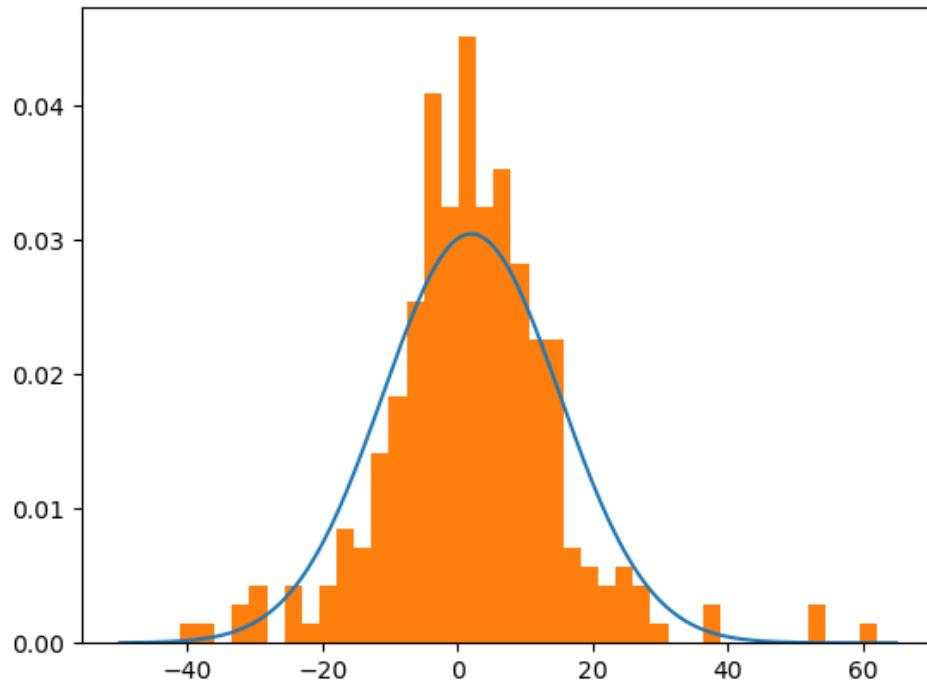
Even though this is not true, it might be a helpful way to think about the data.

Here we match a normal distribution to the Amazon monthly returns by setting the sample mean to the mean of the normal distribution and the sample variance equal to the variance.

Then we plot the density and the histogram.

```
μ = x_amazon.mean()
σ_squared = x_amazon.var()
σ = np.sqrt(σ_squared)
u = scipy.stats.norm(μ, σ)
```

```
x_grid = np.linspace(-50, 65, 200)
fig, ax = plt.subplots()
ax.plot(x_grid, u.pdf(x_grid))
ax.hist(x_amazon, density=True, bins=40)
plt.show()
```



The match between the histogram and the density is not very bad but also not very good.

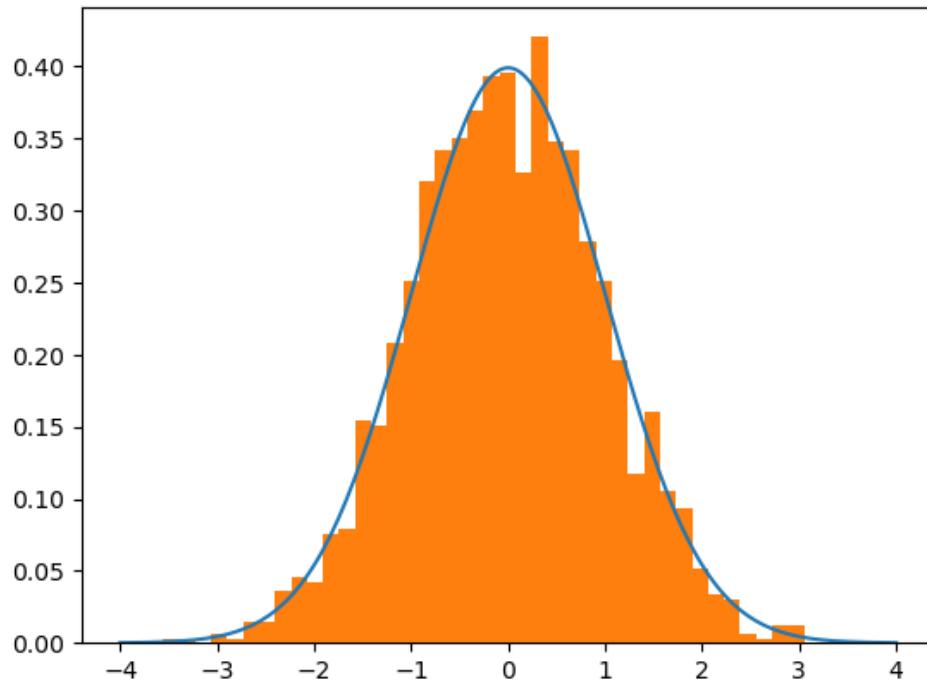
One reason is that the normal distribution is not really a good fit for this observed data — we will discuss this point again when we talk about *heavy tailed distributions*.

Of course, if the data really *is* generated by the normal distribution, then the fit will be better.

Let's see this in action

- first we generate random draws from the normal distribution
- then we histogram them and compare with the density.

```
μ, σ = 0, 1
u = scipy.stats.norm(μ, σ)
N = 2000 # Number of observations
x_draws = u.rvs(N)
x_grid = np.linspace(-4, 4, 200)
fig, ax = plt.subplots()
ax.plot(x_grid, u.pdf(x_grid))
ax.hist(x_draws, density=True, bins=40)
plt.show()
```



Note that if you keep increasing  $N$ , which is the number of observations, the fit will get better and better.

This convergence is a version of the “law of large numbers”, which we will discuss *later*.



## LLN AND CLT

### 16.1 Overview

This lecture illustrates two of the most important results in probability and statistics:

1. the law of large numbers (LLN) and
2. the central limit theorem (CLT).

These beautiful theorems lie behind many of the most fundamental results in econometrics and quantitative economic modeling.

The lecture is based around simulations that show the LLN and CLT in action.

We also demonstrate how the LLN and CLT break down when the assumptions they are based on do not hold.

This lecture will focus on the univariate case (the multivariate case is treated in a more advanced lecture).

We'll need the following imports:

```
import matplotlib.pyplot as plt
import random
import numpy as np
import scipy.stats as st
```

### 16.2 The law of large numbers

We begin with the law of large numbers, which tells us when sample averages will converge to their population means.

#### 16.2.1 The LLN in action

Let's see an example of the LLN in action before we go further.

Consider a Bernoulli random variable  $X$  with parameter  $p$ .

This means that  $X$  takes values in  $\{0, 1\}$  and  $\mathbb{P}\{X = 1\} = p$ .

We can think of drawing  $X$  as tossing a biased coin where

- the coin falls on “heads” with probability  $p$  and
- the coin falls on “tails” with probability  $1 - p$

We set  $X = 1$  if the coin is “heads” and zero otherwise.

The (population) mean of  $X$  is

$$\mathbb{E}X = 0 \cdot \mathbb{P}\{X = 0\} + 1 \cdot \mathbb{P}\{X = 1\} = \mathbb{P}\{X = 1\} = p$$

We can generate a draw of  $X$  with `scipy.stats` (imported as `st`) as follows:

```
p = 0.8
X = st.bernoulli.rvs(p)
print(X)
```

1

In this setting, the LLN tells us if we flip the coin many times, the fraction of heads that we see will be close to the mean  $p$ .

Let's check this:

```
n = 1_000_000
X_draws = st.bernoulli.rvs(p, size=n)
print(X_draws.mean()) # count the number of 1's and divide by n
```

0.799667

If we change  $p$  the claim still holds:

```
p = 0.3
X_draws = st.bernoulli.rvs(p, size=n)
print(X_draws.mean())
```

0.299996

Let's connect this to the discussion above, where we said the sample average converges to the “population mean”.

Think of  $X_1, \dots, X_n$  as independent flips of the coin.

The population mean is the mean in an infinite sample, which equals the expectation  $\mathbb{E}X$ .

The sample mean of the draws  $X_1, \dots, X_n$  is

$$\bar{X}_n := \frac{1}{n} \sum_{i=1}^n X_i$$

In this case, it is the fraction of draws that equal one (the number of heads divided by  $n$ ).

Thus, the LLN tells us that for the Bernoulli trials above

$$\bar{X}_n \rightarrow \mathbb{E}X = p \quad (n \rightarrow \infty) \tag{16.1}$$

This is exactly what we illustrated in the code.

### 16.2.2 Statement of the LLN

Let's state the LLN more carefully.

Let  $X_1, \dots, X_n$  be random variables, all of which have the same distribution.

These random variables can be continuous or discrete.

For simplicity we will

- assume they are continuous and
- let  $f$  denote their common density function

The last statement means that for any  $i$  in  $\{1, \dots, n\}$  and any numbers  $a, b$ ,

$$\mathbb{P}\{a \leq X_i \leq b\} = \int_a^b f(x)dx$$

(For the discrete case, we need to replace densities with probability mass functions and integrals with sums.)

Let  $\mu$  denote the common mean of this sample.

Thus, for each  $i$ ,

$$\mu := \mathbb{E}X_i = \int_{-\infty}^{\infty} xf(x)dx$$

The sample mean is

$$\bar{X}_n := \frac{1}{n} \sum_{i=1}^n X_i$$

The next theorem is called Kolmogorov's strong law of large numbers.

---

#### Theorem 16.2.1

If  $X_1, \dots, X_n$  are IID and  $\mathbb{E}|X|$  is finite, then

$$\mathbb{P}\{\bar{X}_n \rightarrow \mu \text{ as } n \rightarrow \infty\} = 1 \quad (16.2)$$


---

Here

- IID means independent and identically distributed and
- $\mathbb{E}|X| = \int_{-\infty}^{\infty} |x|f(x)dx$

### 16.2.3 Comments on the theorem

What does the probability one statement in the theorem mean?

Let's think about it from a simulation perspective, imagining for a moment that our computer can generate perfect random samples (although this isn't strictly true).

Let's also imagine that we can generate infinite sequences so that the statement  $\bar{X}_n \rightarrow \mu$  can be evaluated.

In this setting, (16.2) should be interpreted as meaning that the probability of the computer producing a sequence where  $\bar{X}_n \rightarrow \mu$  fails to occur is zero.

### 16.2.4 Illustration

Let's illustrate the LLN using simulation.

When we illustrate it, we will use a key idea: the sample mean  $\bar{X}_n$  is itself a random variable.

The reason  $\bar{X}_n$  is a random variable is that it's a function of the random variables  $X_1, \dots, X_n$ .

What we are going to do now is

1. pick some fixed distribution to draw each  $X_i$  from
2. set  $n$  to some large number

and then repeat the following three instructions.

1. generate the draws  $X_1, \dots, X_n$
2. calculate the sample mean  $\bar{X}_n$  and record its value in an array `sample_means`
3. go to step 1.

We will loop over these three steps  $m$  times, where  $m$  is some large integer.

The array `sample_means` will now contain  $m$  draws of the random variable  $\bar{X}_n$ .

If we histogram these observations of  $\bar{X}_n$ , we should see that they are clustered around the population mean  $\mathbb{E}X$ .

Moreover, if we repeat the exercise with a larger value of  $n$ , we should see that the observations are even more tightly clustered around the population mean.

This is, in essence, what the LLN is telling us.

To implement these steps, we will use functions.

Our first function generates a sample mean of size  $n$  given a distribution.

```
def draw_means(X_distribution, # The distribution of each X_i
               n): # The size of the sample mean

    # Generate n draws: X_1, ..., X_n
    X_samples = X_distribution.rvs(size=n)

    # Return the sample mean
    return np.mean(X_samples)
```

Now we write a function to generate  $m$  sample means and histogram them.

```
def generate_histogram(X_distribution, n, m):

    # Compute m sample means

    sample_means = np.empty(m)
    for j in range(m):
        sample_means[j] = draw_means(X_distribution, n)

    # Generate a histogram

    fig, ax = plt.subplots()
    ax.hist(sample_means, bins=30, alpha=0.5, density=True)
    mu = X_distribution.mean() # Get the population mean
    sigma = X_distribution.std() # and the standard deviation
    ax.axvline(x=mu, ls="--", c="k", label=f"$\mu = \{mu\}$")
```

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```

ax.set_xlim( $\mu - \sigma$ ,  $\mu + \sigma$ )
ax.set_xlabel(r'$\bar{X}_n$', size=12)
ax.set_ylabel('density', size=12)
ax.legend()
plt.show()

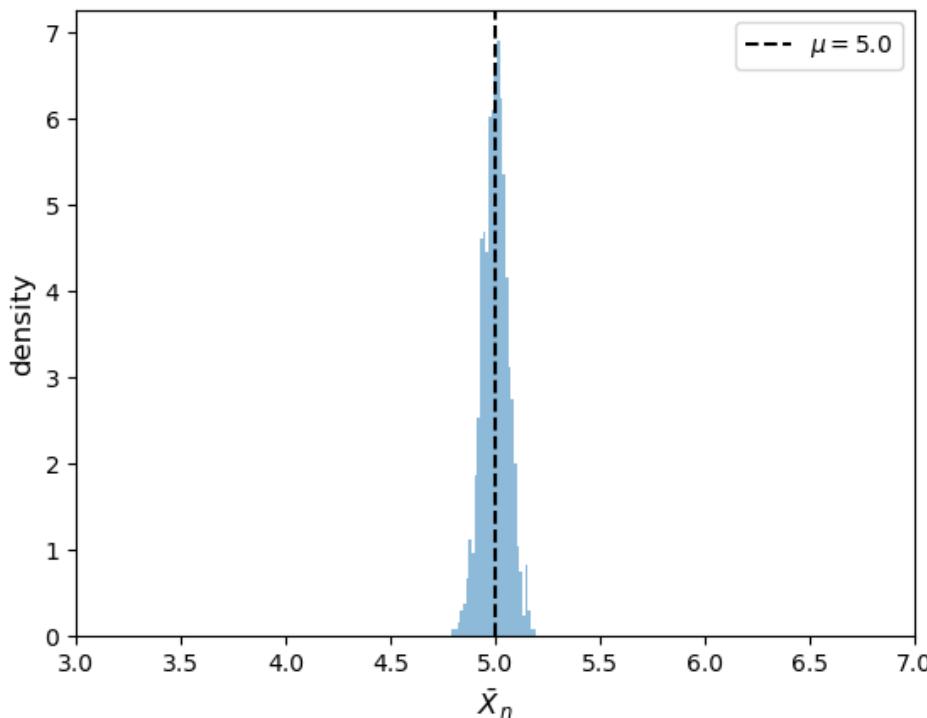
```

Now we call the function.

```

# pick a distribution to draw each  $\bar{X}_i$  from
X_distribution = st.norm(loc=5, scale=2)
# Call the function
generate_histogram(X_distribution, n=1_000, m=1000)

```



We can see that the distribution of  $\bar{X}$  is clustered around  $\mathbb{E}X$  as expected.

Let's vary  $n$  to see how the distribution of the sample mean changes.

We will use a violin plot to show the different distributions.

Each distribution in the violin plot represents the distribution of  $X_n$  for some  $n$ , calculated by simulation.

```

def means_violin_plot(distribution,
                      ns = [1_000, 10_000, 100_000],
                      m = 10_000):

    data = []
    for n in ns:
        sample_means = [draw_means(distribution, n) for i in range(m)]
        data.append(sample_means)

```

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```

fig, ax = plt.subplots()

ax.violinplot(data)
μ = distribution.mean()
ax.axhline(y=μ, ls="--", c="k", label=f"μ = {μ}")

labels=[f'n = {n}' for n in ns]

ax.set_xticks(np.arange(1, len(labels) + 1), labels=labels)
ax.set_xlim(0.25, len(labels) + 0.75)

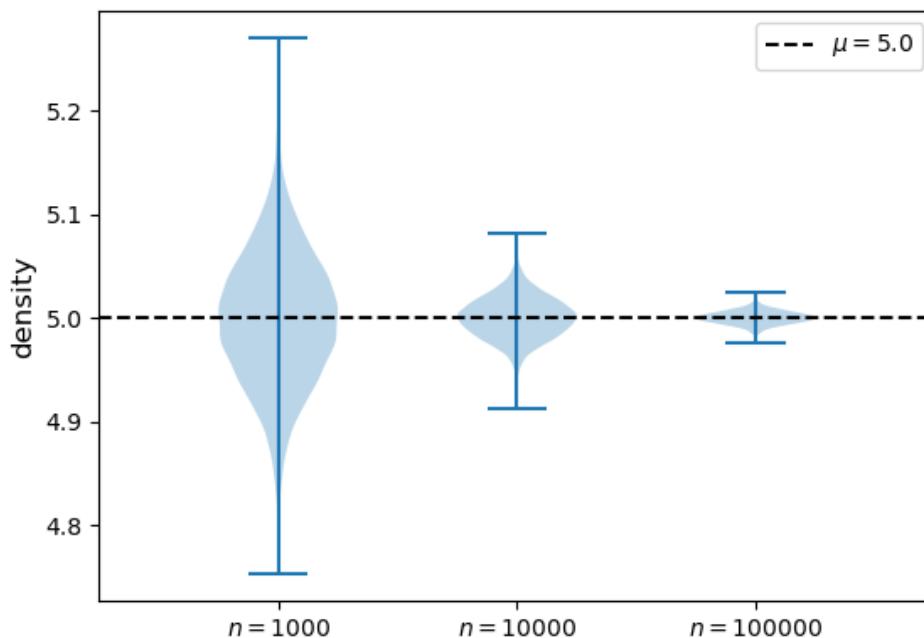
plt.subplots_adjust(bottom=0.15, wspace=0.05)

ax.set_ylabel('density', size=12)
ax.legend()
plt.show()

```

Let's try with a normal distribution.

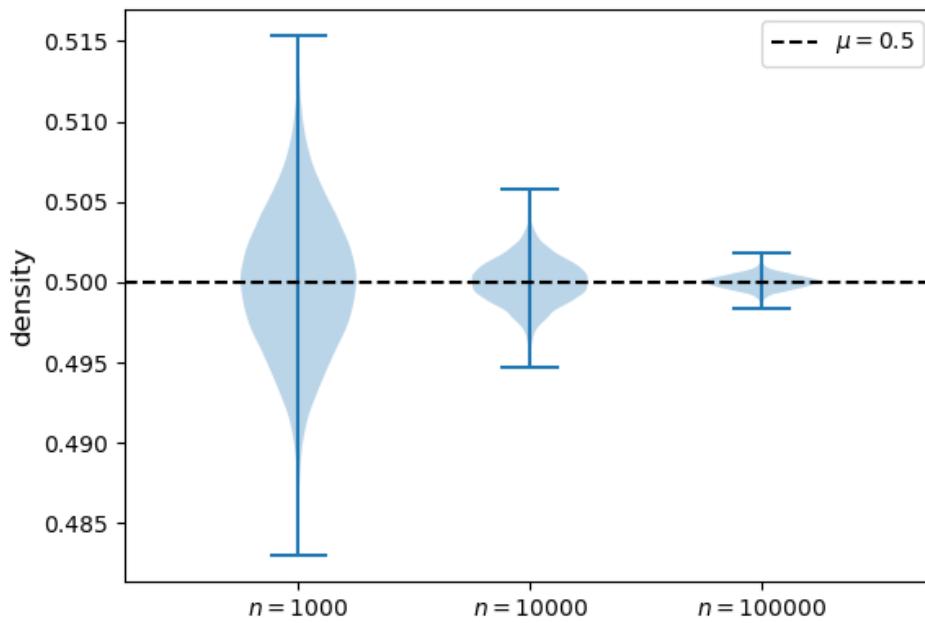
```
means_violin_plot(st.norm(loc=5, scale=2))
```



As  $n$  gets large, more probability mass clusters around the population mean  $\mu$ .

Now let's try with a Beta distribution.

```
means_violin_plot(st.beta(6, 6))
```



We get a similar result.

## 16.3 Breaking the LLN

We have to pay attention to the assumptions in the statement of the LLN.

If these assumptions do not hold, then the LLN might fail.

### 16.3.1 Infinite first moment

As indicated by the theorem, the LLN can break when  $\mathbb{E}|X|$  is not finite.

We can demonstrate this using the [Cauchy distribution](#).

The Cauchy distribution has the following property:

If  $X_1, \dots, X_n$  are IID and Cauchy, then so is  $\bar{X}_n$ .

This means that the distribution of  $\bar{X}_n$  does not eventually concentrate on a single number.

Hence the LLN does not hold.

The LLN fails to hold here because the assumption  $\mathbb{E}|X| = \infty$  is violated by the Cauchy distribution.

### 16.3.2 Failure of the IID condition

The LLN can also fail to hold when the IID assumption is violated.

For example, suppose that

$$X_0 \sim N(0, 1) \quad \text{and} \quad X_i = X_{i-1} \quad \text{for } i = 1, \dots, n$$

In this case,

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i = X_0 \sim N(0, 1)$$

Therefore, the distribution of  $\bar{X}_n$  is  $N(0, 1)$  for all  $n$ !

Does this contradict the LLN, which says that the distribution of  $\bar{X}_n$  collapses to the single point  $\mu$ ?

No, the LLN is correct — the issue is that its assumptions are not satisfied.

In particular, the sequence  $X_1, \dots, X_n$  is not independent.

---

**Note:** Although in this case the violation of IID breaks the LLN, there *are* situations where IID fails but the LLN still holds.

We will show an example in the [exercise](#).

---

## 16.4 Central limit theorem

Next, we turn to the central limit theorem (CLT), which tells us about the distribution of the deviation between sample averages and population means.

### 16.4.1 Statement of the theorem

The central limit theorem is one of the most remarkable results in all of mathematics.

In the IID setting, it tells us the following:

---

#### Theorem 16.4.1

If  $X_1, \dots, X_n$  is IID with common mean  $\mu$  and common variance  $\sigma^2 \in (0, \infty)$ , then

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2) \quad \text{as } n \rightarrow \infty \tag{16.3}$$


---

Here  $\xrightarrow{d} N(0, \sigma^2)$  indicates [convergence in distribution](#) to a centered (i.e., zero mean) normal with standard deviation  $\sigma$ .

The striking implication of the CLT is that for **any** distribution with finite [second moment](#), the simple operation of adding independent copies **always** leads to a Gaussian curve.

## 16.4.2 Simulation 1

Since the CLT seems almost magical, running simulations that verify its implications is one good way to build understanding.

To this end, we now perform the following simulation

1. Choose an arbitrary distribution  $F$  for the underlying observations  $X_i$ .
2. Generate independent draws of  $Y_n := \sqrt{n}(\bar{X}_n - \mu)$ .
3. Use these draws to compute some measure of their distribution — such as a histogram.
4. Compare the latter to  $N(0, \sigma^2)$ .

Here's some code that does exactly this for the exponential distribution  $F(x) = 1 - e^{-\lambda x}$ .

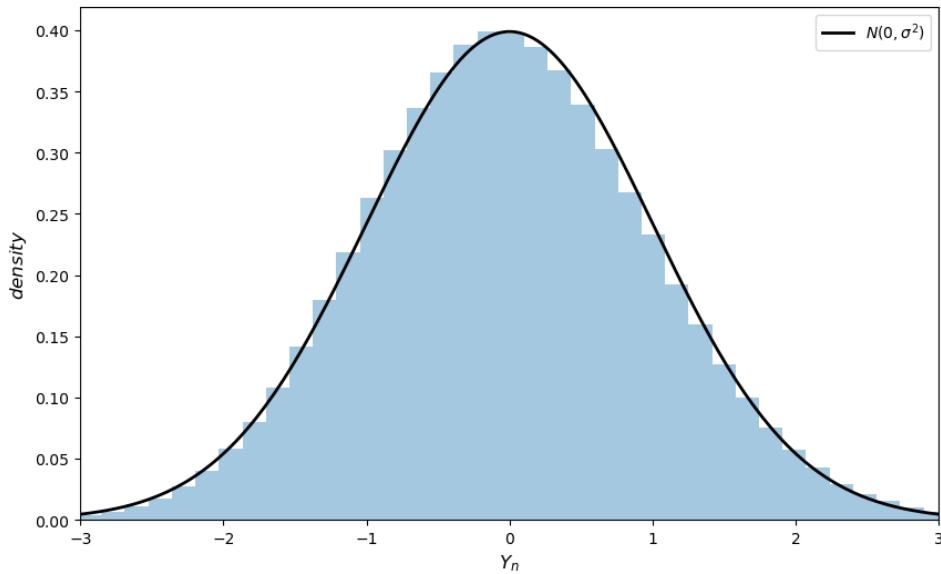
(Please experiment with other choices of  $F$ , but remember that, to conform with the conditions of the CLT, the distribution must have a finite second moment.)

```
# Set parameters
n = 250          # Choice of n
k = 1_000_000    # Number of draws of Y_n
distribution = st.expon(2) # Exponential distribution, λ = 1/2
μ, σ = distribution.mean(), distribution.std()

# Draw underlying RVs. Each row contains a draw of X_1, ..., X_n
data = distribution.rvs((k, n))
# Compute mean of each row, producing k draws of \bar{X}_n
sample_means = data.mean(axis=1)
# Generate observations of Y_n
Y = np.sqrt(n) * (sample_means - μ)

# Plot
fig, ax = plt.subplots(figsize=(10, 6))
xmin, xmax = -3 * σ, 3 * σ
ax.set_xlim(xmin, xmax)
ax.hist(Y, bins=60, alpha=0.4, density=True)
xgrid = np.linspace(xmin, xmax, 200)
ax.plot(xgrid, st.norm.pdf(xgrid, scale=σ),
        'k-', lw=2, label='$N(0, \sigma^2)$')
ax.set_xlabel(r"$Y_n$", size=12)
ax.set_ylabel(r"$density$", size=12)

ax.legend()
plt.show()
```



(Notice the absence of for loops — every operation is vectorized, meaning that the major calculations are all shifted to fast C code.)

The fit to the normal density is already tight and can be further improved by increasing  $n$ .

## 16.5 Exercises

---

### Exercise 16.5.1

Repeat the simulation *above* with the Beta distribution.

You can choose any  $\alpha > 0$  and  $\beta > 0$ .

---



---

### Solution to Exercise 16.5.1

```
# Set parameters
n = 250          # Choice of n
k = 1_000_000    # Number of draws of Y_n
distribution = st.beta(2, 2) # We chose Beta(2, 2) as an example
μ, σ = distribution.mean(), distribution.std()

# Draw underlying RVs. Each row contains a draw of X_1, ..., X_n
data = distribution.rvs((k, n))
# Compute mean of each row, producing k draws of \bar X_n
sample_means = data.mean(axis=1)
# Generate observations of Y_n
Y = np.sqrt(n) * (sample_means - μ)

# Plot
fig, ax = plt.subplots(figsize=(10, 6))
xmin, xmax = -3 * σ, 3 * σ
ax.set_xlim(xmin, xmax)
ax.hist(Y, bins=60, alpha=0.4, density=True)
```

(continues on next page)

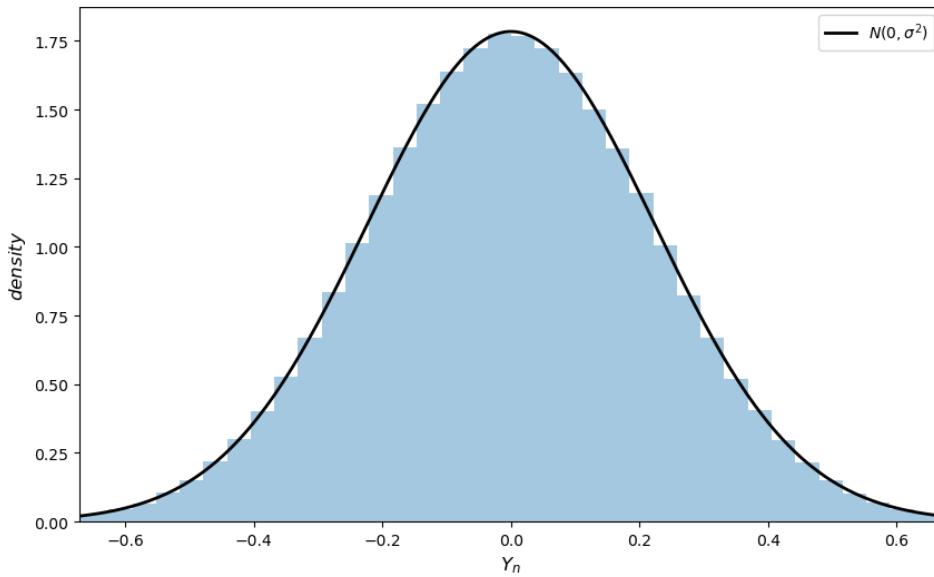
(continued from previous page)

```

ax.set_xlabel(r"$Y_n$", size=12)
ax.set_ylabel(r"$density$", size=12)
xgrid = np.linspace(xmin, xmax, 200)
ax.plot(xgrid, st.norm.pdf(xgrid, scale=o), 'k-', lw=2, label='$N(0, \sigma^2)$')
ax.legend()

plt.show()

```




---

### Exercise 16.5.2

At the start of this lecture we discussed Bernoulli random variables.

NumPy doesn't provide a `bernoulli` function that we can sample from.

However, we can generate a draw of Bernoulli  $X$  using NumPy via

```

U = np.random.rand()
X = 1 if U < p else 0
print(X)

```

Explain why this provides a random variable  $X$  with the right distribution.

---

### Solution to Exercise 16.5.2

We can write  $X$  as  $X = \mathbf{1}\{U < p\}$  where  $\mathbf{1}$  is the indicator function (i.e., 1 if the statement is true and zero otherwise).

Here we generated a uniform draw  $U$  on  $[0, 1]$  and then used the fact that

$$\mathbb{P}\{0 \leq U < p\} = p - 0 = p$$

This means that  $X = \mathbf{1}\{U < p\}$  has the right distribution.

---

---

**Exercise 16.5.3**

We mentioned above that LLN can still hold sometimes when IID is violated.

Let's investigate this claim further.

Consider the AR(1) process

$$X_{t+1} = \alpha + \beta X_t + \sigma \epsilon_{t+1}$$

where  $\alpha, \beta, \sigma$  are constants and  $\epsilon_1, \epsilon_2, \dots$  is IID and standard normal.

Suppose that

$$X_0 \sim N\left(\frac{\alpha}{1-\beta}, \frac{\sigma^2}{1-\beta^2}\right)$$

This process violates the independence assumption of the LLN (since  $X_{t+1}$  depends on the value of  $X_t$ ).

However, the next exercise teaches us that LLN type convergence of the sample mean to the population mean still occurs.

1. Prove that the sequence  $X_1, X_2, \dots$  is identically distributed.
  2. Show that LLN convergence holds using simulations with  $\alpha = 0.8, \beta = 0.2$ .
- 

**Solution to Exercise 16.5.3**
**Q1 Solution**

Regarding part 1, we claim that  $X_t$  has the same distribution as  $X_0$  for all  $t$ .

To construct a proof, we suppose that the claim is true for  $X_t$ .

Now we claim it is also true for  $X_{t+1}$ .

Observe that we have the correct mean:

$$\begin{aligned}\mathbb{E}X_{t+1} &= \alpha + \beta\mathbb{E}X_t \\ &= \alpha + \beta \frac{\alpha}{1-\beta} \\ &= \frac{\alpha}{1-\beta}\end{aligned}$$

We also have the correct variance:

$$\begin{aligned}\text{Var}(X_{t+1}) &= \beta^2\text{Var}(X_t) + \sigma^2 \\ &= \frac{\beta^2\sigma^2}{1-\beta^2} + \sigma^2 \\ &= \frac{\sigma^2}{1-\beta^2}\end{aligned}$$

Finally, since both  $X_t$  and  $\epsilon_0$  are normally distributed and independent from each other, any linear combination of these two variables is also normally distributed.

We have now shown that

$$X_{t+1} \sim N\left(\frac{\alpha}{1-\beta}, \frac{\sigma^2}{1-\beta^2}\right)$$

We can conclude this AR(1) process violates the independence assumption but is identically distributed.

**Q2 Solution**

```

σ = 10
α = 0.8
β = 0.2
n = 100_000

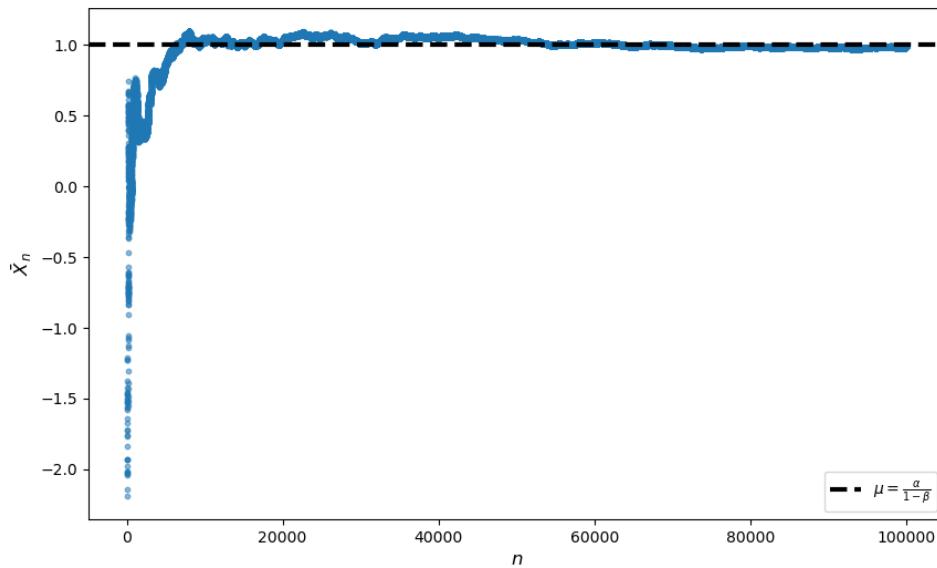
fig, ax = plt.subplots(figsize=(10, 6))
x = np.ones(n)
x[0] = st.norm.rvs(α/(1-β), α**2/(1-β**2))
ε = st.norm.rvs(size=n+1)
means = np.ones(n)
means[0] = x[0]
for t in range(n-1):
    x[t+1] = α + β * x[t] + σ * ε[t+1]
    means[t+1] = np.mean(x[:t+1])

ax.scatter(range(100, n), means[100:n], s=10, alpha=0.5)

ax.set_xlabel(r"$n$", size=12)
ax.set_ylabel(r"$\bar{X}_n$", size=12)
yabs_max = max(ax.get_ylim(), key=abs)
ax.axhline(y=α/(1-β), ls="--", lw=3,
            label=r"$\mu = \frac{\alpha}{1-\beta}$",
            color = 'black')

plt.legend()
plt.show()

```



We see the convergence of  $\bar{x}$  around  $\mu$  even when the independence assumption is violated.



## MONTE CARLO AND OPTION PRICING

### 17.1 Overview

Simple probability calculations can be done either

- with pencil and paper, or
- by looking up facts about well known probability distributions, or
- in our heads.

For example, we can easily work out

- the probability of three heads in five flips of a fair coin
- the expected value of a random variable that equals  $-10$  with probability  $1/2$  and  $100$  with probability  $1/2$ .

But some probability calculations are very complex.

Complex calculations concerning probabilities and expectations occur in many economic and financial problems.

Perhaps the most important tool for handling complicated probability calculations is [Monte Carlo methods](#).

In this lecture we introduce Monte Carlo methods for computing expectations, with some applications in finance.

We will use the following imports.

```
import numpy as np
import matplotlib.pyplot as plt
from numpy.random import randn
```

### 17.2 An introduction to Monte Carlo

In this section we describe how Monte Carlo can be used to compute expectations.

### 17.2.1 Share price with known distribution

Suppose that we are considering buying a share in some company.

Our plan is either to

1. buy the share now, hold it for one year and then sell it, or
2. do something else with our money.

We start by thinking of the share price in one year as a random variable  $S$ .

Before deciding whether or not to buy the share, we need to know some features of the distribution of  $S$ .

For example, suppose the mean of  $S$  is high relative to the price of buying the share.

This suggests we have a good chance of selling at a relatively high price.

Suppose, however, that the variance of  $S$  is also high.

This suggests that buying the share is risky, so perhaps we should refrain.

Either way, this discussion shows the importance of understanding the distribution of  $S$ .

Suppose that, after analyzing the data, we guess that  $S$  is well represented by a lognormal distribution with parameters  $\mu, \sigma$ .

- $S$  has the same distribution as  $\exp(\mu + \sigma Z)$  where  $Z$  is standard normal.
- we write this statement as  $S \sim LN(\mu, \sigma)$ .

Any good reference on statistics (such as [Wikipedia](#)) will tell us that the mean and variance are

$$\mathbb{E}S = \exp\left(\mu + \frac{\sigma^2}{2}\right)$$

and

$$\text{Var } S = [\exp(\sigma^2) - 1] \exp(2\mu + \sigma^2)$$

So far we have no need for a computer.

### 17.2.2 Share price with unknown distribution

But now suppose that we study the distribution of  $S$  more carefully.

We decide that the share price depends on three variables,  $X_1, X_2$ , and  $X_3$  (e.g., sales, inflation, and interest rates).

In particular, our study suggests that

$$S = (X_1 + X_2 + X_3)^p$$

where

- $p$  is a positive number, which is known to us (i.e., has been estimated),
- $X_i \sim LN(\mu_i, \sigma_i)$  for  $i = 1, 2, 3$ ,
- the values  $\mu_i, \sigma_i$  are also known, and
- the random variables  $X_1, X_2$  and  $X_3$  are independent.

How should we compute the mean of  $S$ ?

To do this with pencil and paper is hard (unless, say,  $p = 1$ ).

But fortunately there's an easy way to do this, at least approximately.

This is the Monte Carlo method, which runs as follows:

1. Generate  $n$  independent draws of  $X_1$ ,  $X_2$  and  $X_3$  on a computer,
2. Use these draws to generate  $n$  independent draws of  $S$ , and
3. Take the average value of these draws of  $S$ .

This average will be close to the true mean when  $n$  is large.

This is due to the law of large numbers, which we discussed in [another lecture](#).

We use the following values for  $p$  and each  $\mu_i$  and  $\sigma_i$ .

```
n = 1_000_000
p = 0.5
μ_1, μ_2, μ_3 = 0.2, 0.8, 0.4
σ_1, σ_2, σ_3 = 0.1, 0.05, 0.2
```

## A routine using loops in python

Here's a routine using native Python loops to calculate the desired mean

$$\frac{1}{n} \sum_{i=1}^n S_i \approx \mathbb{E}S$$

```
%time
S = 0.0
for i in range(n):
    X_1 = np.exp(μ_1 + σ_1 * randn())
    X_2 = np.exp(μ_2 + σ_2 * randn())
    X_3 = np.exp(μ_3 + σ_3 * randn())
    S += (X_1 + X_2 + X_3)**p
S / n
```

```
CPU times: user 4.77 s, sys: 0 ns, total: 4.77 s
Wall time: 4.77 s
```

```
2.2296256392841296
```

We can also construct a function that contains these operations:

```
def compute_mean(n=1_000_000):
    S = 0.0
    for i in range(n):
        X_1 = np.exp(μ_1 + σ_1 * randn())
        X_2 = np.exp(μ_2 + σ_2 * randn())
        X_3 = np.exp(μ_3 + σ_3 * randn())
        S += (X_1 + X_2 + X_3)**p
    return (S / n)
```

Now let's call it.

```
compute_mean()
```

```
2.229660657141764
```

### 17.2.3 A vectorized routine

If we want a more accurate estimate we should increase  $n$ .

But the code above runs quite slowly.

To make it faster, let's implement a vectorized routine using NumPy.

```
def compute_mean_vectorized(n=1_000_000):  
    X_1 = np.exp(mu_1 + sigma_1 * randn(n))  
    X_2 = np.exp(mu_2 + sigma_2 * randn(n))  
    X_3 = np.exp(mu_3 + sigma_3 * randn(n))  
    S = (X_1 + X_2 + X_3)**p  
    return S.mean()
```

```
%time
```

```
compute_mean_vectorized()
```

```
CPU times: user 88.9 ms, sys: 12 ms, total: 101 ms  
Wall time: 101 ms
```

```
2.229725389621692
```

Notice that this routine is much faster.

We can increase  $n$  to get more accuracy and still have reasonable speed:

```
%%time
```

```
compute_mean_vectorized(n=10_000_000)
```

```
CPU times: user 939 ms, sys: 92 ms, total: 1.03 s  
Wall time: 1.03 s
```

```
2.2297269881536668
```

## 17.3 Pricing a european call option under risk neutrality

Next we are going to price a European call option under risk neutrality.

Let's first discuss risk neutrality and then consider European options.

### 17.3.1 Risk-Neutral Pricing

When we use risk-neutral pricing, we determine the price of a given asset according to its expected payoff:

$$\text{cost} = \text{expected benefit}$$

For example, suppose someone promises to pay you

- 1,000,000 dollars if “heads” is the outcome of a fair coin flip
- 0 dollars if “tails” is the outcome

Let's denote the payoff as  $G$ , so that

$$\mathbb{P}\{G = 10^6\} = \mathbb{P}\{G = 0\} = \frac{1}{2}$$

Suppose in addition that you can sell this promise to anyone who wants it.

- First they pay you  $P$ , the price at which you sell it
- Then they get  $G$ , which could be either 1,000,000 or 0.

What's a fair price for this asset (this promise)?

The definition of “fair” is ambiguous, but we can say that the **risk-neutral price** is 500,000 dollars.

This is because the risk-neutral price is just the expected payoff of the asset, which is

$$\mathbb{E}G = \frac{1}{2} \times 10^6 + \frac{1}{2} \times 0 = 5 \times 10^5$$

### 17.3.2 A comment on risk

As suggested by the name, the risk-neutral price ignores risk.

To understand this, consider whether you would pay 500,000 dollars for such a promise.

Would you prefer to receive 500,000 for sure or 1,000,000 dollars with 50% probability and nothing with 50% probability?

At least some readers will strictly prefer the first option — although some might prefer the second.

Thinking about this makes us realize that 500,000 is not necessarily the “right” price — or the price that we would see if there was a market for these promises.

Nonetheless, the risk-neutral price is an important benchmark, which economists and financial market participants try to calculate every day.

### 17.3.3 Discounting

Another thing we ignored in the previous discussion was time.

In general, receiving  $x$  dollars now is preferable to receiving  $x$  dollars in  $n$  periods (e.g., 10 years).

After all, if we receive  $x$  dollars now, we could put it in the bank at interest rate  $r > 0$  and receive  $(1+r)^n x$  in  $n$  periods.

Hence future payments need to be discounted when we consider their present value.

We will implement discounting by

- multiplying a payment in one period by  $\beta < 1$
- multiplying a payment in  $n$  periods by  $\beta^n$ , etc.

The same adjustment needs to be applied to our risk-neutral price for the promise described above.

Thus, if  $G$  is realized in  $n$  periods, then the risk-neutral price is

$$P = \beta^n \mathbb{E}G = \beta^n 5 \times 10^5$$

### 17.3.4 European call options

Now let's price a European call option.

The option is described by three things:

2.  $n$ , the **expiry date**,
3.  $K$ , the **strike price**, and
4.  $S_n$ , the price of the **underlying** asset at date  $n$ .

For example, suppose that the underlying is one share in Amazon.

The owner of this option has the right to buy one share in Amazon at price  $K$  after  $n$  days.

If  $S_n > K$ , then the owner will exercise the option, buy at  $K$ , sell at  $S_n$ , and make profit  $S_n - K$ .

If  $S_n \leq K$ , then the owner will not exercise the option and the payoff is zero.

Thus, the payoff is  $\max\{S_n - K, 0\}$ .

Under the assumption of risk neutrality, the price of the option is the expected discounted payoff:

$$P = \beta^n \mathbb{E} \max\{S_n - K, 0\}$$

Now all we need to do is specify the distribution of  $S_n$ , so the expectation can be calculated.

Suppose we know that  $S_n \sim LN(\mu, \sigma)$  and  $\mu$  and  $\sigma$  are known.

If  $S_n^1, \dots, S_n^M$  are independent draws from this lognormal distribution then, by the law of large numbers,

$$\mathbb{E} \max\{S_n - K, 0\} \approx \frac{1}{M} \sum_{m=1}^M \max\{S_n^m - K, 0\}$$

We suppose that

```

μ = 1.0
σ = 0.1
K = 1
n = 10
β = 0.95

```

We set the simulation size to

```
M = 10_000_000
```

Here is our code

```
S = np.exp(mu + sigma * np.random.randn(M))
return_draws = np.maximum(S - K, 0)
P = beta**n * np.mean(return_draws)
print(f"The Monte Carlo option price is approximately {P:.3f}")
```

```
The Monte Carlo option price is approximately 1.036973
```

## 17.4 Pricing via a dynamic model

In this exercise we investigate a more realistic model for the share price  $S_n$ .

This comes from specifying the underlying dynamics of the share price.

First we specify the dynamics.

Then we'll compute the price of the option using Monte Carlo.

### 17.4.1 Simple dynamics

One simple model for  $\{S_t\}$  is

$$\ln \frac{S_{t+1}}{S_t} = \mu + \sigma \xi_{t+1}$$

where

- $S_0$  is normally distributed and
- $\{\xi_t\}$  is IID and standard normal.

Under the stated assumptions,  $S_n$  is lognormally distributed.

To see why, observe that, with  $s_t := \ln S_t$ , the price dynamics become

$$s_{t+1} = s_t + \mu + \sigma \xi_{t+1} \quad (17.1)$$

Since  $s_0$  is normal and  $\xi_1$  is normal and IID, we see that  $s_1$  is normally distributed.

Continuing in this way shows that  $s_n$  is normally distributed.

Hence  $S_n = \exp(s_n)$  is lognormal.

### 17.4.2 Problems with simple dynamics

The simple dynamic model we studied above is convenient, since we can work out the distribution of  $S_n$ .

However, its predictions are counterfactual because, in the real world, volatility (measured by  $\sigma$ ) is not stationary.

Instead it rather changes over time, sometimes high (like during the GFC) and sometimes low.

In terms of our model above, this means that  $\sigma$  should not be constant.

### 17.4.3 More realistic dynamics

This leads us to study the improved version:

$$\ln \frac{S_{t+1}}{S_t} = \mu + \sigma_t \xi_{t+1}$$

where

$$\sigma_t = \exp(h_t), \quad h_{t+1} = \rho h_t + \nu \eta_{t+1}$$

Here  $\{\eta_t\}$  is also IID and standard normal.

### 17.4.4 Default parameters

For the dynamic model, we adopt the following parameter values.

```
μ = 0.0001
ρ = 0.1
ν = 0.001
S0 = 10
h0 = 0
```

(Here  $S_0$  is  $S_0$  and  $h_0$  is  $h_0$ .)

For the option we use the following defaults.

```
K = 100
n = 10
β = 0.95
```

### 17.4.5 Visualizations

With  $s_t := \ln S_t$ , the price dynamics become

$$s_{t+1} = s_t + \mu + \exp(h_t) \xi_{t+1}$$

Here is a function to simulate a path using this equation:

```
def simulate_asset_price_path(μ=μ, S0=S0, h0=h0, n=n, ρ=ρ, ν=ν):
    s = np.empty(n+1)
    s[0] = np.log(S0)
```

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```

h = h0
for t in range(n):
    s[t+1] = s[t] + mu + np.exp(h) * randn()
    h = rho * h + sigma * randn()

return np.exp(s)

```

Here we plot the paths and the log of the paths.

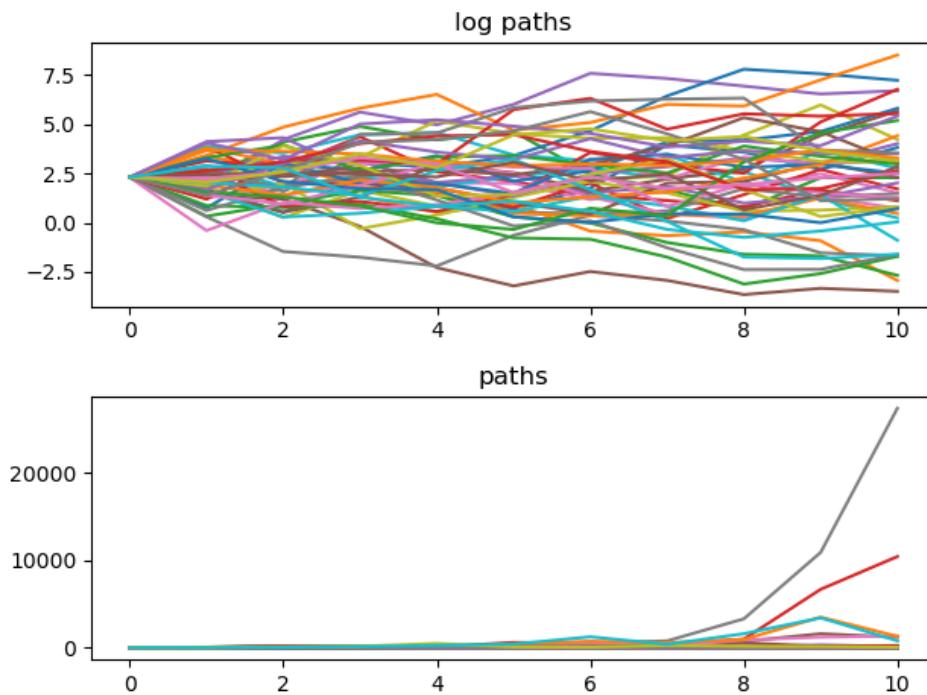
```

fig, axes = plt.subplots(2, 1)

titles = 'log paths', 'paths'
transforms = np.log, lambda x: x
for ax, transform, title in zip(axes, transforms, titles):
    for i in range(50):
        path = simulate_asset_price_path()
        ax.plot(transform(path))
    ax.set_title(title)

fig.tight_layout()
plt.show()

```



### 17.4.6 Computing the price

Now that our model is more complicated, we cannot easily determine the distribution of  $S_n$ .

So to compute the price  $P$  of the option, we use Monte Carlo.

We average over realizations  $S_n^1, \dots, S_n^M$  of  $S_n$  and appealing to the law of large numbers:

$$\mathbb{E} \max\{S_n - K, 0\} \approx \frac{1}{M} \sum_{m=1}^M \max\{S_n^m - K, 0\}$$

Here's a version using Python loops.

```
def compute_call_price(beta=beta,
                      mu=mu,
                      S0=S0,
                      h0=h0,
                      K=K,
                      n=n,
                      rho=rho,
                      v=v,
                      M=10_000):
    current_sum = 0.0
    # For each sample path
    for m in range(M):
        s = np.log(S0)
        h = h0
        # Simulate forward in time
        for t in range(n):
            s = s + mu + np.exp(h) * np.random()
            h = rho * h + v * np.random()
        # And add the value max{S_n - K, 0} to current_sum
        current_sum += np.maximum(np.exp(s) - K, 0)

    return beta**n * current_sum / M
```

```
%time
compute_call_price()
```

```
CPU times: user 247 ms, sys: 0 ns, total: 247 ms
Wall time: 247 ms
```

```
927.0343436175589
```

## 17.5 Exercises

---

### Exercise 17.5.1

We would like to increase  $M$  in the code above to make the calculation more accurate.

But this is problematic because Python loops are slow.

Your task is to write a faster version of this code using NumPy.

---

---

**Solution to Exercise 17.5.1**

```

def compute_call_price ( $\beta=\beta$ ,
                       $\mu=\mu$ ,
                       $S_0=S_0$ ,
                       $h_0=h_0$ ,
                       $K=K$ ,
                       $n=n$ ,
                       $\rho=\rho$ ,
                       $v=v$ ,
                       $M=10_000$ ) :

     $s = np.full(M, np.log(S_0))$ 
     $h = np.full(M, h_0)$ 
    for t in range(n):
         $Z = np.random.randn(2, M)$ 
         $s = s + \mu + np.exp(h) * Z[0, :]$ 
         $h = \rho * h + v * Z[1, :]$ 
    expectation = np.mean(np.maximum(np.exp(s) - K, 0))

    return  $\beta^{**n} * expectation$ 

```

```

%%time
compute_call_price()

```

```

CPU times: user 6.43 ms, sys: 0 ns, total: 6.43 ms
Wall time: 6.32 ms

```

```

543.9459309244429

```

Notice that this version is faster than the one using a Python loop.

Now let's try with larger  $M$  to get a more accurate calculation.

```

%%time
compute_call_price(M=10_000_000)

```

```

CPU times: user 6.24 s, sys: 861 ms, total: 7.1 s
Wall time: 7.1 s

```

```

847.2356604248002

```

---

**Exercise 17.5.2**

Consider that a European call option may be written on an underlying with spot price of \$100 and a knockout barrier of \$120.

This option behaves in every way like a vanilla European call, except if the spot price ever moves above \$120, the option “knocks out” and the contract is null and void.

Note that the option does not reactivate if the spot price falls below \$120 again.

Use the dynamics defined in (17.1) to price the European call option.

---

**Solution to Exercise 17.5.2**

```

μ  = 0.0001
ρ  = 0.1
ν  = 0.001
S0 = 10
h0 = 0
K = 100
n = 10
β = 0.95
bp = 120

```

```

def compute_call_price_with_barrier(β=β,
                                    μ=μ,
                                    S0=S0,
                                    h0=h0,
                                    K=K,
                                    n=n,
                                    ρ=ρ,
                                    ν=ν,
                                    bp=bp,
                                    M=50_000):
    current_sum = 0.0
    # For each sample path
    for m in range(M):
        s = np.log(S0)
        h = h0
        payoff = 0
        option_is_null = False
        # Simulate forward in time
        for t in range(n):
            s = s + μ + np.exp(h) * randn()
            h = ρ * h + ν * randn()
            if np.exp(s) > bp:
                payoff = 0
                option_is_null = True
                break

        if not option_is_null:
            payoff = np.maximum(np.exp(s) - K, 0)
            # And add the payoff to current_sum
        current_sum += payoff

    return β**n * current_sum / M

```

```
%time compute_call_price_with_barrier()
```

```

CPU times: user 1.44 s, sys: 97 µs, total: 1.44 s
Wall time: 1.44 s

```

```
0.040712943176047345
```

Let's look at the vectorized version which is faster than using Python loops.

```
def compute_call_price_with_barrier_vector(β=β,
                                         μ=μ,
                                         S0=S0,
                                         h0=h0,
                                         K=K,
                                         n=n,
                                         ρ=ρ,
                                         v=v,
                                         bp=bp,
                                         M=50_000):
    s = np.full(M, np.log(S0))
    h = np.full(M, h0)
    option_is_null = np.full(M, False)
    for t in range(n):
        Z = np.random.randn(2, M)
        s = s + μ + np.exp(h) * Z[0, :]
        h = ρ * h + v * Z[1, :]
        # Mark all the options null where S_n > barrier price
        option_is_null = np.where(np.exp(s) > bp, True, option_is_null)

    # mark payoff as 0 in the indices where options are null
    payoff = np.where(option_is_null, 0, np.maximum(np.exp(s) - K, 0))
    expectation = np.mean(payoff)
    return β**n * expectation
```

```
%time compute_call_price_with_barrier_vector()
```

```
CPU times: user 32.5 ms, sys: 26 µs, total: 32.5 ms
Wall time: 32.4 ms
```

```
0.037193796496269735
```



## HEAVY-TAILED DISTRIBUTIONS

### Contents

- *Heavy-Tailed Distributions*
  - *Overview*
  - *Visual comparisons*
  - *Heavy tails in economic cross-sections*
  - *Failure of the LLN*
  - *Why do heavy tails matter?*
  - *Classifying tail properties*
  - *Further reading*
  - *Exercises*

In addition to what's in Anaconda, this lecture will need the following libraries:

```
!pip install --upgrade yfinance quantecon pandas_datareader interpolation
```

We use the following imports.

```
import matplotlib.pyplot as plt
import numpy as np
import quantecon as qe
import yfinance as yf
import pandas as pd
import pandas_datareader.data as web
import statsmodels.api as sm

from interpolation import interp
from pandas_datareader import wb
from scipy.stats import norm, cauchy
from pandas.plotting import register_matplotlib_converters
register_matplotlib_converters()
```

```
/usr/share/miniconda3/envs/quantecon/lib/python3.11/site-packages/numba/core/
decorators.py:262: NumbaDeprecationWarning: numba.generated_jit is deprecated.
Please see the documentation at: https://numba.readthedocs.io/en/stable/
reference/deprecation.html#deprecation-of-generated-jit for more information and
advice on a suitable replacement.
```

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```
warnings.warn(msg, NumbaDeprecationWarning)
```

## 18.1 Overview

In this section we give some motivation for the lecture.

### 18.1.1 Introduction: light tails

Most *commonly used probability distributions* in classical statistics and the natural sciences have “light tails.”

To explain this concept, let’s look first at examples.

The classic example is the **normal distribution**, which has density

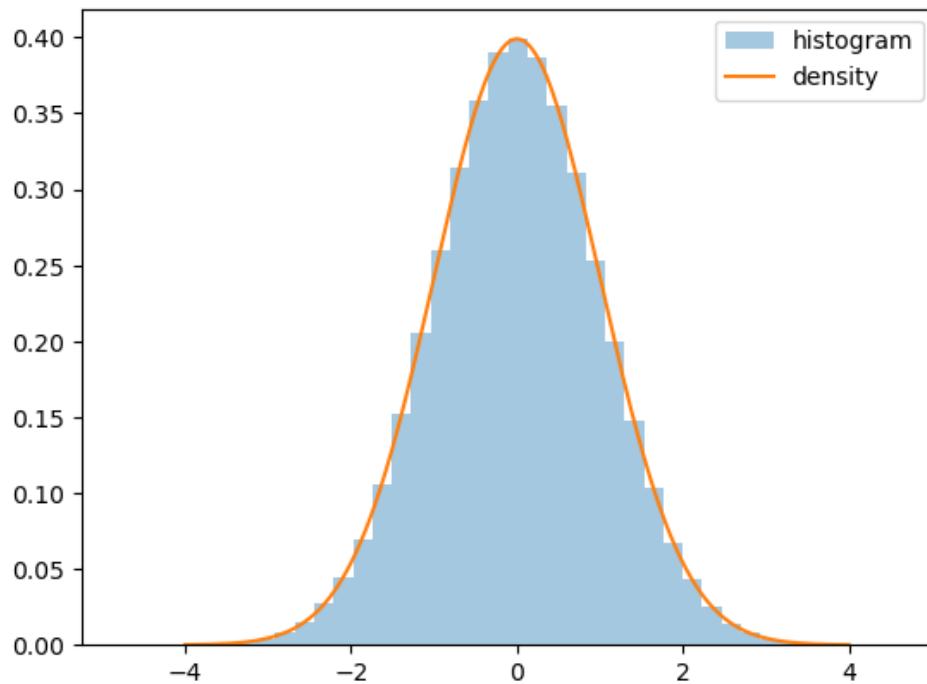
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad (-\infty < x < \infty)$$

The two parameters  $\mu$  and  $\sigma$  are the mean and standard deviation respectively.

As  $x$  deviates from  $\mu$ , the value of  $f(x)$  goes to zero extremely quickly.

We can see this when we plot the density and show a histogram of observations, as with the following code (which assumes  $\mu = 0$  and  $\sigma = 1$ ).

```
fig, ax = plt.subplots()
X = norm.rvs(size=1_000_000)
ax.hist(X, bins=40, alpha=0.4, label='histogram', density=True)
x_grid = np.linspace(-4, 4, 400)
ax.plot(x_grid, norm.pdf(x_grid), label='density')
ax.legend()
plt.show()
```



Notice how

- the density's tails converge quickly to zero in both directions and
- even with 1,000,000 draws, we get no very large or very small observations.

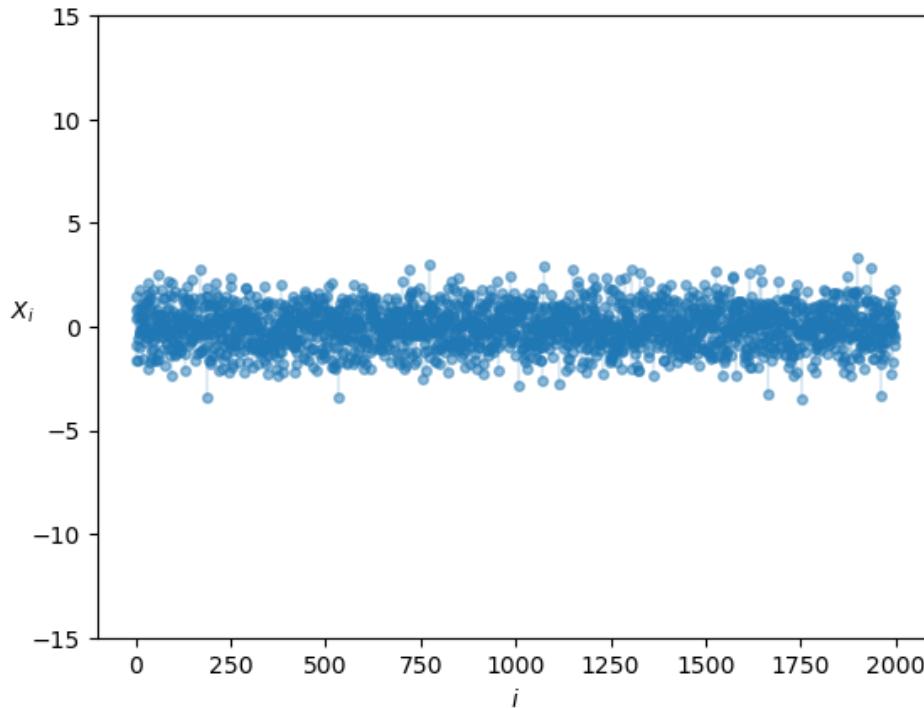
We can see the last point more clearly by executing

```
X.min(), X.max()
```

```
(-4.7806512076216565, 4.568888366413363)
```

Here's another view of draws from the same distribution:

```
n = 2000
fig, ax = plt.subplots()
data = norm.rvs(size=n)
ax.plot(list(range(n)), data, linestyle=' ', marker='o', alpha=0.5, ms=4)
ax.vlines(list(range(n)), 0, data, lw=0.2)
ax.set_xlim(-15, 15)
ax.set_xlabel('$i$')
ax.set_ylabel('$X_i$', rotation=0)
plt.show()
```



We have plotted each individual draw  $X_i$  against  $i$ .

None are very large or very small.

In other words, extreme observations are rare and draws tend not to deviate too much from the mean.

Putting this another way, light-tailed distributions are those that rarely generate extreme values.

(A more formal definition is given below.)

Many statisticians and econometricians use rules of thumb such as “outcomes more than four or five standard deviations from the mean can safely be ignored.”

But this is only true when distributions have light tails.

### 18.1.2 When are light tails valid?

In probability theory and in the real world, many distributions are light-tailed.

For example, human height is light-tailed.

Yes, it's true that we see some very tall people.

- For example, basketballer Sun Mingming is 2.32 meters tall

But have you ever heard of someone who is 20 meters tall? Or 200? Or 2000?

Have you ever wondered why not?

After all, there are 8 billion people in the world!

In essence, the reason we don't see such draws is that the distribution of human height has very light tails.

In fact the distribution of human height obeys a bell-shaped curve similar to the normal distribution.

### 18.1.3 Returns on assets

But what about economic data?

Let's look at some financial data first.

Our aim is to plot the daily change in the price of Amazon (AMZN) stock for the period from 1st January 2015 to 1st July 2022.

This equates to daily returns if we set dividends aside.

The code below produces the desired plot using Yahoo financial data via the `yfinance` library.

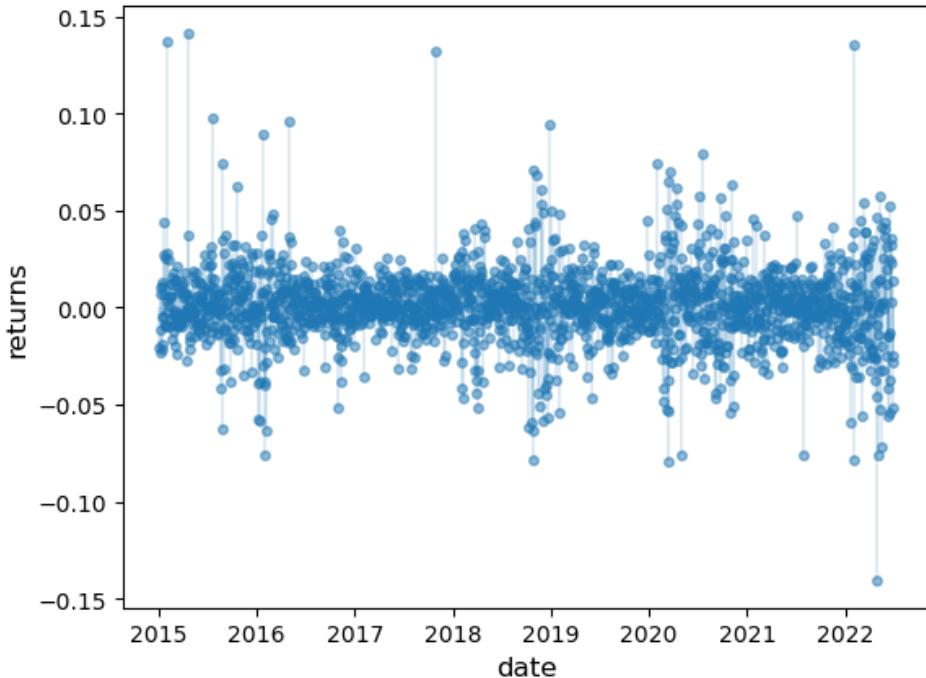
```
s = yf.download('AMZN', '2015-1-1', '2022-7-1')['Adj Close']
r = s.pct_change()

fig, ax = plt.subplots()

ax.plot(r, linestyle='', marker='o', alpha=0.5, ms=4)
ax.vlines(r.index, 0, r.values, lw=0.2)
ax.set_ylabel('returns', fontsize=12)
ax.set_xlabel('date', fontsize=12)

plt.show()
```

[\*\*\*\*\*100%\*\*\*\*\*] 1 of 1 completed



This data looks different to the draws from the normal distribution we saw above.

Several of observations are quite extreme.

We get a similar picture if we look at other assets, such as Bitcoin

```

s = yf.download('BTC-USD', '2015-1-1', '2022-7-1')['Adj Close']
r = s.pct_change()

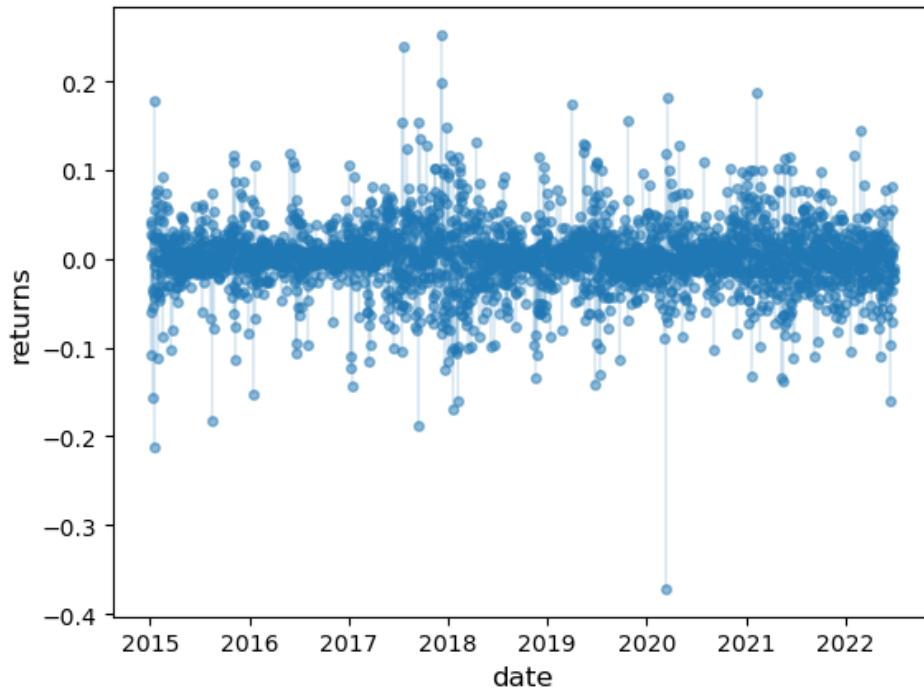
fig, ax = plt.subplots()

ax.plot(r, linestyle='', marker='o', alpha=0.5, ms=4)
ax.vlines(r.index, 0, r.values, lw=0.2)
ax.set_ylabel('returns', fontsize=12)
ax.set_xlabel('date', fontsize=12)

plt.show()

```

```
[*****100%*****] 1 of 1 completed
```

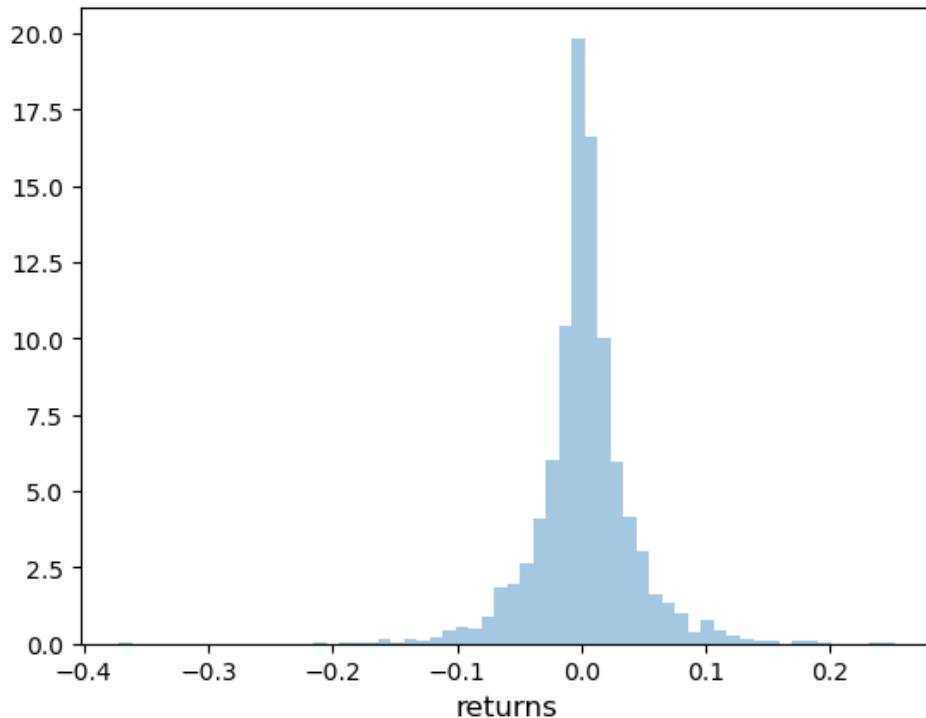


The histogram also looks different to the histogram of the normal distribution:

```

fig, ax = plt.subplots()
ax.hist(r, bins=60, alpha=0.4, label='bitcoin returns', density=True)
ax.set_xlabel('returns', fontsize=12)
plt.show()

```



If we look at higher frequency returns data (e.g., tick-by-tick), we often see even more extreme observations.

See, for example, [Man63] or [Rac03].

#### 18.1.4 Other data

The data we have just seen is said to be “heavy-tailed”.

With heavy-tailed distributions, extreme outcomes occur relatively frequently.

Importantly, there are many examples of heavy-tailed distributions observed in economic and financial settings!

For example, the income and the wealth distributions are heavy-tailed

- You can imagine this: most people have low or modest wealth but some people are extremely rich.

The firm size distribution is also heavy-tailed

- You can imagine this too: most firms are small but some firms are enormous.

The distribution of town and city sizes is heavy-tailed

- Most towns and cities are small but some are very large.

Later in this lecture, we examine heavy tails in these distributions.

### 18.1.5 Why should we care?

Heavy tails are common in economic data but does that mean they are important?

The answer to this question is affirmative!

When distributions are heavy-tailed, we need to think carefully about issues like

- diversification and risk
- forecasting
- taxation (across a heavy-tailed income distribution), etc.

We return to these points below.

## 18.2 Visual comparisons

Later we will provide a mathematical definition of the difference between light and heavy tails.

But for now let's do some visual comparisons to help us build intuition on the difference between these two types of distributions.

### 18.2.1 Simulations

The figure below shows a simulation.

The top two subfigures each show 120 independent draws from the normal distribution, which is light-tailed.

The bottom subfigure shows 120 independent draws from the [Cauchy distribution](#), which is heavy-tailed.

```
n = 120
np.random.seed(11)

fig, axes = plt.subplots(3, 1, figsize=(6, 12))

for ax in axes:
    ax.set_ylim((-120, 120))

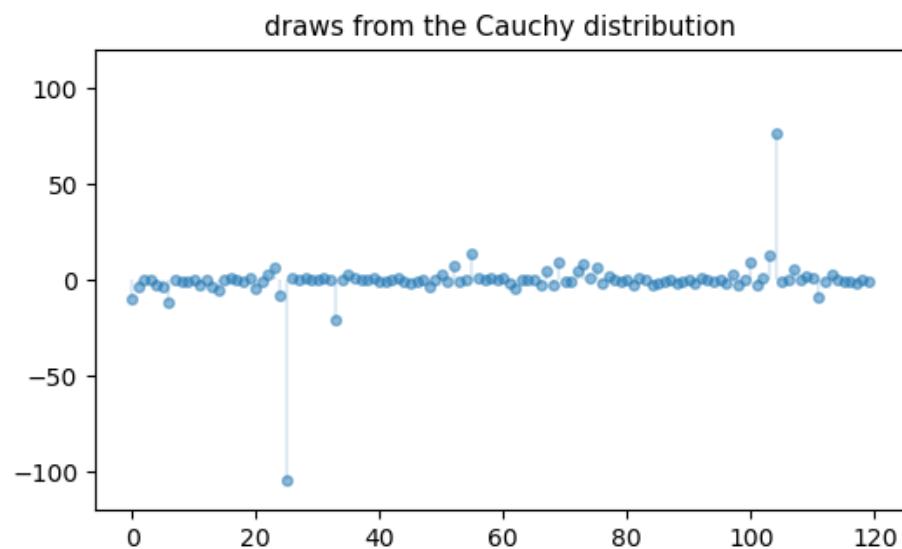
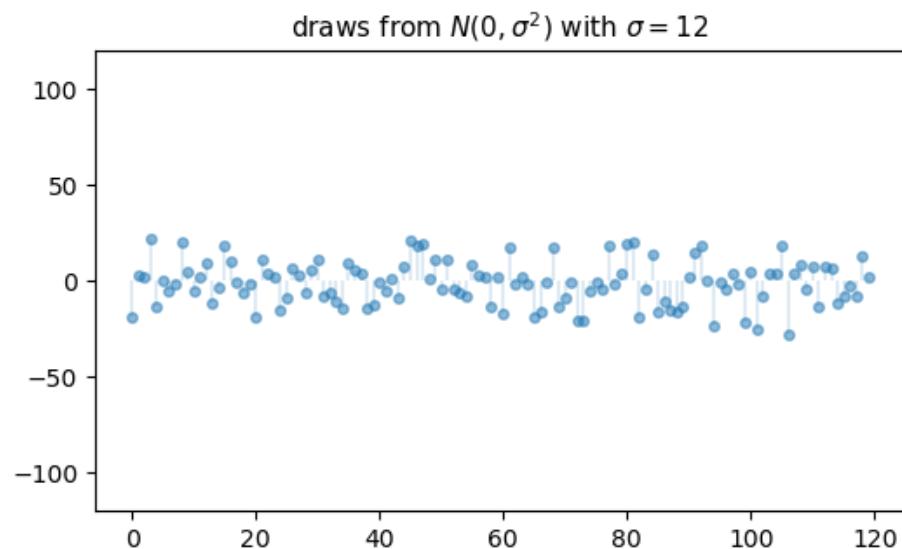
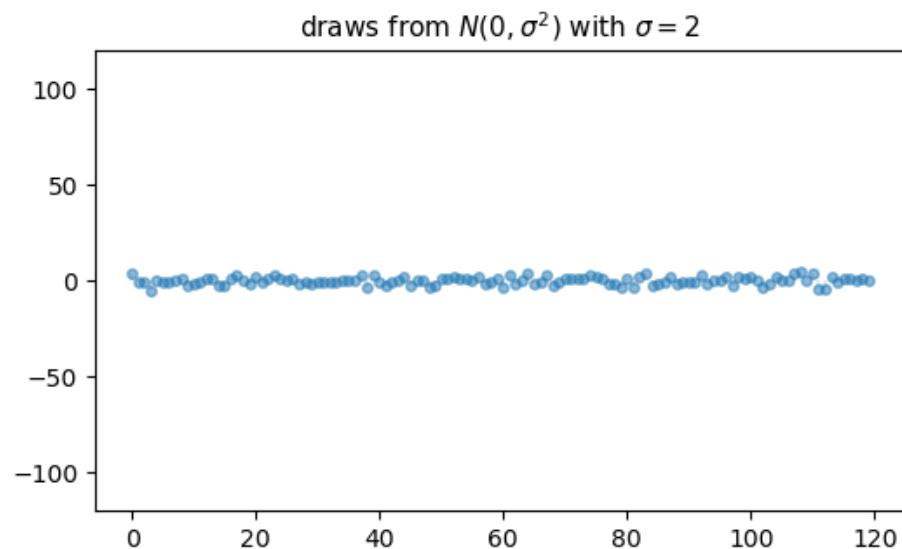
s_vals = 2, 12

for ax, s in zip(axes[:2], s_vals):
    data = np.random.randn(n) * s
    ax.plot(list(range(n)), data, linestyle='', marker='o', alpha=0.5, ms=4)
    ax.vlines(list(range(n)), 0, data, lw=0.2)
    ax.set_title(f"draws from $N(0, \sigma^2)$ with $\sigma = {s}$", fontsize=11)

ax = axes[2]
distribution = cauchy()
data = distribution.rvs(n)
ax.plot(list(range(n)), data, linestyle='', marker='o', alpha=0.5, ms=4)
ax.vlines(list(range(n)), 0, data, lw=0.2)
ax.set_title(f"draws from the Cauchy distribution", fontsize=11)

plt.subplots_adjust(hspace=0.25)

plt.show()
```



In the top subfigure, the standard deviation of the normal distribution is 2, and the draws are clustered around the mean.

In the middle subfigure, the standard deviation is increased to 12 and, as expected, the amount of dispersion rises.

The bottom subfigure, with the Cauchy draws, shows a different pattern: tight clustering around the mean for the great majority of observations, combined with a few sudden large deviations from the mean.

This is typical of a heavy-tailed distribution.

### 18.2.2 Nonnegative distributions

Let's compare some distributions that only take nonnegative values.

One is the exponential distribution, which we discussed in *our lecture on probability and distributions*.

The exponential distribution is a light-tailed distribution.

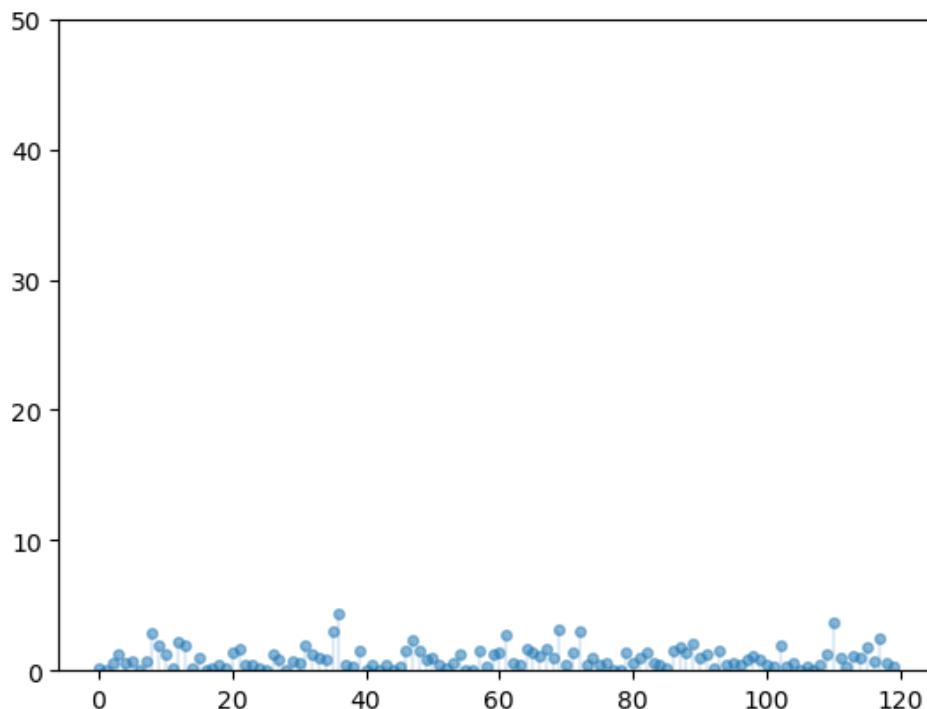
Here are some draws from the exponential distribution.

```
n = 120
np.random.seed(11)

fig, ax = plt.subplots()
ax.set_ylim((0, 50))

data = np.random.exponential(size=n)
ax.plot(list(range(n)), data, linestyle=' ', marker='o', alpha=0.5, ms=4)
ax.vlines(list(range(n)), 0, data, lw=0.2)

plt.show()
```



Another nonnegative distribution is the Pareto distribution.

If  $X$  has the Pareto distribution, then there are positive constants  $\bar{x}$  and  $\alpha$  such that

$$\mathbb{P}\{X > x\} = \begin{cases} (\bar{x}/x)^\alpha & \text{if } x \geq \bar{x} \\ 1 & \text{if } x < \bar{x} \end{cases} \quad (18.1)$$

The parameter  $\alpha$  is called the **tail index** and  $\bar{x}$  is called the **minimum**.

The Pareto distribution is a heavy-tailed distribution.

One way that the Pareto distribution arises is as the exponential of an exponential random variable.

In particular, if  $X$  is exponentially distributed with rate parameter  $\alpha$ , then

$$Y = \bar{x} \exp(X)$$

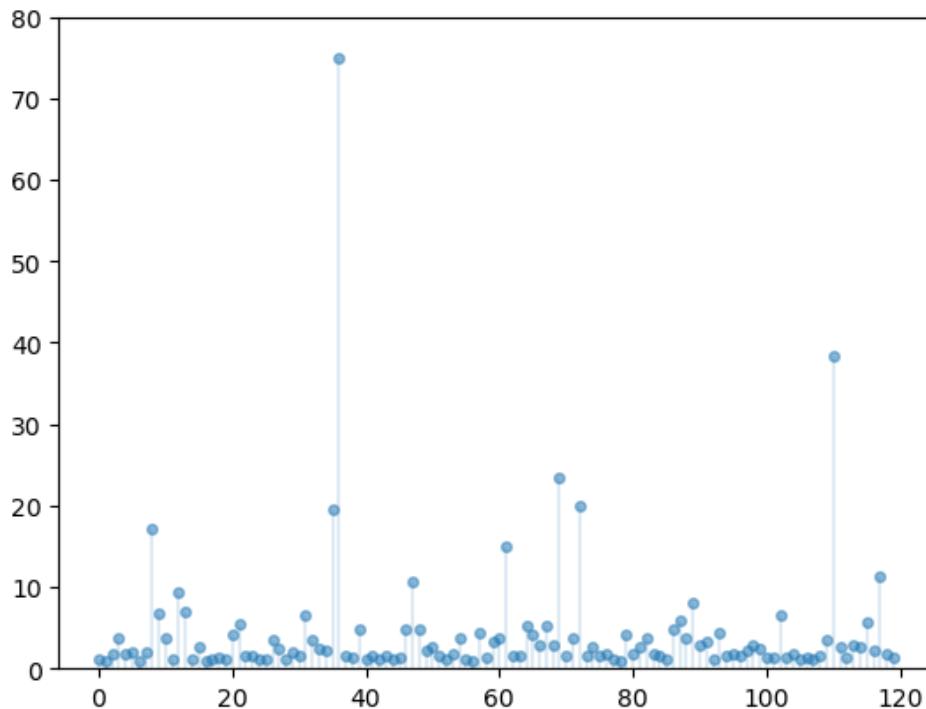
is Pareto-distributed with minimum  $\bar{x}$  and tail index  $\alpha$ .

Here are some draws from the Pareto distribution with tail index 1 and minimum 1.

```
n = 120
np.random.seed(11)

fig, ax = plt.subplots()
ax.set_ylim((0, 80))
exponential_data = np.random.exponential(size=n)
pareto_data = np.exp(exponential_data)
ax.plot(list(range(n)), pareto_data, linestyle='', marker='o', alpha=0.5, ms=4)
ax.vlines(list(range(n)), 0, pareto_data, lw=0.2)

plt.show()
```



Notice how extreme outcomes are more common.

### 18.2.3 Counter CDFs

For nonnegative random variables, one way to visualize the difference between light and heavy tails is to look at the **counter CDF** (CCDF).

For a random variable  $X$  with CDF  $F$ , the CCDF is the function

$$G(x) := 1 - F(x) = \mathbb{P}\{X > x\}$$

(Some authors call  $G$  the “survival” function.)

The CCDF shows how fast the upper tail goes to zero as  $x \rightarrow \infty$ .

If  $X$  is exponentially distributed with rate parameter  $\alpha$ , then the CCDF is

$$G_E(x) = \exp(-\alpha x)$$

This function goes to zero relatively quickly as  $x$  gets large.

The standard Pareto distribution, where  $\bar{x} = 1$ , has CCDF

$$G_P(x) = x^{-\alpha}$$

This function goes to zero as  $x \rightarrow \infty$ , but much slower than  $G_E$ .

---

#### Exercise 18.2.1

Show how the CCDF of the standard Pareto distribution can be derived from the CCDF of the exponential distribution.

---



---

#### Solution to Exercise 18.2.1

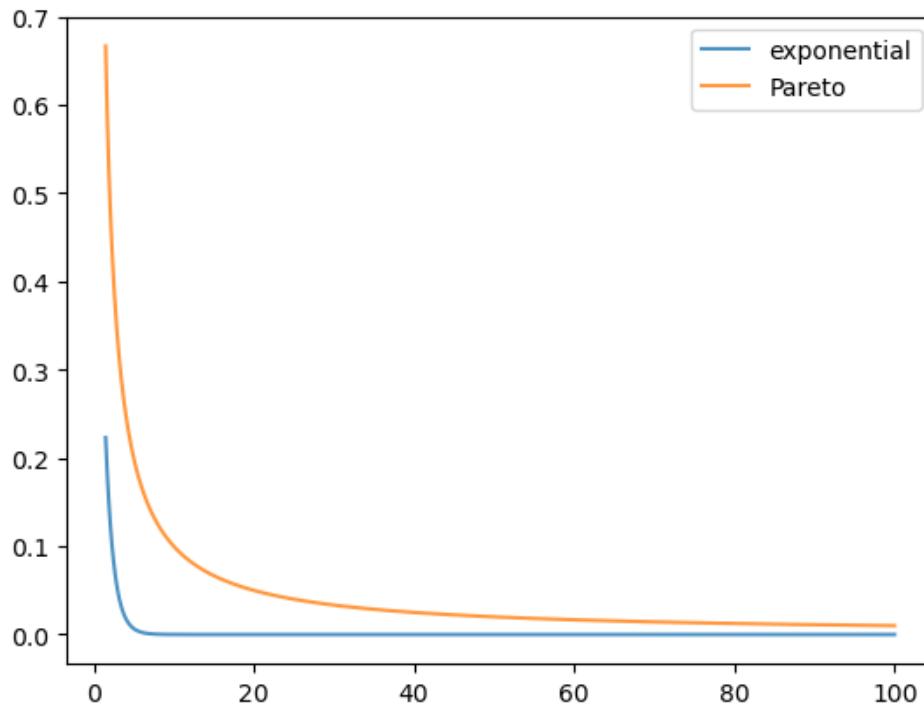
Letting  $G_E$  and  $G_P$  be defined as above, letting  $X$  be exponentially distributed with rate parameter  $\alpha$ , and letting  $Y = \exp(X)$ , we have

$$\begin{aligned} G_P(y) &= \mathbb{P}\{Y > y\} \\ &= \mathbb{P}\{\exp(X) > y\} \\ &= \mathbb{P}\{X > \ln y\} \\ &= G_E(\ln y) \\ &= \exp(-\alpha \ln y) \\ &= y^{-\alpha} \end{aligned}$$


---

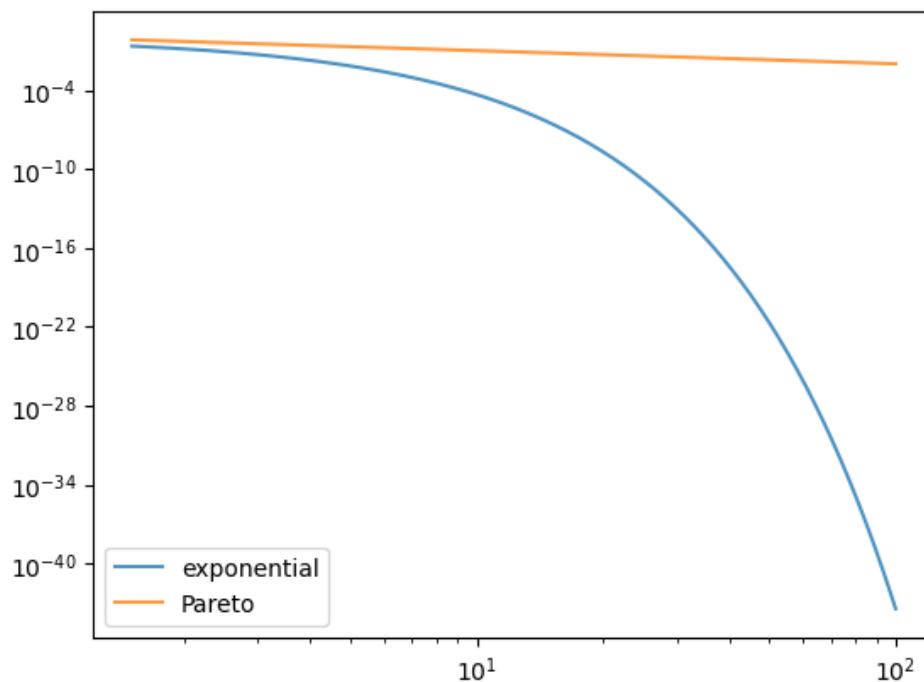
Here's a plot that illustrates how  $G_E$  goes to zero faster than  $G_P$ .

```
x = np.linspace(1.5, 100, 1000)
fig, ax = plt.subplots()
alpha = 1.0
ax.plot(x, np.exp(-alpha * x), label='exponential', alpha=0.8)
ax.plot(x, x**(-alpha), label='Pareto', alpha=0.8)
ax.legend()
plt.show()
```



Here's a log-log plot of the same functions, which makes visual comparison easier.

```
fig, ax = plt.subplots()
alpha = 1.0
ax.loglog(x, np.exp(-alpha * x), label='exponential', alpha=0.8)
ax.loglog(x, x**(-alpha), label='Pareto', alpha=0.8)
ax.legend()
plt.show()
```



In the log-log plot, the Pareto CCDF is linear, while the exponential one is concave.

This idea is often used to separate light- and heavy-tailed distributions in visualisations — we return to this point below.

### 18.2.4 Empirical CCDFs

The sample counterpart of the CCDF function is the **empirical CCDF**.

Given a sample  $x_1, \dots, x_n$ , the empirical CCDF is given by

$$\hat{G}(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{x_i > x\}$$

Thus,  $\hat{G}(x)$  shows the fraction of the sample that exceeds  $x$ .

