Calculating the Electromagnetic Coupling between two Cylindrical Coils

Anubhav Elehence, Prayaag Katta, and Gitansh Kataria

Research Division, Quazar Technologies

I. CALCULATIONS

A. Infinite Coil Approximation

1. *Calculating the Magnetic Field caused by Current in Primary Coil*

Consider a cylinder through which a sinusoidal current with frequency ω is flowing, which results in a corresponding current density $\vec{J}$. From Maxwell's Equations :

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{2}$$

Taking a temporal Fourier transform of $\vec{E}(r,t)$,

$$\vec{E} = \hat{E}_\omega \; e^{i\omega t} \tag{3}$$

$$\mu_0 \epsilon_0 \frac{\partial E}{\partial t} = \mu_0 \epsilon_0 (i\omega \hat{E}_\omega \; e^{i\omega t}) \tag{4}$$

$$= i\frac{\omega}{c^2} \hat{E}_\omega \; e^{i\omega t} \tag{5}$$

For $\omega << L * c$ (the quasi-static approximation), where L is the length scale of the system, Equations 1 and 2 can be written as

$$\vec{\nabla} \times \vec{E} = 0 \tag{6}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \tag{7}$$

Applying Curl Operator on both sides of (7):

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \mu_0 \vec{\nabla} \times \vec{J} \tag{8}$$

$$\vec{\nabla}(\cancelto{0}{\vec{\nabla} \cdot \vec{B}}) - \vec{\nabla}^2 \vec{B} = \mu_0 \vec{\nabla} \times \vec{J} \tag{9}$$

$$-\vec{\nabla}^2 \vec{B} = \mu_0 \vec{\nabla} \times \vec{J} \tag{10}$$

The Laplacian of a vector in cylindrical coordinates is given by :

$$
\begin{aligned}
\vec{\nabla}^2 \vec{B} = &\left(\nabla^2 B_\rho - \frac{B_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial B_\varphi}{\partial \varphi}\right)\hat{\boldsymbol{\rho}} \\
&+\left(\nabla^2 B_\varphi - \frac{B_\varphi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial B_\rho}{\partial \varphi}\right)\hat{\boldsymbol{\varphi}} \\
&+ \nabla^2 B_z \hat{\mathbf{z}}
\end{aligned} \tag{11}
$$

Curl of vector in cylindrical coordinates is given by :

$$
\begin{aligned}
\vec{\nabla} \times \vec{J} = &\left(\frac{1}{\rho}\frac{\partial J_z}{\partial \varphi} - \frac{\partial J_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} \\
&+ \left(\frac{\partial J_\rho}{\partial z} - \frac{\partial J_z}{\partial \rho}\right)\hat{\boldsymbol{\varphi}} \\
&+ \frac{1}{\rho}\left(\frac{\partial\left(\rho J_\varphi\right)}{\partial \rho} - \frac{\partial J_\rho}{\partial \varphi}\right)\hat{\mathbf{z}}
\end{aligned} \tag{12}
$$

We assume that the current flows only along the $\hat{\varphi}$ direction, thus :

$$
J_r = 0, J_z = 0 \tag{13}
$$

Considering a system symmetric about $\hat{\varphi}$ direction and an infinitely long coil stretching along the z-direction, we have

$$
\frac{\partial}{\partial \varphi} = 0 \tag{14}
$$

$$
\frac{\partial}{\partial z} = 0 \tag{15}
$$

Using Equations 13, 14 and 15, Equation 12 can be written as

$$
\begin{aligned}
\vec{\nabla} \times \vec{J} = &\left(\frac{1}{\rho}\cancelto{0}{\frac{\partial J_z}{\partial \varphi}} - \frac{\partial J_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} \\
&+ \left(\cancelto{0}{\frac{\partial J_\rho}{\partial z}} - \frac{\partial \cancelto{0}{J_z}}{\partial \rho}\right)\hat{\boldsymbol{\varphi}} \\
&+ \frac{1}{\rho}\left(\frac{\partial\left(\rho J_\varphi\right)}{\partial \rho} - \cancelto{0}{\frac{\partial J_\rho}{\partial \varphi}}\right)\hat{\mathbf{z}}
\end{aligned} \tag{16}
$$

Equating the $\hat{\mathbf{z}}$ components of Equations 11 and 16, we get :

$$
\nabla^2 B_z = \frac{\mu_0}{\rho}\frac{\partial\left(\rho J_\varphi\right)}{\partial \rho} \tag{17}
$$

The scalar Laplacian of B_z in cylindrical coordinates is given by :

$$\nabla^2 B_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\frac{\partial B_z}{\partial \rho}) + \frac{1}{\rho^2}\frac{\partial^2 B_z}{\partial \varphi^2} + \frac{\partial^2 B_z}{\partial z^2} \tag{18}$$

Using approximations from equations 14 and 15,

$$\nabla^2 B_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial B_z}{\partial \rho}\right) + \frac{1}{\rho^2}\cancelto{0}{\frac{\partial^2 B_z}{\partial \varphi^2}} + \cancelto{0}{\frac{\partial^2 B_z}{\partial z^2}} \tag{19}$$

Thus the equation we have to solve for this case is :

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial B_z}{\partial \rho}\right) = \frac{\mu_0}{\rho}\frac{\partial\left(\rho J_\varphi\right)}{\partial \rho} \tag{20}$$

or

$$\nabla^2 B_z = \vec{\nabla} \times \vec{J} \tag{21}$$

Approximating $\vec{J}$ as a smoothened step function using $tanh$ function, we can write

$$Insert equation Here \tag{22}$$

Thus $\vec{\nabla} \times \vec{J}$ is calculated to be

$$Insert equation Here \tag{23}$$

Expanding $\vec{\nabla} \times \vec{J}$ in terms of Fourier-Bessel series with zeroth order Bessel function,

$$Insert equation Here \tag{24}$$

Plotting the Fourier-Bessel expansion against the analytic $\vec{\nabla} \times \vec{J}$,

Insert Plot Here

Also, expanding B_z in terms of Fourier-Bessel series,

$$Insert equation here \tag{25}$$

From reference [1], we know that the eigenvalue, for the equation,

$$\nabla^2 J_0(\lambda z) = \alpha J_0(\lambda z) \tag{26}$$

is given by

$$\alpha = -\lambda^2 \tag{27}$$

Using this result, we can write,

$$Clearlywritethisequation \nabla^2 B_z = \alpha B_z \tag{28}$$

Using Equation 21, we can equate equations 24 and 28,

$$InsertEquationhere \tag{29}$$

Calculating for the expansion coefficients of B_z,

$$InsertEquationsHere \tag{30}$$

Plotting B_z, for various values of b,

Insert Plot Here

2. Calculating Induced EMF in Secondary Coil

B. Finite Length Coils

II. EXPERIMENT

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[1] Anubhav Elhence, Prayag Katta, Gitansh Kataria, *Bessel Eigenfunction Proof*, Research Division, Quazar Technologies