Every real number is a complex number
1 is also a complex no:- $1+0i$
(lying on x axis)

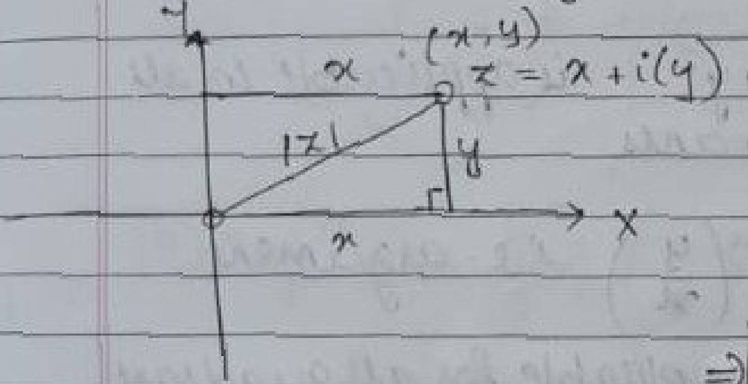
MODULE 3: COMPLEX NUMBER

$z = x + iy$ is a complex number where x, y are real numbers.

where $x = Re(z)$ i.e. real part of z

$y = Im(z)$ imaginary part of z

$\bar{z} = x - iy$ conjugate of z



By pythagoras theorem

$x^2 + y^2 = |z|^2$

$\Rightarrow \boxed{|z| = \sqrt{x^2+y^2}}$ → modulus of complex no.

In right angle Δ

$\tan\theta = \frac{y}{x}$

$\Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$

→ argument of complex no.

① $|z|$ is also denoted as r.
i.e. $|z| = r = \sqrt{x^2+y^2}$

② argument of complex number is denoted by θ or angle(z) or amp(z)

(3) argument of complex no. are of two types.

(i) principle value of argument :- $\boxed{-\pi \le \theta \le \pi}$

(ii) General value of argument :- $\boxed{0 \le \theta \le 2\pi}$

Principle value :- $-\pi \le \theta \le \pi$

General value :- $0 \le \theta \le 2\pi$

Note :- $|z| = \sqrt{x^2+y^2}$ is applicable in all quadrants

But $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ i.e. argument is not applicable for all quadrant

2nd Quadrant

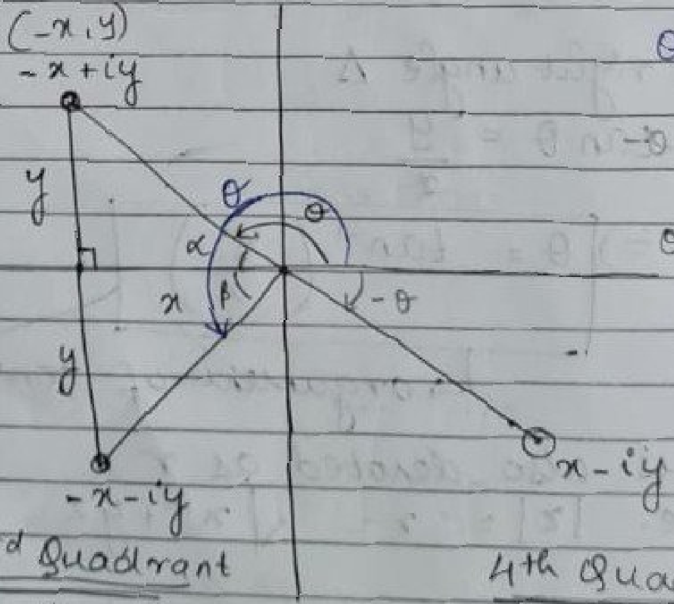
$\tan\alpha = \frac{y}{x}$

$\alpha = \tan^{-1}\frac{y}{x}$

$\pi - \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\boxed{\theta = \pi - \tan^{-1}\left(\frac{y}{x}\right)}$