Here's a figure containing some empirical CCDFs from simulated data.

```
def eccdf(x, data):
    "Simple empirical CCDF function."
    return np.mean(data > x)

x_grid = np.linspace(1, 1000, 1000)
sample_size = 1000
np.random.seed(13)
z = np.random.randn(sample_size)

data_1 = np.random.exponential(size=sample_size)
data_2 = np.exp(z)
data_3 = np.exp(np.random.exponential(size=sample_size))

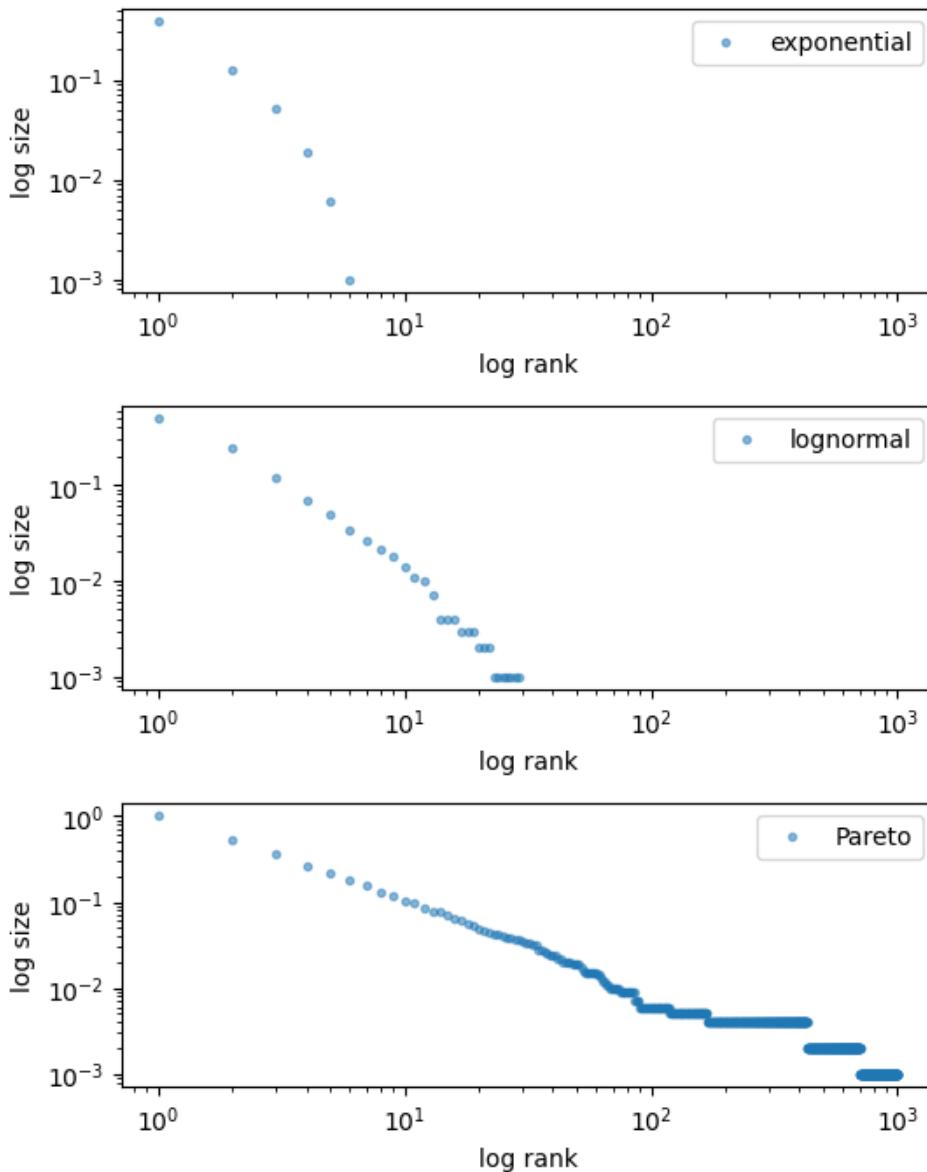
data_list = [data_1, data_2, data_3]

fig, axes = plt.subplots(3, 1, figsize=(6, 8))
axes = axes.flatten()
labels = ['exponential', 'lognormal', 'Pareto']

for data, label, ax in zip(data_list, labels, axes):
    ax.loglog(x_grid, [eccdf(x, data) for x in x_grid],
              'o', markersize=3.0, alpha=0.5, label=label)
    ax.set_xlabel("log rank")
    ax.set_ylabel("log size")

    ax.legend()

fig.subplots_adjust(hspace=0.4)
plt.show()
```



As with the CCDF, the empirical CCDF from the Pareto distributions is approximately linear in a log-log plot.

We will use this idea below when we look at real data.

### 18.2.5 Power laws

One specific class of heavy-tailed distributions has been found repeatedly in economic and social phenomena: the class of so-called power laws.

A random variable  $X$  is said to have a **power law** if, for some  $\alpha > 0$ ,

$$\mathbb{P}\{X > x\} \approx x^{-\alpha} \quad \text{when } x \text{ is large}$$

We can write this more mathematically as

$$\lim_{x \rightarrow \infty} x^\alpha \mathbb{P}\{X > x\} = c \quad \text{for some } c > 0 \tag{18.2}$$

It is also common to say that a random variable  $X$  with this property has a **Pareto tail** with **tail index**  $\alpha$ .

Notice that every Pareto distribution with tail index  $\alpha$  has a **Pareto tail** with **tail index**  $\alpha$ .

We can think of power laws as a generalization of Pareto distributions.

They are distributions that resemble Pareto distributions in their upper right tail.

Another way to think of power laws is a set of distributions with a specific kind of (very) heavy tail.

## 18.3 Heavy tails in economic cross-sections

As mentioned above, heavy tails are pervasive in economic data.

In fact power laws seem to be very common as well.

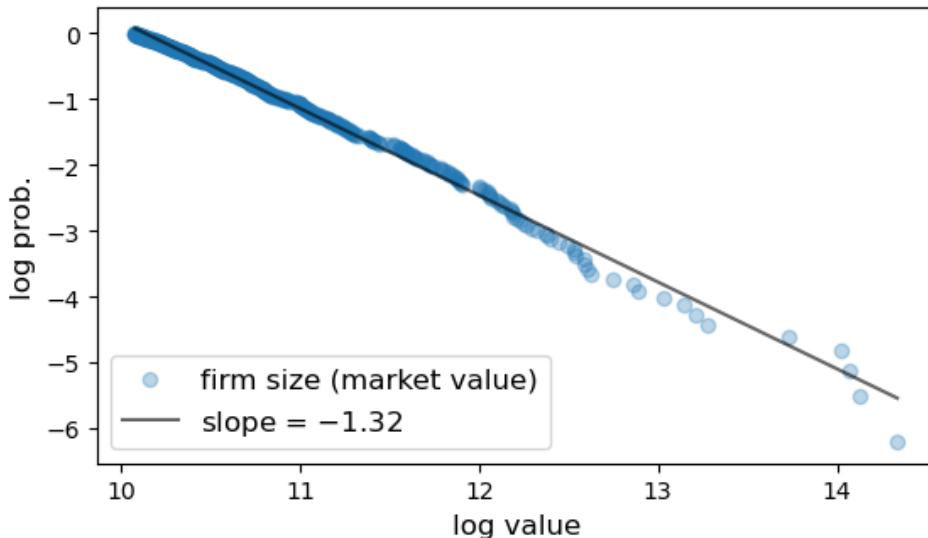
We now illustrate this by showing the empirical CCDF of heavy tails.

All plots are in log-log, so that a power law shows up as a linear log-log plot, at least in the upper tail.

We hide the code that generates the figures, which is somewhat complex, but readers are of course welcome to explore the code (perhaps after examining the figures).

### 18.3.1 Firm size

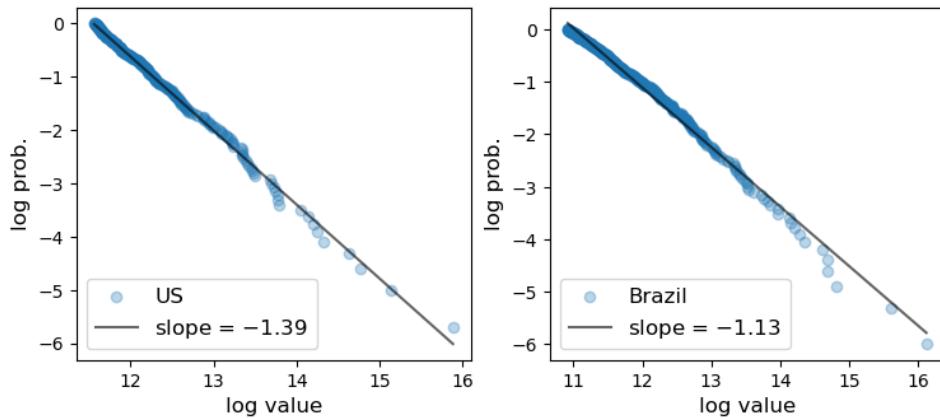
Here is a plot of the firm size distribution for the largest 500 firms in 2020 taken from Forbes Global 2000.



### 18.3.2 City size

Here are plots of the city size distribution for the US and Brazil in 2023 from world population review.

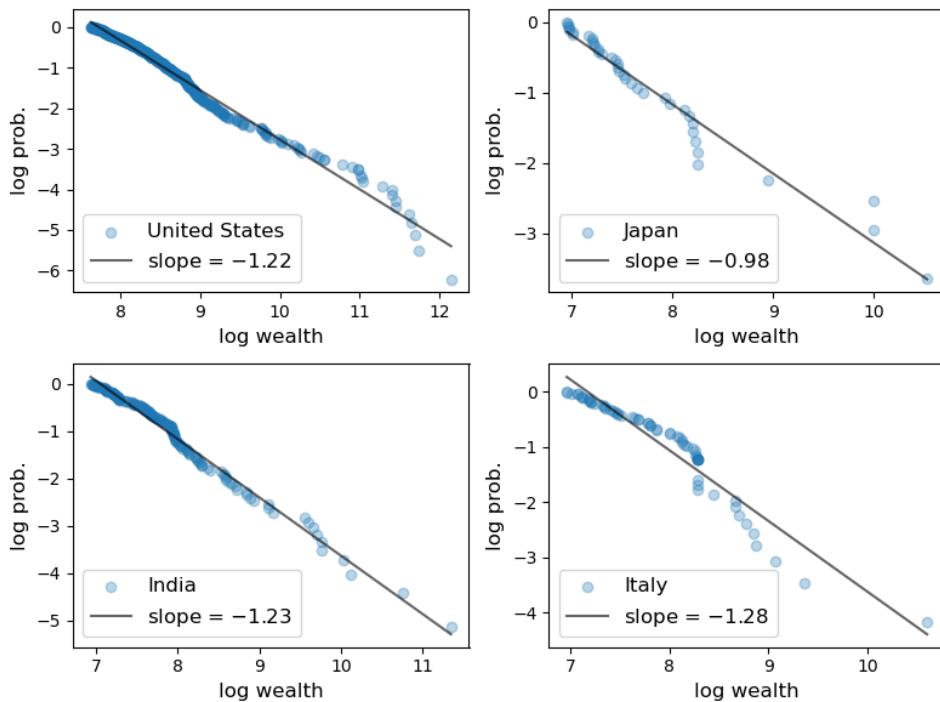
The size is measured by population.



### 18.3.3 Wealth

Here is a plot of the upper tail (top 500) of the wealth distribution.

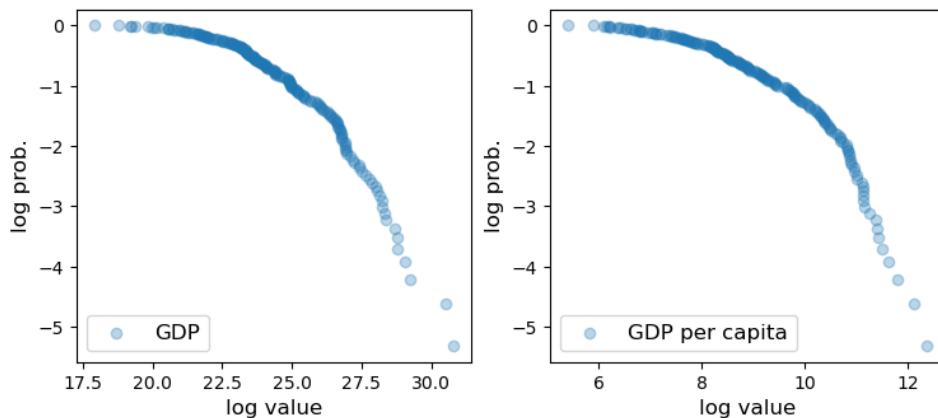
The data is from the Forbes Billionaires list in 2020.



### 18.3.4 GDP

Of course, not all cross-sectional distributions are heavy-tailed.

Here we show cross-country per capita GDP.



The plot is concave rather than linear, so the distribution has light tails.

One reason is that this is data on an aggregate variable, which involves some averaging in its definition.

Averaging tends to eliminate extreme outcomes.

## 18.4 Failure of the LLN

One impact of heavy tails is that sample averages can be poor estimators of the underlying mean of the distribution.

To understand this point better, recall [our earlier discussion](#) of the Law of Large Numbers, which considered IID  $X_1, \dots, X_n$  with common distribution  $F$

If  $\mathbb{E}|X_i|$  is finite, then the sample mean  $\bar{X}_n := \frac{1}{n} \sum_{i=1}^n X_i$  satisfies

$$\mathbb{P}\{\bar{X}_n \rightarrow \mu \text{ as } n \rightarrow \infty\} = 1 \quad (18.3)$$

where  $\mu := \mathbb{E}X_i = \int xF(dx)$  is the common mean of the sample.

The condition  $\mathbb{E}|X_i| = \int |x|F(dx) < \infty$  holds in most cases but can fail if the distribution  $F$  is very heavy tailed.

For example, it fails for the Cauchy distribution.

Let's have a look at the behavior of the sample mean in this case, and see whether or not the LLN is still valid.

```
from scipy.stats import cauchy

np.random.seed(1234)
N = 1_000

distribution = cauchy()

fig, ax = plt.subplots()
data = distribution.rvs(N)

# Compute sample mean at each n
```

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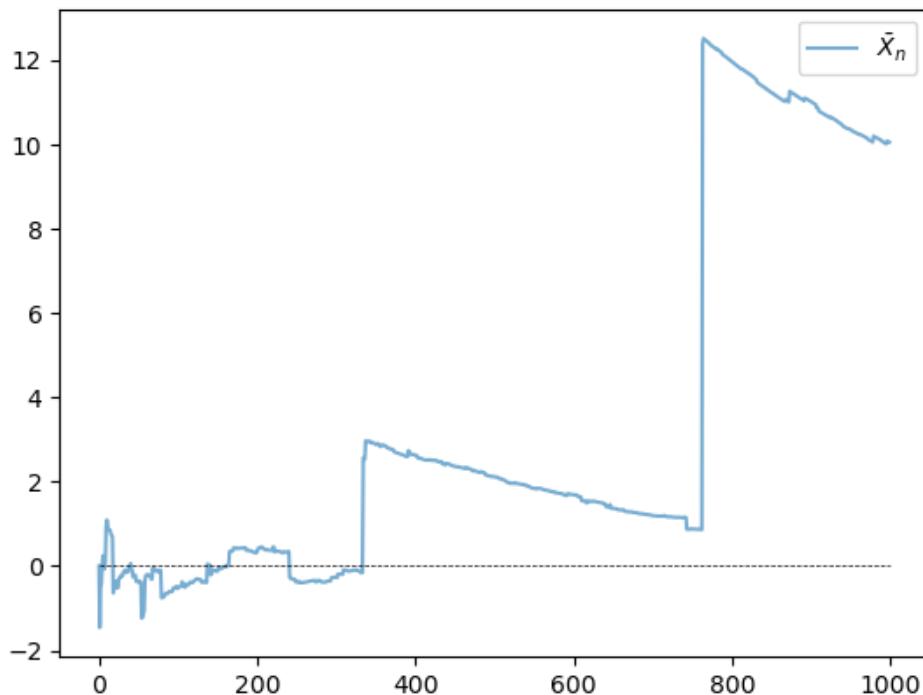
```

sample_mean = np.empty(N)
for n in range(1, N):
    sample_mean[n] = np.mean(data[:n])

# Plot
ax.plot(range(N), sample_mean, alpha=0.6, label='$\bar{X}_n$')
ax.plot(range(N), np.zeros(N), 'k--', lw=0.5)
ax.legend()

plt.show()

```



The sequence shows no sign of converging.

We return to this point in the exercises.

## 18.5 Why do heavy tails matter?

We have now seen that

1. heavy tails are frequent in economics and
2. the Law of Large Numbers fails when tails are very heavy.

But what about in the real world? Do heavy tails matter?

Let's briefly discuss why they do.

### 18.5.1 Diversification

One of the most important ideas in investing is using diversification to reduce risk.

This is a very old idea — consider, for example, the expression “don’t put all your eggs in one basket”.

To illustrate, consider an investor with one dollar of wealth and a choice over  $n$  assets with payoffs  $X_1, \dots, X_n$ .

Suppose that returns on distinct assets are independent and each return has mean  $\mu$  and variance  $\sigma^2$ .

If the investor puts all wealth in one asset, say, then the expected payoff of the portfolio is  $\mu$  and the variance is  $\sigma^2$ .

If instead the investor puts share  $1/n$  of her wealth in each asset, then the portfolio payoff is

$$Y_n = \sum_{i=1}^n \frac{X_i}{n} = \frac{1}{n} \sum_{i=1}^n X_i.$$

Try computing the mean and variance.

You will find that

- The mean is unchanged at  $\mu$ , while
- the variance of the portfolio has fallen to  $\sigma^2/n$ .

Diversification reduces risk, as expected.

But there is a hidden assumption here: the variance of returns is finite.

If the distribution is heavy-tailed and the variance is infinite, then this logic is incorrect.

For example, we saw above that if every  $X_i$  is Cauchy, then so is  $Y_n$ .

This means that diversification doesn’t help at all!

### 18.5.2 Fiscal policy

The heaviness of the tail in the wealth distribution matters for taxation and redistribution policies.

The same is true for the income distribution.

For example, the heaviness of the tail of the income distribution helps determine *how much revenue a given tax policy will raise*.

## 18.6 Classifying tail properties

Up until now we have discussed light and heavy tails without any mathematical definitions.

Let’s now rectify this.

We will focus our attention on the right hand tails of nonnegative random variables and their distributions.

The definitions for left hand tails are very similar and we omit them to simplify the exposition.

### 18.6.1 Light and heavy tails

A distribution  $F$  with density  $f$  on  $\mathbb{R}_+$  is called **heavy-tailed** if

$$\int_0^\infty \exp(tx)f(x)dx = \infty \text{ for all } t > 0. \quad (18.4)$$

We say that a nonnegative random variable  $X$  is **heavy-tailed** if its density is heavy-tailed.

This is equivalent to stating that its **moment generating function**  $m(t) := \mathbb{E} \exp(tX)$  is infinite for all  $t > 0$ .

For example, the [log-normal distribution](#) is heavy-tailed because its moment generating function is infinite everywhere on  $(0, \infty)$ .

The Pareto distribution is also heavy-tailed.

A distribution  $F$  on  $\mathbb{R}_+$  is called **light-tailed** if it is not heavy-tailed.

A nonnegative random variable  $X$  is **light-tailed** if its distribution  $F$  is light-tailed.

For example, every random variable with bounded support is light-tailed. (Why?)

As another example, if  $X$  has the [exponential distribution](#), with cdf  $F(x) = 1 - \exp(-\lambda x)$  for some  $\lambda > 0$ , then its moment generating function is

$$m(t) = \frac{\lambda}{\lambda - t} \text{ when } t < \lambda$$

In particular,  $m(t)$  is finite whenever  $t < \lambda$ , so  $X$  is light-tailed.

One can show that if  $X$  is light-tailed, then all of its [moments](#) are finite.

Conversely, if some moment is infinite, then  $X$  is heavy-tailed.

The latter condition is not necessary, however.

For example, the lognormal distribution is heavy-tailed but every moment is finite.

## 18.7 Further reading

For more on heavy tails in the wealth distribution, see e.g., [\[Vil96\]](#) and [\[BB18\]](#).

For more on heavy tails in the firm size distribution, see e.g., [\[Axt01\]](#), [\[Gab16\]](#).

For more on heavy tails in the city size distribution, see e.g., [\[RRGM11\]](#), [\[Gab16\]](#).

There are other important implications of heavy tails, aside from those discussed above.

For example, heavy tails in income and wealth affect productivity growth, business cycles, and political economy.

For further reading, see, for example, [\[AR02\]](#), [\[GSS03\]](#), [\[BEGS18\]](#) or [\[AKM+18\]](#).

## 18.8 Exercises

---

### Exercise 18.8.1

Prove: If  $X$  has a Pareto tail with tail index  $\alpha$ , then  $\mathbb{E}[X^r] = \infty$  for all  $r \geq \alpha$ .

---



---

### Solution to Exercise 18.8.1

Let  $X$  have a Pareto tail with tail index  $\alpha$  and let  $F$  be its cdf.

Fix  $r \geq \alpha$ .

In view of (18.2), we can take positive constants  $b$  and  $\bar{x}$  such that

$$\mathbb{P}\{X > x\} \geq bx^{-\alpha} \text{ whenever } x \geq \bar{x}$$

But then

$$\mathbb{E}X^r = r \int_0^\infty x^{r-1} \mathbb{P}\{X > x\} dx \geq r \int_0^{\bar{x}} x^{r-1} \mathbb{P}\{X > x\} dx + r \int_{\bar{x}}^\infty x^{r-1} bx^{-\alpha} dx.$$

We know that  $\int_{\bar{x}}^\infty x^{r-\alpha-1} dx = \infty$  whenever  $r - \alpha - 1 \geq -1$ .

Since  $r \geq \alpha$ , we have  $\mathbb{E}X^r = \infty$ .

---



---

### Exercise 18.8.2

Repeat exercise 1, but replace the three distributions (two normal, one Cauchy) with three Pareto distributions using different choices of  $\alpha$ .

For  $\alpha$ , try 1.15, 1.5 and 1.75.

Use `np.random.seed(11)` to set the seed.

---



---

### Solution to Exercise 18.8.2

```
from scipy.stats import pareto

np.random.seed(11)

n = 120
alphas = [1.15, 1.50, 1.75]

fig, axes = plt.subplots(3, 1, figsize=(6, 8))

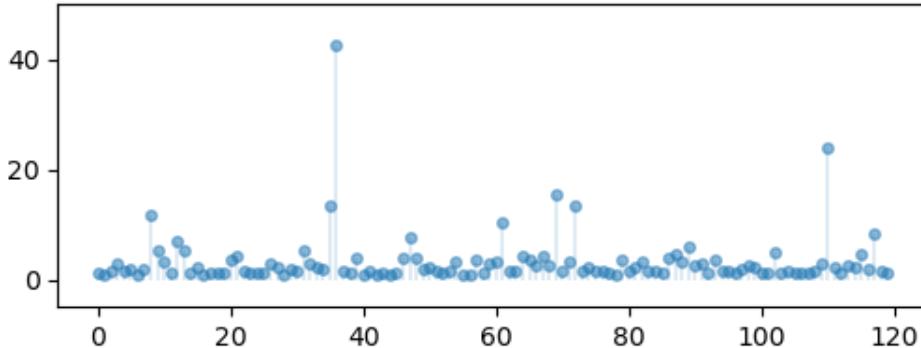
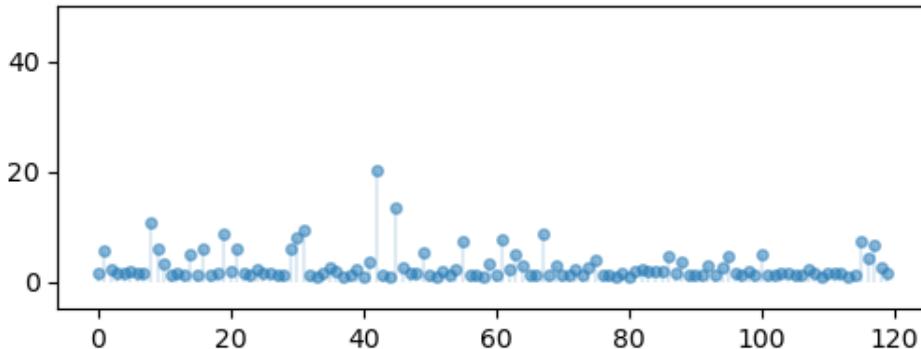
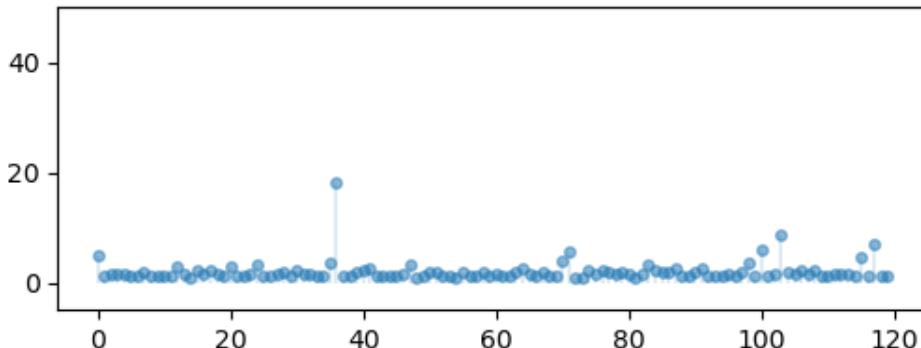
for (a, ax) in zip(alphas, axes):
    ax.set_ylimit((-5, 50))
    data = pareto.rvs(size=n, scale=1, b=a)
    ax.plot(list(range(n)), data, linestyle=':', marker='o', alpha=0.5, ms=4)
    ax.vlines(list(range(n)), 0, data, lw=0.2)
    ax.set_title(f"Pareto draws with \u03b1 = {a}", fontsize=11)

plt.subplots_adjust(hspace=0.4)
```

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```
plt.show()
```

Pareto draws with  $\alpha = 1.15$ Pareto draws with  $\alpha = 1.5$ Pareto draws with  $\alpha = 1.75$ 


---

### Exercise 18.8.3

There is an ongoing argument about whether the firm size distribution should be modeled as a Pareto distribution or a lognormal distribution (see, e.g., [FDGA+04], [KLS18] or [ST19]).

This sounds esoteric but has real implications for a variety of economic phenomena.

To illustrate this fact in a simple way, let us consider an economy with 100,000 firms, an interest rate of  $r = 0.05$  and

a corporate tax rate of 15%.

Your task is to estimate the present discounted value of projected corporate tax revenue over the next 10 years.

Because we are forecasting, we need a model.

We will suppose that

1. the number of firms and the firm size distribution (measured in profits) remain fixed and
2. the firm size distribution is either lognormal or Pareto.

Present discounted value of tax revenue will be estimated by

1. generating 100,000 draws of firm profit from the firm size distribution,
2. multiplying by the tax rate, and
3. summing the results with discounting to obtain present value.

The Pareto distribution is assumed to take the form (18.1) with  $\bar{x} = 1$  and  $\alpha = 1.05$ .

(The value the tail index  $\alpha$  is plausible given the data [Gab16].)

To make the lognormal option as similar as possible to the Pareto option, choose its parameters such that the mean and median of both distributions are the same.

Note that, for each distribution, your estimate of tax revenue will be random because it is based on a finite number of draws.

To take this into account, generate 100 replications (evaluations of tax revenue) for each of the two distributions and compare the two samples by

- producing a [violin plot](#) visualizing the two samples side-by-side and
- printing the mean and standard deviation of both samples.

For the seed use `np.random.seed(1234)`.

What differences do you observe?

(Note: a better approach to this problem would be to model firm dynamics and try to track individual firms given the current distribution. We will discuss firm dynamics in later lectures.)

---

### Solution to Exercise 18.8.3

To do the exercise, we need to choose the parameters  $\mu$  and  $\sigma$  of the lognormal distribution to match the mean and median of the Pareto distribution.

Here we understand the lognormal distribution as that of the random variable  $\exp(\mu + \sigma Z)$  when  $Z$  is standard normal.

The mean and median of the Pareto distribution (18.1) with  $\bar{x} = 1$  are

$$\text{mean} = \frac{\alpha}{\alpha - 1} \quad \text{and} \quad \text{median} = 2^{1/\alpha}$$

Using the corresponding expressions for the lognormal distribution leads us to the equations

$$\frac{\alpha}{\alpha - 1} = \exp(\mu + \sigma^2/2) \quad \text{and} \quad 2^{1/\alpha} = \exp(\mu)$$

which we solve for  $\mu$  and  $\sigma$  given  $\alpha = 1.05$ .

Here is code that generates the two samples, produces the violin plot and prints the mean and standard deviation of the two samples.

```

num_firms = 100_000
num_years = 10
tax_rate = 0.15
r = 0.05

β = 1 / (1 + r)      # discount factor

x_bar = 1.0
a = 1.05

def pareto_rvs(n):
    "Uses a standard method to generate Pareto draws."
    u = np.random.uniform(size=n)
    y = x_bar / (u**(1/a))
    return y

```

Let's compute the lognormal parameters:

```

μ = np.log(2) / a
σ_sq = 2 * (np.log(a/(a - 1)) - np.log(2)/a)
σ = np.sqrt(σ_sq)

```

Here's a function to compute a single estimate of tax revenue for a particular choice of distribution `dist`.

```

def tax_rev(dist):
    tax_raised = 0
    for t in range(num_years):
        if dist == 'pareto':
            π = pareto_rvs(num_firms)
        else:
            π = np.exp(μ + σ * np.random.randn(num_firms))
        tax_raised += β**t * np.sum(π * tax_rate)
    return tax_raised

```

Now let's generate the violin plot.

```

num_reps = 100
np.random.seed(1234)

tax_rev_lognorm = np.empty(num_reps)
tax_rev_pareto = np.empty(num_reps)

for i in range(num_reps):
    tax_rev_pareto[i] = tax_rev('pareto')
    tax_rev_lognorm[i] = tax_rev('lognorm')

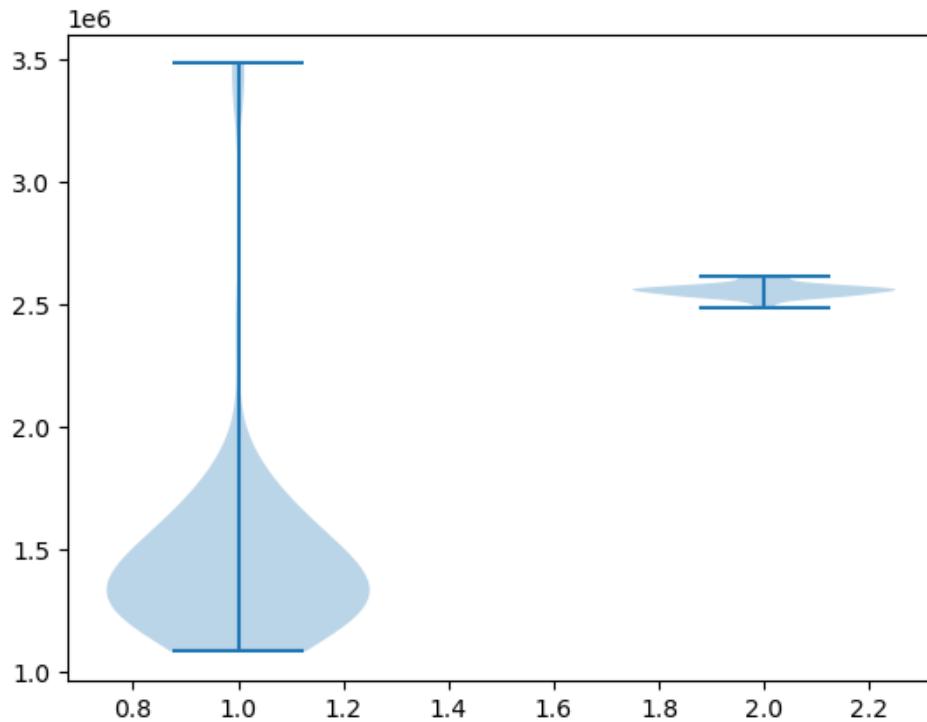
fig, ax = plt.subplots()

data = tax_rev_pareto, tax_rev_lognorm

ax.violinplot(data)

plt.show()

```



Finally, let's print the means and standard deviations.

```
tax_rev_pareto.mean(), tax_rev_pareto.std()
```

```
(1458729.0546623734, 406089.36136615666)
```

```
tax_rev_lognorm.mean(), tax_rev_lognorm.std()
```

```
(2556174.8615230713, 25586.444565139616)
```

Looking at the output of the code, our main conclusion is that the Pareto assumption leads to a lower mean and greater dispersion.

---

#### Exercise 18.8.4

The characteristic function of the Cauchy distribution is

$$\phi(t) = \mathbb{E}e^{itX} = \int e^{itx} f(x) dx = e^{-|t|} \quad (18.5)$$

Prove that the sample mean  $\bar{X}_n$  of  $n$  independent draws  $X_1, \dots, X_n$  from the Cauchy distribution has the same characteristic function as  $X_1$ .

(This means that the sample mean never converges.)

---

#### Solution to Exercise 18.8.4

By independence, the characteristic function of the sample mean becomes

$$\begin{aligned}\mathbb{E}e^{it\bar{X}_n} &= \mathbb{E} \exp \left\{ i \frac{t}{n} \sum_{j=1}^n X_j \right\} \\ &= \mathbb{E} \prod_{j=1}^n \exp \left\{ i \frac{t}{n} X_j \right\} \\ &= \prod_{j=1}^n \mathbb{E} \exp \left\{ i \frac{t}{n} X_j \right\} = [\phi(t/n)]^n\end{aligned}$$

In view of (18.5), this is just  $e^{-|t|}$ .

Thus, in the case of the Cauchy distribution, the sample mean itself has the very same Cauchy distribution, regardless of  $n$ !

---



## RACIAL SEGREGATION

### Contents

- *Racial Segregation*
  - *Outline*
  - *The model*
  - *Results*
  - *Exercises*

### 19.1 Outline

In 1969, Thomas C. Schelling developed a simple but striking model of racial segregation [Sch69].

His model studies the dynamics of racially mixed neighborhoods.

Like much of Schelling's work, the model shows how local interactions can lead to surprising aggregate outcomes.

It studies a setting where agents (think of households) have relatively mild preference for neighbors of the same race.

For example, these agents might be comfortable with a mixed race neighborhood but uncomfortable when they feel “surrounded” by people from a different race.

Schelling illustrated the follow surprising result: in such a setting, mixed race neighborhoods are likely to be unstable, tending to collapse over time.

In fact the model predicts strongly divided neighborhoods, with high levels of segregation.

In other words, extreme segregation outcomes arise even though people's preferences are not particularly extreme.

These extreme outcomes happen because of *interactions* between agents in the model (e.g., households in a city) that drive self-reinforcing dynamics in the model.

These ideas will become clearer as the lecture unfolds.

In recognition of his work on segregation and other research, Schelling was awarded the 2005 Nobel Prize in Economic Sciences (joint with Robert Aumann).

Let's start with some imports:

```
%matplotlib inline
import matplotlib.pyplot as plt
from random import uniform, seed
from math import sqrt
import numpy as np
```

## 19.2 The model

In this section we will build a version of Schelling's model.

### 19.2.1 Set-Up

We will cover a variation of Schelling's model that is different from the original but also easy to program and, at the same time, captures his main idea.

Suppose we have two types of people: orange people and green people.

Assume there are  $n$  of each type.

These agents all live on a single unit square.

Thus, the location (e.g, address) of an agent is just a point  $(x, y)$ , where  $0 < x, y < 1$ .

- The set of all points  $(x, y)$  satisfying  $0 < x, y < 1$  is called the **unit square**
- Below we denote the unit square by  $S$

### 19.2.2 Preferences

We will say that an agent is *happy* if 5 or more of her 10 nearest neighbors are of the same type.

An agent who is not happy is called *unhappy*.

For example,

- if an agent is orange and 5 of her 10 nearest neighbors are orange, then she is happy.
- if an agent is green and 8 of her 10 nearest neighbors are orange, then she is unhappy.

'Nearest' is in terms of *Euclidean distance*.

An important point to note is that agents are **not** averse to living in mixed areas.

They are perfectly happy if half of their neighbors are of the other color.

### 19.2.3 Behavior

Initially, agents are mixed together (integrated).

In particular, we assume that the initial location of each agent is an independent draw from a bivariate uniform distribution on the unit square  $S$ .

- First their  $x$  coordinate is drawn from a uniform distribution on  $(0, 1)$
- Then, independently, their  $y$  coordinate is drawn from the same distribution.

Now, cycling through the set of all agents, each agent is now given the chance to stay or move.

Each agent stays if they are happy and moves if they are unhappy.

The algorithm for moving is as follows

---

**Algorithm 19.2.1 (Jump Chain Algorithm)**

1. Draw a random location in  $S$
  2. If happy at new location, move there
  3. Otherwise, go to step 1
- 

We cycle continuously through the agents, each time allowing an unhappy agent to move.

We continue to cycle until no one wishes to move.

## 19.3 Results

Let's now implement and run this simulation.

In what follows, agents are modeled as `objects`.

Here's an indication of their structure:

```
* Data:
  * type (green or orange)
  * location

* Methods:
  * determine whether happy or not given locations of other agents
  * If not happy, move
    * find a new location where happy
```

Let's build them.

```
class Agent:

    def __init__(self, type):
        self.type = type
        self.draw_location()

    def draw_location(self):
        self.location = uniform(0, 1), uniform(0, 1)

    def get_distance(self, other):
        "Computes the euclidean distance between self and other agent."
        a = (self.location[0] - other.location[0])**2
        b = (self.location[1] - other.location[1])**2
        return sqrt(a + b)

    def happy(self,
              agents,                      # List of other agents
```

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```

        num_neighbors=10,          # No. of agents viewed as neighbors
        require_same_type=5):    # How many neighbors must be same type
    """
        True if a sufficient number of nearest neighbors are of the same
        type.
    """
    distances = []

    # Distances is a list of pairs (d, agent), where d is distance from
    # agent to self
    for agent in agents:
        if self != agent:
            distance = self.get_distance(agent)
            distances.append((distance, agent))

    # Sort from smallest to largest, according to distance
    distances.sort()

    # Extract the neighboring agents
    neighbors = [agent for d, agent in distances[:num_neighbors]]

    # Count how many neighbors have the same type as self
    num_same_type = sum(self.type == agent.type for agent in neighbors)
    return num_same_type >= require_same_type

def update(self, agents):
    "If not happy, then randomly choose new locations until happy."
    while not self.happy(agents):
        self.draw_location()

```

Here's some code that takes a list of agents and produces a plot showing their locations on the unit square.

Orange agents are represented by orange dots and green ones are represented by green dots.

```

def plot_distribution(agents, cycle_num):
    "Plot the distribution of agents after cycle_num rounds of the loop."
    x_values_0, y_values_0 = [], []
    x_values_1, y_values_1 = [], []
    # == Obtain locations of each type == #
    for agent in agents:
        x, y = agent.location
        if agent.type == 0:
            x_values_0.append(x)
            y_values_0.append(y)
        else:
            x_values_1.append(x)
            y_values_1.append(y)
    fig, ax = plt.subplots()
    plot_args = {'markersize': 8, 'alpha': 0.8}
    ax.set_facecolor('azure')
    ax.plot(x_values_0, y_values_0,
            'o', markerfacecolor='orange', **plot_args)
    ax.plot(x_values_1, y_values_1,
            'o', markerfacecolor='green', **plot_args)
    ax.set_title(f'Cycle {cycle_num-1}')
    plt.show()

```

And here's some pseudocode for the main loop, where we cycle through the agents until no one wishes to move.

The pseudocode is

```
plot the distribution
while agents are still moving
    for agent in agents
        give agent the opportunity to move
plot the distribution
```

The real code is below

```
def run_simulation(num_of_type_0=600,
                   num_of_type_1=600,
                   max_iter=100_000,          # Maximum number of iterations
                   set_seed=1234):

    # Set the seed for reproducibility
    seed(set_seed)

    # Create a list of agents of type 0
    agents = [Agent(0) for i in range(num_of_type_0)]
    # Append a list of agents of type 1
    agents.extend(Agent(1) for i in range(num_of_type_1))

    # Initialize a counter
    count = 1

    # Plot the initial distribution
    plot_distribution(agents, count)

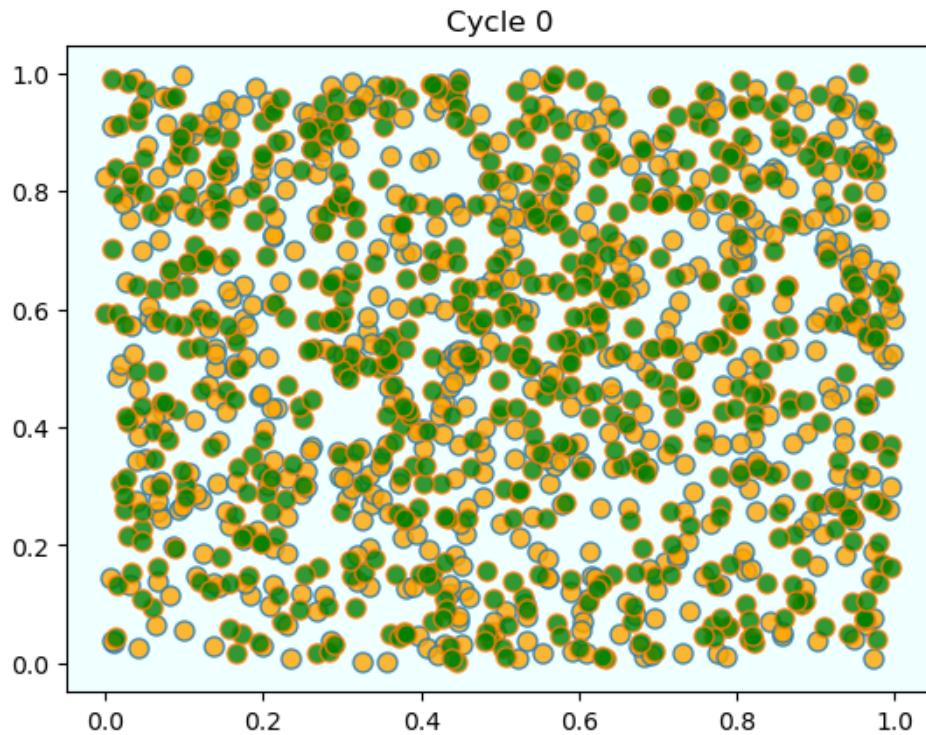
    # Loop until no agent wishes to move
    while count < max_iter:
        print('Entering loop ', count)
        count += 1
        no_one_moved = True
        for agent in agents:
            old_location = agent.location
            agent.update(agents)
            if agent.location != old_location:
                no_one_moved = False
        if no_one_moved:
            break

    # Plot final distribution
    plot_distribution(agents, count)

    if count < max_iter:
        print(f'Converged after {count} iterations.')
    else:
        print('Hit iteration bound and terminated.')
```

Let's have a look at the results.

```
run_simulation()
```



Entering loop 1

Entering loop 2

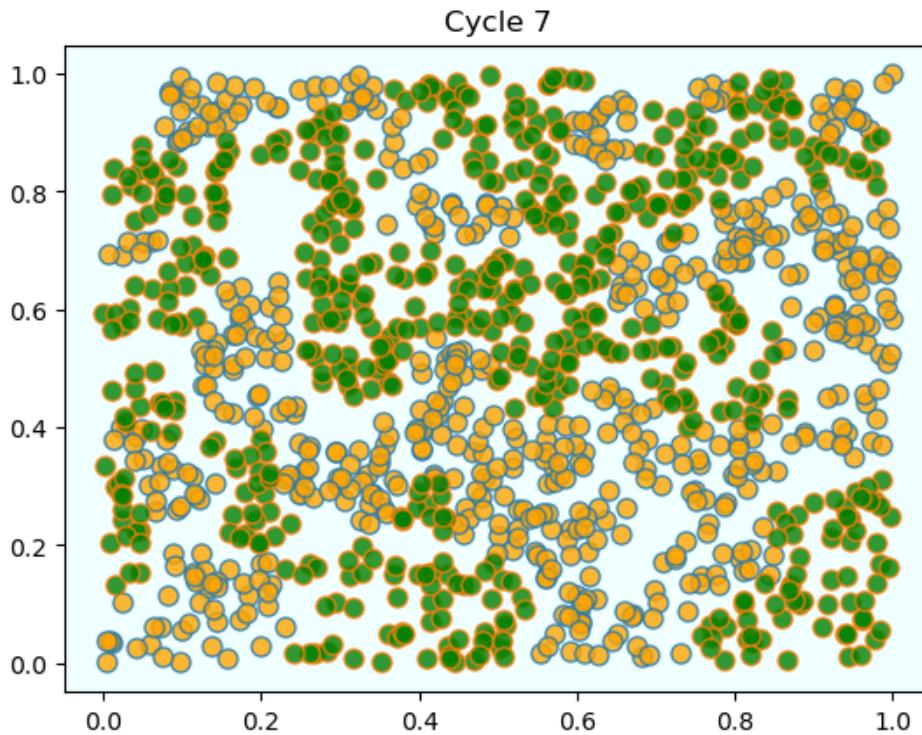
Entering loop 3

Entering loop 4

Entering loop 5

Entering loop 6

Entering loop 7



Converged after 8 iterations.

As discussed above, agents are initially mixed randomly together.

But after several cycles, they become segregated into distinct regions.

In this instance, the program terminated after a small number of cycles through the set of agents, indicating that all agents had reached a state of happiness.

What is striking about the pictures is how rapidly racial integration breaks down.

This is despite the fact that people in the model don't actually mind living mixed with the other type.

Even with these preferences, the outcome is a high degree of segregation.

## 19.4 Exercises

### Exercise 19.4.1

The object oriented style that we used for coding above is neat but harder to optimize than procedural code (i.e., code based around functions rather than objects and methods).

Try writing a new version of the model that stores

- the locations of all agents as a 2D NumPy array of floats.
- the types of all agents as a flat NumPy array of integers.

Write functions that act on this data to update the model using the logic similar to that described above.

However, implement the following two changes:

1. Agents are offered a move at random (i.e., selected randomly and given the opportunity to move).
2. After an agent has moved, flip their type with probability 0.01

The second change introduces extra randomness into the model.

(We can imagine that, every so often, an agent moves to a different city and, with small probability, is replaced by an agent of the other type.)

---

### Solution to Exercise 19.4.1

solution here

```
from numpy.random import uniform, randint

n = 1000          # number of agents (agents = 0, ..., n-1)
k = 10            # number of agents regarded as neighbors
require_same_type = 5 # want >= require_same_type neighbors of the same type

def initialize_state():
    locations = uniform(size=(n, 2))
    types = randint(0, high=2, size=n) # label zero or one
    return locations, types

def compute_distances_from_loc(loc, locations):
    """ Compute distance from location loc to all other points. """
    return np.linalg.norm(loc - locations, axis=1)

def get_neighbors(loc, locations):
    " Get all neighbors of a given location. "
    all_distances = compute_distances_from_loc(loc, locations)
    indices = np.argsort(all_distances) # sort agents by distance to loc
    neighbors = indices[:k]           # keep the k closest ones
    return neighbors

def is_happy(i, locations, types):
    happy = True
    agent_loc = locations[i, :]
    agent_type = types[i]
    neighbors = get_neighbors(agent_loc, locations)
    neighbor_types = types[neighbors]
    if sum(neighbor_types == agent_type) < require_same_type:
        happy = False
    return happy

def count_happy(locations, types):
    " Count the number of happy agents. "
    happy_sum = 0
    for i in range(n):
        happy_sum += is_happy(i, locations, types)
    return happy_sum

def update_agent(i, locations, types):
    " Move agent if unhappy. "
    moved = False
    while not is_happy(i, locations, types):
        # Move agent to a random location
        new_loc = uniform(size=(2))
        # Flip agent's type with probability 0.01
        if np.random.rand() < 0.01:
            types[i] = 1 - types[i]
        locations[i] = new_loc
        moved = True
    return moved
```

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```

moved = True
locations[i, :] = uniform(), uniform()
return moved

def plot_distribution(locations, types, title, savepdf=False):
    " Plot the distribution of agents after cycle_num rounds of the loop."
    fig, ax = plt.subplots()
    colors = 'orange', 'green'
    for agent_type, color in zip((0, 1), colors):
        idx = (types == agent_type)
        ax.plot(locations[idx, 0],
                locations[idx, 1],
                'o',
                markersize=8,
                markerfacecolor=color,
                alpha=0.8)
    ax.set_title(title)
    plt.show()

def sim_random_select(max_iter=100_000, flip_prob=0.01, test_freq=10_000):
    """
    Simulate by randomly selecting one household at each update.

    Flip the color of the household with probability `flip_prob`.
    """
    locations, types = initialize_state()
    current_iter = 0

    while current_iter <= max_iter:

        # Choose a random agent and update them
        i = randint(0, n)
        moved = update_agent(i, locations, types)

        if flip_prob > 0:
            # flip agent i's type with probability epsilon
            U = uniform()
            if U < flip_prob:
                current_type = types[i]
                types[i] = 0 if current_type == 1 else 1

        # Every so many updates, plot and test for convergence
        if current_iter % test_freq == 0:
            cycle = current_iter / n
            plot_distribution(locations, types, f'iteration {current_iter}')
            if count_happy(locations, types) == n:
                print(f"Converged at iteration {current_iter}")
                break

        current_iter += 1

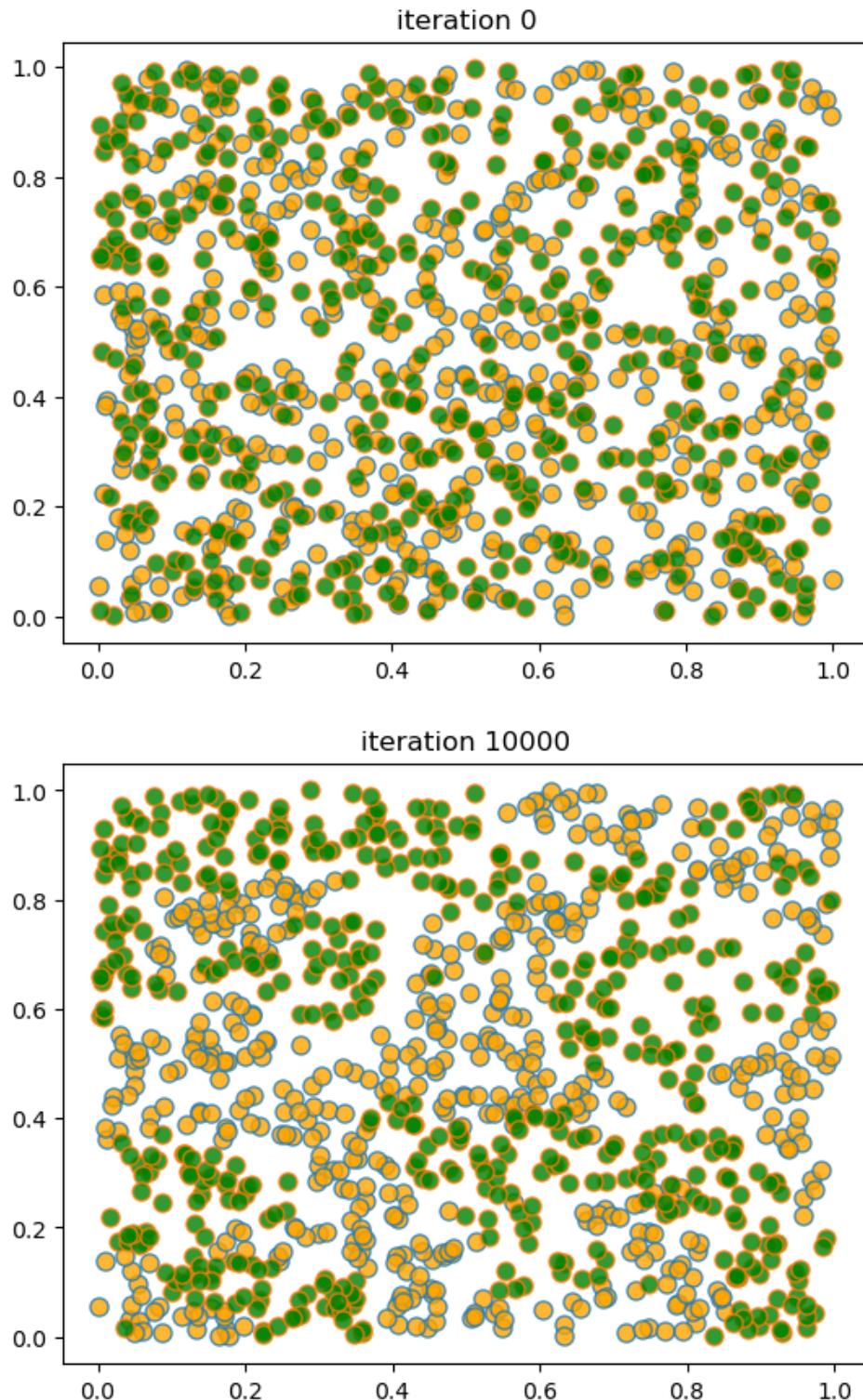
    if current_iter > max_iter:
        print(f"Terminating at iteration {current_iter}")

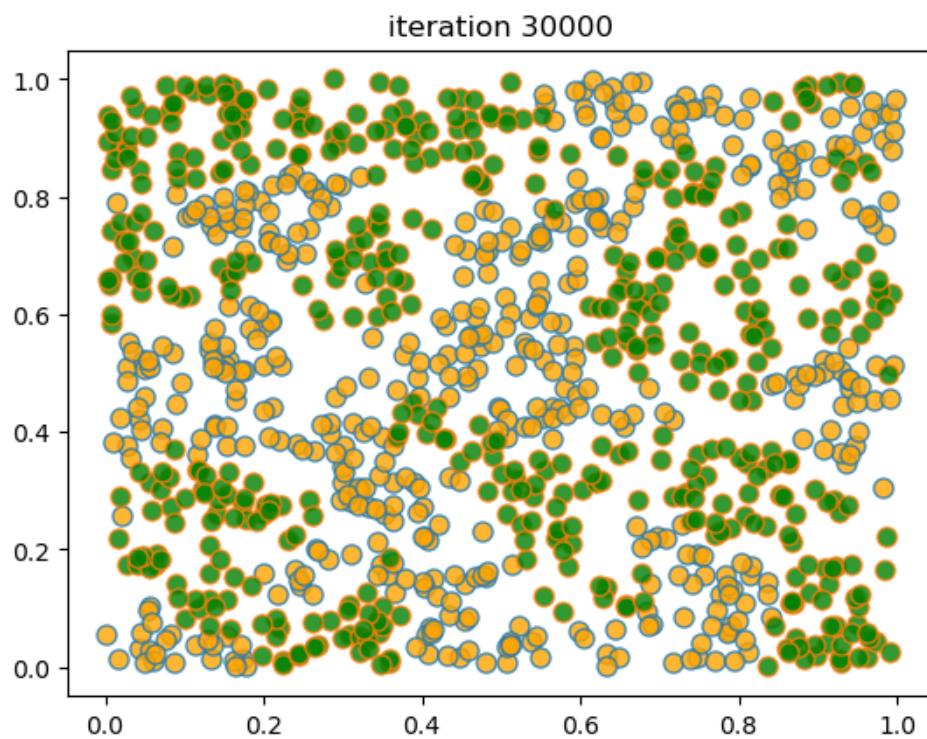
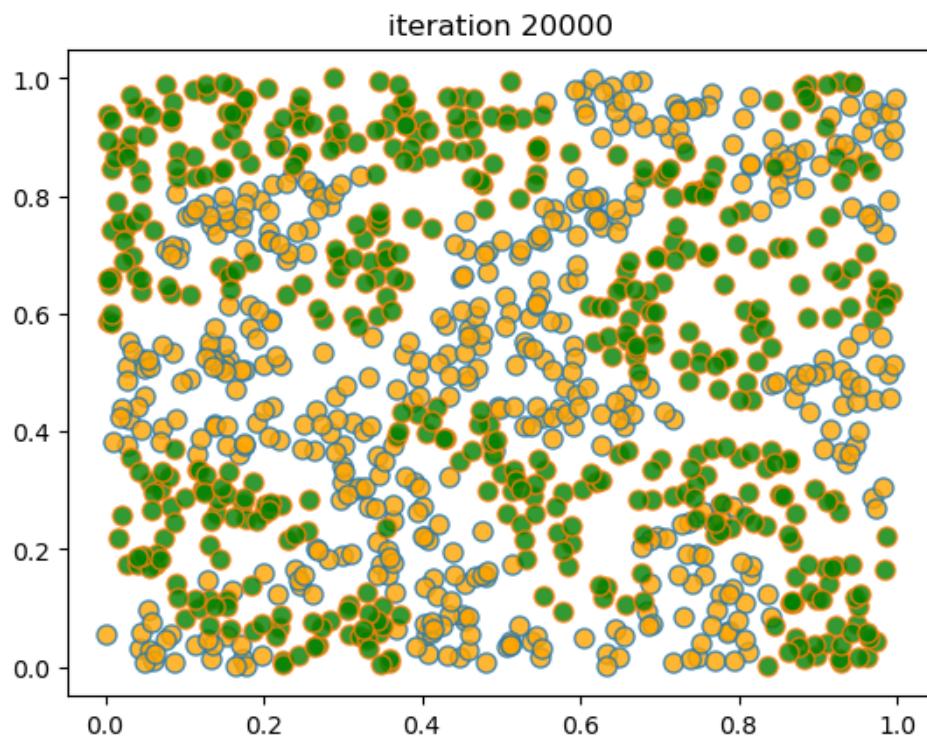
```

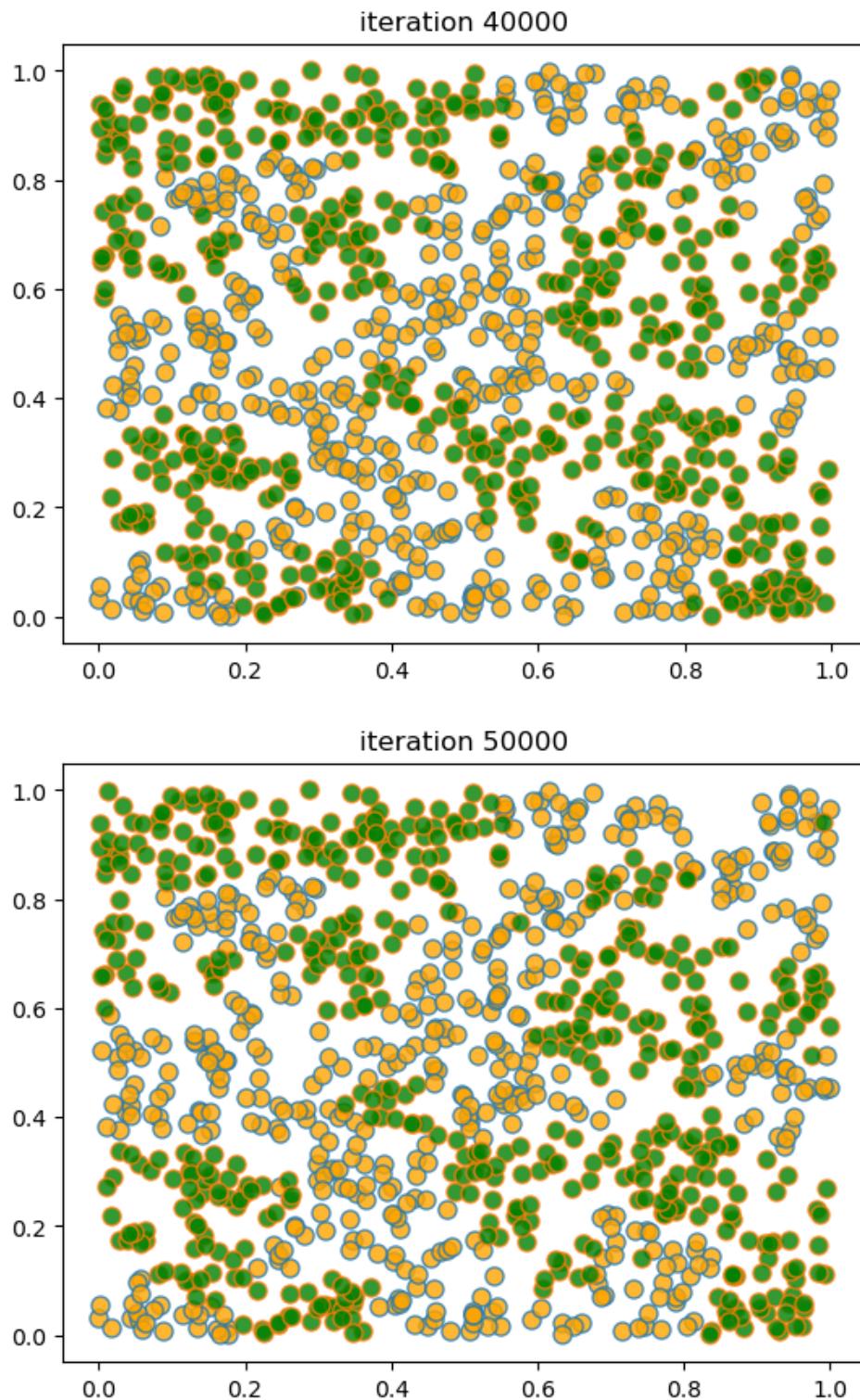
When we run this we again find that mixed neighborhoods break down and segregation emerges.

Here's a sample run.

```
sim_random_select(max_iter=50_000, flip_prob=0.01, test_freq=10_000)
```







Terminating at iteration 50001

## **Part VI**

# **Nonlinear Dynamics**



## THE SOLOW-SWAN GROWTH MODEL

In this lecture we review a famous model due to Robert Solow (1925–2014) and Trevor Swan (1918–1989).

The model is used to study growth over the long run.

Although the model is simple, it contains some interesting lessons.

We will use the following imports

```
import matplotlib.pyplot as plt
import numpy as np
```

### 20.1 The model

In a Solow–Swan economy, agents save a fixed fraction of their current incomes.

Savings sustain or increase the stock of capital.

Capital is combined with labor to produce output, which in turn is paid out to workers and owners of capital.

To keep things simple, we ignore population and productivity growth.

For each integer  $t \geq 0$ , output  $Y_t$  in period  $t$  is given by  $Y_t = F(K_t, L_t)$ , where  $K_t$  is capital,  $L_t$  is labor and  $F$  is an aggregate production function.

The function  $F$  is assumed to be nonnegative and **homogeneous of degree one**, meaning that

$$F(\lambda K, \lambda L) = \lambda F(K, L) \quad \text{for all } \lambda \geq 0$$

Production functions with this property include

- the **Cobb-Douglas** function  $F(K, L) = AK^\alpha L^{1-\alpha}$  with  $0 \leq \alpha \leq 1$  and
- the **CES** function  $F(K, L) = \{aK^\rho + bL^\rho\}^{1/\rho}$  with  $a, b, \rho > 0$ .

We assume a closed economy, so domestic investment equals aggregate domestic saving.

The saving rate is a constant  $s$  satisfying  $0 \leq s \leq 1$ , so that aggregate investment and saving both equal  $sY_t$ .

Capital depreciates: without replenishing through investment, one unit of capital today becomes  $1 - \delta$  units tomorrow.

Thus,

$$K_{t+1} = sF(K_t, L_t) + (1 - \delta)K_t$$

Without population growth,  $L_t$  equals some constant  $L$ .

Setting  $k_t := K_t/L$  and using homogeneity of degree one now yields

$$k_{t+1} = s \frac{F(K_t, L)}{L} + (1 - \delta)k_t = sF(k_t, 1) + (1 - \delta)k_t$$

With  $f(k) := F(k, 1)$ , the final expression for capital dynamics is

$$k_{t+1} = g(k_t) \text{ where } g(k) := sf(k) + (1 - \delta)k \quad (20.1)$$

Our aim is to learn about the evolution of  $k_t$  over time, given an exogenous initial capital stock  $k_0$ .

## 20.2 A graphical perspective

To understand the dynamics of the sequence  $(k_t)_{t \geq 0}$  we use a 45 degree diagram.

To do so, we first need to specify the functional form for  $f$  and assign values to the parameters.

We choose the Cobb–Douglas specification  $f(k) = Ak^\alpha$  and set  $A = 2.0$ ,  $\alpha = 0.3$ ,  $s = 0.3$  and  $\delta = 0.4$ .

The function  $g$  from (20.1) is then plotted, along with the 45 degree line.

Let's define the constants.

```
A, s, alpha, delta = 2, 0.3, 0.3, 0.4
x0 = 0.25
xmin, xmax = 0, 3
```

Now, we define the function  $g$

```
def g(A, s, alpha, delta, k):
    return A * s * k**alpha + (1 - delta) * k
```

Let's plot the 45 degree diagram of  $g$

```
def plot45(kstar=None):
    xgrid = np.linspace(xmin, xmax, 12000)

    fig, ax = plt.subplots()

    ax.set_xlim(xmin, xmax)

    g_values = g(A, s, alpha, delta, xgrid)

    ymin, ymax = np.min(g_values), np.max(g_values)
    ax.set_ylim(ymin, ymax)

    lb = r'$g(k) = sAk^{\alpha} + (1 - \delta)k$'
    ax.plot(xgrid, g_values, lw=2, alpha=0.6, label=lb)
    ax.plot(xgrid, xgrid, 'k-', lw=1, alpha=0.7, label='45')

    if kstar:
        fps = (kstar,)

        ax.plot(fps, fps, 'go', ms=10, alpha=0.6)

        ax.annotate(r'$k^* = (sA / \delta)^{(1/(1-\alpha))}$',
```

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```

xy=(kstar, kstar),
xycoords='data',
xytext=(-40, -60),
textcoords='offset points',
fontsize=14,
arrowprops=dict (arrowstyle="->"))

ax.legend(loc='upper left', frameon=False, fontsize=12)

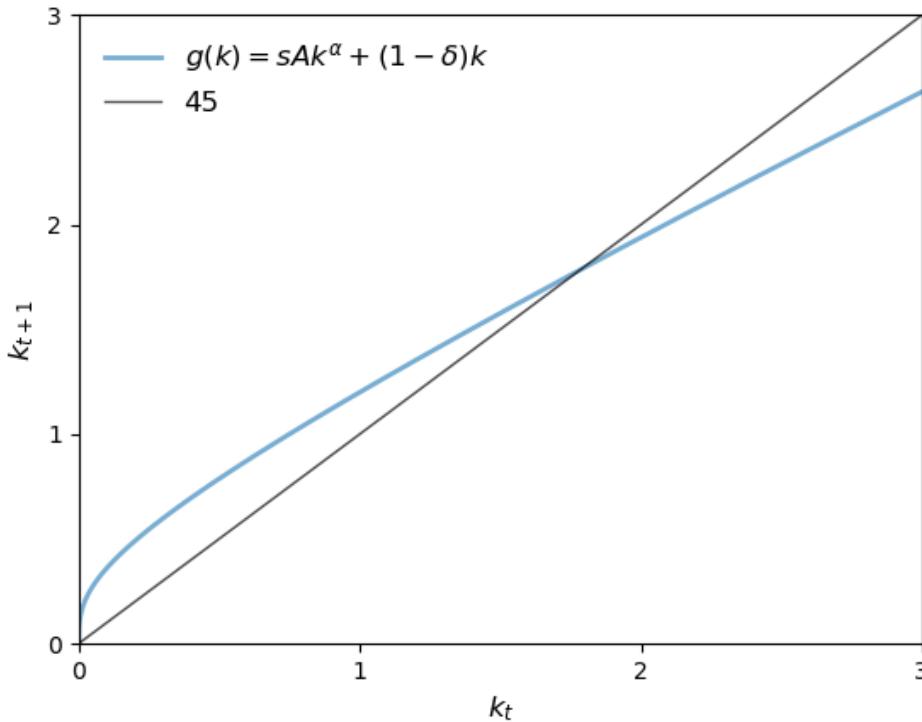
ax.set_xticks((0, 1, 2, 3))
ax.set_yticks((0, 1, 2, 3))

ax.set_xlabel('$k_t$', fontsize=12)
ax.set_ylabel('$k_{t+1}$', fontsize=12)

plt.show()

```

plot45()



Suppose, at some  $k_t$ , the value  $g(k_t)$  lies strictly above the 45 degree line.

Then we have  $k_{t+1} = g(k_t) > k_t$  and capital per worker rises.

If  $g(k_t) < k_t$  then capital per worker falls.

If  $g(k_t) = k_t$ , then we are at a **steady state** and  $k_t$  remains constant.

(A steady state of the model is a **fixed point** of the mapping  $g$ .)

From the shape of the function  $g$  in the figure, we see that there is a unique steady state in  $(0, \infty)$ .

It solves  $k = sAk^\alpha + (1 - \delta)k$  and hence is given by

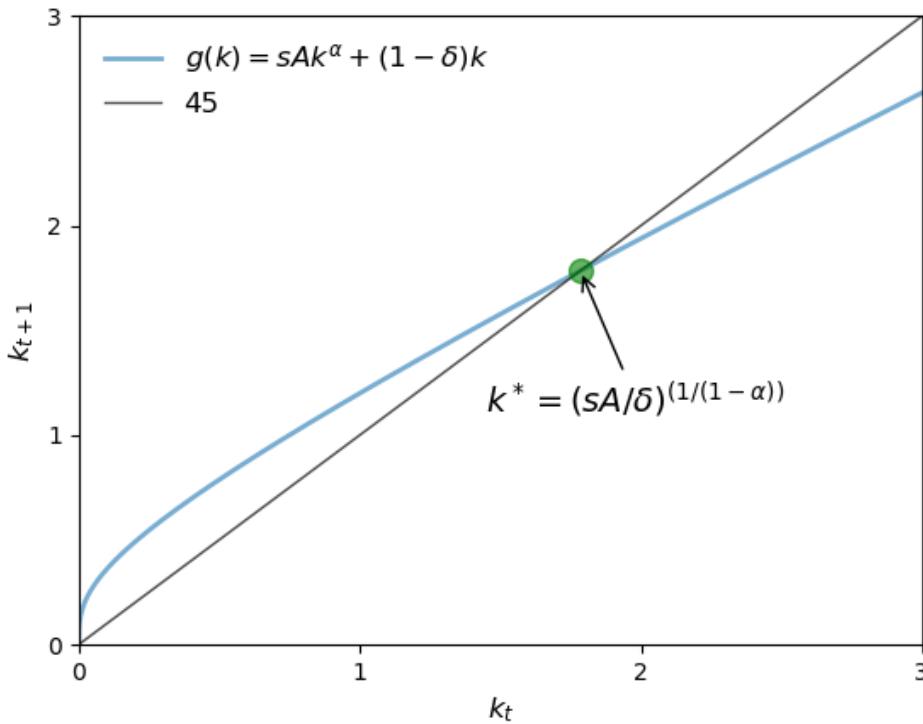
$$k^* := \left( \frac{sA}{\delta} \right)^{1/(1-\alpha)} \quad (20.2)$$

If initial capital is below  $k^*$ , then capital increases over time.

If initial capital is above this level, then the reverse is true.

Let's plot the 45 degree diagram to show the  $k^*$  in the plot

```
kstar = ((s * A) / delta)**(1 / (1 - alpha))
plot45(kstar)
```



From our graphical analysis, it appears that  $(k_t)$  converges to  $k^*$ , regardless of initial capital  $k_0$ .

This is a form of global stability.

The next figure shows three time paths for capital, from three distinct initial conditions, under the parameterization listed above.

At this parameterization,  $k^* \approx 1.78$ .

Let's define the constants and three distinct initial conditions

```
A, s, alpha, delta = 2, 0.3, 0.3, 0.4
x0 = np.array([.25, 1.25, 3.25])

ts_length = 20
xmin, xmax = 0, ts_length
ymin, ymax = 0, 3.5
```

```

def simulate_ts(x0_values, ts_length):

    k_star = (s * A / delta)**(1/(1-alpha))
    fig, ax = plt.subplots(figsize=[11, 5])
    ax.set_xlim(xmin, xmax)
    ax.set_ylimits(ymin, ymax)

    ts = np.zeros(ts_length)

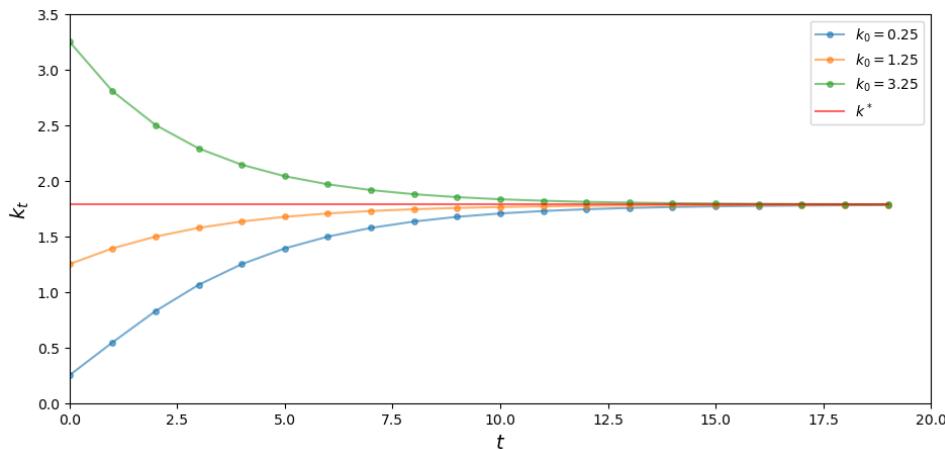
    # simulate and plot time series
    for x_init in x0_values:
        ts[0] = x_init
        for t in range(1, ts_length):
            ts[t] = g(A, s, alpha, delta, ts[t-1])
        ax.plot(np.arange(ts_length), ts, '-o', ms=4, alpha=0.6,
                label=r'$k_0=%g$' % x_init)
    ax.plot(np.arange(ts_length), np.full(ts_length, k_star),
            alpha=0.6, color='red', label=r'$k^*$')
    ax.legend(fontsize=10)

    ax.set_xlabel(r'$t$', fontsize=14)
    ax.set_ylabel(r'$k_t$', fontsize=14)

    plt.show()

```

```
simulate_ts(x0, ts_length)
```



As expected, the time paths in the figure both converge to this value.

## 20.3 Growth in continuous time

In this section we investigate a continuous time version of the Solow–Swan growth model.

We will see how the smoothing provided by continuous time can simplify analysis.

Recall that the discrete time dynamics for capital are given by  $k_{t+1} = sf(k_t) + (1 - \delta)k_t$ .

A simple rearrangement gives the rate of change per unit of time:

$$\Delta k_t = sf(k_t) - \delta k_t \quad \text{where} \quad \Delta k_t := k_{t+1} - k_t$$

Taking the time step to zero gives the continuous time limit

$$k'_t = sf(k_t) - \delta k_t \quad \text{with} \quad k'_t := \frac{d}{dt} k_t \quad (20.3)$$

Our aim is to learn about the evolution of  $k_t$  over time, given initial stock  $k_0$ .

A **steady state** for (20.3) is a value  $k^*$  at which capital is unchanging, meaning  $k'_t = 0$  or, equivalently,  $sf(k^*) = \delta k^*$ .

We assume  $f(k) = Ak^\alpha$ , so  $k^*$  solves  $sAk^\alpha = \delta k$ .

The solution is the same as the discrete time case—see (20.2).

The dynamics are represented in the next figure, maintaining the parameterization we used above.

Writing  $k'_t = g(k_t)$  with  $g(k) = sAk^\alpha - \delta k$ , values of  $k$  with  $g(k) > 0$  imply that  $k'_t > 0$ , so capital is increasing.

When  $g(k) < 0$ , the opposite occurs. Once again, high marginal returns to savings at low levels of capital combined with low rates of return at high levels of capital combine to yield global stability.

To see this in a figure, let's define the constants

```
A, s, alpha, delta = 2, 0.3, 0.3, 0.4
```

Next we define the function  $g$  for growth in continuous time

```
def g_con(A, s, alpha, delta, k):
    return A * s * k**alpha - delta * k
```

```
def plot_gcon(kstar=None):
    k_grid = np.linspace(0, 2.8, 10000)

    fig, ax = plt.subplots(figsize=[11, 5])
    ax.plot(k_grid, g_con(A, s, alpha, delta, k_grid), label='$g(k)$')
    ax.plot(k_grid, 0 * k_grid, label="$k'=0$")

    if kstar:
        fps = (kstar,)

        ax.plot(fps, 0, 'go', ms=10, alpha=0.6)

        ax.annotate(r'$k^* = (sA / \delta)^{(1/(1-\alpha))}$',
                    xy=(kstar, 0),
                    xycoords='data',
                    xytext=(0, 60),
```

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```

textcoords='offset points',
fontsize=12,
arrowprops=dict(arrowstyle="->"))

ax.legend(loc='lower left', fontsize=12)

ax.set_xlabel("$k$", fontsize=10)
ax.set_ylabel("$k'$", fontsize=10)

ax.set_xticks((0, 1, 2, 3))
ax.set_yticks((-0.3, 0, 0.3))

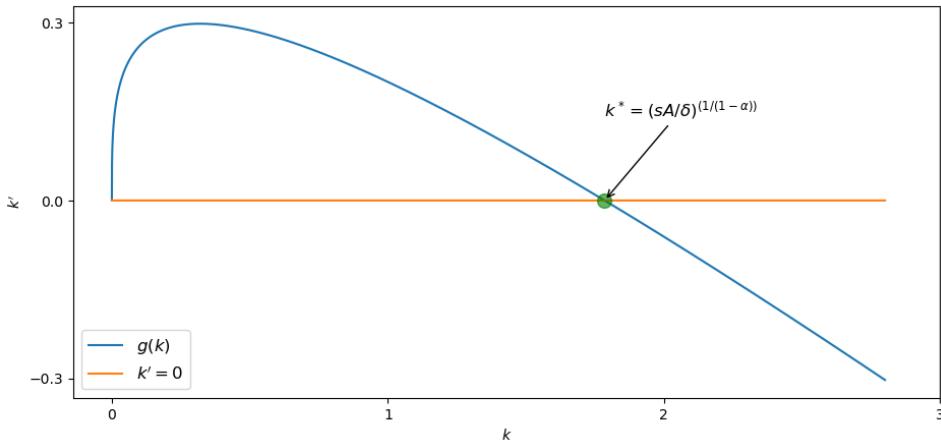
plt.show()

```

```

kstar = ((s * A) / delta)**(1 / (1 - alpha))
plot_gcon(kstar)

```



This shows global stability heuristically for a fixed parameterization, but how would we show the same thing formally for a continuum of plausible parameters?

In the discrete time case, a neat expression for  $k_t$  is hard to obtain.

In continuous time the process is easier: we can obtain a relatively simple expression for  $k_t$  that specifies the entire path.

The first step is to set  $x_t := k_t^{1-\alpha}$ , so that  $x'_t = (1-\alpha)k_t^{-\alpha}k'_t$ .

Substituting into  $k'_t = sAk_t^\alpha - \delta k_t$  leads to the linear differential equation

$$x'_t = (1-\alpha)(sA - \delta x_t) \quad (20.4)$$

This equation has the exact solution

$$x_t = \left( k_0^{1-\alpha} - \frac{sA}{\delta} \right) e^{-\delta(1-\alpha)t} + \frac{sA}{\delta}$$

(You can confirm that this function  $x_t$  satisfies (20.4) by differentiating it with respect to  $t$ .)

Converting back to  $k_t$  yields

$$k_t = \left[ \left( k_0^{1-\alpha} - \frac{sA}{\delta} \right) e^{-\delta(1-\alpha)t} + \frac{sA}{\delta} \right]^{1/(1-\alpha)} \quad (20.5)$$

Since  $\delta > 0$  and  $\alpha \in (0, 1)$ , we see immediately that  $k_t \rightarrow k^*$  as  $t \rightarrow \infty$  independent of  $k_0$ . Thus, global stability holds.

## 20.4 Exercises

---

### Exercise 20.4.1

Plot per capita consumption  $c$  at the steady state, as a function of the savings rate  $s$ , where  $0 \leq s \leq 1$ .

Use the Cobb–Douglas specification  $f(k) = Ak^\alpha$ .

Set  $A = 2.0$ ,  $\alpha = 0.3$ , and  $\delta = 0.5$

Also, find the approximate value of  $s$  that maximizes the  $c^*(s)$  and show it in the plot.

---



---

### Solution to Exercise 20.4.1

Steady state consumption at savings rate  $s$  is given by

$$c^*(s) = (1 - s)f(k^*) = (1 - s)A(k^*)^\alpha$$

```
A = 2.0
alpha = 0.3
delta = 0.5
```

```
s_grid = np.linspace(0, 1, 1000)
k_star = ((s_grid * A) / delta)**(1/(1 - alpha))
c_star = (1 - s_grid) * A * k_star ** alpha
```

Let's find the value of  $s$  that maximizes  $c^*$  using `scipy.optimize.minimize_scalar`. We will use  $-c^*(s)$  since `minimize_scalar` finds the minimum value.

```
from scipy.optimize import minimize_scalar

def calc_c_star(s):
    k = ((s * A) / delta)**(1/(1 - alpha))
    return - (1 - s) * A * k ** alpha

return_values = minimize_scalar(calc_c_star, bounds=(0, 1))
s_star_max = return_values.x
c_star_max = -return_values.fun
print(f"Function is maximized at s = {round(s_star_max, 4)}")
```

Function is maximized at s = 0.3

```
x_s_max = np.array([s_star_max, s_star_max])
y_s_max = np.array([0, c_star_max])

fig, ax = plt.subplots(figsize=[11, 5])
```

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```

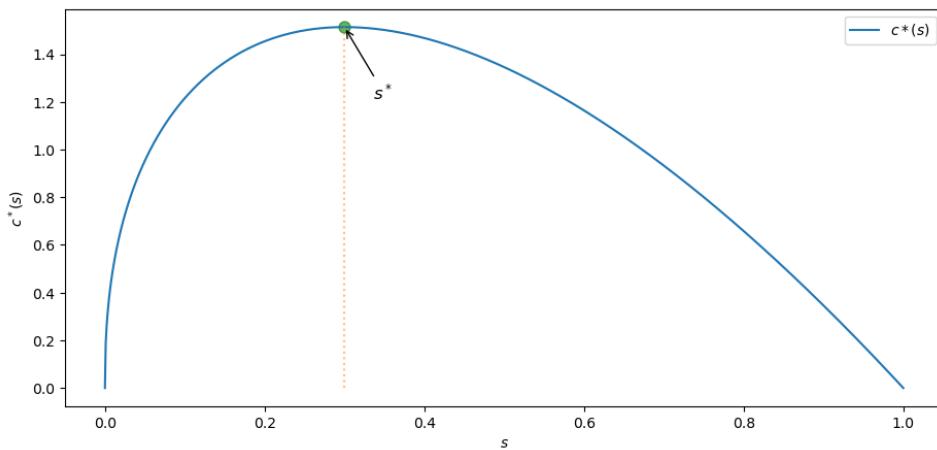
fps = (c_star_max,)

# Highlight the maximum point with a marker
ax.plot((s_star_max, ), (c_star_max, ), 'go', ms=8, alpha=0.6)

ax.annotate(r'$s^*$',
            xy=(s_star_max, c_star_max),
            xycoords='data',
            xytext=(20, -50),
            textcoords='offset points',
            fontsize=12,
            arrowprops=dict(arrowstyle="->"))
ax.plot(s_grid, c_star, label=r'$c^*(s)$')
ax.plot(x_s_max, y_s_max, alpha=0.5, ls='dotted')
ax.set_xlabel(r'$s$')
ax.set_ylabel(r'$c^*(s)$')
ax.legend()

plt.show()

```



One can also try to solve this mathematically by differentiating  $c^*(s)$  and solve for  $\frac{d}{ds}c^*(s) = 0$  using `sympy`.

```
from sympy import solve, Symbol
```

```

s_symbol = Symbol('s', real=True)
k = ((s_symbol * A) / delta)**(1/(1 - alpha))
c = (1 - s_symbol) * A * k ** alpha

```

Let's differentiate  $c$  and solve using `sympy.solve`

```

# Solve using sympy
s_star = solve(c.diff())[0]
print(f"s_star = {s_star}")

```

```
s_star = 0.300000000000000
```

Incidentally, the rate of savings which maximizes steady state level of per capita consumption is called the **Golden Rule**

savings rate.

---

### Exercise 20.4.2

#### Stochastic Productivity

To bring the Solow–Swan model closer to data, we need to think about handling random fluctuations in aggregate quantities.

Among other things, this will eliminate the unrealistic prediction that per-capita output  $y_t = Ak_t^\alpha$  converges to a constant  $y^* := A(k^*)^\alpha$ .

We shift to discrete time for the following discussion.

One approach is to replace constant productivity with some stochastic sequence  $(A_t)_{t \geq 1}$ .

Dynamics are now

$$k_{t+1} = sA_{t+1}f(k_t) + (1 - \delta)k_t \quad (20.6)$$

We suppose  $f$  is Cobb–Douglas and  $(A_t)$  is IID and lognormal.

Now the long run convergence obtained in the deterministic case breaks down, since the system is hit with new shocks at each point in time.

Consider  $A = 2.0$ ,  $s = 0.6$ ,  $\alpha = 0.3$ , and  $\delta = 0.5$

Generate and plot the time series  $k_t$ .

---

#### Solution to Exercise 20.4.2

Let's define the constants for lognormal distribution and initial values used for simulation

```
# Define the constants
sig = 0.2
mu = np.log(2) - sig**2 / 2
A = 2.0
s = 0.6
alpha = 0.3
delta = 0.5
x0 = [.25, 3.25] # list of initial values used for simulation
```

Let's define the function  $k\_next$  to find the next value of  $k$

```
def lgnorm():
    return np.exp(mu + sig * np.random.randn())

def k_next(s, alpha, delta, k):
    return lgnorm() * s * k**alpha + (1 - delta) * k
```

```
def ts_plot(x_values, ts_length):
    fig, ax = plt.subplots(figsize=[11, 5])
    ts = np.zeros(ts_length)

    # simulate and plot time series
    for x_init in x_values:
```

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```

ts[0] = x_init
for t in range(1, ts_length):
    ts[t] = k_next(s, alpha, delta, ts[t-1])
ax.plot(np.arange(ts_length), ts, '-o', ms=4,
        alpha=0.6, label=r'$k_0=%g$' %x_init)

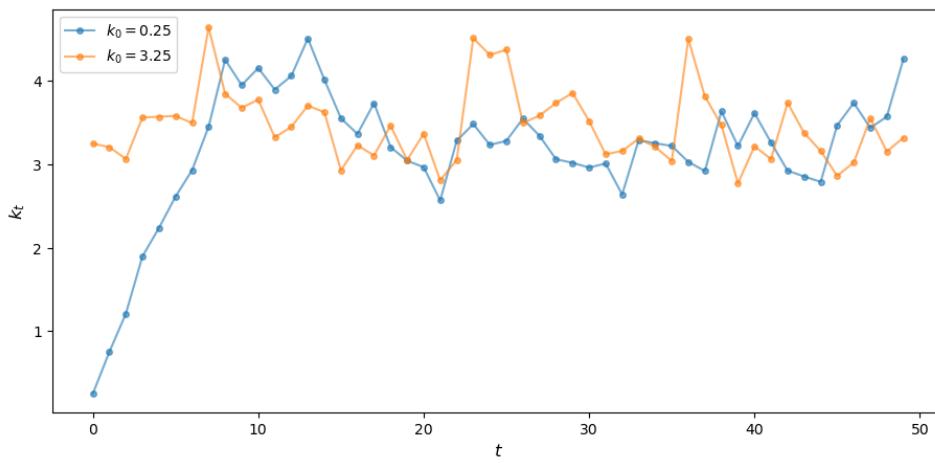
ax.legend(loc='best', fontsize=10)

ax.set_xlabel(r'$t$', fontsize=12)
ax.set_ylabel(r'$k_t$', fontsize=12)

plt.show()

```

```
ts_plot(x0, 50)
```





## DYNAMICS IN ONE DIMENSION

### Contents

- *Dynamics in One Dimension*
  - *Overview*
  - *Some definitions*
  - *Stability*
  - *Graphical analysis*
  - *Exercises*

## 21.1 Overview

In this lecture we give a quick introduction to discrete time dynamics in one dimension.

- In one-dimensional models, the state of the system is described by a single variable.
- The variable is a number (that is, a point in  $\mathbb{R}$ ).

While most quantitative models have two or more state variables, the one-dimensional setting is a good place to learn the foundations of dynamics and understand key concepts.

Let's start with some standard imports:

```
%matplotlib inline
import matplotlib.pyplot as plt
import numpy as np
```

## 21.2 Some definitions

This section sets out the objects of interest and the kinds of properties we study.

### 21.2.1 Composition of functions

For this lecture you should know the following.

If

- $g$  is a function from  $A$  to  $B$  and
- $f$  is a function from  $B$  to  $C$ ,

then the **composition**  $f \circ g$  of  $f$  and  $g$  is defined by

$$(f \circ g)(x) = f(g(x))$$

For example, if

- $A = B = C = \mathbb{R}$ , the set of real numbers,
- $g(x) = x^2$  and  $f(x) = \sqrt{x}$ , then  $(f \circ g)(x) = \sqrt{x^2} = |x|$ .

If  $f$  is a function from  $A$  to itself, then  $f^2$  is the composition of  $f$  with itself.

For example, if  $A = (0, \infty)$ , the set of positive numbers, and  $f(x) = \sqrt{x}$ , then

$$f^2(x) = \sqrt{\sqrt{x}} = x^{1/4}$$

Similarly, if  $n$  is an integer, then  $f^n$  is  $n$  compositions of  $f$  with itself.

In the example above,  $f^n(x) = x^{1/(2^n)}$ .

### 21.2.2 Dynamic systems

A **(discrete time) dynamic system** is a set  $S$  and a function  $g$  that sends set  $S$  back into to itself.

Examples of dynamic systems include

- $S = (0, 1)$  and  $g(x) = \sqrt{x}$
- $S = (0, 1)$  and  $g(x) = x^2$
- $S = \mathbb{Z}$  (the integers) and  $g(x) = 2x$

On the other hand, if  $S = (-1, 1)$  and  $g(x) = x + 1$ , then  $S$  and  $g$  do not form a dynamic system, since  $g(1) = 2$ .

- $g$  does not always send points in  $S$  back into  $S$ .

### 21.2.3 Dynamic systems

We care about dynamic systems because we can use them to study dynamics!

Given a dynamic system consisting of set  $S$  and function  $g$ , we can create a sequence  $\{x_t\}$  of points in  $S$  by setting

$$x_{t+1} = g(x_t) \quad \text{with } x_0 \text{ given.} \quad (21.1)$$

This means that we choose some number  $x_0$  in  $S$  and then take

$$x_0, \quad x_1 = g(x_0), \quad x_2 = g(x_1) = g(g(x_0)), \quad \text{etc.} \quad (21.2)$$

This sequence  $\{x_t\}$  is called the **trajectory** of  $x_0$  under  $g$ .

In this setting,  $S$  is called the **state space** and  $x_t$  is called the **state variable**.

Recalling that  $g^n$  is the  $n$  compositions of  $g$  with itself, we can write the trajectory more simply as

$$x_t = g^t(x_0) \quad \text{for } t \geq 0.$$

In all of what follows, we are going to assume that  $S$  is a subset of  $\mathbb{R}$ , the real numbers.

Equation (21.1) is sometimes called a **first order difference equation**

- first order means dependence on only one lag (i.e., earlier states such as  $x_{t-1}$  do not enter into (21.1)).

#### 21.2.4 Example: A Linear Model

One simple example of a dynamic system is when  $S = \mathbb{R}$  and  $g(x) = ax + b$ , where  $a, b$  are fixed constants.

This leads to the **linear difference equation**

$$x_{t+1} = ax_t + b \quad \text{with } x_0 \text{ given.}$$

The trajectory of  $x_0$  is

$$x_0, \quad ax_0 + b, \quad a^2x_0 + ab + b, \quad \text{etc.} \quad (21.3)$$

Continuing in this way, and using our knowledge of *geometric series*, we find that, for any  $t \geq 0$ ,

$$x_t = a^t x_0 + b \frac{1 - a^t}{1 - a} \quad (21.4)$$

We have an exact expression for  $x_t$  for all  $t$  and hence a full understanding of the dynamics.

Notice in particular that  $|a| < 1$ , then, by (21.4), we have

$$x_t \rightarrow \frac{b}{1 - a} \text{ as } t \rightarrow \infty \quad (21.5)$$

regardless of  $x_0$

This is an example of what is called global stability, a topic we return to below.

#### 21.2.5 Example: A Nonlinear Model

In the linear example above, we obtained an exact analytical expression for  $x_t$  in terms of arbitrary  $t$  and  $x_0$ .

This made analysis of dynamics very easy.

When models are nonlinear, however, the situation can be quite different.

For example, recall how we [previously studied](#) the law of motion for the Solow growth model, a simplified version of which is

$$k_{t+1} = szk_t^\alpha + (1 - \delta)k_t \quad (21.6)$$

Here  $k$  is capital stock and  $s, z, \alpha, \delta$  are positive parameters with  $0 < \alpha, \delta < 1$ .

If you try to iterate like we did in (21.3), you will find that the algebra gets messy quickly.

Analyzing the dynamics of this model requires a different method (see below).

## 21.3 Stability

Consider a fixed dynamic system consisting of set  $S \subset \mathbb{R}$  and  $g$  mapping  $S$  to  $S$ .

### 21.3.1 Steady states

A **steady state** of this system is a point  $x^*$  in  $S$  such that  $x^* = g(x^*)$ .

In other words,  $x^*$  is a **fixed point** of the function  $g$  in  $S$ .

For example, for the linear model  $x_{t+1} = ax_t + b$ , you can use the definition to check that

- $x^* := b/(1 - a)$  is a steady state whenever  $a \neq 1$ .
- if  $a = 1$  and  $b = 0$ , then every  $x \in \mathbb{R}$  is a steady state.
- if  $a = 1$  and  $b \neq 0$ , then the linear model has no steady state in  $\mathbb{R}$ .

### 21.3.2 Global stability

A steady state  $x^*$  of the dynamic system is called **globally stable** if, for all  $x_0 \in S$ ,

$$x_t = g^t(x_0) \rightarrow x^* \text{ as } t \rightarrow \infty$$

For example, in the linear model  $x_{t+1} = ax_t + b$  with  $a \neq 1$ , the steady state  $x^*$

- is globally stable if  $|a| < 1$  and
- fails to be globally stable otherwise.

This follows directly from (21.4).

### 21.3.3 Local stability

A steady state  $x^*$  of the dynamic system is called **locally stable** if there exists an  $\epsilon > 0$  such that

$$|x_0 - x^*| < \epsilon \implies x_t = g^t(x_0) \rightarrow x^* \text{ as } t \rightarrow \infty$$

Obviously every globally stable steady state is also locally stable.

We will see examples below where the converse is not true.

## 21.4 Graphical analysis

As we saw above, analyzing the dynamics for nonlinear models is nontrivial.

There is no single way to tackle all nonlinear models.

However, there is one technique for one-dimensional models that provides a great deal of intuition.

This is a graphical approach based on **45 degree diagrams**.

Let's look at an example: the Solow model with dynamics given in (21.6).

We begin with some plotting code that you can ignore at first reading.

The function of the code is to produce 45 degree diagrams and time series plots.

Let's create a 45 degree diagram for the Solow model with a fixed set of parameters

```
A, s, alpha, delta = 2, 0.3, 0.3, 0.4
```

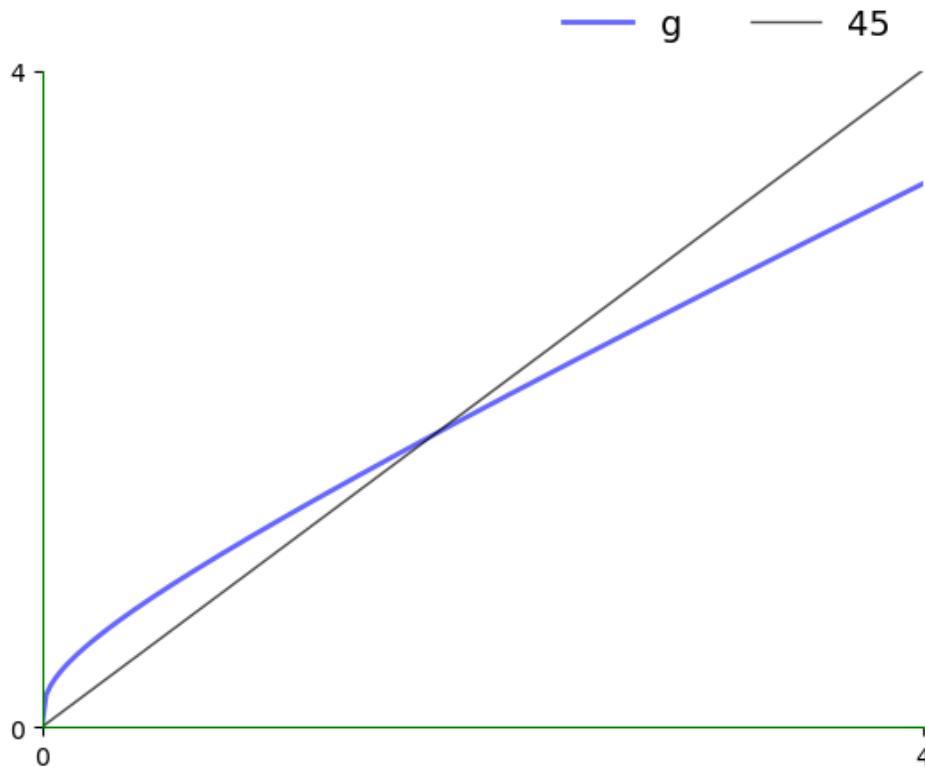
Here's the update function corresponding to the model.