distance is never negative dist = x not $-x$



θ → θ for 3rd Quad

-θ → θ for 4th quad

θ → θ for 2nd quad

3rd Quadrant

$\tan\beta = \frac{y}{x}$

$\beta = \tan^{-1}\frac{y}{x}$

Now $\theta = \pi + \beta$

$\boxed{\theta = \pi + \tan^{-1}\frac{y}{x}}$

4th Quadrant

$\arg(z) = -\theta$

$= -\tan^{-1}\left(\frac{y}{x}\right)$

Eg1) Find modulus and arg of $\frac{1}{2}+\frac{i\sqrt{3}}{2}$,

$-\frac{1}{2}+\frac{i\sqrt{3}}{2}$,

$-\frac{1}{2}-\frac{i\sqrt{3}}{2}$ and $\frac{1}{2}-\frac{i\sqrt{3}}{2}$

(i) If $z=\frac{1}{2}+\frac{i\sqrt{3}}{2}$

$$|z|=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$$

(ii) $\arg(z)=\arg\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

$$=\tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)=\frac{\pi}{3}$$

(ii) If $z=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$

$$\arg(z)=\pi-\tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$$

mode / setup :- 2 :- complex.

type the number

shift → 2. [complex]

3 :- arg z θ

(*) Forms of complex numbers

I Cartesian form :-

$\boxed{z = x + iy}$ where x, y are real

II Polar form :-

put $x = r\cos\theta$ and $y = r\sin\theta$

$\therefore z = x + iy$ becomes ~~$z = r\cos\theta + i\sin\theta$~~

$z = r\cos\theta + ir\sin\theta$

$\therefore \boxed{z = r(\cos\theta + i\sin\theta)}$

where $r = |z| = \cancel{\sqrt{x^2+y^2}}$ $\sqrt{x^2+y^2}$

$\theta = \arg(z) = \tan^{-1}\frac{y}{x}$

III Exponential form

$\boxed{z = re^{i\theta}}$ where $e^{i\theta} = \cos\theta + i\sin\theta$

this is known as Euler's expression

$\therefore e^{-i\theta} = \cos\theta - i\sin\theta$

Eg 1) Find $(1+i)^{100}$ (we'll prefer exponential form)

$\rightarrow$ $1+i = re^{i\theta}$ by exp form

where $r = |1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \arg(1+i) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

directly calculate in scientific calculator

$$\therefore (1+i)^{100} = \left(\sqrt{2}\, e^{i\pi/4}\right)^{100} \quad ①$$

unknowingly we have applied De Morvre theorem

$$= \left(\sqrt{2}\right)^{100} \left(e^{i\pi/4}\right)^{100} \quad ②$$

$$= \cancel{2^{50} e^{i25\pi}} \quad 2^{50} e^{i25\pi} \quad ③$$

$(e^{i\theta})^n = e^{in\theta}$

$$= 2^{50}\left[\cos(25\pi) + i\sin(25\pi)\right] \quad ④$$

step ③ to step ④ is not De movire theorem

use calculator
calculator is in degree form
write 25 x 180 for 25 x π

$$= 2^{50}\left[-1 + i0\right]$$

$$= -\left(2^{50}\right)$$

* <u>De' Morvre's theorem</u>

For any real value of n, the one of the value of $(\cos\theta + i\sin\theta)^n$ is given by $\cos(n\theta) + i\sin(n\theta)$

i.e. $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

where n is a real number

$$(\cos\theta + i\sin\theta)^5 = \cos(5\theta) + i\sin(5\theta)$$

$$(\cos\theta + i\sin\theta)^{-3/4} = \cos\left(\frac{-3\theta}{4}\right) + i\sin\left(\frac{-3\theta}{4}\right)$$

irrational no

$$(\cos\theta + i\sin\theta)^{\sqrt{2}} = \cos\left(\sqrt{2}\,\theta\right) + i\sin\left(\sqrt{2}\,\theta\right)$$

$(\cos\theta + i\sin\theta)^n$ will have multiple values out of which one value is given by $\cos(n\theta) + i\sin(n\theta)$