```
def g(k):
    return A * s * k**alpha + (1 - delta) * k
```

Here is the 45 degree plot.

```
xmin, xmax = 0, 4 # Suitable plotting region.
```

```
plot45(g, xmin, xmax, 0, num_arrows=0)
```



The plot shows the function  $g$  and the 45 degree line.

Think of  $k_t$  as a value on the horizontal axis.

To calculate  $k_{t+1}$ , we can use the graph of  $g$  to see its value on the vertical axis.

Clearly,

- If  $g$  lies above the 45 degree line at this point, then we have  $k_{t+1} > k_t$ .
- If  $g$  lies below the 45 degree line at this point, then we have  $k_{t+1} < k_t$ .
- If  $g$  hits the 45 degree line at this point, then we have  $k_{t+1} = k_t$ , so  $k_t$  is a steady state.

For the Solow model, there are two steady states when  $S = \mathbb{R}_+ = [0, \infty)$ .

- the origin  $k = 0$

- the unique positive number such that  $k = szk^\alpha + (1 - \delta)k$ .

By using some algebra, we can show that in the second case, the steady state is

$$k^* = \left(\frac{sz}{\delta}\right)^{1/(1-\alpha)}$$

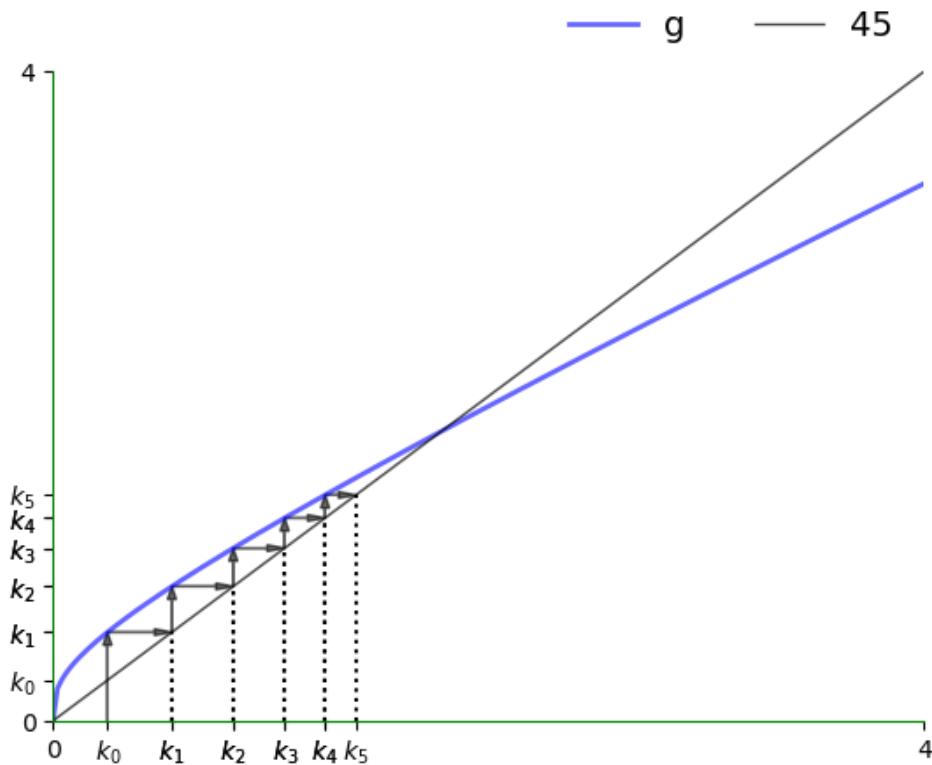
### 21.4.1 Trajectories

By the preceding discussion, in regions where  $g$  lies above the 45 degree line, we know that the trajectory is increasing.

The next figure traces out a trajectory in such a region so we can see this more clearly.

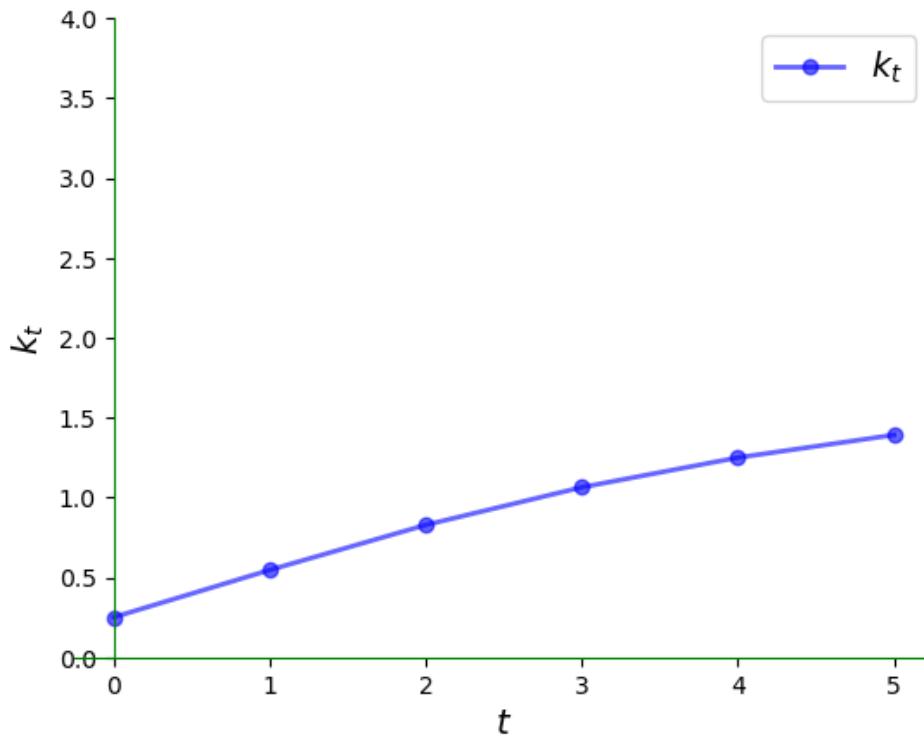
The initial condition is  $k_0 = 0.25$ .

```
k0 = 0.25
plot45(g, xmin, xmax, k0, num_arrows=5, var='k')
```



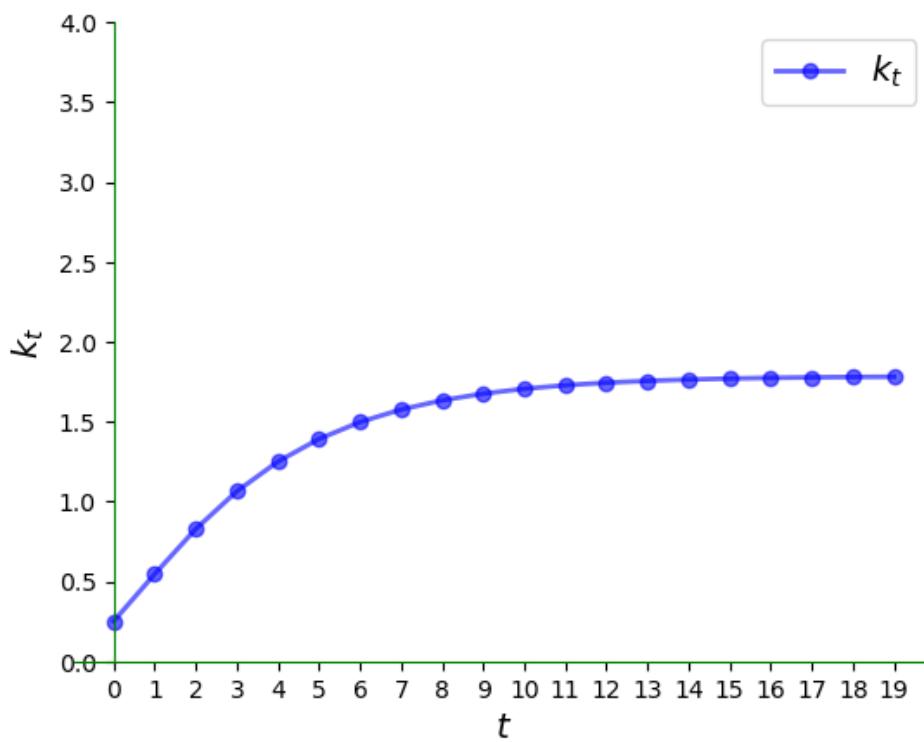
We can plot the time series of capital corresponding to the figure above as follows:

```
ts_plot(g, xmin, xmax, k0, var='k')
```



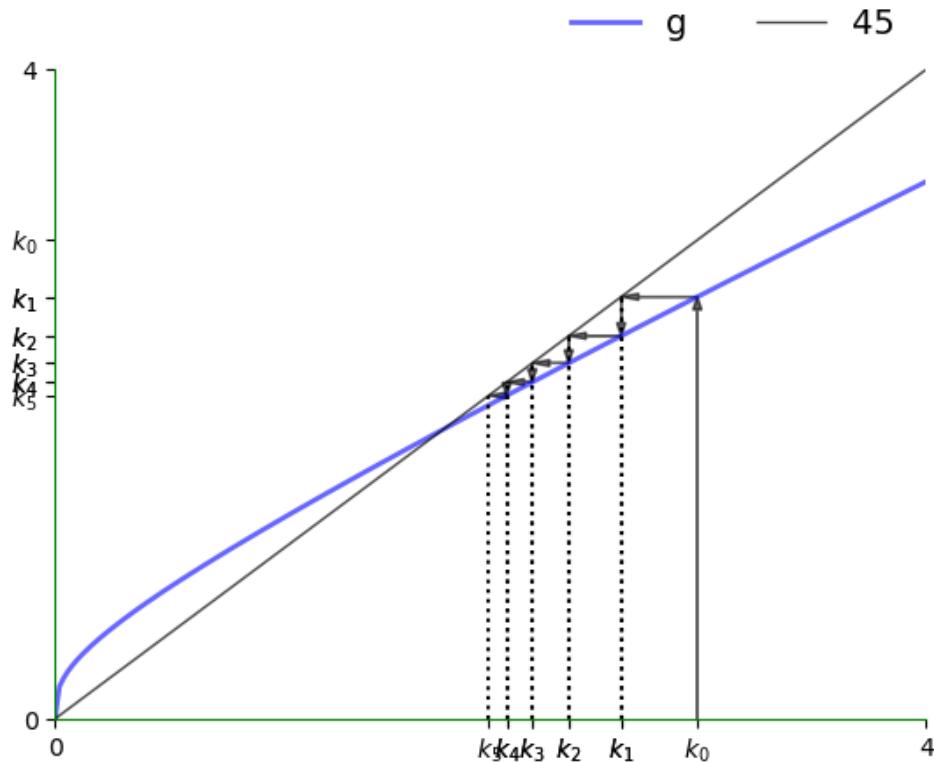
Here's a somewhat longer view:

```
ts_plot(g, xmin, xmax, k0, ts_length=20, var='k')
```



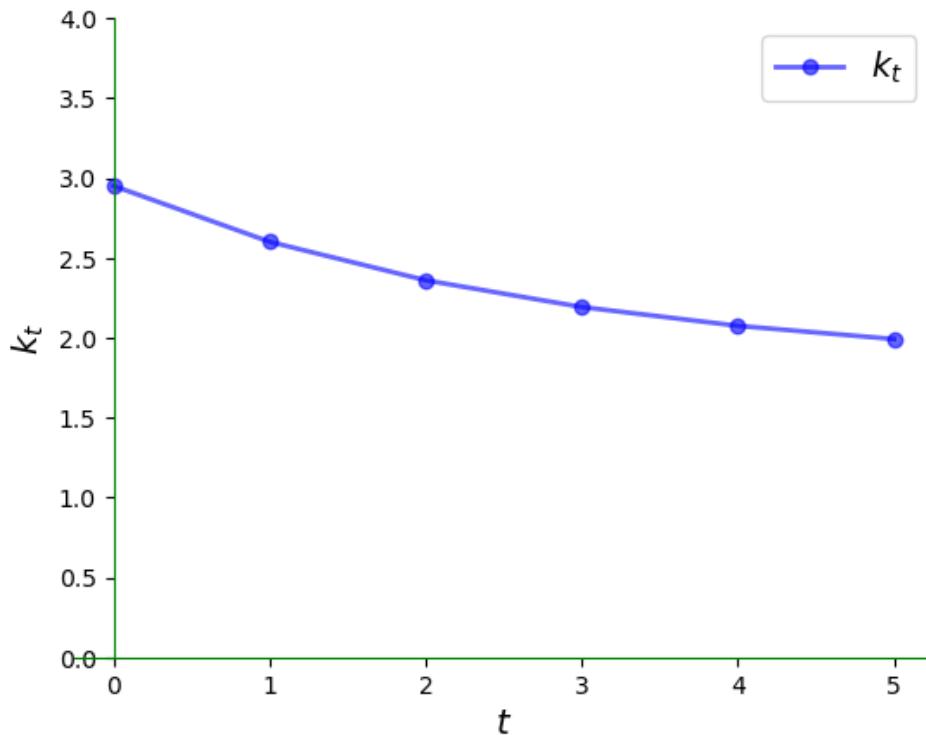
When capital stock is higher than the unique positive steady state, we see that it declines:

```
k0 = 2.95
plot45(g, xmin, xmax, k0, num_arrows=5, var='k')
```



Here is the time series:

```
ts_plot(g, xmin, xmax, k0, var='k')
```



## 21.4.2 Complex dynamics

The Solow model is nonlinear but still generates very regular dynamics.

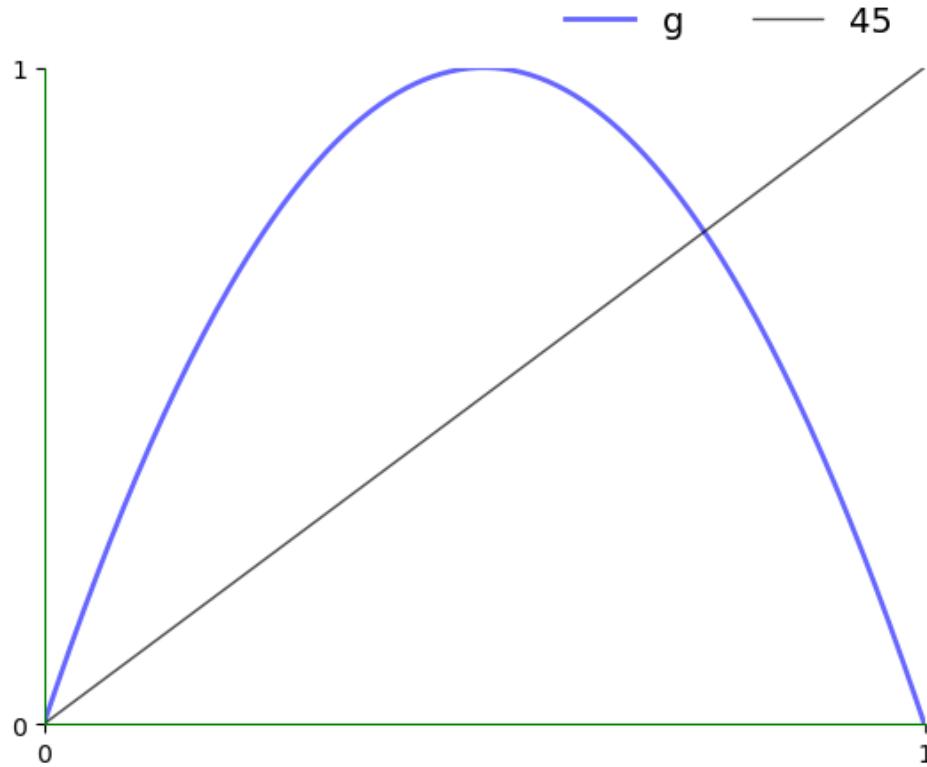
One model that generates irregular dynamics is the **quadratic map**

$$g(x) = 4x(1 - x), \quad x \in [0, 1]$$

Let's have a look at the 45 degree diagram.

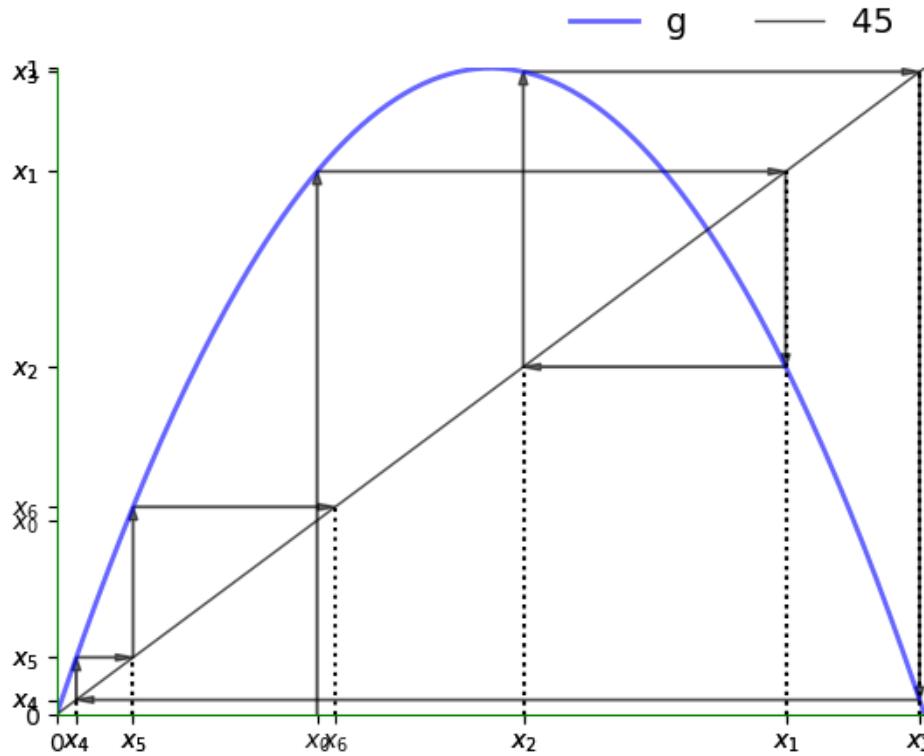
```
xmin, xmax = 0, 1
g = lambda x: 4 * x * (1 - x)

x0 = 0.3
plot45(g, xmin, xmax, x0, num_arrows=0)
```



Now let's look at a typical trajectory.

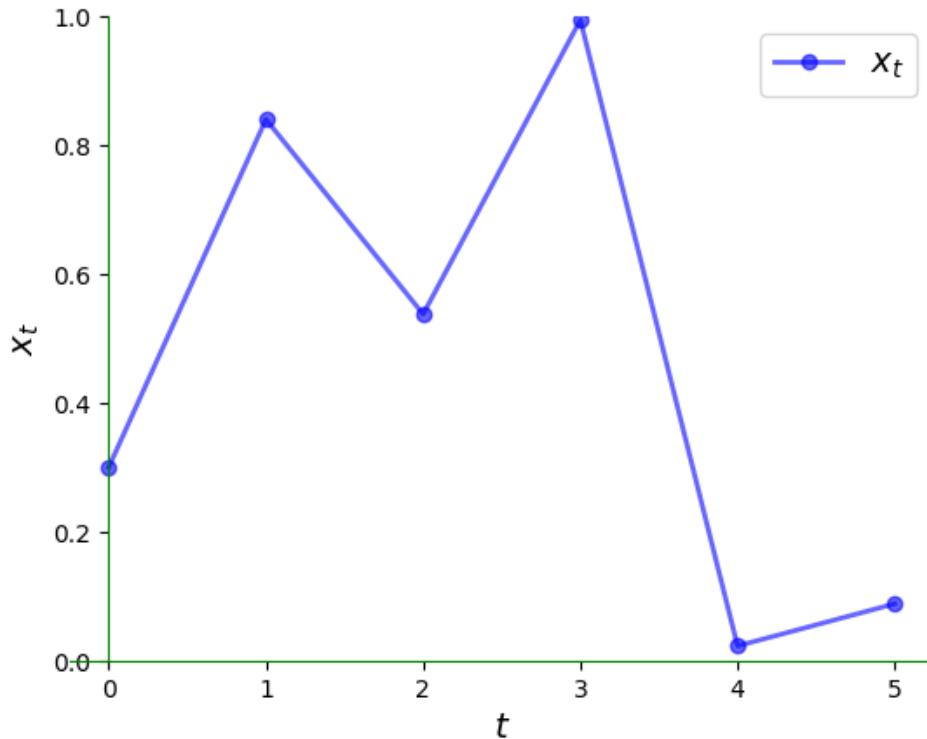
```
plot45(g, xmin, xmax, x0, num_arrows=6)
```



Notice how irregular it is.

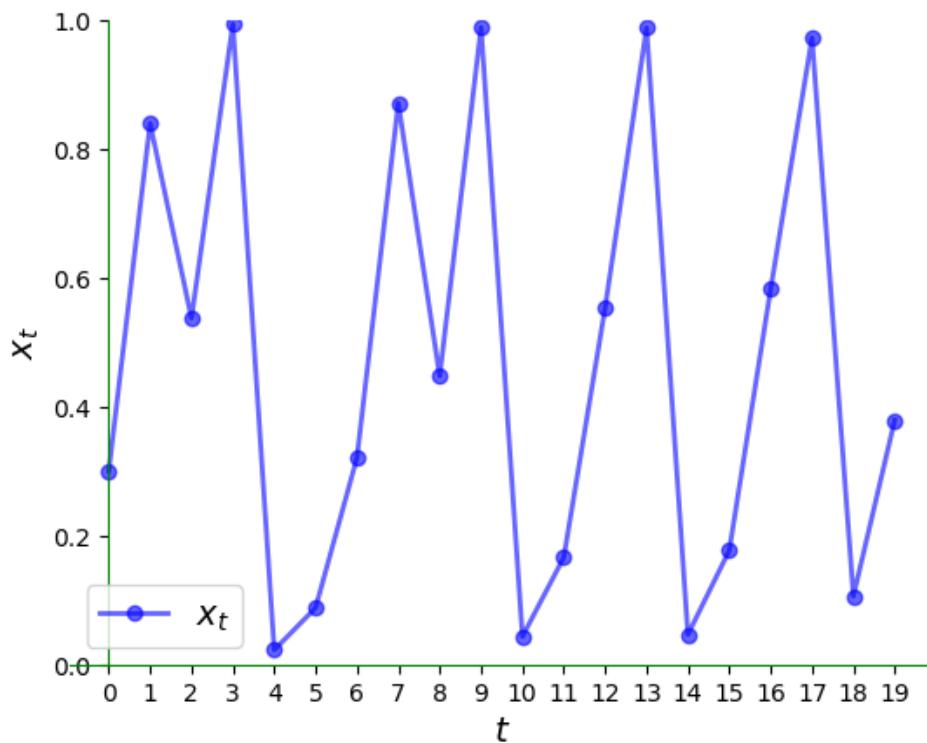
Here is the corresponding time series plot.

```
ts_plot(g, xmin, xmax, x0, ts_length=6)
```



The irregularity is even clearer over a longer time horizon:

```
ts_plot(g, xmin, xmax, x0, ts_length=20)
```



## 21.5 Exercises

---

### Exercise 21.5.1

Consider again the linear model  $x_{t+1} = ax_t + b$  with  $a \neq 1$ .

The unique steady state is  $b/(1 - a)$ .

The steady state is globally stable if  $|a| < 1$ .

Try to illustrate this graphically by looking at a range of initial conditions.

What differences do you notice in the cases  $a \in (-1, 0)$  and  $a \in (0, 1)$ ?

Use  $a = 0.5$  and then  $a = -0.5$  and study the trajectories

Set  $b = 1$  throughout.

---

### Solution to Exercise 21.5.1

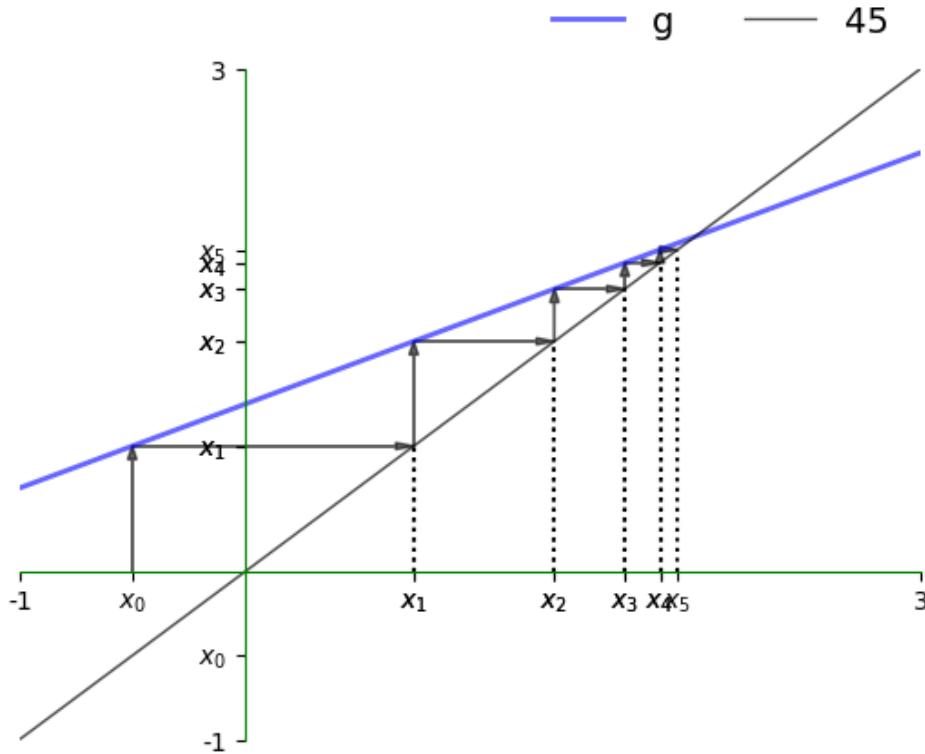
We will start with the case  $a = 0.5$ .

Let's set up the model and plotting region:

```
a, b = 0.5, 1
xmin, xmax = -1, 3
g = lambda x: a * x + b
```

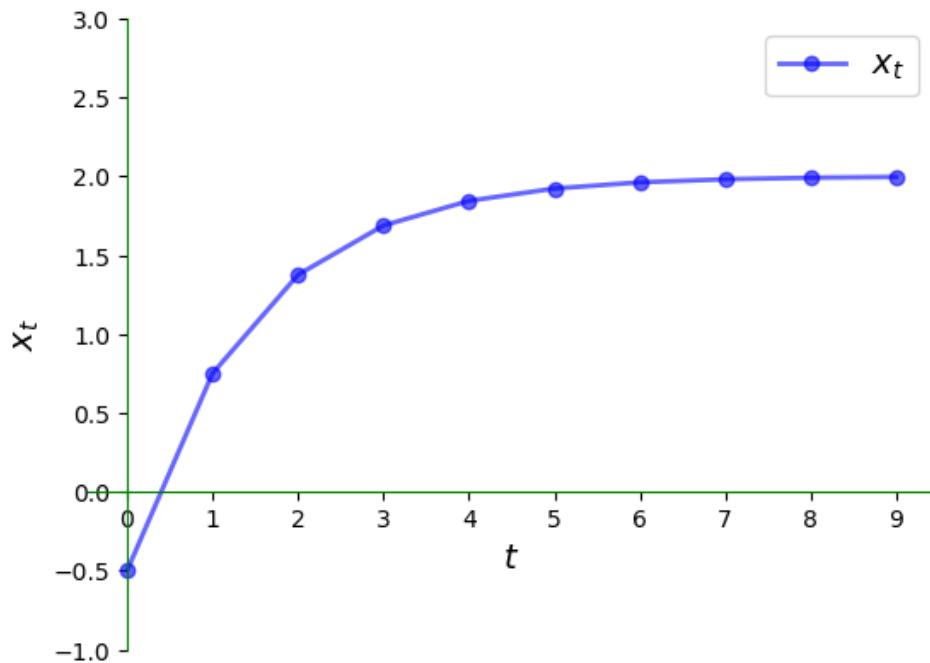
Now let's plot a trajectory:

```
x0 = -0.5
plot45(g, xmin, xmax, x0, num_arrows=5)
```



Here is the corresponding time series, which converges towards the steady state.

```
ts_plot(g, xmin, xmax, x0, ts_length=10)
```



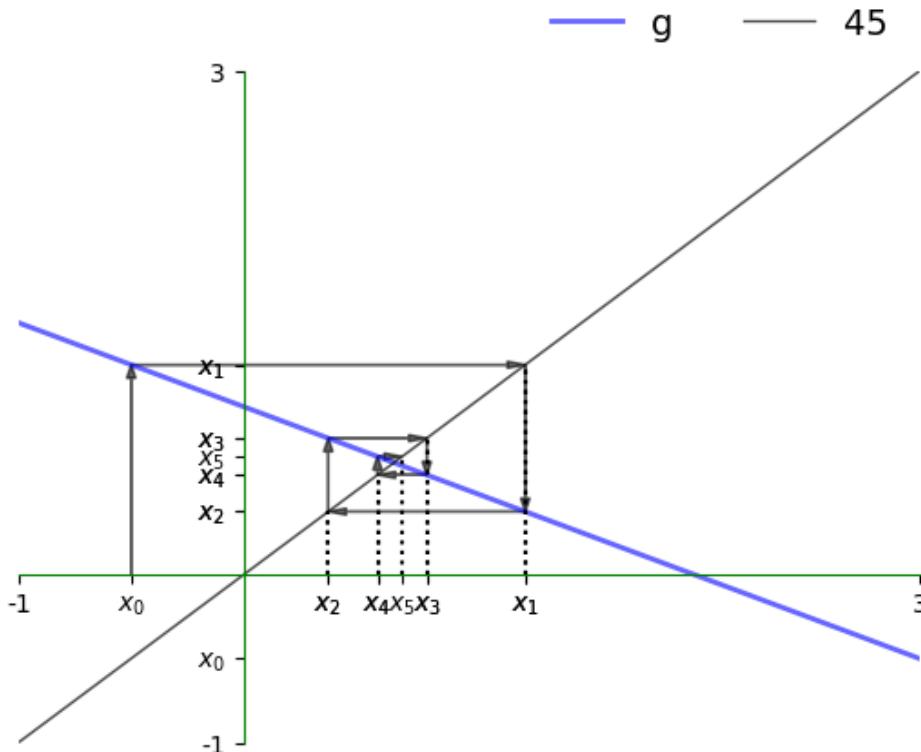
Now let's try  $a = -0.5$  and see what differences we observe.

Let's set up the model and plotting region:

```
a, b = -0.5, 1
xmin, xmax = -1, 3
g = lambda x: a * x + b
```

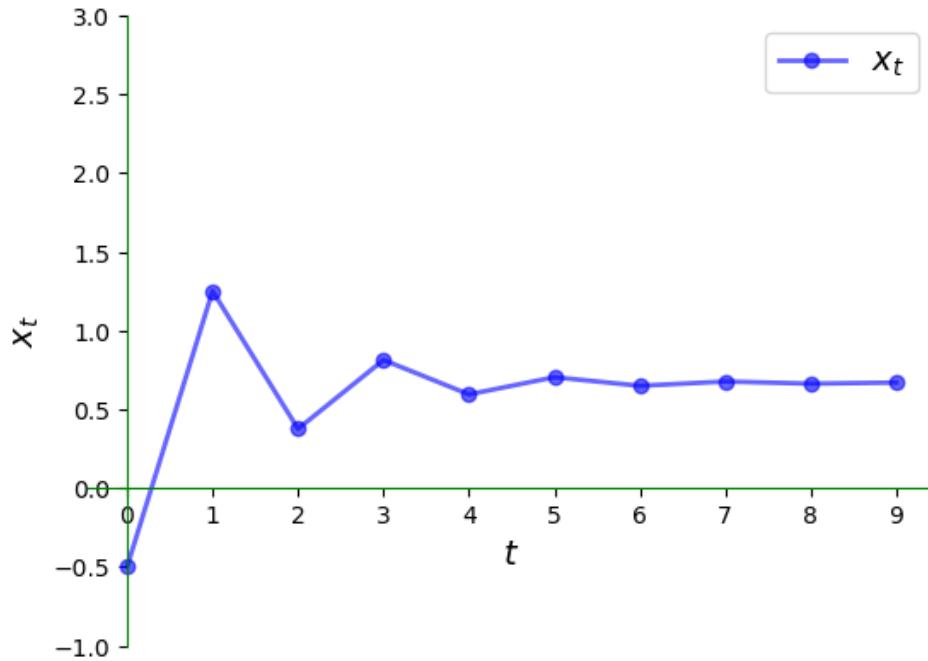
Now let's plot a trajectory:

```
x0 = -0.5
plot45(g, xmin, xmax, x0, num_arrows=5)
```



Here is the corresponding time series, which converges towards the steady state.

```
ts_plot(g, xmin, xmax, x0, ts_length=10)
```



Once again, we have convergence to the steady state but the nature of convergence differs.

In particular, the time series jumps from above the steady state to below it and back again.

In the current context, the series is said to exhibit **damped oscillations**.

---

---

CHAPTER  
**TWENTYTWO**

---

## THE COBWEB MODEL

The cobweb model is a model of prices and quantities in a given market, and how they evolve over time.

### 22.1 Overview

The cobweb model dates back to the 1930s and, while simple, it remains significant because it shows the fundamental importance of *expectations*.

To give some idea of how the model operates, and why expectations matter, imagine the following scenario.

There is a market for soy beans, say, where prices and traded quantities depend on the choices of buyers and sellers.

The buyers are represented by a demand curve — they buy more at low prices and less at high prices.

The sellers have a supply curve — they wish to sell more at high prices and less at low prices.

However, the sellers (who are farmers) need time to grow their crops.

Suppose now that the price is currently high.

Seeing this high price, and perhaps expecting that the high price will remain for some time, the farmers plant many fields with soy beans.

Next period the resulting high supply floods the market, causing the price to drop.

Seeing this low price, the farmers now shift out of soy beans, restricting supply and causing the price to climb again.

You can imagine how these dynamics could cause cycles in prices and quantities that persist over time.

The cobweb model puts these ideas into equations so we can try to quantify them, and to study conditions under which cycles persist (or disappear).

In this lecture, we investigate and simulate the basic model under different assumptions regarding the way that produces form expectations.

Our discussion and simulations draw on [high quality lectures by Cars Hommes](#).

We will use the following imports.

```
import numpy as np
import matplotlib.pyplot as plt
```

## 22.2 History

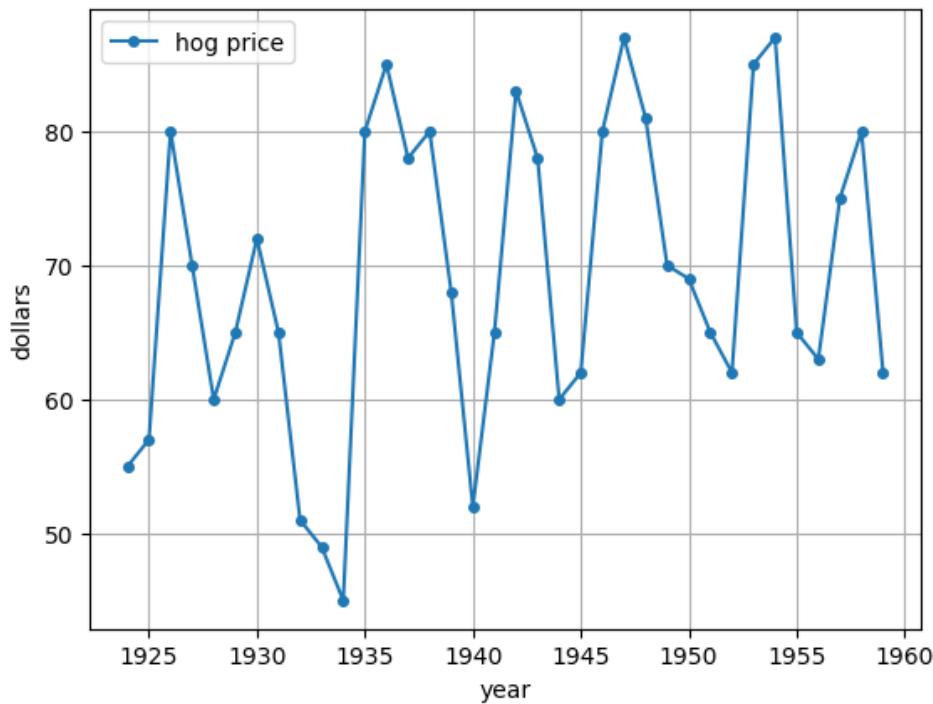
Early papers on the cobweb cycle include [Wau64] and [Har60].

The paper [Har60] uses the cobweb theorem to explain the prices of hog in the US over 1920–1950

The next plot replicates part of Figure 2 from that paper, which plots the price of hogs at yearly frequency.

Notice the cyclical price dynamics, which match the kind of cyclical soybean price dynamics discussed above.

```
hog_prices = [55, 57, 80, 70, 60, 65, 72, 65, 51, 49, 45, 80, 85,
              78, 80, 68, 52, 65, 83, 78, 60, 62, 80, 87, 81, 70,
              69, 65, 62, 85, 87, 65, 63, 75, 80, 62]
years = np.arange(1924, 1960)
fig, ax = plt.subplots()
ax.plot(years, hog_prices, '-o', ms=4, label='hog price')
ax.set_xlabel('year')
ax.set_ylabel('dollars')
ax.legend()
ax.grid()
plt.show()
```



## 22.3 The model

Let's return to our discussion of a hypothetical soy bean market, where price is determined by supply and demand.

We suppose that demand for soy beans is given by

$$D(p_t) = a - bp_t$$

where  $a, b$  are nonnegative constants and  $p_t$  is the spot (i.e, current market) price at time  $t$ .

( $D(p_t)$  is the quantity demanded in some fixed unit, such as thousands of tons.)

Because the crop of soy beans for time  $t$  is planted at  $t - 1$ , supply of soy beans at time  $t$  depends on *expected* prices at time  $t$ , which we denote  $p_{t-1}^e$ .

We suppose that supply is nonlinear in expected prices, and takes the form

$$S(p_{t-1}^e) = \tanh(\lambda(p_{t-1}^e - c)) + d$$

where  $\lambda$  is a positive constant and  $c, d \geq 0$ .

Let's make a plot of supply and demand for particular choices of the parameter values.

First we store the parameters in a class and define the functions above as methods.

```
class Market:
    def __init__(self,
                 a=8,          # demand parameter
                 b=1,          # demand parameter
                 c=6,          # supply parameter
                 d=1,          # supply parameter
                 λ=2.0):       # supply parameter
        self.a, self.b, self.c, self.d = a, b, c, d
        self.λ = λ

    def demand(self, p):
        a, b = self.a, self.b
        return a - b * p

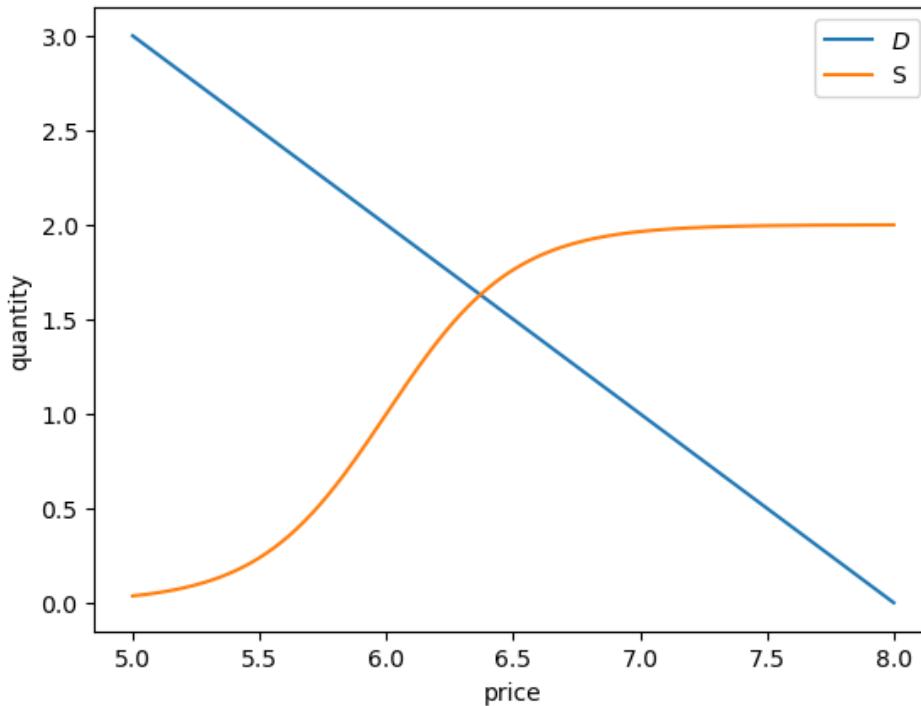
    def supply(self, p):
        c, d, λ = self.c, self.d, self.λ
        return np.tanh(λ * (p - c)) + d
```

Now let's plot.

```
p_grid = np.linspace(5, 8, 200)
m = Market()
fig, ax = plt.subplots()

ax.plot(p_grid, m.demand(p_grid), label="$D$")
ax.plot(p_grid, m.supply(p_grid), label="S")
ax.set_xlabel("price")
ax.set_ylabel("quantity")
ax.legend()

plt.show()
```



Market equilibrium requires that supply equals demand, or

$$a - bp_t = S(p_{t-1}^e)$$

Rewriting in terms of  $p_t$  gives

$$p_t = -\frac{1}{b}[S(p_{t-1}^e) - a]$$

Finally, to complete the model, we need to describe how price expectations are formed.

We will assume that expected prices at time  $t$  depend on past prices.

In particular, we suppose that

$$p_{t-1}^e = f(p_{t-1}, p_{t-2}) \quad (22.1)$$

where  $f$  is some function.

Thus, we are assuming that producers expect the time- $t$  price to be some function of lagged prices, up to 2 lags.

(We could of course add additional lags and readers are encouraged to experiment with such cases.)

Combining the last two equations gives the dynamics for prices:

$$p_t = -\frac{1}{b}[S(f(p_{t-1}, p_{t-2})) - a] \quad (22.2)$$

The price dynamics depend on the parameter values and also on the function  $f$  that determines how producers form expectations.

## 22.4 Naive expectations

To go further in our analysis we need to specify the function  $f$ ; that is, how expectations are formed.

Let's start with naive expectations, which refers to the case where producers expect the next period spot price to be whatever the price is in the current period.

In other words,

$$p_{t-1}^e = p_{t-1}$$

Using (22.2), we then have

$$p_t = -\frac{1}{b}[S(p_{t-1}) - a]$$

We can write this as

$$p_t = g(p_{t-1})$$

where  $g$  is the function defined by

$$g(p) = -\frac{1}{b}[S(p) - a] \quad (22.3)$$

Here we represent the function  $g$

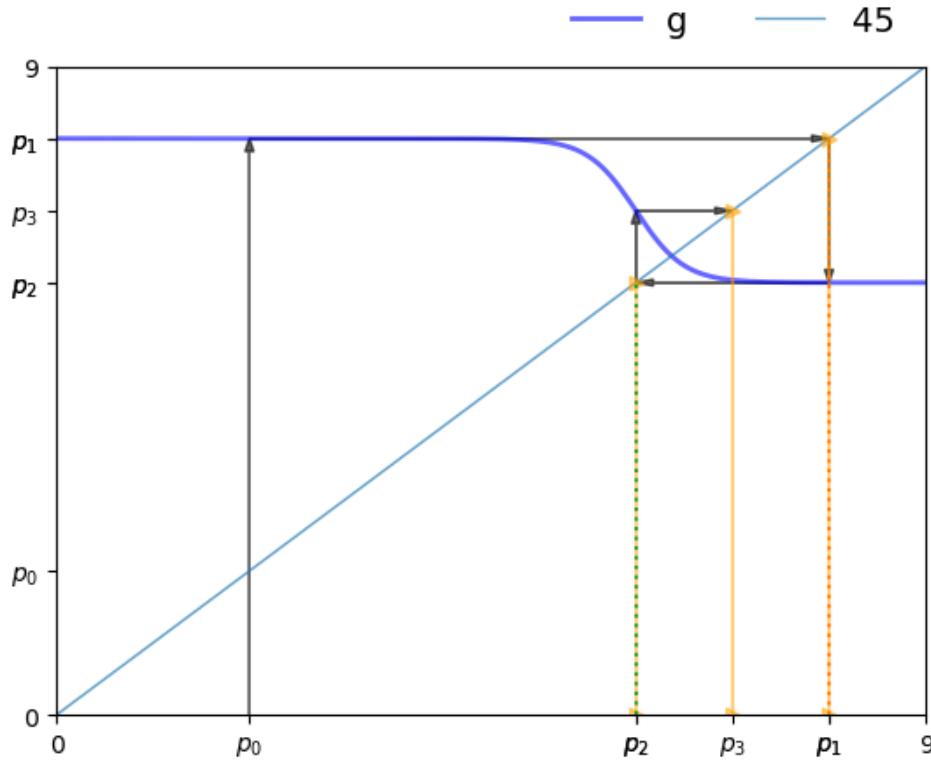
```
def g(model, current_price):
    """
    Function to find the next price given the current price
    and Market model
    """
    a, b = model.a, model.b
    next_price = - (model.supply(current_price) - a) / b
    return next_price
```

Let's try to understand how prices will evolve using a 45 degree diagram, which is a tool for studying one-dimensional dynamics.

The function `plot45` defined below helps us draw the 45 degree diagram.

Now we can set up a market and plot the 45 degree diagram.

```
m = Market()
plot45(m, 0, 9, 2, num_arrows=3)
```



The plot shows the function  $g$  defined in (22.3) and the 45 degree line.

Think of  $p_t$  as a value on the horizontal axis.

Since  $p_{t+1} = g(p_t)$ , we use the graph of  $g$  to see  $p_{t+1}$  on the vertical axis.

Clearly,

- If  $g$  lies above the 45 degree line at  $p_t$ , then we have  $p_{t+1} > p_t$ .
- If  $g$  lies below the 45 degree line at  $p_t$ , then we have  $p_{t+1} < p_t$ .
- If  $g$  hits the 45 degree line at  $p_t$ , then we have  $p_{t+1} = p_t$ , so  $p_t$  is a steady state.

Consider the sequence of prices starting at  $p_0$ , as shown in the figure.

We find  $p_1$  on the vertical axis and then shift it to the horizontal axis using the 45 degree line (where values on the two axes are equal).

Then from  $p_1$  we obtain  $p_2$  and continue.

We can see the start of a cycle.

To confirm this, let's plot a time series.

```

def ts_plot_price(model,
                  p0,                      # Market model
                  p0,                      # Initial price
                  y_a=3, y_b= 12,          # Controls y-axis
                  ts_length=10):           # Length of time series
    """
    Function to simulate and plot the time series of price.
    """

```

(continues on next page)

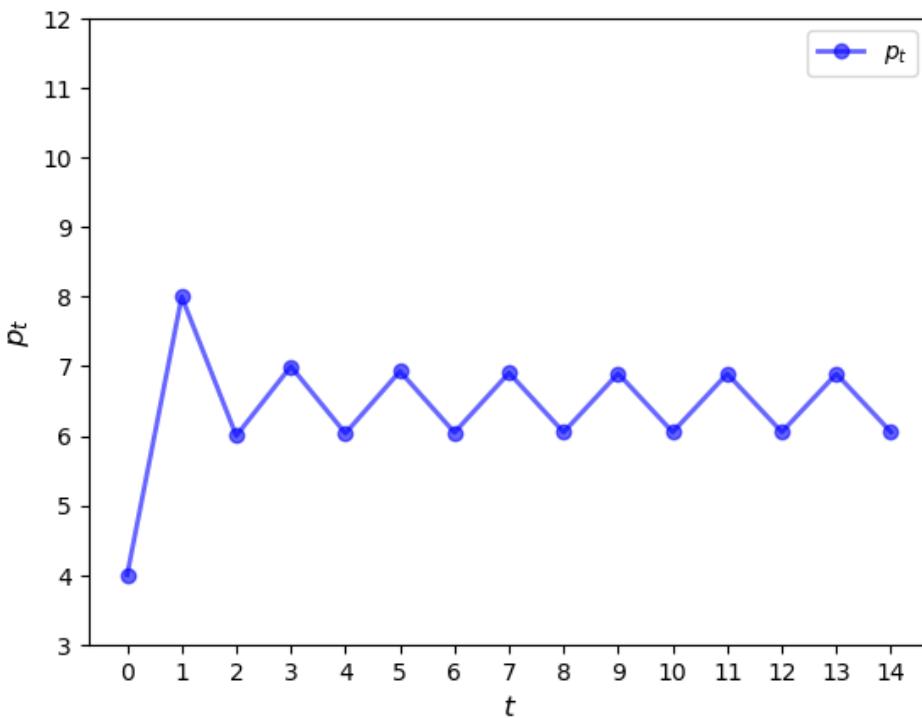
(continued from previous page)

```

fig, ax = plt.subplots()
ax.set_xlabel(r'$t$', fontsize=12)
ax.set_ylabel(r'$p_t$', fontsize=12)
p = np.empty(ts_length)
p[0] = p0
for t in range(1, ts_length):
    p[t] = g(model, p[t-1])
ax.plot(np.arange(ts_length),
        p,
        'bo-',
        alpha=0.6,
        lw=2,
        label=r'$p_t$')
ax.legend(loc='best', fontsize=10)
ax.set_xlim(y_a, y_b)
ax.set_xticks(np.arange(ts_length))
plt.show()

```

```
ts_plot_price(m, 4, ts_length=15)
```



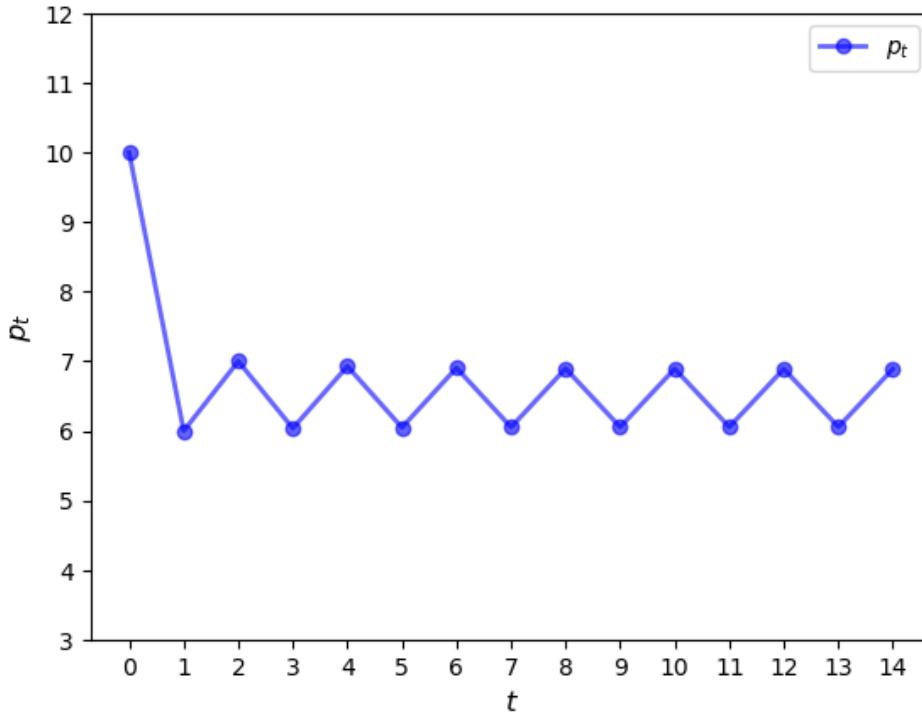
We see that a cycle has formed and the cycle is persistent.

(You can confirm this by plotting over a longer time horizon.)

The cycle is “stable”, in the sense that prices converge to it from most starting conditions.

For example,

```
ts_plot_price(m, 10, ts_length=15)
```



## 22.5 Adaptive expectations

Naive expectations are quite simple and also important in driving the cycle that we found.

What if expectations are formed in a different way?

Next we consider adaptive expectations.

This refers to the case where producers form expectations for the next period price as a weighted average of their last guess and the current spot price.

That is,

$$p_{t-1}^e = \alpha p_{t-1} + (1 - \alpha)p_{t-2}^e \quad (0 \leq \alpha \leq 1) \quad (22.4)$$

Another way to write this is

$$p_{t-1}^e = p_{t-2}^e + \alpha(p_{t-1} - p_{t-2}^e) \quad (22.5)$$

This equation helps to show that expectations shift

1. up when prices last period were above expectations
2. down when prices last period were below expectations

Using (22.4), we obtain the dynamics

$$p_t = -\frac{1}{b}[S(\alpha p_{t-1} + (1 - \alpha)p_{t-2}^e) - a]$$

Let's try to simulate the price and observe the dynamics using different values of  $\alpha$ .

```

def find_next_price_adaptive(model, curr_price_exp):
    """
    Function to find the next price given the current price expectation
    and Market model
    """
    return - (model.supply(curr_price_exp) - model.a) / model.b

```

The function below plots price dynamics under adaptive expectations for different values of  $\alpha$ .

```

def ts_price_plot_adaptive(model, p0, ts_length=10, a=[1.0, 0.9, 0.75]):
    fig, axs = plt.subplots(1, len(a), figsize=(12, 5))
    for i_plot, a in enumerate(a):
        pe_last = p0
        p_values = np.empty(ts_length)
        p_values[0] = p0
        for i in range(1, ts_length):
            p_values[i] = find_next_price_adaptive(model, pe_last)
            pe_last = a*p_values[i] + (1 - a)*pe_last

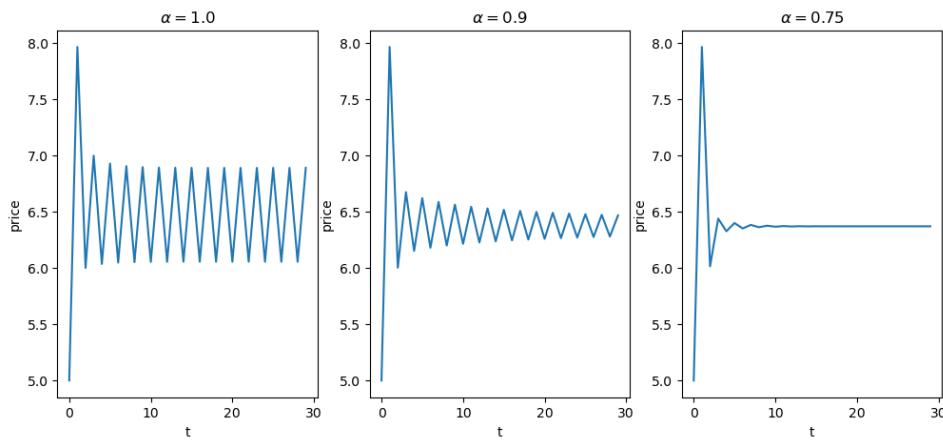
        axs[i_plot].plot(np.arange(ts_length), p_values)
        axs[i_plot].set_title(r'$\alpha=' + str(a) + '$')
        axs[i_plot].set_xlabel('t')
        axs[i_plot].set_ylabel('price')
    plt.show()

```

Let's call the function with prices starting at  $p_0 = 5$ .

TODO does this fit well in the page, even in the pdf? If not should it be stacked vertically?

```
ts_price_plot_adaptive(m, 5, ts_length=30)
```



Note that if  $\alpha = 1$ , then adaptive expectations are just naive expectation.

Decreasing the value of  $\alpha$  shifts more weight to the previous expectations, which stabilizes expected prices.

This increased stability can be seen in the figures.

TODO check / fix exercises

## 22.6 Exercises

---

### Exercise 22.6.1

Using the default `Market` class and naive expectations, plot a time series simulation of supply (rather than the price). Show, in particular, that supply also cycles.

---

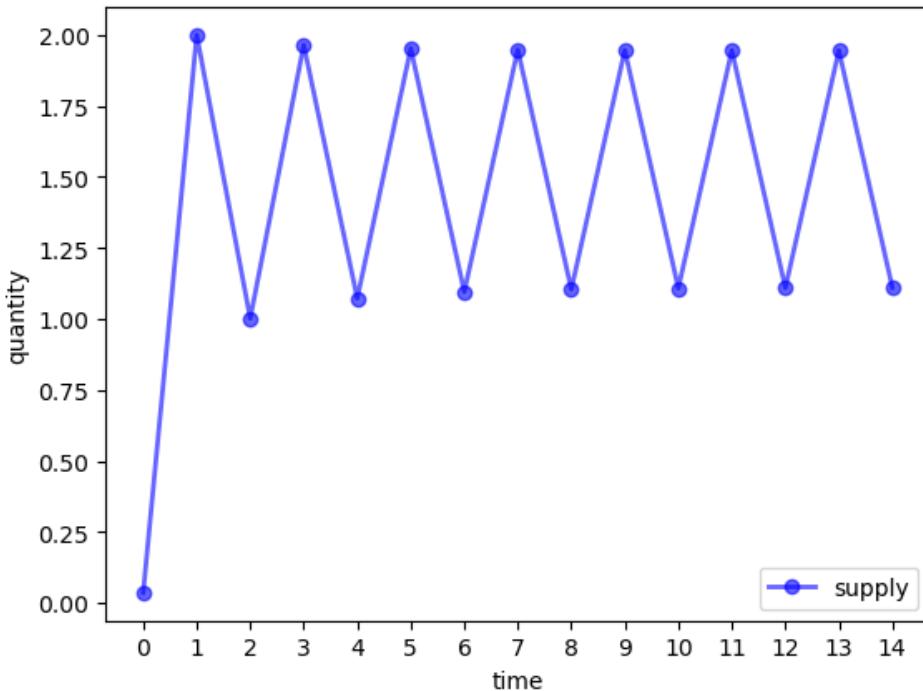
### Solution to Exercise 22.6.1

```
def ts_plot_supply(model, p0, ts_length=10):
    """
    Function to simulate and plot the supply function
    given the initial price.
    """
    pe_last = p0
    s_values = np.empty(ts_length)
    for i in range(ts_length):
        # store quantity
        s_values[i] = model.supply(pe_last)
        # update price
        pe_last = - (s_values[i] - model.a) / model.b

    fig, ax = plt.subplots()
    ax.plot(np.arange(ts_length),
            s_values,
            'bo-',
            alpha=0.6,
            lw=2,
            label=r'supply')

    ax.legend(loc='best', fontsize=10)
    ax.set_xticks(np.arange(ts_length))
    ax.set_xlabel("time")
    ax.set_ylabel("quantity")
    plt.show()
```

```
m = Market()
ts_plot_supply(m, 5, 15)
```




---

**Exercise 22.6.2**
**Backward looking average expectations**

Backward looking average expectations refers to the case where producers form expectations for the next period price as a linear combination of their last guess and the second last guess.

That is,

$$p_{t-1}^e = \alpha p_{t-1} + (1 - \alpha)p_{t-2} \quad (22.6)$$

Simulate and plot the price dynamics for  $\alpha \in \{0.1, 0.3, 0.5, 0.8\}$  where  $p_0 = 1$  and  $p_1 = 2.5$ .

---

**Solution to Exercise 22.6.2**

```
def find_next_price_blae(model, curr_price_exp):
    """
    Function to find the next price given the current price expectation
    and Market model
    """
    return - (model.supply(curr_price_exp) - model.a) / model.b
```

```
def ts_plot_price_blae(model, p0, p1, alphas, ts_length=15):
    """
    Function to simulate and plot the time series of price
    using backward looking average expectations.
    """
    fig, axes = plt.subplots(len(alphas), 1, figsize=(8, 16))
```

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```

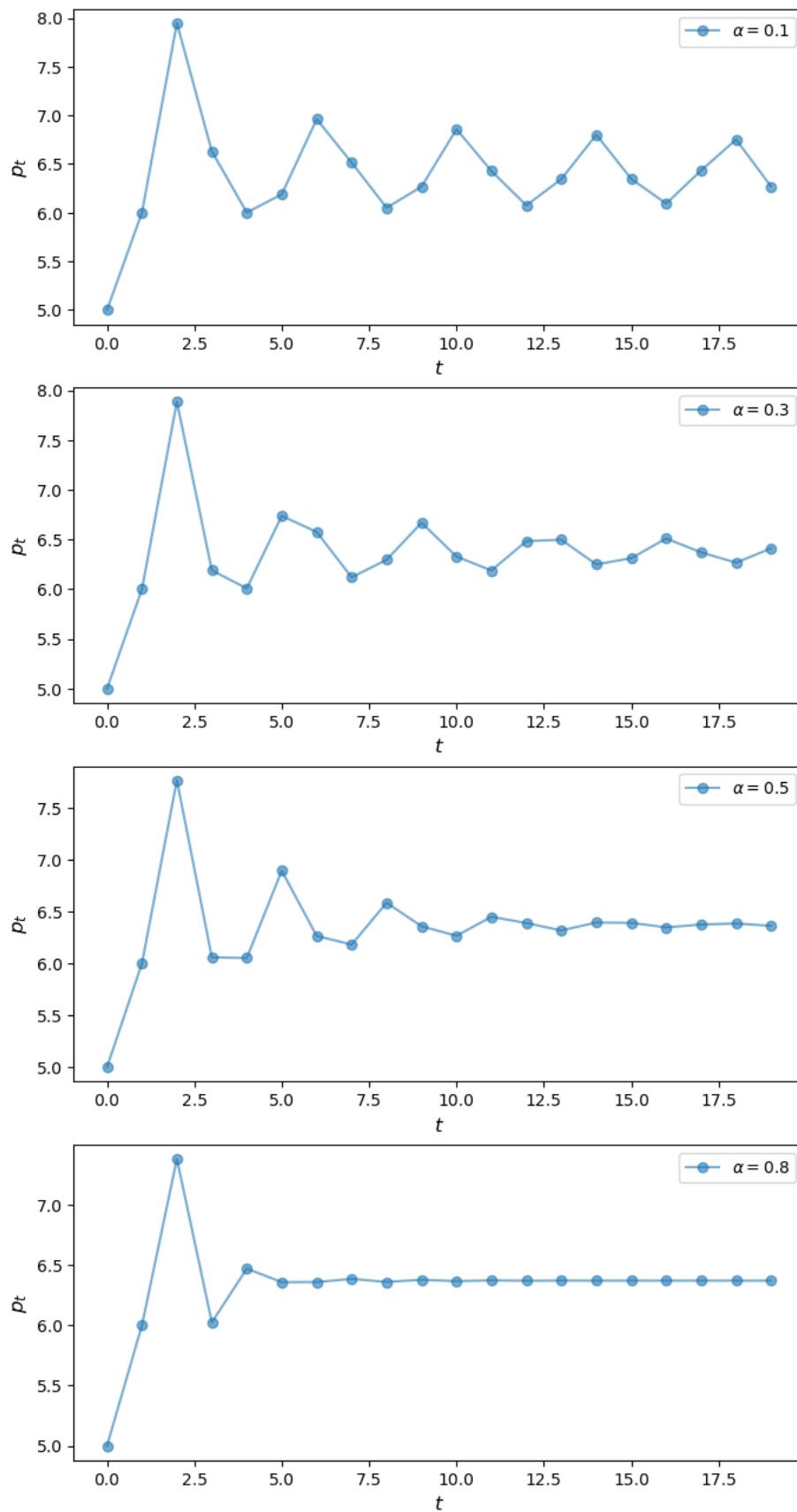
for ax, a in zip(axes.flatten(), alphas):
    p = np.empty(ts_length)
    p[0] = p0
    p[1] = p1
    for t in range(2, ts_length):
        pe = a*p[t-1] + (1 - a)*p[t-2]
        p[t] = -(model.supply(pe) - model.a) / model.b
    ax.plot(np.arange(ts_length),
            p,
            'o-',
            alpha=0.6,
            label=r'$\alpha={}$'.format(a))
ax.legend(loc='best', fontsize=10)
ax.set_xlabel(r'$t$', fontsize=12)
ax.set_ylabel(r'$p_t$', fontsize=12)
plt.show()

```

```

m = Market()
ts_plot_price_blae(m,
                    p0=5,
                    p1=6,
                    alphas=[0.1, 0.3, 0.5, 0.8],
                    ts_length=20)

```





---

CHAPTER  
**TWENTYTHREE**

---

## THE OVERLAPPING GENERATIONS MODEL

In this lecture we study the famous overlapping generations (OLG) model, which is used by policy makers and researchers to examine

- fiscal policy
- monetary policy
- long run growth

and many other topics.

The first rigorous version of the OLG model was developed by Paul Samuelson [Sam58].

Our aim is to gain a good understanding of a simple version of the OLG model.

### 23.1 Overview

The dynamics of the OLG model are quite similar to those of the Solow-Swan growth model.

At the same time, the OLG model adds an important new feature: the choice of how much to save is endogenous.

To see why this is important, suppose, for example, that we are interested in predicting the effect of a new tax on long-run growth.

We could add a tax to the Solow-Swan model and look at the change in the steady state.

But this ignores the fact that households will change their savings and consumption behavior when they face the new tax rate.

Such changes can substantially alter the predictions of the model.

Hence, if we care about accurate predictions, we should model the decision problems of the agents.

In particular, households in the model should decide how much to save and how much to consume, given the environment that they face (technology, taxes, prices, etc.)

The OLG model takes up this challenge.

We will present a simple version of the OLG model that clarifies the decision problem of households and studies the implications for long run growth.

Let's start with some imports.

```
import numpy as np
from scipy import optimize
from collections import namedtuple
```

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```
from functools import partial
import matplotlib.pyplot as plt
```

## 23.2 Environment

We assume that time is discrete, so that  $t = 0, 1, \dots$ ,

An individual born at time  $t$  lives for two periods,  $t$  and  $t + 1$ .

We call an agent

- “young” during the first period of their lives and
- “old” during the second period of their lives

Young agents work, supplying labor and earning labor income.

They also decide how much to save.

Old agents do not work, so all income is financial.

Their financial income is from interest on their savings from wage income, which is then combined with the labor of the new young generation at  $t + 1$ .

The wage and interest rates are determined in equilibrium by supply and demand.

To make the algebra slightly easier, we are going to assume a constant population size.

We normalize the constant population size in each period to 1.

We also suppose that each agent supplies one “unit” of labor hours, so total labor supply is 1.

## 23.3 Supply of capital

First let's consider the household side.

### 23.3.1 Consumer's problem

Suppose that utility for individuals born at time  $t$  takes the form

$$U_t = u(c_t) + \beta u(c_{t+1}) \quad (23.1)$$

Here

- $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  is called the “flow” utility function
- $\beta \in (0, 1)$  is the discount factor
- $c_t$  is time  $t$  consumption of the individual born at time  $t$
- $c_{t+1}$  is time  $t + 1$  consumption of the same individual

We assume that  $u$  is strictly increasing.

Savings behavior is determined by the optimization problem

$$\max_{c_t, c_{t+1}} \{u(c_t) + \beta u(c_{t+1})\} \quad (23.2)$$

subject to

$$c_t + s_t \leq w_t \quad \text{and} \quad c_{t+1} \leq R_{t+1}s_t$$

Here

- $s_t$  is savings by an individual born at time  $t$
- $w_t$  is the wage rate at time  $t$
- $R_{t+1}$  is the interest rate on savings invested at time  $t$ , paid at time  $t+1$

Since  $u$  is strictly increasing, both of these constraints will hold as equalities at the maximum.

Using this fact and substituting  $s_t$  from the first constraint into the second we get  $c_{t+1} = R_{t+1}(w_t - c_t)$ .

The first-order condition for a maximum can be obtained by plugging  $c_{t+1}$  into the objective function, taking the derivative with respect to  $c_t$ , and setting it to zero.

This leads to the **Euler equation** of the OLG model, which is

$$u'(c_t) = \beta R_{t+1} u'(R_{t+1}(w_t - c_t)) \quad (23.3)$$

From the first constraint we get  $c_t = w_t - s_t$ , so the Euler equation can also be expressed as

$$u'(w_t - s_t) = \beta R_{t+1} u'(R_{t+1}s_t) \quad (23.4)$$

Suppose that, for each  $w_t$  and  $R_{t+1}$ , there is exactly one  $s_t$  that solves (23.3.4).

Then savings can be written as a fixed function of  $w_t$  and  $R_{t+1}$ .

We write this as

$$s_t = s(w_t, R_{t+1}) \quad (23.5)$$

The precise form of the function  $s$  will depend on the choice of flow utility function  $u$ .

Together,  $w_t$  and  $R_{t+1}$  represent the *prices* in the economy (price of labor and rental rate of capital).

Thus, (23.3.5) states the quantity of savings given prices.

### 23.3.2 Example: log preferences

In the special case  $u(c) = \log c$ , the Euler equation simplifies to  $s_t = \beta(w_t - s_t)$ .

Solving for saving, we get

$$s_t = s(w_t, R_{t+1}) = \frac{\beta}{1 + \beta} w_t \quad (23.6)$$

In this special case, savings does not depend on the interest rate.

### 23.3.3 Savings and investment

Since the population size is normalized to 1,  $s_t$  is also total savings in the economy at time  $t$ .

In our closed economy, there is no foreign investment, so net savings equals total investment, which can be understood as supply of capital to firms.

In the next section we investigate demand for capital.

Equating supply and demand will allow us to determine equilibrium in the OLG economy.

## 23.4 Demand for capital

First we describe the firm problem and then we write down an equation describing demand for capital given prices.

### 23.4.1 Firm's problem

For each integer  $t \geq 0$ , output  $y_t$  in period  $t$  is given by the **Cobb-Douglas production function**

$$y_t = k_t^\alpha \ell_t^{1-\alpha} \quad (23.7)$$

Here  $k_t$  is capital,  $\ell_t$  is labor, and  $\alpha$  is a parameter (sometimes called the “output elasticity of capital”).

The profit maximization problem of the firm is

$$\max_{k_t, \ell_t} \{k_t^\alpha \ell_t^{1-\alpha} - R_t k_t - \ell_t w_t\} \quad (23.8)$$

The first-order conditions are obtained by taking the derivative of the objective function with respect to capital and labor respectively and setting them to zero:

$$(1-\alpha)(k_t/\ell_t)^\alpha = w_t \quad \text{and} \quad \alpha(k_t/\ell_t)^{\alpha-1} = R_t$$

### 23.4.2 Demand

Using our assumption  $\ell_t = 1$  allows us to write

$$w_t = (1-\alpha)k_t^\alpha \quad (23.9)$$

and

$$R_t = \alpha k_t^{\alpha-1} \quad (23.10)$$

Rearranging (23.4.4) gives the aggregate demand for capital at time  $t+1$

$$k^d(R_{t+1}) := \left( \frac{\alpha}{R_{t+1}} \right)^{1/(1-\alpha)} \quad (23.11)$$

In Python code this is

```
def capital_demand(R, a):
    return (a/R) ** (1 / (1-a))
```

The next figure plots the supply of capital, as in (23.3.6), as well as the demand for capital, as in (23.4.5), as functions of the interest rate  $R_{t+1}$ .

(For the special case of log utility, supply does not depend on the interest rate, so we have a constant function.)

```
R_vals = np.linspace(0.3, 1)
a, b = 0.5, 0.9
w = 2.0

fig, ax = plt.subplots()

ax.plot(R_vals, capital_demand(R_vals, a),
```

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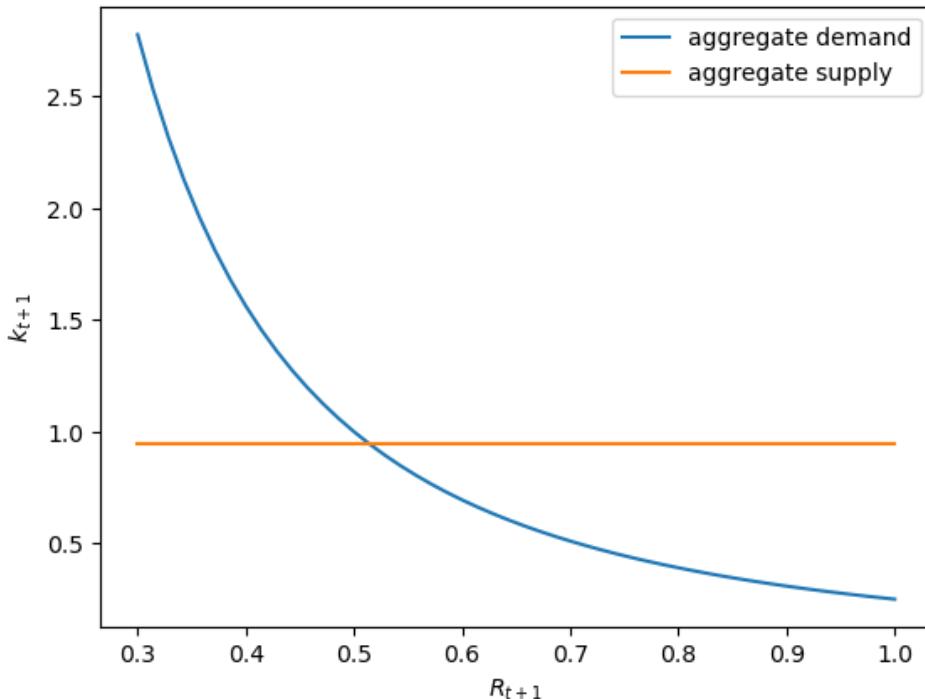
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```

label="aggregate demand")
ax.plot(R_vals, np.ones_like(R_vals) * (β / (1 + β)) * w,
        label="aggregate supply")

ax.set_xlabel("$R_{t+1}$")
ax.set_ylabel("$k_{t+1}$")
ax.legend()
plt.show()

```



## 23.5 Equilibrium

In this section we derive equilibrium conditions and investigate an example.

### 23.5.1 Equilibrium conditions

In equilibrium, savings at time  $t$  equals investment at time  $t$ , which equals capital supply at time  $t + 1$ .

Equilibrium is computed by equating these quantities, setting

$$s(w_t, R_{t+1}) = k^d(R_{t+1}) = \left( \frac{\alpha}{R_{t+1}} \right)^{1/(1-\alpha)} \quad (23.12)$$

In principle, we can now solve for the equilibrium price  $R_{t+1}$  given  $w_t$ .

(In practice, we first need to specify the function  $u$  and hence  $s$ .)

When we solve this equation, which concerns time  $t + 1$  outcomes, time  $t$  quantities are already determined, so we can treat  $w_t$  as a constant.

From equilibrium  $R_{t+1}$  and (23.4.5), we can obtain the equilibrium quantity  $k_{t+1}$ .

### 23.5.2 Example: log utility

In the case of log utility, we can use (23.5.1) and (23.3.6) to obtain

$$\frac{\beta}{1+\beta} w_t = \left( \frac{\alpha}{R_{t+1}} \right)^{1/(1-\alpha)} \quad (23.13)$$

Solving for the equilibrium interest rate gives

$$R_{t+1} = \alpha \left( \frac{\beta}{1+\beta} w_t \right)^{\alpha-1} \quad (23.14)$$

In Python we can compute this via

```
def equilibrium_R_log_utility(a, beta, w):
    R = a * ((beta * w) / (1 + beta)) ** (a - 1)
    return R
```

In the case of log utility, since capital supply does not depend on the interest rate, the equilibrium quantity is fixed by supply.

That is,

$$k_{t+1} = s(w_t, R_{t+1}) = \frac{\beta}{1+\beta} w_t \quad (23.15)$$

Let's redo our plot above but now inserting the equilibrium quantity and price.

```
R_vals = np.linspace(0.3, 1)
a, beta = 0.5, 0.9
w = 2.0

fig, ax = plt.subplots()

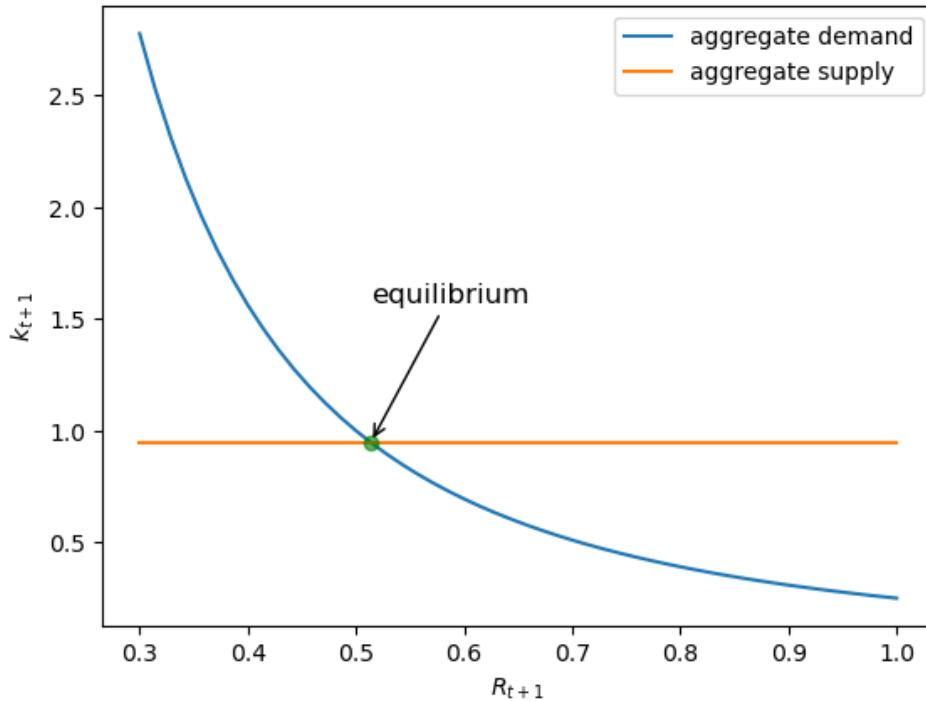
ax.plot(R_vals, capital_demand(R_vals, a),
        label="aggregate demand")
ax.plot(R_vals, np.ones_like(R_vals) * (beta / (1 + beta)) * w,
        label="aggregate supply")

R_e = equilibrium_R_log_utility(a, beta, w)
k_e = (beta / (1 + beta)) * w

ax.plot(R_e, k_e, 'go', ms=6, alpha=0.6)

ax.annotate(r'equilibrium',
            xy=(R_e, k_e),
            xycoords='data',
            xytext=(0, 60),
            textcoords='offset points',
            fontsize=12,
            arrowprops=dict(arrowstyle="->"))

ax.set_xlabel("$R_{t+1}$")
ax.set_ylabel("$k_{t+1}$")
ax.legend()
plt.show()
```



## 23.6 Dynamics

In this section we discuss dynamics.

For now we will focus on the case of log utility, so that the equilibrium is determined by (23.5.4).

### 23.6.1 Evolution of capital

The discussion above shows how equilibrium  $k_{t+1}$  is obtained given  $w_t$ .

From (23.4.3) we can translate this into  $k_{t+1}$  as a function of  $k_t$

In particular, since  $w_t = (1 - \alpha)k_t^\alpha$ , we have

$$k_{t+1} = \frac{\beta}{1 + \beta}(1 - \alpha)(k_t)^\alpha \quad (23.16)$$

If we iterate on this equation, we get a sequence for capital stock.

Let's plot the 45 degree diagram of these dynamics, which we write as

$$k_{t+1} = g(k_t) \quad \text{where } g(k) := \frac{\beta}{1 + \beta}(1 - \alpha)(k)^\alpha$$

```
def k_update(k, alpha, beta):
    return beta * (1 - alpha) * k**alpha / (1 + beta)
```

```

a, β = 0.5, 0.9
kmin, kmax = 0, 0.1
x = 1000
k_grid = np.linspace(kmin, kmax, x)
k_grid_next = np.empty_like(k_grid)

for i in range(x):
    k_grid_next[i] = k_update(k_grid[i], a, β)

fig, ax = plt.subplots(figsize=(6, 6))

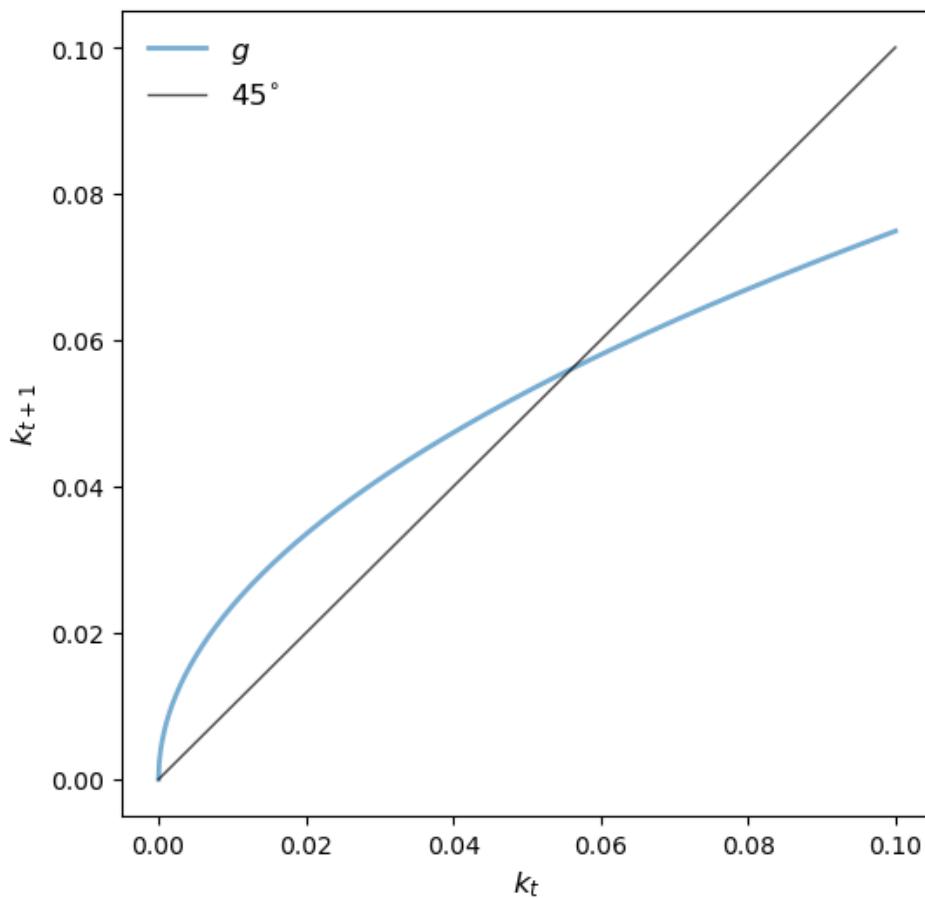
ymin, ymax = np.min(k_grid_next), np.max(k_grid_next)

ax.plot(k_grid, k_grid_next, lw=2, alpha=0.6, label='g')
ax.plot(k_grid, k_grid, 'k-', lw=1, alpha=0.7, label='45°')

ax.legend(loc='upper left', frameon=False, fontsize=12)
ax.set_xlabel('$k_t$', fontsize=12)
ax.set_ylabel('$k_{t+1}$', fontsize=12)

plt.show()

```



### 23.6.2 Steady state (log case)

The diagram shows that the model has a unique positive steady state, which we denote by  $k^*$ .

We can solve for  $k^*$  by setting  $k^* = g(k^*)$ , or

$$k^* = \frac{\beta(1-\alpha)(k^*)^\alpha}{(1+\beta)} \quad (23.17)$$

Solving this equation yields

$$k^* = \left( \frac{\beta(1-\alpha)}{1+\beta} \right)^{1/(1-\alpha)} \quad (23.18)$$

We can get the steady state interest rate from (23.4.4), which yields

$$R^* = \alpha(k^*)^{\alpha-1} = \frac{\alpha}{1-\alpha} \frac{1+\beta}{\beta}$$

In Python we have

```
k_star = ((beta * (1 - alpha)) / (1 + beta)) ** (1 / (1 - alpha))
R_star = (alpha / (1 - alpha)) * ((1 + beta) / beta)
```

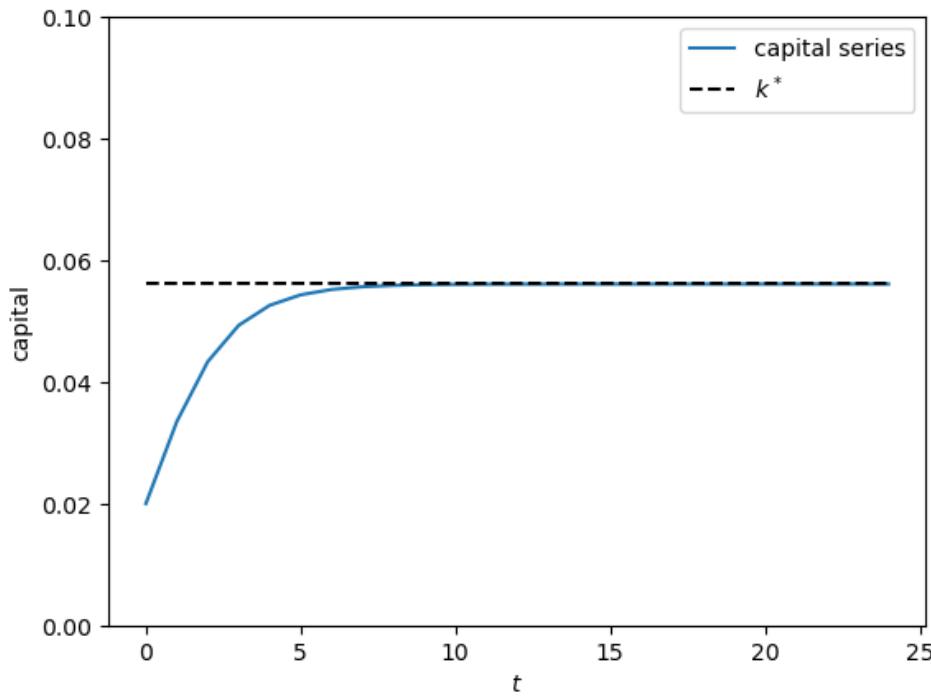
### 23.6.3 Time series

The 45 degree diagram above shows that time series of capital with positive initial conditions converge to this steady state.

Let's plot some time series that visualize this.

```
ts_length = 25
k_series = np.empty(ts_length)
k_series[0] = 0.02
for t in range(ts_length - 1):
    k_series[t+1] = k_update(k_series[t], alpha, beta)

fig, ax = plt.subplots()
ax.plot(k_series, label="capital series")
ax.plot(range(ts_length), np.full(ts_length, k_star), 'k--', label="$k^*$")
ax.set_xlim(0, 0.1)
ax.set_ylabel("capital")
ax.set_xlabel("$t$")
ax.legend()
plt.show()
```

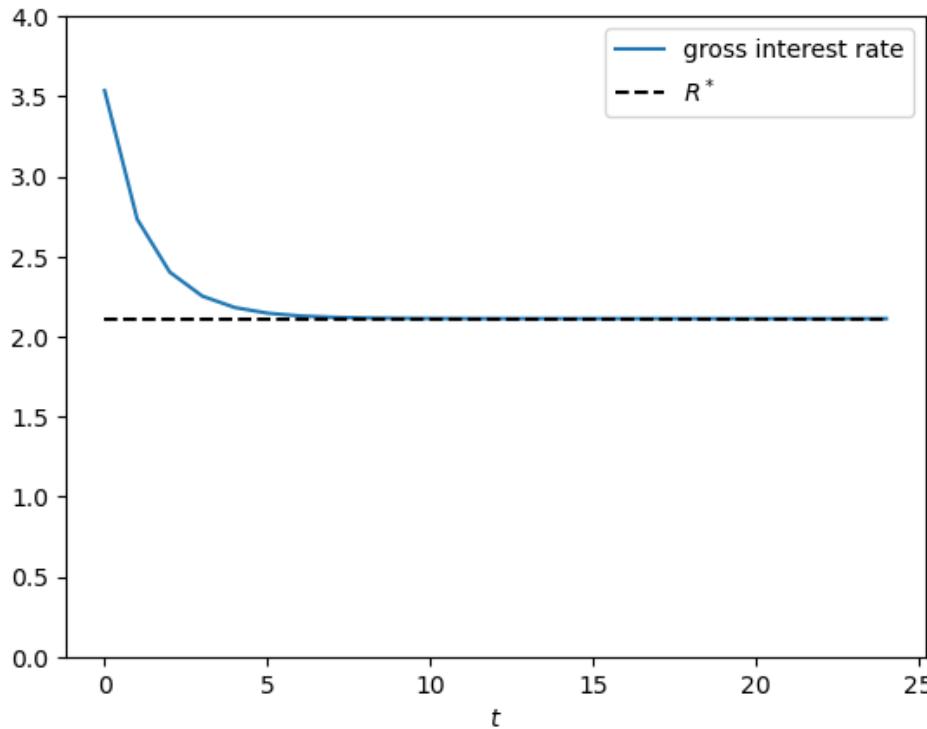


If you experiment with different positive initial conditions, you will see that the series always converges to  $k^*$ .

Below we also plot the gross interest rate over time.

```
R_series = a * k_series**(a - 1)

fig, ax = plt.subplots()
ax.plot(R_series, label="gross interest rate")
ax.plot(range(ts_length), np.full(ts_length, R_star), 'k--', label="$R^*$")
ax.set_xlim(0, 4)
ax.set_xlabel("$t$")
ax.legend()
plt.show()
```



The interest rate reflects the marginal product of capital, which is high when capital stock is low.

## 23.7 CRRA preferences

Previously, in our examples, we looked at the case of log utility.

Log utility is a rather special case.

In this section, we are going to assume that  $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$ , where  $\gamma > 0, \gamma \neq 1$ .

This function is called the CRRA utility function.

In other respects, the model is the same.

Below we define the utility function in Python and construct a namedtuple to store the parameters.

```

def crra(c, y):
    return c**(1 - y) / (1 - y)

Model = namedtuple('Model', ['alpha',           # Cobb-Douglas parameter
                            'beta',            # discount factor
                            'gamma']          # parameter in CRRA utility
                  )

def create_olg_model(alpha=0.4, beta=0.9, gamma=0.5):
    return Model(alpha=alpha, beta=beta, gamma=gamma)

```

Let's also redefine the capital demand function to work with this namedtuple.

```

def capital_demand(r, model):
    return (alpha/r)**(1/(1-model.alpha))

```

### 23.7.1 Supply

For households, the Euler equation becomes

$$(w_t - s_t)^{-\gamma} = \beta R_{t+1}^{1-\gamma} (s_t)^{-\gamma} \quad (23.19)$$

Solving for savings, we have

$$s_t = s(w_t, R_{t+1}) = w_t \left[ 1 + \beta^{-1/\gamma} R_{t+1}^{(\gamma-1)/\gamma} \right]^{-1} \quad (23.20)$$

Notice how, unlike the log case, savings now depends on the interest rate.

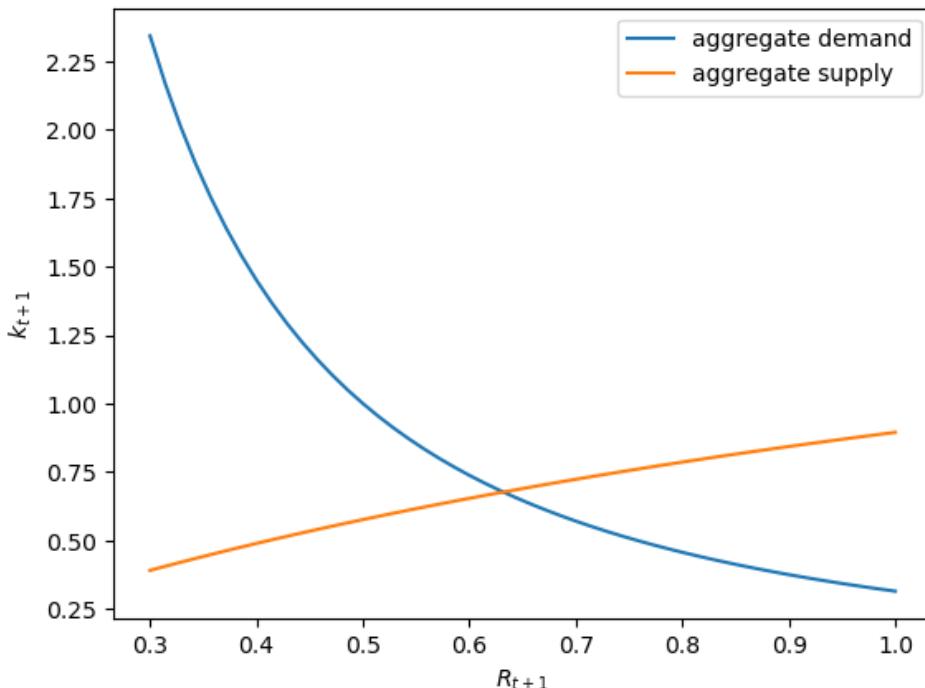
```
def savings_crra(w, R, model):
    a, beta, Y = model
    return w / (1 + beta**(-1/Y) * R**((Y-1)/Y))
```

```
R_vals = np.linspace(0.3, 1)
model = create_olg_model()
w = 2.0

fig, ax = plt.subplots()

ax.plot(R_vals, capital_demand(R_vals, model),
        label="aggregate demand")
ax.plot(R_vals, savings_crra(w, R_vals, model),
        label="aggregate supply")

ax.set_xlabel("$R_{t+1}$")
ax.set_ylabel("$k_{t+1}$")
ax.legend()
plt.show()
```



### 23.7.2 Equilibrium

Equating aggregate demand for capital (see (23.4.5)) with our new aggregate supply function yields equilibrium capital.

Thus, we set

$$w_t \left[ 1 + \beta^{-1/\gamma} R_{t+1}^{(\gamma-1)/\gamma} \right]^{-1} = \left( \frac{R_{t+1}}{\alpha} \right)^{1/(\alpha-1)} \quad (23.21)$$

This expression is quite complex and we cannot solve for  $R_{t+1}$  analytically.

Combining (23.4.4) and (23.7.3) yields

$$k_{t+1} = \left[ 1 + \beta^{-1/\gamma} (\alpha k_{t+1}^{\alpha-1})^{(\gamma-1)/\gamma} \right]^{-1} (1 - \alpha) (k_t)^\alpha \quad (23.22)$$

Again, with this equation and  $k_t$  as given, we cannot solve for  $k_{t+1}$  by pencil and paper.

In the exercise below, you will be asked to solve these equations numerically.

## 23.8 Exercises

### Exercise 23.8.1

Solve for the dynamics of equilibrium capital stock in the CRRA case numerically using (23.7.4).

Visualize the dynamics using a 45 degree diagram.

### Solution to Exercise 23.8.1

To solve for  $k_{t+1}$  given  $k_t$  we use Newton's method.

Let

$$f(k_{t+1}, k_t) = k_{t+1} \left[ 1 + \beta^{-1/\gamma} (\alpha k_{t+1}^{\alpha-1})^{(\gamma-1)/\gamma} \right] - (1 - \alpha) k_t^\alpha = 0 \quad (23.23)$$

If  $k_t$  is given then  $f$  is a function of unknown  $k_{t+1}$ .

Then we can use `scipy.optimize.newton` to solve  $f(k_{t+1}, k_t) = 0$  for  $k_{t+1}$ .

First let's define  $f$ .

```
def f(k_prime, k, model):
    alpha, beta, gamma = model.alpha, model.beta, model.gamma
    z = (1 - alpha) * k**alpha
    a = alpha**((1-gamma)/gamma)
    b = k_prime**((alpha * gamma - alpha + 1) / gamma)
    p = k_prime + k_prime * beta**((1-gamma)/gamma) * a * b
    return p - z
```

Now let's define a function that finds the value of  $k_{t+1}$ .

```
def k_update(k, model):
    return optimize.newton(lambda k_prime: f(k_prime, k, model), 0.1)
```

Finally, here is the 45 degree diagram.

```

kmin, kmax = 0, 0.5
x = 1000
k_grid = np.linspace(kmin, kmax, x)
k_grid_next = np.empty_like(k_grid)

for i in range(x):
    k_grid_next[i] = k_update(k_grid[i], model)

fig, ax = plt.subplots(figsize=(6, 6))

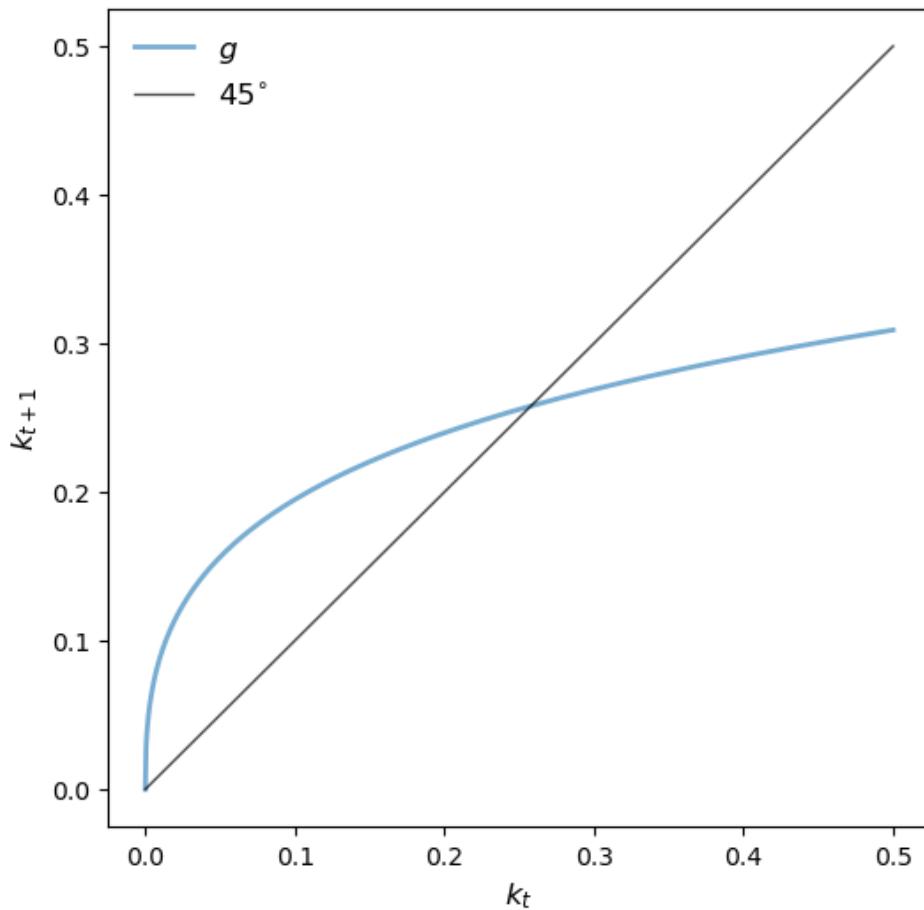
ymin, ymax = np.min(k_grid_next), np.max(k_grid_next)

ax.plot(k_grid, k_grid_next, lw=2, alpha=0.6, label='g')
ax.plot(k_grid, k_grid, 'k-', lw=1, alpha=0.7, label='45°\circ')

ax.legend(loc='upper left', frameon=False, fontsize=12)
ax.set_xlabel('$k_t$', fontsize=12)
ax.set_ylabel('$k_{t+1}$', fontsize=12)

plt.show()

```




---

Exercise 23.8.2

---

The 45 degree diagram from the last exercise shows that there is a unique positive steady state.

The positive steady state can be obtained by setting  $k_{t+1} = k_t = k^*$  in (23.7.4), which yields

$$k^* = \frac{(1-\alpha)(k^*)^\alpha}{1 + \beta^{-1/\gamma}(\alpha(k^*)^{\alpha-1})^{(\gamma-1)/\gamma}}$$

Unlike the log preference case, the CRRA utility steady state  $k^*$  cannot be obtained analytically.

Instead, we solve for  $k^*$  using Newton's method.

---

### Solution to Exercise 23.8.2

We introduce a function  $h$  such that positive steady state is the root of  $h$ .

$$h(k^*) = k^* [1 + \beta^{-1/\gamma}(\alpha(k^*)^{\alpha-1})^{(\gamma-1)/\gamma}] - (1-\alpha)(k^*)^\alpha \quad (23.24)$$

Here it is in Python

```
def h(k_star, model):
    α, β, γ = model.α, model.β, model.γ
    z = (1 - α) * k_star**α
    R1 = α ** (1-1/γ)
    R2 = k_star**((α * γ - α + 1) / γ)
    p = k_star + k_star * β**(-1/γ) * R1 * R2
    return p - z
```

Let's apply Newton's method to find the root:

```
k_star = optimize.newton(h, 0.2, args=(model,))
print(f"k_star = {k_star}")
```

```
k_star = 0.25788950250843484
```

---

### Exercise 23.8.3

Generate three time paths for capital, from three distinct initial conditions, under the parameterization listed above.

Use initial conditions for  $k_0$  of 0.001, 1.2, 2.6 and time series length 10.

---

### Solution to Exercise 23.8.3

Let's define the constants and three distinct initial conditions

```
ts_length = 10
k0 = np.array([0.001, 1.2, 2.6])
```

```
def simulate_ts(model, k0_values, ts_length):
    fig, ax = plt.subplots()
    ts = np.zeros(ts_length)
```

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```

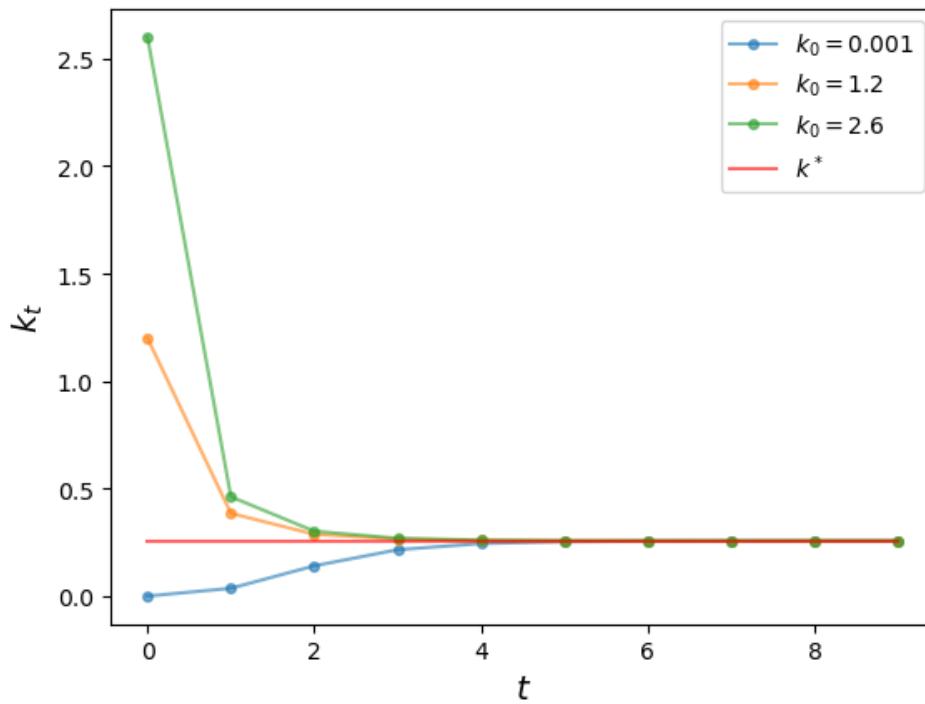
# simulate and plot time series
for k_init in k0_values:
    ts[0] = k_init
    for t in range(1, ts_length):
        ts[t] = k_update(ts[t-1], model)
    ax.plot(np.arange(ts_length), ts, '-o', ms=4, alpha=0.6,
            label=r'$k_0=%g$' %k_init)
    ax.plot(np.arange(ts_length), np.full(ts_length, k_star),
            alpha=0.6, color='red', label=r'$k^*$')
    ax.legend(fontsize=10)

    ax.set_xlabel(r'$t$', fontsize=14)
    ax.set_ylabel(r'$k_t$', fontsize=14)

plt.show()

```

```
simulate_ts(model, k0, ts_length)
```



---

CHAPTER  
**TWENTYFOUR**

---

## COMMODITY PRICES

### 24.1 Outline

For more than half of all countries around the globe, commodities account for the majority of total exports.

Examples of commodities include copper, diamonds, iron ore, lithium, cotton and coffee beans.

In this lecture we give an introduction to the theory of commodity prices.

The lecture is quite advanced relative to other lectures in this series.

We need to compute an equilibrium, and that equilibrium is described by a price function.

We will solve an equation where the price function is the unknown.

This is harder than solving an equation for an unknown number, or vector.

The lecture will discuss one way to solve a “functional equation” for an unknown function

For this lecture we need the `yfinance` library.

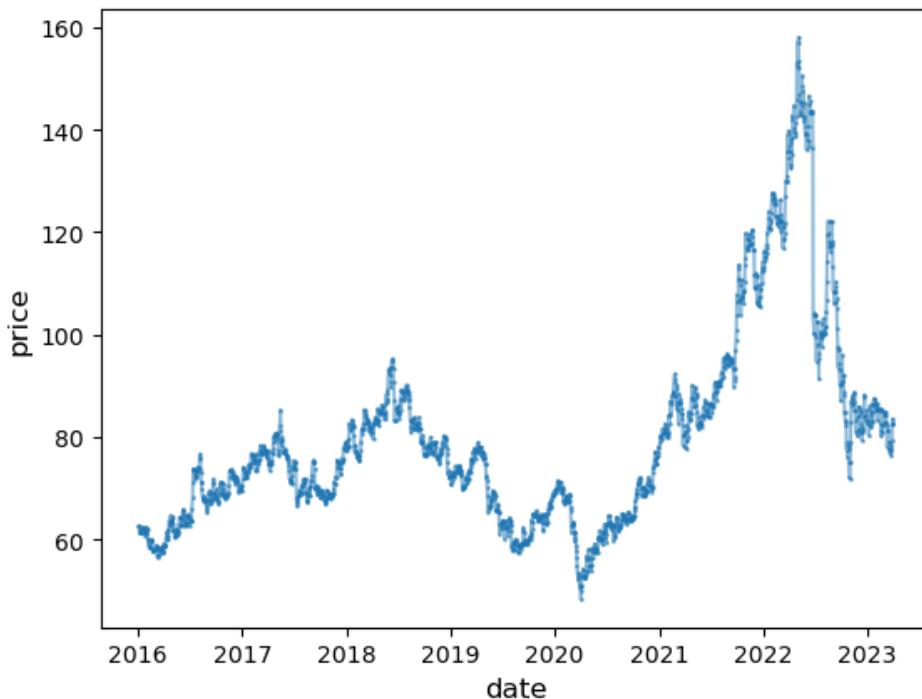
```
!pip install yfinance
```

We will use the following imports

```
import numpy as np
import yfinance as yf
import matplotlib.pyplot as plt
from scipy.interpolate import interp1d
from scipy.optimize import minimize_scalar, brentq
from scipy.stats import beta
```

### 24.2 Data

The figure below shows the price of cotton in USD since the start of 2016.



The figure shows surprisingly large movements in the price of cotton.

What causes these movements?

In general, prices depend on the choices and actions of

1. suppliers,
2. consumers, and
3. speculators.

Our focus will be on the interaction between these parties.

We will connect them together in a dynamic model of supply and demand, called the *competitive storage model*.

This model was developed by [Sam71], [WW82], [SS83], [DL92], [DL96], and [CB96].

## 24.3 The competitive storage model

In the competitive storage model, commodities are assets that

1. can be traded by speculators and
2. have intrinsic value to consumers.

Total demand is the sum of consumer demand and demand by speculators.

Supply is exogenous, depending on “harvests”.

---

**Note:** These days, goods such as basic computer chips and integrated circuits are often treated as commodities in financial markets, being highly standardized, and, for these kinds of commodities, the word “harvest” is not appropriate.

Nonetheless, we maintain it for simplicity.

---

The equilibrium price is determined competitively.

It is a function of the current state (which determines current harvests and predicts future harvests).

## 24.4 The model

Consider a market for a single commodity, whose price is given at  $t$  by  $p_t$ .

The harvest of the commodity at time  $t$  is  $Z_t$ .

We assume that the sequence  $\{Z_t\}_{t \geq 1}$  is IID with common density function  $\phi$ .

Speculators can store the commodity between periods, with  $I_t$  units purchased in the current period yielding  $\alpha I_t$  units in the next.

Here  $\alpha \in (0, 1)$  is a depreciation rate for the commodity.

For simplicity, the risk free interest rate is taken to be zero, so expected profit on purchasing  $I_t$  units is

$$\mathbb{E}_t p_{t+1} \cdot \alpha I_t - p_t I_t = (\alpha \mathbb{E}_t p_{t+1} - p_t) I_t$$

Here  $\mathbb{E}_t p_{t+1}$  is the expectation of  $p_{t+1}$  taken at time  $t$ .

## 24.5 Equilibrium

In this section we define the equilibrium and discuss how to compute it.

### 24.5.1 Equilibrium conditions

Speculators are assumed to be risk neutral, which means that they buy the commodity whenever expected profits are positive.

As a consequence, if expected profits are positive, then the market is not in equilibrium.

Hence, to be in equilibrium, prices must satisfy the “no-arbitrage” condition

$$\alpha \mathbb{E}_t p_{t+1} - p_t \leq 0 \tag{24.1}$$

Profit maximization gives the additional condition

$$\alpha \mathbb{E}_t p_{t+1} - p_t < 0 \text{ implies } I_t = 0 \tag{24.2}$$

We also require that the market clears in each period.

We assume that consumers generate demand quantity  $D(p)$  corresponding to price  $p$ .

Let  $P := D^{-1}$  be the inverse demand function.

Regarding quantities,

- supply is the sum of carryover by speculators and the current harvest
- demand is the sum of purchases by consumers and purchases by speculators.

Mathematically,

- supply =  $X_t = \alpha I_{t-1} + Z_t$ , which takes values in  $S := \mathbb{R}_+$ , while

- demand =  $D(p_t) + I_t$

Thus, the market equilibrium condition is

$$\alpha I_{t-1} + Z_t = D(p_t) + I_t \quad (24.3)$$

The initial condition  $X_0 \in S$  is treated as given.

## 24.5.2 An equilibrium function

How can we find an equilibrium?

Our path of attack will be to seek a system of prices that depend only on the current state.

In other words, we take a function  $p$  on  $S$  and set  $p_t = p(X_t)$  for every  $t$ .

Prices and quantities then follow

$$p_t = p(X_t), \quad I_t = X_t - D(p_t), \quad X_{t+1} = \alpha I_t + Z_{t+1} \quad (24.4)$$

We choose  $p$  so that these prices and quantities satisfy the equilibrium conditions above.

More precisely, we seek a  $p$  such that (24.5.1) and (24.5.2) hold for the corresponding system (24.5.4).

To this end, suppose that there exists a function  $p^*$  on  $S$  satisfying

$$p^*(x) = \max \left\{ \alpha \int_0^\infty p^*(\alpha I(x) + z) \phi(z) dz, P(x) \right\} \quad (x \in S) \quad (24.5)$$

where

$$I(x) := x - D(p^*(x)) \quad (x \in S) \quad (24.6)$$

It turns out that such a  $p^*$  will suffice, in the sense that (24.5.1) and (24.5.2) hold for the corresponding system (24.5.4).

To see this, observe first that

$$\mathbb{E}_t p_{t+1} = \mathbb{E}_t p^*(X_{t+1}) = \mathbb{E}_t p^*(\alpha I(X_t) + Z_{t+1}) = \int_0^\infty p^*(\alpha I(X_t) + z) \phi(z) dz$$

Thus (24.5.1) requires that

$$\alpha \int_0^\infty p^*(\alpha I(X_t) + z) \phi(z) dz \leq p^*(X_t)$$

This inequality is immediate from (24.5.5).

Second, regarding (24.5.2), suppose that

$$\alpha \int_0^\infty p^*(\alpha I(X_t) + z) \phi(z) dz < p^*(X_t)$$

Then by (24.5.5) we have  $p^*(X_t) = P(X_t)$

But then  $D(p^*(X_t)) = X_t$  and  $I_t = I(X_t) = 0$ .

As a consequence, both (24.5.1) and (24.5.2) hold.

We have found an equilibrium.

### 24.5.3 Computing the equilibrium

We now know that an equilibrium can be obtained by finding a function  $p^*$  that satisfies (24.5.5).

It can be shown that, under mild conditions there is exactly one function on  $S$  satisfying (24.5.5).

Moreover, we can compute this function using successive approximation.

This means that we start with a guess of the function and then update it using (24.5.5).

This generates a sequence of functions  $p_1, p_2, \dots$

We continue until this process converges, in the sense that  $p_k$  and  $p_{k+1}$  are very close together.

Then we take the final  $p_k$  that we computed as our approximation of  $p^*$ .

To implement our update step, it is helpful if we put (24.5.5) and (24.5.6) together.

This leads us to the update rule

$$p_{k+1}(x) = \max \left\{ \alpha \int_0^\infty p_k(\alpha(x - D(p_{k+1}(x))) + z) \phi(z) dz, P(x) \right\} \quad (24.7)$$

In other words, we take  $p_k$  as given and, at each  $x$ , solve for  $q$  in

$$q = \max \left\{ \alpha \int_0^\infty p_k(\alpha(x - D(q)) + z) \phi(z) dz, P(x) \right\} \quad (24.8)$$

Actually we can't do this at every  $x$ , so instead we do it on a grid of points  $x_1, \dots, x_n$ .

Then we get the corresponding values  $q_1, \dots, q_n$ .

Then we compute  $p_{k+1}$  as the linear interpolation of the values  $q_1, \dots, q_n$  over the grid  $x_1, \dots, x_n$ .

Then we repeat, seeking convergence.

## 24.6 Code

The code below implements this iterative process, starting from  $p_0 = P$ .

The distribution  $\phi$  is set to a shifted Beta distribution (although many other choices are possible).

The integral in (24.5.8) is computed via Monte Carlo.

```
a, a, c = 0.8, 1.0, 2.0
beta_a, beta_b = 5, 5
mc_draw_size = 250
gridsize = 150
grid_max = 35
grid = np.linspace(a, grid_max, gridsize)

beta_dist = beta(5, 5)
Z = a + beta_dist.rvs(mc_draw_size) * c      # Shock observations
D = P = lambda x: 1.0 / x
tol = 1e-4

def T(p_array):
    new_p = np.empty_like(p_array)
```

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```
# Interpolate to obtain p as a function.
p = interp1d(grid,
             p_array,
             fill_value=(p_array[0], p_array[-1]),
             bounds_error=False)

# Update
for i, x in enumerate(grid):

    h = lambda q: q - max(a * np.mean(p(a * (x - D(q)) + Z)), P(x))
    new_p[i] = brentq(h, 1e-8, 100)

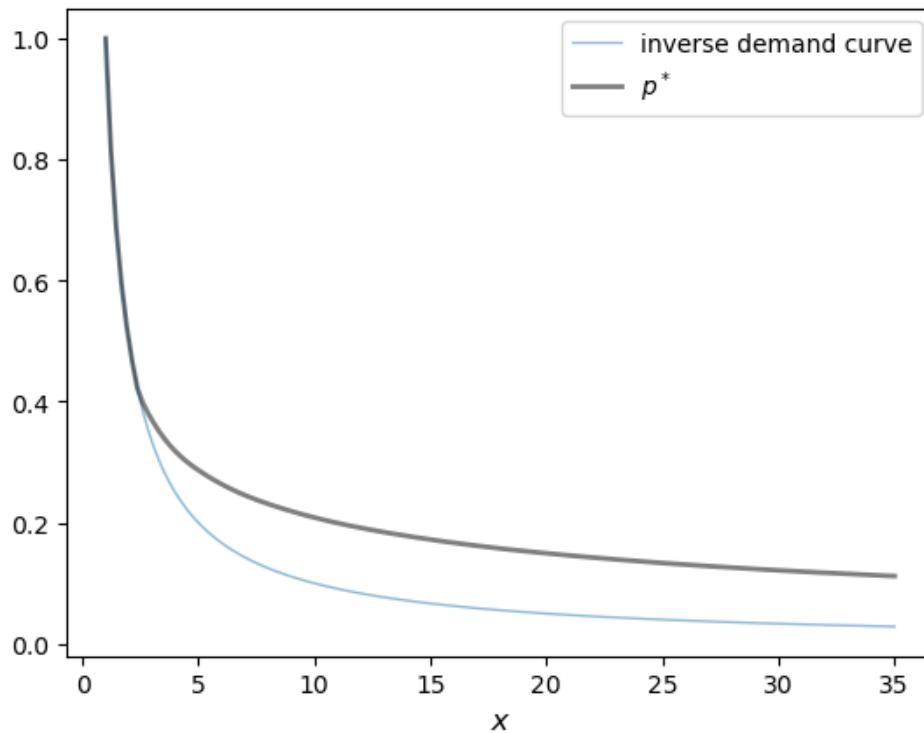
return new_p

fig, ax = plt.subplots()

price = P(grid)
ax.plot(grid, price, alpha=0.5, lw=1, label="inverse demand curve")
error = tol + 1
while error > tol:
    new_price = T(price)
    error = max(np.abs(new_price - price))
    price = new_price

ax.plot(grid, price, 'k-', alpha=0.5, lw=2, label=r'$p^*$')
ax.legend()
ax.set_xlabel('$x$', fontsize=12)

plt.show()
```



The figure above shows the inverse demand curve  $P$ , which is also  $p_0$ , as well as our approximation of  $p^*$ .

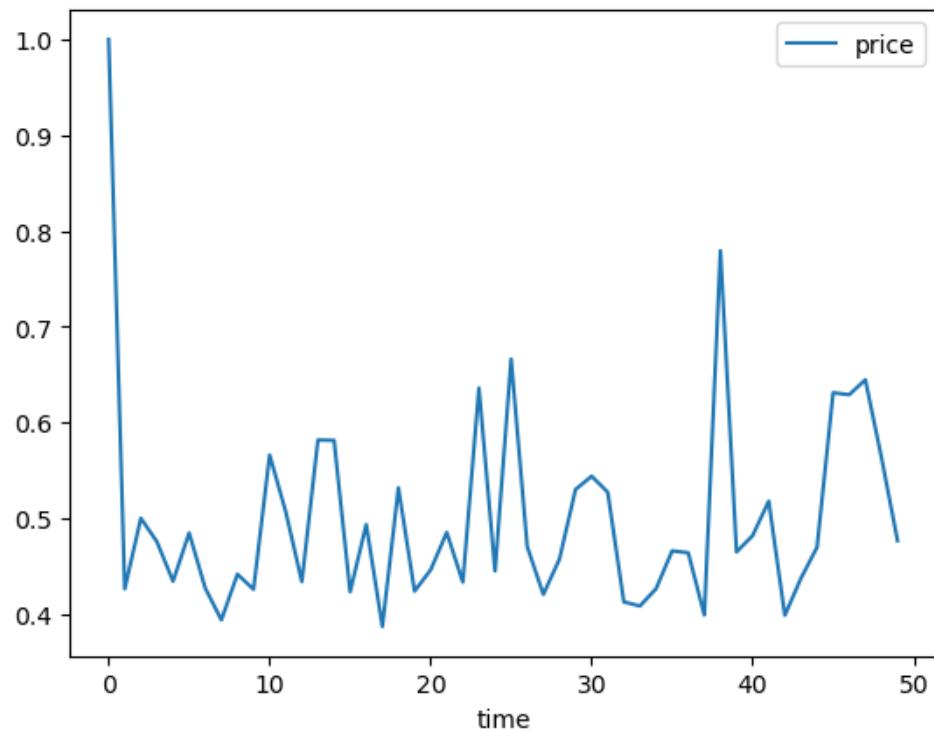
Once we have an approximation of  $p^*$ , we can simulate a time series of prices.

```
# Turn the price array into a price function
p_star = interp1d(grid,
                  price,
                  fill_value=(price[0], price[-1]),
                  bounds_error=False)

def carry_over(x):
    return a * (x - D(p_star(x)))

def generate_cp_ts(init=1, n=50):
    X = np.empty(n)
    X[0] = init
    for t in range(n-1):
        Z = a + c * beta_dist.rvs()
        X[t+1] = carry_over(X[t]) + Z
    return p_star(X)

fig, ax = plt.subplots()
ax.plot(generate_cp_ts(), label="price")
ax.set_xlabel("time")
ax.legend()
plt.show()
```



# **Part VII**

# **Stochastic Dynamics**



---

CHAPTER  
**TWENTYFIVE**

---

## MARKOV CHAINS: BASIC CONCEPTS

### Contents

- *Markov Chains: Basic Concepts*
  - *Overview*
  - *Definitions and examples*
  - *Simulation*
  - *Distributions over time*
  - *Stationary distributions*
  - *Computing expectations*

In addition to what's in Anaconda, this lecture will need the following libraries:

```
!pip install quantecon
```

### 25.1 Overview

Markov chains are a standard way to model time series with some dependence between observations.

For example,

- inflation next year depends on inflation this year
- unemployment next month depends on unemployment this month

Markov chains are one of the workhorse models of economics and finance.

The theory of Markov chains is beautiful and provides many insights into probability and dynamics.

In this introductory lecture, we will

- review some of the key ideas from the theory of Markov chains and
- show how Markov chains appear in some economic applications.

Let's start with some standard imports:

```

import matplotlib.pyplot as plt
import quantecon as qe
import numpy as np
import networkx as nx
from matplotlib import cm
import matplotlib as mpl
from itertools import cycle

```

## 25.2 Definitions and examples

In this section we provide the basic definitions and some elementary examples.

### 25.2.1 Stochastic matrices

Recall that a **probability mass function** over  $n$  possible outcomes is a nonnegative  $n$ -vector  $p$  that sums to one.

For example,  $p = (0.2, 0.2, 0.6)$  is a probability mass function over 3 outcomes.

A **stochastic matrix** (or **Markov matrix**) is an  $n \times n$  square matrix  $P$  such that each row of  $P$  is a probability mass function over  $n$  outcomes.

In other words,

1. each element of  $P$  is nonnegative, and
2. each row of  $P$  sums to one

If  $P$  is a stochastic matrix, then so is the  $k$ -th power  $P^k$  for all  $k \in \mathbb{N}$ .

Checking this in the first exercises below.

### 25.2.2 Markov chains

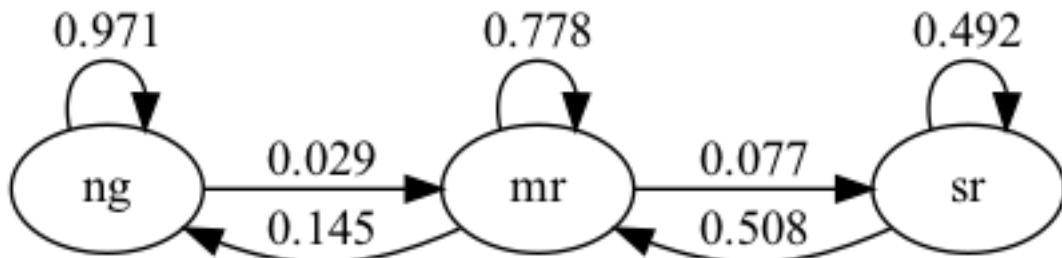
Now we can introduce Markov chains.

First we will give some examples and then we will define them more carefully.

At that time, the connection between stochastic matrices and Markov chains will become clear.

#### Example 1

From US unemployment data, Hamilton [Ham05] estimated the following dynamics.



Here there are three **states**

- “ng” represents normal growth
- “mr” represents mild recession
- “sr” represents severe recession

The arrows represent **transition probabilities** over one month.

For example, the arrow from mild recession to normal growth has 0.145 next to it.

This tells us that, according to past data, there is a 14.5% probability of transitioning from mild recession to normal growth in one month.

The arrow from normal growth back to normal growth tells us that there is a 97% probability of transitioning from normal growth to normal growth (staying in the same state).

Note that these are *conditional* probabilities — the probability of transitioning from one state to another (or staying at the same one) conditional on the current state.

To make the problem easier to work with numerically, let’s convert states to numbers.

In particular, we agree that

- state 0 represents normal growth
- state 1 represents mild recession
- state 2 represents severe recession

Let  $X_t$  record the value of the state at time  $t$ .

Now we can write the statement “there is a 14.5% probability of transitioning from mild recession to normal growth in one month” as

$$\mathbb{P}\{X_{t+1} = 0 \mid X_t = 1\} = 0.145$$

We can collect all of these conditional probabilities into a matrix, as follows

$$P = \begin{bmatrix} 0.971 & 0.029 & 0 \\ 0.145 & 0.778 & 0.077 \\ 0 & 0.508 & 0.492 \end{bmatrix}$$

Notice that  $P$  is a stochastic matrix.

Now we have the following relationship

$$P(i, j) = \mathbb{P}\{X_{t+1} = j \mid X_t = i\}$$

This holds for any  $i, j$  between 0 and 2.

In particular,  $P(i, j)$  is the probability of transitioning from state  $i$  to state  $j$  in one month.

## Example 2

Consider a worker who, at any given time  $t$ , is either unemployed (state 0) or employed (state 1).

Suppose that, over a one-month period,

1. the unemployed worker finds a job with probability  $\alpha \in (0, 1)$ .
2. the employed worker loses her job and becomes unemployed with probability  $\beta \in (0, 1)$ .

Given the above information, we can write out the transition probabilities in matrix form as

$$P = \begin{bmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{bmatrix} \quad (25.1)$$

For example,

$$\begin{aligned} P(0,1) &= \text{probability of transitioning from state 0 to state 1 in one month} \\ &= \text{probability finding a job next month} \\ &= \alpha \end{aligned}$$

Suppose we can estimate the values  $\alpha$  and  $\beta$ .

Then we can address a range of questions, such as

- What is the average duration of unemployment?
- Over the long-run, what fraction of the time does a worker find herself unemployed?
- Conditional on employment, what is the probability of becoming unemployed at least once over the next 12 months?

We'll cover some of these applications below.

### Example 3

Imam and Temple [IT23] categorize political institutions into three types: democracy (D), autocracy (A), and an intermediate state called anocracy (N).

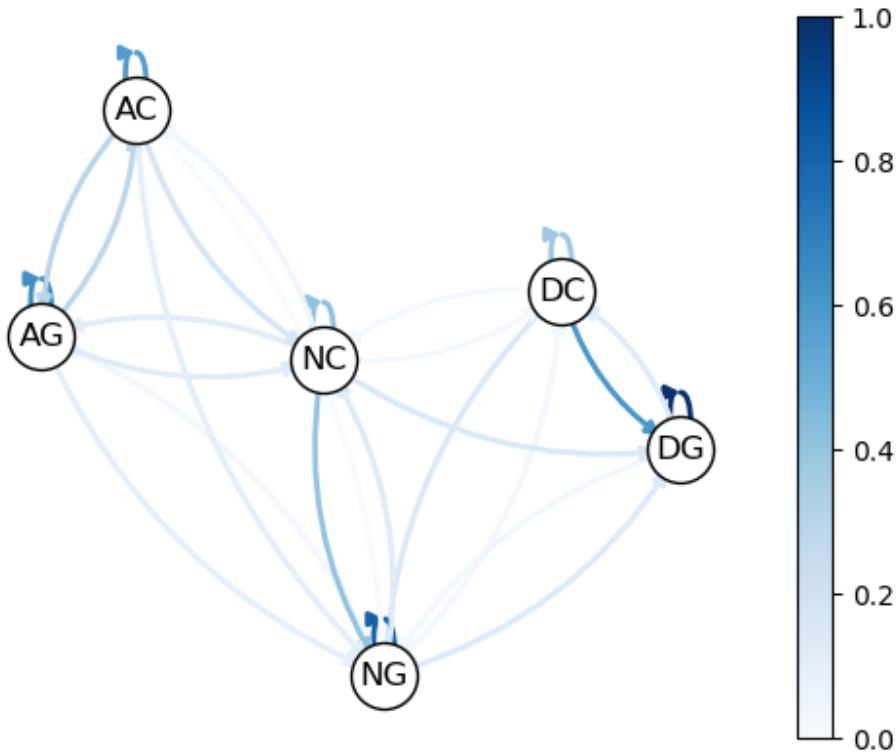
Each institution can have two potential development regimes: collapse (C) and growth (G). This results in six possible states: DG, DC, NG, NC, AG and AC.

Imam and Temple [IT23] estimate the following transition probabilities:

$$P := \begin{bmatrix} 0.86 & 0.11 & 0.03 & 0.00 & 0.00 & 0.00 \\ 0.52 & 0.33 & 0.13 & 0.02 & 0.00 & 0.00 \\ 0.12 & 0.03 & 0.70 & 0.11 & 0.03 & 0.01 \\ 0.13 & 0.02 & 0.35 & 0.36 & 0.10 & 0.04 \\ 0.00 & 0.00 & 0.09 & 0.11 & 0.55 & 0.25 \\ 0.00 & 0.00 & 0.09 & 0.15 & 0.26 & 0.50 \end{bmatrix}$$

```
nodes = ['DG', 'DC', 'NG', 'NC', 'AG', 'AC']
P = [[0.86, 0.11, 0.03, 0.00, 0.00, 0.00],
      [0.52, 0.33, 0.13, 0.02, 0.00, 0.00],
      [0.12, 0.03, 0.70, 0.11, 0.03, 0.01],
      [0.13, 0.02, 0.35, 0.36, 0.10, 0.04],
      [0.00, 0.00, 0.09, 0.11, 0.55, 0.25],
      [0.00, 0.00, 0.09, 0.15, 0.26, 0.50]]
```

Here is a visualization, with darker colors indicating higher probability.



Looking at the data, we see that democracies tend to have longer-lasting growth regimes compared to autocracies (as indicated by the lower probability of transitioning from growth to growth in autocracies).

We can also find a higher probability from collapse to growth in democratic regimes

### 25.2.3 Defining Markov chains

So far we've given examples of Markov chains but now let's define them more carefully.

To begin, let  $S$  be a finite set  $\{x_1, \dots, x_n\}$  with  $n$  elements.

The set  $S$  is called the **state space** and  $x_1, \dots, x_n$  are the **state values**.

A **distribution**  $\psi$  on  $S$  is a probability mass function of length  $n$ , where  $\psi(i)$  is the amount of probability allocated to state  $x_i$ .

A **Markov chain**  $\{X_t\}$  on  $S$  is a sequence of random variables taking values in  $S$  that have the **Markov property**.

This means that, for any date  $t$  and any state  $y \in S$ ,

$$\mathbb{P}\{X_{t+1} = y \mid X_t\} = \mathbb{P}\{X_{t+1} = y \mid X_t, X_{t-1}, \dots\} \quad (25.2)$$

In other words, knowing the current state is enough to know probabilities for the future states.

In particular, the dynamics of a Markov chain are fully determined by the set of values

$$P(x, y) := \mathbb{P}\{X_{t+1} = y \mid X_t = x\} \quad (x, y \in S) \quad (25.3)$$

By construction,

- $P(x, y)$  is the probability of going from  $x$  to  $y$  in one unit of time (one step)
- $P(x, \cdot)$  is the conditional distribution of  $X_{t+1}$  given  $X_t = x$

We can view  $P$  as a stochastic matrix where

$$P_{ij} = P(x_i, x_j) \quad 1 \leq i, j \leq n$$

Going the other way, if we take a stochastic matrix  $P$ , we can generate a Markov chain  $\{X_t\}$  as follows:

- draw  $X_0$  from a distribution  $\psi_0$  on  $S$
- for each  $t = 0, 1, \dots$ , draw  $X_{t+1}$  from  $P(X_t, \cdot)$

By construction, the resulting process satisfies (25.3).

## 25.3 Simulation

One natural way to answer questions about Markov chains is to simulate them.

Let's start by doing this ourselves and then look at libraries that can help us.

In these exercises, we'll take the state space to be  $S = 0, \dots, n - 1$ .

(We start at 0 because Python arrays are indexed from 0.)

### 25.3.1 Writing our own simulation code

To simulate a Markov chain, we need

1. a stochastic matrix  $P$  and
2. a probability mass function  $\psi_0$  of length  $n$  from which to draw an initial realization of  $X_0$ .

The Markov chain is then constructed as follows:

1. At time  $t = 0$ , draw a realization of  $X_0$  from the distribution  $\psi_0$ .
2. At each subsequent time  $t$ , draw a realization of the new state  $X_{t+1}$  from  $P(X_t, \cdot)$ .

(That is, draw from row  $X_t$  of  $P$ .)

To implement this simulation procedure, we need a method for generating draws from a discrete distribution.

For this task, we'll use `random.draw` from `QuantEcon.py`.

To use `random.draw`, we first need to convert the probability mass function to a cumulative distribution

```
ψ_0 = (0.3, 0.7)          # probabilities over {0, 1}
cdf = np.cumsum(ψ_0)        # convert into cumulative distribution
qe.random.draw(cdf, 5)      # generate 5 independent draws from ψ
```

```
array([0, 1, 1, 0, 0])
```

We'll write our code as a function that accepts the following three arguments

- A stochastic matrix  $P$ .
- An initial distribution  $\psi_0$ .
- A positive integer `ts_length` representing the length of the time series the function should return.

```

def mc_sample_path(P,  $\psi_0$ =None, ts_length=1_000):

    # set up
    P = np.asarray(P)
    X = np.empty(ts_length, dtype=int)

    # Convert each row of P into a cdf
    n = len(P)
    P_dist = np.cumsum(P, axis=1) # Convert rows into cdfs

    # draw initial state, defaulting to 0
    if  $\psi_0$  is not None:
        X_0 = qe.random.draw(np.cumsum( $\psi_0$ ))
    else:
        X_0 = 0

    # simulate
    X[0] = X_0
    for t in range(ts_length - 1):
        X[t+1] = qe.random.draw(P_dist[X[t], :])

    return X

```

Let's see how it works using the small matrix

```

P = [[0.4, 0.6],
      [0.2, 0.8]]

```

Here's a short time series.

```
mc_sample_path(P,  $\psi_0$ =[1.0, 0.0], ts_length=10)
```

```
array([0, 1, 0, 0, 1, 1, 0, 1, 1, 0])
```

It can be shown that for a long series drawn from  $P$ , the fraction of the sample that takes value 0 will be about 0.25.

(We will explain why *later*.)

Moreover, this is true regardless of the initial distribution from which  $X_0$  is drawn.

The following code illustrates this

```

X = mc_sample_path(P,  $\psi_0$ =[0.1, 0.9], ts_length=1_000_000)
np.mean(X == 0)

```

```
0.24986
```

You can try changing the initial distribution to confirm that the output is always close to 0.25 (for the  $P$  matrix above).

### 25.3.2 Using QuantEcon's routines

QuantEcon.py has routines for handling Markov chains, including simulation.

Here's an illustration using the same  $P$  as the preceding example

```
mc = qe.MarkovChain(P)
X = mc.simulate(ts_length=1_000_000)
np.mean(X == 0)
```

```
0.250113
```

The `simulate` routine is faster (because it is JIT compiled).

```
%time mc_sample_path(P, ts_length=1_000_000) # Our homemade code version
```

```
CPU times: user 1.35 s, sys: 0 ns, total: 1.35 s
Wall time: 1.35 s
```

```
array([0, 0, 0, ..., 1, 0, 1])
```

```
%time mc.simulate(ts_length=1_000_000) # qe code version
```

```
CPU times: user 16.9 ms, sys: 4.04 ms, total: 20.9 ms
Wall time: 20.7 ms
```

```
array([1, 1, 1, ..., 1, 1, 1])
```

### Adding state values and initial conditions

If we wish to, we can provide a specification of state values to `MarkovChain`.

These state values can be integers, floats, or even strings.

The following code illustrates

```
mc = qe.MarkovChain(P, state_values=('unemployed', 'employed'))
mc.simulate(ts_length=4, init='employed')
```

```
array(['employed', 'employed', 'employed', 'employed'], dtype='<U10')
```

```
mc.simulate(ts_length=4, init='unemployed')
```

```
array(['unemployed', 'unemployed', 'employed', 'employed'], dtype='<U10')
```

```
mc.simulate(ts_length=4) # Start at randomly chosen initial state
```

```
array(['employed', 'unemployed', 'unemployed', 'unemployed'], dtype='<U10')
```

If we want to see indices rather than state values as outputs as we can use

```
mc.simulate_indices(ts_length=4)
```

```
array([0, 1, 0, 1])
```

## 25.4 Distributions over time

We learned that

1.  $\{X_t\}$  is a Markov chain with stochastic matrix  $P$
2. the distribution of  $X_t$  is known to be  $\psi_t$

What then is the distribution of  $X_{t+1}$ , or, more generally, of  $X_{t+m}$ ?

To answer this, we let  $\psi_t$  be the distribution of  $X_t$  for  $t = 0, 1, 2, \dots$

Our first aim is to find  $\psi_{t+1}$  given  $\psi_t$  and  $P$ .

To begin, pick any  $y \in S$ .

To get the probability of being at  $y$  tomorrow (at  $t+1$ ), we account for all ways this can happen and sum their probabilities.

This leads to

$$\mathbb{P}\{X_{t+1} = y\} = \sum_{x \in S} \mathbb{P}\{X_{t+1} = y \mid X_t = x\} \cdot \mathbb{P}\{X_t = x\}$$

(We are using the [law of total probability](#).)

Rewriting this statement in terms of marginal and conditional probabilities gives

$$\psi_{t+1}(y) = \sum_{x \in S} P(x, y) \psi_t(x)$$

There are  $n$  such equations, one for each  $y \in S$ .

If we think of  $\psi_{t+1}$  and  $\psi_t$  as row vectors, these  $n$  equations are summarized by the matrix expression

$$\psi_{t+1} = \psi_t P \tag{25.4}$$

Thus, we postmultiply by  $P$  to move a distribution forward one unit of time.

By postmultiplying  $m$  times, we move a distribution forward  $m$  steps into the future.

Hence, iterating on (25.4), the expression  $\psi_{t+m} = \psi_t P^m$  is also valid — here  $P^m$  is the  $m$ -th power of  $P$ .

As a special case, we see that if  $\psi_0$  is the initial distribution from which  $X_0$  is drawn, then  $\psi_0 P^m$  is the distribution of  $X_m$ .

This is very important, so let's repeat it

$$X_0 \sim \psi_0 \implies X_m \sim \psi_0 P^m \tag{25.5}$$

The general rule is that post-multiplying a distribution by  $P^m$  shifts it forward  $m$  units of time.

Hence the following is also valid.

$$X_t \sim \psi_t \implies X_{t+m} \sim \psi_t P^m \tag{25.6}$$

### 25.4.1 Multiple step transition probabilities

We know that the probability of transitioning from  $x$  to  $y$  in one step is  $P(x, y)$ .

It turns out that the probability of transitioning from  $x$  to  $y$  in  $m$  steps is  $P^m(x, y)$ , the  $(x, y)$ -th element of the  $m$ -th power of  $P$ .

To see why, consider again (25.6), but now with a  $\psi_t$  that puts all probability on state  $x$ .

Then  $\psi_t$  is a vector with 1 in position  $x$  and zero elsewhere.

Inserting this into (25.6), we see that, conditional on  $X_t = x$ , the distribution of  $X_{t+m}$  is the  $x$ -th row of  $P^m$ .

In particular

$$\mathbb{P}\{X_{t+m} = y \mid X_t = x\} = P^m(x, y) = (x, y)\text{-th element of } P^m$$

### 25.4.2 Example: probability of recession

Recall the stochastic matrix  $P$  for recession and growth *considered above*.

Suppose that the current state is unknown — perhaps statistics are available only at the *end* of the current month.

We guess that the probability that the economy is in state  $x$  is  $\psi_t(x)$  at time  $t$ .

The probability of being in recession (either mild or severe) in 6 months time is given by

$$(\psi_t P^6)(1) + (\psi_t P^6)(2)$$

### 25.4.3 Example 2: Cross-sectional distributions

The distributions we have been studying can be viewed either

1. as probabilities or
2. as cross-sectional frequencies that the Law of Large Numbers leads us to anticipate for large samples.

To illustrate, recall our model of employment/unemployment dynamics for a given worker *discussed above*.

Consider a large population of workers, each of whose lifetime experience is described by the specified dynamics, with each worker's outcomes being realizations of processes that are statistically independent of all other workers' processes.

Let  $\psi_t$  be the current *cross-sectional* distribution over  $\{0, 1\}$ .

The cross-sectional distribution records fractions of workers employed and unemployed at a given moment  $t$ .

- For example,  $\psi_t(0)$  is the unemployment rate.

What will the cross-sectional distribution be in 10 periods hence?

The answer is  $\psi_t P^{10}$ , where  $P$  is the stochastic matrix in (25.1).

This is because each worker's state evolves according to  $P$ , so  $\psi_t P^{10}$  is a marginal distribution for a single randomly selected worker.

But when the sample is large, outcomes and probabilities are roughly equal (by an application of the Law of Large Numbers).

So for a very large (tending to infinite) population,  $\psi_t P^{10}$  also represents fractions of workers in each state.

This is exactly the cross-sectional distribution.

## 25.5 Stationary distributions

As seen in (25.4), we can shift a distribution forward one unit of time via postmultiplication by  $P$ .

Some distributions are invariant under this updating process — for example,

```
P = np.array([[0.4, 0.6],
              [0.2, 0.8]])
ψ = (0.25, 0.75)
ψ @ P
```

```
array([0.25, 0.75])
```

Notice that  $\psi @ P$  is the same as  $\psi$

Such distributions are called **stationary** or **invariant**.

Formally, a distribution  $\psi^*$  on  $S$  is called **stationary** for  $P$  if  $\psi^*P = \psi^*$ .

Notice that, post-multiplying by  $P$ , we have  $\psi^*P^2 = \psi^*P = \psi^*$ .

Continuing in the same way leads to  $\psi^* = \psi^*P^t$  for all  $t$ .

This tells us an important fact: If the distribution of  $\psi_0$  is a stationary distribution, then  $\psi_t$  will have this same distribution for all  $t$ .

The following theorem is proved in Chapter 4 of [SS23] and numerous other sources.

---

### Theorem 25.5.1

Every stochastic matrix  $P$  has at least one stationary distribution.

---

Note that there can be many stationary distributions corresponding to a given stochastic matrix  $P$ .

- For example, if  $P$  is the identity matrix, then all distributions on  $S$  are stationary.

To get uniqueness, we need the Markov chain to “mix around,” so that the state doesn’t get stuck in some part of the state space.

This gives some intuition for the following theorem.

---

### Theorem 25.5.2

If  $P$  is everywhere positive, then  $P$  has exactly one stationary distribution.

---

We will come back to this when we introduce irreducibility in the [next lecture](#) on Markov chains.

### 25.5.1 Example

Recall our model of the employment/unemployment dynamics of a particular worker *discussed above*.

If  $\alpha \in (0, 1)$  and  $\beta \in (0, 1)$ , then the transition matrix is everywhere positive.

Let  $\psi^* = (p, 1 - p)$  be the stationary distribution, so that  $p$  corresponds to unemployment (state 0).

Using  $\psi^* = \psi^* P$  and a bit of algebra yields

$$p = \frac{\beta}{\alpha + \beta}$$

This is, in some sense, a steady state probability of unemployment.

Not surprisingly it tends to zero as  $\beta \rightarrow 0$ , and to one as  $\alpha \rightarrow 0$ .

### 25.5.2 Calculating stationary distributions

A stable algorithm for computing stationary distributions is implemented in `QuantEcon.py`.

Here's an example

```
P = [[0.4, 0.6],
     [0.2, 0.8]]

mc = qe.MarkovChain(P)
mc.stationary_distributions # Show all stationary distributions
```

```
array([[0.25, 0.75]])
```

### 25.5.3 Asymptotic stationarity

Consider an everywhere positive stochastic matrix with unique stationary distribution  $\psi^*$ .

Sometimes the distribution  $\psi_t = \psi_0 P^t$  of  $X_t$  converges to  $\psi^*$  regardless of  $\psi_0$ .

For example, we have the following result

---

#### Theorem 25.5.3

Theorem: If there exists an integer  $m$  such that all entries of  $P^m$  are strictly positive, with unique stationary distribution  $\psi^*$ , and

$$\psi_0 P^t \rightarrow \psi^* \quad \text{as } t \rightarrow \infty$$


---

See, for example, [SS23] Chapter 4.

### Example: Hamilton's chain

Hamilton's chain satisfies the conditions of the theorem because  $P^2$  is everywhere positive:

```
P = np.array([[0.971, 0.029, 0.000],
              [0.145, 0.778, 0.077],
              [0.000, 0.508, 0.492]])
```

```
P @ P
```

```
array([[0.947046, 0.050721, 0.002233],
       [0.253605, 0.648605, 0.09779],
       [0.07366, 0.64516, 0.28118]])
```

Let's pick an initial distribution  $\psi_0$  and trace out the sequence of distributions  $\psi_0 P^t$  for  $t = 0, 1, 2, \dots$

First, we write a function to iterate the sequence of distributions for `ts_length` period

```
def iterate_psi(psi_0, P, ts_length):
    n = len(P)
    psi_t = np.empty((ts_length, n))
    psi = psi_0
    for t in range(ts_length):
        psi_t[t] = psi
        psi = psi @ P
    return np.array(psi_t)
```

Now we plot the sequence

```
psi_0 = (0.0, 0.2, 0.8)           # Initial condition

fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')

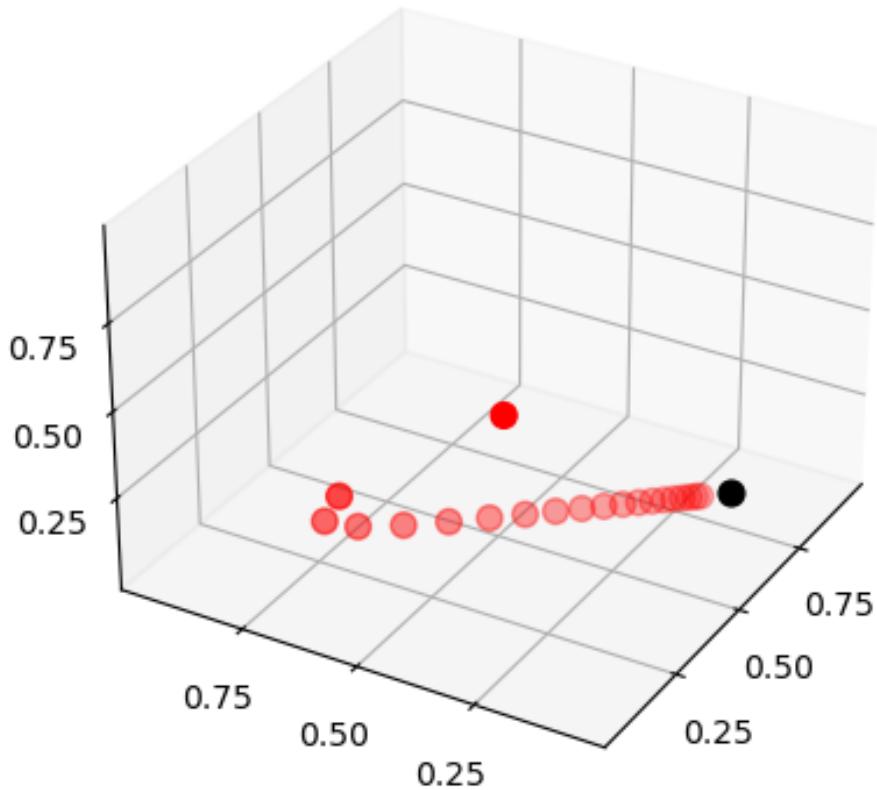
ax.set(xlim=(0, 1), ylim=(0, 1), zlim=(0, 1),
       xticks=(0.25, 0.5, 0.75),
       yticks=(0.25, 0.5, 0.75),
       zticks=(0.25, 0.5, 0.75))

psi_t = iterate_psi(psi_0, P, 20)

ax.scatter(psi_t[:,0], psi_t[:,1], psi_t[:,2], c='r', s=60)
ax.view_init(30, 210)

mc = qe.MarkovChain(P)
psi_star = mc.stationary_distributions[0]
ax.scatter(psi_star[0], psi_star[1], psi_star[2], c='k', s=60)

plt.show()
```



Here

- $P$  is the stochastic matrix for recession and growth *considered above*.
- The highest red dot is an arbitrarily chosen initial marginal probability distribution  $\psi_0$ , represented as a vector in  $\mathbb{R}^3$ .
- The other red dots are the marginal distributions  $\psi_0 P^t$  for  $t = 1, 2, \dots$ .
- The black dot is  $\psi^*$ .

You might like to try experimenting with different initial conditions.

### An alternative illustration

We can show this in a slightly different way by focusing on the probability that  $\psi_t$  puts on each state.

First, we write a function to draw initial distributions  $\psi_0$  of size num\_distributions

```
def generate_initial_values(num_distributions):
    n = len(P)
    psi_0s = np.empty((num_distributions, n))

    for i in range(num_distributions):
        draws = np.random.randint(1, 10_000_000, size=n)

        # Scale them so that they add up into 1
```

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```

ψ_0s[i,:] = np.array(draws/sum(draws))

return ψ_0s

```

We then write a function to plot the dynamics of  $(\psi_0 P^t)(i)$  as  $t$  gets large, for each state  $i$  with different initial distributions

```

def plot_distribution(P, ts_length, num_distributions):

    # Get parameters of transition matrix
    n = len(P)
    mc = qe.MarkovChain(P)
    ψ_star = mc.stationary_distributions[0]

    ## Draw the plot
    fig, axes = plt.subplots(nrows=1, ncols=n, figsize=[11, 5])
    plt.subplots_adjust(wspace=0.35)

    ψ_0s = generate_initial_values(num_distributions)

    # Get the path for each starting value
    for ψ_0 in ψ_0s:
        ψ_t = iterate_ψ(ψ_0, P, ts_length)

        # Obtain and plot distributions at each state
        for i in range(n):
            axes[i].plot(range(0, ts_length), ψ_t[:,i], alpha=0.3)

    # Add labels
    for i in range(n):
        axes[i].axhline(ψ_star[i], linestyle='dashed', lw=2, color = 'black',
                        label = fr'$\psi^*(i)$')
        axes[i].set_xlabel('t')
        axes[i].set_ylabel(fr'$\psi_t(i)$')
        axes[i].legend()

    plt.show()

```

The following figure shows

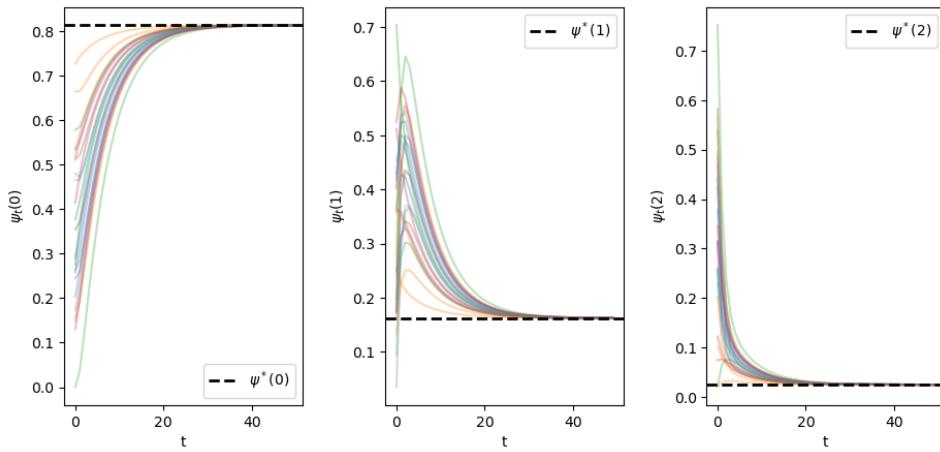
```

# Define the number of iterations
# and initial distributions
ts_length = 50
num_distributions = 25

P = np.array([[0.971, 0.029, 0.000],
              [0.145, 0.778, 0.077],
              [0.000, 0.508, 0.492]])

plot_distribution(P, ts_length, num_distributions)

```



The convergence to  $\psi^*$  holds for different initial distributions.

### Example: Failure of convergence

In the case of a periodic chain, with

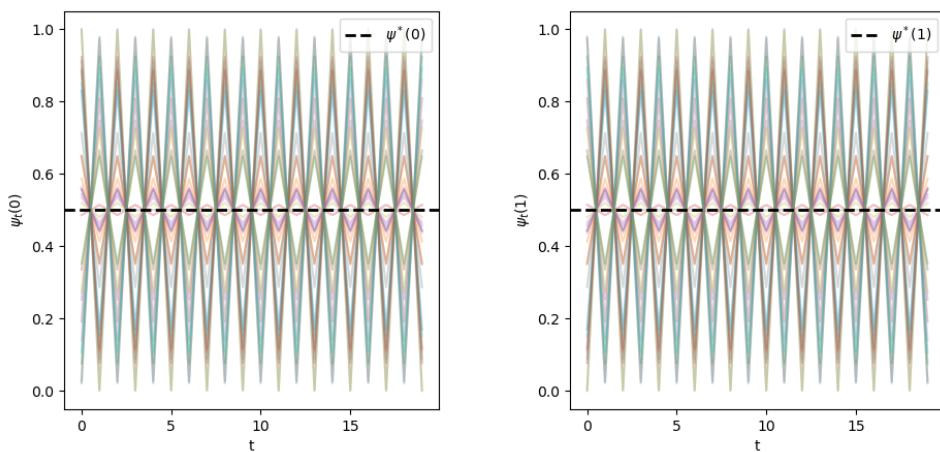
$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

we find the distribution oscillates

```
P = np.array([[0, 1],
              [1, 0]])

ts_length = 20
num_distributions = 30

plot_distribution(P, ts_length, num_distributions)
```



Indeed, this  $P$  fails our asymptotic stationarity condition, since, as you can verify,  $P^t$  is not everywhere positive for any  $t$ .

## 25.6 Computing expectations

We sometimes want to compute mathematical expectations of functions of  $X_t$  of the form

$$\mathbb{E}[h(X_t)] \quad (25.7)$$

and conditional expectations such as

$$\mathbb{E}[h(X_{t+k}) \mid X_t = x] \quad (25.8)$$

where

- $\{X_t\}$  is a Markov chain generated by  $n \times n$  stochastic matrix  $P$ .
- $h$  is a given function, which, in terms of matrix algebra, we'll think of as the column vector

$$h = \begin{bmatrix} h(x_1) \\ \vdots \\ h(x_n) \end{bmatrix}.$$

Computing the unconditional expectation (25.7) is easy.

We just sum over the marginal distribution of  $X_t$  to get

$$\mathbb{E}[h(X_t)] = \sum_{x \in S} (\psi P^t)(x) h(x)$$

Here  $\psi$  is the distribution of  $X_0$ .

Since  $\psi$  and hence  $\psi P^t$  are row vectors, we can also write this as

$$\mathbb{E}[h(X_t)] = \psi P^t h$$

For the conditional expectation (25.8), we need to sum over the conditional distribution of  $X_{t+k}$  given  $X_t = x$ .

We already know that this is  $P^k(x, \cdot)$ , so

$$\mathbb{E}[h(X_{t+k}) \mid X_t = x] = (P^k h)(x) \quad (25.9)$$

### 25.6.1 Expectations of geometric sums

Sometimes we want to compute the mathematical expectation of a geometric sum, such as  $\sum_t \beta^t h(X_t)$ .

In view of the preceding discussion, this is

$$\mathbb{E} \left[ \sum_{j=0}^{\infty} \beta^j h(X_{t+j}) \mid X_t = x \right] = x + \beta(P h)(x) + \beta^2(P^2 h)(x) + \dots$$

By the *Neumann series lemma*, this sum can be calculated using

$$I + \beta P + \beta^2 P^2 + \dots = (I - \beta P)^{-1}$$

The vector  $P^k h$  stores the conditional expectation  $\mathbb{E}[h(X_{t+k}) \mid X_t = x]$  over all  $x$ .

---

#### Exercise 25.6.1

Imam and Temple [IT23] used a three-state transition matrix to describe the transition of three states of a regime: growth, stagnation, and collapse

$$P := \begin{bmatrix} 0.68 & 0.12 & 0.20 \\ 0.50 & 0.24 & 0.26 \\ 0.36 & 0.18 & 0.46 \end{bmatrix}$$

where rows, from top to down, correspond to growth, stagnation, and collapse.

In this exercise,

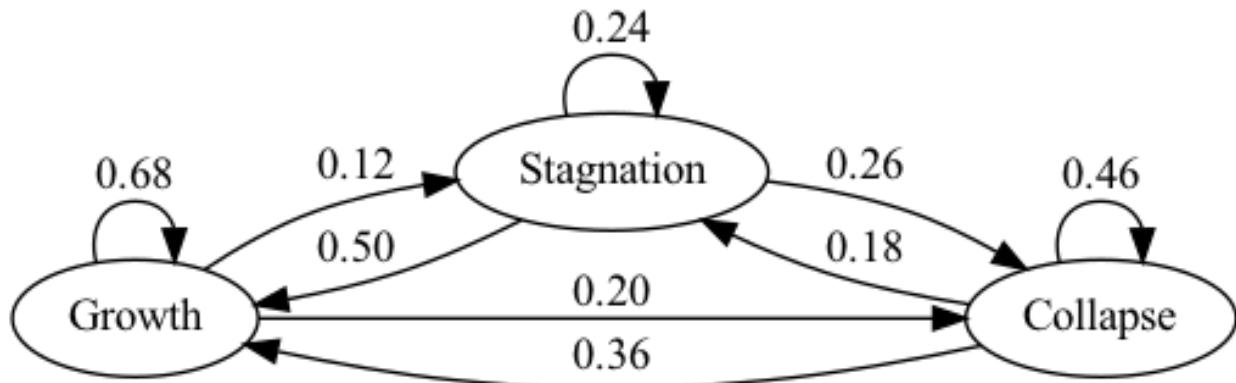
1. visualize the transition matrix and show this process is asymptotically stationary
2. calculate the stationary distribution using simulations
3. visualize the dynamics of  $(\psi_0 P^t)(i)$  where  $t \in 0, \dots, 25$  and compare the convergent path with the previous transition matrix

Compare your solution to the paper.

---

### Solution to Exercise 25.6.1

Solution 1:



Since the matrix is everywhere positive, there is a unique stationary distribution.

Solution 2:

One simple way to calculate the stationary distribution is to take the power of the transition matrix as we have shown before

```

P = np.array([[0.68, 0.12, 0.20],
              [0.50, 0.24, 0.26],
              [0.36, 0.18, 0.46]])
P_power = np.linalg.matrix_power(P, 20)
P_power
  
```

```

array([[0.56145769, 0.15565164, 0.28289067],
       [0.56145769, 0.15565164, 0.28289067],
       [0.56145769, 0.15565164, 0.28289067]])
  
```

Note that rows of the transition matrix converge to the stationary distribution.

```
ψ_star_p = P_power[0]
ψ_star_p
```

```
array([0.56145769, 0.15565164, 0.28289067])
```

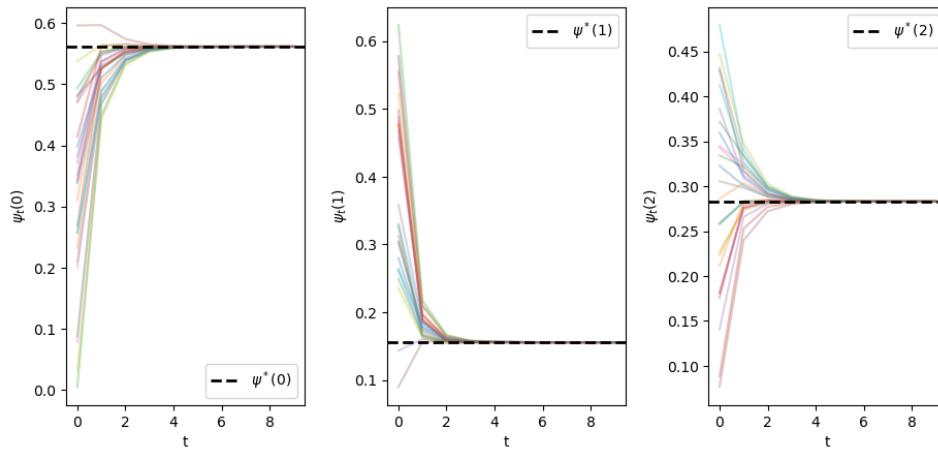
```
mc = qe.MarkovChain(P)
ψ_star = mc.stationary_distributions[0]
ψ_star
```

```
array([0.56145769, 0.15565164, 0.28289067])
```

Solution 3:

We find the distribution  $\psi$  converges to the stationary distribution more quickly compared to the *hamilton's chain*.

```
ts_length = 10
num_distributions = 25
plot_distribution(P, ts_length, num_distributions)
```



In fact, the rate of convergence is governed by *eigenvalues* [SS23].

```
P_eigenvals = np.linalg.eigvals(P)
P_eigenvals
```

```
array([1.0, 0.28219544, 0.09780456])
```

```
P_hamilton = np.array([[0.971, 0.029, 0.000],
[0.145, 0.778, 0.077],
[0.000, 0.508, 0.492]])
```

```
hamilton_eigenvals = np.linalg.eigvals(P_hamilton)
hamilton_eigenvals
```

```
array([1.0, 0.85157412, 0.38942588])
```

More specifically, it is governed by the spectral gap, the difference between the largest and the second largest eigenvalue.

```

sp_gap_P = P_eigenvalues[0] - np.diff(P_eigenvalues)[0]
sp_gap_hamilton = hamilton_eigenvalues[0] - np.diff(hamilton_eigenvalues)[0]

sp_gap_P > sp_gap_hamilton

```

True

We will come back to this when we discuss *spectral theory*.

---

### Exercise 25.6.2

We discussed the six-state transition matrix estimated by Imam & Temple [IT23] *before*.

```

nodes = ['DG', 'DC', 'NG', 'NC', 'AG', 'AC']
P = [[0.86, 0.11, 0.03, 0.00, 0.00, 0.00],
      [0.52, 0.33, 0.13, 0.02, 0.00, 0.00],
      [0.12, 0.03, 0.70, 0.11, 0.03, 0.01],
      [0.13, 0.02, 0.35, 0.36, 0.10, 0.04],
      [0.00, 0.00, 0.09, 0.11, 0.55, 0.25],
      [0.00, 0.00, 0.09, 0.15, 0.26, 0.50]]

```

In this exercise,

1. show this process is asymptotically stationary without simulation
  2. simulate and visualize the dynamics starting with a uniform distribution across states (each state will have a probability of 1/6)
  3. change the initial distribution to  $P(DG) = 1$ , while all other states have a probability of 0
- 

### Solution to Exercise 25.6.2

Solution 1:

Although  $P$  is not every positive,  $P^m$  when  $m = 3$  is everywhere positive.

```

P = np.array([[0.86, 0.11, 0.03, 0.00, 0.00, 0.00],
              [0.52, 0.33, 0.13, 0.02, 0.00, 0.00],
              [0.12, 0.03, 0.70, 0.11, 0.03, 0.01],
              [0.13, 0.02, 0.35, 0.36, 0.10, 0.04],
              [0.00, 0.00, 0.09, 0.11, 0.55, 0.25],
              [0.00, 0.00, 0.09, 0.15, 0.26, 0.50]])

np.linalg.matrix_power(P, 3)

```

```

array([[0.764927, 0.133481, 0.085949, 0.011481, 0.002956, 0.001206],
       [0.658861, 0.131559, 0.161367, 0.031703, 0.011296, 0.005214],
       [0.291394, 0.057788, 0.439702, 0.113408, 0.062707, 0.035001],
       [0.272459, 0.051361, 0.365075, 0.132207, 0.108152, 0.070746],
       [0.064129, 0.012533, 0.232875, 0.154385, 0.299243, 0.236835],
       [0.072865, 0.014081, 0.244139, 0.160905, 0.265846, 0.242164]])

```

So it satisfies the requirement.

Solution 2:

We find the distribution  $\psi$  converges to the stationary distribution quickly regardless of the initial distributions

```

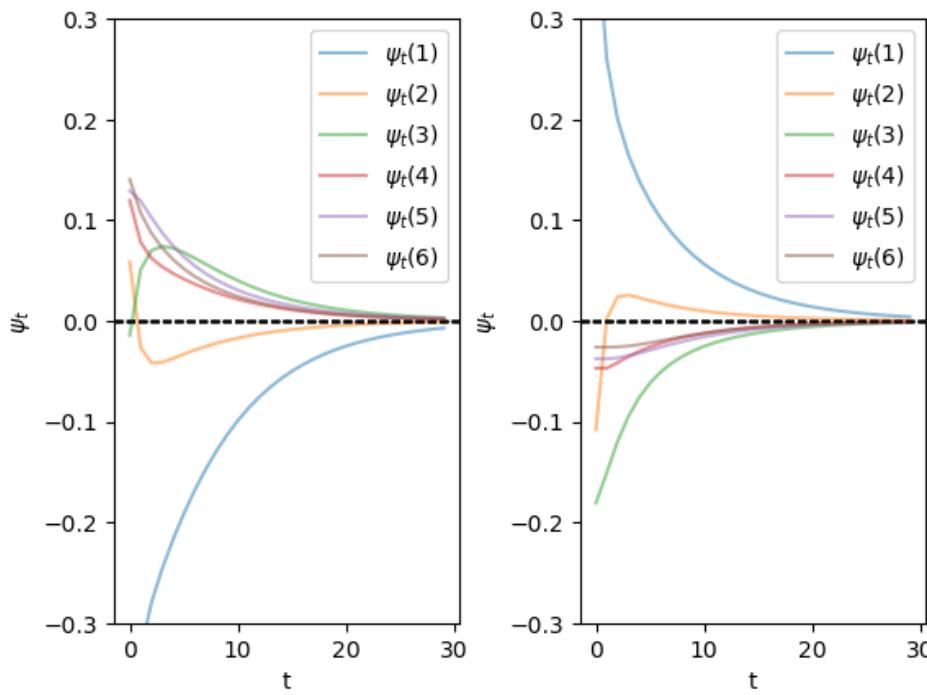
ts_length = 30
num_distributions = 20
nodes = ['DG', 'DC', 'NG', 'NC', 'AG', 'AC']

# Get parameters of transition matrix
n = len(P)
mc = qe.MarkovChain(P)
psi_star = mc.stationary_distributions[0]
psi_0 = np.array([[1/6 for i in range(6)],
                 [0 if i != 0 else 1 for i in range(6)]])

## Draw the plot
fig, axes = plt.subplots(ncols=2)
plt.subplots_adjust(wspace=0.35)
for idx in range(2):
    psi_t = iterate_psi(psi_0[idx], P, ts_length)
    for i in range(n):
        axes[idx].plot(psi_t[:, i] - psi_star[i], alpha=0.5, label=fr'$\psi_t({i+1})$')
    axes[idx].set_xlim([-0.3, 0.3])
    axes[idx].set_xlabel('t')
    axes[idx].set_ylabel(fr'$\psi_t$')
    axes[idx].legend()
    axes[idx].axhline(0, linestyle='dashed', lw=1, color = 'black')

plt.show()

```



Exercise 25.6.3

Prove the following: If  $P$  is a stochastic matrix, then so is the  $k$ -th power  $P^k$  for all  $k \in \mathbb{N}$ .

---

---

**Solution to Exercise 25.6.3**

Suppose that  $P$  is stochastic and, moreover, that  $P^k$  is stochastic for some integer  $k$ .

We will prove that  $P^{k+1} = PP^k$  is also stochastic.

(We are doing proof by induction — we assume the claim is true at  $k$  and now prove it is true at  $k + 1$ .)

To see this, observe that, since  $P^k$  is stochastic and the product of nonnegative matrices is nonnegative,  $P^{k+1} = PP^k$  is nonnegative.

Also, if  $\mathbf{1}$  is a column vector of ones, then, since  $P^k$  is stochastic we have  $P^k \mathbf{1} = \mathbf{1}$  (rows sum to one).

Therefore  $P^{k+1} \mathbf{1} = PP^k \mathbf{1} = P \mathbf{1} = \mathbf{1}$

The proof is done.

---

## MARKOV CHAINS: IRREDUCIBILITY AND ERGODICITY

### Contents

- *Markov Chains: Irreducibility and Ergodicity*
  - *Overview*
  - *Irreducibility*
  - *Ergodicity*
  - *Exercises*

In addition to what's in Anaconda, this lecture will need the following libraries:

```
!pip install quantecon
```

### 26.1 Overview

This lecture continues on from our *earlier lecture on Markov chains*.

Specifically, we will introduce the concepts of irreducibility and ergodicity, and see how they connect to stationarity.

Irreducibility describes the ability of a Markov chain to move between any two states in the system.

Ergodicity is a sample path property that describes the behavior of the system over long periods of time.

As we will see,

- an irreducible Markov chain guarantees the existence of a unique stationary distribution, while
- an ergodic Markov chain generates time series that satisfy a version of the law of large numbers.

Together, these concepts provide a foundation for understanding the long-term behavior of Markov chains.

Let's start with some standard imports:

```
import matplotlib.pyplot as plt
plt.rcParams["figure.figsize"] = (11, 5) # set default figure size
import quantecon as qe
import numpy as np
import networkx as nx
from matplotlib import cm
import matplotlib as mpl
```

## 26.2 Irreducibility

To explain irreducibility, let's take  $P$  to be a fixed stochastic matrix.

Two states  $x$  and  $y$  are said to **communicate** with each other if there exist positive integers  $j$  and  $k$  such that

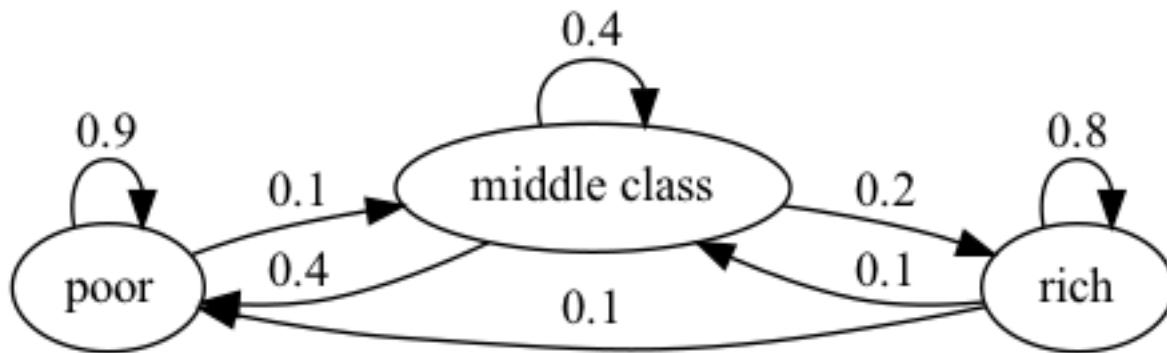
$$P^j(x, y) > 0 \quad \text{and} \quad P^k(y, x) > 0$$

In view of our discussion *above*, this means precisely that

- state  $x$  can eventually be reached from state  $y$ , and
- state  $y$  can eventually be reached from state  $x$

The stochastic matrix  $P$  is called **irreducible** if all states communicate; that is, if  $x$  and  $y$  communicate for all  $(x, y)$  in  $S \times S$ .

For example, consider the following transition probabilities for wealth of a fictitious set of households



We can translate this into a stochastic matrix, putting zeros where there's no edge between nodes

$$P := \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$$

It's clear from the graph that this stochastic matrix is irreducible: we can eventually reach any state from any other state.

We can also test this using `QuantEcon.py`'s `MarkovChain` class

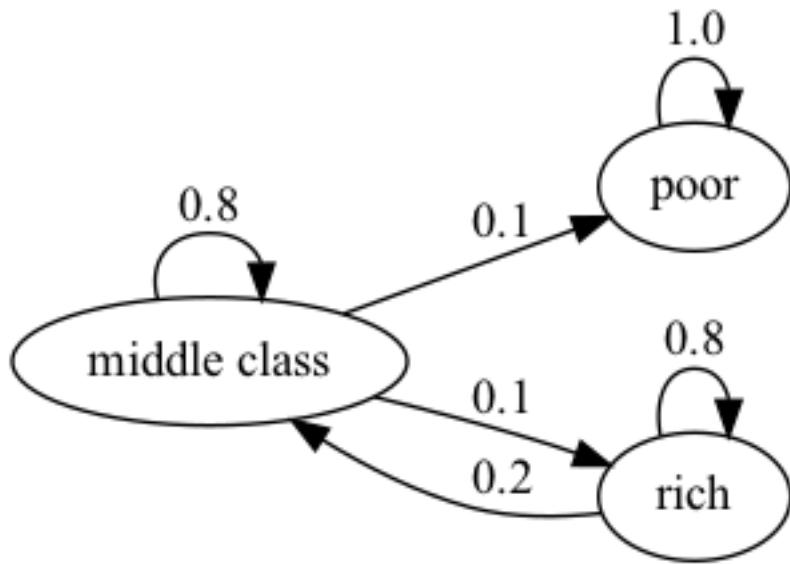
```

P = [[0.9, 0.1, 0.0],
      [0.4, 0.4, 0.2],
      [0.1, 0.1, 0.8]]

mc = qe.MarkovChain(P, ('poor', 'middle', 'rich'))
mc.is_irreducible
  
```

True

Here's a more pessimistic scenario in which poor people remain poor forever



This stochastic matrix is not irreducible since, for example, rich is not accessible from poor.

Let's confirm this

```

P = [[1.0, 0.0, 0.0],
      [0.1, 0.8, 0.1],
      [0.0, 0.2, 0.8]]
mc = qe.MarkovChain(P, ('poor', 'middle', 'rich'))
mc.is_irreducible
  
```

False

It might be clear to you already that irreducibility is going to be important in terms of long-run outcomes.

For example, poverty is a life sentence in the second graph but not the first.

We'll come back to this a bit later.

### 26.2.1 Irreducibility and stationarity

We discussed the uniqueness of the stationary in the [previous lecture](#) requires the transition matrix to be everywhere positive.

In fact irreducibility is enough for the uniqueness of the stationary distribution to hold if the distribution exists.

We can revise the [theorem](#) into the following fundamental theorem:

---

#### Theorem 26.2.1

If  $P$  is irreducible, then  $P$  has exactly one stationary distribution.

---

For proof, see Chapter 4 of [SS23] or Theorem 5.2 of [Haggstrom02].

## 26.3 Ergodicity

Please note that we use  $\mathbf{1}$  for a vector of ones in this lecture.

Under irreducibility, yet another important result obtains:

---

### Theorem 26.3.1

If  $P$  is irreducible and  $\psi^*$  is the unique stationary distribution, then, for all  $x \in S$ ,

$$\frac{1}{m} \sum_{t=1}^m \mathbf{1}\{X_t = x\} \rightarrow \psi^*(x) \quad \text{as } m \rightarrow \infty \quad (26.1)$$


---

Here

- $\{X_t\}$  is a Markov chain with stochastic matrix  $P$  and initial. distribution  $\psi_0$
- $\mathbf{1}\{X_t = x\} = 1$  if  $X_t = x$  and zero otherwise.

The result in (26.1) is sometimes called **ergodicity**.

The theorem tells us that the fraction of time the chain spends at state  $x$  converges to  $\psi^*(x)$  as time goes to infinity.

This gives us another way to interpret the stationary distribution (provided irreducibility holds).

Importantly, the result is valid for any choice of  $\psi_0$ .

The theorem is related to *the Law of Large Numbers*.

It tells us that, in some settings, the law of large numbers sometimes holds even when the sequence of random variables is *not IID*.

### 26.3.1 Example 1

Recall our cross-sectional interpretation of the employment/unemployment model *discussed above*.

Assume that  $\alpha \in (0, 1)$  and  $\beta \in (0, 1)$ , so that irreducibility holds.

We saw that the stationary distribution is  $(p, 1 - p)$ , where

$$p = \frac{\beta}{\alpha + \beta}$$

In the cross-sectional interpretation, this is the fraction of people unemployed.

In view of our latest (ergodicity) result, it is also the fraction of time that a single worker can expect to spend unemployed.

Thus, in the long run, cross-sectional averages for a population and time-series averages for a given person coincide.

This is one aspect of the concept of ergodicity.

### 26.3.2 Example 2

Another example is the Hamilton dynamics we *discussed before*.

The *graph* of the Markov chain shows it is irreducible

Therefore, we can see the sample path averages for each state (the fraction of time spent in each state) converges to the stationary distribution regardless of the starting state

Let's denote the fraction of time spent in state  $x$  at time  $t$  in our sample path as  $\hat{p}_t(x)$  where

$$\hat{p}_t(x) := \frac{1}{t} \sum_{t=1}^t \mathbf{1}\{X_t = x\}$$

Here we compare  $\hat{p}_t(x)$  with the stationary distribution  $\psi^*(x)$  for different starting points  $x_0$ .

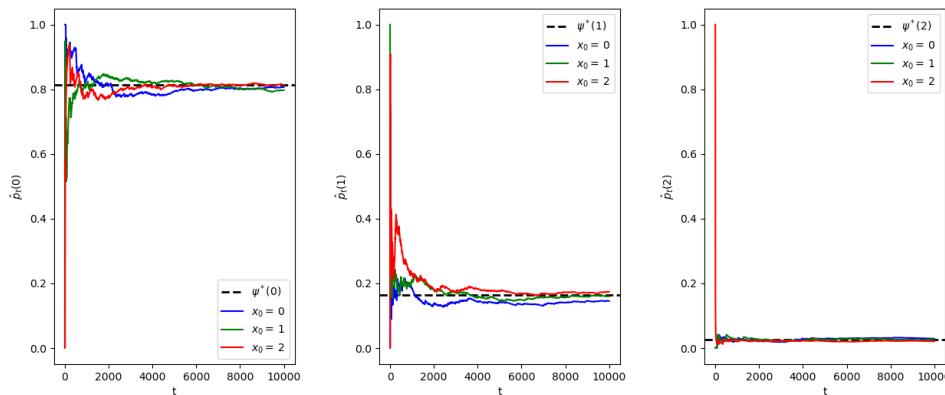
```

P = np.array([[0.971, 0.029, 0.000],
              [0.145, 0.778, 0.077],
              [0.000, 0.508, 0.492]])
ts_length = 10_000
mc = qe.MarkovChain(P)
n = len(P)
fig, axes = plt.subplots(nrows=1, ncols=n, figsize=(15, 6))
psi_star = mc.stationary_distributions[0]
plt.subplots_adjust(wspace=0.35)

for i in range(n):
    axes[i].axhline(psi_star[i], linestyle='dashed', lw=2, color='black',
                     label=fr'$\psi^*({i})$')
    axes[i].set_xlabel('t')
    axes[i].set_ylabel(fr'$\hat{p}_t({i})$')

    # Compute the fraction of time spent, starting from different x_0s
    for x0, col in ((0, 'blue'), (1, 'green'), (2, 'red')):
        # Generate time series that starts at different x0
        X = mc.simulate(ts_length, init=x0)
        p_hat = (X == i).cumsum() / (1 + np.arange(ts_length, dtype=float))
        axes[i].plot(p_hat, color=col, label=f'$x_0 = {x0}$')
    axes[i].legend()
plt.show()

```



Note the convergence to the stationary distribution regardless of the starting point  $x_0$ .

### 26.3.3 Example 3

Let's look at one more example with six states *discussed before*.

$$P := \begin{bmatrix} 0.86 & 0.11 & 0.03 & 0.00 & 0.00 & 0.00 \\ 0.52 & 0.33 & 0.13 & 0.02 & 0.00 & 0.00 \\ 0.12 & 0.03 & 0.70 & 0.11 & 0.03 & 0.01 \\ 0.13 & 0.02 & 0.35 & 0.36 & 0.10 & 0.04 \\ 0.00 & 0.00 & 0.09 & 0.11 & 0.55 & 0.25 \\ 0.00 & 0.00 & 0.09 & 0.15 & 0.26 & 0.50 \end{bmatrix}$$

The *graph* for the chain shows all states are reachable, indicating that this chain is irreducible.

Here we visualize the difference between  $\hat{p}_t(x)$  and the stationary distribution  $\psi^*(x)$  for each state  $x$

```

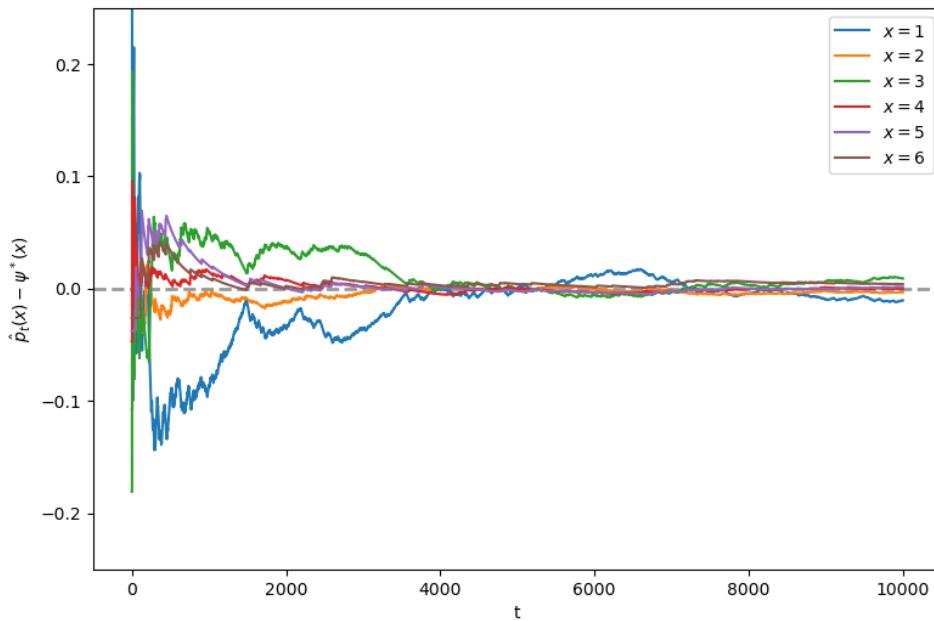
P = [[0.86, 0.11, 0.03, 0.00, 0.00, 0.00],
      [0.52, 0.33, 0.13, 0.02, 0.00, 0.00],
      [0.12, 0.03, 0.70, 0.11, 0.03, 0.01],
      [0.13, 0.02, 0.35, 0.36, 0.10, 0.04],
      [0.00, 0.00, 0.09, 0.11, 0.55, 0.25],
      [0.00, 0.00, 0.09, 0.15, 0.26, 0.50]]

ts_length = 10_000
mc = qe.MarkovChain(P)
psi_star = mc.stationary_distributions[0]
fig, ax = plt.subplots(figsize=(9, 6))
X = mc.simulate(ts_length)
# Center the plot at 0
ax.set_xlim(-0.25, 0.25)
ax.axhline(0, linestyle='dashed', lw=2, color='black', alpha=0.4)

for x0 in range(6):
    # Calculate the fraction of time for each state
    p_hat = (X == x0).cumsum() / (1 + np.arange(ts_length, dtype=float))
    ax.plot(p_hat - psi_star[x0], label=f'$x = {x0+1}$')
    ax.set_xlabel('t')
    ax.set_ylabel(r'$\hat{p}_t(x) - \psi^*(x)$')

ax.legend()
plt.show()

```



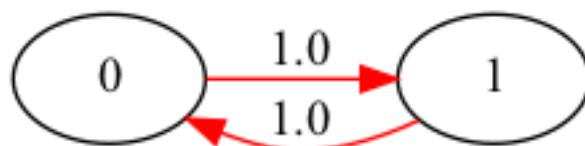
Similar to previous examples, the sample path averages for each state converge to the stationary distribution.

#### 26.3.4 Example 4

Let's look at another example with two states: 0 and 1.

$$P := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The diagram of the Markov chain shows that it is **irreducible**



In fact it has a periodic cycle — the state cycles between the two states in a regular way.

This is called **periodicity**.

It is still irreducible so ergodicity holds.