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$\underbrace{(\cos\theta + i\sin\theta)^n}_{A} = \cos(n\theta) + i\sin(n\theta)$

this is actually

$(e^{i\theta})^n = e^{in\theta}$ [According to law of indices]

Difference is Laws of indices says A form has only one value i.e $e^{in\theta}$

But de Morve's theorem says A form has multiple values out of which one value is $\cos(n\theta) + i\sin(n\theta)$

2) Find the value of $(\sqrt{3} - i)^4$ using De Morre's theorem

$\rightarrow \sqrt{3} - i = re^{i\theta}$ .... by exp form

where $r = |\sqrt{3} - i| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = 2$

this is in 4th quadrant

$\theta = \arg(\sqrt{3} - i) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

$(\sqrt{3} - i)^4 = (re^{i\theta})^4 = \underbrace{\left(2e^{i(-\pi/6)}\right)^4}_{(e^{i\theta})^n}$

unknowingly we have applied De Morre's theorem

$= 2^4 \underbrace{e^{-i4\pi/6}}_{\rightarrow e^{in\theta}}$

$= 16\, e^{-i\,2\pi/3}$

$= 16\left[\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)\right]$

$= 16\left[-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] = -8(1 + i\sqrt{3})$

$i^2 = -1$

3) P.T. using D'M Theorem $(4n)^{th}$ power of $\frac{1+7i}{(2-i)^2}$ is equal to $(-4)^n$

Numerator is in cartesian form
Denominator is in Cartesian form
But ratio $\frac{Num}{Den}$ is not in cartesian form

$$\Rightarrow \frac{1+7i}{(2-i)^2} = \frac{1+7i}{4-4i+i^2} = \frac{1+7i}{3-i4} \times \frac{3+i4}{3+i4}$$

Take conjugate of denominator

$$= \frac{3+i4+i21+28i^2}{3^2-(i4)^2} = \frac{3+25i-28}{3^2+16}$$

$$= \frac{-25+i25}{25} = -1+i$$

Let $-1+i = re^{i\theta}$ ... by exp form

where $r = |-1+i| = \sqrt{(-1)^2+(1)^2} = \sqrt{2}$

$$\theta = \arg(-1+i) = \frac{3\pi}{4}$$

using calculator :- $135 \times \frac{\pi}{180} = \frac{3\pi}{4}$

~~Let $-1+i = re^{i\theta}$~~ $= \sqrt{2}\, e^{i3\pi/4}$

$$\therefore \left[\frac{1+7i}{(2-i)^2}\right]^{4n} = (-1+i)^{4n} = \left(\sqrt{2}\, e^{i3\pi/4}\right)^{4n}$$

$$= (\sqrt{2})^{4n} \left(e^{i3\pi/4}\right)^{4n}$$

we are keeping n of question outside bcoz RHS of question has $(\ )^n$ form

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$$\left\{\frac{1+7i}{(2-i)^2}\right\}^{4n} = \left\{\left(2^{1/2}\right)^4\right\}^n \cdot \left\{e^{i\frac{3\pi}{4}\cdot 4}\right\}^n$$

$$= \left(2^2\right)^n \cdot \left\{e^{i\cdot 3\pi}\right\}^n$$

$$= 4^n \cdot (\cos 3\pi + i\sin 3\pi)^n$$

$$= 4^n \cdot (-1+i0)^n$$

$$= 4^n \cdot (-1)^n$$

$$= (-4)^n$$

4) If α and β are roots of Equation $x^2-2x+4=0$ then using D'M theorem show that $\alpha^n + \beta^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$ and hence find the value of $\alpha^{10}+\beta^{10}$

(If Question me D'M theorem was not mentioned how can we identify that it is a question of complex?

Ans:- x^2-2x+4 ka roots will be complex (Finding Discriminant)

$\Rightarrow$ $x^2-2x+4=0 \Rightarrow x = \dfrac{-(-2) \pm \sqrt{(-2)^2-4(1)(4)}}{2(1)}$

$$= \frac{2 \pm \sqrt{4-16}}{2}$$

you can directly write using calculator

$$x = \frac{2 \pm i\sqrt{12}}{2} = \frac{2 \pm i\,2\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$ roots

Let $\alpha = 1 + i\sqrt{3}$ and ~~β = i~~ $\beta = 1 - i\sqrt{3}$

$1 + i\sqrt{3} = re^{i\theta}$ .... by exp form

Use calculator and directly write value
No need to show calculation of r and θ

$= 2e^{i\pi/3}$, $\parallel^{ly}$ $1 - i\sqrt{3} = 2e^{-i\pi/3}$

No need to show calculation $1+i\sqrt{3}$ & $1-i\sqrt{3}$ are conjugate of each other ∴ you can directly write

$$\alpha^n + \beta^n = (1 + i\sqrt{3})^n + (1 - i\sqrt{3})^n$$

$$= \left(2e^{i\pi/3}\right)^n + \left(2e^{-i\pi/3}\right)^n$$

$$= 2^n e^{in\pi/3} + 2^n e^{-in\pi/3}$$

$$= 2^n\left(e^{in\pi/3} + e^{-i\frac{n\pi}{3}}\right)$$

$$= 2^n\left[\cos\left(\frac{n\pi}{3}\right) + \cancel{i\sin\left(\frac{n\pi}{3}\right)} + \cos\left(\frac{n\pi}{3}\right) - \cancel{i\sin\left(\frac{n\pi}{3}\right)}\right]$$

$$= 2^n\, 2\cos\left(\frac{n\pi}{3}\right)$$

$$= 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$$

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To find $\alpha^{10}+\beta^{10}$, put $\boxed{n=10}$

$$\Rightarrow \alpha^{10}+\beta^{10} = 2^{10+1}\cos\left(\frac{10\pi}{3}\right)$$

$$= 2^{11} \cdot -\frac{1}{2}$$

$$= -(2^{10})$$

30/12/22. Solve $x^6+1=0$ using De'Moivre's theorem

$$\Rightarrow x^6 = -1 = -1+i0 = re^{i\theta} = 1e^{i\pi}$$

$$x = (1e^{i\pi})^{1/6}$$

($x^6+1=0$ can have 6 roots (maximum))

we can directly see that $\theta=\pi$ without using calculator

$\Rightarrow$ If we apply De Moivre's theorem here, we will get only 1 root

$\therefore$ Don't apply De Moivre's theorem in this stage.

$$x = (1e^{i\pi})^{1/6} = (\cos\pi + i\sin\pi)^{1/6}$$

$$x = \left[\cos(2\pi k+\pi) + i\sin(2\pi k+\pi)\right]^{1/6} \ldots \text{by general polar form}$$

where k = ~~non~~ non negative integer

(non negative means +ve + 0 i.e. whole nos.)

(~~General form of $\cos\theta$~~ $= \cos(2n\pi+\theta)$)

Similarly general form of $\sin\theta = \sin(2n\pi+\theta)$

$$x = \cos\left((2K+1)\frac{\pi}{6}\right) + i\sin\left((2K+1)\frac{\pi}{6}\right)$$

.... by D'M theorem
where $k = 0, 1, 2, 3, 4, 5$

6 answers

$$\Rightarrow \boxed{x = e^{i(2k+1)\pi/6}}, \quad k = 0, 1, 2, 3, 4, 5$$

~~ans is only upto here~~ ~~represents 6 distinct roots of~~

For $k = 0$, $x = e^{i\pi/6}$ → (If we had applied D'M at * step then we would have got only this ans)

For $k = 1$, $x = e^{i3\pi/6}$

For $k = 2$, $x = e^{i5\pi/6}$

For $k = 3$, $x = e^{i7\pi/6} = e^{i(12-5)\pi/6} = e^{i2\pi}e^{-i5\pi/6} = 1e^{-\frac{i5\pi}{6}}$

For $k = 4$, $x = e^{i9\pi/6} = e^{i(12-3)\pi/6} = e^{i2\pi}e^{-i3\pi/6} = 1e^{-i3\pi/6}$