```
P = np.array([[0, 1],
              [1, 0]])
ts_length = 10_000
mc = qe.MarkovChain(P)
n = len(P)
fig, axes = plt.subplots(nrows=1, ncols=n)
psi_star = mc.stationary_distributions[0]

for i in range(n):
    axes[i].set_ylim(0.45, 0.55)
    axes[i].axhline(psi_star[i], linestyle='dashed', lw=2, color='black',
```

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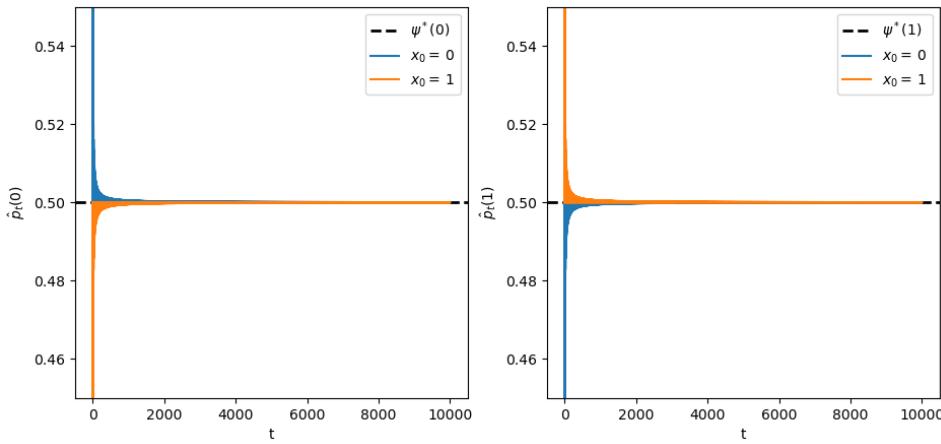
```

label = fr'$\psi^*(i)$'
axes[i].set_xlabel('t')
axes[i].set_ylabel(fr'$\hat{p}_t(i)$')

# Compute the fraction of time spent, for each x
for x0 in range(n):
    # Generate time series starting at different x_0
    X = mc.simulate(ts_length, init=x0)
    p_hat = (X == i).cumsum() / (1 + np.arange(ts_length, dtype=float))
    axes[i].plot(p_hat, label=f'$x_0 = {x0}$')

axes[i].legend()
plt.show()

```



This example helps to emphasize that asymptotic stationarity is about the distribution, while ergodicity is about the sample path.

The proportion of time spent in a state can converge to the stationary distribution with periodic chains.

However, the distribution at each state does not.

### 26.3.5 Expectations of geometric sums

Sometimes we want to compute the mathematical expectation of a geometric sum, such as  $\sum_t \beta^t h(X_t)$ .

In view of the preceding discussion, this is

$$\mathbb{E} \left[ \sum_{j=0}^{\infty} \beta^j h(X_{t+j}) \mid X_t = x \right] = x + \beta(Ph)(x) + \beta^2(P^2h)(x) + \dots$$

By the *Neumann series lemma*, this sum can be calculated using

$$I + \beta P + \beta^2 P^2 + \dots = (I - \beta P)^{-1}$$

## 26.4 Exercises

### Exercise 26.4.1

Benhabib et al. [BBL19] estimated that the transition matrix for social mobility as the following

$$P := \begin{bmatrix} 0.222 & 0.222 & 0.215 & 0.187 & 0.081 & 0.038 & 0.029 & 0.006 \\ 0.221 & 0.22 & 0.215 & 0.188 & 0.082 & 0.039 & 0.029 & 0.006 \\ 0.207 & 0.209 & 0.21 & 0.194 & 0.09 & 0.046 & 0.036 & 0.008 \\ 0.198 & 0.201 & 0.207 & 0.198 & 0.095 & 0.052 & 0.04 & 0.009 \\ 0.175 & 0.178 & 0.197 & 0.207 & 0.11 & 0.067 & 0.054 & 0.012 \\ 0.182 & 0.184 & 0.2 & 0.205 & 0.106 & 0.062 & 0.05 & 0.011 \\ 0.123 & 0.125 & 0.166 & 0.216 & 0.141 & 0.114 & 0.094 & 0.021 \\ 0.084 & 0.084 & 0.142 & 0.228 & 0.17 & 0.143 & 0.121 & 0.028 \end{bmatrix}$$

where each state 1 to 8 corresponds to a percentile of wealth shares

0 – 20%, 20 – 40%, 40 – 60%, 60 – 80%, 80 – 90%, 90 – 95%, 95 – 99%, 99 – 100%

The matrix is recorded as  $P$  below

```
P = [
    [0.222, 0.222, 0.215, 0.187, 0.081, 0.038, 0.029, 0.006],
    [0.221, 0.22, 0.215, 0.188, 0.082, 0.039, 0.029, 0.006],
    [0.207, 0.209, 0.21, 0.194, 0.09, 0.046, 0.036, 0.008],
    [0.198, 0.201, 0.207, 0.198, 0.095, 0.052, 0.04, 0.009],
    [0.175, 0.178, 0.197, 0.207, 0.11, 0.067, 0.054, 0.012],
    [0.182, 0.184, 0.2, 0.205, 0.106, 0.062, 0.05, 0.011],
    [0.123, 0.125, 0.166, 0.216, 0.141, 0.114, 0.094, 0.021],
    [0.084, 0.084, 0.142, 0.228, 0.17, 0.143, 0.121, 0.028]
]

P = np.array(P)
codes_B = ('1', '2', '3', '4', '5', '6', '7', '8')
```

In this exercise,

1. show this process is asymptotically stationary and calculate the stationary distribution using simulations.
2. use simulations to demonstrate ergodicity of this process.

### Solution to Exercise 26.4.1

Solution 1:

Use the technique we learnt before, we can take the power of the transition matrix

```
P = [[0.222, 0.222, 0.215, 0.187, 0.081, 0.038, 0.029, 0.006],
    [0.221, 0.22, 0.215, 0.188, 0.082, 0.039, 0.029, 0.006],
    [0.207, 0.209, 0.21, 0.194, 0.09, 0.046, 0.036, 0.008],
    [0.198, 0.201, 0.207, 0.198, 0.095, 0.052, 0.04, 0.009],
    [0.175, 0.178, 0.197, 0.207, 0.11, 0.067, 0.054, 0.012],
    [0.182, 0.184, 0.2, 0.205, 0.106, 0.062, 0.05, 0.011],
    [0.123, 0.125, 0.166, 0.216, 0.141, 0.114, 0.094, 0.021],
    [0.084, 0.084, 0.142, 0.228, 0.17, 0.143, 0.121, 0.028]]
```

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```
P = np.array(P)
codes_B = ('1', '2', '3', '4', '5', '6', '7', '8')

np.linalg.matrix_power(P, 10)
```

```
array([[0.20254451, 0.20379879, 0.20742102, 0.19505842, 0.09287832,
       0.0503871 , 0.03932382, 0.00858802],
       [0.20254451, 0.20379879, 0.20742102, 0.19505842, 0.09287832,
       0.0503871 , 0.03932382, 0.00858802],
       [0.20254451, 0.20379879, 0.20742102, 0.19505842, 0.09287832,
       0.0503871 , 0.03932382, 0.00858802],
       [0.20254451, 0.20379879, 0.20742102, 0.19505842, 0.09287832,
       0.0503871 , 0.03932382, 0.00858802],
       [0.20254451, 0.20379879, 0.20742102, 0.19505842, 0.09287832,
       0.0503871 , 0.03932382, 0.00858802],
       [0.20254451, 0.20379879, 0.20742102, 0.19505842, 0.09287832,
       0.0503871 , 0.03932382, 0.00858802],
       [0.20254451, 0.20379879, 0.20742102, 0.19505842, 0.09287832,
       0.0503871 , 0.03932382, 0.00858802],
       [0.20254451, 0.20379879, 0.20742102, 0.19505842, 0.09287832,
       0.0503871 , 0.03932382, 0.00858802],
       [0.20254451, 0.20379879, 0.20742102, 0.19505842, 0.09287832,
       0.0503871 , 0.03932382, 0.00858802],
       [0.20254451, 0.20379879, 0.20742102, 0.19505842, 0.09287832,
       0.0503871 , 0.03932382, 0.00858802]])
```

We find again that rows of the transition matrix converge to the stationary distribution

```
mc = qe.MarkovChain(P)
psi_star = mc.stationary_distributions[0]
psi_star
```

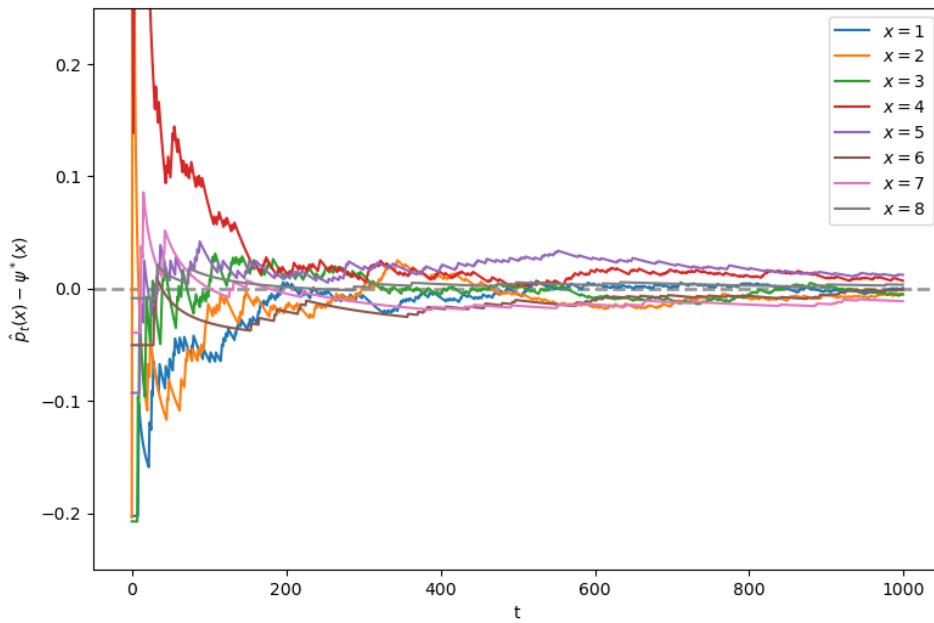
```
array([0.20254451, 0.20379879, 0.20742102, 0.19505842, 0.09287832,
       0.0503871 , 0.03932382, 0.00858802])
```

Solution 2:

```
ts_length = 1000
mc = qe.MarkovChain(P)
fig, ax = plt.subplots(figsize=(9, 6))
X = mc.simulate(ts_length)
ax.set_xlim(-0.25, 0.25)
ax.axhline(0, linestyle='dashed', lw=2, color='black', alpha=0.4)

for x0 in range(8):
    # Calculate the fraction of time for each worker
    p_hat = (X == x0).cumsum() / (1 + np.arange(ts_length, dtype=float))
    ax.plot(p_hat - psi_star[x0], label=f'$x = {x0+1}$')
    ax.set_xlabel('t')
    ax.set_ylabel(r'$\hat{p}_t(x) - \psi^*(x)$')

ax.legend()
plt.show()
```



Note that the fraction of time spent at each state quickly converges to the probability assigned to that state by the stationary distribution.

---

### Exercise 26.4.2

According to the discussion [above](#), if a worker's employment dynamics obey the stochastic matrix

$$P := \begin{bmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{bmatrix}$$

with  $\alpha \in (0, 1)$  and  $\beta \in (0, 1)$ , then, in the long run, the fraction of time spent unemployed will be

$$p := \frac{\beta}{\alpha + \beta}$$

In other words, if  $\{X_t\}$  represents the Markov chain for employment, then  $\bar{X}_m \rightarrow p$  as  $m \rightarrow \infty$ , where

$$\bar{X}_m := \frac{1}{m} \sum_{t=1}^m \mathbf{1}\{X_t = 0\}$$

This exercise asks you to illustrate convergence by computing  $\bar{X}_m$  for large  $m$  and checking that it is close to  $p$ .

You will see that this statement is true regardless of the choice of initial condition or the values of  $\alpha, \beta$ , provided both lie in  $(0, 1)$ .

The result should be similar to the plot we plotted [here](#)

---

### Solution to Exercise 26.4.2

We will address this exercise graphically.

The plots show the time series of  $\bar{X}_m - p$  for two initial conditions.

As  $m$  gets large, both series converge to zero.

```

a = β = 0.1
ts_length = 10000
p = β / (a + β)

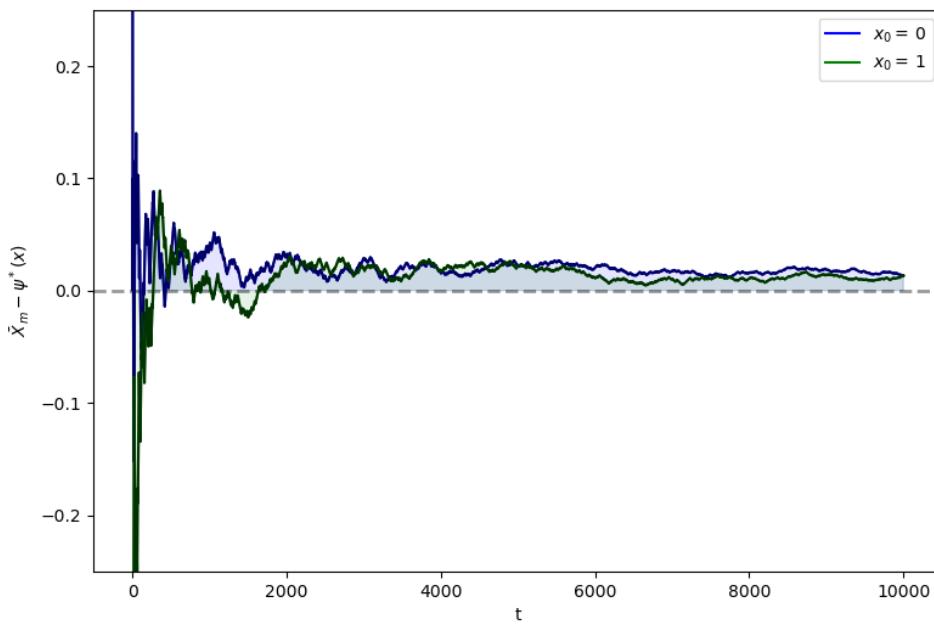
P = ((1 - a, a),          # Careful: P and p are distinct
      (β, 1 - β))
mc = qe.MarkovChain(P)

fig, ax = plt.subplots(figsize=(9, 6))
ax.set_xlim(-0.25, 0.25)
ax.axhline(0, linestyle='dashed', lw=2, color='black', alpha=0.4)

for x0, col in ((0, 'blue'), (1, 'green')):
    # Generate time series for worker that starts at x0
    X = mc.simulate(ts_length, init=x0)
    # Compute fraction of time spent unemployed, for each n
    X_bar = (X == 0).cumsum() / (1 + np.arange(ts_length, dtype=float))
    # Plot
    ax.fill_between(range(ts_length), np.zeros(ts_length), X_bar - p, color=col, alpha=0.1)
    ax.plot(X_bar - p, color=col, label=f'$x_0 = {x0}$')
    # Overlay in black--make lines clearer
    ax.plot(X_bar - p, 'k-', alpha=0.6)
    ax.set_xlabel('t')
    ax.set_ylabel(r'$\bar{X}_m - \psi^*(x)$')

ax.legend(loc='upper right')
plt.show()

```




---

### Exercise 26.4.3

In `quantecon` library, irreducibility is tested by checking whether the chain forms a `strongly connected component`. However, another way to verify irreducibility is by checking whether  $A$  satisfies the following statement:

Assume  $A$  is an  $n \times n$ .  $A$  is irreducible if and only if  $\sum_{k=0}^{n-1} A^k$  is a positive matrix.

(see more: [Zha12] and [here](#))

Based on this claim, write a function to test irreducibility.

---

---

### Solution to Exercise 26.4.3

```
def is_irreducible(P):
    n = len(P)
    result = np.zeros((n, n))
    for i in range(n):
        result += np.linalg.matrix_power(P, i)
    return np.all(result > 0)
```

```
P1 = np.array([[0, 1],
              [1, 0]])
P2 = np.array([[1.0, 0.0, 0.0],
              [0.1, 0.8, 0.1],
              [0.0, 0.2, 0.8]])
P3 = np.array([[0.971, 0.029, 0.000],
              [0.145, 0.778, 0.077],
              [0.000, 0.508, 0.492]])

for P in (P1, P2, P3):
    result = lambda P: 'irreducible' if is_irreducible(P) else 'reducible'
    print(f'{P}: {result(P)}')
```

```
[[0 1]
 [1 0]]: irreducible
[[1. 0. 0. ]
 [0.1 0.8 0.1]
 [0. 0.2 0.8]]: reducible
[[0.971 0.029 0.    ]
 [0.145 0.778 0.077]
 [0.      0.508 0.492]]: irreducible
```

---



## UNIVARIATE TIME SERIES WITH MATRIX ALGEBRA

### Contents

- *Univariate Time Series with Matrix Algebra*
  - *Overview*
  - *Samuelson's model*
  - *Adding a random term*
  - *Computing population moments*
  - *Moving average representation*
  - *A forward looking model*

### 27.1 Overview

This lecture uses matrices to solve some linear difference equations.

As a running example, we'll study a **second-order linear difference equation** that was the key technical tool in Paul Samuelson's 1939 article [Sam39] that introduced the **multiplier-accelerator** model.

This model became the workhorse that powered early econometric versions of Keynesian macroeconomic models in the United States.

You can read about the details of that model in [this QuantEcon lecture](#).

(That lecture also describes some technicalities about second-order linear difference equations.)

In this lecture, we'll also learn about an **autoregressive** representation and a **moving average** representation of a non-stationary univariate time series  $\{y_t\}_{t=0}^T$ .

We'll also study a “perfect foresight” model of stock prices that involves solving a “forward-looking” linear difference equation.

We will use the following imports:

```
import numpy as np
%matplotlib inline
import matplotlib.pyplot as plt
from matplotlib import cm
plt.rcParams["figure.figsize"] = (11, 5) #set default figure size
```

## 27.2 Samuelson's model

Let  $t = 0, \pm 1, \pm 2, \dots$  index time.

For  $t = 1, 2, 3, \dots, T$  suppose that

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} \quad (27.1)$$

where we assume that  $y_0$  and  $y_{-1}$  are given numbers that we take as **initial conditions**.

In Samuelson's model,  $y_t$  stood for **national income** or perhaps a different measure of aggregate activity called **gross domestic product** (GDP) at time  $t$ .

Equation (27.1) is called a **second-order linear difference equation**.

But actually, it is a collection of  $T$  simultaneous linear equations in the  $T$  variables  $y_1, y_2, \dots, y_T$ .

---

**Note:** To be able to solve a second-order linear difference equation, we require two **boundary conditions** that can take the form either of two **initial conditions** or two **terminal conditions** or possibly one of each.

---

Let's write our equations as a stacked system

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\alpha_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\alpha_2 & -\alpha_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\alpha_2 & -\alpha_1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\alpha_2 & -\alpha_1 & 1 \end{bmatrix}}_{\equiv A} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_T \end{bmatrix}}_{\equiv y} = \underbrace{\begin{bmatrix} \alpha_0 + \alpha_1 y_0 + \alpha_2 y_{-1} \\ \alpha_0 + \alpha_2 y_0 \\ \alpha_0 \\ \alpha_0 \\ \vdots \\ \alpha_0 \end{bmatrix}}_{\equiv b}$$

or

$$Ay = b$$

where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix}$$

Evidently  $y$  can be computed from

$$y = A^{-1}b$$

The vector  $y$  is a complete time path  $\{y_t\}_{t=1}^T$ .

Let's put Python to work on an example that captures the flavor of Samuelson's multiplier-accelerator model.

We'll set parameters equal to the same values we used in this QuantEcon lecture.

```
T = 80

# parameters
alpha0 = 10.0
alpha1 = 1.53
alpha2 = -.9

y_1 = 28. # y_{-1}
y_0 = 24.
```

Now we construct  $A$  and  $b$ .

```

A = np.identity(T)    # The T x T identity matrix

for i in range(T):
    if i-1 >= 0:
        A[i, i-1] = -21

    if i-2 >= 0:
        A[i, i-2] = -22

b = np.full(T, 20)
b[0] = 20 + 21 * y0 + 22 * y_1
b[1] = 20 + 22 * y0

```

Let's look at the matrix  $A$  and the vector  $b$  for our example.

A, b

Now let's solve for the path of  $y$ .

If  $y_t$  is GNP at time  $t$ , then we have a version of Samuelson's model of the dynamics for GNP.

To solve  $y = A^{-1}b$  we can either invert  $A$  directly, as in

```
A_inv = np.linalg.inv(A)
```

or we can use `np.linalg.solve`:

```
y_second_method = np.linalg.solve(A, b)
```

Here make sure the two methods give the same result, at least up to floating point precision:

```
np.allclose(y, y_second_method)
```

```
True
```

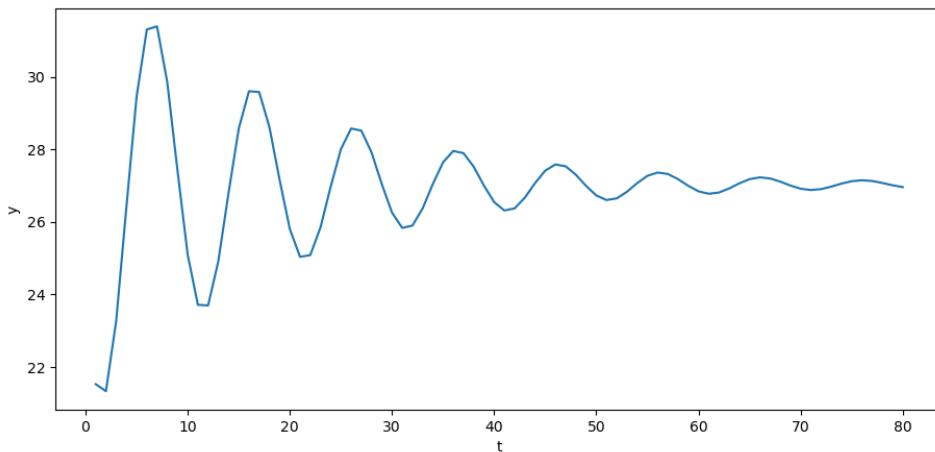
**Note:** In general, `np.linalg.solve` is more numerically stable than using `np.linalg.inv` directly. However, stability is not an issue for this small example. Moreover, we will repeatedly use `A_inv` in what follows, so there is added value in computing it directly.

---

Now we can plot.

```
plt.plot(np.arange(T)+1, y)
plt.xlabel('t')
plt.ylabel('y')

plt.show()
```



The **steady state** value  $y^*$  of  $y_t$  is obtained by setting  $y_t = y_{t-1} = y_{t-2} = y^*$  in (27.1), which yields

$$y^* = \frac{\alpha_0}{1 - \alpha_1 - \alpha_2}$$

If we set the initial values to  $y_0 = y_{-1} = y^*$ , then  $y_t$  will be constant:

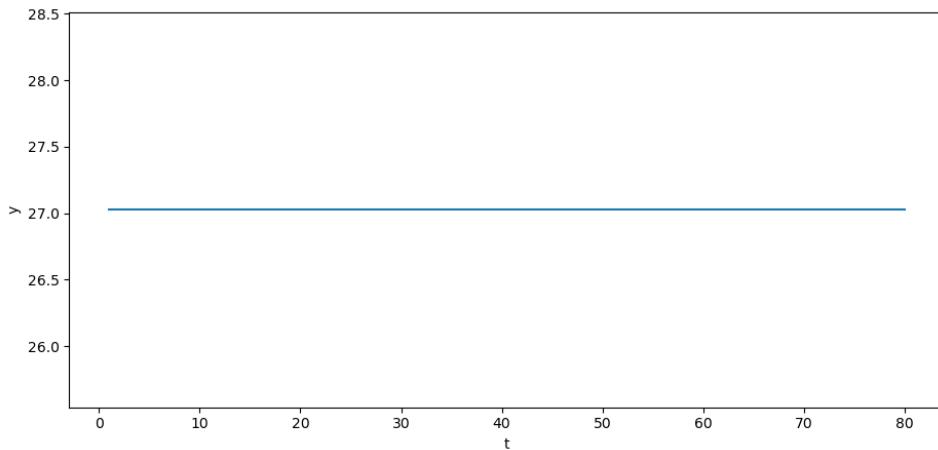
```
y_star = 20 / (1 - 0.1 - 0.2)
y_1_steady = y_star # y_{-1}
y0_steady = y_star

b_steady = np.full(T, 20)
b_steady[0] = 20 + 0.1 * y0_steady + 0.2 * y_1_steady
b_steady[1] = 20 + 0.2 * y0_steady
```

```
y_steady = A_inv @ b_steady
```

```
plt.plot(np.arange(T)+1, y_steady)
plt.xlabel('t')
plt.ylabel('y')

plt.show()
```



### 27.3 Adding a random term

To generate some excitement, we'll follow in the spirit of the great economists Eugen Slutsky and Ragnar Frisch and replace our original second-order difference equation with the following **second-order stochastic linear difference equation**:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + u_t \quad (27.2)$$

where  $u_t \sim N(0, \sigma_u^2)$  and is IID, meaning **independent** and **identically** distributed.

We'll stack these  $T$  equations into a system cast in terms of matrix algebra.

Let's define the random vector

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_T \end{bmatrix}$$

Where  $A, b, y$  are defined as above, now assume that  $y$  is governed by the system

$$Ay = b + u \quad (27.3)$$

The solution for  $y$  becomes

$$y = A^{-1}(b + u) \quad (27.4)$$

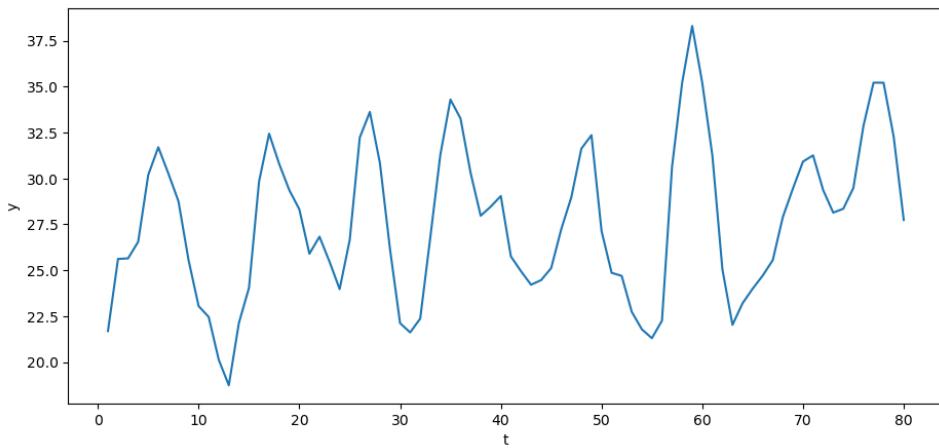
Let's try it out in Python.

```
u = 2.
```

```
u = np.random.normal(0, u, size=T)
y = A_inv @ (b + u)
```

```
plt.plot(np.arange(T)+1, y)
plt.xlabel('t')
plt.ylabel('y')
```

```
plt.show()
```



The above time series looks a lot like (detrended) GDP series for a number of advanced countries in recent decades.

We can simulate  $N$  paths.

```

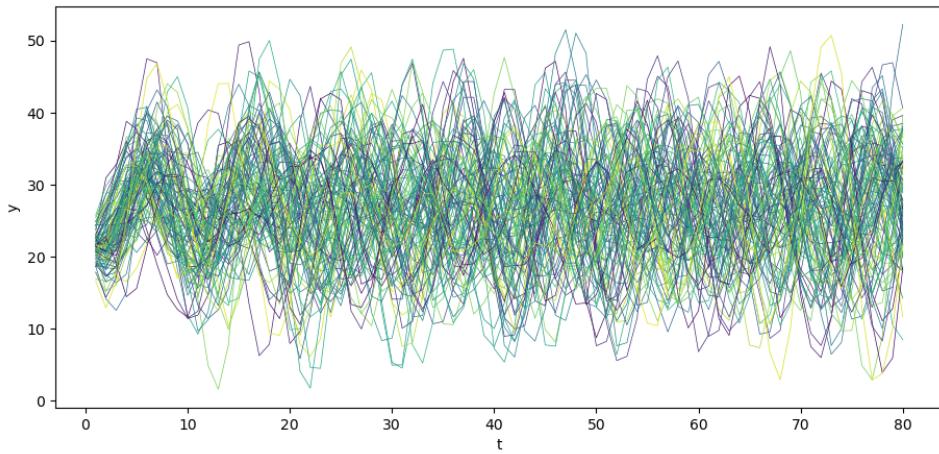
N = 100

for i in range(N):
    col = cm.viridis(np.random.rand()) # Choose a random color from viridis
    u = np.random.normal(0, 2, size=T)
    y = A_inv @ (b + u)
    plt.plot(np.arange(T)+1, y, lw=0.5, color=col)

plt.xlabel('t')
plt.ylabel('y')

plt.show()

```



Also consider the case when  $y_0$  and  $y_{-1}$  are at steady state.

```

N = 100

for i in range(N):
    col = cm.viridis(np.random.rand()) # Choose a random color from viridis
    u = np.random.normal(0, 2, size=T)
    y_steady = A_inv @ (b_steady + u)

```

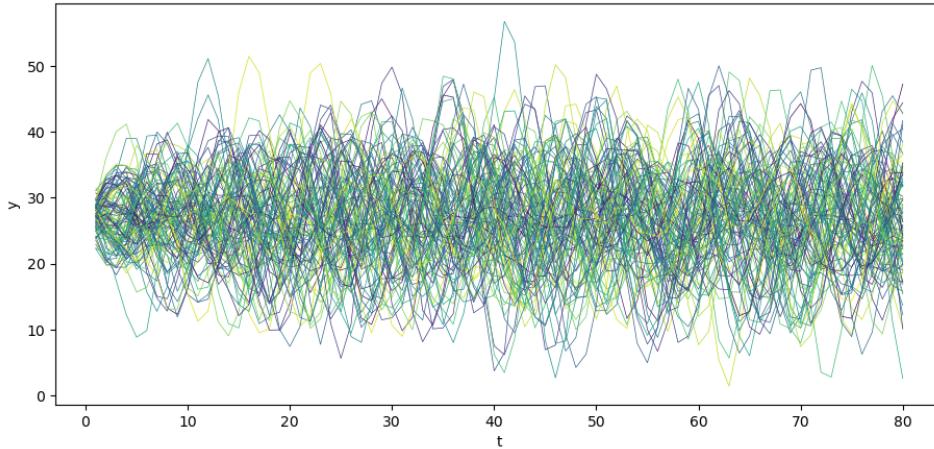
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```
plt.plot(np.arange(T)+1, y_steady, lw=0.5, color=col)

plt.xlabel('t')
plt.ylabel('y')

plt.show()
```



## 27.4 Computing population moments

We can apply standard formulas for multivariate normal distributions to compute the mean vector and covariance matrix for our time series model

$$y = A^{-1}(b + u).$$

You can read about multivariate normal distributions in this lecture [Multivariate Normal Distribution](#).

Let's write our model as

$$y = \tilde{A}(b + u)$$

where  $\tilde{A} = A^{-1}$ .

Because linear combinations of normal random variables are normal, we know that

$$y \sim \mathcal{N}(\mu_y, \Sigma_y)$$

where

$$\mu_y = \tilde{A}b$$

and

$$\Sigma_y = \tilde{A}(\sigma_u^2 I_{T \times T}) \tilde{A}^T$$

Let's write a Python class that computes the mean vector  $\mu_y$  and covariance matrix  $\Sigma_y$ .

```

class population_moments:
    """
    Compute population moments mu_y, Sigma_y.
    -----
    Parameters:
    alpha0, alpha1, alpha2, T, y_1, y0
    """
    def __init__(self, alpha0, alpha1, alpha2, T, y_1, y0, sigma_u):
        # compute A
        A = np.identity(T)

        for i in range(T):
            if i-1 >= 0:
                A[i, i-1] = -alpha1

            if i-2 >= 0:
                A[i, i-2] = -alpha2

        # compute b
        b = np.full(T, alpha0)
        b[0] = alpha0 + alpha1 * y0 + alpha2 * y_1
        b[1] = alpha0 + alpha2 * y0

        # compute A inverse
        A_inv = np.linalg.inv(A)

        self.A, self.b, self.A_inv, self.sigma_u, self.T = A, b, A_inv, sigma_u, T

    def sample_y(self, n):
        """
        Give a sample of size n of y.
        """
        A_inv, sigma_u, b, T = self.A_inv, self.sigma_u, self.b, self.T
        us = np.random.normal(0, sigma_u, size=[n, T])
        ys = np.vstack([A_inv @ (b + u) for u in us])

        return ys

    def get_moments(self):
        """
        Compute the population moments of y.
        """
        A_inv, sigma_u, b = self.A_inv, self.sigma_u, self.b

        # compute mu_y
        self.mu_y = A_inv @ b
        self.Sigma_y = sigma_u**2 * (A_inv @ A_inv.T)

        return self.mu_y, self.Sigma_y

my_process = population_moments(
    alpha0=10.0, alpha1=1.53, alpha2=-.9, T=80, y_1=28., y0=24., sigma_u=1)

mu_y, Sigma_y = my_process.get_moments()
A_inv = my_process.A_inv

```

It is enlightening to study the  $\mu_y, \Sigma_y$ 's implied by various parameter values.

Among other things, we can use the class to exhibit how **statistical stationarity** of  $y$  prevails only for very special initial conditions.

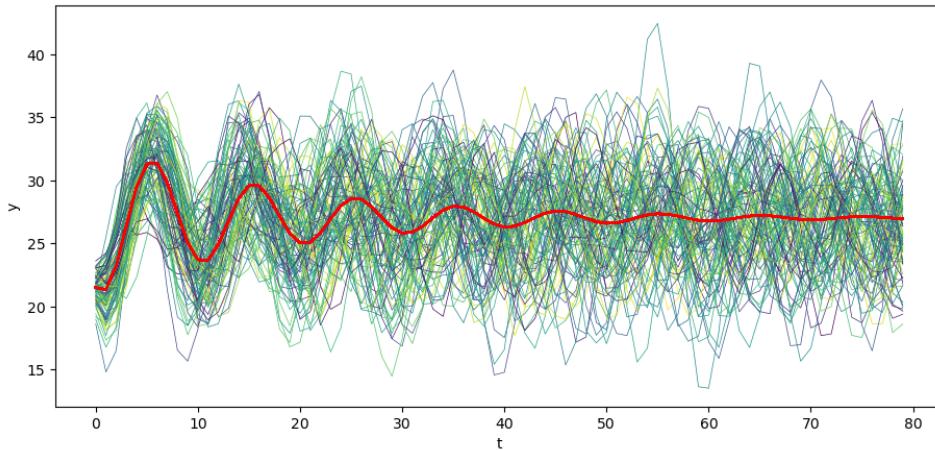
Let's begin by generating  $N$  time realizations of  $y$  plotting them together with population mean  $\mu_y$ .

```
# plot mean
N = 100

for i in range(N):
    col = cm.viridis(np.random.rand()) # Choose a random color from viridis
    ys = my_process.sample_y(N)
    plt.plot(ys[i,:], lw=0.5, color=col)
    plt.plot(mu_y, color='red')

plt.xlabel('t')
plt.ylabel('y')

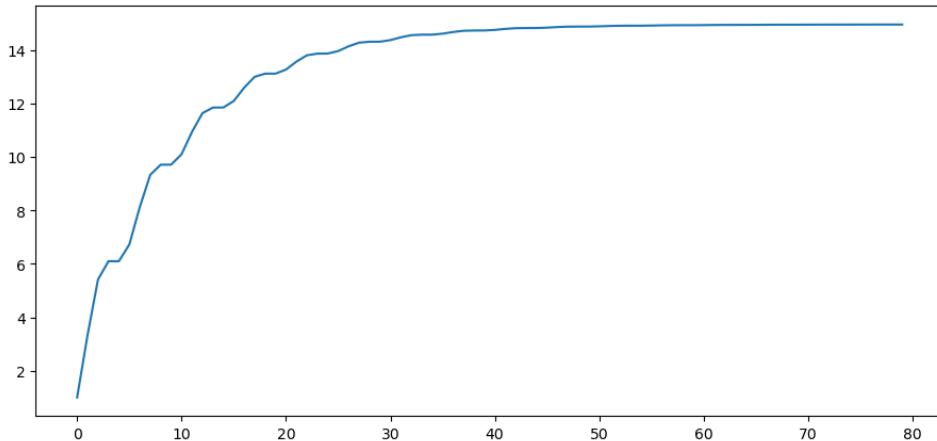
plt.show()
```



Visually, notice how the variance across realizations of  $y_t$  decreases as  $t$  increases.

Let's plot the population variance of  $y_t$  against  $t$ .

```
# plot variance
plt.plot(Sigma_y.diagonal())
plt.show()
```



Notice how the population variance increases and asymptotes

Let's print out the covariance matrix  $\Sigma_y$  for a time series  $y$

```
my_process = population_moments(alpha0=0, alpha1=.8, alpha2=0, T=6, y_1=0., y0=0.,  
                                   sigma_u=1)

mu_y, Sigma_y = my_process.get_moments()
print("mu_y = ", mu_y)
print("Sigma_y = ", Sigma_y)
```

```
mu_y = [0. 0. 0. 0. 0. 0.]
Sigma_y = [[1. 0.64 0.512 0.4096 0.32768
            0.8 1.64 1.312 1.0496 0.83968 0.671744
           0.64 1.312 2.0496 1.63968 1.311744 1.0493952
           0.512 1.0496 1.63968 2.311744 1.8493952 1.47951616
           0.4096 0.83968 1.311744 1.8493952 2.47951616 1.98361293
           0.32768 0.671744 1.0493952 1.47951616 1.98361293 2.58689034]]
```

Notice that the covariance between  $y_t$  and  $y_{t-1}$  – the elements on the superdiagonal – are **not** identical.

This is an indication that the time series represented by our  $y$  vector is not **stationary**.

To make it stationary, we'd have to alter our system so that our **initial conditions**  $(y_1, y_0)$  are not fixed numbers but instead a jointly normally distributed random vector with a particular mean and covariance matrix.

We describe how to do that in another lecture in this lecture [Linear State Space Models](#).

But just to set the stage for that analysis, let's print out the bottom right corner of  $\Sigma_y$ .

```
mu_y, Sigma_y = my_process.get_moments()
print("bottom right corner of Sigma_y = \n", Sigma_y[72:, 72:])
```

```
bottom right corner of Sigma_y =
[]
```

Please notice how the sub diagonal and super diagonal elements seem to have converged.

This is an indication that our process is asymptotically stationary.

You can read about stationarity of more general linear time series models in this lecture [Linear State Space Models](#).

There is a lot to be learned about the process by staring at the off diagonal elements of  $\Sigma_y$  corresponding to different time periods  $t$ , but we resist the temptation to do so here.

## 27.5 Moving average representation

Let's print out  $A^{-1}$  and stare at its structure

- is it triangular or almost triangular or ... ?

To study the structure of  $A^{-1}$ , we shall print just up to 3 decimals.

Let's begin by printing out just the upper left hand corner of  $A^{-1}$

```
with np.printoptions(precision=3, suppress=True):
    print(A_inv[0:7, 0:7])
```

```
[[ 1.      -0.      0.      -0.      -0.      0.      -0.      ]
 [ 1.53     1.      0.      -0.      -0.      0.      -0.      ]
 [ 1.441    1.53    1.      0.      0.      0.      0.      ]
 [ 0.828    1.441   1.53    1.      0.      0.      0.      ]
 [-0.031   0.828   1.441   1.53    1.      0.      -0.      ]
 [-0.792  -0.031   0.828   1.441   1.53    1.      0.      ]
 [-1.184  -0.792  -0.031   0.828   1.441   1.53    1.      ]]
```

Evidently,  $A^{-1}$  is a lower triangular matrix.

Let's print out the lower right hand corner of  $A^{-1}$  and stare at it.

```
with np.printoptions(precision=3, suppress=True):
    print(A_inv[72:, 72:])
```

```
[[ 1.      0.      0.      0.      0.      0.      0.      0.      ]
 [ 1.53     1.      0.      -0.      -0.      0.      -0.      -0.      ]
 [ 1.441    1.53    1.      0.      0.      0.      0.      0.      ]
 [ 0.828    1.441   1.53    1.      0.      0.      0.      0.      ]
 [-0.031   0.828   1.441   1.53    1.      0.      -0.      -0.      ]
 [-0.792  -0.031   0.828   1.441   1.53    1.      0.      0.      ]
 [-1.184  -0.792  -0.031   0.828   1.441   1.53    1.      0.      ]
 [-1.099  -1.184  -0.792  -0.031   0.828   1.441   1.53    1.      ]]
```

Notice how every row ends with the previous row's pre-diagonal entries.

Since  $A^{-1}$  is lower triangular, each row represents  $y_t$  for a particular  $t$  as the sum of

- a time-dependent function  $A^{-1}b$  of the initial conditions incorporated in  $b$ , and
- a weighted sum of current and past values of the IID shocks  $\{u_t\}$

Thus, let  $\tilde{A} = A^{-1}$ .

Evidently, for  $t \geq 0$ ,

$$y_{t+1} = \sum_{i=1}^{t+1} \tilde{A}_{t+1,i} b_i + \sum_{i=1}^t \tilde{A}_{t+1,i} u_i + u_{t+1}$$

This is a **moving average** representation with time-varying coefficients.



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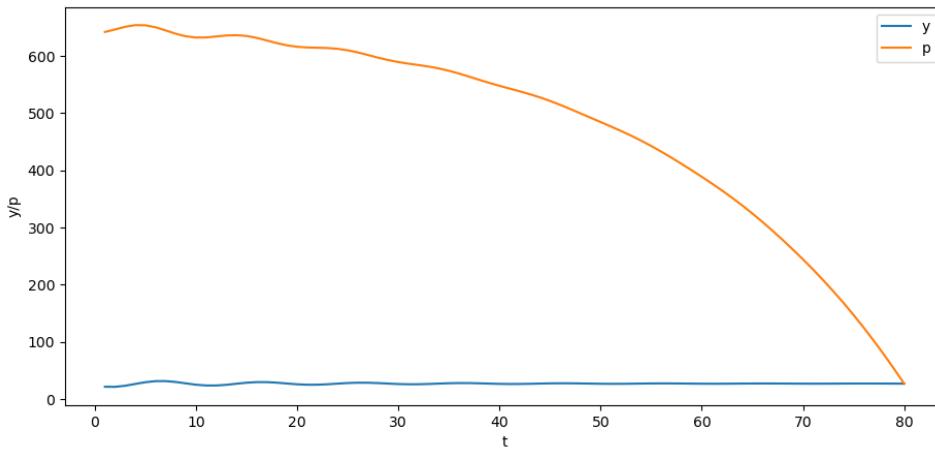
```
[0.         , 0.         , 0.         , ... , 1.         , 0.96       ,
 0.9216     ],
[0.         , 0.         , 0.         , ... , 0.         , 1.         ,
 0.96     ],
[0.         , 0.         , 0.         , ... , 0.         , 0.         ,
 1.         ]])
```

```
bu = 0.
u = np.random.normal(0, bu, size=T)
y = A_inv @ (b + u)
y_steady = A_inv @ (b_steady + u)
```

```
p = B @ y
```

```
plt.plot(np.arange(0, T)+1, y, label='y')
plt.plot(np.arange(0, T)+1, p, label='p')
plt.xlabel('t')
plt.ylabel('y/p')
plt.legend()

plt.show()
```



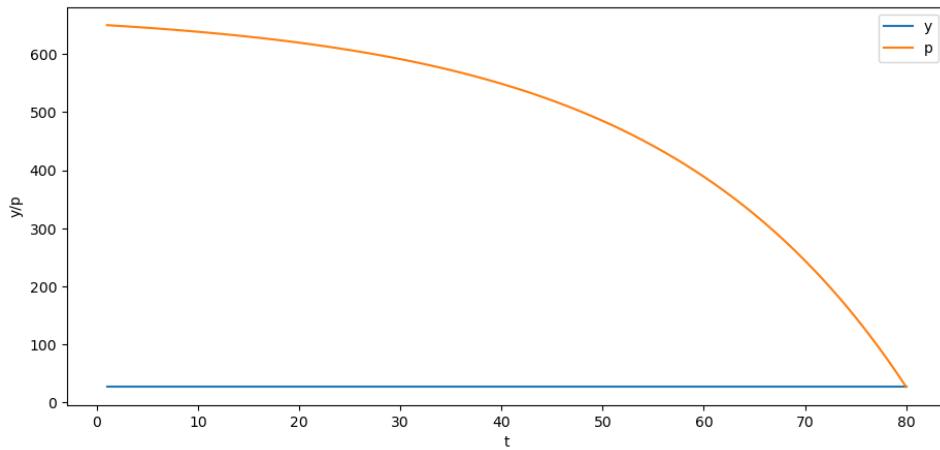
Can you explain why the trend of the price is downward over time?

Also consider the case when  $y_0$  and  $y_{-1}$  are at the steady state.

```
p_steady = B @ y_steady

plt.plot(np.arange(0, T)+1, y_steady, label='y')
plt.plot(np.arange(0, T)+1, p_steady, label='p')
plt.xlabel('t')
plt.ylabel('y/p')
plt.legend()

plt.show()
```



# **Part VIII**

# **Optimization**



In this lecture, we will need the following library. Install `ortools` using `pip`.

```
!pip install ortools
```



## LINEAR PROGRAMMING

### 28.1 Overview

**Linear programming** problems either maximize or minimize a linear objective function subject to a set of linear equality and/or inequality constraints.

Linear programs come in pairs:

- an original **primal** problem, and
- an associated **dual** problem.

If a primal problem involves **maximization**, the dual problem involves **minimization**.

If a primal problem involves **minimization**, the dual problem involves **maximization**.

We provide a standard form of a linear program and methods to transform other forms of linear programming problems into a standard form.

We tell how to solve a linear programming problem using **SciPy** and **Google OR-Tools**.

We describe the important concept of complementary slackness and how it relates to the dual problem.

Let's start with some standard imports.

```
import numpy as np
from ortools.linear_solver import pywraplp
from scipy.optimize import linprog
import matplotlib.pyplot as plt
from matplotlib.patches import Polygon
%matplotlib inline
```

Let's start with some examples of linear programming problem.

### 28.2 Example 1: Production Problem

This example was created by [Ber97]

Suppose that a factory can produce two goods called Product 1 and Product 2.

To produce each product requires both material and labor.

Selling each product generates revenue.

Required per unit material and labor inputs and revenues are shown in table below:

	Product 1	Product 2
Material	2	5
Labor	4	2
Revenue	3	4

30 units of material and 20 units of labor available.

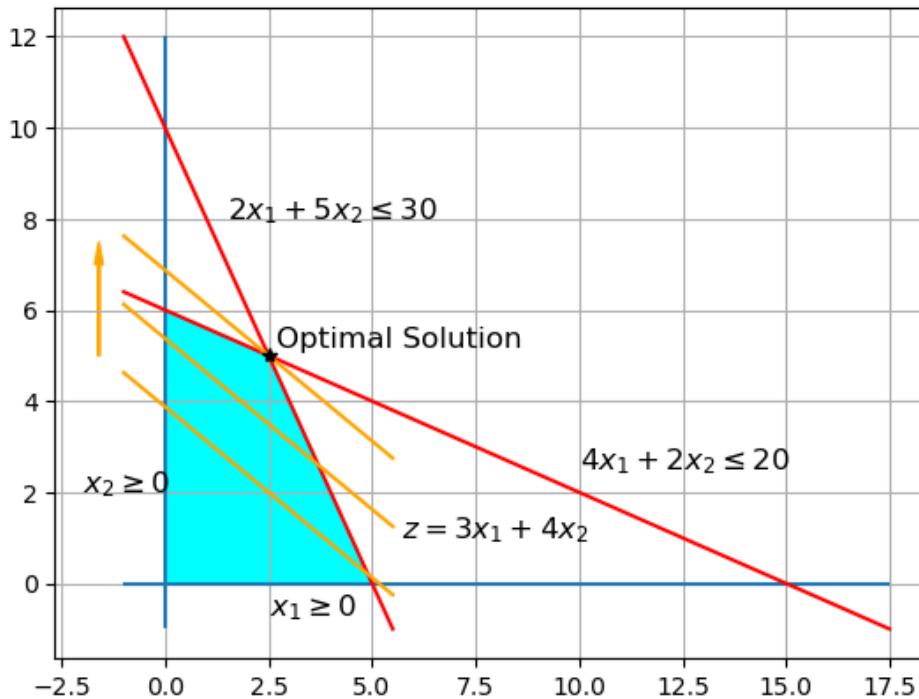
A firm's problem is to construct a production plan that uses its 30 units of materials and 20 units of labor to maximize its revenue.

Let  $x_i$  denote the quantity of Product  $i$  that the firm produces.

This problem can be formulated as:

$$\begin{aligned} \max_{x_1, x_2} \quad & z = 3x_1 + 4x_2 \\ \text{subject to} \quad & 2x_1 + 5x_2 \leq 30 \\ & 4x_1 + 2x_2 \leq 20 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The following graph illustrates the firm's constraints and iso-revenue lines.



The blue region is the feasible set within which all constraints are satisfied.

Parallel orange lines are iso-revenue lines.

The firm's objective is to find the parallel orange lines to the upper boundary of the feasible set.

The intersection of the feasible set and the highest orange line delineates the optimal set.

In this example, the optimal set is the point (2.5, 5).

### 28.2.1 Computation: Using OR-Tools

Let's try to solve the same problem using the package `ortools.linear_solver`

The following cell instantiates a solver and creates two variables specifying the range of values that they can have.

```
# Instantiate a GLOP (Google Linear Optimization Package) solver
solver = pywraplp.Solver.CreateSolver('GLOP')
```

Let's us create two variables  $x_1$  and  $x_2$  such that they can only have nonnegative values.

```
# Create the two variables and let them take on any non-negative value.
x1 = solver.NumVar(0, solver.infinity(), 'x1')
x2 = solver.NumVar(0, solver.infinity(), 'x2')
```

Add the constraints to the problem.

```
# Constraint 1: 2x_1 + 5x_2 <= 30.0
solver.Add(2 * x1 + 5 * x2 <= 30.0)

# Constraint 2: 4x_1 + 2x_2 <= 20.0
solver.Add(4 * x1 + 2 * x2 <= 20.0)
```

```
<ortools.linear_solver.pywraplp.Constraint; proxy of <Swig Object of type
  ↳'operations_research::MPConstraint *' at 0x7f0939f38ba0> >
```

Let's specify the objective function. We use `solver.Maximize` method in the case when we want to maximize the objective function and in the case of minimization we can use `solver.Minimize`.

```
# Objective function: 3x_1 + 4x_2
solver.Maximize(3 * x1 + 4 * x2)
```

Once we solve the problem, we can check whether the solver was successful in solving the problem using it's status. If it's successful, then the status will be equal to `pywraplp.Solver.OPTIMAL`.

```
# Solve the system.
status = solver.Solve()

if status == pywraplp.Solver.OPTIMAL:
    print('Objective value =', solver.Objective().Value())
    x1_sol = round(x1.solution_value(), 2)
    x2_sol = round(x2.solution_value(), 2)
    print(f'(x1, x2): ({x1_sol}, {x2_sol})')
else:
    print('The problem does not have an optimal solution.')
```

```
Objective value = 27.5
(x1, x2): (2.5, 5.0)
```

## 28.3 Example 2: Investment Problem

We now consider a problem posed and solved by [Hu18].

A mutual fund has \$100,000 to be invested over a three year horizon.

Three investment options are available:

1. **Annuity:** the fund can pay a same amount of new capital at the beginning of each of three years and receive a payoff of 130% of **total capital** invested at the end of the third year. Once the mutual fund decides to invest in this annuity, it has to keep investing in all subsequent years in the three year horizon.
2. **Bank account:** the fund can deposit any amount into a bank at the beginning of each year and receive its capital plus 6% interest at the end of that year. In addition, the mutual fund is permitted to borrow no more than \$20,000 at the beginning of each year and is asked to pay back the amount borrowed plus 6% interest at the end of the year. The mutual fund can choose whether to deposit or borrow at the beginning of each year.
3. **Corporate bond:** At the beginning of the second year, a corporate bond becomes available. The fund can buy an amount that is no more than \$50,000 of this bond at the beginning of the second year and at the end of the third year receive a payout of 130% of the amount invested in the bond.

The mutual fund's objective is to maximize total payout that it owns at the end of the third year.

We can formulate this as a linear programming problem.

Let  $x_1$  be the amount of put in the annuity,  $x_2, x_3, x_4$  be bank deposit balances at the beginning of the three years, and  $x_5$  be the amount invested in the corporate bond.

When  $x_2, x_3, x_4$  are negative, it means that the mutual fund has borrowed from bank.

The table below shows the mutual fund's decision variables together with the timing protocol described above:

	Year 1	Year 2	Year 3
Annuity	$x_1$	$x_1$	$x_1$
Bank account	$x_2$	$x_3$	$x_4$
Corporate bond	0	$x_5$	0

The mutual fund's decision making proceeds according to the following timing protocol:

1. At the beginning of the first year, the mutual fund decides how much to invest in the annuity and how much to deposit in the bank. This decision is subject to the constraint:

$$x_1 + x_2 = 100,000$$

2. At the beginning of the second year, the mutual fund has a bank balance of  $1.06x_2$ . It must keep  $x_1$  in the annuity. It can choose to put  $x_5$  into the corporate bond, and put  $x_3$  in the bank. These decisions are restricted by

$$x_1 + x_5 = 1.06x_2 - x_3$$

3. At the beginning of the third year, the mutual fund has a bank account balance equal to  $1.06x_3$ . It must again invest  $x_1$  in the annuity, leaving it with a bank account balance equal to  $x_4$ . This situation is summarized by the restriction:

$$x_1 = 1.06x_3 - x_4$$

The mutual fund's objective function, i.e., its wealth at the end of the third year is:

$$1.30 \cdot 3x_1 + 1.06x_4 + 1.30x_5$$

Thus, the mutual fund confronts the linear program:

$$\begin{aligned} & \max_x 1.30 \cdot 3x_1 + 1.06x_4 + 1.30x_5 \\ & \text{subject to } x_1 + x_2 = 100,000 \\ & \quad x_1 - 1.06x_2 + x_3 + x_5 = 0 \\ & \quad x_1 - 1.06x_3 + x_4 = 0 \\ & \quad x_2 \geq -20,000 \\ & \quad x_3 \geq -20,000 \\ & \quad x_4 \geq -20,000 \\ & \quad x_5 \leq 50,000 \\ & \quad x_j \geq 0, \quad j = 1, 5 \\ & \quad x_j \text{ unrestricted, } \quad j = 2, 3, 4 \end{aligned}$$

### 28.3.1 Computation: Using OR-Tools

Let's try to solve the above problem using the package `ortools.linear_solver`.

The following cell instantiates a solver and creates two variables specifying the range of values that they can have.

```
# Instantiate a GLOP (Google Linear Optimization Package) solver
solver = pywraplp.Solver.CreateSolver('GLOP')
```

Let's us create five variables  $x_1, x_2, x_3, x_4$ , and  $x_5$  such that they can only have the values defined in the above constraints.

```
# Create the variables using the ranges available from constraints
x1 = solver.NumVar(0, solver.infinity(), 'x1')
x2 = solver.NumVar(-20_000, solver.infinity(), 'x2')
x3 = solver.NumVar(-20_000, solver.infinity(), 'x3')
x4 = solver.NumVar(-20_000, solver.infinity(), 'x4')
x5 = solver.NumVar(0, 50_000, 'x5')
```

Add the constraints to the problem.

```
# Constraint 1: x1 + x2 = 100,000
solver.Add(x1 + x2 == 100_000.0)

# Constraint 2: x1 - 1.06 * x2 + x3 + x5 = 0
solver.Add(x1 - 1.06 * x2 + x3 + x5 == 0.0)

# Constraint 3: x1 - 1.06 * x3 + x4 = 0
solver.Add(x1 - 1.06 * x3 + x4 == 0.0)
```

```
<ortools.linear_solver.pywraplp.Constraint; proxy of <Swig Object of type
'operations_research::MPConstraint *' at 0x7f0939f38030> >
```

Let's specify the objective function.

```
# Objective function: 1.30 * 3 * x_1 + 1.06 * x_4 + 1.30 * x_5
solver.Maximize(1.30 * 3 * x1 + 1.06 * x4 + 1.30 * x5)
```

Let's solve the problem and check the status using `pywraplp.Solver.OPTIMAL`.

```
# Solve the system.
status = solver.Solve()

if status == pywraplp.Solver.OPTIMAL:
    print('Objective value =', solver.Objective().Value())
    x1_sol = round(x1.solution_value(), 3)
    x2_sol = round(x2.solution_value(), 3)
    x3_sol = round(x3.solution_value(), 3)
    x4_sol = round(x4.solution_value(), 3)
    x5_sol = round(x5.solution_value(), 3)
    print(f'(x1, x2, x3, x4, x5): ({x1_sol}, {x2_sol}, {x3_sol}, {x4_sol}, {x5_sol})')
else:
    print('The problem does not have an optimal solution.')
```

```
Objective value = 141018.24349792692
(x1, x2, x3, x4, x5): (24927.755, 75072.245, 24927.755, 75072.245, 24927.755)
```

OR-Tools tells us that the best investment strategy is:

1. At the beginning of the first year, the mutual fund should buy \$24,927.755 of the annuity. Its bank account balance should be \$75,072.245.
2. At the beginning of the second year, the mutual fund should buy \$24,927.755 of the corporate bond and keep invest in the annuity. Its bank balance should be \$24,927.755.
3. At the beginning of the third year, the bank balance should be \$75,072.245.
4. At the end of the third year, the mutual fund will get payouts from the annuity and corporate bond and repay its loan from the bank. At the end it will own \$141018.24, so that it's total net rate of return over the three periods is 41.02%.

## 28.4 Standard form

For purposes of

- unifying linear programs that are initially stated in superficially different forms, and
- having a form that is convenient to put into black-box software packages,

it is useful to devote some effort to describe a **standard form**.

Our standard form is:

$$\begin{aligned}
& \min_x c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
& \text{subject to } a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
& \quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
& \quad \vdots \\
& \quad a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\
& \quad x_1, x_2, \dots, x_n \geq 0
\end{aligned}$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The standard form LP problem can be expressed concisely as:

$$\begin{aligned} & \min_x c' x \\ & \text{subject to } Ax = b \\ & \quad x \geq 0 \end{aligned} \tag{28.1}$$

Here,  $Ax = b$  means that the  $i$ -th entry of  $Ax$  equals the  $i$ -th entry of  $b$  for every  $i$ .

Similarly,  $x \geq 0$  means that  $x_j$  is greater than equal to 0 for every  $j$ .

### 28.4.1 Useful transformations

It is useful to know how to transform a problem that initially is not stated in the standard form into one that is.

By deploying the following steps, any linear programming problem can be transformed into an equivalent standard form linear programming problem.

1. **Objective Function:** If a problem is originally a constrained **maximization** problem, we can construct a new objective function that is the additive inverse of the original objective function. The transformed problem is then a **minimization** problem.
2. **Decision Variables:** Given a variable  $x_j$  satisfying  $x_j \leq 0$ , we can introduce a new variable  $x'_j = -x_j$  and substitute it into original problem. Given a free variable  $x_i$  with no restriction on its sign, we can introduce two new variables  $x_j^+$  and  $x_j^-$  satisfying  $x_j^+, x_j^- \geq 0$  and replace  $x_j$  by  $x_j^+ - x_j^-$ .
3. **Inequality constraints:** Given an inequality constraint  $\sum_{j=1}^n a_{ij}x_j \leq 0$ , we can introduce a new variable  $s_i$ , called a **slack variable** that satisfies  $s_i \geq 0$  and replace the original constraint by  $\sum_{j=1}^n a_{ij}x_j + s_i = 0$ .

Let's apply the above steps to the two examples described above.

### 28.4.2 Example 1: Production Problem

The original problem is:

$$\begin{aligned} & \max_{x_1, x_2} 3x_1 + 4x_2 \\ & \text{subject to } 2x_1 + 5x_2 \leq 30 \\ & \quad 4x_1 + 2x_2 \leq 20 \\ & \quad x_1, x_2 \geq 0 \end{aligned}$$

This problem is equivalent to the following problem with a standard form:

$$\begin{aligned} & \min_{x_1, x_2} -(3x_1 + 4x_2) \\ & \text{subject to } 2x_1 + 5x_2 + s_1 = 30 \\ & \quad 4x_1 + 2x_2 + s_2 = 20 \\ & \quad x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

### 28.4.3 Computation: Using SciPy

The package `scipy.optimize` provides a function `linprog` to solve linear programming problems with a form below:

$$\begin{aligned} & \min_x c' x \\ & \text{subject to } A_{ub}x \leq b_{ub} \\ & \quad A_{eq}x = b_{eq} \\ & \quad l \leq x \leq u \end{aligned}$$

---

**Note:** By default  $l = 0$  and  $u = \text{None}$  unless explicitly specified with the argument ‘bounds’.

---

Let's now try to solve the Problem 1 using SciPy.

```
# Construct parameters
c_ex1 = np.array([3, 4])

# Inequality constraints
A_ex1 = np.array([[2, 5],
                  [4, 2]])
b_ex1 = np.array([30, 20])
```

Once we solve the problem, we can check whether the solver was successful in solving the problem using the boolean attribute `success`. If it's successful, then the `success` attribute is set to `True`.

```
# Solve the problem
# we put a negative sign on the objective as linprog does minimization
res_ex1 = linprog(-c_ex1, A_ub=A_ex1, b_ub=b_ex1)

if res_ex1.success:
    # We use negative sign to get the optimal value (maximized value)
    print('Optimal Value:', -res_ex1.fun)
    print(f'(x1, x2): {res_ex1.x[0], res_ex1.x[1]}')
else:
    print('The problem does not have an optimal solution.')
```

```
Optimal Value: 27.5
(x1, x2): (2.5, 5.0)
```

The optimal plan tells the factory to produce 2.5 units of Product 1 and 5 units of Product 2; that generates a maximizing value of revenue of 27.5.

We are using the `linprog` function as a **black box**.

Inside it, Python first transforms the problem into standard form.

To do that, for each inequality constraint it generates one slack variable.

Here the vector of slack variables is a two-dimensional NumPy array that equals  $b_{ub} - A_{ub}x$ .

See the [official documentation](#) for more details.

---

**Note:** This problem is to maximize the objective, so that we need to put a minus sign in front of parameter vector `c`.

---

#### 28.4.4 Example 2: Investment Problem

The original problem is:

$$\begin{aligned}
& \max_x 1.30 \cdot 3x_1 + 1.06x_4 + 1.30x_5 \\
& \text{subject to } x_1 + x_2 = 100,000 \\
& \quad x_1 - 1.06x_2 + x_3 + x_5 = 0 \\
& \quad x_1 - 1.06x_3 + x_4 = 0 \\
& \quad x_2 \geq -20,000 \\
& \quad x_3 \geq -20,000 \\
& \quad x_4 \geq -20,000 \\
& \quad x_5 \leq 50,000 \\
& \quad x_j \geq 0, \quad j = 1, 5 \\
& \quad x_j \text{ unrestricted, } \quad j = 2, 3, 4
\end{aligned}$$

This problem is equivalent to the following problem with a standard form:

$$\begin{aligned}
& \min_x -(1.30 \cdot 3x_1 + 1.06x_4^+ - 1.06x_4^- + 1.30x_5) \\
& \text{subject to } x_1 + x_2^+ - x_2^- = 100,000 \\
& \quad x_1 - 1.06(x_2^+ - x_2^-) + x_3^+ - x_3^- + x_5 = 0 \\
& \quad x_1 - 1.06(x_3^+ - x_3^-) + x_4^+ - x_4^- = 0 \\
& \quad x_2^- - x_2^+ + s_1 = 20,000 \\
& \quad x_3^- - x_3^+ + s_2 = 20,000 \\
& \quad x_4^- - x_4^+ + s_3 = 20,000 \\
& \quad x_5 + s_4 = 50,000 \\
& \quad x_j \geq 0, \quad j = 1, 5 \\
& \quad x_j^+, x_j^- \geq 0, \quad j = 2, 3, 4 \\
& \quad s_j \geq 0, \quad j = 1, 2, 3, 4
\end{aligned}$$

```

# Construct parameters
rate = 1.06

# Objective function parameters
c_ex2 = np.array([1.30*3, 0, 0, 1.06, 1.30])

# Inequality constraints
A_ex2 = np.array([[1, 1, 0, 0, 0],
                  [1, -rate, 1, 0, 1],
                  [1, 0, -rate, 1, 0]])
b_ex2 = np.array([100000, 0, 0])

# Bounds on decision variables
bounds_ex2 = [(0, None),
               (-20000, None),
               (-20000, None),
               (-20000, None),
               (0, 50000)]

```

Let's solve the problem and check the status using `success` attribute.

```

# Solve the problem
res_ex2 = linprog(-c_ex2, A_eq=A_ex2, b_eq=b_ex2,
                  bounds=bounds_ex2)

if res_ex2.success:
    # We use negative sign to get the optimal value (maximized value)
    print('Optimal Value:', -res_ex2.fun)
    x1_sol = round(res_ex2.x[0], 3)
    x2_sol = round(res_ex2.x[1], 3)
    x3_sol = round(res_ex2.x[2], 3)
    x4_sol = round(res_ex2.x[3], 3)
    x5_sol = round(res_ex2.x[4], 3)
    print(f'(x1, x2, x3, x4, x5): {x1_sol, x2_sol, x3_sol, x4_sol, x5_sol}')
else:
    print('The problem does not have an optimal solution.')

```

```

Optimal Value: 141018.24349792697
(x1, x2, x3, x4, x5): (24927.755, 75072.245, 4648.825, -20000.0, 50000.0)

```

SciPy tells us that the best investment strategy is:

1. At the beginning of the first year, the mutual fund should buy \$24,927.75 of the annuity. Its bank account balance should be \$75,072.25.
2. At the beginning of the second year, the mutual fund should buy \$50,000 of the corporate bond and keep invest in the annuity. Its bank account balance should be \$4,648.83.
3. At the beginning of the third year, the mutual fund should borrow \$20,000 from the bank and invest in the annuity.
4. At the end of the third year, the mutual fund will get payouts from the annuity and corporate bond and repay its loan from the bank. At the end it will own \$141018.24, so that it's total net rate of return over the three periods is 41.02%.

---

**Note:** You might notice the difference in the values of optimal solution using OR-Tools and SciPy but the optimal value is the same. It is because there can be many optimal solutions for the same problem.

---

## 28.5 Exercises

---

### Exercise 28.5.1

Implement a new extended solution for the Problem 1 where in the factory owner decides that number of units of Product 1 should not be less than the number of units of Product 2.

---



---

### Solution to Exercise 28.5.1

So we can reformulate the problem as:

$$\begin{aligned} \max_{x_1, x_2} \quad & z = 3x_1 + 4x_2 \\ \text{subject to} \quad & 2x_1 + 5x_2 \leq 30 \\ & 4x_1 + 2x_2 \leq 20 \\ & x_1 \geq x_2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

```
# Instantiate a GLOP (Google Linear Optimization Package) solver
solver = pywraplp.Solver.CreateSolver('GLOP')

# Create the two variables and let them take on any non-negative value.
x1 = solver.NumVar(0, solver.infinity(), 'x1')
x2 = solver.NumVar(0, solver.infinity(), 'x2')
```

```
# Constraint 1: 2x_1 + 5x_2 <= 30.0
solver.Add(2 * x1 + 5 * x2 <= 30.0)

# Constraint 2: 4x_1 + 2x_2 <= 20.0
solver.Add(4 * x1 + 2 * x2 <= 20.0)

# Constraint 3: x_1 >= x_2
solver.Add(x1 >= x2)
```

```
<ortools.linear_solver.pywraplp.Constraint; proxy of <Swig Object of type
 ↴'operations_research::MPConstraint *' at 0x7f0939f38e70> >
```

```
# Objective function: 3x_1 + 4x_2
solver.Maximize(3 * x1 + 4 * x2)
```

```
# Solve the system.
status = solver.Solve()

if status == pywraplp.Solver.OPTIMAL:
    print('Objective value =', solver.Objective().Value())
    x1_sol = round(x1.solution_value(), 2)
    x2_sol = round(x2.solution_value(), 2)
    print(f'(x1, x2): ({x1_sol}, {x2_sol})')
else:
    print('The problem does not have an optimal solution.')
```

```
Objective value = 23.333333333333336
(x1, x2): (3.33, 3.33)
```

---

### Exercise 28.5.2

A carpenter manufactures 2 products - *A* and *B*.

Product *A* generates a profit of 23 and product *B* generates a profit of 10.

It takes 2 hours for the carpenter to produce *A* and 0.8 hours to produce *B*.

Moreover, he can't spend more than 25 hours per week and the total number of units of  $A$  and  $B$  should not be greater than 20.

Find the number of units of  $A$  and product  $B$  that he should manufacture in order to maximise his profit.

---

### Solution to Exercise 28.5.2

Let us assume the carpenter produces  $x$  units of  $A$  and  $y$  units of  $B$ .

So we can formulate the problem as:

$$\begin{aligned} \max_{x,y} z &= 23x + 10y \\ \text{subject to } x + y &\leq 20 \\ 2x + 0.8y &\leq 25 \end{aligned}$$

```
# Instantiate a GLOP (Google Linear Optimization Package) solver
solver = pywraplp.Solver.CreateSolver('GLOP')
```

Let's us create two variables  $x_1$  and  $x_2$  such that they can only have nonnegative values.

```
# Create the two variables and let them take on any non-negative value.
x = solver.NumVar(0, solver.infinity(), 'x')
y = solver.NumVar(0, solver.infinity(), 'y')
```

```
# Constraint 1: x + y <= 20.0
solver.Add(x + y <= 20.0)

# Constraint 2: 2x + 0.8y <= 25.0
solver.Add(2 * x + 0.8 * y <= 25.0)
```

```
<ortools.linear_solver.pywraplp.Constraint; proxy of <Swig Object of type
`operations_research::MPConstraint *` at 0x7f0939f38480> >
```

```
# Objective function: 23x + 10y
solver.Maximize(23 * x + 10 * y)
```

```
# Solve the system.
status = solver.Solve()

if status == pywraplp.Solver.OPTIMAL:
    print('Maximum Profit =', solver.Objective().Value())
    x_sol = round(x.solution_value(), 3)
    y_sol = round(y.solution_value(), 3)
    print(f'(x, y): ({x_sol}, {y_sol})')
else:
    print('The problem does not have an optimal solution.')
```

```
Maximum Profit = 297.5
(x, y): (7.5, 12.5)
```

## SHORTEST PATHS

### Contents

- *Shortest Paths*
  - *Overview*
  - *Outline of the problem*
  - *Finding least-cost paths*
  - *Solving for minimum cost-to-go*
  - *Exercises*

## 29.1 Overview

The shortest path problem is a [classic problem](#) in mathematics and computer science with applications in

- Economics (sequential decision making, analysis of social networks, etc.)
- Operations research and transportation
- Robotics and artificial intelligence
- Telecommunication network design and routing
- etc., etc.

Variations of the methods we discuss in this lecture are used millions of times every day, in applications such as

- Google Maps
- routing packets on the internet

For us, the shortest path problem also provides a nice introduction to the logic of **dynamic programming**.

Dynamic programming is an extremely powerful optimization technique that we apply in many lectures on this site.

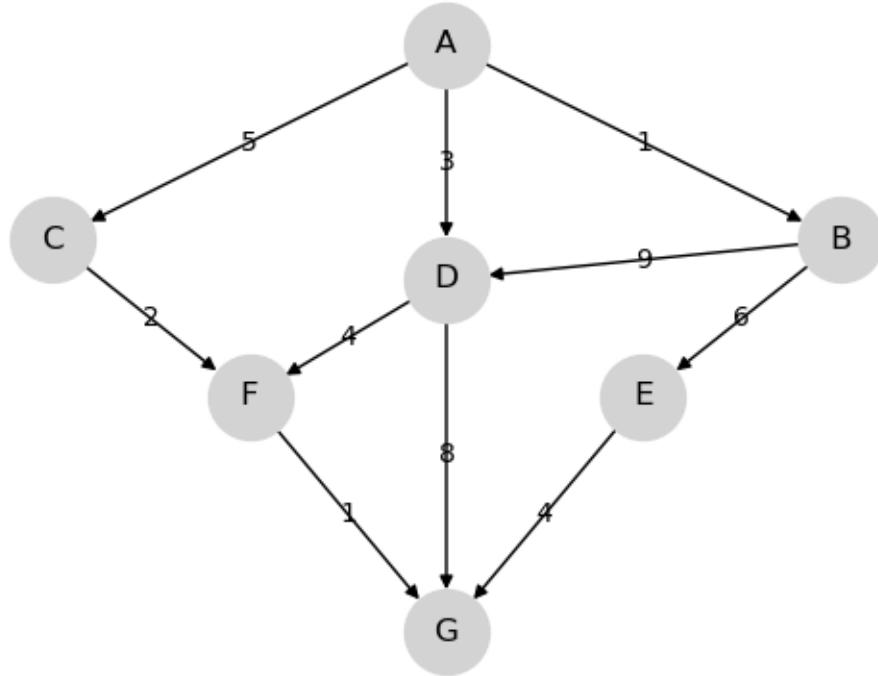
The only scientific library we'll need in what follows is NumPy:

```
import numpy as np
```

## 29.2 Outline of the problem

The shortest path problem is one of finding how to traverse a [graph](#) from one specified node to another at minimum cost.

Consider the following graph



We wish to travel from node (vertex) A to node G at minimum cost

- Arrows (edges) indicate the movements we can take.
- Numbers on edges indicate the cost of traveling that edge.

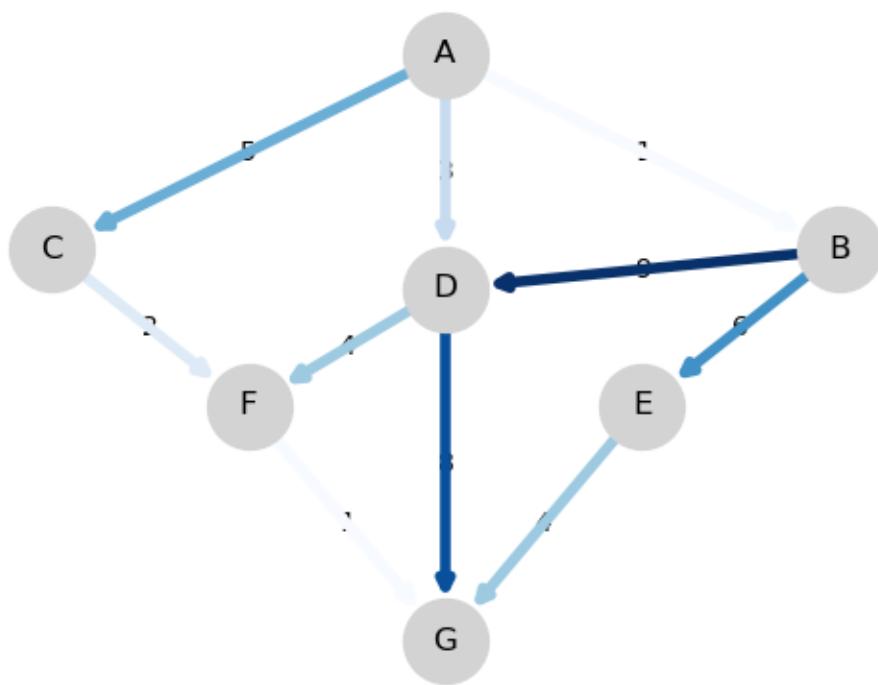
(Graphs such as the one above are called [weighted directed graphs](#).)

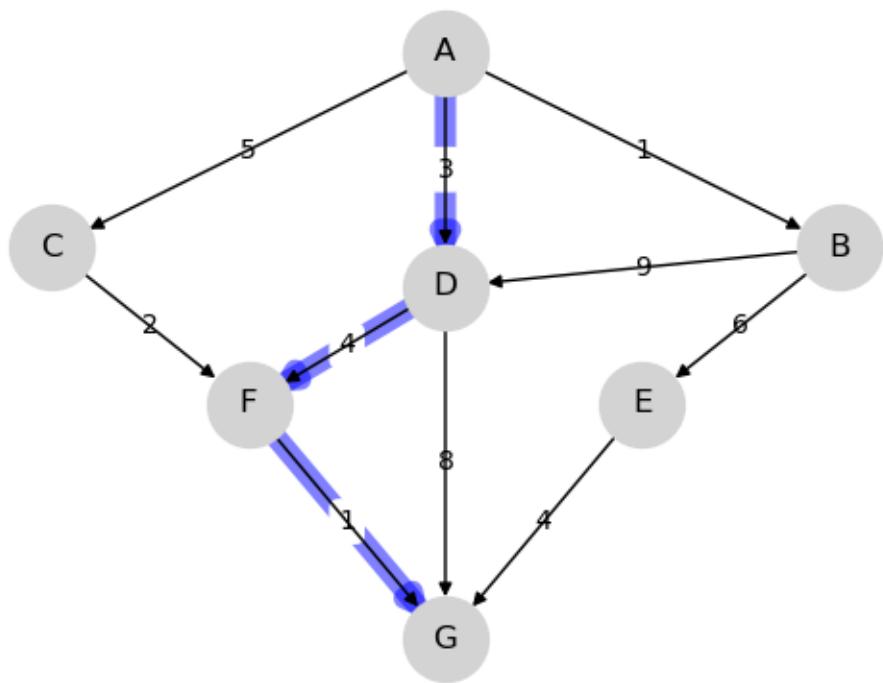
Possible interpretations of the graph include

- Minimum cost for supplier to reach a destination.
- Routing of packets on the internet (minimize time).
- Etc., etc.

For this simple graph, a quick scan of the edges shows that the optimal paths are

- A, C, F, G at cost 8
- A, D, F, G at cost 8



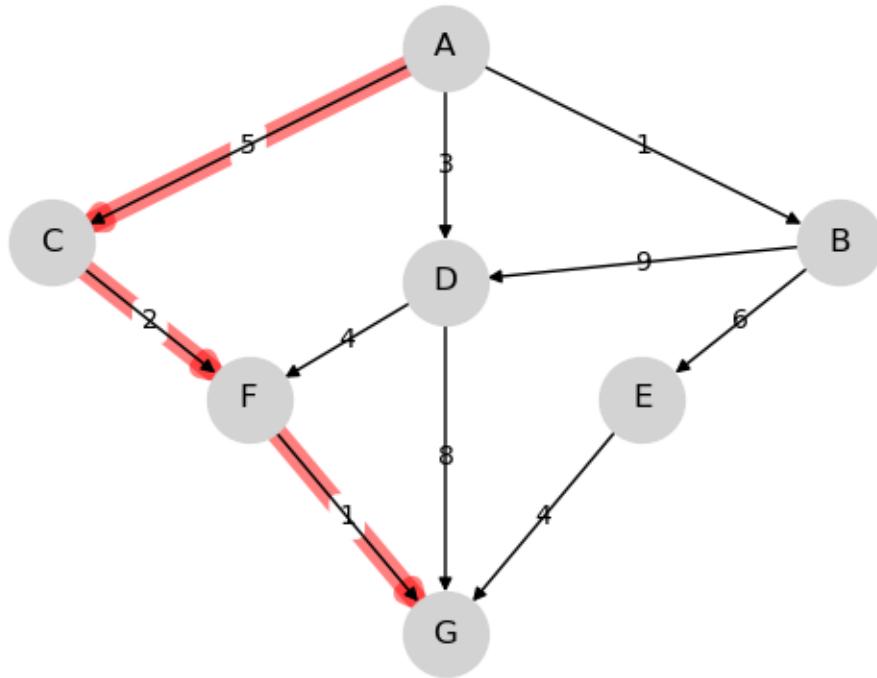


## 29.3 Finding least-cost paths

For large graphs, we need a systematic solution.

Let  $J(v)$  denote the minimum cost-to-go from node  $v$ , understood as the total cost from  $v$  if we take the best route.

Suppose that we know  $J(v)$  for each node  $v$ , as shown below for the graph from the preceding example



Note that  $J(G) = 0$ .

The best path can now be found as follows

1. Start at node  $v = A$
2. From current node  $v$ , move to any node that solves

$$\min_{w \in F_v} \{c(v, w) + J(w)\} \quad (29.1)$$

where

- $F_v$  is the set of nodes that can be reached from  $v$  in one step.
- $c(v, w)$  is the cost of traveling from  $v$  to  $w$ .

Hence, if we know the function  $J$ , then finding the best path is almost trivial.

But how can we find the cost-to-go function  $J$ ?

Some thought will convince you that, for every node  $v$ , the function  $J$  satisfies

$$J(v) = \min_{w \in F_v} \{c(v, w) + J(w)\} \quad (29.2)$$

This is known as the *Bellman equation*, after the mathematician Richard Bellman.

The Bellman equation can be thought of as a restriction that  $J$  must satisfy.

What we want to do now is use this restriction to compute  $J$ .

## 29.4 Solving for minimum cost-to-go

Let's look at an algorithm for computing  $J$  and then think about how to implement it.

### 29.4.1 The algorithm

The standard algorithm for finding  $J$  is to start an initial guess and then iterate.

This is a standard approach to solving nonlinear equations, often called the method of **successive approximations**.

Our initial guess will be

$$J_0(v) = 0 \text{ for all } v \quad (29.3)$$

Now

1. Set  $n = 0$
2. Set  $J_{n+1}(v) = \min_{w \in F_v} \{c(v, w) + J_n(w)\}$  for all  $v$
3. If  $J_{n+1}$  and  $J_n$  are not equal then increment  $n$ , go to 2

This sequence converges to  $J$ .

Although we omit the proof, we'll prove similar claims in our other lectures on dynamic programming.

### 29.4.2 Implementation

Having an algorithm is a good start, but we also need to think about how to implement it on a computer.

First, for the cost function  $c$ , we'll implement it as a matrix  $Q$ , where a typical element is

$$Q(v, w) = \begin{cases} c(v, w) & \text{if } w \in F_v \\ +\infty & \text{otherwise} \end{cases}$$

In this context  $Q$  is usually called the **distance matrix**.

We're also numbering the nodes now, with  $A = 0$ , so, for example

$$Q(1, 2) = \text{ the cost of traveling from B to C}$$

For example, for the simple graph above, we set

```
from numpy import inf

Q = np.array([[inf, 1, 5, 3, inf, inf, inf],
              [inf, inf, inf, 9, 6, inf, inf],
              [inf, inf, inf, inf, inf, 2, inf],
              [inf, inf, inf, inf, inf, 4, 8],
              [inf, inf, inf, inf, inf, inf, 4],
              [inf, inf, inf, inf, inf, inf, 1],
              [inf, inf, inf, inf, inf, inf, 0]])
```

Notice that the cost of staying still (on the principle diagonal) is set to

- `np.inf` for non-destination nodes — moving on is required.
- 0 for the destination node — here is where we stop.

For the sequence of approximations  $\{J_n\}$  of the cost-to-go functions, we can use NumPy arrays.

Let's try with this example and see how we go:

```
nodes = range(7)                                # Nodes = 0, 1, ..., 6
J = np.zeros_like(nodes, dtype=int)               # Initial guess
next_J = np.empty_like(nodes, dtype=int)           # Stores updated guess

max_iter = 500
i = 0

while i < max_iter:
    for v in nodes:
        # minimize  $Q[v, w] + J[w]$  over all choices of  $w$ 
        lowest_cost = inf
        for w in nodes:
            cost = Q[v, w] + J[w]
            if cost < lowest_cost:
                lowest_cost = cost
        next_J[v] = lowest_cost
    if np.equal(next_J, J).all():
        break
    else:
        J[:] = next_J    # Copy contents of next_J to J
        i += 1

print("The cost-to-go function is", J)
```

The cost-to-go function is [ 8 10 3 5 4 1 0]

This matches with the numbers we obtained by inspection above.

But, importantly, we now have a methodology for tackling large graphs.

## 29.5 Exercises

---

### Exercise 29.5.1

The text below describes a weighted directed graph.

The line `node0, node1 0.04, node8 11.11, node14 72.21` means that from node0 we can go to

- node1 at cost 0.04
- node8 at cost 11.11
- node14 at cost 72.21

No other nodes can be reached directly from node0.

Other lines have a similar interpretation.

Your task is to use the algorithm given above to find the optimal path and its cost.

---

**Note:** You will be dealing with floating point numbers now, rather than integers, so consider replacing `np.equal()` with `np.allclose()`.

---

```
%%file graph.txt
node0, node1 0.04, node8 11.11, node14 72.21
node1, node46 1247.25, node6 20.59, node13 64.94
node2, node66 54.18, node31 166.80, node45 1561.45
node3, node20 133.65, node6 2.06, node11 42.43
node4, node75 3706.67, node5 0.73, node7 1.02
node5, node45 1382.97, node7 3.33, node11 34.54
node6, node31 63.17, node9 0.72, node10 13.10
node7, node50 478.14, node9 3.15, node10 5.85
node8, node69 577.91, node11 7.45, node12 3.18
node9, node70 2454.28, node13 4.42, node20 16.53
node10, node89 5352.79, node12 1.87, node16 25.16
node11, node94 4961.32, node18 37.55, node20 65.08
node12, node84 3914.62, node24 34.32, node28 170.04
node13, node60 2135.95, node38 236.33, node40 475.33
node14, node67 1878.96, node16 2.70, node24 38.65
node15, node91 3597.11, node17 1.01, node18 2.57
node16, node36 392.92, node19 3.49, node38 278.71
node17, node76 783.29, node22 24.78, node23 26.45
node18, node91 3363.17, node23 16.23, node28 55.84
node19, node26 20.09, node20 0.24, node28 70.54
node20, node98 3523.33, node24 9.81, node33 145.80
node21, node56 626.04, node28 36.65, node31 27.06
node22, node72 1447.22, node39 136.32, node40 124.22
node23, node52 336.73, node26 2.66, node33 22.37
node24, node66 875.19, node26 1.80, node28 14.25
node25, node70 1343.63, node32 36.58, node35 45.55
node26, node47 135.78, node27 0.01, node42 122.00
node27, node65 480.55, node35 48.10, node43 246.24
node28, node82 2538.18, node34 21.79, node36 15.52
node29, node64 635.52, node32 4.22, node33 12.61
node30, node98 2616.03, node33 5.61, node35 13.95
node31, node98 3350.98, node36 20.44, node44 125.88
node32, node97 2613.92, node34 3.33, node35 1.46
```

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```

node33, node81 1854.73, node41 3.23, node47 111.54
node34, node73 1075.38, node42 51.52, node48 129.45
node35, node52 17.57, node41 2.09, node50 78.81
node36, node71 1171.60, node54 101.08, node57 260.46
node37, node75 269.97, node38 0.36, node46 80.49
node38, node93 2767.85, node40 1.79, node42 8.78
node39, node50 39.88, node40 0.95, node41 1.34
node40, node75 548.68, node47 28.57, node54 53.46
node41, node53 18.23, node46 0.28, node54 162.24
node42, node59 141.86, node47 10.08, node72 437.49
node43, node98 2984.83, node54 95.06, node60 116.23
node44, node91 807.39, node46 1.56, node47 2.14
node45, node58 79.93, node47 3.68, node49 15.51
node46, node52 22.68, node57 27.50, node67 65.48
node47, node50 2.82, node56 49.31, node61 172.64
node48, node99 2564.12, node59 34.52, node60 66.44
node49, node78 53.79, node50 0.51, node56 10.89
node50, node85 251.76, node53 1.38, node55 20.10
node51, node98 2110.67, node59 23.67, node60 73.79
node52, node94 1471.80, node64 102.41, node66 123.03
node53, node72 22.85, node56 4.33, node67 88.35
node54, node88 967.59, node59 24.30, node73 238.61
node55, node84 86.09, node57 2.13, node64 60.80
node56, node76 197.03, node57 0.02, node61 11.06
node57, node86 701.09, node58 0.46, node60 7.01
node58, node83 556.70, node64 29.85, node65 34.32
node59, node90 820.66, node60 0.72, node71 0.67
node60, node76 48.03, node65 4.76, node67 1.63
node61, node98 1057.59, node63 0.95, node64 4.88
node62, node91 132.23, node64 2.94, node76 38.43
node63, node66 4.43, node72 70.08, node75 56.34
node64, node80 47.73, node65 0.30, node76 11.98
node65, node94 594.93, node66 0.64, node73 33.23
node66, node98 395.63, node68 2.66, node73 37.53
node67, node82 153.53, node68 0.09, node70 0.98
node68, node94 232.10, node70 3.35, node71 1.66
node69, node99 247.80, node70 0.06, node73 8.99
node70, node76 27.18, node72 1.50, node73 8.37
node71, node89 104.50, node74 8.86, node91 284.64
node72, node76 15.32, node84 102.77, node92 133.06
node73, node83 52.22, node76 1.40, node90 243.00
node74, node81 1.07, node76 0.52, node78 8.08
node75, node92 68.53, node76 0.81, node77 1.19
node76, node85 13.18, node77 0.45, node78 2.36
node77, node80 8.94, node78 0.98, node86 64.32
node78, node98 355.90, node81 2.59
node79, node81 0.09, node85 1.45, node91 22.35
node80, node92 121.87, node88 28.78, node98 264.34
node81, node94 99.78, node89 39.52, node92 99.89
node82, node91 47.44, node88 28.05, node93 11.99
node83, node94 114.95, node86 8.75, node88 5.78
node84, node89 19.14, node94 30.41, node98 121.05
node85, node97 94.51, node87 2.66, node89 4.90
node86, node97 85.09
node87, node88 0.21, node91 11.14, node92 21.23
node88, node93 1.31, node91 6.83, node98 6.12

```

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```

node89, node97 36.97, node99 82.12
node90, node96 23.53, node94 10.47, node99 50.99
node91, node97 22.17
node92, node96 10.83, node97 11.24, node99 34.68
node93, node94 0.19, node97 6.71, node99 32.77
node94, node98 5.91, node96 2.03
node95, node98 6.17, node99 0.27
node96, node98 3.32, node97 0.43, node99 5.87
node97, node98 0.30
node98, node99 0.33
node99,

```

Overwriting graph.txt

---

### Solution to Exercise 29.5.1

First let's write a function that reads in the graph data above and builds a distance matrix.