For $k = 5$, $x = e^{i11\pi/6} = e^{i(12-1)\frac{\pi}{6}} = e^{i2\pi}e^{-i\pi/6} = 1e^{-i\pi/6}$

If all coeffn of eqn are real ~~complex roots should be conjugate of each other~~

Here $x = e^{i\pi/6}$ ka conjugate is $e^{-i\pi/6}$

But in the list of roots $e^{-i\pi/6}$ is not here BUT they are actually conjugate of each other

$e^{i7\pi/6}$ can be written as $e^{i(12-5)\pi/6}$

12 because it is the next multiple of 6

$$= e^{i2\pi}e^{-i5\pi/6} = e^{-i5\pi/6}$$

$\cos 2\pi + i\sin 2\pi$
$= 1$

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Q2 Solve $x^7 + x^4 + i(x^3+1) = 0$

$\Rightarrow x^7 + x^4 + i(x^3+1) = 0$

$x^4(x^3+1) + i(x^3+1) = 0$

$(x^3+1)(x^4+i) = 0$

$\Rightarrow \boxed{x^3+1=0}$ or $\boxed{x^4+i=0}$ → this will give 4 roots

this eqn will give 3 roots

3+4 = 7 roots

consider $x^3+1=0$

$(x+1)(x^2-x+1)=0$ ... using $a^3+b^3 = (a+b)(a^2-ab+b^2)$

$\therefore (x+1)=0$ or $x^2-x+1=0$

i.e. $\boxed{x=-1}$

i.e. $x = \dfrac{-(-1) \pm \sqrt{(-1)-4(1)(1)}}{2(1)}$

You can use calculator as well

$$\boxed{x = \frac{1 \pm i\sqrt{3}}{2}}$$

or $x = \dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}$

$\therefore x = -1, \dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}$ are roots of $x^3+1=0$

Now consider $x^4+i=0$

$x^4 = -i = re^{i\theta} = 1e^{i3\pi/2}$

or

$1e^{-i\pi/2}$

You can write $\frac{3\pi}{2}$ or $-\pi/2$

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$$x = \left(1e^{i3\pi/2}\right)^{1/4} \text{ or } \left(1e^{-i\pi/2}\right)^{1/4}$$

$$x = \left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]^{1/4}$$

(we can write this as $\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

$[\because \cos(-\theta) = \cos\theta \quad \sin(-\theta) = -\sin\theta]$

But we have to apply D'M theorem later

D'M theorem is applicable only for $\cos\theta + i\sin\theta$ NOT for $\cos\theta - i\sin\theta$)

$$x = \left[\cos\left(2\pi k - \frac{\pi}{2}\right) + i\sin\left(2\pi k - \frac{\pi}{2}\right)\right]^{1/4}$$

.... by general polar form

$$x = \left[\cos\left((4k-1)\frac{\pi}{2}\right) + i\sin\left((4k-1)\frac{\pi}{2}\right)\right]^{1/4}$$

$$x = \left(\cos\left((4k-1)\frac{\pi}{8}\right) + i\sin\left(4k-1\right)\frac{\pi}{8}\right)$$

.... by D'M theorem

where $k = 0, 1, 2, 3$

$\because$ we are solving $x^4 + i = 0$

4 degree equation

$$\boxed{x = e^{i(4k-1)\pi/8}}, \quad k = 0, 1, 2, 3$$

represents 4 distinct roots of $x^4 + i = 0$

$\therefore$ All roots of eq^n $x^7+x^4+i(x^3+1)=0$ are given by
$\underbrace{-1, \frac{1}{2} \pm \frac{i\sqrt{3}}{2}}$, $\underbrace{e^{-i\pi/8}, e^{i3\pi/8}, e^{i7\pi/8}, e^{i11\pi/8}}$ //

$x^3+1=0$ ka roots are conj pairs bcoz all coefficient in x^3+1 are real

But x^4+i me all coeff are not real. Here constant is complex $\therefore$ these roots won't be conj pairs