```

num_nodes = 100
destination_node = 99

def map_graph_to_distance_matrix(in_file):

    # First let's set of the distance matrix Q with inf everywhere
    Q = np.full((num_nodes, num_nodes), np.inf)

    # Now we read in the data and modify Q
    with open(in_file) as infile:
        for line in infile:
            elements = line.split(',')
            node = elements.pop(0)
            node = int(node[4:])    # convert node description to integer
            if node != destination_node:
                for element in elements:
                    destination, cost = element.split()
                    destination = int(destination[4:])
                    Q[node, destination] = float(cost)
    Q[destination_node, destination_node] = 0
    return Q

```

In addition, let's write

1. a “Bellman operator” function that takes a distance matrix and current guess of  $J$  and returns an updated guess of  $J$ , and
2. a function that takes a distance matrix and returns a cost-to-go function.

We'll use the algorithm described above.

The minimization step is vectorized to make it faster.

```

def bellman(J, Q):
    num_nodes = Q.shape[0]
    next_J = np.empty_like(J)

```

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```

for v in range(num_nodes):
    next_J[v] = np.min(Q[v, :] + J)
return next_J

def compute_cost_to_go(Q):
    num_nodes = Q.shape[0]
    J = np.zeros(num_nodes)           # Initial guess
    max_iter = 500
    i = 0

    while i < max_iter:
        next_J = bellman(J, Q)
        if np.allclose(next_J, J):
            break
        else:
            J[:] = next_J      # Copy contents of next_J to J
            i += 1

    return (J)

```

We used `np.allclose()` rather than testing exact equality because we are dealing with floating point numbers now.

Finally, here's a function that uses the cost-to-go function to obtain the optimal path (and its cost).

```

def print_best_path(J, Q):
    sum_costs = 0
    current_node = 0
    while current_node != destination_node:
        print(current_node)
        # Move to the next node and increment costs
        next_node = np.argmin(Q[current_node, :] + J)
        sum_costs += Q[current_node, next_node]
        current_node = next_node

    print(destination_node)
    print('Cost: ', sum_costs)

```

Okay, now we have the necessary functions, let's call them to do the job we were assigned.

```

Q = map_graph_to_distance_matrix('graph.txt')
J = compute_cost_to_go(Q)
print_best_path(J, Q)

```

```

0
8
11
18
23
33
41
53
56
57
60

```

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```
67
70
73
76
85
87
88
93
94
96
97
98
99
Cost: 160.55000000000007
```

The total cost of the path should agree with  $J[0]$  so let's check this.

```
J[0]
```

```
160.55
```

## **Part IX**

# **Modeling in Higher Dimensions**



## THE PERRON-FROBENIUS THEOREM

### Contents

- *The Perron-Frobenius Theorem*
  - *Nonnegative matrices*
  - *Exercises*

In addition to what's in Anaconda, this lecture will need the following libraries:

```
!pip install quantecon
```

In this lecture we will begin with the foundational concepts in spectral theory.

Then we will explore the Perron-Frobenius Theorem and connect it to applications in Markov chains and networks.

We will use the following imports:

```
import matplotlib.pyplot as plt
import numpy as np
from numpy.linalg import eig
import scipy as sp
import quantecon as qe
```

### 30.1 Nonnegative matrices

Often, in economics, the matrix that we are dealing with is nonnegative.

Nonnegative matrices have several special and useful properties.

In this section we will discuss some of them — in particular, the connection between nonnegativity and eigenvalues.

An  $n \times m$  matrix  $A$  is called **nonnegative** if every element of  $A$  is nonnegative, i.e.,  $a_{ij} \geq 0$  for every  $i, j$ .

We denote this as  $A \geq 0$ .

### 30.1.1 Irreducible matrices

We introduced irreducible matrices in the [Markov chain lecture](#).

Here we generalize this concept:

Let  $a_{ij}^k$  be element  $(i, j)$  of  $A^k$ .

An  $n \times n$  nonnegative matrix  $A$  is called irreducible if  $A + A^2 + A^3 + \dots \gg 0$ , where  $\gg 0$  indicates that every element in  $A$  is strictly positive.

In other words, for each  $i, j$  with  $1 \leq i, j \leq n$ , there exists a  $k \geq 0$  such that  $a_{ij}^k > 0$ .

Here are some examples to illustrate this further:

$$A = \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.2 \end{bmatrix}$$

$A$  is irreducible since  $a_{ij} > 0$  for all  $(i, j)$ .

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$B$  is irreducible since  $B + B^2$  is a matrix of ones.

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$C$  is not irreducible since  $C^k = C$  for all  $k \geq 0$  and thus  $c_{12}^k, c_{21}^k = 0$  for all  $k \geq 0$ .

### 30.1.2 Left eigenvectors

Recall that we previously discussed eigenvectors in [Eigenvalues and Eigenvectors](#).

In particular,  $\lambda$  is an eigenvalue of  $A$  and  $v$  is an eigenvector of  $A$  if  $v$  is nonzero and satisfy

$$Av = \lambda v.$$

In this section we introduce left eigenvectors.

To avoid confusion, what we previously referred to as “eigenvectors” will be called “right eigenvectors”.

Left eigenvectors will play important roles in what follows, including that of stochastic steady states for dynamic models under a Markov assumption.

A vector  $w$  is called a left eigenvector of  $A$  if  $w$  is a right eigenvector of  $A^\top$ .

In other words, if  $w$  is a left eigenvector of matrix  $A$ , then  $A^\top w = \lambda w$ , where  $\lambda$  is the eigenvalue associated with the left eigenvector  $v$ .

This hints at how to compute left eigenvectors

```
A = np.array([[3, 2],
             [1, 4]])

# Compute eigenvalues and right eigenvectors
λ, v = eig(A)

# Compute eigenvalues and left eigenvectors
```

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```

λ, w = eig(A.T)

# Keep 5 decimals
np.set_printoptions(precision=5)

print(f"The eigenvalues of A are:\n {λ}\n")
print(f"The corresponding right eigenvectors are: \n {v[:,0]} and {-v[:,1]}\n")
print(f"The corresponding left eigenvectors are: \n {w[:,0]} and {-w[:,1]}\n")

```

```

The eigenvalues of A are:
[2. 5.]
The corresponding right eigenvectors are:
[-0.89443  0.44721] and [0.70711  0.70711]
The corresponding left eigenvectors are:
[-0.70711  0.70711] and [0.44721  0.89443]

```

We can also use `scipy.linalg.eig` with argument `left=True` to find left eigenvectors directly

```

eigenvals, ε, e = sp.linalg.eig(A, left=True)

print(f"The eigenvalues of A are:\n {eigenvals.real}\n")
print(f"The corresponding right eigenvectors are: \n {e[:,0]} and {-e[:,1]}\n")
print(f"The corresponding left eigenvectors are: \n {ε[:,0]} and {-ε[:,1]}\n")

```

```

The eigenvalues of A are:
[2. 5.]
The corresponding right eigenvectors are:
[-0.89443  0.44721] and [0.70711  0.70711]
The corresponding left eigenvectors are:
[-0.70711  0.70711] and [0.44721  0.89443]

```

The eigenvalues are the same while the eigenvectors themselves are different.

(Also note that we are taking the nonnegative value of the eigenvector of *dominant eigenvalue*, this is because `eig` automatically normalizes the eigenvectors.)

We can then take transpose to obtain  $A^T w = \lambda w$  and obtain  $w^T A = \lambda w^T$ .

This is a more common expression and where the name left eigenvectors originates.

### 30.1.3 The Perron-Frobenius theorem

For a square nonnegative matrix  $A$ , the behavior of  $A^k$  as  $k \rightarrow \infty$  is controlled by the eigenvalue with the largest absolute value, often called the **dominant eigenvalue**.

For any such matrix  $A$ , the Perron-Frobenius Theorem characterizes certain properties of the dominant eigenvalue and its corresponding eigenvector.

---

#### Theorem (Perron-Frobenius Theorem)

If a matrix  $A \geq 0$  then,

1. the dominant eigenvalue of  $A$ ,  $r(A)$ , is real-valued and nonnegative.
2. for any other eigenvalue (possibly complex)  $\lambda$  of  $A$ ,  $|\lambda| \leq r(A)$ .
3. we can find a nonnegative and nonzero eigenvector  $v$  such that  $Av = r(A)v$ .

Moreover if  $A$  is also irreducible then,

4. the eigenvector  $v$  associated with the eigenvalue  $r(A)$  is strictly positive.
5. there exists no other positive eigenvector  $v$  (except scalar multiples of  $v$ ) associated with  $r(A)$ .

(More of the Perron-Frobenius theorem about primitive matrices will be introduced [below](#).)

---

(This is a relatively simple version of the theorem — for more details see [here](#)).

We will see applications of the theorem below.

Let's build our intuition for the theorem using a simple example we have seen [before](#).

Now let's consider examples for each case.

### Example: Irreducible matrix

Consider the following irreducible matrix  $A$ :

```
A = np.array([[0, 1, 0],
              [.5, 0, .5],
              [0, 1, 0]])
```

We can compute the dominant eigenvalue and the corresponding eigenvector

```
eig(A)
```

```
(array([-1.00000e+00, -3.30139e-18, 1.00000e+00]),
 array([[ 5.77350e-01,  7.07107e-01,  5.77350e-01],
        [-5.77350e-01,  3.16979e-18,  5.77350e-01],
        [ 5.77350e-01, -7.07107e-01,  5.77350e-01]]))
```

Now we can see the claims of the Perron-Frobenius Theorem holds for the irreducible matrix  $A$ :

1. The dominant eigenvalue is real-valued and non-negative.
2. All other eigenvalues have absolute values less than or equal to the dominant eigenvalue.
3. A non-negative and nonzero eigenvector is associated with the dominant eigenvalue.
4. As the matrix is irreducible, the eigenvector associated with the dominant eigenvalue is strictly positive.
5. There exists no other positive eigenvector associated with the dominant eigenvalue.

### 30.1.4 Primitive matrices

We know that in real world situations it's hard for a matrix to be everywhere positive (although they have nice properties).

The primitive matrices, however, can still give us helpful properties with looser definitions.

Let  $A$  be a square nonnegative matrix and let  $A^k$  be the  $k^{th}$  power of  $A$ .

A matrix is called **primitive** if there exists a  $k \in \mathbb{N}$  such that  $A^k$  is everywhere positive.

Recall the examples given in irreducible matrices:

$$A = \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.2 \end{bmatrix}$$

$A$  here is also a primitive matrix since  $A^k$  is everywhere nonnegative for  $k \in \mathbb{N}$ .

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$B$  is irreducible but not primitive since there are always zeros in either principal diagonal or secondary diagonal.

We can see that if a matrix is primitive, then it implies the matrix is irreducible but not vice versa.

Now let's step back to the primitive matrices part of the Perron-Frobenius Theorem

---

#### Theorem (Continous of Perron-Frobenius Theorem)

If  $A$  is primitive then,

6. the inequality  $|\lambda| \leq r(A)$  is **strict** for all eigenvalues  $\lambda$  of  $A$  distinct from  $r(A)$ , and
  7. with  $v$  and  $w$  normalized so that the inner product of  $w$  and  $v = 1$ , we have  $r(A)^{-m}A^m$  converges to  $vw^\top$  when  $m \rightarrow \infty$ . The matrix  $vw^\top$  is called the **Perron projection** of  $A$ .
- 

#### Example 1: Primitive matrix

Consider the following primitive matrix  $B$ :

```
B = np.array([[0, 1, 1],
              [1, 0, 1],
              [1, 1, 0]])
np.linalg.matrix_power(B, 2)
```

```
array([[2, 1, 1],
       [1, 2, 1],
       [1, 1, 2]])
```

We compute the dominant eigenvalue and the corresponding eigenvector

```
eig(B)
```

```
(array([-1.,  2., -1.]),
 array([[-0.8165 ,  0.57735, -0.09266],
        [ 0.40825,  0.57735, -0.65621],
        [ 0.40825,  0.57735,  0.74887]]))
```

Now let's give some examples to see if the claims of the Perron-Frobenius Theorem hold for the primitive matrix  $B$ :

1. The dominant eigenvalue is real-valued and non-negative.
2. All other eigenvalues have absolute values strictly less than the dominant eigenvalue.
3. A non-negative and nonzero eigenvector is associated with the dominant eigenvalue.
4. The eigenvector associated with the dominant eigenvalue is strictly positive.
5. There exists no other positive eigenvector associated with the dominant eigenvalue.
6. The inequality  $|\lambda| < r(B)$  holds for all eigenvalues  $\lambda$  of  $B$  distinct from the dominant eigenvalue.

Furthermore, we can verify the convergence property (7) of the theorem on the following examples:

```

def compute_perron_projection (M) :

    eigval, v = eig(M)
    eigval, w = eig(M.T)

    r = np.max(eigval)

    # Find the index of the dominant (Perron) eigenvalue
    i = np.argmax(eigval)

    # Get the Perron eigenvectors
    v_P = v[:, i].reshape(-1, 1)
    w_P = w[:, i].reshape(-1, 1)

    # Normalize the left and right eigenvectors
    norm_factor = w_P.T @ v_P
    v_norm = v_P / norm_factor

    # Compute the Perron projection matrix
    P = v_norm @ w_P.T
    return P, r

def check_convergence (M) :
    P, r = compute_perron_projection(M)
    print("Perron projection:")
    print(P)

    # Define a list of values for n
    n_list = [1, 10, 100, 1000, 10000]

    for n in n_list:

        # Compute (A/r)^n
        M_n = np.linalg.matrix_power(M/r, n)

        # Compute the difference between A^n / r^n and the Perron projection
        diff = np.abs(M_n - P)

        # Calculate the norm of the difference matrix
        diff_norm = np.linalg.norm(diff, 'fro')
        print(f"n = {n}, error = {diff_norm:.10f}")

A1 = np.array([[1, 2],

```

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```

[1, 4]])

A2 = np.array([[0, 1, 1],
               [1, 0, 1],
               [1, 1, 0]])

A3 = np.array([[0.971, 0.029, 0.1, 1],
               [0.145, 0.778, 0.077, 0.59],
               [0.1, 0.508, 0.492, 1.12],
               [0.2, 0.8, 0.71, 0.95]])

for M in A1, A2, A3:
    print("Matrix:")
    print(M)
    check_convergence(M)
    print()
    print("-"*36)
    print()

```

```

Matrix:
[[1 2]
 [1 4]]
Perron projection:
[[0.1362 0.48507]
 [0.24254 0.8638 ]]
n = 1, error = 0.0989045731
n = 10, error = 0.0000000001
n = 100, error = 0.0000000000
n = 1000, error = 0.0000000000
n = 10000, error = 0.0000000000
-----
```

```

Matrix:
[[0 1 1]
 [1 0 1]
 [1 1 0]]
Perron projection:
[[0.33333 0.33333 0.33333]
 [0.33333 0.33333 0.33333]
 [0.33333 0.33333 0.33333]]
n = 1, error = 0.7071067812
n = 10, error = 0.0013810679
n = 100, error = 0.0000000000
n = 1000, error = 0.0000000000
n = 10000, error = 0.0000000000
-----
```

```

Matrix:
[[0.971 0.029 0.1 1.]
 [0.145 0.778 0.077 0.59]
 [0.1 0.508 0.492 1.12]
 [0.2 0.8 0.71 0.95]]
Perron projection:
```

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```
[[0.12506 0.31949 0.20233 0.43341]
 [0.07714 0.19707 0.1248 0.26735]
 [0.12158 0.31058 0.19669 0.42133]
 [0.13885 0.3547 0.22463 0.48118]]
n = 1, error = 0.5361031549
n = 10, error = 0.0000434043
n = 100, error = 0.0000000000
n = 1000, error = 0.0000000000
n = 10000, error = 0.0000000000
```

-----

The convergence is not observed in cases of non-primitive matrices.

Let's go through an example

```
B = np.array([[0, 1, 1],
             [1, 0, 0],
             [1, 0, 0]])

# This shows that the matrix is not primitive
print("Matrix:")
print(B)
print("100th power of matrix B:")
print(np.linalg.matrix_power(B, 100))

check_convergence(B)
```

```
Matrix:
[[0 1 1]
 [1 0 0]
 [1 0 0]]
100th power of matrix B:
[[1125899906842624 0 0]
 [0 562949953421312 562949953421312]
 [0 562949953421312 562949953421312]]
Perron projection:
[[0.5 0.35355 0.35355]
 [0.35355 0.25 0.25]
 [0.35355 0.25 0.25]]
n = 1, error = 1.0000000000
n = 10, error = 1.0000000000
n = 100, error = 1.0000000000
n = 1000, error = 1.0000000000
n = 10000, error = 1.0000000000
```

The result shows that the matrix is not primitive as it is not everywhere positive.

These examples show how the Perron-Frobenius Theorem relates to the eigenvalues and eigenvectors of positive matrices and the convergence of the power of matrices.

In fact we have already seen the theorem in action before in *the Markov chain lecture*.

### Example 2: Connection to Markov chains

We are now prepared to bridge the languages spoken in the two lectures.

A primitive matrix is both irreducible and aperiodic.

So Perron-Frobenius Theorem explains why both *Imam and Temple matrix* and *Hamilton matrix* converge to a stationary distribution, which is the Perron projection of the two matrices

```
P = np.array([[0.68, 0.12, 0.20],
             [0.50, 0.24, 0.26],
             [0.36, 0.18, 0.46]])

print(compute_perron_projection(P)[0])
```

```
[[0.56146 0.15565 0.28289]
 [0.56146 0.15565 0.28289]
 [0.56146 0.15565 0.28289]]
```

```
mc = qe.MarkovChain(P)
psi_star = mc.stationary_distributions[0]
psi_star
```

```
array([0.56146, 0.15565, 0.28289])
```

```
P_hamilton = np.array([[0.971, 0.029, 0.000],
                      [0.145, 0.778, 0.077],
                      [0.000, 0.508, 0.492]])

print(compute_perron_projection(P_hamilton)[0])
```

```
[[0.8128 0.16256 0.02464]
 [0.8128 0.16256 0.02464]
 [0.8128 0.16256 0.02464]]
```

```
mc = qe.MarkovChain(P_hamilton)
psi_star = mc.stationary_distributions[0]
psi_star
```

```
array([0.8128, 0.16256, 0.02464])
```

We can also verify other properties hinted by Perron-Frobenius in these stochastic matrices.

Another example is the relationship between convergence gap and convergence rate.

In the *exercise*, we stated that the convergence rate is determined by the spectral gap, the difference between the largest and the second largest eigenvalue.

This can be proven using what we have learned here.

Please note that we use  $\mathbb{1}$  for a vector of ones in this lecture.

With Markov model  $M$  with state space  $S$  and transition matrix  $P$ , we can write  $P^t$  as

$$P^t = \sum_{i=1}^{n-1} \lambda_i^t v_i w_i^\top + \mathbb{1} \psi^*,$$

This is proven in [SS23] and a nice discussion can be found [here](#).

In this formula  $\lambda_i$  is an eigenvalue of  $P$  with corresponding right and left eigenvectors  $v_i$  and  $w_i$ .

Premultiplying  $P^t$  by arbitrary  $\psi \in \mathcal{D}(S)$  and rearranging now gives

$$\psi P^t - \psi^* = \sum_{i=1}^{n-1} \lambda_i^t \psi v_i w_i^\top$$

Recall that eigenvalues are ordered from smallest to largest from  $i = 1 \dots n$ .

As we have seen, the largest eigenvalue for a primitive stochastic matrix is one.

This can be proven using [Gershgorin Circle Theorem](#), but it is out of the scope of this lecture.

So by the statement (6) of Perron-Frobenius Theorem,  $\lambda_i < 1$  for all  $i < n$ , and  $\lambda_n = 1$  when  $P$  is primitive.

Hence, after taking the Euclidean norm deviation, we obtain

$$\|\psi P^t - \psi^*\| = O(\eta^t) \quad \text{where} \quad \eta := |\lambda_{n-1}| < 1$$

Thus, the rate of convergence is governed by the modulus of the second largest eigenvalue.

## 30.2 Exercises

---

### Exercise 30.2.1 (Leontief's Input-Output Model)

[Wassily Leontief](#) developed a model of an economy with  $n$  sectors producing  $n$  different commodities representing the interdependencies of different sectors of an economy.

Under this model some of the output is consumed internally by the industries and the rest is consumed by external consumers.

We define a simple model with 3 sectors - agriculture, industry, and service.

The following table describes how output is distributed within the economy:

	Total output	Agriculture	Industry	Service	Consumer
Agriculture	$x_1$	$0.3x_1$	$0.2x_2$	$0.3x_3$	4
Industry	$x_2$	$0.2x_1$	$0.4x_2$	$0.3x_3$	5
Service	$x_3$	$0.2x_1$	$0.5x_2$	$0.1x_3$	12

The first row depicts how agriculture's total output  $x_1$  is distributed

- $0.3x_1$  is used as inputs within agriculture itself,
- $0.2x_2$  is used as inputs by the industry sector to produce  $x_2$  units,
- $0.3x_3$  is used as inputs by the service sector to produce  $x_3$  units and
- 4 units is the external demand by consumers.

We can transform this into a system of linear equations for the 3 sectors as given below:

$$\begin{aligned} x_1 &= 0.3x_1 + 0.2x_2 + 0.3x_3 + 4 \\ x_2 &= 0.2x_1 + 0.4x_2 + 0.3x_3 + 5 \\ x_3 &= 0.2x_1 + 0.5x_2 + 0.1x_3 + 12 \end{aligned}$$

This can be transformed into the matrix equation  $x = Ax + d$  where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, A = \begin{bmatrix} 0.3 & 0.2 & 0.3 \\ 0.2 & 0.4 & 0.3 \\ 0.2 & 0.5 & 0.1 \end{bmatrix} \text{ and } d = \begin{bmatrix} 4 \\ 5 \\ 12 \end{bmatrix}$$

The solution  $x^*$  is given by the equation  $x^* = (I - A)^{-1}d$

1. Since  $A$  is a nonnegative irreducible matrix, find the Perron-Frobenius eigenvalue of  $A$ .
  2. Use the *Neumann Series Lemma* to find the solution  $x^*$  if it exists.
- 

### Solution to Exercise 30.2.1 (Leontief's Input-Output Model)

```
A = np.array([[0.3, 0.2, 0.3],
             [0.2, 0.4, 0.3],
             [0.2, 0.5, 0.1]])

evals, evecs = eig(A)

r = max(abs(lam) for lam in evals)    #dominant eigenvalue/spectral radius
print(r)
```

0.8444086477164563

Since we have  $r(A) < 1$  we can thus find the solution using the Neumann Series Lemma.

```
I = np.identity(3)
B = I - A

d = np.array([4, 5, 12])
d.shape = (3,1)

B_inv = np.linalg.inv(B)
x_star = B_inv @ d
print(x_star)
```

[38.30189]  
[44.33962]  
[46.47799]

---



## INPUT-OUTPUT MODELS

### 31.1 Overview

This lecture requires the following imports and installs before we proceed.

```
!pip install quantecon_book_networks
!pip install quantecon
!pip install pandas-datareader
```

```
import numpy as np
import pandas as pd
import networkx as nx
import matplotlib.pyplot as plt
import quantecon_book_networks
import quantecon_book_networks.input_output as qbn_io
import quantecon_book_networks.plotting as qbn_plt
import quantecon_book_networks.data as qbn_data
import matplotlib as mpl
from matplotlib.patches import Polygon

quantecon_book_networks.config("matplotlib")
mpl.rcParams.update(mpl.rcParamsDefault)
```

The following figure illustrates a network of linkages among 15 sectors obtained from the US Bureau of Economic Analysis's 2021 Input-Output Accounts Data.

Label	Sector	Label	Sector	Label	Sector
ag	Agriculture	wh	Wholesale	pr	Professional Services
mi	Mining	re	Retail	ed	Education & Health
ut	Utilities	tr	Transportation	ar	Arts & Entertainment
co	Construction	in	Information	ot	Other Services (exc govt)
ma	Manufacturing	fi	Finance	go	Government

An arrow from  $i$  to  $j$  means that some of sector  $i$ 's output serves as an input to production of sector  $j$ .

Economies are characterised by many such links.

A basic framework for their analysis is Leontief's input-output model.

After introducing the input-output model, we describe some of its connections to *linear programming lecture*.

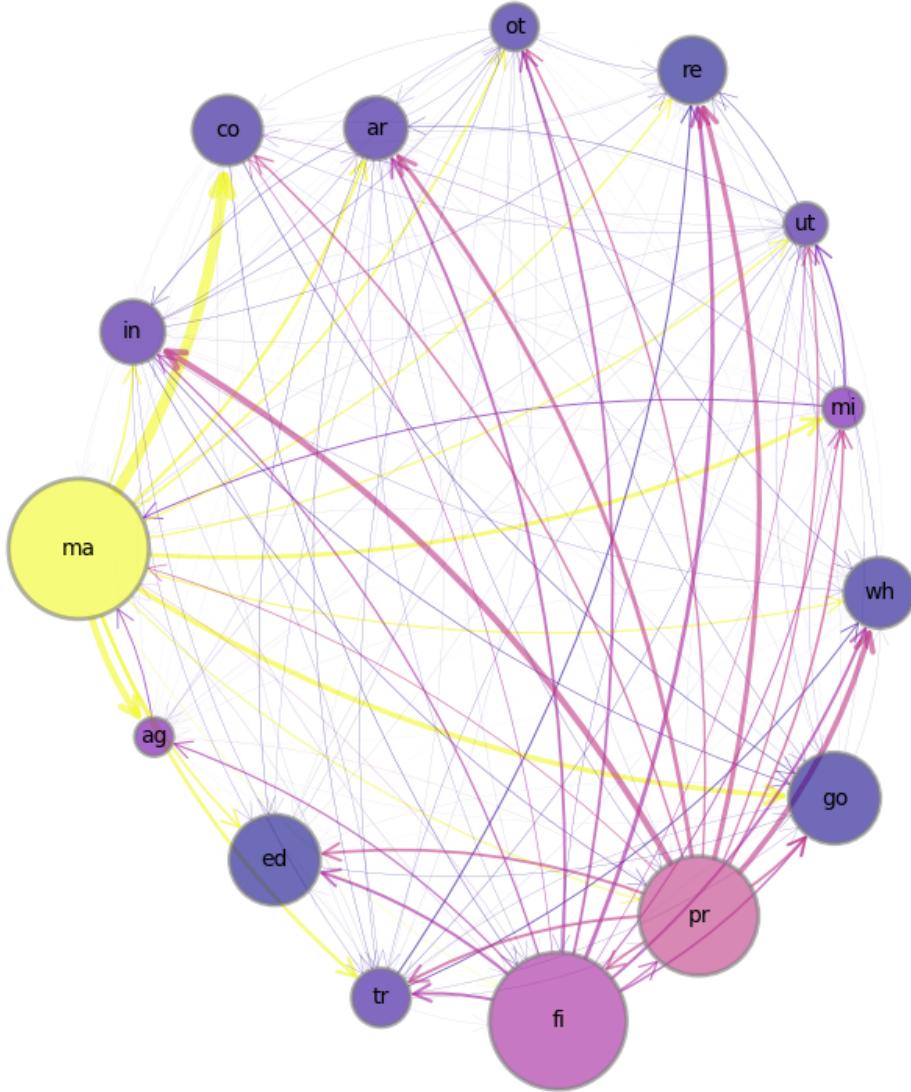


Fig. 31.1: US 15 sector production network

## 31.2 Input output analysis

Let

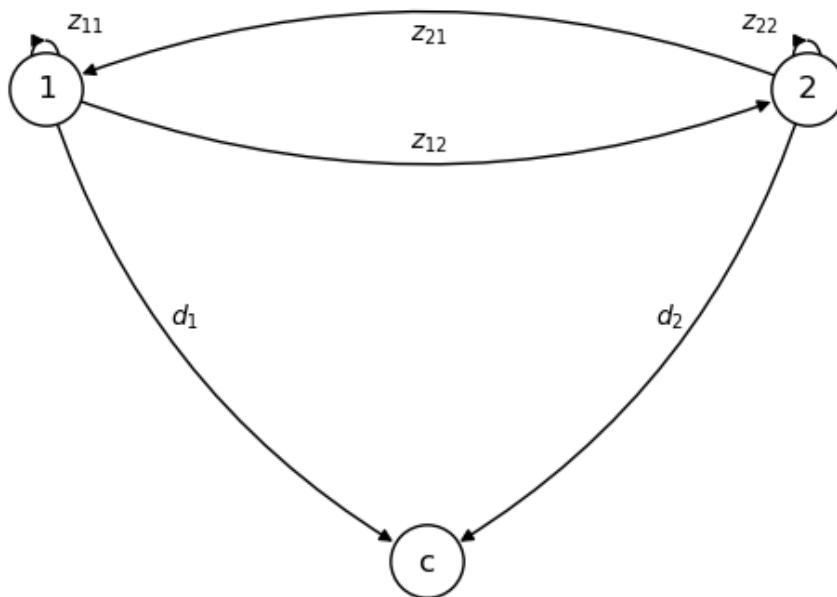
- $x_0$  be the amount of a single exogenous input to production, say labor
- $x_j, j = 1, \dots, n$  be the gross output of final good  $j$
- $d_j, j = 1, \dots, n$  be the net output of final good  $j$  that is available for final consumption
- $z_{ij}$  be the quantity of good  $i$  allocated to be an input to producing good  $j$  for  $i = 1, \dots, n, j = 1, \dots, n$
- $z_{0j}$  be the quantity of labor allocated to producing good  $j$ .
- $a_{ij}$  be the number of units of good  $i$  required to produce one unit of good  $j$ ,  $i = 0, \dots, n, j = 1, \dots, n$ .
- $w > 0$  be an exogenous wage of labor, denominated in dollars per unit of labor
- $p$  be an  $n \times 1$  vector of prices of produced goods  $i = 1, \dots, n$ .

The technology for producing good  $j \in \{1, \dots, n\}$  is described by the **Leontief** function

$$x_j = \min_{i \in \{0, \dots, n\}} \left( \frac{z_{ij}}{a_{ij}} \right)$$

### 31.2.1 Two goods

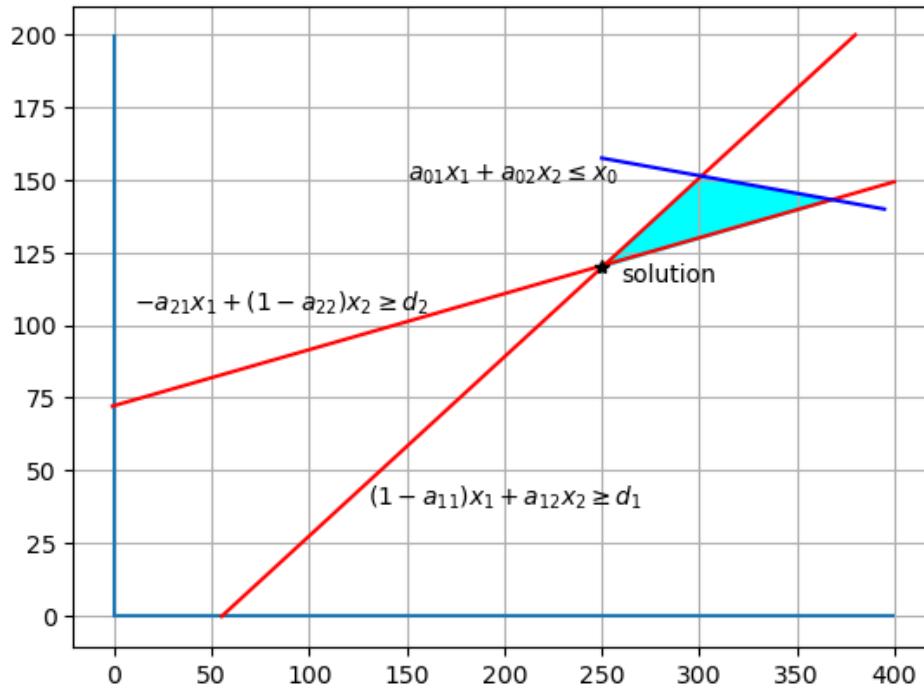
To illustrate, we begin by setting  $n = 2$  and formulating the following network.



**Feasible allocations must satisfy**

$$\begin{aligned} (1 - a_{11})x_1 - a_{12}x_2 &\geq d_1 \\ -a_{21}x_1 + (1 - a_{22})x_2 &\geq d_2 \\ a_{01}x_1 + a_{02}x_2 &\leq x_0 \end{aligned}$$

This can be graphically represented as follows.



More generally, constraints on production are

$$\begin{aligned} (I - A)x &\geq d \\ a_0^\top x &\leq x_0 \end{aligned} \tag{31.1}$$

where  $A$  is the  $n \times n$  matrix with typical element  $a_{ij}$  and  $a_0^\top = [a_{01} \quad \dots \quad a_{0n}]$ .

If we solve the first block of equations of (31.1) for gross output  $x$  we get

$$x = (I - A)^{-1}d \equiv Ld \tag{31.2}$$

where the matrix  $L = (I - A)^{-1}$  is sometimes called a **Leontief Inverse**.

To assure that the solution  $X$  of (31.2) is a positive vector, the following **Hawkins-Simon conditions** suffice:

$$\begin{aligned} \det(I - A) &> 0 \text{ and} \\ (I - A)_{ij} &> 0 \text{ for all } i = j \end{aligned}$$

For example a two good economy described by

$$A = \begin{bmatrix} 0.1 & 40 \\ 0.01 & 0 \end{bmatrix} \text{ and } d = \begin{bmatrix} 50 \\ 2 \end{bmatrix} \tag{31.3}$$

```
A = np.array([[0.1, 40],
             [0.01, 0]])
d = np.array([50, 2]).reshape((2, 1))
```

```
I = np.identity(2)
B = I - A
B
```

```
array([[ 9.e-01, -4.e+01],
       [-1.e-02,  1.e+00]])
```

Let's check the **Hawkins-Simon conditions**

```
np.linalg.det(B) > 0 # checking Hawkins-Simon conditions
```

```
True
```

Now, let's compute the **Leontief inverse** matrix

```
L = np.linalg.inv(B) # obtaining Leontief inverse matrix
L
```

```
array([[2.0e+00, 8.0e+01],
       [2.0e-02, 1.8e+00]])
```

```
x = L @ d # solving for gross output
x
```

```
array([[260. ],
       [ 4.6]])
```

### 31.3 Production possibility frontier

The second equation of (31.1) can be written

$$a_0^\top x = x_0$$

or

$$A_0^\top d = x_0 \tag{31.4}$$

where

$$A_0^\top = a_0^\top (I - A)^{-1}$$

For  $i \in \{1, \dots, n\}$ , the  $i$ th component of  $A_0$  is the amount of labor that is required to produce one unit of final output of good  $i$ .

Equation (31.4) sweeps out a **production possibility frontier** of final consumption bundles  $d$  that can be produced with exogenous labor input  $x_0$ .

Consider the example in (31.3).

Suppose we are now given

$$a_0^\top = [4 \ 100]$$

Then we can find  $A_0^\top$  by

```
a0 = np.array([4, 100])
A0 = a0 @ L
A0
```

```
array([ 10., 500.])
```

Thus, the production possibility frontier for this economy is

$$10d_1 + 500d_2 = x_0$$

## 31.4 Prices

[DSS58] argue that relative prices of the  $n$  produced goods must satisfy

$$\begin{aligned} p_1 &= a_{11}p_1 + a_{21}p_2 + a_{01}w \\ p_2 &= a_{12}p_1 + a_{22}p_2 + a_{02}w \end{aligned}$$

More generally,

$$p = A^\top p + a_0 w$$

which states that the price of each final good equals the total cost of production, which consists of costs of intermediate inputs  $A^\top p$  plus costs of labor  $a_0 w$ .

This equation can be written as

$$(I - A^\top)p = a_0 w \tag{31.5}$$

which implies

$$p = (I - A^\top)^{-1}a_0 w$$

Notice how (31.5) with (31.1) forms a **conjugate pair** through the appearance of operators that are transposes of one another.

This connection surfaces again in a classic linear program and its dual.

## 31.5 Linear programs

A **primal** problem is

$$\min_x w a_0^\top x$$

subject to

$$(I - A)x \geq d$$

The associated **dual** problem is

$$\max_p p^\top d$$

subject to

$$(I - A)^\top p \leq a_0 w$$

The primal problem chooses a feasible production plan to minimize costs for delivering a pre-assigned vector of final goods consumption  $d$ .

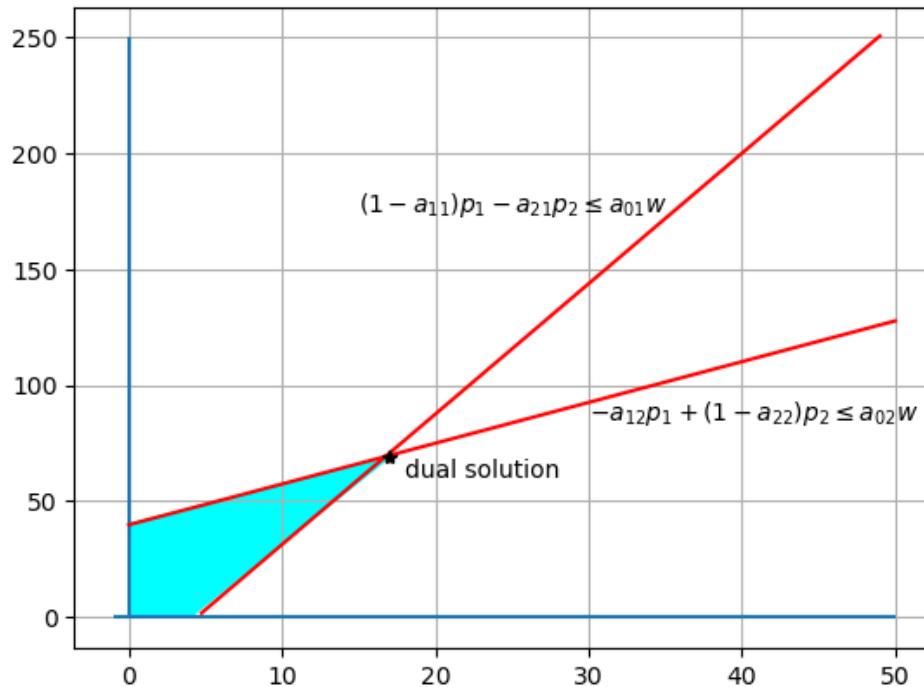
The dual problem chooses prices to maximize the value of a pre-assigned vector of final goods  $d$  subject to prices covering costs of production.

By the [strong duality theorem](#), optimal value of the primal and dual problems coincide:

$$w a_0^\top x^* = p^* d$$

where  $^*$ 's denote optimal choices for the primal and dual problems.

The dual problem can be graphically represented as follows.



## 31.6 Leontief inverse

We have discussed that gross output  $x$  is given by (31.2), where  $L$  is called the Leontief Inverse.

Recall the [Neumann Series Lemma](#) which states that  $L$  exists if the spectral radius  $r(A) < 1$ .

In fact

$$L = \sum_{i=0}^{\infty} A^i$$

### 31.6.1 Demand shocks

Consider the impact of a demand shock  $\Delta d$  which shifts demand from  $d_0$  to  $d_1 = d_0 + \Delta d$ .

Gross output shifts from  $x_0 = Ld_0$  to  $x_1 = Ld_1$ .

If  $r(A) < 1$  then a solution exists and

$$\Delta x = L\Delta d = \Delta d + A(\Delta d) + A^2(\Delta d) + \dots$$

This illustrates that an element  $l_{ij}$  of  $L$  shows the total impact on sector  $i$  of a unit change in demand of good  $j$ .

## 31.7 Applications of graph theory

We can further study input output networks through applications of *graph theory*.

An input output network can be represented by a weighted directed graph induced by the adjacency matrix  $A$ .

The set of nodes  $V = [n]$  is the list of sectors and the set of edges is given by

$$E = \{(i, j) \in V \times V : a_{ij} > 0\}$$

In Fig. 31.1 weights are indicated by the widths of the arrows, which are proportional to the corresponding input-output coefficients.

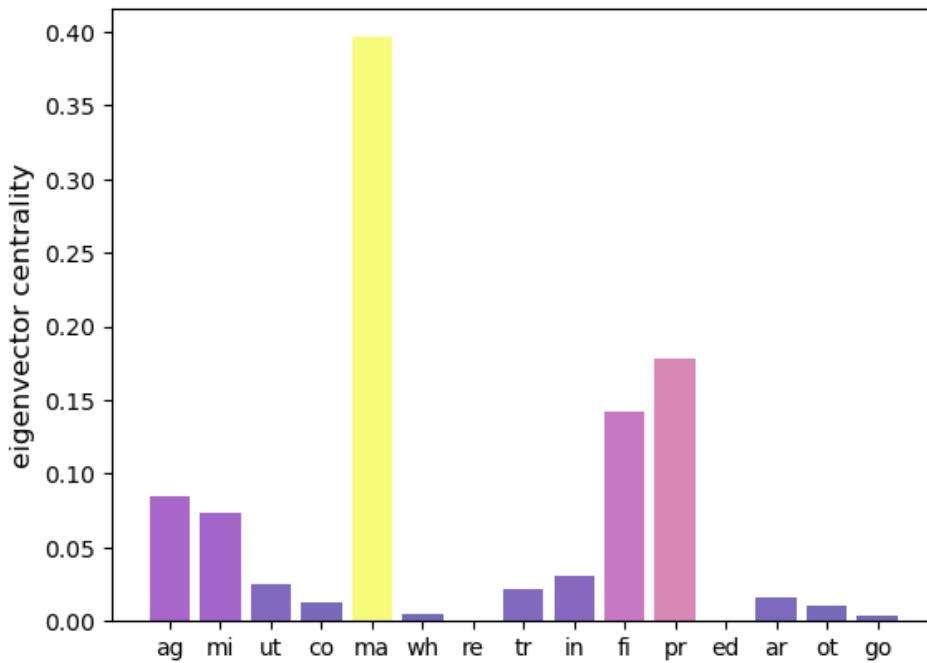
We can now use centrality measures to rank sectors and discuss their importance relative to the other sectors.

### 31.7.1 Eigenvector centrality

Eigenvector centrality of a node  $i$  is measured by

$$e_i = \frac{1}{r(A)} \sum_{1 \leq j \leq n} a_{ij} e_j$$

We plot a bar graph of hub-based eigenvector centrality for the sectors represented in Fig. 31.1.



A higher measure indicates higher importance as a supplier.

As a result demand shocks in most sectors will significantly impact activity in sectors with high eigenvector centrality.

The above figure indicates that manufacturing is the most dominant sector in the US economy.

### 31.7.2 Output multipliers

Another way to rank sectors in input output networks is via output multipliers.

The **output multiplier** of sector  $j$  denoted by  $\mu_j$  is usually defined as the total sector-wide impact of a unit change of demand in sector  $j$ .

Earlier when discussing demand shocks we concluded that for  $L = (l_{ij})$  the element  $l_{ij}$  represents the impact on sector  $i$  of a unit change in demand in sector  $j$ .

Thus,

$$\mu_j = \sum_{i=1}^n l_{ij}$$

This can be written as  $\mu^\top = \mathbb{1}^\top L$  or

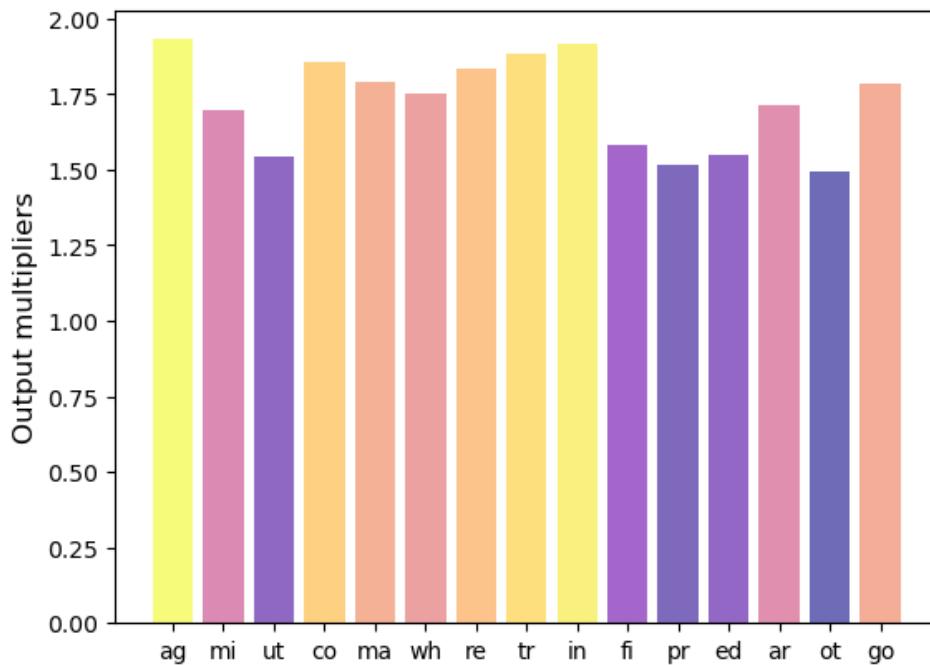
$$\mu^\top = \mathbb{1}^\top (I - A)^{-1}$$

Please note that here we use  $\mathbb{1}$  to represent a vector of ones.

High ranking sectors within this measure are important buyers of intermediate goods.

A demand shock in such sectors will cause a large impact on the whole production network.

The following figure displays the output multipliers for the sectors represented in Fig. 31.1.



We observe that manufacturing and agriculture are highest ranking sectors.

## 31.8 Exercises

---

### Exercise 31.8.1

[DSS58] Chapter 9 discusses an example with the following parameter settings:

$$A = \begin{bmatrix} 0.1 & 1.46 \\ 0.16 & 0.17 \end{bmatrix} \text{ and } a_0 = [.04 \quad .33]$$

$$x = \begin{bmatrix} 250 \\ 120 \end{bmatrix} \text{ and } x_0 = 50$$

$$d = \begin{bmatrix} 50 \\ 60 \end{bmatrix}$$

Describe how they infer the input-output coefficients in  $A$  and  $a_0$  from the following hypothetical underlying “data” on agricultural and manufacturing industries:

$$z = \begin{bmatrix} 25 & 175 \\ 40 & 20 \end{bmatrix} \text{ and } z_0 = [10 \quad 40]$$

where  $z_0$  is a vector of labor services used in each industry.

---

### Solution to Exercise 31.8.1

For each  $i = 0, 1, 2$  and  $j = 1, 2$

$$a_{ij} = \frac{z_{ij}}{x_j}$$

---

### Exercise 31.8.2

Derive the production possibility frontier for the economy characterized in the previous exercise.

---

---

### Solution to Exercise 31.8.2

```
A = np.array([[0.1, 1.46],  
             [0.16, 0.17]])  
a_0 = np.array([0.04, 0.33])
```

```
I = np.identity(2)  
B = I - A  
L = np.linalg.inv(B)
```

```
A_0 = a_0 @ L  
A_0
```

```
array([0.16751071, 0.69224776])
```

Thus the production possibility frontier is given by

$$0.17d_1 + 0.69d_2 = 50$$

---



## A LAKE MODEL OF EMPLOYMENT

### 32.1 Outline

In addition to what's in Anaconda, this lecture will need the following libraries:

```
import numpy as np
import matplotlib.pyplot as plt
```

### 32.2 The Lake model

This model is sometimes called the **lake model** because there are two pools of workers:

1. those who are currently employed.
2. those who are currently unemployed but are seeking employment.

The “flows” between the two lakes are as follows:

1. workers exit the labor market at rate  $d$ .
2. new workers enter the labor market at rate  $b$ .
3. employed workers separate from their jobs at rate  $\alpha$ .
4. unemployed workers find jobs at rate  $\lambda$ .

The below graph illustrates the lake model.

### 32.3 Dynamics

Let  $e_t$  and  $u_t$  be the number of employed and unemployed workers at time  $t$  respectively.

The total population of workers is  $n_t = e_t + u_t$ .

The number of unemployed and employed workers thus evolves according to:

$$\begin{aligned} u_{t+1} &= (1-d)(1-\lambda)u_t + \alpha(1-d)e_t + bn_t \\ &= ((1-d)(1-\lambda) + b)u_t + (\alpha(1-d) + b)e_t \\ e_{t+1} &= (1-d)\lambda u_t + (1-\alpha)(1-d)e_t \end{aligned} \tag{32.1}$$

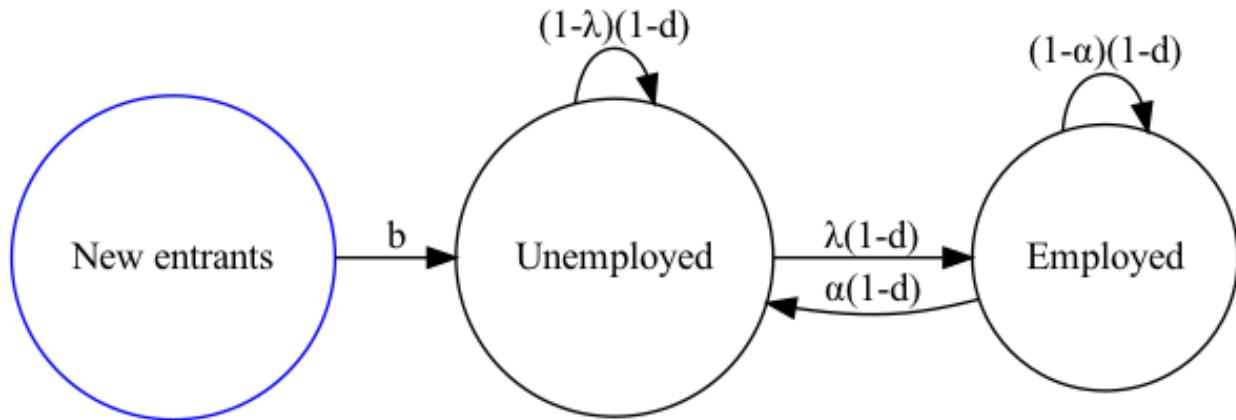


Fig. 32.1: An illustration of the lake model

We can arrange (32.1) as a linear system of equations in matrix form  $x_{t+1} = Ax_t$  where

$$x_{t+1} = \begin{bmatrix} u_{t+1} \\ e_{t+1} \end{bmatrix} \quad A = \begin{bmatrix} (1-d)(1-\lambda) + b & \alpha(1-d) + b \\ (1-d)\lambda & (1-\alpha)(1-d) \end{bmatrix} \quad \text{and} \quad x_t = \begin{bmatrix} u_t \\ e_t \end{bmatrix}.$$

Suppose at  $t = 0$  we have  $x_0 = [u_0 \ e_0]^\top$ .

Then,  $x_1 = Ax_0$ ,  $x_2 = Ax_1 = A^2x_0$  and thus  $x_t = A^t x_0$ .

Thus the long-run outcomes of this system may depend on the initial condition  $x_0$  and the matrix  $A$ .

We are interested in how  $u_t$  and  $e_t$  evolve over time.

What long-run unemployment rate and employment rate should we expect?

Do long-run outcomes depend on the initial values  $(u_0, e_0)$ ?

### 32.3.1 Visualising the long-run outcomes

Let us first plot the time series of unemployment  $u_t$ , employment  $e_t$ , and labor force  $n_t$ .

```

class LakeModel:
    """
    Solves the lake model and computes dynamics of the unemployment stocks and
    rates.

    Parameters:
    -----
    lambda : scalar
        The job finding rate for currently unemployed workers
    alpha : scalar
        The dismissal rate for currently employed workers
    b : scalar
        Entry rate into the labor force
    d : scalar
        Exit rate from the labor force
    """

    def __init__(self, lambda=0.1, alpha=0.013, b=0.0124, d=0.00822):
        """
        Initialize the LakeModel with parameters: lambda, alpha, b, d.
        """
        self.lambda = lambda
        self.alpha = alpha
        self.b = b
        self.d = d
  
```

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```

self.λ, self.α, self.β, self.δ = λ, α, β, δ

λ, α, β, δ = self.λ, self.α, self.β, self.δ
self.g = β - δ
g = self.g

self.A = np.array([[ (1-d)*(1-λ) + β,      α*(1-d) + β],
                  [          (1-d)*λ,      (1-α)*(1-d) ]])

self.ū = (1 + g - (1 - d) * (1 - α)) / (1 + g - (1 - d) * (1 - α) + (1 - d) * λ)
self.ē = 1 - self.ū

def simulate_path(self, x0, T=1000):
    """
    Simulates the sequence of employment and unemployment

    Parameters
    -----
    x0 : array
        Contains initial values (u0, e0)
    T : int
        Number of periods to simulate

    Returns
    -----
    x : iterator
        Contains sequence of employment and unemployment rates
    """
    x0 = np.atleast_1d(x0) # Recast as array just in case
    x_ts = np.zeros((2, T))
    x_ts[:, 0] = x0
    for t in range(1, T):
        x_ts[:, t] = self.A @ x_ts[:, t-1]
    return x_ts

```

```

lm = LakeModel()
e_0 = 0.92          # Initial employment
u_0 = 1 - e_0       # Initial unemployment, given initial n_0 = 1

lm = LakeModel()
T = 100             # Simulation length

x_0 = (u_0, e_0)
x_path = lm.simulate_path(x_0, T)

fig, axes = plt.subplots(3, 1, figsize=(10, 8))

axes[0].plot(x_path[0, :], lw=2)
axes[0].set_title('Unemployment')

```

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```

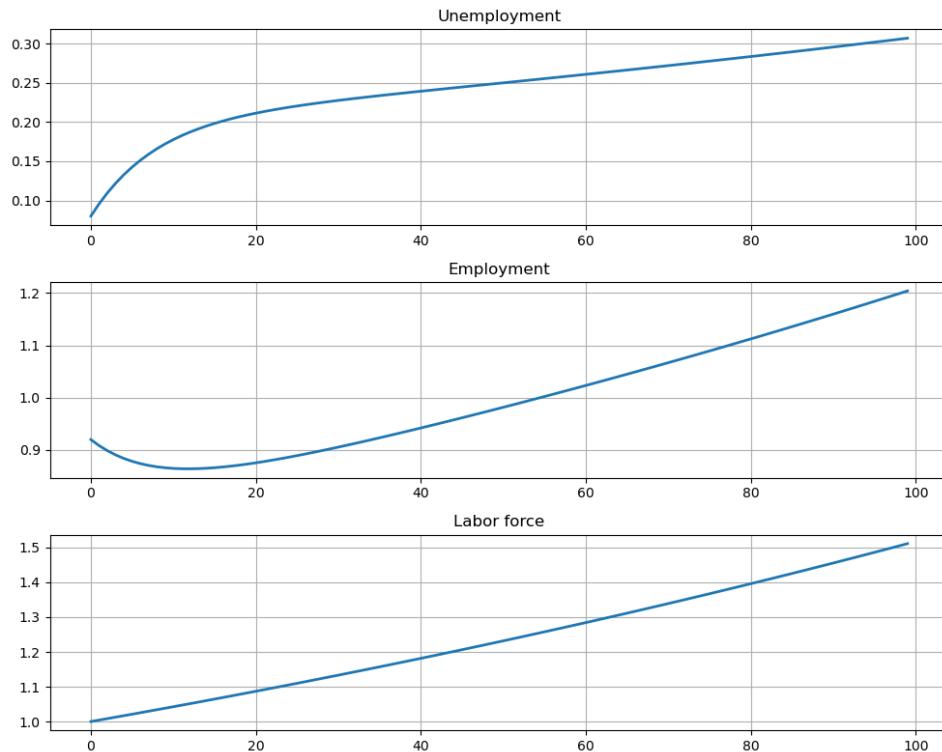
axes[1].plot(x_path[1, :], lw=2)
axes[1].set_title('Employment')

axes[2].plot(x_path.sum(0), lw=2)
axes[2].set_title('Labor force')

for ax in axes:
    ax.grid()

plt.tight_layout()
plt.show()

```



Not surprisingly, we observe that labor force  $n_t$  increases at a constant rate.

This coincides with the fact there is only one inflow source (new entrants pool) to unemployment and employment pools.

The inflow and outflow of labor market system is determined by constant exit rate and entry rate of labor market in the long run.

In detail, let  $\mathbb{1} = [1, 1]^\top$  be a vector of ones.

Observe that

$$\begin{aligned}
n_{t+1} &= u_{t+1} + e_{t+1} \\
&= \mathbb{1}^\top x_{t+1} \\
&= \mathbb{1}^\top A x_t \\
&= (1 + b - d)(u_t + e_t) \\
&= (1 + b - d)n_t.
\end{aligned}$$

Hence, the growth rate of  $n_t$  is fixed at  $1 + b - d$ .

Moreover, the times series of unemployment and employment seems to grow at some stable rates in the long run.

### 32.3.2 The application of Perron-Frobenius theorem

Since by intuition if we consider unemployment pool and employment pool as a closed system, the growth should be similar to the labor force.

We next ask whether the long run growth rates of  $e_t$  and  $u_t$  also dominated by  $1 + b - d$  as labor force.

The answer will be clearer if we appeal to *Perron-Frobenius theorem*.

The importance of the Perron-Frobenius theorem stems from the fact that firstly in the real world most matrices we encounter are nonnegative matrices.

Secondly, many important models are simply linear iterative models that begin with an initial condition  $x_0$  and then evolve recursively by the rule  $x_{t+1} = Ax_t$  or in short  $x_t = A^t x_0$ .

This theorem helps characterise the dominant eigenvalue  $r(A)$  which determines the behavior of this iterative process.

#### Dominant eigenvector

We now illustrate the power of the Perron-Frobenius theorem by showing how it helps us to analyze the lake model.

Since  $A$  is a nonnegative and irreducible matrix, the Perron-Frobenius theorem implies that:

- the spectral radius  $r(A)$  is an eigenvalue of  $A$ , where

$$r(A) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$$

- any other eigenvalue  $\lambda$  in absolute value is strictly smaller than  $r(A)$ :  $|\lambda| < r(A)$ ,
- there exist unique and everywhere positive right eigenvector  $\phi$  (column vector) and left eigenvector  $\psi$  (row vector):

$$A\phi = r(A)\phi, \quad \psi A = r(A)\psi$$

- if further  $A$  is positive, then with  $\langle \psi, \phi \rangle = \psi\phi = 1$  we have

$$r(A)^{-t} A^t \rightarrow \phi\psi$$

The last statement implies that the magnitude of  $A^t$  is identical to the magnitude of  $r(A)^t$  in the long run, where  $r(A)$  can be considered as the dominant eigenvalue in this lecture.

Therefore, the magnitude  $x_t = A^t x_0$  is also dominated by  $r(A)^t$  in the long run.

Recall that the spectral radius is bounded by column sums: for  $A \geq 0$ , we have

$$\min_j \text{colsum}_j(A) \leq r(A) \leq \max_j \text{colsum}_j(A) \quad (32.2)$$

Note that  $\text{colsum}_j(A) = 1 + b - d$  for  $j = 1, 2$  and by (32.2) we can thus conclude that the dominant eigenvalue is  $r(A) = 1 + b - d$ .

Denote  $g = b - d$  as the overall growth rate of the total labor force, so that  $r(A) = 1 + g$ .

The Perron-Frobenius implies that there is a unique positive eigenvector  $\bar{x} = \begin{bmatrix} \bar{u} \\ \bar{e} \end{bmatrix}$  such that  $A\bar{x} = r(A)\bar{x}$  and  $[1 \ 1] \bar{x} = 1$ :

$$\begin{aligned} \bar{u} &= \frac{b + \alpha(1 - d)}{b + (\alpha + \lambda)(1 - d)} \\ \bar{e} &= \frac{\lambda(1 - d)}{b + (\alpha + \lambda)(1 - d)} \end{aligned} \quad (32.3)$$

Since  $\bar{x}$  is the eigenvector corresponding to the dominant eigenvalue  $r(A)$ , we call  $\bar{x}$  the dominant eigenvector.

This dominant eigenvector plays an important role in determining long-run outcomes as illustrated below.

```

def plot_time_paths(lm, x0=None, T=1000, ax=None):
    """
    Plots the simulated time series.

    Parameters
    -----
    lm : class
        Lake Model
    x0 : array
        Contains some different initial values.
    T : int
        Number of periods to simulate
    """

    if x0 is None:
        x0 = np.array([[5.0, 0.1]])

    u, e = lm.u, lm.e

    x0 = np.atleast_2d(x0)

    if ax is None:
        fig, ax = plt.subplots(figsize=(10, 8))
        # Plot line D
        s = 10
        ax.plot([0, s * u], [0, s * e], "k--", lw=1, label='set $D$')

        # Set the axes through the origin
        for spine in ["left", "bottom"]:
            ax.spines[spine].set_position("zero")
        for spine in ["right", "top"]:
            ax.spines[spine].set_color("none")

        ax.set_xlim(-2, 6)
        ax.set_ylim(-2, 6)
        ax.set_xlabel("unemployed workforce")
        ax.set_ylabel("employed workforce")
        ax.set_xticks((0, 6))
        ax.set_yticks((0, 6))

    # Plot time series
    for x in x0:
        x_ts = lm.simulate_path(x0=x)

        ax.scatter(x_ts[0, :], x_ts[1, :], s=4,)

        u0, e0 = x
        ax.plot([u0], [e0], "ko", ms=2, alpha=0.6)
        ax.annotate(f'$x_0 = ({u0}, {e0})$', xy=(u0, e0),
                    xycoords="data",
                    xytext=(0, 20),

```

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```

textcoords="offset points",
arrowprops=dict(arrowstyle = "->"))

ax.plot([ū], [ē], "ko", ms=4, alpha=0.6)
ax.annotate(r'$\bar{x}$',
            xy=(ū, ē),
            xycoords="data",
            xytext=(20, -20),
            textcoords="offset points",
            arrowprops=dict(arrowstyle = "->"))

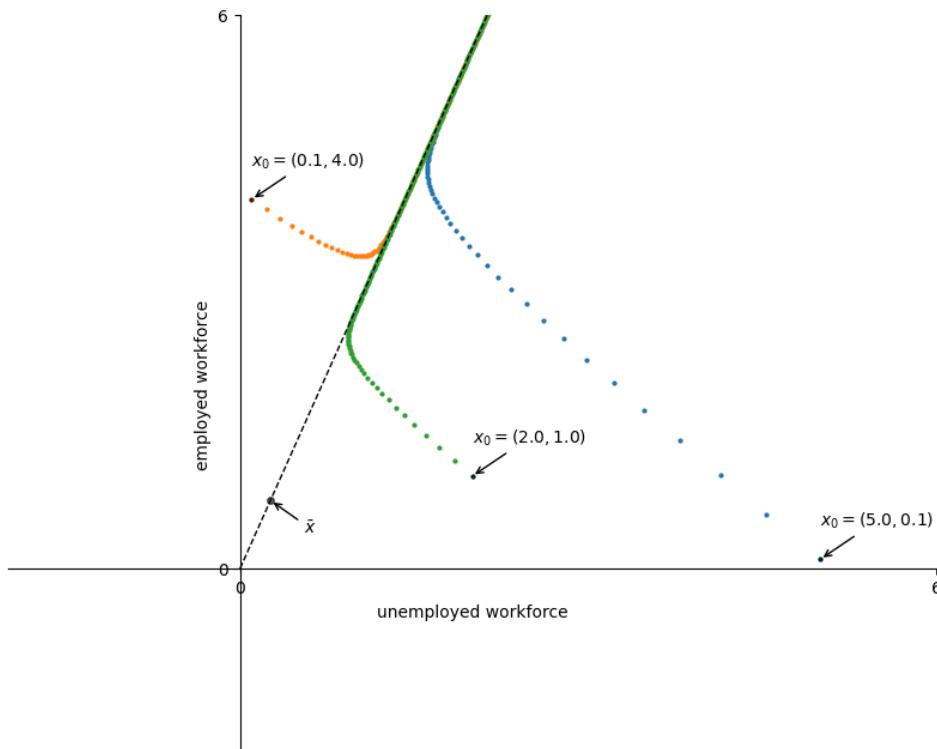
if ax is None:
    plt.show()

```

```

lm = LakeModel(a=0.01, λ=0.1, d=0.02, b=0.025)
x0 = ((5.0, 0.1), (0.1, 4.0), (2.0, 1.0))
plot_time_paths(lm, x0=x0)

```



Since  $\bar{x}$  is an eigenvector corresponding to the eigenvalue  $r(A)$ , all the vectors in the set  $D := \{x \in \mathbb{R}^2 : x = \alpha \bar{x} \text{ for some } \alpha > 0\}$  are also eigenvectors corresponding to  $r(A)$ .

This set  $D$  is represented by a dashed line in the above figure.

The graph illustrates that for two distinct initial conditions  $x_0$  the sequences of iterates  $(A^t x_0)_{t \geq 0}$  move towards  $D$  over time.

This suggests that all such sequences share strong similarities in the long run, determined by the dominant eigenvector  $\bar{x}$ .

### Negative growth rate

In the example illustrated above we considered parameters such that overall growth rate of the labor force  $g > 0$ .

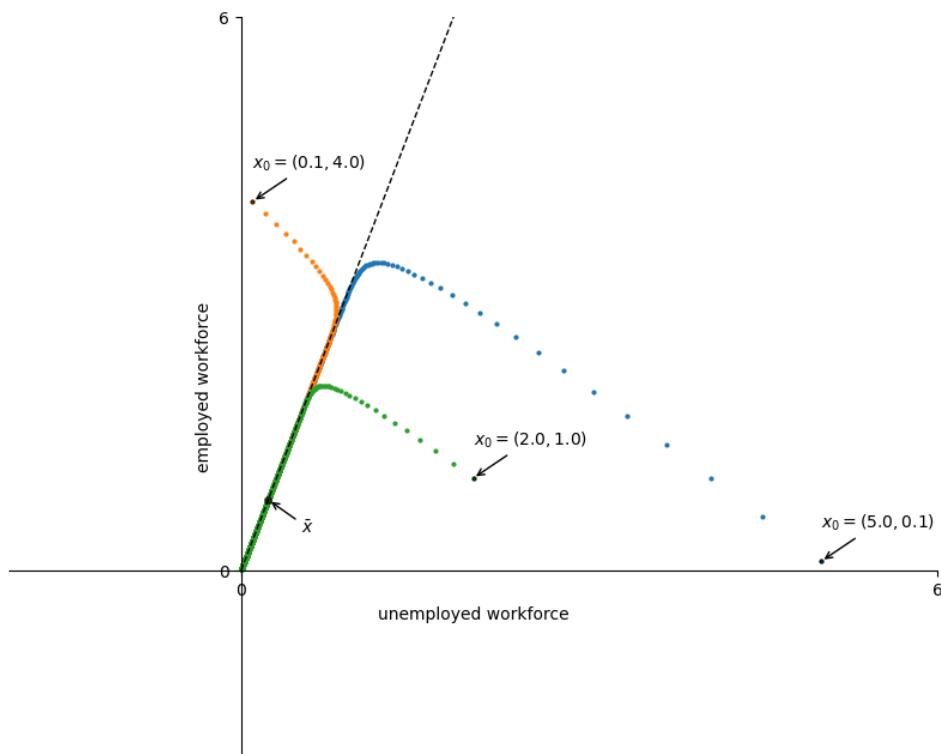
Suppose now we are faced with a situation where the  $g < 0$ , i.e., negative growth in the labor force.

This means that  $b - d < 0$ , i.e., workers exit the market faster than they enter.

What would the behavior of the iterative sequence  $x_{t+1} = Ax_t$  be now?

This is visualised below.

```
lm = LakeModel(a=0.01, λ=0.1, d=0.025, b=0.02)
plot_time_paths(lm, x0=x0)
```



Thus, while the sequence of iterates still moves towards the dominant eigenvector  $\bar{x}$ , in this case they converge to the origin.

This is a result of the fact that  $r(A) < 1$ , which ensures that the iterative sequence  $(A^t x_0)_{t \geq 0}$  will converge to some point, in this case to  $(0, 0)$ .

This leads us to the next result.

### 32.3.3 Properties

Since the column sums of  $A$  are  $r(A) = 1$ , the left eigenvector is  $\mathbb{1}^\top = [1, 1]$ .

Perron-Frobenius theory implies that

$$r(A)^{-t} A^t \approx \bar{x} \mathbb{1}^\top = \begin{bmatrix} \bar{u} & \bar{u} \\ \bar{e} & \bar{e} \end{bmatrix}.$$

As a result, for any  $x_0 = (u_0, e_0)^\top$ , we have

$$\begin{aligned} x_t &= A^t x_0 \approx r(A)^t \begin{bmatrix} \bar{u} & \bar{u} \\ \bar{e} & \bar{e} \end{bmatrix} \begin{bmatrix} u_0 \\ e_0 \end{bmatrix} \\ &= (1+g)^t (u_0 + e_0) \begin{bmatrix} \bar{u} \\ \bar{e} \end{bmatrix} \\ &= (1+g)^t n_0 \bar{x} \\ &= n_t \bar{x}. \end{aligned}$$

as  $t$  is large enough.

We see that the growth of  $u_t$  and  $e_t$  also dominated by  $r(A) = 1+g$  in the long run:  $x_t$  grows along  $D$  as  $r(A) > 1$  and converges to  $(0, 0)$  as  $r(A) < 1$ .

Moreover, the long-run unemployment and employment are steady fractions of  $n_t$ .

The latter implies that  $\bar{u}$  and  $\bar{e}$  are long-run unemployment rate and employment rate, respectively.

In detail, we have the unemployment rates and employment rates:  $x_t/n_t = A^t n_0/n_t \rightarrow \bar{x}$  as  $t \rightarrow \infty$ .

To illustrate the dynamics of the rates, let  $\hat{A} := A/(1+g)$  be the transition matrix of  $r_t := x_t/n_t$ .

The dynamics of the rates follow

$$r_{t+1} = \frac{x_{t+1}}{n_{t+1}} = \frac{x_{t+1}}{(1+g)n_t} = \frac{Ax_t}{(1+g)n_t} = \hat{A} \frac{x_t}{n_t} = \hat{A} r_t.$$

Observe that the column sums of  $\hat{A}$  are all one so that  $r(\hat{A}) = 1$ .

One can check that  $\bar{x}$  is also the right eigenvector of  $\hat{A}$  corresponding to  $r(\hat{A})$  that  $\bar{x} = \hat{A} \bar{x}$ .

Moreover,  $\hat{A}^t r_0 \rightarrow \bar{x}$  as  $t \rightarrow \infty$  for any  $r_0 = x_0/n_0$ , since the above discussion implies

$$r_t = \hat{A}^t r_0 = (1+g)^{-t} A^t r_0 = r(A)^{-t} A^t r_0 \rightarrow \begin{bmatrix} \bar{u} & \bar{u} \\ \bar{e} & \bar{e} \end{bmatrix} r_0 = \begin{bmatrix} \bar{u} \\ \bar{e} \end{bmatrix}.$$

This is illustrated below.

```
lm = LakeModel()
e_0 = 0.92          # Initial employment
u_0 = 1 - e_0      # Initial unemployment, given initial n_0 = 1

lm = LakeModel()
T = 100             # Simulation length

x_0 = (u_0, e_0)

x_path = lm.simulate_path(x_0, T)
rate_path = x_path / x_path.sum(0)
```

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```

fig, axes = plt.subplots(2, 1, figsize=(10, 8))

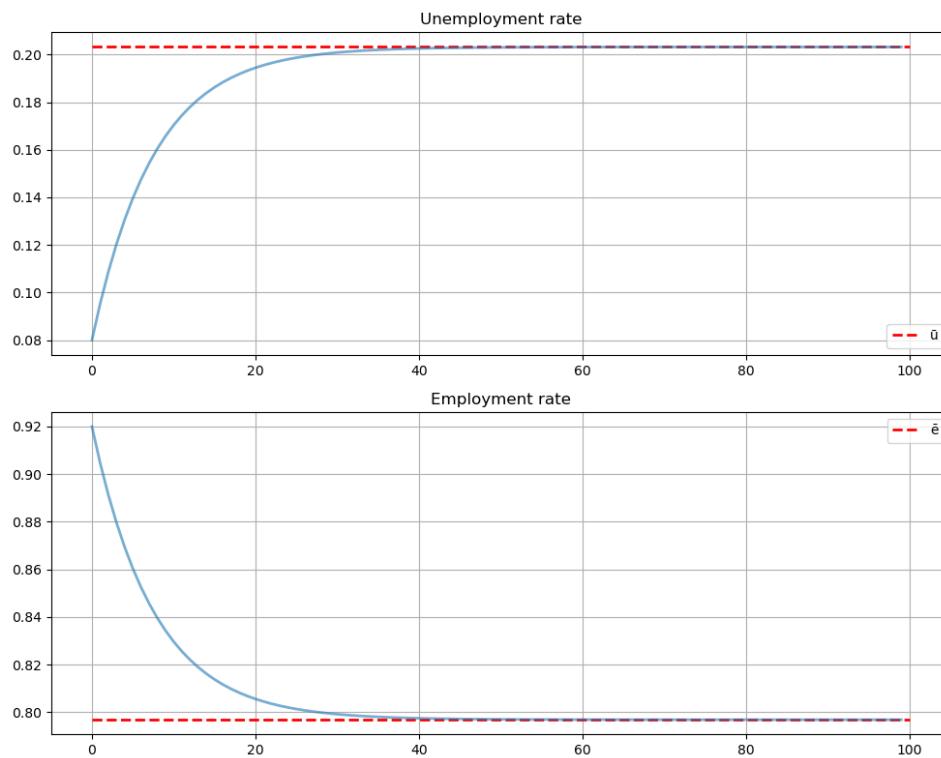
# Plot steady ū and ē
axes[0].hlines(lm.ū, 0, T, 'r', '--', lw=2, label='ū')
axes[1].hlines(lm.ē, 0, T, 'r', '--', lw=2, label='ē')

titles = ['Unemployment rate', 'Employment rate']
locations = ['lower right', 'upper right']

# Plot unemployment rate and employment rate
for i, ax in enumerate(axes):
    ax.plot(rate_path[i, :], lw=2, alpha=0.6)
    ax.set_title(titles[i])
    ax.grid()
    ax.legend(loc=locations[i])

plt.tight_layout()
plt.show()

```



To provide more intuition for convergence, we further explain the convergence below without the Perron-Frobenius theorem.

Suppose that  $\hat{A} = PDP^{-1}$  is diagonalizable, where  $P = [v_1, v_2]$  consists of eigenvectors  $v_1$  and  $v_2$  of  $\hat{A}$  corresponding to eigenvalues  $\gamma_1$  and  $\gamma_2$  respectively, and  $D = \text{diag}(\gamma_1, \gamma_2)$ .

Let  $\gamma_1 = r(\hat{A}) = 1$  and  $|\gamma_2| < \gamma_1$ , so that the spectral radius is a dominant eigenvalue.

The dynamics of the rates follow  $r_{t+1} = \hat{A}r_t$ , where  $r_0$  is a probability vector:  $\sum_j r_{0,j} = 1$ .

Consider  $z_t = P^{-1}r_t$ .

Then, we have  $z_{t+1} = P^{-1}r_{t+1} = P^{-1}\hat{A}r_t = P^{-1}\hat{A}Pz_t = Dz_t$ .

Hence, we obtain  $z_t = D^t z_0$ , and for some  $z_0 = (c_1, c_2)^\top$  we have

$$r_t = Pz_t = [v_1 \ v_2] \begin{bmatrix} \gamma_1^t & 0 \\ 0 & \gamma_2^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 \gamma_1^t v_1 + c_2 \gamma_2^t v_2.$$

Since  $|\gamma_2| < |\gamma_1| = 1$ , the second term in the right hand side converges to zero.

Therefore, the convergence follows  $r_t \rightarrow c_1 v_1$ .

Since the column sums of  $\hat{A}$  are one and  $r_0$  is a probability vector,  $r_t$  must be a probability vector.

In this case,  $c_1 v_1$  must be a normalized eigenvector, so  $c_1 v_1 = \bar{x}$  and then  $r_t \rightarrow \bar{x}$ .

## 32.4 Exercise

---

### Exercise 32.4.1 (Evolution of unemployment and employment rate)

How do the long-run unemployment rate and employment rate evolve if there is an increase in the separation rate  $\alpha$  or a decrease in job finding rate  $\lambda$ ?

Is the result compatible with your intuition?

Plot the graph to illustrate how the line  $D := \{x \in \mathbb{R}^2 : x = \alpha \bar{x} \text{ for some } \alpha > 0\}$  shifts in the unemployment-employment space.

---



---

### Solution to Exercise 32.4.1 (Evolution of unemployment and employment rate)

Eq. (32.3) implies that the long-run unemployment rate will increase, and the employment rate will decrease if  $\alpha$  increases or  $\lambda$  decreases.

Suppose first that  $\alpha = 0.01$ ,  $\lambda = 0.1$ ,  $d = 0.02$ ,  $b = 0.025$ . Assume that  $\alpha$  increases to 0.04.

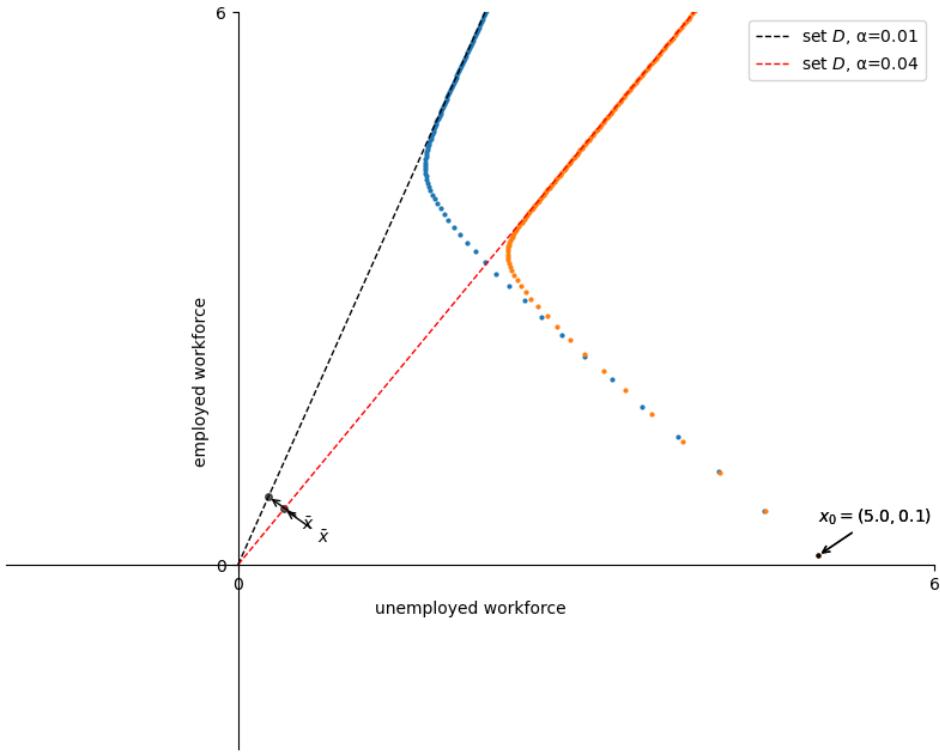
The below graph illustrates that the line  $D$  shifts clockwise downward, which indicates that the fraction of unemployment rises as the separation rate increases.

```
fig, ax = plt.subplots(figsize=(10, 8))

lm = LakeModel(a=0.01, λ=0.1, d=0.02, b=0.025)
plot_time_paths(lm, ax=ax)
s=10
ax.plot([0, s * lm.ū], [0, s * lm.ē], "k--", lw=1, label='set $D$, a=0.01')

lm = LakeModel(a=0.04, λ=0.1, d=0.02, b=0.025)
plot_time_paths(lm, ax=ax)
ax.plot([0, s * lm.ū], [0, s * lm.ē], "r--", lw=1, label='set $D$, a=0.04')

ax.legend(loc='best')
plt.show()
```



---

CHAPTER  
**THIRTYTHREE**

---

## NETWORKS

```
!pip install quantecon-book-networks pandas-datareader
```

### 33.1 Outline

In recent years there has been rapid growth in a field called [network science](#).

Network science studies relationships between groups of objects.

One important example is the [world wide web](#) , where web pages are connected by hyperlinks.

Another is the [human brain](#): studies of brain function emphasize the network of connections between nerve cells (neurons).

[Artificial neural networks](#) are based on this idea, using data to build intricate connections between simple processing units.

Epidemiologists studying [transmission of diseases](#) like COVID-19 analyze interactions between groups of human hosts.

In operations research, network analysis is used to study fundamental problems as on minimum cost flow, the traveling salesman, [shortest paths](#), and assignment.

This lecture gives an introduction to economic and financial networks.

Some parts of this lecture are drawn from the text <https://networks.quantecon.org/> but the level of this lecture is more introductory.

We will need the following imports.

```
import numpy as np
import networkx as nx
import matplotlib.pyplot as plt
import pandas as pd
import quantecon as qe

import matplotlib.cm as cm
import quantecon_book_networks
import quantecon_book_networks.input_output as qbn_io
import quantecon_book_networks.data as qbn_data

import matplotlib.patches as mpatches
```

## 33.2 Economic and financial networks

Within economics, important examples of networks include

- financial networks
- production networks
- trade networks
- transport networks and
- social networks

Social networks affect trends in market sentiment and consumer decisions.

The structure of financial networks helps to determine relative fragility of the financial system.

The structure of production networks affects trade, innovation and the propagation of local shocks.

To better understand such networks, let's look at some examples in more depth.

### 33.2.1 Example: Aircraft Exports

The following figure shows international trade in large commercial aircraft in 2019 based on International Trade Data SITC Revision 2.

The circles in the figure are called **nodes** or **vertices** – in this case they represent countries.

The arrows in the figure are called **edges** or **links**.

Node size is proportional to total exports and edge width is proportional to exports to the target country.

(The data is for trade in commercial aircraft weighing at least 15,000kg and was sourced from CID Dataverse.)

The figure shows that the US, France and Germany are major export hubs.

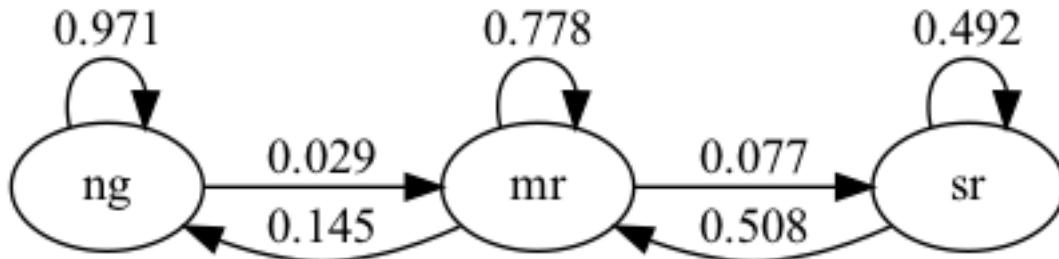
In the discussion below, we learn to quantify such ideas.

### 33.2.2 Example: A Markov Chain

Recall that, in our lecture on *Markov chains* we studied a dynamic model of business cycles where the states are

- “ng” = “normal growth”
- “mr” = “mild recession”
- “sr” = “severe recession”

Let's examine the following figure



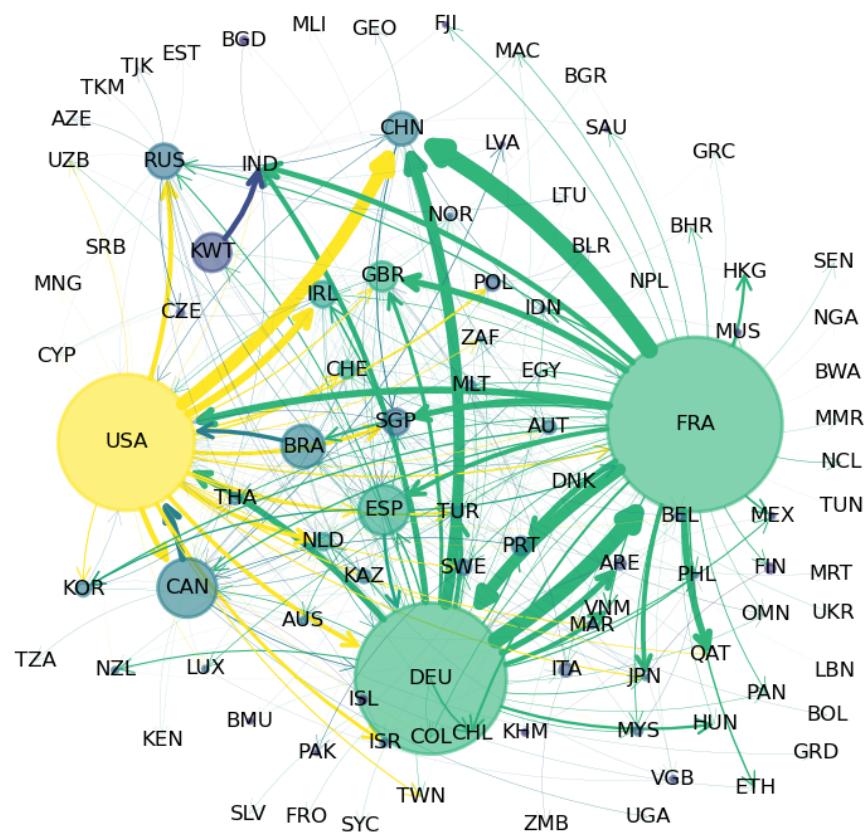


Fig. 33.1: Commercial Aircraft Network

This is an example of a network, where the set of nodes  $V$  equals the states:

$$V = \{"ng", "mr", "sr"\}$$

The edges between the nodes show the one month transition probabilities.

### 33.3 An introduction to graph theory

Now we've looked at some examples, let's move on to theory.

This theory will allow us to better organize our thoughts.

The theoretical part of network science is constructed using a major branch of mathematics called [graph theory](#).

Graph theory can be complicated and we will cover only the basics.

However, these concepts will already be enough for us to discuss interesting and important ideas on economic and financial networks.

We focus on “directed” graphs, where connections are, in general, asymmetric (arrows typically point one way, not both ways).

E.g.,

- bank  $A$  lends money to bank  $B$
- firm  $A$  supplies goods to firm  $B$
- individual  $A$  “follows” individual  $B$  on a given social network

(“Undirected” graphs, where connections are symmetric, are a special case of directed graphs — we just need to insist that each arrow pointing from  $A$  to  $B$  is paired with another arrow pointing from  $B$  to  $A$ .)

#### 33.3.1 Key definitions

A **directed graph** consists of two things:

1. a finite set  $V$  and
2. a collection of pairs  $(u, v)$  where  $u$  and  $v$  are elements of  $V$ .

The elements of  $V$  are called the **vertices** or **nodes** of the graph.

The pairs  $(u, v)$  are called the **edges** of the graph and the set of all edges will usually be denoted by  $E$ .

Intuitively and visually, an edge  $(u, v)$  is understood as an arrow from node  $u$  to node  $v$ .

(A neat way to represent an arrow is to record the location of the tail and head of the arrow, and that's exactly what an edge does.)

In the aircraft export example shown in [Fig. 33.1](#)

- $V$  is all countries included in the data set.
- $E$  is all the arrows in the figure, each indicating some positive amount of aircraft exports from one country to another.

Let's look at more examples.

Two graphs are shown below, each with three nodes.

We now construct a graph with the same nodes but different edges.

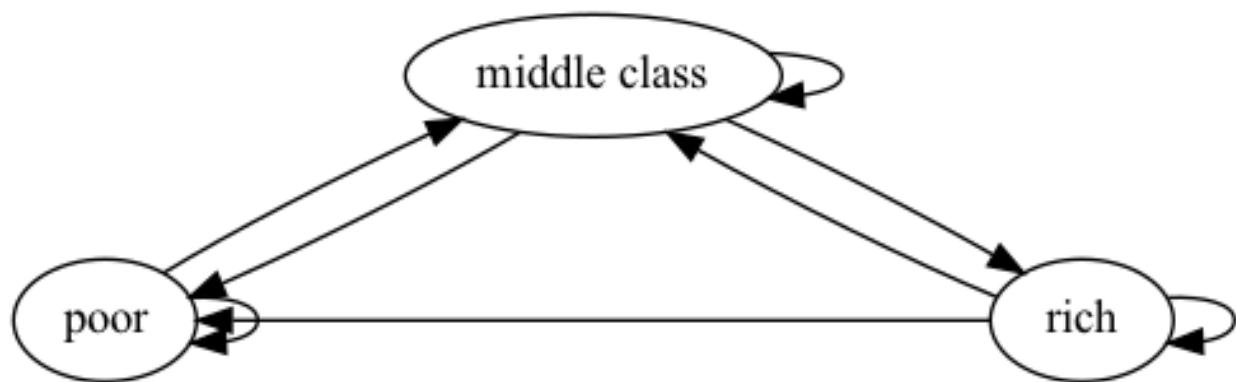


Fig. 33.2: Poverty Trap

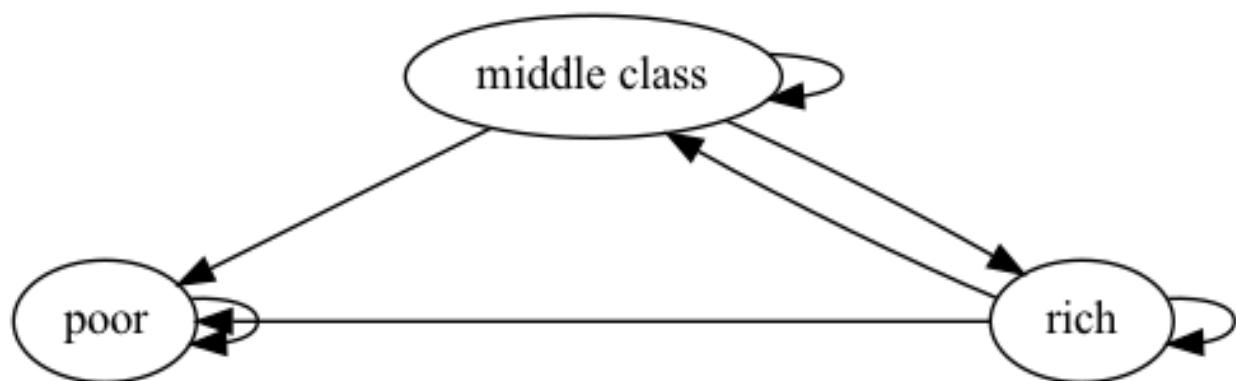


Fig. 33.3: Poverty Trap

For these graphs, the arrows (edges) can be thought of as representing positive transition probabilities over a given unit of time.

In general, if an edge  $(u, v)$  exists, then the node  $u$  is called a **direct predecessor** of  $v$  and  $v$  is called a **direct successor** of  $u$ .

Also, for  $v \in V$ ,

- the **in-degree** is  $i_d(v)$  = the number of direct predecessors of  $v$  and
- the **out-degree** is  $o_d(v)$  = the number of direct successors of  $v$ .

### 33.3.2 Digraphs in Networkx

The Python package `Networkx` provides a convenient data structure for representing directed graphs and implements many common routines for analyzing them.

As an example, let us recreate Fig. 33.3 using Networkx.

To do so, we first create an empty `DiGraph` object:

```
G_p = nx.DiGraph()
```

Next we populate it with nodes and edges.

To do this we write down a list of all edges, with *poor* represented by  $p$  and so on:

```
edge_list = [('p', 'p'),
              ('m', 'p'), ('m', 'm'), ('m', 'r'),
              ('r', 'p'), ('r', 'm'), ('r', 'r')]
```

Finally, we add the edges to our `DiGraph` object:

```
for e in edge_list:
    u, v = e
    G_p.add_edge(u, v)
```

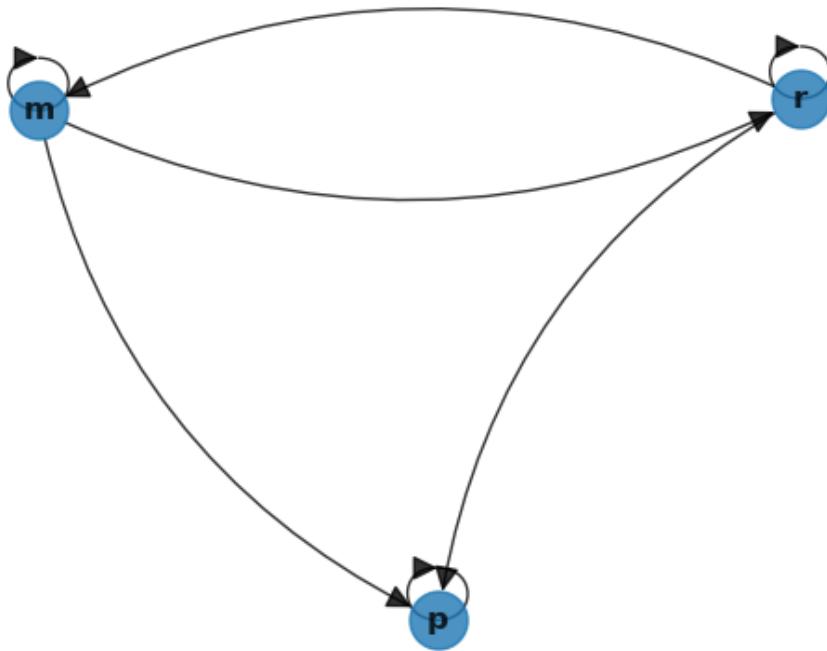
Alternatively, we can use the method `add_edges_from`.

```
G_p.add_edges_from(edge_list)
```

Adding the edges automatically adds the nodes, so `G_p` is now a correct representation of our graph.

We can verify this by plotting the graph via Networkx with the following code:

```
fig, ax = plt.subplots()
nx.draw_spring(G_p, ax=ax, node_size=500, with_labels=True,
                font_weight='bold', arrows=True, alpha=0.8,
                connectionstyle='arc3,rad=0.25', arrowsize=20)
plt.show()
```



The figure obtained above matches the original directed graph in [Fig. 33.3](#).

DiGraph objects have methods that calculate in-degree and out-degree of nodes.

For example,

```
G_p.in_degree('p')
```

```
3
```

### 33.3.3 Communication

Next, we study communication and connectedness, which have important implications for economic networks.

Node  $v$  is called **accessible** from node  $u$  if either  $u = v$  or there exists a sequence of edges that lead from  $u$  to  $v$ .

- in this case, we write  $u \rightarrow v$

(Visually, there is a sequence of arrows leading from  $u$  to  $v$ .)

For example, suppose we have a directed graph representing a production network, where

- elements of  $V$  are industrial sectors and
- existence of an edge  $(i, j)$  means that  $i$  supplies products or services to  $j$ .

Then  $m \rightarrow \ell$  means that sector  $m$  is an upstream supplier of sector  $\ell$ .

Two nodes  $u$  and  $v$  are said to **communicate** if both  $u \rightarrow v$  and  $v \rightarrow u$ .

A graph is called **strongly connected** if all nodes communicate.

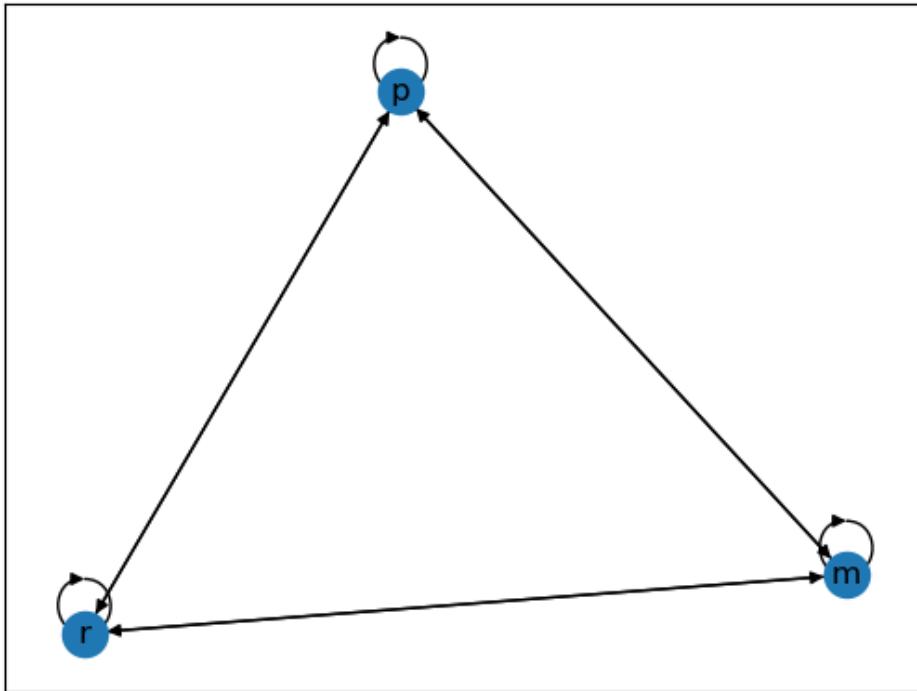
For example, [Fig. 33.2](#) is strongly connected however in [Fig. 33.3](#) rich is not accessible from poor, thus it is not strongly connected.

We can verify this by first constructing the graphs using Networkx and then using `nx.is_strongly_connected`.

```
fig, ax = plt.subplots()
G1 = nx.DiGraph()

G1.add_edges_from([('p', 'p'), ('p', 'm'), ('p', 'r'),
                   ('m', 'p'), ('m', 'm'), ('m', 'r'),
                   ('r', 'p'), ('r', 'm'), ('r', 'r')])

nx.draw_networkx(G1, with_labels = True)
```



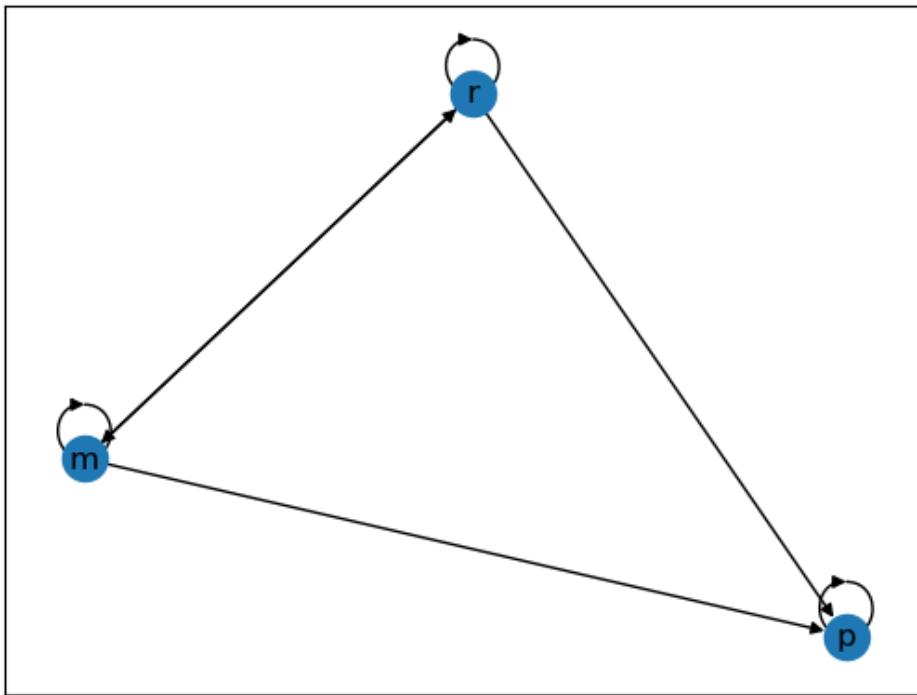
```
nx.is_strongly_connected(G1)      #checking if above graph is strongly connected
```

```
True
```

```
fig, ax = plt.subplots()
G2 = nx.DiGraph()

G2.add_edges_from([('p', 'p'),
                   ('m', 'p'), ('m', 'm'), ('m', 'r'),
                   ('r', 'p'), ('r', 'm'), ('r', 'r')])

nx.draw_networkx(G2, with_labels = True)
```



```
nx.is_strongly_connected(G2)      #checking if above graph is strongly connected
```

```
False
```

## 33.4 Weighted graphs

We now introduce weighted graphs, where weights (numbers) are attached to each edge.

### 33.4.1 International private credit flows by country

To motivate the idea, consider the following figure which shows flows of funds (i.e., loans) between private banks, grouped by country of origin.

The country codes are given in the following table

Code	Country	Code	Country	Code	Country	Code	Country
AU	Australia	DE	Germany	CL	Chile	ES	Spain
PT	Portugal	FR	France	TR	Turkey	GB	United Kingdom
US	United States	IE	Ireland	AT	Austria	IT	Italy
BE	Belgium	JP	Japan	SW	Switzerland	SE	Sweden

An arrow from Japan to the US indicates aggregate claims held by Japanese banks on all US-registered banks, as collected by the Bank of International Settlements (BIS).

The size of each node in the figure is increasing in the total foreign claims of all other nodes on this node.

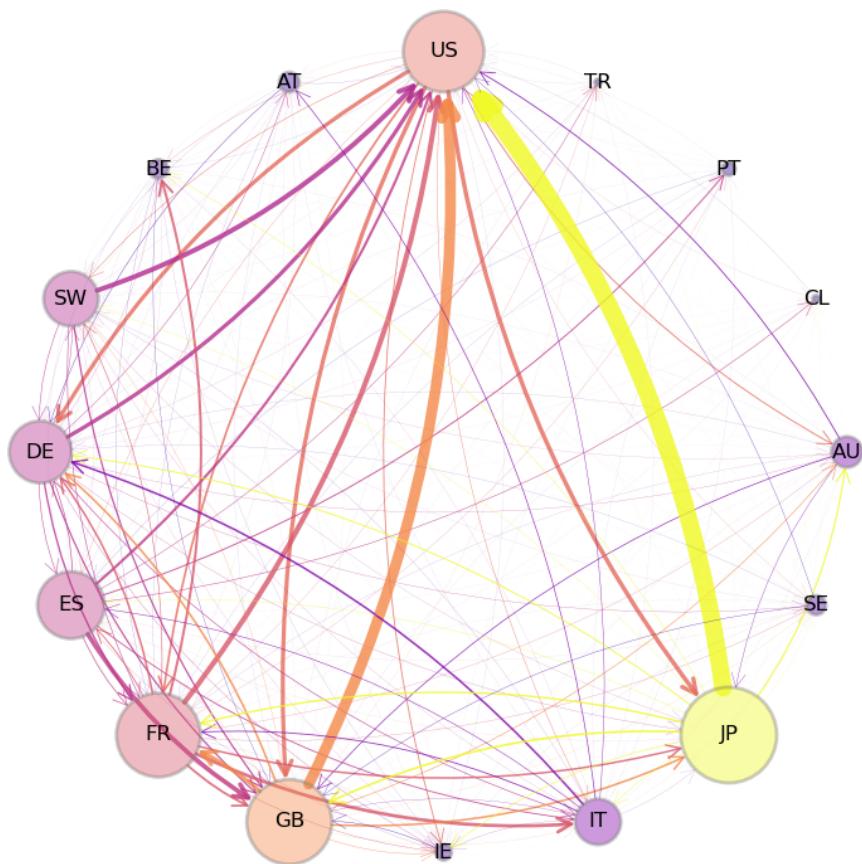


Fig. 33.4: International Credit Network

The widths of the arrows are proportional to the foreign claims they represent.

Notice that, in this network, an edge  $(u, v)$  exists for almost every choice of  $u$  and  $v$  (i.e., almost every country in the network).

(In fact, there are even more small arrows, which we have dropped for clarity.)

Hence the existence of an edge from one node to another is not particularly informative.

To understand the network, we need to record not just the existence or absence of a credit flow, but also the size of the flow.

The correct data structure for recording this information is a “weighted directed graph”.

### 33.4.2 Definitions

A **weighted directed graph** is a directed graph to which we have added a **weight function**  $w$  that assigns a positive number to each edge.

The figure above shows one weighted directed graph, where the weights are the size of fund flows.

The following figure shows a weighted directed graph, with arrows representing edges of the induced directed graph.

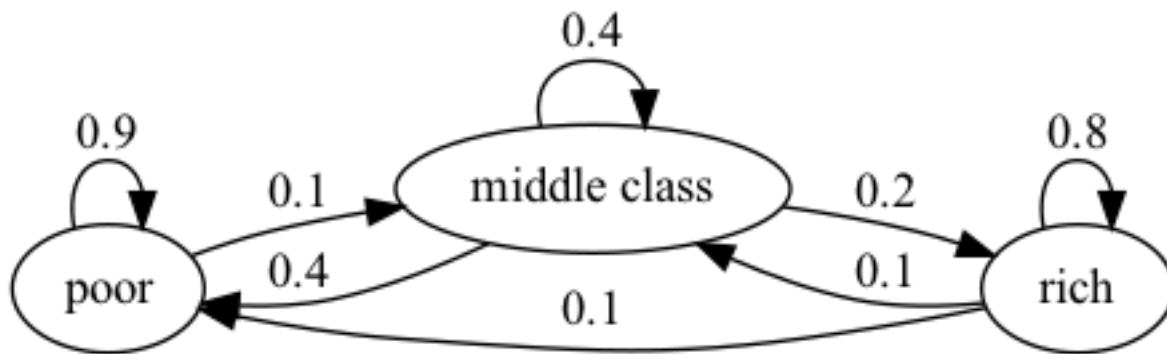


Fig. 33.5: Weighted Poverty Trap

The numbers next to the edges are the weights.

In this case, you can think of the numbers on the arrows as transition probabilities for a household over, say, one year.

We see that a rich household has a 10% chance of becoming poor in one year.

## 33.5 Adjacency matrices

Another way that we can represent weights, which turns out to be very convenient for numerical work, is via a matrix.

The **adjacency matrix** of a weighted directed graph with nodes  $\{v_1, \dots, v_n\}$ , edges  $E$  and weight function  $w$  is the matrix

$$A = (a_{ij})_{1 \leq i, j \leq n} \quad \text{with} \quad a_{ij} = \begin{cases} w(v_i, v_j) & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Once the nodes in  $V$  are enumerated, the weight function and adjacency matrix provide essentially the same information.

For example, with  $\{\text{poor, middle, rich}\}$  mapped to  $\{1, 2, 3\}$  respectively, the adjacency matrix corresponding to the weighted directed graph in Fig. 33.5 is

$$\begin{pmatrix} 0.9 & 0.1 & 0 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}.$$

In QuantEcon's `DiGraph` implementation, weights are recorded via the keyword `weighted`:

```
A = ((0.9, 0.1, 0.0),
      (0.4, 0.4, 0.2),
      (0.1, 0.1, 0.8))
A = np.array(A)
G = qe.DiGraph(A, weighted=True)      # store weights
```

One of the key points to remember about adjacency matrices is that taking the transpose *reverses all the arrows* in the associated directed graph.

For example, the following directed graph can be interpreted as a stylized version of a financial network, with nodes as banks and edges showing the flow of funds.

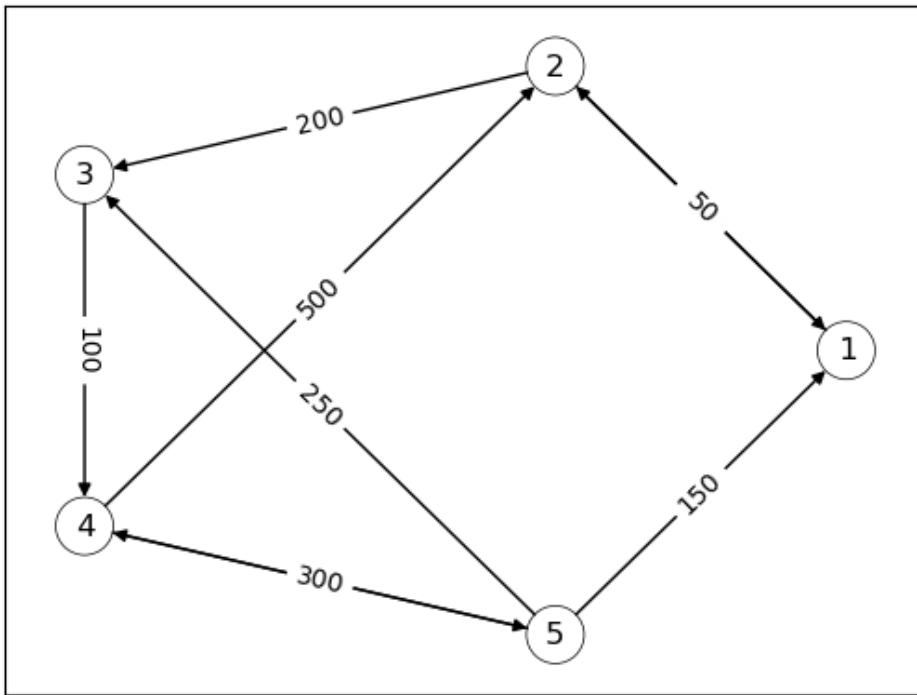
```
G4 = nx.DiGraph()

G4.add_edges_from([('1', '2'),
                  ('2', '1'), ('2', '3'),
                  ('3', '4'),
                  ('4', '2'), ('4', '5'),
                  ('5', '1'), ('5', '3'), ('5', '4')])
pos = nx.circular_layout(G4)

edge_labels={('1','2'): '100',
            ('2','1'): '50', ('2','3'): '200',
            ('3','4'): '100',
            ('4','2'): '500', ('4','5'): '50',
            ('5','1'): '150', ('5','3'): '250', ('5','4'): '300'}

nx.draw_networkx(G4, pos, node_color = 'none', node_size = 500)
nx.draw_networkx_edge_labels(G4, pos, edge_labels=edge_labels)
nx.draw_networkx_nodes(G4, pos, linewidths= 0.5, edgecolors = 'black',
                      node_color = 'none', node_size = 500)

plt.show()
```



We see that bank 2 extends a loan of size 200 to bank 3.

The corresponding adjacency matrix is

$$A = \begin{pmatrix} 0 & 100 & 0 & 0 & 0 \\ 50 & 0 & 200 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 \\ 0 & 500 & 0 & 0 & 50 \\ 150 & 0 & 250 & 300 & 0 \end{pmatrix}.$$

The transpose is

$$A^T = \begin{pmatrix} 0 & 50 & 0 & 0 & 150 \\ 100 & 0 & 0 & 500 & 0 \\ 0 & 200 & 0 & 0 & 250 \\ 0 & 0 & 100 & 0 & 300 \\ 0 & 0 & 0 & 50 & 0 \end{pmatrix}.$$

The corresponding network is visualized in the following figure which shows the network of liabilities after the loans have been granted.

Both of these networks (original and transpose) are useful for analyzing financial markets.

```
G5 = nx.DiGraph()

G5.add_edges_from([('1', '2'), ('1', '5'),
                  ('2', '1'), ('2', '4'),
                  ('3', '2'), ('3', '5'),
                  ('4', '3'), ('4', '5'),
                  ('5', '4')])

edge_labels={(('1', '2'): '50', ('1', '5'): '150',
             ('2', '1'): '100', ('2', '4'): '500',
```

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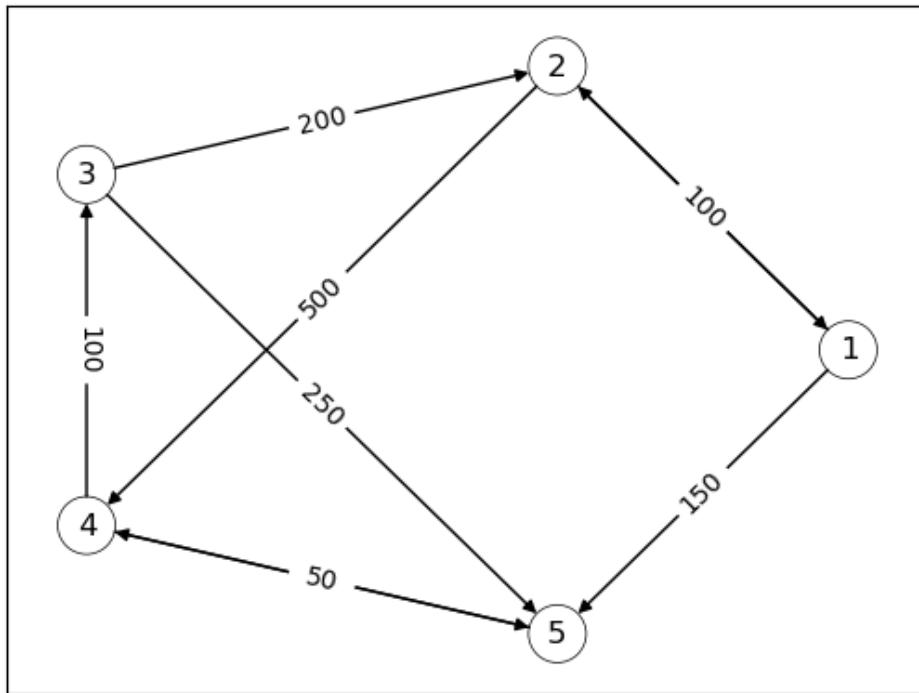
```

('3','2'): '200', ('3','5'): '250',
('4','3'): '100', ('4','5'): '300',
('5','4'): '50'}

nx.draw_networkx(G5, pos, node_color = 'none', node_size = 500)
nx.draw_networkx_edge_labels(G5, pos, edge_labels=edge_labels)
nx.draw_networkx_nodes(G5, pos, linewidths= 0.5, edgecolors = 'black',
node_color = 'none', node_size = 500)

plt.show()

```



In general, every nonnegative  $n \times n$  matrix  $A = (a_{ij})$  can be viewed as the adjacency matrix of a weighted directed graph.

To build the graph we set  $V = 1, \dots, n$  and take the edge set  $E$  to be all  $(i, j)$  such that  $a_{ij} > 0$ .

For the weight function we set  $w(i, j) = a_{ij}$  for all edges  $(i, j)$ .

We call this graph the weighted directed graph induced by  $A$ .

## 33.6 Properties

Consider a weighted directed graph with adjacency matrix  $A$ .

Let  $a_{ij}^k$  be element  $i, j$  of  $A^k$ , the  $k$ -th power of  $A$ .

The following result is useful in many applications:

---

### Theorem 33.6.1

For distinct nodes  $i, j$  in  $V$  and any integer  $k$ , we have

$$a_{ij}^k > 0 \quad \text{if and only if} \quad j \text{ is accessible from } i.$$

The above result is obvious when  $k = 1$  and a proof of the general case can be found in [SS22].

Now recall from the eigenvalues lecture that a nonnegative matrix  $A$  is called *irreducible* if for each  $(i, j)$  there is an integer  $k \geq 0$  such that  $a_{ij}^k > 0$ .

From the preceding theorem, it is not too difficult (see [SS22] for details) to get the next result.

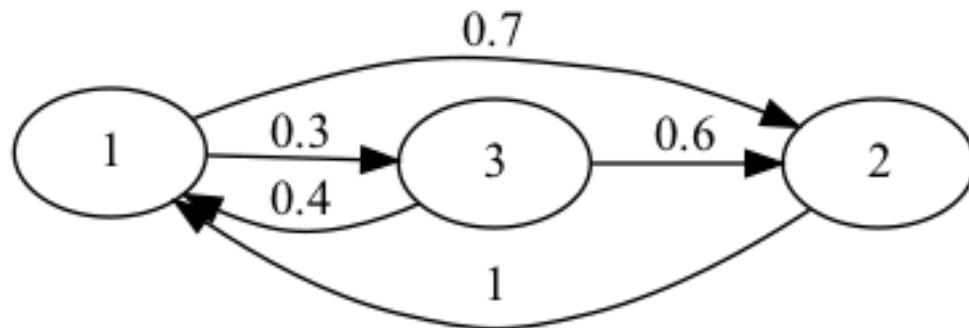
### Theorem 33.6.2

For a weighted directed graph the following statements are equivalent:

1. The directed graph is strongly connected.
2. The adjacency matrix of the graph is irreducible.

We illustrate the above theorem with a simple example.

Consider the following weighted directed graph.



We first create the above network as a Networkx DiGraph object.

```

G6 = nx.DiGraph()

G6.add_edges_from([('1', '2'), ('1', '3'),
                   ('2', '1'),
                   ('3', '1'), ('3', '2')])

```

Then we construct the associated adjacency matrix  $A$ .

```

A = np.array([[0, 0.7, 0.3],      # adjacency matrix A
              [1, 0, 0],
              [0.4, 0.6, 0]])

```

```

is_irreducible(A)      # check irreducibility of A

```

```

True

```

```
nx.is_strongly_connected(G6)      # check connectedness of graph
```

```
True
```

## 33.7 Network centrality

When studying networks of all varieties, a recurring topic is the relative “centrality” or “importance” of different nodes.

Examples include

- ranking of web pages by search engines
- determining the most important bank in a financial network (which one a central bank should rescue if there is a financial crisis)
- determining the most important industrial sector in an economy.

In what follows, a **centrality measure** associates to each weighted directed graph a vector  $m$  where the  $m_i$  is interpreted as the centrality (or rank) of node  $v_i$ .

### 33.7.1 Degree centrality

Two elementary measures of “importance” of a node in a given directed graph are its in-degree and out-degree.

Both of these provide a centrality measure.

In-degree centrality is a vector containing the in-degree of each node in the graph.

Consider the following simple example.

```
G7 = nx.DiGraph()

G7.add_nodes_from(['1','2','3','4','5','6','7'])

G7.add_edges_from([(1,2),(1,6),
                  (2,1),(2,4),
                  (3,2),
                  (4,2),
                  (5,3),(5,4),
                  (6,1),
                  (7,4),(7,6)])

pos = nx.planar_layout(G7)

nx.draw_networkx(G7, pos, node_color='none', node_size=500)
nx.draw_networkx_nodes(G7, pos, linewidths=0.5, edgecolors='black',
                      node_color='none', node_size=500)

plt.show()
```

The following code displays the in-degree centrality of all nodes.

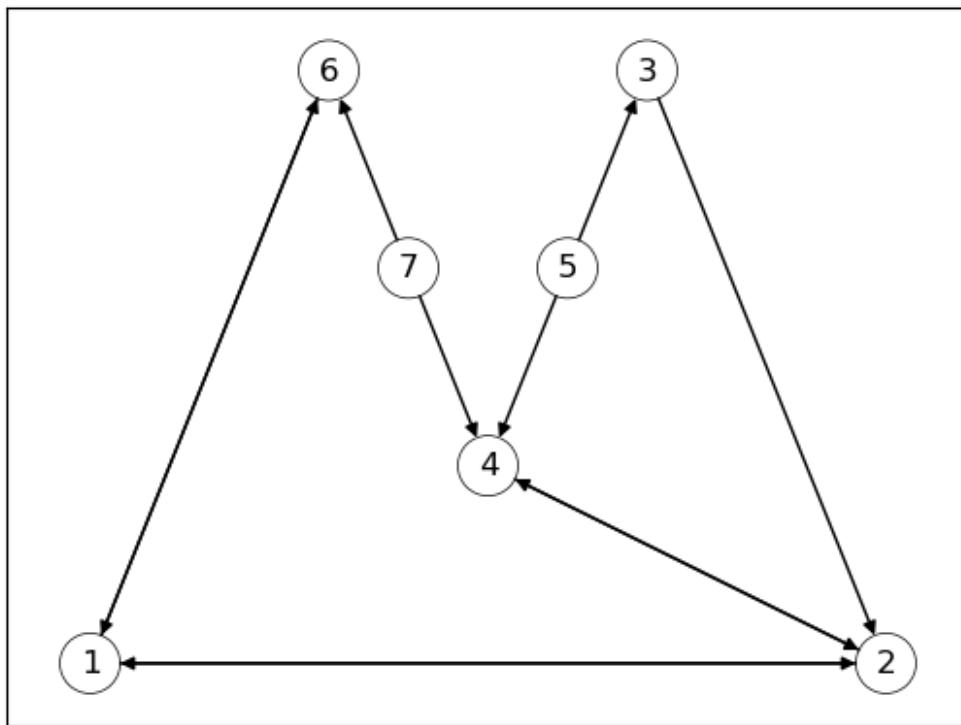


Fig. 33.6: Sample Graph

```

iG7 = [G7.in_degree(v) for v in G7.nodes()] # computing in-degree centrality

for i, d in enumerate(iG7):
    print(i+1, d)
  
```

```

1 2
2 3
3 1
4 3
5 0
6 2
7 0
  
```

Consider the international credit network displayed in Fig. 33.4.

The following plot displays the in-degree centrality of each country.

```

D = qbn_io.build_unweighted_matrix(Z)
indegree = D.sum(axis=0)
  
```

```

def centrality_plot_data(countries, centrality_measures):
    df = pd.DataFrame({'code': countries,
                       'centrality': centrality_measures,
                       'color': qbn_io.colorise_weights(centrality_measures).tolist()
                      })
    return df.sort_values('centrality')
  
```

```

fig, ax = plt.subplots()

df = centrality_plot_data(countries, indegree)

ax.bar('code', 'centrality', data=df, color=df["color"], alpha=0.6)

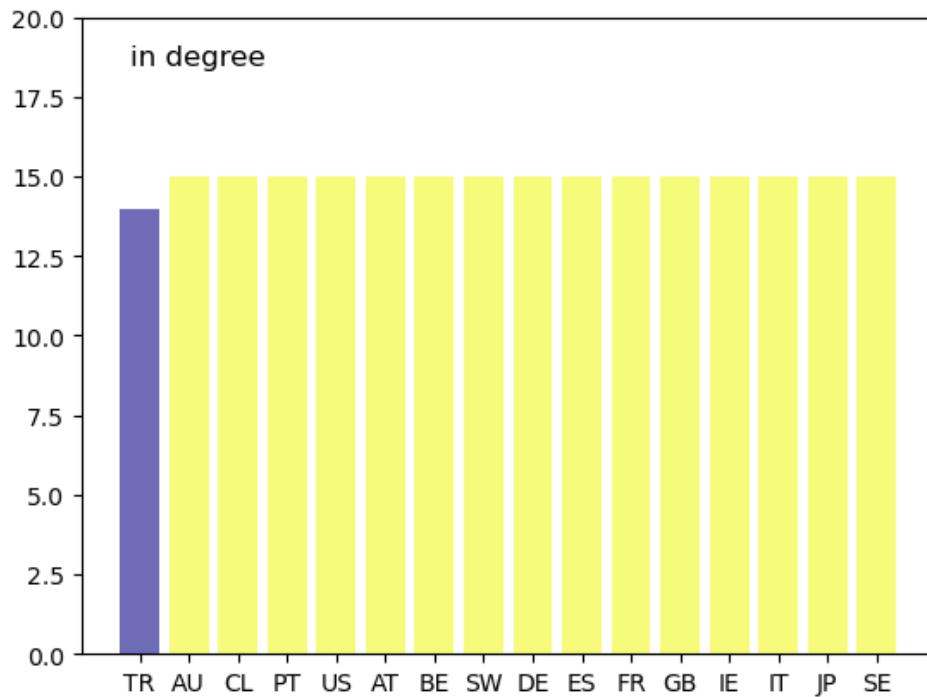
patch = mpatches.Patch(color=None, label='in degree', visible=False)
ax.legend(handles=[patch], fontsize=12, loc="upper left", handlelength=0,   

    frameon=False)

ax.set_ylim((0,20))

plt.show()

```



Unfortunately, while in-degree and out-degree centrality are simple to calculate, they are not always informative.

In Fig. 33.4, an edge exists between almost every node, so the in- or out-degree based centrality ranking fails to effectively separate the countries.

This can be seen in the above graph as well.

Another example is the task of a web search engine, which ranks pages by relevance whenever a user enters a search.

Suppose web page A has twice as many inbound links as page B.

In-degree centrality tells us that page A deserves a higher rank.

But in fact, page A might be less important than page B.

To see why, suppose that the links to A are from pages that receive almost no traffic, while the links to B are from pages that receive very heavy traffic.

In this case, page B probably receives more visitors, which in turn suggests that page B contains more valuable (or entertaining) content.

Thinking about this point suggests that importance might be *recursive*.

This means that the importance of a given node depends on the importance of other nodes that link to it.

As another example, we can imagine a production network where the importance of a given sector depends on the importance of the sectors that it supplies.

This reverses the order of the previous example: now the importance of a given node depends on the importance of other nodes that *it links to*.

The next centrality measures will have these recursive features.

### 33.7.2 Eigenvector centrality

Suppose we have a weighted directed graph with adjacency matrix  $A$ .

For simplicity, we will suppose that the nodes  $V$  of the graph are just the integers  $1, \dots, n$ .

Let  $r(A)$  denote the *spectral radius* of  $A$ .

The **eigenvector centrality** of the graph is defined as the  $n$ -vector  $e$  that solves

$$e = \frac{1}{r(A)} Ae. \quad (33.1)$$

In other words,  $e$  is the dominant eigenvector of  $A$  (the eigenvector of the largest eigenvalue — see the discussion of the *Perron-Frobenius theorem* in the eigenvalue lecture).

To better understand (33.1), we write out the full expression for some element  $e_i$

$$e_i = \frac{1}{r(A)} \sum_{1 \leq j \leq n} a_{ij} e_j \quad (33.2)$$

Note the recursive nature of the definition: the centrality obtained by node  $i$  is proportional to a sum of the centrality of all nodes, weighted by the *rates of flow* from  $i$  into these nodes.

A node  $i$  is highly ranked if

1. there are many edges leaving  $i$ ,
2. these edges have large weights, and
3. the edges point to other highly ranked nodes.

Later, when we study demand shocks in production networks, there will be a more concrete interpretation of eigenvector centrality.

We will see that, in production networks, sectors with high eigenvector centrality are important *suppliers*.

In particular, they are activated by a wide array of demand shocks once orders flow backwards through the network.

To compute eigenvector centrality we can use the following function.

```
def eigenvector_centrality(A, k=40, authority=False):
    """
    Computes the dominant eigenvector of A. Assumes A is
    primitive and uses the power method.

    """
    A_temp = A.T if authority else A
    n = len(A_temp)
    r = np.max(np.abs(np.linalg.eigvals(A_temp)))
```

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```
e = r**(-k) * (np.linalg.matrix_power(A_temp, k) @ np.ones(n))
return e / np.sum(e)
```

Let's compute eigenvector centrality for the graph generated in Fig. 33.6.

```
A = nx.to_numpy_array(G7)          # compute adjacency matrix of graph
```

```
e = eigenvector_centrality(A)
n = len(e)

for i in range(n):
    print(i+1,e[i])
```

```
1 0.18580570704268035
2 0.18580570704268035
3 0.11483424225608216
4 0.11483424225608216
5 0.14194292957319637
6 0.11483424225608216
7 0.14194292957319637
```

While nodes 2 and 4 had the highest in-degree centrality, we can see that nodes 1 and 2 have the highest eigenvector centrality.

Let's revisit the international credit network in Fig. 33.4.

```
eig_central = eigenvector_centrality(Z)

fig, ax = plt.subplots()

df = centrality_plot_data(countries, eig_central)

ax.bar('code', 'centrality', data=df, color=df["color"], alpha=0.6)

patch = mpatches.Patch(color=None, visible=False)
ax.legend(handles=[patch], fontsize=12, loc="upper left", handlelength=0,_
frameon=False)

plt.show()
```

Countries that are rated highly according to this rank tend to be important players in terms of supply of credit.

Japan takes the highest rank according to this measure, although countries with large financial sectors such as Great Britain and France are not far behind.

The advantage of eigenvector centrality is that it measures a node's importance while considering the importance of its neighbours.

A variant of eigenvector centrality is at the core of Google's PageRank algorithm, which is used to rank web pages.

The main principle is that links from important nodes (as measured by degree centrality) are worth more than links from unimportant nodes.

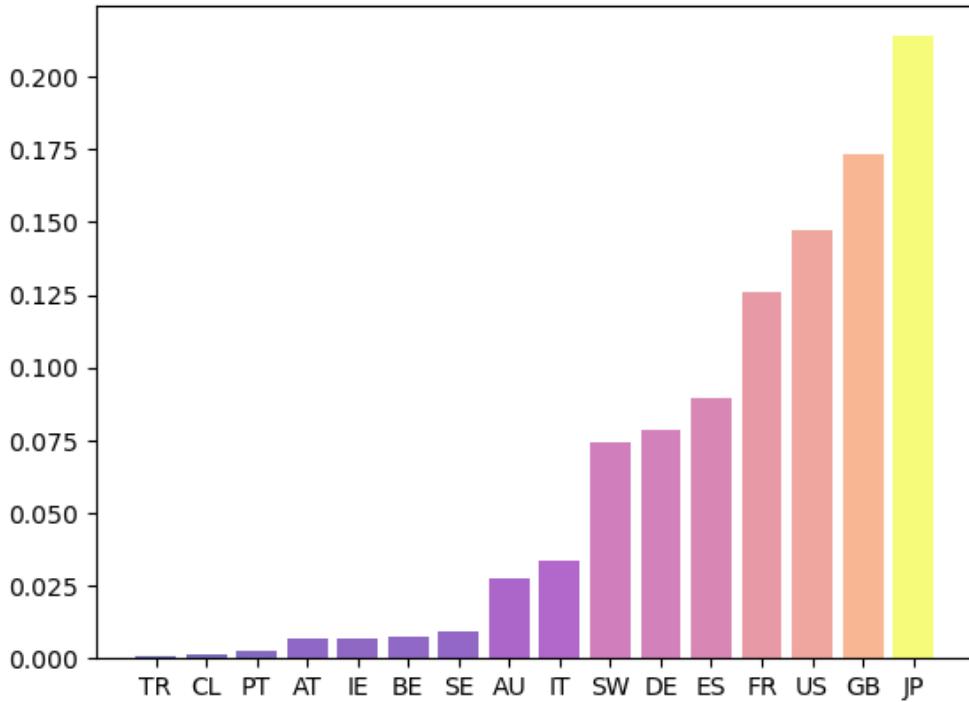


Fig. 33.7: Eigenvector centrality

### 33.7.3 Katz centrality

One problem with eigenvector centrality is that  $r(A)$  might be zero, in which case  $1/r(A)$  is not defined.

For this and other reasons, some researchers prefer another measure of centrality for networks called Katz centrality.

Fixing  $\beta$  in  $(0, 1/r(A))$ , the **Katz centrality** of a weighted directed graph with adjacency matrix  $A$  is defined as the vector  $\kappa$  that solves

$$\kappa_i = \beta \sum_{1 \leq j \leq n} a_{ij} \kappa_j + 1 \quad \text{for all } i \in \{0, \dots, n-1\}. \quad (33.3)$$

Here  $\beta$  is a parameter that we can choose.

In vector form we can write

$$\kappa = \mathbf{1} + \beta A \kappa \quad (33.4)$$

where  $\mathbf{1}$  is a column vector of ones.

The intuition behind this centrality measure is similar to that provided for eigenvector centrality: high centrality is conferred on  $i$  when it is linked to by nodes that themselves have high centrality.

Provided that  $0 < \beta < 1/r(A)$ , Katz centrality is always finite and well-defined because then  $r(\beta A) < 1$ .

This means that (33.4) has the unique solution

$$\kappa = (I - \beta A)^{-1} \mathbf{1}$$

This follows from the *Neumann series theorem*.

The parameter  $\beta$  is used to ensure that  $\kappa$  is finite

When  $r(A) < 1$ , we use  $\beta = 1$  as the default for Katz centrality computations.

### 33.7.4 Authorities vs hubs

Search engine designers recognize that web pages can be important in two different ways.

Some pages have high **hub centrality**, meaning that they link to valuable sources of information (e.g., news aggregation sites).

Other pages have high **authority centrality**, meaning that they contain valuable information, as indicated by the number and significance of incoming links (e.g., websites of respected news organizations).

Similar ideas can and have been applied to economic networks (often using different terminology).

The eigenvector centrality and Katz centrality measures we discussed above measure hub centrality.

(Nodes have high centrality if they point to other nodes with high centrality.)

If we care more about authority centrality, we can use the same definitions except that we take the transpose of the adjacency matrix.

This works because taking the transpose reverses the direction of the arrows.

(Now nodes will have high centrality if they receive links from other nodes with high centrality.)

For example, the **authority-based eigenvector centrality** of a weighted directed graph with adjacency matrix  $A$  is the vector  $e$  solving

$$e = \frac{1}{r(A)} A^\top e. \quad (33.5)$$

The only difference from the original definition is that  $A$  is replaced by its transpose.

(Transposes do not affect the spectral radius of a matrix so we wrote  $r(A)$  instead of  $r(A^\top)$ .)

Element-by-element, this is given by

$$e_j = \frac{1}{r(A)} \sum_{1 \leq i \leq n} a_{ij} e_i \quad (33.6)$$

We see  $e_j$  will be high if many nodes with high authority rankings link to  $j$ .

The following figure shows the authority-based eigenvector centrality ranking for the international credit network shown in Fig. 33.4.

```
ecentral_authority = eigenvector_centrality(Z, authority=True)

fig, ax = plt.subplots()

df = centrality_plot_data(countries, ecentral_authority)

ax.bar('code', 'centrality', data=df, color=df["color"], alpha=0.6)

patch = mpatches.Patch(color=None, visible=False)
ax.legend(handles=[patch], fontsize=12, loc="upper left", handlelength=0, handlewidth=0, frameon=False)

plt.show()
```

Highly ranked countries are those that attract large inflows of credit, or credit inflows from other major players.

In this case the US clearly dominates the rankings as a target of interbank credit.

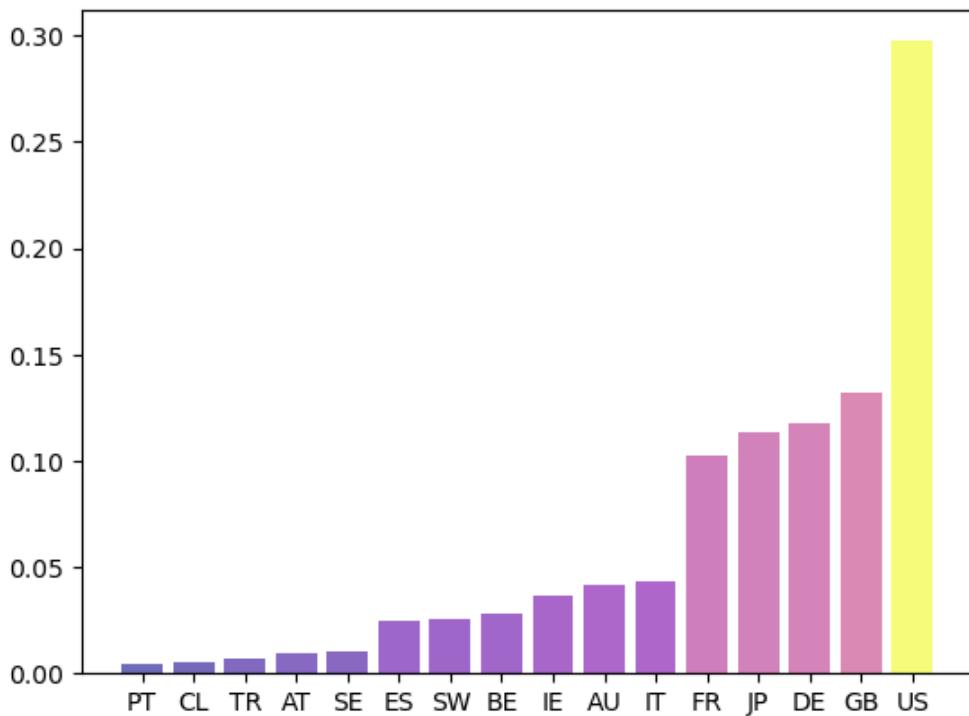


Fig. 33.8: Eigenvector authority

## 33.8 Further reading

We apply the ideas discussed in this lecture to:

Textbooks on economic and social networks include [Jac10], [EK+10], [BEJ18], [SS22] and [Goy23].

Within the realm of network science, the texts by [New18], [MFD20] and [Cos21] are excellent.

## 33.9 Exercises

---

### Exercise 33.9.1

Here is a mathematical exercise for those who like proofs.

Let  $(V, E)$  be a directed graph and write  $u \sim v$  if  $u$  and  $v$  communicate.

Show that  $\sim$  is an equivalence relation on  $V$ .

---



---

### Solution to Exercise 33.9.1

**Reflexivity:**

Trivially,  $u = v \Rightarrow u \rightarrow v$ .

Thus,  $u \sim u$ .

**Symmetry:** Suppose,  $u \sim v$

$\Rightarrow u \rightarrow v$  and  $v \rightarrow u$ .

By definition, this implies  $v \sim u$ .

### Transitivity:

Suppose,  $u \sim v$  and  $v \sim w$

This implies,  $u \rightarrow v$  and  $v \rightarrow u$  and also  $v \rightarrow w$  and  $w \rightarrow v$ .

Thus, we can conclude  $u \rightarrow v \rightarrow w$  and  $w \rightarrow v \rightarrow u$ .

Which means  $u \sim w$ .

---

### Exercise 33.9.2

Consider a directed graph  $G$  with the set of nodes

$$V = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

and the set of edges

$$E = \{(0, 1), (0, 3), (1, 0), (2, 4), (3, 2), (3, 4), (3, 7), (4, 3), (5, 4), (5, 6), (6, 3), (6, 5), (7, 0)\}$$

1. Use Networkx to draw graph  $G$ .
  2. Find the associated adjacency matrix  $A$  for  $G$ .
  3. Use the functions defined above to compute in-degree centrality, out-degree centrality and eigenvector centrality of  $G$ .
- 

### Solution to Exercise 33.9.2

```
# First, let's plot the given graph

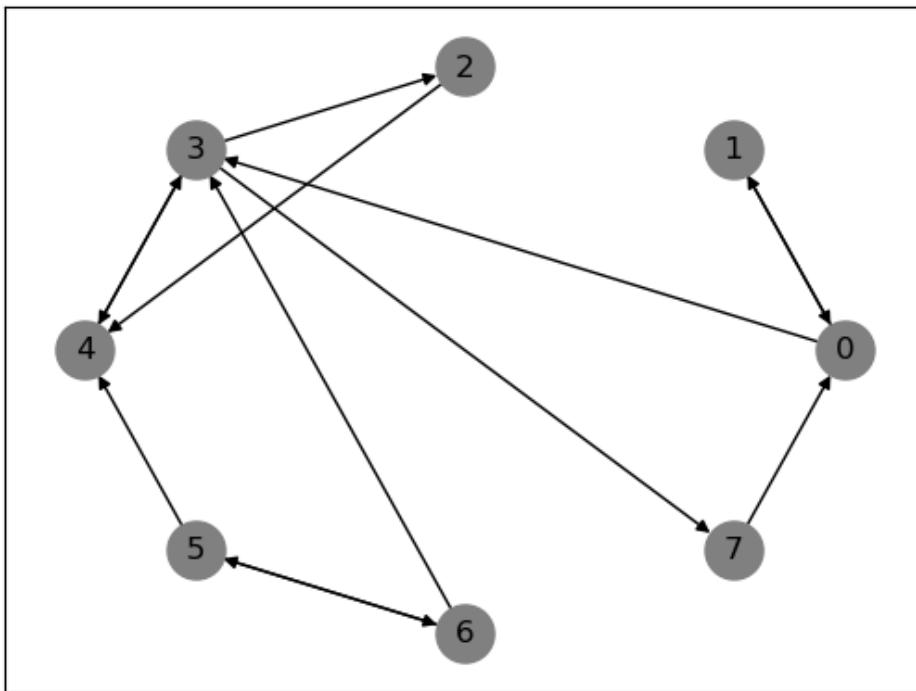
G = nx.DiGraph()

G.add_nodes_from(np.arange(8))    # adding nodes

G.add_edges_from([(0, 1), (0, 3),           # adding edges
                  (1, 0),
                  (2, 4),
                  (3, 2), (3, 4), (3, 7),
                  (4, 3),
                  (5, 4), (5, 6),
                  (6, 3), (6, 5),
                  (7, 0)])]

nx.draw_networkx(G, pos=nx.circular_layout(G), node_color='gray', node_size=500, with_
                  labels=True)

plt.show()
```



```
A = nx.to_numpy_array(G)      # find adjacency matrix associated with G
```

```
A
```

```
array([[0.,  1.,  0.,  1.,  0.,  0.,  0.,  0.],
       [1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.],
       [0.,  0.,  0.,  0.,  1.,  0.,  0.,  0.],
       [0.,  0.,  1.,  0.,  1.,  0.,  0.,  1.],
       [0.,  0.,  0.,  1.,  0.,  0.,  0.,  0.],
       [0.,  0.,  0.,  0.,  1.,  0.,  1.,  0.],
       [0.,  0.,  0.,  1.,  0.,  1.,  0.,  0.],
       [1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.]])
```

```
oG = [G.out_degree(v) for v in G.nodes()]      # computing in-degree centrality
```

```
for i, d in enumerate(oG):
    print(i, d)
```

```
0 2
1 1
2 1
3 3
4 1
5 2
6 2
7 1
```

```
e = eigenvector_centrality(A)      # computing eigenvector centrality
n = len(e)
```

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```

for i in range(n):
    print(i+1, e[i])

1 0.1458980838002507
2 0.0901698980074874
3 0.05572805602479352
4 0.14589810100962305
5 0.0901699482402499
6 0.1803397955498566
7 0.20162621936025152
8 0.0901698980074874

```

---

### Exercise 33.9.3

Consider a graph  $G$  with  $n$  nodes and  $n \times n$  adjacency matrix  $A$ .

Let  $S = \sum_{k=0}^{n-1} A^k$

We can say for any two nodes  $i$  and  $j$ ,  $j$  is accessible from  $i$  if and only if  $S_{ij} > 0$ .

Devise a function `is_accessible` that checks if any two nodes of a given graph are accessible.

Consider the graph in [Exercise 33.9.2](#) and use this function to check if

1. 1 is accessible from 2
  2. 6 is accessible from 3
- 

### Solution to Exercise 33.9.3

```

def is_accessible(G, i, j):
    A = nx.to_numpy_array(G)
    n = len(A)
    result = np.zeros((n, n))
    for i in range(n):
        result += np.linalg.matrix_power(A, i)
    if result[i, j] > 0:
        return True
    else:
        return False

```

```

G = nx.DiGraph()

G.add_nodes_from(np.arange(8))    # adding nodes

G.add_edges_from([(0, 1), (0, 3),           # adding edges
                  (1, 0),
                  (2, 4),
                  (3, 2), (3, 4), (3, 7),
                  (4, 3),
                  (5, 4), (5, 6),

```

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```
(6, 3), (6, 5),  
(7, 0)])
```

```
is_accessible(G, 2, 1)
```

```
True
```

```
is_accessible(G, 3, 6)
```

```
False
```

---



## **Part X**

# **Markets and Competitive Equilibrium**



## SUPPLY AND DEMAND WITH MANY GOODS

### 34.1 Overview

In a *previous lecture* we studied supply, demand and welfare in a market with a single consumption good.

In this lecture, we study a setting with  $n$  goods and  $n$  corresponding prices.

Key infrastructure concepts that we'll encounter in this lecture are

- inverse demand curves
- marginal utilities of wealth
- inverse supply curves
- consumer surplus
- producer surplus
- social welfare as a sum of consumer and producer surpluses
- competitive equilibrium

We will provide a version of the *first fundamental welfare theorem*, which was formulated by

- Leon Walras
- Francis Ysidro Edgeworth
- Vilfredo Pareto

Important extensions to the key ideas were obtained by

- Abba Lerner
- Harold Hotelling
- Paul Samuelson
- Kenneth Arrow
- Gerard Debreu

We shall describe two classic welfare theorems:

- **first welfare theorem:** for a given distribution of wealth among consumers, a competitive equilibrium allocation of goods solves a social planning problem.
- **second welfare theorem:** An allocation of goods to consumers that solves a social planning problem can be supported by a competitive equilibrium with an appropriate initial distribution of wealth.

As usual, we start by importing some Python modules.

```
# import some packages
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from scipy.linalg import inv
```

## 34.2 Formulas from linear algebra

We shall apply formulas from linear algebra that

- differentiate an inner product with respect to each vector
- differentiate a product of a matrix and a vector with respect to the vector
- differentiate a quadratic form in a vector with respect to the vector

Where  $a$  is an  $n \times 1$  vector,  $A$  is an  $n \times n$  matrix, and  $x$  is an  $n \times 1$  vector:

$$\begin{aligned}\frac{\partial a^\top x}{\partial x} &= \frac{\partial x^\top a}{\partial x} = a \\ \frac{\partial Ax}{\partial x} &= A \\ \frac{\partial x^\top Ax}{\partial x} &= (A + A^\top)x\end{aligned}$$

## 34.3 From utility function to demand curve

Our study of consumers will use the following primitives

- $\Pi$  be an  $m \times n$  matrix,
- $b$  be an  $m \times 1$  vector of bliss points,
- $e$  be an  $n \times 1$  vector of endowments, and

We will analyze endogenous objects  $c$  and  $p$ , where

- $c$  is an  $n \times 1$  vector of consumptions of various goods,
- $p$  is an  $n \times 1$  vector of prices

The matrix  $\Pi$  describes a consumer's willingness to substitute one good for every other good.

We assume that  $\Pi$  has linearly independent columns, which implies that  $\Pi^\top \Pi$  is a positive definite matrix.

- it follows that  $\Pi^\top \Pi$  has an inverse.

We shall see below that  $(\Pi^\top \Pi)^{-1}$  is a matrix of slopes of (compensated) demand curves for  $c$  with respect to a vector of prices:

$$\frac{\partial c}{\partial p} = (\Pi^\top \Pi)^{-1}$$

A consumer faces  $p$  as a price taker and chooses  $c$  to maximize the utility function

$$-\frac{1}{2}(\Pi c - b)^\top (\Pi c - b) \tag{34.1}$$

subject to the budget constraint

$$p^\top(c - e) = 0 \quad (34.2)$$

We shall specify examples in which  $\Pi$  and  $b$  are such that it typically happens that

$$\Pi c \ll b \quad (34.3)$$

This means that the consumer has much less of each good than he wants.

The deviation in (34.3) will ultimately assure us that competitive equilibrium prices are positive.

### 34.3.1 Demand curve implied by constrained utility maximization

For now, we assume that the budget constraint is (34.2).

So we'll be deriving what is known as a **Marshallian** demand curve.

Our aim is to maximize (34.3.1) subject to (34.3.2).

Form a Lagrangian

$$L = -\frac{1}{2}(\Pi c - b)^\top(\Pi c - b) + \mu[p^\top(e - c)]$$

where  $\mu$  is a Lagrange multiplier that is often called a **marginal utility of wealth**.

The consumer chooses  $c$  to maximize  $L$  and  $\mu$  to minimize it.

First-order conditions for  $c$  are

$$\frac{\partial L}{\partial c} = -\Pi^\top \Pi c + \Pi^\top b - \mu p = 0$$

so that, given  $\mu$ , the consumer chooses

$$c = (\Pi^\top \Pi)^{-1}(\Pi^\top b - \mu p) \quad (34.4)$$

Substituting (34.4) into budget constraint (34.2) and solving for  $\mu$  gives

$$\mu(p, e) = \frac{p^\top(\Pi^\top \Pi)^{-1}\Pi^\top b - p^\top e}{p^\top(\Pi^\top \Pi)^{-1}p}. \quad (34.5)$$

Equation (34.5) tells how marginal utility of wealth depends on the endowment vector  $e$  and the price vector  $p$ .

---

**Note:** Equation (34.5) is a consequence of imposing that  $p^\top(c - e) = 0$ .

We could instead take  $\mu$  as a parameter and use (34.4) and the budget constraint (34.6) to solve for wealth.

Which way we proceed determines whether we are constructing a **Marshallian** or **Hicksian** demand curve.

---

## 34.4 Endowment economy

We now study a pure-exchange economy, or what is sometimes called an endowment economy.

Consider a single-consumer, multiple-goods economy without production.

The only source of goods is the single consumer's endowment vector  $e$ .

A competitive equilibrium price vector induces the consumer to choose  $c = e$ .

This implies that the equilibrium price vector satisfies

$$p = \mu^{-1}(\Pi^\top b - \Pi^\top \Pi e)$$

In the present case where we have imposed budget constraint in the form (34.2), we are free to normalize the price vector by setting the marginal utility of wealth  $\mu = 1$  (or any other value for that matter).

This amounts to choosing a common unit (or numeraire) in which prices of all goods are expressed.

(Doubling all prices will affect neither quantities nor relative prices.)

We'll set  $\mu = 1$ .

---

#### Exercise 34.4.1

Verify that setting  $\mu = 1$  in (34.4) implies that formula (34.5) is satisfied.

---

#### Exercise 34.4.2

Verify that setting  $\mu = 2$  in (34.4) also implies that formula (34.5) is satisfied.

---

Here is a class that computes competitive equilibria for our economy.

```
class ExchangeEconomy:

    def __init__(self,
                 Pi,
                 b,
                 e,
                 thres=1.5):
        """
        Set up the environment for an exchange economy

        Args:
            Pi (np.array): shared matrix of substitution
            b (list): the consumer's bliss point
            e (list): the consumer's endowment
            thres (float): a threshold to check p >> Pi e condition
        """

        # check non-satiation
        if np.min(b / np.max(Pi @ e)) <= thres:
            raise Exception('set bliss points further away')

        self.Pi, self.b, self.e = Pi, b, e

    def competitive_equilibrium(self):
        """
        Compute the competitive equilibrium prices and allocation
        """
        Pi, b, e = self.Pi, self.b, self.e
```

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```

# compute price vector with p=1
p = Π.T @ b - Π.T @ Π @ e

# compute consumption vector
slope_dc = inv(Π.T @ Π)
Π_inv = inv(Π)
c = Π_inv @ b - slope_dc @ p

if any(c < 0):
    print('allocation: ', c)
    raise Exception('negative allocation: equilibrium does not exist')

return p, c

```

## 34.5 Digression: Marshallian and Hicksian demand curves

Sometimes we'll use budget constraint (34.2) in situations in which a consumer's endowment vector  $e$  is his **only** source of income.

Other times we'll instead assume that the consumer has another source of income (positive or negative) and write his budget constraint as

$$p^\top(c - e) = w \quad (34.6)$$

where  $w$  is measured in "dollars" (or some other **numeraire**) and component  $p_i$  of the price vector is measured in dollars per unit of good  $i$ .

Whether the consumer's budget constraint is (34.2) or (34.6) and whether we take  $w$  as a free parameter or instead as an endogenous variable will affect the consumer's marginal utility of wealth.

Consequently, how we set  $\mu$  determines whether we are constructing

- a **Marshallian** demand curve, as when we use (34.2) and solve for  $\mu$  using equation (34.5) above, or
- a **Hicksian** demand curve, as when we treat  $\mu$  as a fixed parameter and solve for  $w$  from (34.6).

Marshallian and Hicksian demand curves contemplate different mental experiments:

For a Marshallian demand curve, hypothetical changes in a price vector have both **substitution** and **income** effects

- income effects are consequences of changes in  $p^\top e$  associated with the change in the price vector

For a Hicksian demand curve, hypothetical price vector changes have only **substitution** effects

- changes in the price vector leave the  $p^\top e + w$  unaltered because we freeze  $\mu$  and solve for  $w$

Sometimes a Hicksian demand curve is called a **compensated** demand curve in order to emphasize that, to disarm the income (or wealth) effect associated with a price change, the consumer's wealth  $w$  is adjusted.

We'll discuss these distinct demand curves more below.

## 34.6 Dynamics and risk as special cases

Special cases of our  $n$ -good pure exchange model can be created to represent

- **dynamics** — by putting different dates on different commodities
- **risk** — by interpreting delivery of goods as being contingent on states of the world whose realizations are described by a *known probability distribution*

Let's illustrate how.

### 34.6.1 Dynamics

Suppose that we want to represent a utility function

$$-\frac{1}{2}[(c_1 - b_1)^2 + \beta(c_2 - b_2)^2]$$

where  $\beta \in (0, 1)$  is a discount factor,  $c_1$  is consumption at time 1 and  $c_2$  is consumption at time 2.

To capture this with our quadratic utility function (34.1), set

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\beta} \end{bmatrix}$$

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

and

$$b = \begin{bmatrix} b_1 \\ \sqrt{\beta}b_2 \end{bmatrix}$$

The budget constraint (34.2) becomes

$$p_1 c_1 + p_2 c_2 = p_1 e_1 + p_2 e_2$$

The left side is the **discounted present value** of consumption.

The right side is the **discounted present value** of the consumer's endowment.

The relative price  $\frac{p_1}{p_2}$  has units of time 2 goods per unit of time 1 goods.

Consequently,

$$(1 + r) := R := \frac{p_1}{p_2}$$

is the **gross interest rate** and  $r$  is the **net interest rate**.

Here is an example.

```
beta = 0.95

Pi = np.array([[1, 0],
              [0, np.sqrt(beta)]])

b = np.array([5, np.sqrt(beta) * 5])
```

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```
e = np.array([1, 1])

dynamics = ExchangeEconomy(Pi, b, e)
p, c = dynamics.competitive_equilibrium()

print('Competitive equilibrium price vector:', p)
print('Competitive equilibrium allocation:', c)
```

```
Competitive equilibrium price vector: [4. 3.8]
Competitive equilibrium allocation: [1. 1.]
```

### 34.6.2 Risk and state-contingent claims

We study risk in the context of a **static** environment, meaning that there is only one period.

By **risk** we mean that an outcome is not known in advance, but that it is governed by a known probability distribution.

As an example, our consumer confronts **risk** means in particular that

- there are two states of nature, 1 and 2.
- the consumer knows that the probability that state 1 occurs is  $\lambda$ .
- the consumer knows that the probability that state 2 occurs is  $(1 - \lambda)$ .

Before the outcome is realized, the consumer's **expected utility** is

$$-\frac{1}{2}[\lambda(c_1 - b_1)^2 + (1 - \lambda)(c_2 - b_2)^2]$$

where

- $c_1$  is consumption in state 1
- $c_2$  is consumption in state 2

To capture these preferences we set

$$\begin{aligned}\Pi &= \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{1 - \lambda} \end{bmatrix} \\ e &= \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \\ b &= \begin{bmatrix} \sqrt{\lambda}b_1 \\ \sqrt{1 - \lambda}b_2 \end{bmatrix}\end{aligned}$$

A consumer's endowment vector is

$$c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

A price vector is

$$p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

where  $p_i$  is the price of one unit of consumption in state  $i \in \{1, 2\}$ .

The state-contingent goods being traded are often called **Arrow securities**.

Before the random state of the world  $i$  is realized, the consumer sells his/her state-contingent endowment bundle and purchases a state-contingent consumption bundle.

Trading such state-contingent goods is one way economists often model **insurance**.

We use the tricks described above to interpret  $c_1, c_2$  as “Arrow securities” that are state-contingent claims to consumption goods.

Here is an instance of the risk economy:

```
prob = 0.2

Pi = np.array([[np.sqrt(prob), 0],
               [0, np.sqrt(1 - prob)]])

b = np.array([np.sqrt(prob) * 5, np.sqrt(1 - prob) * 5])

e = np.array([1, 1])

risk = ExchangeEconomy(Pi, b, e)
p, c = risk.competitive_equilibrium()

print('Competitive equilibrium price vector:', p)
print('Competitive equilibrium allocation:', c)
```

```
Competitive equilibrium price vector: [0.8 3.2]
Competitive equilibrium allocation: [1. 1.]
```

---

### Exercise 34.6.1

Consider the instance above.

Please numerically study how each of the following cases affects the equilibrium prices and allocations:

- the consumer gets poorer,
- they like the first good more, or
- the probability that state 1 occurs is higher.

Hints. For each case choose some parameter  $e, b$ , or  $\lambda$  different from the instance.

---

---

### Solution to Exercise 34.6.1

First consider when the consumer is poorer.

Here we just decrease the endowment.

```
risk.e = np.array([0.5, 0.5])

p, c = risk.competitive_equilibrium()

print('Competitive equilibrium price vector:', p)
print('Competitive equilibrium allocation:', c)
```

```
Competitive equilibrium price vector: [0.9 3.6]
Competitive equilibrium allocation: [0.5 0.5]
```

If the consumer likes the first (or second) good more, then we can set a larger bliss value for good 1.

```
risk.b = np.array([np.sqrt(prob) * 6, np.sqrt(1 - prob) * 5])
p, c = risk.competitive_equilibrium()

print('Competitive equilibrium price vector:', p)
print('Competitive equilibrium allocation:', c)
```

```
Competitive equilibrium price vector: [1.1 3.6]
Competitive equilibrium allocation: [0.5 0.5]
```

Increase the probability that state 1 occurs.

```
prob = 0.8

Pi = np.array([[np.sqrt(prob), 0],
              [0, np.sqrt(1 - prob)]])

b = np.array([np.sqrt(prob) * 5, np.sqrt(1 - prob) * 5])

e = np.array([1, 1])

risk = ExchangeEconomy(Pi, b, e)
p, c = risk.competitive_equilibrium()

print('Competitive equilibrium price vector:', p)
print('Competitive equilibrium allocation:', c)
```

```
Competitive equilibrium price vector: [3.2 0.8]
Competitive equilibrium allocation: [1. 1.]
```

## 34.7 Economies with endogenous supplies of goods

Up to now we have described a pure exchange economy in which endowments of goods are exogenous, meaning that they are taken as given from outside the model.

### 34.7.1 Supply curve of a competitive firm

A competitive firm that can produce goods takes a price vector  $p$  as given and chooses a quantity  $q$  to maximize total revenue minus total costs.

The firm's total revenue equals  $p^\top q$  and its total cost equals  $C(q)$  where  $C(q)$  is a total cost function

$$C(q) = h^\top q + \frac{1}{2} q^\top J q$$

and  $J$  is a positive definite matrix.

So the firm's profits are

$$p^\top q - C(q) \quad (34.7)$$

An  $n \times 1$  vector of **marginal costs** is

$$\frac{\partial C(q)}{\partial q} = h + Hq$$

where

$$H = \frac{1}{2}(J + J^\top)$$

The firm maximizes total profits by setting **marginal revenue to marginal costs**.

An  $n \times 1$  vector of marginal revenues for the price-taking firm is  $\frac{\partial p^\top q}{\partial q} = p$ .

So **price equals marginal revenue** for our price-taking competitive firm.

This leads to the following **inverse supply curve** for the competitive firm:

$$p = h + Hq$$

### 34.7.2 Competitive equilibrium

To compute a competitive equilibrium for a production economy where demand curve is pinned down by the marginal utility of wealth  $\mu$ , we first compute an allocation by solving a planning problem.

Then we compute the equilibrium price vector using the inverse demand or supply curve.

$\mu = 1$  **warmup**

As a special case, let's pin down a demand curve by setting the marginal utility of wealth  $\mu = 1$ .

Equating supply price to demand price and letting  $q = c$  we get

$$p = h + Hc = \Pi^\top b - \Pi^\top \Pi c,$$

which implies the equilibrium quantity vector

$$c = (\Pi^\top \Pi + H)^{-1}(\Pi^\top b - h) \quad (34.8)$$

This equation is the counterpart of equilibrium quantity (7.3) for the scalar  $n = 1$  model with which we began.

**General  $\mu \neq 1$  case**

Now let's extend the preceding analysis to a more general case by allowing  $\mu \neq 1$ .

Then the inverse demand curve is

$$p = \mu^{-1}[\Pi^\top b - \Pi^\top \Pi c] \quad (34.9)$$

Equating this to the inverse supply curve, letting  $q = c$  and solving for  $c$  gives

$$c = [\Pi^\top \Pi + \mu H]^{-1}[\Pi^\top b - \mu h] \quad (34.10)$$

### 34.7.3 Implementation

A Production Economy will consist of

- a single **person** that we'll interpret as a representative consumer
- a single set of **production costs**
- a multiplier  $\mu$  that weights “consumers” versus “producers” in a planner’s welfare function, as described above in the main text
- an  $n \times 1$  vector  $p$  of competitive equilibrium prices
- an  $n \times 1$  vector  $c$  of competitive equilibrium quantities
- **consumer surplus**
- **producer surplus**

Here we define a class `ProductionEconomy`.

```
class ProductionEconomy:

    def __init__(self,
                 Π,
                 b,
                 h,
                 J,
                 μ):
        """
        Set up the environment for a production economy

        Args:
            Π (np.ndarray): matrix of substitution
            b (np.array): bliss points
            h (np.array): h in cost func
            J (np.ndarray): J in cost func
            μ (float): welfare weight of the corresponding planning problem
        """
        self.n = len(b)
        self.Π, self.b, self.h, self.J, self.μ = Π, b, h, J, μ

    def competitive_equilibrium(self):
        """
        Compute a competitive equilibrium of the production economy
        """
        Π, b, h, μ, J = self.Π, self.b, self.h, self.μ, self.J
        H = .5 * (J + J.T)

        # allocation
        c = inv(Π.T @ Π + μ * H) @ (Π.T @ b - μ * h)

        # price
        p = 1 / μ * (Π.T @ b - Π.T @ Π @ c)

        # check non-satiation
        if any(Π @ c - b >= 0):
            raise Exception('invalid result: set bliss points further away')

        return c, p
```

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```

def compute_surplus(self):
    """
    Compute consumer and producer surplus for single good case
    """
    if self.n != 1:
        raise Exception('not single good')
    h, J, Π, b, μ = self.h.item(), self.J.item(), self.Π.item(), self.b.item(), self.μ
    H = J

    # supply/demand curve coefficients
    s0, s1 = h, H
    d0, d1 = 1 / μ * Π * b, 1 / μ * Π**2

    # competitive equilibrium
    c, p = self.competitive_equilibrium()

    # calculate surplus
    c_surplus = d0 * c - .5 * d1 * c**2 - p * c
    p_surplus = p * c - s0 * c - .5 * s1 * c**2

    return c_surplus, p_surplus

```

Then define a function that plots demand and supply curves and labels surpluses and equilibrium.

### Example: single agent with one good and production

Now let's construct an example of a production economy with one good.

To do this we

- specify a single **person** and a **cost curve** in a way that let's us replicate the simple single-good supply demand example with which we started
- compute equilibrium  $p$  and  $c$  and consumer and producer surpluses
- draw graphs of both surpluses
- do experiments in which we shift  $b$  and watch what happens to  $p, c$ .

```

Π = np.array([[1]])  # the matrix now is a singleton
b = np.array([10])
h = np.array([0.5])
J = np.array([[1]])
μ = 1

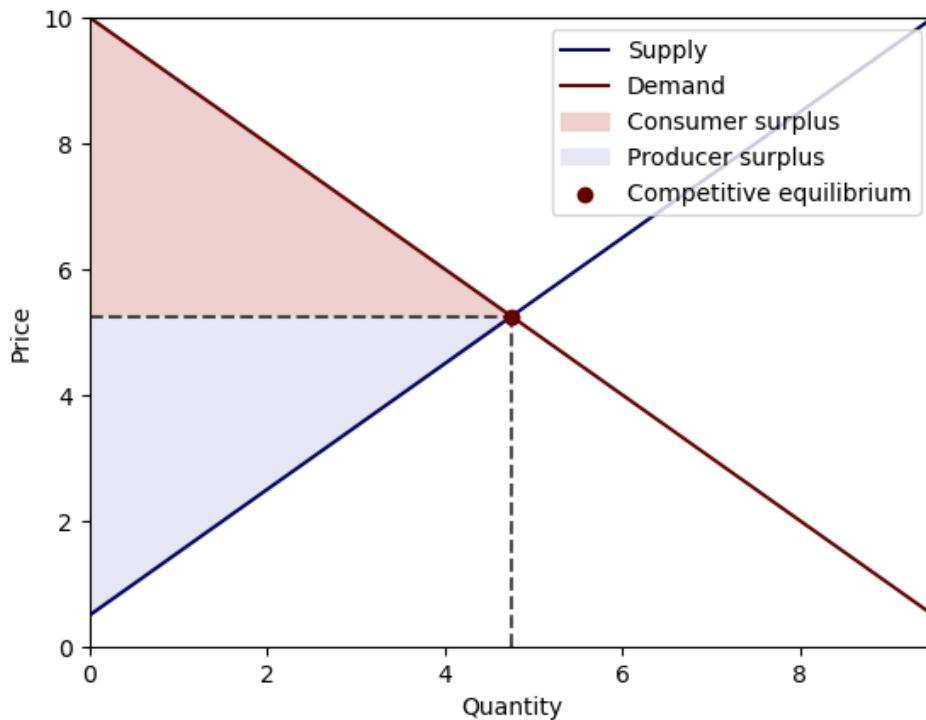
PE = ProductionEconomy(Π, b, h, J, μ)
c, p = PE.competitive_equilibrium()

print('Competitive equilibrium price:', p.item())
print('Competitive equilibrium allocation:', c.item())

# plot
plot_competitive_equilibrium(PE)

```

```
Competitive equilibrium price: 5.25
Competitive equilibrium allocation: 4.75
```



```
c_surplus, p_surplus = PE.compute_surplus()

print('Consumer surplus:', c_surplus.item())
print('Producer surplus:', p_surplus.item())
```

```
Consumer surplus: 11.28125
Producer surplus: 11.28125
```

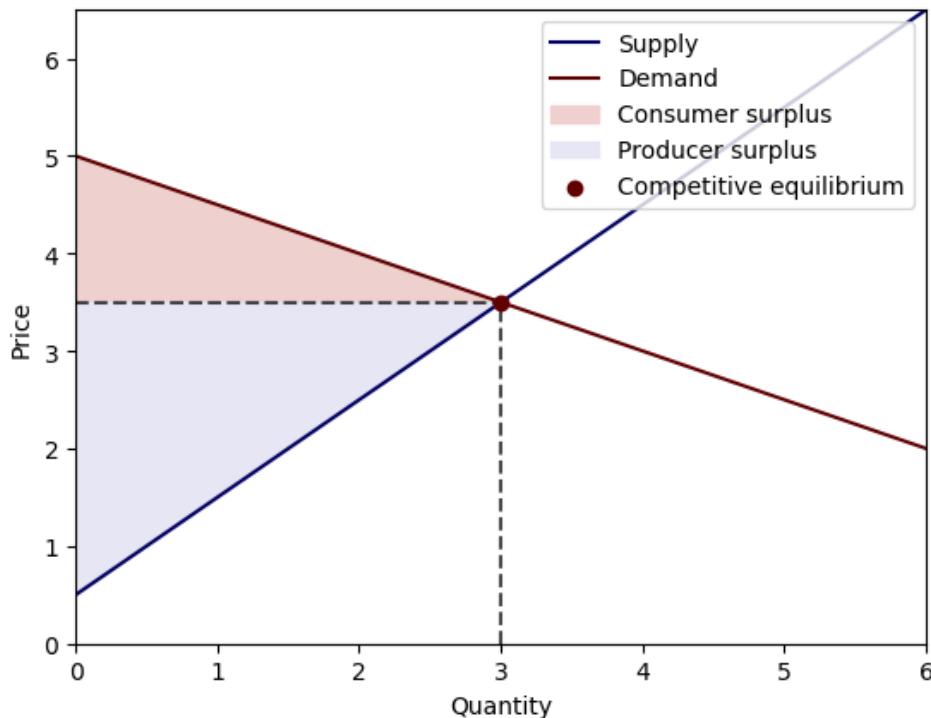
Let's give the consumer a lower welfare weight by raising  $\mu$ .

```
PE.mu = 2
c, p = PE.competitive_equilibrium()

print('Competitive equilibrium price:', p.item())
print('Competitive equilibrium allocation:', c.item())

# plot
plot_competitive_equilibrium(PE)
```

```
Competitive equilibrium price: 3.5
Competitive equilibrium allocation: 3.0
```



```
c_surplus, p_surplus = PE.compute_surplus()

print('Consumer surplus:', c_surplus.item())
print('Producer surplus:', p_surplus.item())
```

```
Consumer surplus: 2.25
Producer surplus: 4.5
```

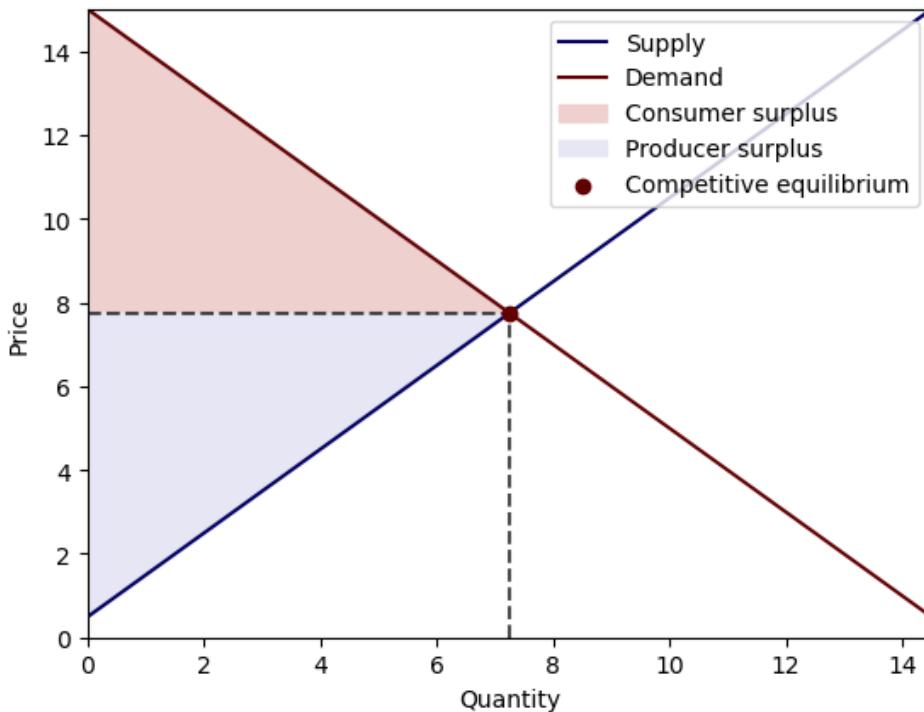
Now we change the bliss point so that the consumer derives more utility from consumption.

```
PE.u = 1
PE.b = PE.b * 1.5
c, p = PE.competitive_equilibrium()

print('Competitive equilibrium price:', p.item())
print('Competitive equilibrium allocation:', c.item())

# plot
plot_competitive_equilibrium(PE)
```

```
Competitive equilibrium price: 7.75
Competitive equilibrium allocation: 7.25
```



This raises both the equilibrium price and quantity.

### Example: single agent two-good economy with production

- we'll do some experiments like those above
- we can do experiments with a **diagonal**  $\Pi$  and also with a **non-diagonal**  $\Pi$  matrices to study how cross-slopes affect responses of  $p$  and  $c$  to various shifts in  $b$  (TODO)

```

Pi = np.array([[1, 0],
              [0, 1]])

b = np.array([10, 10])

h = np.array([0.5, 0.5])

J = np.array([[1, 0.5],
              [0.5, 1]])

mu = 1

PE = ProductionEconomy(Pi, b, h, J, mu)
c, p = PE.competitive_equilibrium()

print('Competitive equilibrium price:', p)
print('Competitive equilibrium allocation:', c)

```

```

Competitive equilibrium price: [6.2 6.2]
Competitive equilibrium allocation: [3.8 3.8]

```

```
PE.b = np.array([12, 10])

c, p = PE.competitive_equilibrium()

print('Competitive equilibrium price:', p)
print('Competitive equilibrium allocation:', c)
```

```
Competitive equilibrium price: [7.13333333 6.46666667]
Competitive equilibrium allocation: [4.86666667 3.53333333]
```

```
PE.Pi = np.array([[1, 0.5],
                  [0.5, 1]])

PE.b = np.array([10, 10])

c, p = PE.competitive_equilibrium()

print('Competitive equilibrium price:', p)
print('Competitive equilibrium allocation:', c)
```

```
Competitive equilibrium price: [6.3 6.3]
Competitive equilibrium allocation: [3.86666667 3.86666667]
```

```
PE.b = np.array([12, 10])

c, p = PE.competitive_equilibrium()

print('Competitive equilibrium price:', p)
print('Competitive equilibrium allocation:', c)
```

```
Competitive equilibrium price: [7.23333333 6.56666667]
Competitive equilibrium allocation: [4.93333333 3.6]
```

### 34.7.4 Digression: a supplier who is a monopolist

A competitive firm is a **price-taker** who regards the price and therefore its marginal revenue as being beyond its control.

A monopolist knows that it has no competition and can influence the price and its marginal revenue by setting quantity.

A monopolist takes a **demand curve** and not the **price** as beyond its control.

Thus, instead of being a price-taker, a monopolist sets prices to maximize profits subject to the inverse demand curve (34.9).

So the monopolist's total profits as a function of its output  $q$  is

$$[\mu^{-1}\Pi^\top(b - \Pi q)]^\top q - h^\top q - \frac{1}{2}q^\top J q \quad (34.11)$$

After finding first-order necessary conditions for maximizing monopoly profits with respect to  $q$  and solving them for  $q$ , we find that the monopolist sets

$$q = (H + 2\mu^{-1}\Pi^\top\Pi)^{-1}(\mu^{-1}\Pi^\top b - h) \quad (34.12)$$

We'll soon see that a monopolist sets a **lower output**  $q$  than does either a

- planner who chooses  $q$  to maximize social welfare
- a competitive equilibrium

---

**Exercise 34.7.1**

Please verify the monopolist's supply curve (34.12).

---

### 34.7.5 A monopolist

Let's consider a monopolist supplier.

We have included a method in our `ProductionEconomy` class to compute an equilibrium price and allocation when the supplier is a monopolist.

Since the supplier now has the price-setting power

- we first compute the optimal quantity that solves the monopolist's profit maximization problem.
- Then we back out an equilibrium price from the consumer's inverse demand curve.

Next, we use a graph for the single good case to illustrate the difference between a competitive equilibrium and an equilibrium with a monopolist supplier.

Recall that in a competitive equilibrium, a price-taking supplier equates marginal revenue  $p$  to marginal cost  $h + Hq$ .

This yields a competitive producer's inverse supply curve.

A monopolist's marginal revenue is not constant but instead is a non-trivial function of the quantity it sets.

The monopolist's marginal revenue is

$$MR(q) = -2\mu^{-1}\Pi^\top\Pi q + \mu^{-1}\Pi^\top b,$$

which the monopolist equates to its marginal cost.

The plot indicates that the monopolist's sets output lower than either the competitive equilibrium quantity.

In a single good case, this equilibrium is associated with a higher price of the good.

```
class Monopoly(ProductionEconomy):
    def __init__(self,
                 Π,
                 b,
                 h,
                 J,
                 μ):
        """
        Inherit all properties and methods from class ProductionEconomy
        """
        super().__init__(Π, b, h, J, μ)

    def equilibrium_with_monopoly(self):
        """
        Compute the equilibrium price and allocation when there is a monopolist
        supplier
        """

```

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```

Π, b, h, μ, J = self.Π, self.b, self.h, self.μ, self.J
H = .5 * (J + J.T)

# allocation
q = inv(μ * H + 2 * Π.T @ Π) @ (Π.T @ b - μ * h)

# price
p = 1 / μ * (Π.T @ b - Π.T @ Π @ q)

if any(Π @ q - b >= 0):
    raise Exception('invalid result: set bliss points further away')

return q, p

```

Define a function that plots the demand, marginal cost and marginal revenue curves with surpluses and equilibrium labelled.

### A multiple good example

Let's compare competitive equilibrium and monopoly outcomes in a multiple goods economy.

```

Π = np.array([[1, 0],
              [0, 1.2]])

b = np.array([10, 10])

h = np.array([0.5, 0.5])

J = np.array([[1, 0.5],
              [0.5, 1]])

μ = 1

M = Monopoly(Π, b, h, J, μ)
c, p = M.competitive_equilibrium()
q, pm = M.equilibrium_with_monopoly()

print('Competitive equilibrium price:', p)
print('Competitive equilibrium allocation:', c)

print('Equilibrium with monopolist supplier price:', pm)
print('Equilibrium with monopolist supplier allocation:', q)

```

```

Competitive equilibrium price: [6.23542117 6.32397408]
Competitive equilibrium allocation: [3.76457883 3.94168467]
Equilibrium with monopolist supplier price: [7.26865672 8.23880597]
Equilibrium with monopolist supplier allocation: [2.73134328 2.6119403 ]

```

### A single-good example

```

Pi = np.array([[1]]) # the matrix now is a singleton
b = np.array([10])
h = np.array([0.5])
J = np.array([[1]])
mu = 1

M = Monopoly(Pi, b, h, J, mu)
c, p = M.competitive_equilibrium()
q, pm = M.equilibrium_with_monopoly()

print('Competitive equilibrium price:', p.item())
print('Competitive equilibrium allocation:', c.item())

print('Equilibrium with monopolist supplier price:', pm.item())
print('Equilibrium with monopolist supplier allocation:', q.item())

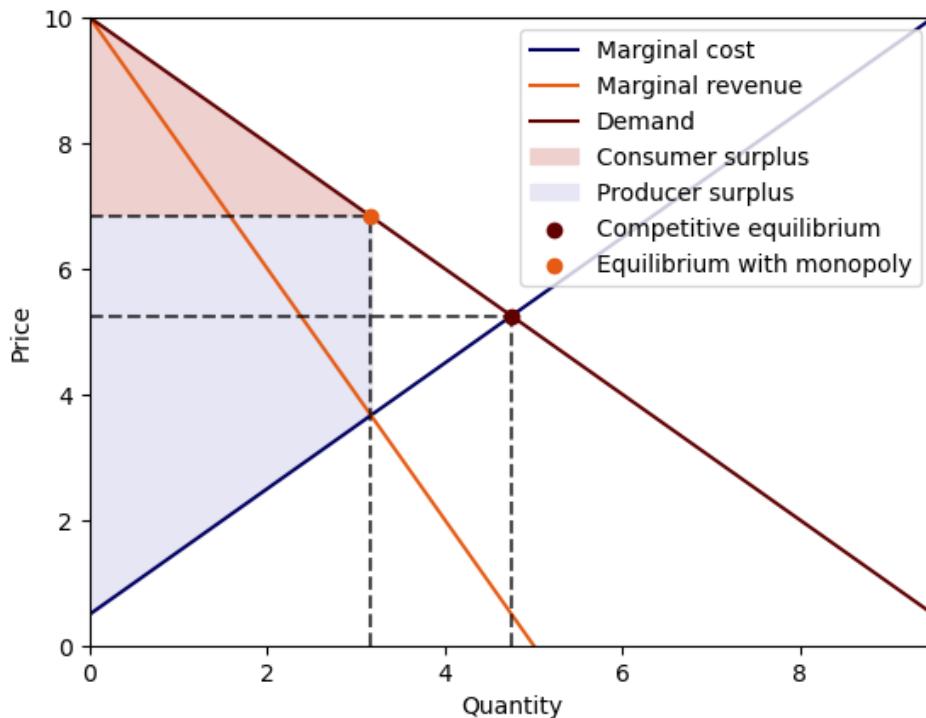
# plot
plot_monopoly(M)

```

```

Competitive equilibrium price: 5.25
Competitive equilibrium allocation: 4.75
Equilibrium with monopolist supplier price: 6.833333333333334
Equilibrium with monopolist supplier allocation: 3.1666666666666665

```



## 34.8 Multi-good welfare maximization problem

Our welfare maximization problem – also sometimes called a social planning problem – is to choose  $c$  to maximize

$$-\frac{1}{2}\mu^{-1}(\Pi c - b)^\top(\Pi c - b)$$

minus the area under the inverse supply curve, namely,

$$hc + \frac{1}{2}c^\top Jc$$

So the welfare criterion is

$$-\frac{1}{2}\mu^{-1}(\Pi c - b)^\top(\Pi c - b) - hc - \frac{1}{2}c^\top Jc$$

In this formulation,  $\mu$  is a parameter that describes how the planner weighs interests of outside suppliers and our representative consumer.

The first-order condition with respect to  $c$  is

$$-\mu^{-1}\Pi^\top\Pi c + \mu^{-1}\Pi^\top b - h - Hc = 0$$

which implies (34.10).

Thus, as for the single-good case, with multiple goods a competitive equilibrium quantity vector solves a planning problem. (This is another version of the first welfare theorem.)

We can deduce a competitive equilibrium price vector from either

- the inverse demand curve, or
- the inverse supply curve

## MARKET EQUILIBRIUM WITH HETEROGENEITY

### Contents

- *Market Equilibrium with Heterogeneity*
  - *Overview*
  - *An simple example*
  - *Pure exchange economy*
  - *Implementation*
  - *Deducing a representative consumer*

### 35.1 Overview

In the *previous lecture*, we studied competitive equilibria in an economy with many goods.

While the results of the study were informative, we used a strong simplifying assumption: all of the agents in the economy are identical.

In the real world, households, firms and other economic agents differ from one another along many dimensions.

In this lecture, we introduce heterogeneity across consumers by allowing their preferences and endowments to differ.

We will examine competitive equilibrium in this setting.

We will also show how a “representative consumer” can be constructed.

Here are some imports:

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from scipy.linalg import inv
```

## 35.2 An simple example

Let's study a simple example of **pure exchange** economy without production.

There are two consumers who differ in their endowment vectors  $e_i$  and their bliss-point vectors  $b_i$  for  $i = 1, 2$ .

The total endowment is  $e_1 + e_2$ .

A competitive equilibrium requires that

$$c_1 + c_2 = e_1 + e_2$$

Assume the demand curves

$$c_i = (\Pi^\top \Pi)^{-1} (\Pi^\top b_i - \mu_i p)$$

Competitive equilibrium then requires that

$$e_1 + e_2 = (\Pi^\top \Pi)^{-1} (\Pi^\top (b_1 + b_2) - (\mu_1 + \mu_2) p)$$

which, after a line or two of linear algebra, implies that

$$(\mu_1 + \mu_2) p = \Pi^\top (b_1 + b_2) - \Pi^\top \Pi (e_1 + e_2) \quad (35.1)$$

We can normalize prices by setting  $\mu_1 + \mu_2 = 1$  and then solving

$$\mu_i(p, e) = \frac{p^\top (\Pi^{-1} b_i - e_i)}{p^\top (\Pi^\top \Pi)^{-1} p} \quad (35.2)$$

for  $\mu_i, i = 1, 2$ .

---

### Exercise 35.2.1

Show that, up to normalization by a positive scalar, the same competitive equilibrium price vector that you computed in the preceding two-consumer economy would prevail in a single-consumer economy in which a single **representative consumer** has utility function

$$-.5(\Pi c - b)^\top (\Pi c - b)$$

and endowment vector  $e$ , where

$$b = b_1 + b_2$$

and

$$e = e_1 + e_2.$$


---

## 35.3 Pure exchange economy

Let's further explore a pure exchange economy with  $n$  goods and  $m$  people.

### 35.3.1 Competitive equilibrium

We'll compute a competitive equilibrium.

To compute a competitive equilibrium of a pure exchange economy, we use the fact that

- Relative prices in a competitive equilibrium are the same as those in a special single person or representative consumer economy with preference  $\Pi$  and  $b = \sum_i b_i$ , and endowment  $e = \sum_i e_i$ .

We can use the following steps to compute a competitive equilibrium:

- First we solve the single representative consumer economy by normalizing  $\mu = 1$ . Then, we renormalize the price vector by using the first consumption good as a numeraire.
- Next we use the competitive equilibrium prices to compute each consumer's marginal utility of wealth:

$$\mu_i = \frac{-W_i + p^\top (\Pi^{-1} b_i - e_i)}{p^\top (\Pi^\top \Pi)^{-1} p}$$

- Finally we compute a competitive equilibrium allocation by using the demand curves:

$$c_i = \Pi^{-1} b_i - (\Pi^\top \Pi)^{-1} \mu_i p$$

### 35.3.2 Designing some Python code

Below we shall construct a Python class with the following attributes:

- **Preferences** in the form of
  - an  $n \times n$  positive definite matrix  $\Pi$
  - an  $n \times 1$  vector of bliss points  $b$
- **Endowments** in the form of
  - an  $n \times 1$  vector  $e$
  - a scalar “wealth”  $W$  with default value 0

The class will include a test to make sure that  $b \gg \Pi e$  and raise an exception if it is violated (at some threshold level we'd have to specify).

- **A Person** in the form of a pair that consists of
  - **Preferences** and **Endowments**
- **A Pure Exchange Economy** will consist of
  - a collection of  $m$  **persons**
    - \*  $m = 1$  for our single-agent economy
    - \*  $m = 2$  for our illustrations of a pure exchange economy
  - an equilibrium price vector  $p$  (normalized somehow)
  - an equilibrium allocation  $c_1, c_2, \dots, c_m$  – a collection of  $m$  vectors of dimension  $n \times 1$

Now let's proceed to code.

```
class ExchangeEconomy:
    def __init__(self,
                 Pi,
                 bs,
```

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```

        es,
        Ws=None,
        thres=1.5):
"""
Set up the environment for an exchange economy

Args:
    Π (np.array): shared matrix of substitution
    bs (list): all consumers' bliss points
    es (list): all consumers' endowments
    Ws (list): all consumers' wealth
    thres (float): a threshold set to test  $b >> \Pi e$  violated
"""
n, m = Π.shape[0], len(bs)

# check non-satiation
for b, e in zip(bs, es):
    if np.min(b / np.max(Π @ e)) <= thres:
        raise Exception('set bliss points further away')

if Ws == None:
    Ws = np.zeros(m)
else:
    if sum(Ws) != 0:
        raise Exception('invalid wealth distribution')

self.Π, self.bs, self.es, self.Ws, self.n, self.m = Π, bs, es, Ws, n, m

def competitive_equilibrium(self):
"""
Compute the competitive equilibrium prices and allocation
"""
Π, bs, es, Ws = self.Π, self.bs, self.es, self.Ws
n, m = self.n, self.m
slope_dc = inv(Π.T @ Π)
Π_inv = inv(Π)

# aggregate
b = sum(bs)
e = sum(es)

# compute price vector with mu=1 and renormalize
p = Π.T @ b - Π.T @ Π @ e
p = p / p[0]

# compute marginal utility of wealth
μ_s = []
c_s = []
A = p.T @ slope_dc @ p

for i in range(m):
    μ_i = (-Ws[i] + p.T @ (Π_inv @ bs[i] - es[i])) / A
    c_i = Π_inv @ bs[i] - μ_i * slope_dc @ p
    μ_s.append(μ_i)
    c_s.append(c_i)

```

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```

for c_i in c_s:
    if any(c_i < 0):
        print('allocation: ', c_s)
        raise Exception('negative allocation: equilibrium does not exist')

return p, c_s, mu_s

```

## 35.4 Implementation

Next we use the class `ExchangeEconomy` defined above to study

- a two-person economy without production,
- a dynamic economy, and
- an economy with risk and arrow securities.

### 35.4.1 Two-person economy without production

Here we study how competitive equilibrium  $p, c_1, c_2$  respond to different  $b_i$  and  $e_i, i \in \{1, 2\}$ .

```

Pi = np.array([[1, 0],
              [0, 1]])

bs = [np.array([5, 5]),  # first consumer's bliss points
      np.array([5, 5])]  # second consumer's bliss points

es = [np.array([0, 2]),  # first consumer's endowment
      np.array([2, 0])]  # second consumer's endowment

EE = ExchangeEconomy(Pi, bs, es)
p, c_s, mu_s = EE.competitive_equilibrium()

print('Competitive equilibrium price vector:', p)
print('Competitive equilibrium allocation:', c_s)

```

```

Competitive equilibrium price vector: [1. 1.]
Competitive equilibrium allocation: [array([1., 1.]), array([1., 1.])]

```

What happens if the first consumer likes the first good more and the second consumer likes the second good more?

```

EE.bs = [np.array([6, 5]),  # first consumer's bliss points
         np.array([5, 6])]  # second consumer's bliss points

p, c_s, mu_s = EE.competitive_equilibrium()

print('Competitive equilibrium price vector:', p)
print('Competitive equilibrium allocation:', c_s)

```

```

Competitive equilibrium price vector: [1. 1.]
Competitive equilibrium allocation: [array([1.5, 0.5]), array([0.5, 1.5])]

```

Let the first consumer be poorer.

```
EE.es = [np.array([0.5, 0.5]), # first consumer's endowment
         np.array([1, 1])] # second consumer's endowment

p, c_s, mu_s = EE.competitive_equilibrium()

print('Competitive equilibrium price vector:', p)
print('Competitive equilibrium allocation:', c_s)
```

```
Competitive equilibrium price vector: [1. 1.]
Competitive equilibrium allocation: [array([1., 0.]), array([0.5, 1.5])]
```

Now let's construct an autarky (i.e., no-trade) equilibrium.

```
EE.bs = [np.array([4, 6]), # first consumer's bliss points
         np.array([6, 4])] # second consumer's bliss points

EE.es = [np.array([0, 2]), # first consumer's endowment
         np.array([2, 0])] # second consumer's endowment

p, c_s, mu_s = EE.competitive_equilibrium()

print('Competitive equilibrium price vector:', p)
print('Competitive equilibrium allocation:', c_s)
```

```
Competitive equilibrium price vector: [1. 1.]
Competitive equilibrium allocation: [array([0., 2.]), array([2., 0.])]
```

Now let's redistribute endowments before trade.

```
bs = [np.array([5, 5]), # first consumer's bliss points
      np.array([5, 5])] # second consumer's bliss points

es = [np.array([1, 1]), # first consumer's endowment
      np.array([1, 1])] # second consumer's endowment

Ws = [0.5, -0.5]
EE_new = ExchangeEconomy(Pi, bs, es, Ws)
p, c_s, mu_s = EE_new.competitive_equilibrium()

print('Competitive equilibrium price vector:', p)
print('Competitive equilibrium allocation:', c_s)
```

```
Competitive equilibrium price vector: [1. 1.]
Competitive equilibrium allocation: [array([1.25, 1.25]), array([0.75, 0.75])]
```

### 35.4.2 A dynamic economy

Now let's use the tricks described above to study a dynamic economy, one with two periods.

```
beta = 0.95

Pi = np.array([[1, 0],
              [0, np.sqrt(beta)]])

bs = [np.array([5, np.sqrt(beta) * 5])]

es = [np.array([1, 1])]

EE_DE = ExchangeEconomy(Pi, bs, es)
p, c_s, mu_s = EE_DE.competitive_equilibrium()

print('Competitive equilibrium price vector:', p)
print('Competitive equilibrium allocation:', c_s)
```

```
Competitive equilibrium price vector: [1. 0.95]
Competitive equilibrium allocation: [array([1., 1.])]
```

### 35.4.3 Risk economy with arrow securities

We use the tricks described above to interpret  $c_1, c_2$  as “Arrow securities” that are state-contingent claims to consumption goods.

```
prob = 0.7

Pi = np.array([[np.sqrt(prob), 0],
              [0, np.sqrt(1 - prob)]])

bs = [np.array([np.sqrt(prob) * 5, np.sqrt(1 - prob) * 5]),
      np.array([np.sqrt(prob) * 5, np.sqrt(1 - prob) * 5])]

es = [np.array([1, 0]),
      np.array([0, 1])]

EE_AS = ExchangeEconomy(Pi, bs, es)
p, c_s, mu_s = EE_AS.competitive_equilibrium()

print('Competitive equilibrium price vector:', p)
print('Competitive equilibrium allocation:', c_s)
```

```
Competitive equilibrium price vector: [1. 0.42857143]
Competitive equilibrium allocation: [array([0.7, 0.7]), array([0.3, 0.3])]
```

## 35.5 Deducing a representative consumer

In the class of multiple consumer economies that we are studying here, it turns out that there exists a single **representative consumer** whose preferences and endowments can be deduced from lists of preferences and endowments for separate individual consumers.

Consider a multiple consumer economy with initial distribution of wealth  $W_i$  satisfying  $\sum_i W_i = 0$

We allow an initial redistribution of wealth.

We have the following objects

- The demand curve:

$$c_i = \Pi^{-1}b_i - (\Pi^\top \Pi)^{-1}\mu_i p$$

- The marginal utility of wealth:

$$\mu_i = \frac{-W_i + p^\top (\Pi^{-1}b_i - e_i)}{p^\top (\Pi^\top \Pi)^{-1}p}$$

- Market clearing:

$$\sum c_i = \sum e_i$$

Denote aggregate consumption  $\sum_i c_i = c$  and  $\sum_i \mu_i = \mu$ .

Market clearing requires

$$\Pi^{-1} \left( \sum_i b_i \right) - (\Pi^\top \Pi)^{-1} p \left( \sum_i \mu_i \right) = \sum_i e_i$$

which, after a few steps, leads to

$$p = \mu^{-1} (\Pi^\top b - \Pi^\top \Pi e)$$

where

$$\mu = \sum_i \mu_i = \frac{0 + p^\top (\Pi^{-1}b - e)}{p^\top (\Pi^\top \Pi)^{-1}p}.$$

Now consider the representative consumer economy specified above.

Denote the marginal utility of wealth of the representative consumer by  $\tilde{\mu}$ .

The demand function is

$$c = \Pi^{-1}b - (\Pi^\top \Pi)^{-1}\tilde{\mu}p$$

Substituting this into the budget constraint gives

$$\tilde{\mu} = \frac{p^\top (\Pi^{-1}b - e)}{p^\top (\Pi^\top \Pi)^{-1}p}$$

In an equilibrium  $c = e$ , so

$$p = \tilde{\mu}^{-1} (\Pi^\top b - \Pi^\top \Pi e)$$

Thus, we have verified that, up to the choice of a numeraire in which to express absolute prices, the price vector in our representative consumer economy is the same as that in an underlying economy with multiple consumers.

## **Part XI**

# **Estimation**



## SIMPLE LINEAR REGRESSION MODEL

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
```

The simple regression model estimates the relationship between two variables  $x_i$  and  $y_i$

$$y_i = \alpha + \beta x_i + \epsilon_i, i = 1, 2, \dots, N$$

where  $\epsilon_i$  represents the error between the line of best fit and the sample values for  $y_i$  given  $x_i$ .

Our goal is to choose values for  $\alpha$  and  $\beta$  to build a line of “best” fit for some data that is available for variables  $x_i$  and  $y_i$ .

Let us consider a simple dataset of 10 observations for variables  $x_i$  and  $y_i$ :

	$y_i$	$x_i$
1	2000	32
2	1000	21
3	1500	24
4	2500	35
5	500	10
6	900	11
7	1100	22
8	1500	21
9	1800	27
10	250	2

Let us think about  $y_i$  as sales for an ice-cream cart, while  $x_i$  is a variable that records the day’s temperature in Celsius.

```
x = [32, 21, 24, 35, 10, 11, 22, 21, 27, 2]
y = [2000, 1000, 1500, 2500, 500, 900, 1100, 1500, 1800, 250]
df = pd.DataFrame([x,y]).T
df.columns = ['X', 'Y']
df
```

	X	Y
0	32	2000
1	21	1000
2	24	1500
3	35	2500

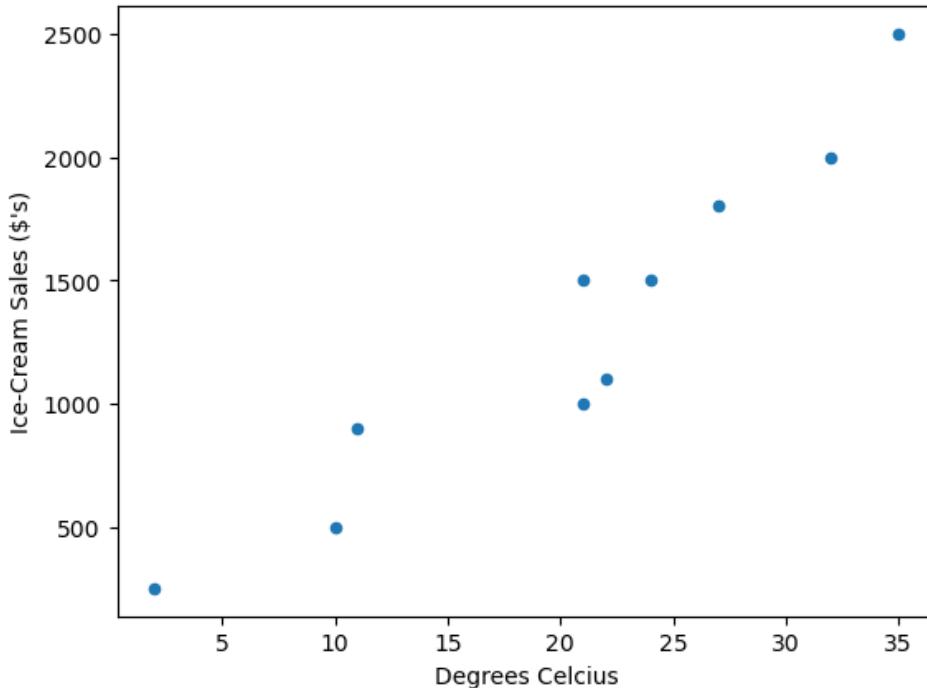
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4	10	500
5	11	900
6	22	1100
7	21	1500
8	27	1800
9	2	250

We can use a scatter plot of the data to see the relationship between  $y_i$  (ice-cream sales in dollars (\$'s)) and  $x_i$  (degrees Celsius).

```
ax = df.plot(
    x='X',
    y='Y',
    kind='scatter',
    ylabel='Ice-Cream Sales ($\'s)',
    xlabel='Degrees Celcius'
)
```



as you can see the data suggests that more ice-cream is typically sold on hotter days.

To build a linear model of the data we need to choose values for  $\alpha$  and  $\beta$  that represents a line of “best” fit such that

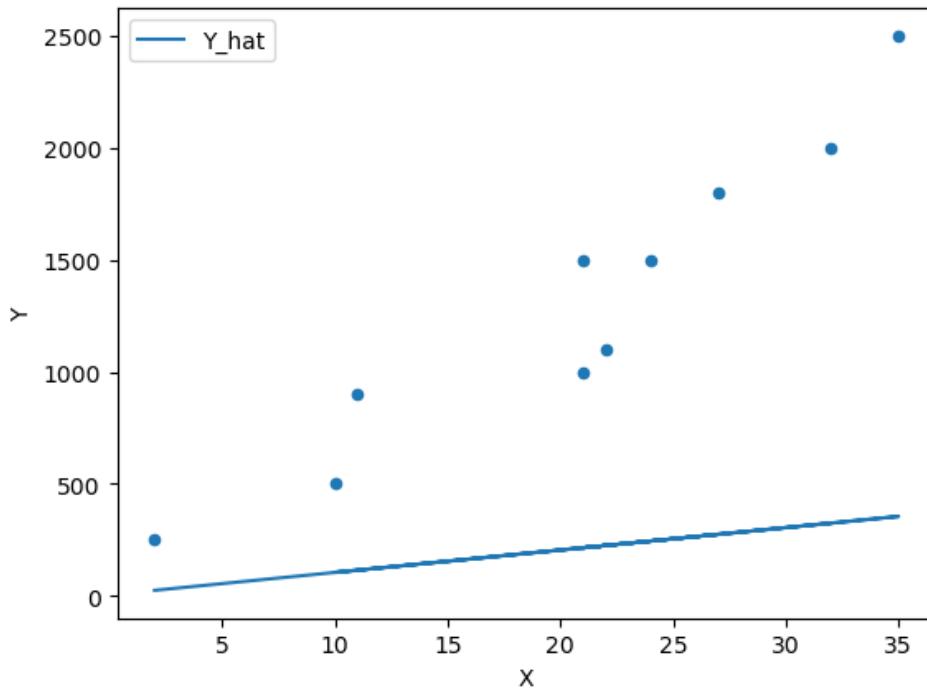
$$\hat{y}_i = \hat{\alpha} + \hat{\beta}x_i$$

Let's start with  $\alpha = 5$  and  $\beta = 10$

```
α = 5
β = 10
df['Y_hat'] = α + β * df['X']
```

```
fig, ax = plt.subplots()
df.plot(x='X', y='Y', kind='scatter', ax=ax)
df.plot(x='X', y='Y_hat', kind='line', ax=ax)
```

```
<Axes: xlabel='X', ylabel='Y'>
```



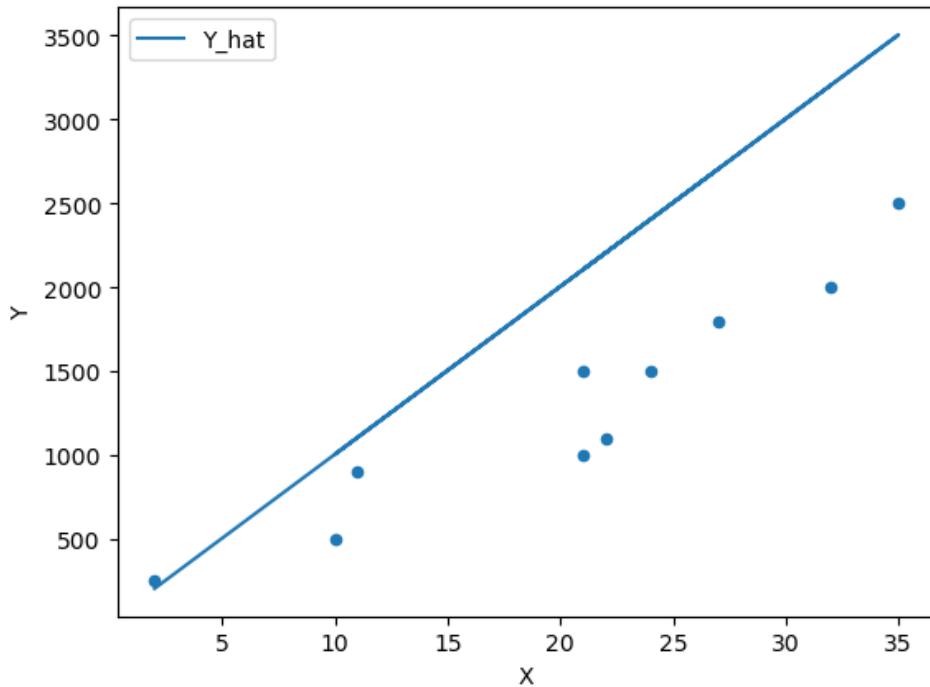
We can see that this model does a poor job of estimating the relationship.

We can continue to guess and iterate towards a line of “best” fit by adjusting the parameters

```
β = 100
df['Y_hat'] = α + β * df['X']
```

```
fig, ax = plt.subplots()
df.plot(x='X', y='Y', kind='scatter', ax=ax)
df.plot(x='X', y='Y_hat', kind='line', ax=ax)
```

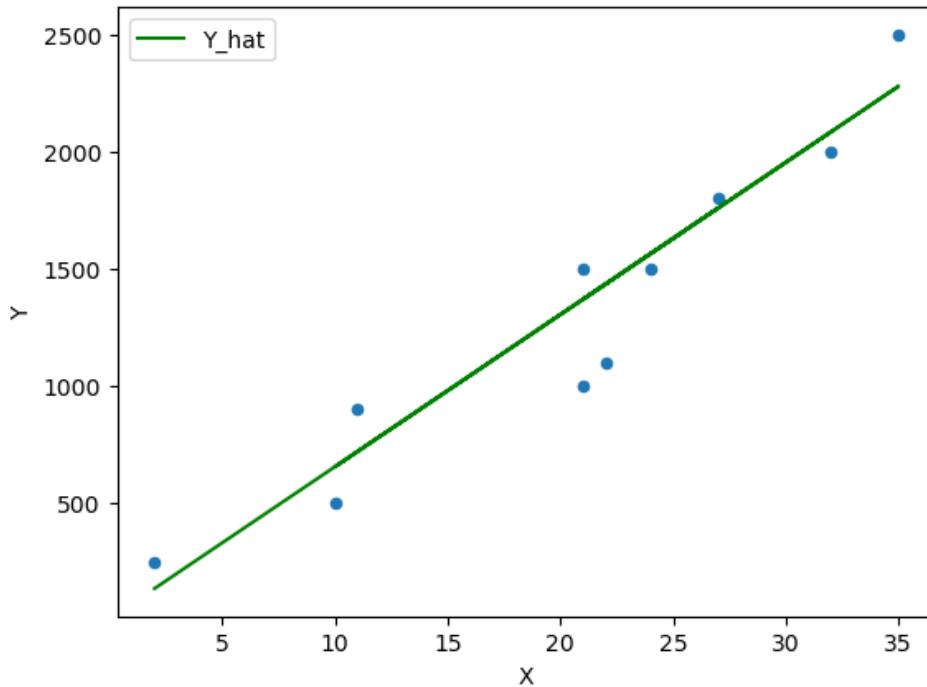
```
<Axes: xlabel='X', ylabel='Y'>
```



```
β = 65
df['Y_hat'] = α + β * df['X']
```

```
fig, ax = plt.subplots()
df.plot(x='X', y='Y', kind='scatter', ax=ax)
df.plot(x='X', y='Y_hat', kind='line', ax=ax, color='g')
```

```
<Axes: xlabel='X', ylabel='Y'>
```



However we need to think about formalising this guessing process by thinking of this problem as an optimization problem.

Let's consider the error  $\epsilon_i$  and define the difference between the observed values  $y_i$  and the estimated values  $\hat{y}_i$  which we will call the residuals

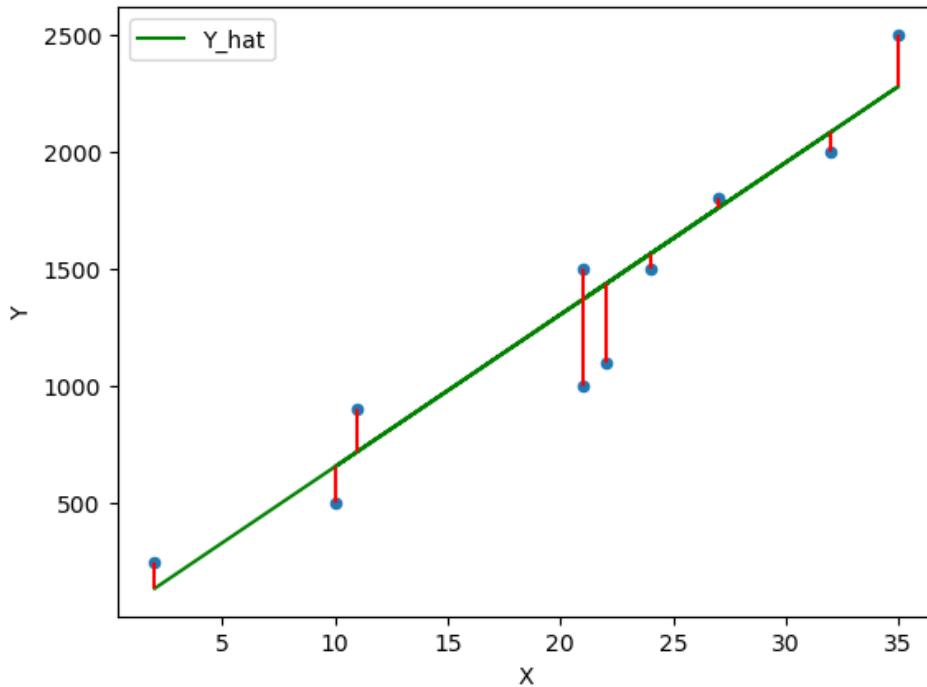
$$\begin{aligned}\hat{e}_i &= y_i - \hat{y}_i \\ &= y_i - \hat{\alpha} - \hat{\beta}x_i\end{aligned}$$

```
df['error'] = df['Y_hat'] - df['Y']
```

```
df
```

	X	Y	Y_hat	error
0	32	2000	2085	85
1	21	1000	1370	370
2	24	1500	1565	65
3	35	2500	2280	-220
4	10	500	655	155
5	11	900	720	-180
6	22	1100	1435	335
7	21	1500	1370	-130
8	27	1800	1760	-40
9	2	250	135	-115

```
fig, ax = plt.subplots()
df.plot(x='X', y='Y', kind='scatter', ax=ax)
df.plot(x='X', y='Y_hat', kind='line', ax=ax, color='g')
plt.vlines(df['X'], df['Y_hat'], df['Y'], color='r');
```



The Ordinary Least Squares (OLS) method, as the name suggests, chooses  $\alpha$  and  $\beta$  in such a way that **minimises** the Sum of the Squared Residuals (SSR).

$$\min_{\alpha, \beta} \sum_{i=1}^N \hat{e}_i^2 = \min_{\alpha, \beta} \sum_{i=1}^N (y_i - \alpha - \beta x_i)^2$$

Let's call this a cost function

$$C = \sum_{i=1}^N (y_i - \alpha - \beta x_i)^2$$

that we would like to minimise with parameters  $\alpha$  and  $\beta$ .

## 36.1 How does error change with respect to $\alpha$ and $\beta$

Let us first look at how the total error changes with respect to  $\beta$  (holding the intercept  $\alpha$  constant)

We know from *the next section* the optimal values for  $\alpha$  and  $\beta$  are:

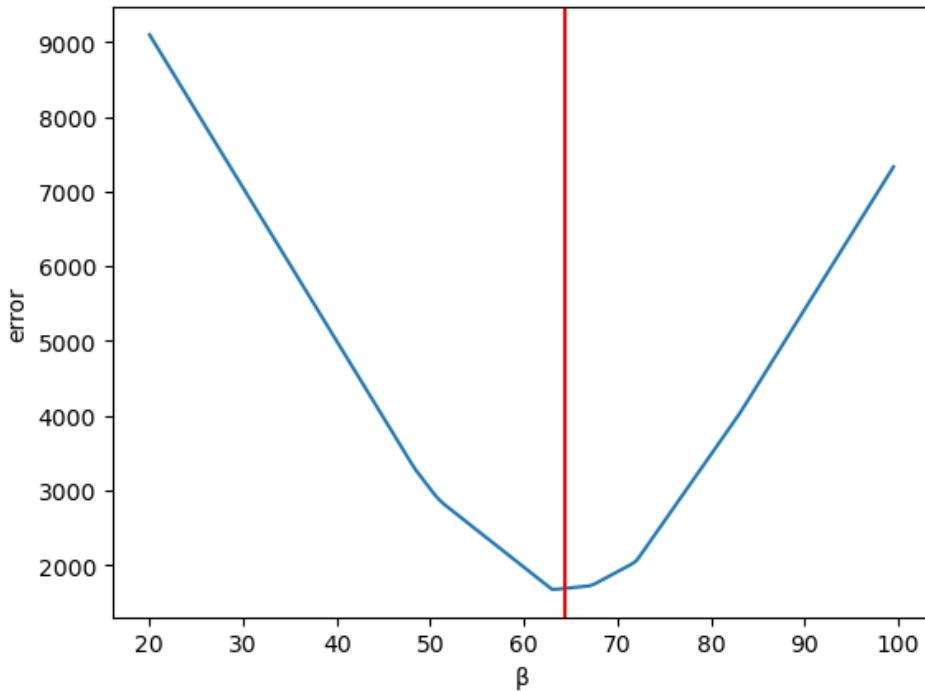
```
beta_optimal = 64.38
alpha_optimal = -14.72
```

We can then calculate the error for a range of  $\beta$  values

```
errors = []
for beta in np.arange(20, 100, 0.5):
    errors[beta] = abs((alpha_optimal + beta * df['X']) - df['Y']).sum()
```

Plotting the error

```
ax = pd.Series(errors).plot(xlabel='β', ylabel='error')
plt.axvline(β_optimal, color='r');
```

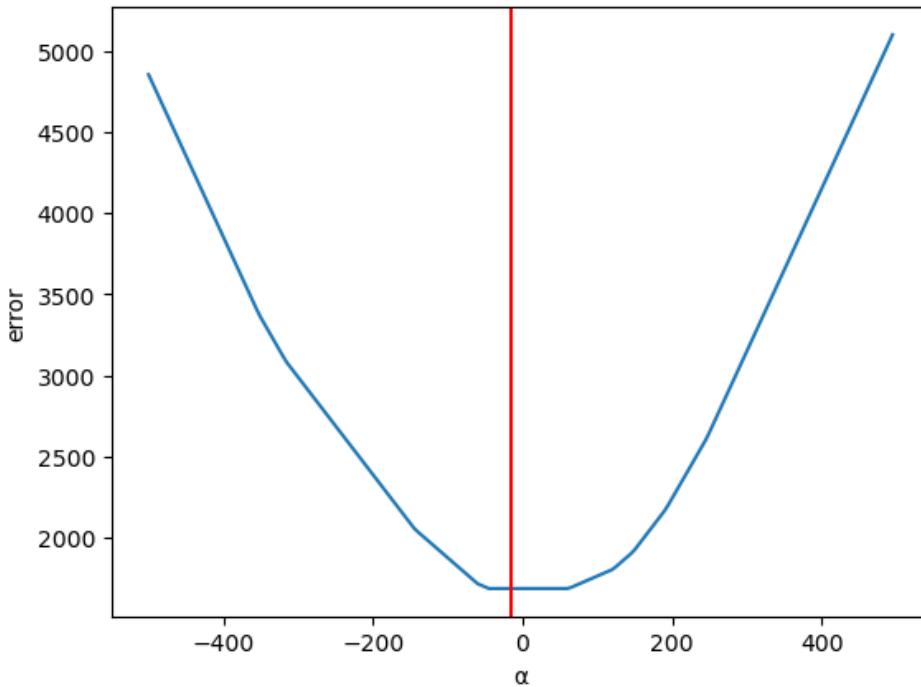


Now let us vary  $\alpha$  (holding  $\beta$  constant)

```
errors = []
for a in np.arange(-500, 500, 5):
    errors[a] = abs((a + β_optimal * df['X']) - df['Y']).sum()
```

Plotting the error

```
ax = pd.Series(errors).plot(xlabel='α', ylabel='error')
plt.axvline(a_optimal, color='r');
```



## 36.2 Calculating optimal values

Now let us use calculus to solve the optimization problem and compute the optimal values for  $\alpha$  and  $\beta$  to find the ordinary least squares solution.

First taking the partial derivative with respect to  $\alpha$

$$\frac{\partial C}{\partial \alpha} \left[ \sum_{i=1}^N (y_i - \alpha - \beta x_i)^2 \right]$$

and setting it equal to 0

$$0 = \sum_{i=1}^N -2(y_i - \alpha - \beta x_i)$$

we can remove the constant  $-2$  from the summation by dividing both sides by  $-2$

$$0 = \sum_{i=1}^N (y_i - \alpha - \beta x_i)$$

Now we can split this equation up into the components

$$0 = \sum_{i=1}^N y_i - \sum_{i=1}^N \alpha - \beta \sum_{i=1}^N x_i$$

The middle term is a straight forward sum from  $i = 1, \dots, N$  by a constant  $\alpha$

$$0 = \sum_{i=1}^N y_i - N * \alpha - \beta \sum_{i=1}^N x_i$$

and rearranging terms

$$\alpha = \frac{\sum_{i=1}^N y_i - \beta \sum_{i=1}^N x_i}{N}$$

We observe that both fractions resolve to the means  $\bar{y}_i$  and  $\bar{x}_i$

$$\alpha = \bar{y}_i - \beta \bar{x}_i \quad (36.1)$$

Now let's take the partial derivative of the cost function  $C$  with respect to  $\beta$

$$\frac{\partial C}{\partial \beta} \left[ \sum_{i=1}^N (y_i - \alpha - \beta x_i)^2 \right]$$

and setting it equal to 0

$$0 = \sum_{i=1}^N -2x_i(y_i - \alpha - \beta x_i)$$

we can again take the constant outside of the summation and divide both sides by  $-2$

$$0 = \sum_{i=1}^N x_i(y_i - \alpha - \beta x_i)$$

which becomes

$$0 = \sum_{i=1}^N (x_i y_i - \alpha x_i - \beta x_i^2)$$

now substituting for  $\alpha$

$$0 = \sum_{i=1}^N (x_i y_i - (\bar{y}_i - \beta \bar{x}_i) x_i - \beta x_i^2)$$

and rearranging terms

$$0 = \sum_{i=1}^N (x_i y_i - \bar{y}_i x_i - \beta \bar{x}_i x_i - \beta x_i^2)$$

This can be split into two summations

$$0 = \sum_{i=1}^N (x_i y_i - \bar{y}_i x_i) + \beta \sum_{i=1}^N (\bar{x}_i x_i - x_i^2)$$

and solving for  $\beta$  yields

$$\beta = \frac{\sum_{i=1}^N (x_i y_i - \bar{y}_i x_i)}{\sum_{i=1}^N (x_i^2 - \bar{x}_i x_i)} \quad (36.2)$$

We can now use (36.1) and (36.2) to calculate the optimal values for  $\alpha$  and  $\beta$

Calculating  $\beta$

```
df = df[['X', 'Y']].copy() # Original Data

# Calculate the sample means
x_bar = df['X'].mean()
y_bar = df['Y'].mean()
```

Now computing across the 10 observations and then summing the numerator and denominator

```
# Compute the Sums
df['num'] = df['X'] * df['Y'] - y_bar * df['X']
df['den'] = pow(df['X'], 2) - x_bar * df['X']
β = df['num'].sum() / df['den'].sum()
print(β)
```

64.37665782493369

Calculating  $\alpha$ 

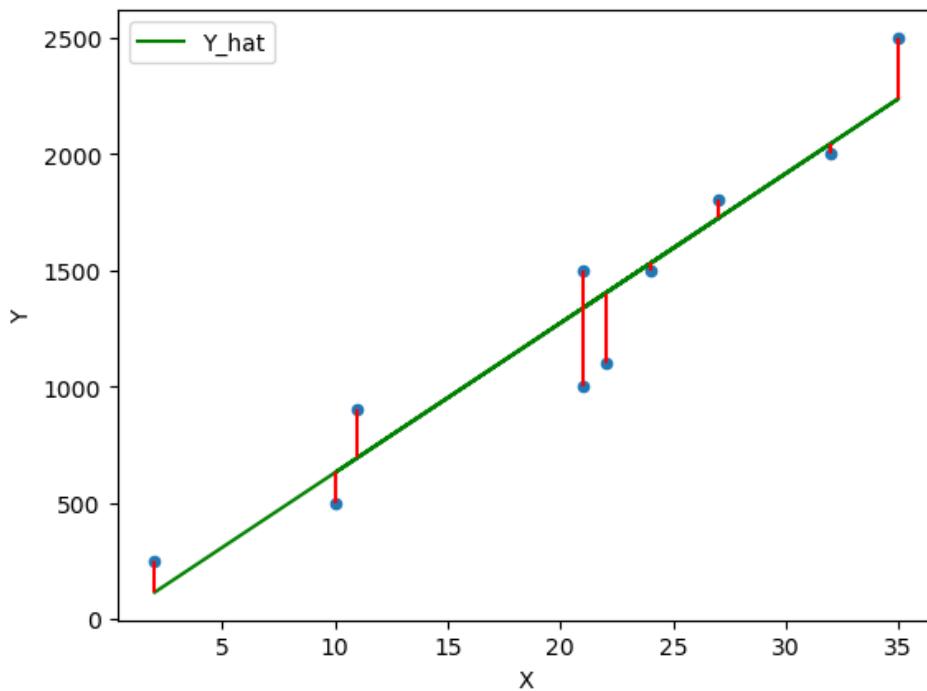
```
a = y_bar - β * x_bar
print(a)
```

-14.72148541114052

Now we can plot the OLS solution

```
df['Y_hat'] = a + β * df['X']
df['error'] = df['Y_hat'] - df['Y']

fig, ax = plt.subplots()
df.plot(x='X', y='Y', kind='scatter', ax=ax)
df.plot(x='X', y='Y_hat', kind='line', ax=ax, color='g')
plt.vlines(df['X'], df['Y_hat'], df['Y'], color='r');
```




---

### Why use OLS?

TODO

1. Discuss mathematical properties for why we have chosen OLS

---

**Exercise 36.2.1**

Now that you know the equations that solve the simple linear regression model using OLS you can now run your own regressions to build a model between  $y$  and  $x$ .

Let's consider two economic variables GDP per capita and Life Expectancy.

1. What do you think their relationship would be?
  2. Gather some data from our world in data
  3. Use pandas to import the csv formated data and plot a few different countries of interest
  4. Use (36.1) and (36.2) to compute optimal values for  $\alpha$  and  $\beta$
  5. Plot the line of best fit found using OLS
  6. Interpret the coefficients and write a summary sentence of the relationship between GDP per capita and Life Expectancy
- 

**Solution to Exercise 36.2.1**

**Q2:** Gather some data from our world in data

You can download a copy of the data here if you get stuck

**Q3:** Use pandas to import the csv formated data and plot a few different countries of interest

```
f1 = "_static/lecture_specific/simple_linear_regression/life-expectancy-vs-gdp-per-
    -capita.csv" # TODO: Replace with GitHub link
df = pd.read_csv(f1, nrows=10)
```

df

	Entity	Code	Year	Life expectancy at birth (historical)	\
0	Abkhazia	OWID_ABK	2015	NaN	
1	Afghanistan	AFG	1950	27.7	
2	Afghanistan	AFG	1951	28.0	
3	Afghanistan	AFG	1952	28.4	
4	Afghanistan	AFG	1953	28.9	
5	Afghanistan	AFG	1954	29.2	
6	Afghanistan	AFG	1955	29.9	
7	Afghanistan	AFG	1956	30.4	
8	Afghanistan	AFG	1957	30.9	
9	Afghanistan	AFG	1958	31.5	

	GDP per capita	417485-annotations	Population (historical estimates)	\
0	NaN	NaN	NaN	
1	1156.0	NaN	7480464.0	
2	1170.0	NaN	7571542.0	
3	1189.0	NaN	7667534.0	
4	1240.0	NaN	7764549.0	
5	1245.0	NaN	7864289.0	
6	1246.0	NaN	7971933.0	
7	1278.0	NaN	8087730.0	

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```

8          1253.0          NaN          8210207.0
9          1298.0          NaN          8333827.0

   Continent
0      Asia
1      NaN
2      NaN
3      NaN
4      NaN
5      NaN
6      NaN
7      NaN
8      NaN
9      NaN

```

You can see that the data downloaded from Our World in Data has provided a global set of countries with the GDP per capita and Life Expectancy Data.

It is often a good idea to at first import a few lines of data from a csv to understand its structure so that you can then choose the columns that you want to read into your DataFrame.

You can observe that there are a bunch of columns we won't need to import such as `Continent`

So let's built a list of the columns we want to import

```

cols = ['Code', 'Year', 'Life expectancy at birth (historical)', 'GDP per capita']
df = pd.read_csv(f1, usecols=cols)
df

```

```

      Code  Year  Life expectancy at birth (historical)  GDP per capita
0    OWID_ABK  2015                      NaN          NaN
1      AFG  1950                      27.7          1156.0
2      AFG  1951                      28.0          1170.0
3      AFG  1952                      28.4          1189.0
4      AFG  1953                      28.9          1240.0
...
...
62151    ZWE  1946                      NaN          NaN
62152    ZWE  1947                      NaN          NaN
62153    ZWE  1948                      NaN          NaN
62154    ZWE  1949                      NaN          NaN
62155    ALA  2015                      NaN          NaN

[62156 rows x 4 columns]

```

Sometimes it can be useful to rename your columns to make it easier to work with in the DataFrame

```

df.columns = ["cntry", "year", "life_expectancy", "gdppc"]
df

```

```

      cntry  year  life_expectancy  gdppc
0    OWID_ABK  2015              NaN      NaN
1      AFG  1950              27.7  1156.0
2      AFG  1951              28.0  1170.0
3      AFG  1952              28.4  1189.0
4      AFG  1953              28.9  1240.0

```

(continues on next page)

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```
...
62151    ZWE  1946      NaN      NaN
62152    ZWE  1947      NaN      NaN
62153    ZWE  1948      NaN      NaN
62154    ZWE  1949      NaN      NaN
62155    ALA  2015      NaN      NaN
[62156 rows x 4 columns]
```

We can see there are NaN values which represents missing data so let us go ahead and drop those

```
df.dropna(inplace=True)
```

```
df
```

```
      cntry  year  life_expectancy      gdppc
1      AFG  1950        27.7  1156.0000
2      AFG  1951        28.0  1170.0000
3      AFG  1952        28.4  1189.0000
4      AFG  1953        28.9  1240.0000
5      AFG  1954        29.2  1245.0000
...
61960    ZWE  2014        58.8  1594.0000
61961    ZWE  2015        59.6  1560.0000
61962    ZWE  2016        60.3  1534.0000
61963    ZWE  2017        60.7  1582.3662
61964    ZWE  2018        61.4  1611.4052
```

```
[12445 rows x 4 columns]
```

We have now dropped the number of rows in our DataFrame from 62156 to 12445 removing a lot of empty data relationships.

Now we have a dataset containing life expectancy and GDP per capita for a range of years.

It is always a good idea to spend a bit of time understanding what data you actually have.

For example, you may want to explore this data to see if there is consistent reporting for all countries across years

Let's first look at the Life Expectancy Data

```
le_years = df[['cntry', 'year', 'life_expectancy']].set_index(['cntry', 'year']).  
         unstack()['life_expectancy']
le_years
```

cntry	year	1543	1548	1553	1558	1563	1568	1573	1578	1583	1588	...	2009	\
AFG	NaN	...	60.4	...										
AGO	NaN	...	55.8	...										
ALB	NaN	...	77.8	...										
ARE	NaN	...	78.0	...										
ARG	NaN	...	75.9	...										
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
VNM	NaN	...	73.5	...										

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YEM	NaN	NaN	NaN	...	67.2								
ZAF	NaN	NaN	NaN	...	57.4								
ZMB	NaN	NaN	NaN	...	55.3								
ZWE	NaN	NaN	NaN	...	48.1								
year	2010	2011	2012	2013	2014	2015	2016	2017	2018				
cntry													
AFG	60.9	61.4	61.9	62.4	62.5	62.7	63.1	63.0	63.1				
AGO	56.7	57.6	58.6	59.3	60.0	60.7	61.1	61.7	62.1				
ALB	77.9	78.1	78.1	78.1	78.4	78.6	78.9	79.0	79.2				
ARE	78.3	78.5	78.7	78.9	79.0	79.2	79.3	79.5	79.6				
ARG	75.7	76.1	76.5	76.5	76.8	76.8	76.3	76.8	77.0				
...	...	...	...	...	...	...	...	...	...				
VNM	73.5	73.7	73.7	73.8	73.9	73.9	73.9	74.0	74.0				
YEM	67.3	67.4	67.3	67.5	67.4	65.9	66.1	66.0	64.6				
ZAF	58.9	60.7	61.8	62.5	63.4	63.9	64.7	65.4	65.7				
ZMB	56.8	57.8	58.9	59.9	60.7	61.2	61.8	62.1	62.3				
ZWE	50.7	53.3	55.6	57.5	58.8	59.6	60.3	60.7	61.4				

[166 rows x 310 columns]

As you can see there are a lot of countries where data is not available for the Year 1543!

Which country does report this data?

le\_years[~le\_years[1543].isna()]

year	1543	1548	1553	1558	1563	1568	1573	1578	1583	1588	\
cntry											
GBR	33.94	38.82	39.59	22.38	36.66	39.67	41.06	41.56	42.7	37.05	
year	...	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
cntry	...										
GBR	...	80.2	80.4	80.8	80.9	80.9	81.2	80.9	81.1	81.2	81.1

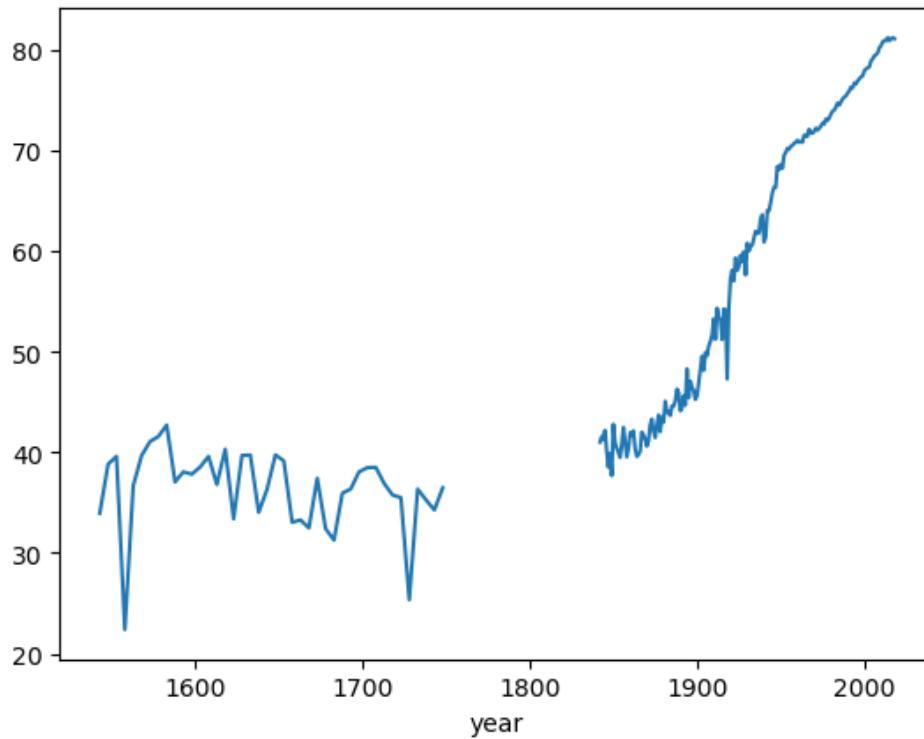
[1 rows x 310 columns]

You can see that Great Britain (GBR) is the only one available

You can also take a closer look at the time series to find that it is also non-continuous, even for GBR.

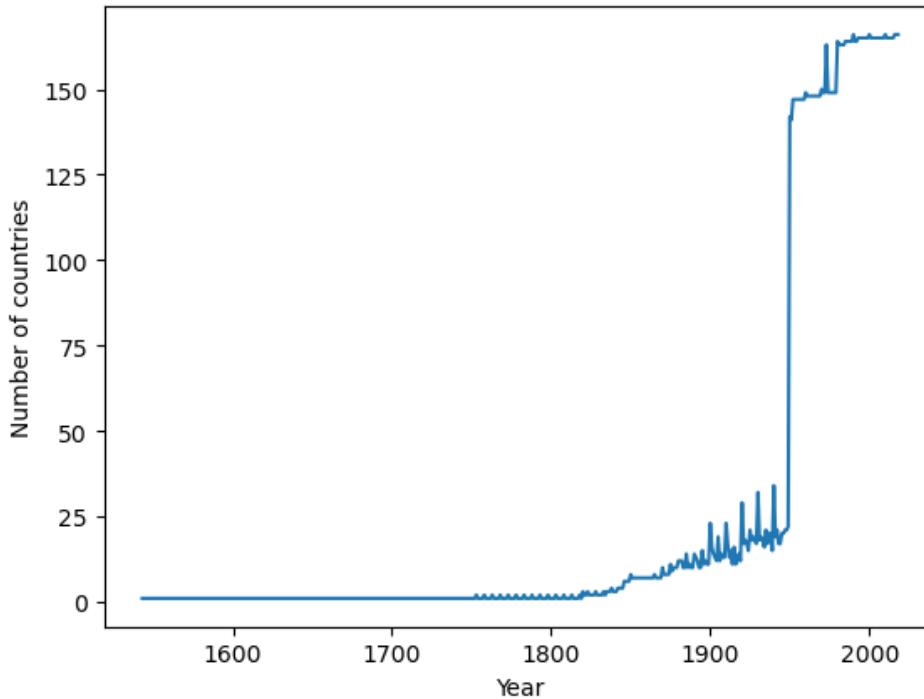
le\_years.loc['GBR'].plot()

&lt;Axes: xlabel='year'&gt;



In fact we can use pandas to quickly check how many countries are captured in each year

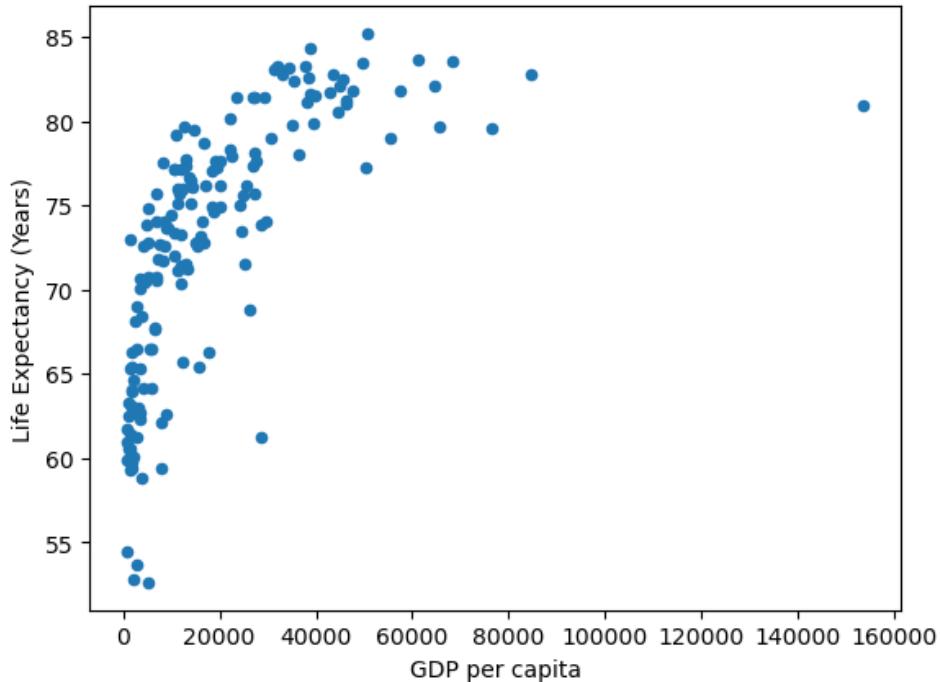
```
le_years.stack().unstack(level=0).count(axis=1).plot(xlabel="Year", ylabel="Number of countries");
```



So it is clear that if you are doing cross-sectional comparisons then more recent data will include a wider set of countries

Now let us consider the most recent year in the dataset 2018

```
df = df[df.year == 2018].reset_index(drop=True).copy()  
  
df.plot(x='gdppc', y='life_expectancy', kind='scatter', xlabel="GDP per capita",  
ylabel="Life Expectancy (Years);");
```



This data shows a couple of interesting relationships.

1. there are a number of countries with similar GDP per capita levels but a wide range in Life Expectancy
2. there appears to be a positive relationship between GDP per capita and life expectancy. Countries with higher GDP per capita tend to have higher life expectancy outcomes

Even though OLS is solving linear equations – one option we have is to transform the variables, such as through a log transform, and then use OLS to estimate the transformed variables

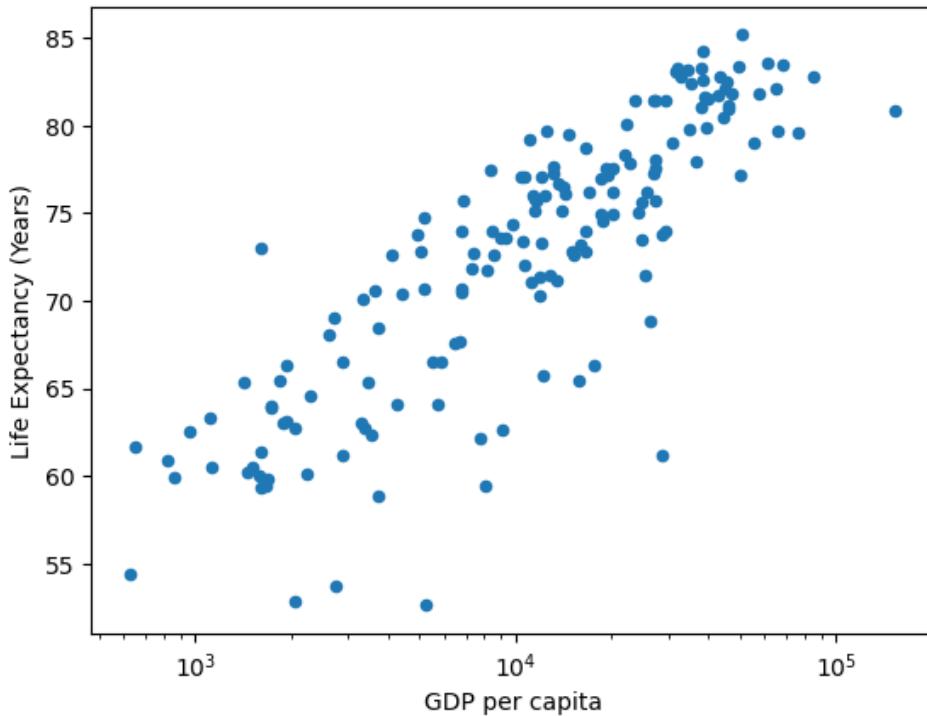
---

**Tip:**  $\ln \rightarrow \ln$  == elasticities

---

By specifying `logx` you can plot the GDP per Capita data on a log scale

```
df.plot(x='gdppc', y='life_expectancy', kind='scatter', xlabel="GDP per capita",  
ylabel="Life Expectancy (Years)", logx=True);
```



As you can see from this transformation – a linear model fits the shape of the data more closely.

```
df['log_gdppc'] = df['gdppc'].apply(np.log10)
```

```
df
```

	cntry	year	life_expectancy	gdppc	log_gdppc
0	AFG	2018	63.1	1934.5550	3.286581
1	ALB	2018	79.2	11104.1660	4.045486
2	DZA	2018	76.1	14228.0250	4.153145
3	AGO	2018	62.1	7771.4420	3.890502
4	ARG	2018	77.0	18556.3830	4.268493
..	...	...	...	...	...
161	VNM	2018	74.0	6814.1420	3.833411
162	OWID_WRL	2018	72.6	15212.4150	4.182198
163	YEM	2018	64.6	2284.8900	3.358865
164	ZMB	2018	62.3	3534.0337	3.548271
165	ZWE	2018	61.4	1611.4052	3.207205

[166 rows x 5 columns]

**Q4:** Use (36.1) and (36.2) to compute optimal values for  $\alpha$  and  $\beta$

```
data = df[['log_gdppc', 'life_expectancy']].copy() # Get Data from DataFrame

# Calculate the sample means
x_bar = data['log_gdppc'].mean()
y_bar = data['life_expectancy'].mean()
```

```
data
```

```

      log_gdppc  life_expectancy
0        3.286581        63.1
1        4.045486        79.2
2        4.153145        76.1
3        3.890502        62.1
4        4.268493        77.0
...
161      3.833411        74.0
162      4.182198        72.6
163      3.358865        64.6
164      3.548271        62.3
165      3.207205        61.4

[166 rows x 2 columns]
```

```

# Compute the Sums
data['num'] = data['log_gdppc'] * data['life_expectancy'] - y_bar * data['log_gdppc']
data['den'] = pow(data['log_gdppc'], 2) - x_bar * data['log_gdppc']
β = data['num'].sum() / data['den'].sum()
print(β)
```

```
12.643730292819699
```

```
a = y_bar - β * x_bar
print(a)
```

```
21.702096701389074
```

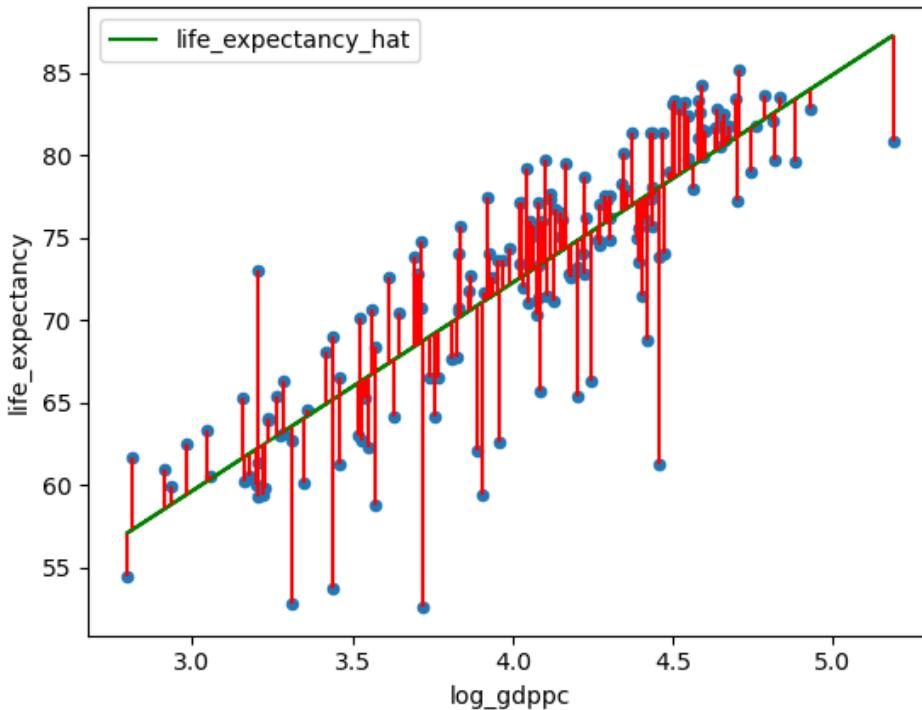
### Q5: Plot the line of best fit found using OLS

```

data['life_expectancy_hat'] = a + β * df['log_gdppc']
data['error'] = data['life_expectancy_hat'] - data['life_expectancy']

fig, ax = plt.subplots()
data.plot(x='log_gdppc', y='life_expectancy', kind='scatter', ax=ax)
data.plot(x='log_gdppc', y='life_expectancy_hat', kind='line', ax=ax, color='g')
plt.vlines(data['log_gdppc'], data['life_expectancy_hat'], data['life_expectancy'], color='r')
```

```
<matplotlib.collections.LineCollection at 0x7fa6627195d0>
```



---

**Exercise 36.2.2**

Minimising the sum of squares is not the **only** way to generate the line of best fit.

For example, we could also consider minimising the sum of the **absolute values**, that would give less weight to outliers.

Solve for  $\alpha$  and  $\beta$  using the least absolute values

---



## MAXIMUM LIKELIHOOD ESTIMATION

```
from scipy.stats import lognorm, pareto, expon, norm
import numpy as np
from scipy.integrate import quad
import matplotlib.pyplot as plt
import pandas as pd
from math import exp, log
```

### 37.1 Introduction

Consider a situation where a policymaker is trying to estimate how much revenue a proposed wealth tax will raise.

The proposed tax is

$$h(w) = \begin{cases} aw & \text{if } w \leq \bar{w} \\ a\bar{w} + b(w - \bar{w}) & \text{if } w > \bar{w} \end{cases}$$

where  $w$  is wealth.

For example, if  $a = 0.05$ ,  $b = 0.1$ , and  $\bar{w} = 2.5$ , this means

- a 5% tax on wealth up to 2.5 and
- a 10% tax on wealth in excess of 2.5.

The unit is 100,000, so  $w = 2.5$  means 250,000 dollars.

Let's go ahead and define  $h$ :

```
def h(w, a=0.05, b=0.1, w_bar=2.5):
    if w <= w_bar:
        return a * w
    else:
        return a * w_bar + b * (w - w_bar)
```

For a population of size  $N$ , where individual  $i$  has wealth  $w_i$ , total revenue raised by the tax will be

$$T = \sum_{i=1}^N h(w_i)$$

We wish to calculate this quantity.

The problem we face is that, in most countries, wealth is not observed for all individuals.

Collecting and maintaining accurate wealth data for all individuals or households in a country is just too hard.

So let's suppose instead that we obtain a sample  $w_1, w_2, \dots, w_n$  telling us the wealth of  $n$  randomly selected individuals.

For our exercise we are going to use a sample of  $n = 10,000$  observations from wealth data in the US in 2016.

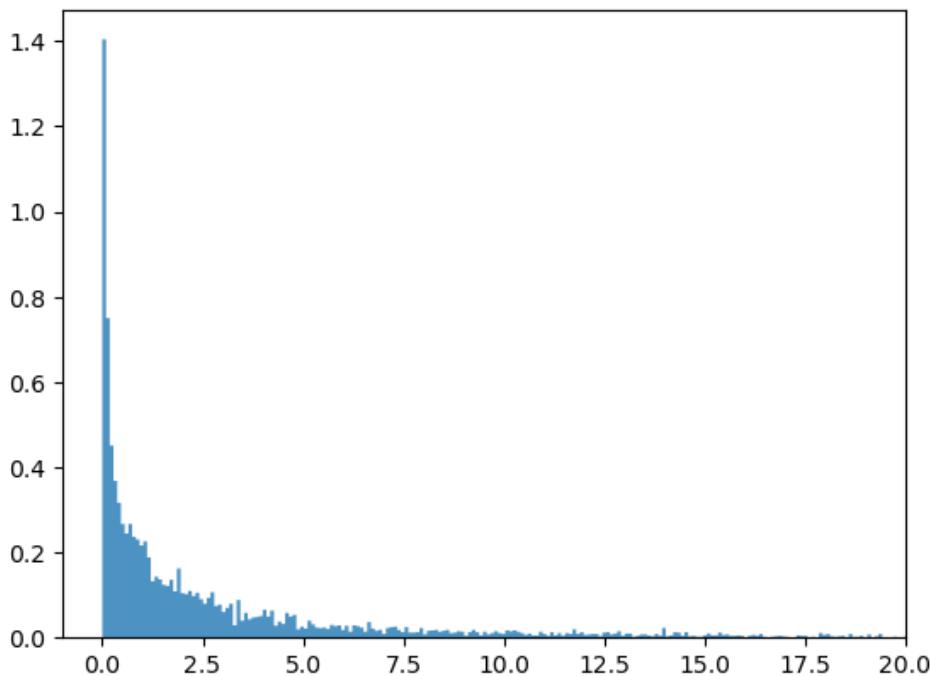
```
n = 10_000
```

The data is derived from the Survey of Consumer Finances (SCF).

The following code imports this data and reads it into an array called `sample`.

Let's histogram this sample.

```
fig, ax = plt.subplots()
ax.set_xlim(-1,20)
ax.hist(sample, density=True, bins=5_000, histtype='stepfilled', alpha=0.8)
plt.show()
```



The histogram shows that many people have very low wealth and a few people have very high wealth.

We will take the full population size to be

```
N = 100_000_000
```

How can we estimate total revenue from the full population using only the sample data?

Our plan is to assume that wealth of each individual is a draw from a distribution with density  $f$ .

If we obtain an estimate of  $f$  we can then approximate  $T$  as follows:

$$T = \sum_{i=1}^N h(w_i) = N \frac{1}{N} \sum_{i=1}^N h(w_i) \approx N \int_0^\infty h(w) f(w) dw \quad (37.1)$$

(The sample mean should be close to the mean by the law of large numbers.)

The problem now is: how do we estimate  $f$ ?

## 37.2 Maximum likelihood estimation

Maximum likelihood estimation is a method of estimating an unknown distribution.

Maximum likelihood estimation has two steps:

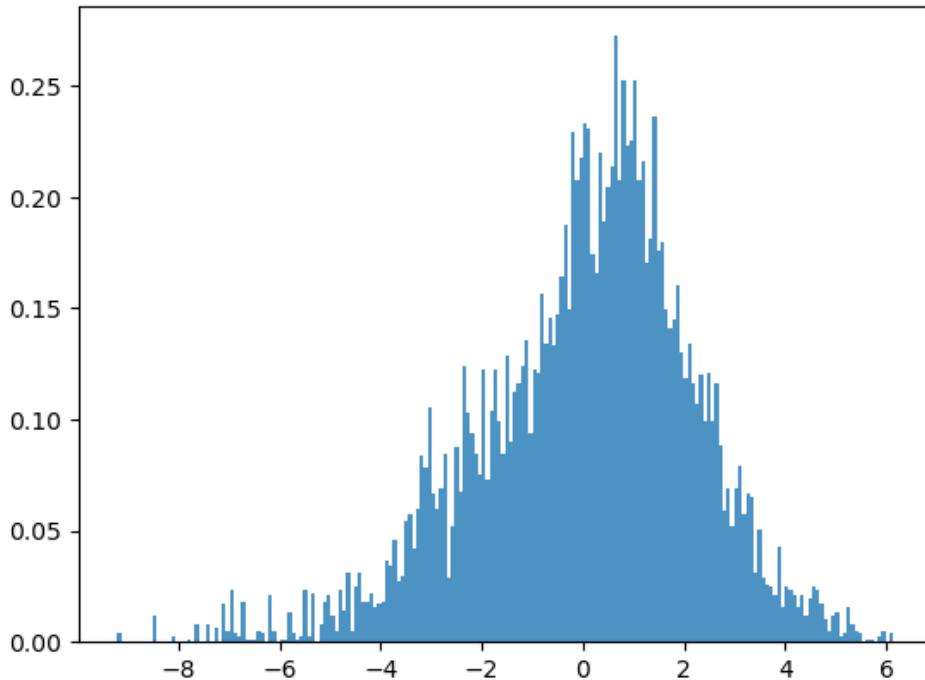
1. Guess what the underlying distribution is (e.g., normal with mean  $\mu$  and standard deviation  $\sigma$ ).
2. Estimate the parameter values (e.g., estimate  $\mu$  and  $\sigma$  for the normal distribution)

One possible assumption for the wealth is that each  $w_i$  is log-normally distributed, with parameters  $\mu \in (-\infty, \infty)$  and  $\sigma \in (0, \infty)$ .

(This means that  $\ln w_i$  is normally distributed with mean  $\mu$  and standard deviation  $\sigma$ .)

You can see that this assumption is not completely unreasonable because, if we histogram log wealth instead of wealth, the picture starts to look something like a bell-shaped curve.

```
ln_sample = np.log(sample)
fig, ax = plt.subplots()
ax.hist(ln_sample, density=True, bins=200, histtype='stepfilled', alpha=0.8)
plt.show()
```



Now our job is to obtain the maximum likelihood estimates of  $\mu$  and  $\sigma$ , which we denote by  $\hat{\mu}$  and  $\hat{\sigma}$ .

These estimates can be found by maximizing the likelihood function given the data.

The pdf of a lognormally distributed random variable  $X$  is given by:

$$f(x, \mu, \sigma) = \frac{1}{x} \frac{1}{\sigma \sqrt{2\pi}} \exp\left(\frac{-1}{2} \left(\frac{\ln x - \mu}{\sigma}\right)^2\right)$$

For our sample  $w_1, w_2, \dots, w_n$ , the likelihood function is given by

$$L(\mu, \sigma|w_i) = \prod_{i=1}^n f(w_i, \mu, \sigma)$$

The likelihood function can be viewed as both

- the joint distribution of the sample (which is assumed to be IID) and
- the “likelihood” of parameters  $(\mu, \sigma)$  given the data.

Taking logs on both sides gives us the log likelihood function, which is

$$\begin{aligned}\ell(\mu, \sigma|w_i) &= \ln \left[ \prod_{i=1}^n f(w_i, \mu, \sigma) \right] \\ &= - \sum_{i=1}^n \ln w_i - \frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^n (\ln w_i - \mu)^2\end{aligned}$$

To find where this function is maximised we find its partial derivatives wrt  $\mu$  and  $\sigma^2$  and equate them to 0.

Let's first find the maximum likelihood estimate (MLE) of  $\mu$

$$\begin{aligned}\frac{\delta \ell}{\delta \mu} &= -\frac{1}{2\sigma^2} \times 2 \sum_{i=1}^n (\ln w_i - \mu) = 0 \\ \implies \sum_{i=1}^n \ln w_i - n\mu &= 0 \\ \implies \hat{\mu} &= \frac{\sum_{i=1}^n \ln w_i}{n}\end{aligned}$$

Now let's find the MLE of  $\sigma$

$$\begin{aligned}\frac{\delta \ell}{\delta \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n (\ln w_i - \mu)^2 = 0 \\ \implies \frac{n}{2\sigma^2} &= \frac{1}{2\sigma^4} \sum_{i=1}^n (\ln w_i - \mu)^2 \\ \implies \hat{\sigma} &= \left( \frac{\sum_{i=1}^n (\ln w_i - \hat{\mu})^2}{n} \right)^{1/2}\end{aligned}$$

Now that we have derived the expressions for  $\hat{\mu}$  and  $\hat{\sigma}$ , let's compute them for our wealth sample.

```
μ_hat = np.mean(ln_sample)
μ_hat
```

```
0.0634375526654064
```

```
num = (ln_sample - μ_hat)**2
σ_hat = (np.mean(num))**(1/2)
σ_hat
```

```
2.1507346258433424
```

Let's plot the log-normal pdf using the estimated parameters against our sample data.

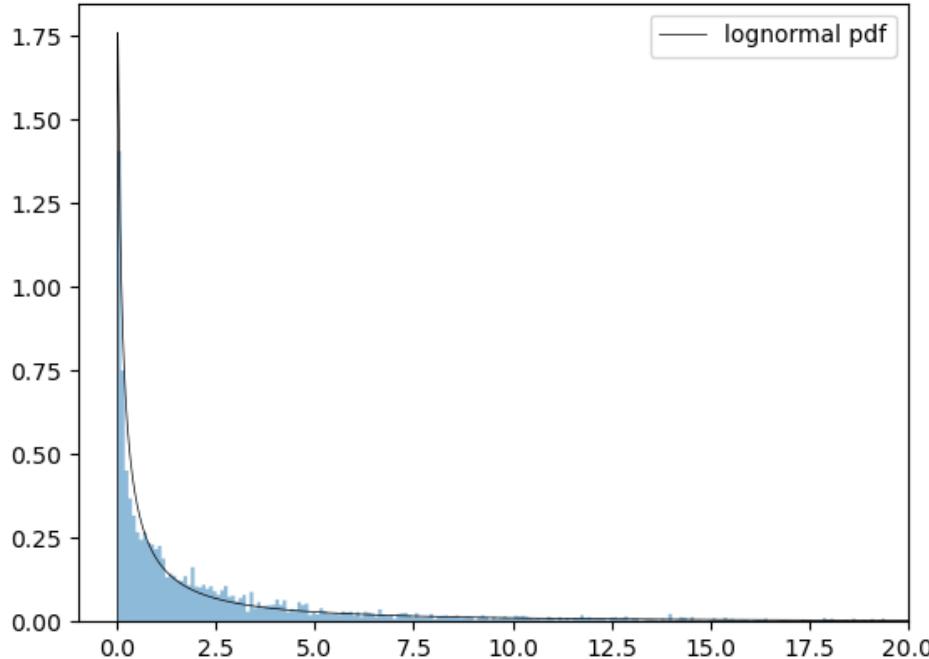
```

dist_lognorm = lognorm(sigma_hat, scale = exp(mu_hat))
x = np.linspace(0, 50, 10000)

fig, ax = plt.subplots()
ax.set_xlim(-1, 20)

ax.hist(sample, density=True, bins=5_000, histtype='stepfilled', alpha=0.5)
ax.plot(x, dist_lognorm.pdf(x), 'k-', lw=0.5, label='lognormal pdf')
ax.legend()
plt.show()

```



Our estimated lognormal distribution appears to be a reasonable fit for the overall data.

We now use (37.1) to calculate total revenue.

We will compute the integral using numerical integration via SciPy's `quad` function

```

def total_revenue(dist):
    integral, _ = quad(lambda x: h(x) * dist.pdf(x), 0, 100_000)
    T = N * integral
    return T

```

```

tr_lognorm = total_revenue(dist_lognorm)
tr_lognorm

```

```
101105326.82814863
```

(Our unit was 100,000 dollars, so this means that actual revenue is 100,000 times as large.)

### 37.3 Pareto distribution

We mentioned above that using maximum likelihood estimation requires us to make a prior assumption of the underlying distribution.

Previously we assumed that the distribution is lognormal.

Suppose instead we assume that  $w_i$  are drawn from the [Pareto Distribution](#) with parameters  $b$  and  $x_m$ .

In this case, the maximum likelihood estimates are known to be

$$\hat{b} = \frac{n}{\sum_{i=1}^n \ln(w_i/\hat{x}_m)} \quad \text{and} \quad \hat{x}_m = \min_i w_i$$

Let's calculate them.

```
xm_hat = min(sample)
xm_hat
```

```
0.0001
```

```
den = np.log(sample/xm_hat)
b_hat = 1/np.mean(den)
b_hat
```

```
0.10783091940803055
```

Now let's recompute total revenue.

```
dist_pareto = pareto(b = b_hat, scale = xm_hat)
tr_pareto = total_revenue(dist_pareto)
tr_pareto
```

```
12933168365.762571
```

The number is very different!

```
tr_pareto / tr_lognorm
```

```
127.91777418162562
```

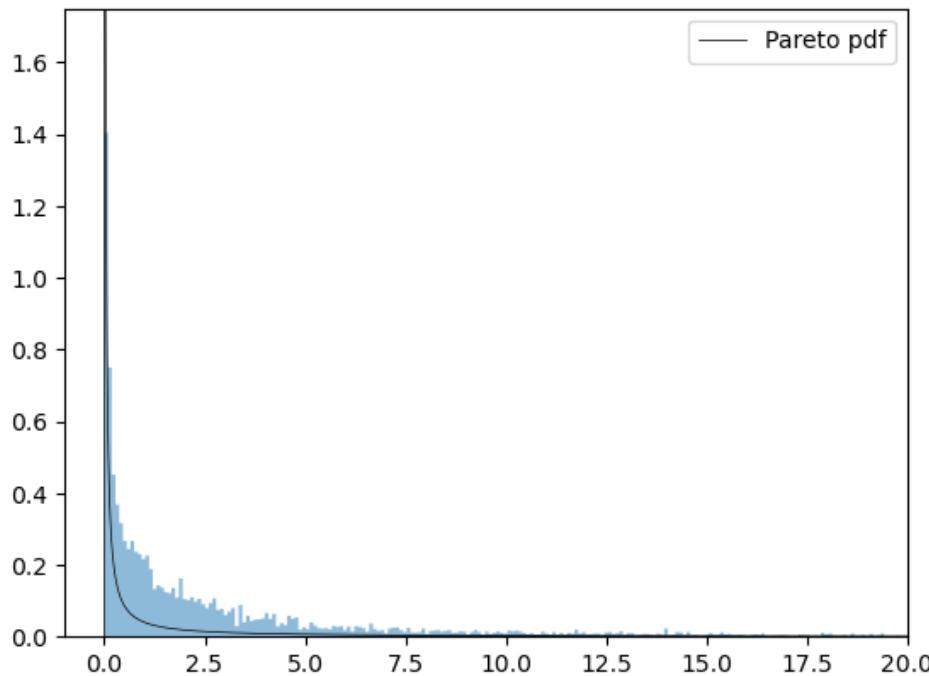
We see that choosing the right distribution is extremely important.

Let's compare the fitted Pareto distribution to the histogram:

```
fig, ax = plt.subplots()
ax.set_xlim(-1, 20)
ax.set_ylim(0, 1.75)

ax.hist(sample, density=True, bins=5_000, histtype='stepfilled', alpha=0.5)
ax.plot(x, dist_pareto.pdf(x), 'k-', lw=0.5, label='Pareto pdf')
ax.legend()

plt.show()
```



We observe that in this case the fit for the Pareto distribution is not very good, so we can probably reject it.

## 37.4 What is the best distribution?

There is no “best” distribution — every choice we make is an assumption.

All we can do is try to pick a distribution that fits the data well.

The plots above suggested that the lognormal distribution is optimal.

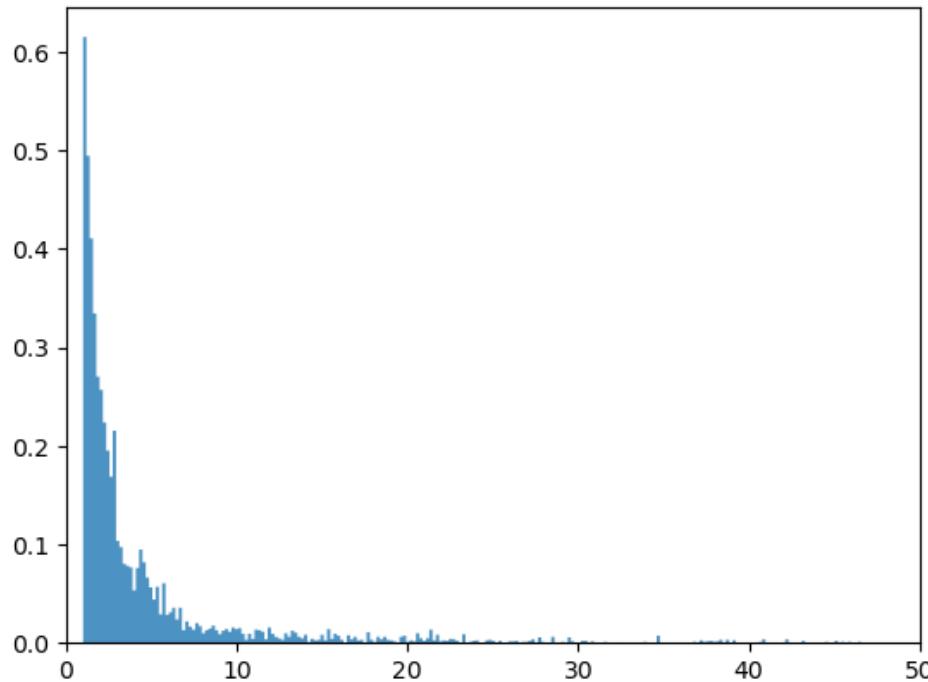
However when we inspect the upper tail (the richest people), the Pareto distribution may be a better fit.

To see this, let’s now set a minimum threshold of net worth in our dataset.

We set an arbitrary threshold of \$500,000 and read the data into `sample_tail`.

Let’s plot this data.

```
fig, ax = plt.subplots()
ax.set_xlim(0, 50)
ax.hist(sample_tail, density=True, bins=500, histtype='stepfilled', alpha=0.8)
plt.show()
```



Now let's try fitting some distributions to this data.

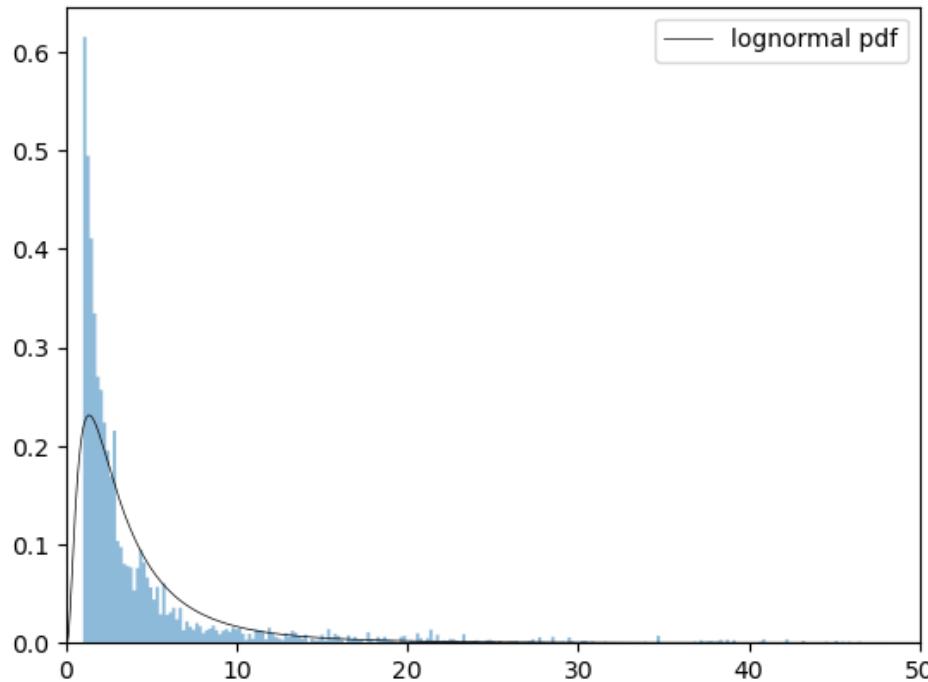
### 37.4.1 Lognormal distribution for the right hand tail

Let's start with the lognormal distribution

We estimate the parameters again and plot the density against our data.

```
ln_sample_tail = np.log(sample_tail)
μ_hat_tail = np.mean(ln_sample_tail)
num_tail = (ln_sample_tail - μ_hat_tail)**2
σ_hat_tail = (np.mean(num_tail))**(1/2)
dist_lognorm_tail = lognorm(σ_hat_tail, scale = exp(μ_hat_tail))

fig, ax = plt.subplots()
ax.set_xlim(0,50)
ax.hist(sample_tail, density=True, bins=500, histtype='stepfilled', alpha=0.5)
ax.plot(x, dist_lognorm_tail.pdf(x), 'k-', lw=0.5, label='lognormal pdf')
ax.legend()
plt.show()
```



While the lognormal distribution was a good fit for the entire dataset, it is not a good fit for the right hand tail.

### 37.4.2 Pareto distribution for the right hand tail

Let's now assume the truncated dataset has a Pareto distribution.

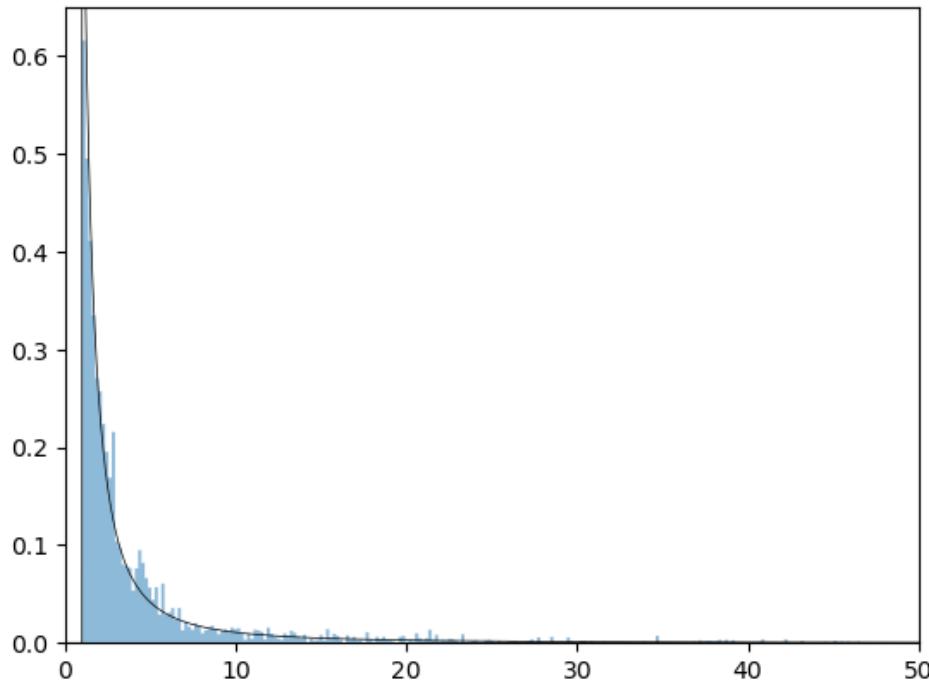
We estimate the parameters again and plot the density against our data.

```

xm_hat_tail = min(sample_tail)
den_tail = np.log(sample_tail/xm_hat_tail)
b_hat_tail = 1/np.mean(den_tail)
dist_pareto_tail = pareto(b = b_hat_tail, scale = xm_hat_tail)

fig, ax = plt.subplots()
ax.set_xlim(0, 50)
ax.set_ylim(0, 0.65)
ax.hist(sample_tail, density=True, bins= 500, histtype='stepfilled', alpha=0.5)
ax.plot(x, dist_pareto_tail.pdf(x), 'k-', lw=0.5, label='pareto pdf')
plt.show()

```



The Pareto distribution is a better fit for the right hand tail of our dataset.

### 37.4.3 So what is the best distribution?

As we said above, there is no “best” distribution — each choice is an assumption.

We just have to test what we think are reasonable distributions.

One test is to plot the data against the fitted distribution, as we did.

There are other more rigorous tests, such as the [Kolmogorov-Smirnov test](#).

We omit such advanced topics (but encourage readers to study them once they have completed these lectures).

## 37.5 Exercises

---

### Exercise 37.5.1

Suppose we assume wealth is [exponentially](#) distributed with parameter  $\lambda > 0$ .

The maximum likelihood estimate of  $\lambda$  is given by

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^n w_i}$$

1. Compute  $\hat{\lambda}$  for our initial sample.
  2. Use  $\hat{\lambda}$  to find the total revenue
- 

---

### Solution to Exercise 37.5.1

```
λ_hat = 1/np.mean(sample)
λ_hat
```

```
0.15234120963403971
```

```
dist_expo = expon(scale = 1/λ_hat)
tr_expo = total_revenue(dist_expo)
tr_expo
```

```
55246978.53427645
```

### Exercise 37.5.2

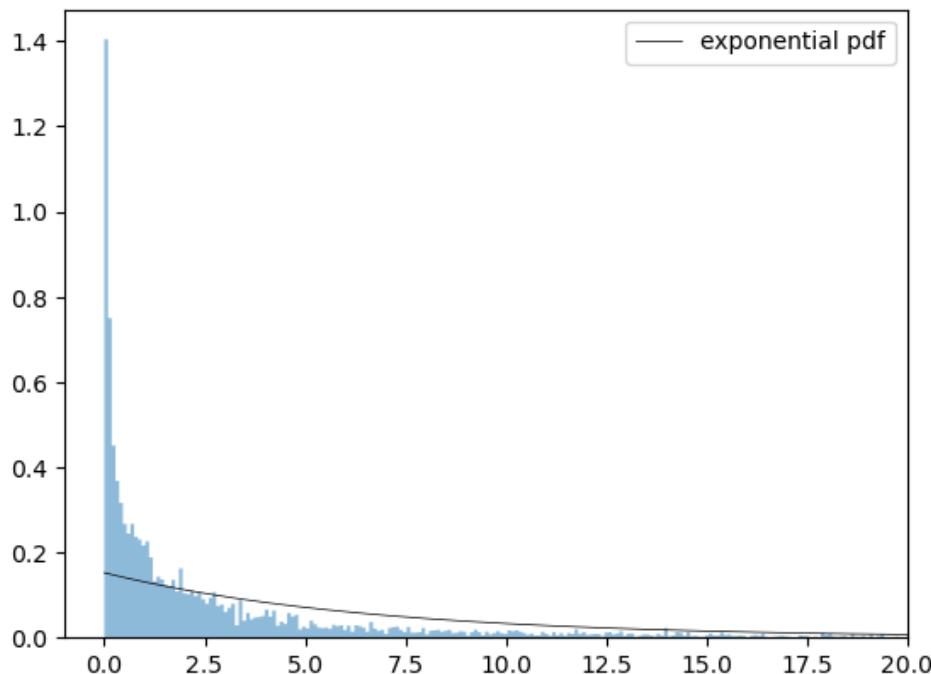
Plot the exponential distribution against the sample and check if it is a good fit or not.

### Solution to Exercise 37.5.2

```
fig, ax = plt.subplots()
ax.set_xlim(-1, 20)

ax.hist(sample, density=True, bins=5000, histtype='stepfilled', alpha=0.5)
ax.plot(x, dist_expo.pdf(x), 'k-', lw=0.5, label='exponential pdf')
ax.legend()

plt.show()
```



Clearly, this distribution is not a good fit for our data.

---

## **Part XII**

### **Other**



## **TROUBLESHOOTING**

### **Contents**

- *Troubleshooting*
  - *Fixing your local environment*
  - *Reporting an issue*

This page is for readers experiencing errors when running the code from the lectures.

### **38.1 Fixing your local environment**

The basic assumption of the lectures is that code in a lecture should execute whenever

1. it is executed in a Jupyter notebook and
2. the notebook is running on a machine with the latest version of Anaconda Python.

You have installed Anaconda, haven't you, following the instructions in [this lecture](#)?

Assuming that you have, the most common source of problems for our readers is that their Anaconda distribution is not up to date.

Here's a [useful article](#) on how to update Anaconda.

Another option is to simply remove Anaconda and reinstall.

You also need to keep the external code libraries, such as `QuantEcon.py` up to date.

For this task you can either

- use `conda install -y quantecon` on the command line, or
- execute `!conda install -y quantecon` within a Jupyter notebook.

If your local environment is still not working you can do two things.

First, you can use a remote machine instead, by clicking on the Launch Notebook icon available for each lecture



**Launch Notebook**

Second, you can report an issue, so we can try to fix your local set up.

We like getting feedback on the lectures so please don't hesitate to get in touch.

## 38.2 Reporting an issue

One way to give feedback is to raise an issue through our [issue tracker](#).

Please be as specific as possible. Tell us where the problem is and as much detail about your local set up as you can provide.

Another feedback option is to use our [discourse forum](#).

Finally, you can provide direct feedback to [contact@quantecon.org](mailto:contact@quantecon.org)

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CHAPTER  
**THIRTYNINE**

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**REFERENCES**



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CHAPTER  
**FORTY**

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## EXECUTION STATISTICS

This table contains the latest execution statistics.

Document	Modified	Method	Run Time (s)	Status
<i>business_cycle</i>	2023-10-11 03:33	cache	18.35	✓
<i>cagan_adaptive</i>	2023-10-11 03:33	cache	3.09	✓
<i>cagan_ree</i>	2023-10-11 03:33	cache	4.43	✓
<i>cobweb</i>	2023-10-11 03:33	cache	3.33	✓
<i>commod_price</i>	2023-10-11 03:34	cache	22.86	✓
<i>cons_smooth</i>	2023-10-11 03:34	cache	2.25	✓
<i>eigen_I</i>	2023-10-11 03:34	cache	5.8	✓
<i>eigen_II</i>	2023-10-11 03:34	cache	9.74	✓
<i>equalizing_difference</i>	2023-10-11 03:34	cache	2.94	✓
<i>geom_series</i>	2023-10-11 03:34	cache	3.55	✓
<i>heavy_tails</i>	2023-10-11 03:34	cache	20.07	✓
<i>inequality</i>	2023-10-11 03:41	cache	404.76	✓
<i>inflation_history</i>	2023-10-11 03:41	cache	11.44	✓
<i>input_output</i>	2023-10-11 03:42	cache	18.74	✓
<i>intro</i>	2023-10-11 03:42	cache	1.04	✓
<i>intro_supply_demand</i>	2023-10-11 03:42	cache	2.3	✓
<i>lake_model</i>	2023-10-11 03:42	cache	3.02	✓
<i>linear_equations</i>	2023-10-11 03:42	cache	2.26	✓
<i>lln_clt</i>	2023-10-11 03:45	cache	202.49	✓
<i>long_run_growth</i>	2023-10-11 03:45	cache	9.99	✓
<i>lp_intro</i>	2023-10-11 03:45	cache	8.4	✓
<i>markov_chains_I</i>	2023-10-11 03:46	cache	12.65	✓
<i>markov_chains_II</i>	2023-10-11 03:46	cache	8.92	✓
<i>mle</i>	2023-10-11 03:46	cache	7.55	✓
<i>monte_carlo</i>	2023-10-11 03:46	cache	21.35	✓
<i>networks</i>	2023-10-11 03:46	cache	11.8	✓
<i>olg</i>	2023-10-11 03:47	cache	3.14	✓
<i>prob_dist</i>	2023-10-11 03:47	cache	10.74	✓
<i>pv</i>	2023-10-11 03:47	cache	2.16	✓
<i>scalar_dynam</i>	2023-10-11 03:47	cache	3.59	✓
<i>schelling</i>	2023-10-11 03:47	cache	14.72	✓
<i>short_path</i>	2023-10-11 03:47	cache	1.15	✓
<i>simple_linear_regression</i>	2023-10-11 03:47	cache	4.4	✓
<i>solow</i>	2023-10-11 03:47	cache	5.0	✓
<i>status</i>	2023-10-11 03:42	cache	1.04	✓

continues on next page

Table 40.1 – continued from previous page

Document	Modified	Method	Run Time (s)	Status
<a href="#"><i>supply_demand_heterogeneity</i></a>	2023-10-11 03:47	cache	1.83	✓
<a href="#"><i>supply_demand_multiple_goods</i></a>	2023-10-11 03:47	cache	2.81	✓
<a href="#"><i>time_series_with_matrices</i></a>	2023-10-11 03:47	cache	3.3	✓
<a href="#"><i>troubleshooting</i></a>	2023-10-11 03:42	cache	1.04	✓
<a href="#"><i>zreferences</i></a>	2023-10-11 03:42	cache	1.04	✓

These lectures are built on linux instances through github actions and amazon web services (aws) to enable access to a gpu. These lectures are built on a p3.2xlarge that has access to 8 vcpu's, a V100 NVIDIA Tesla GPU, and 61 Gb of memory.

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