3) Solve $x^4-x^3+x^2-x+1=0$

$\rightarrow$ 1, $-x$, x^2, $-x^3$, x^4 follows Geometric Progr. G.P. with 1^{st} term $\boxed{a=1}$ and common ratio $\boxed{r=-x}$

$\therefore$ $1-x+x^2-x^3+x^4=$ Sum of 5 terms of G.P. (S_5)

$$= \frac{a}{\cancel{(1-r)}}\cancel{(1-r^5)} \; \frac{a}{(1-r)}(1-r^5)$$

$$= \frac{1}{[1-(-x)]}\left[1-(-x)^5\right]$$

$$1-x+x^2-x^3+x^4 = \frac{(1+x^5)}{(1+x)}$$

$$\boxed{(x^5+1) = (x+1)(x^4-x^3+x^2-x+1)}$$

$\therefore x^4-x^3+x^2-x+1$ is a root of x^5+1

$\therefore$ To solve this it is enough to solve x^5+1 which will have 5 roots

Ignore this bcoz question me it is not asked $\leftarrow$ 1 root of $x+1$

and another of $x^4-x^3+x^2-x+1$

$$\boxed{(x^5+1) = (x+1)(x^4-x^3+x^2-x+1)}$$

Enough to solve $x^5+1=0$

Consider $x^5+1=0$

$x^5 = -1 = -1+i0 = re^{i\theta} = 1e^{i\pi}$

$x = (e^{i\pi})^{1/5} = (\cos\pi + i\sin\pi)^{1/5}$

$x = [\cos(2\pi k+\pi) + i\sin(2\pi k+\pi)]^{1/5}$

$x = \cos\left((2k+1)\frac{\pi}{5}\right) + i\sin\left((2k+1)\frac{\pi}{5}\right)$ ... by D'M

$\boxed{x = e^{i(2k+1)\pi/5}}$; $k=0,1,2,3,4$

$\Rightarrow$ 5 distinct roots of $x^5+1=0$

$k=0$ $x = e^{i\pi/5}$

$k=1$; $x = e^{i3\pi/5}$

$k=2$; $x = e^{i5\pi/5} = e^{i\pi} = -1$ which belongs to $(x+1)=0$

$k=3$ $x = e^{i7\pi/5}$

$k=4$; $x = e^{i9\pi/5}$

$\Rightarrow e^{i\pi/5}, e^{i3\pi/5}, e^{i7\pi/5}, e^{i9\pi/5}$ roots belong to $x^4-x^3+x^2-x+1=0$ //

$\rightarrow$ they are in conjugate pairs

$e^{i\pi/5}$ & $e^{i9\pi/5}$ } are conj pairs

$e^{i3\pi/5}$ & $e^{i7\pi/5}$ } $e^{i9\pi/5} = e^{(10-1)i\pi/5}$

4) Find the continued product of $(1+i)^{4/5}$ using De' Moivre's theorem.

$\Rightarrow$ Let $x = (1+i)^{4/5}$

Here degree is not 4/5 ($\because$ i is not an integer)

i.e. $x^5 = (1+i)^4 = \cancel{re^{i\theta}} (re^{i\theta})^4$

$= \left(\sqrt{2}\, e^{i\pi/4}\right)^4$

$\therefore$ degree of equation = 5

$\therefore$ It will have 5 roots

$= 4e^{i\pi}$

i.e. $x = \left(4e^{i\pi}\right)^{1/5}$

Don't apply D'M directly here or else we'll get only 1 answer

$x = \left(4e^{i\pi}\right)^{1/5} = \left[4(\cos\pi + i\sin\pi)\right]^{1/5}$

$= \left\{4\left[\cos(2\pi k+\pi) + i\sin(2\pi k+\pi)\right]\right\}^{1/5}$

$= 4^{1/5}\left[\cos\left((2k+1)\frac{\pi}{5}\right) + i\sin\left((2k+1)\frac{\pi}{5}\right)\right]$ ..... by D'M theorem

i.e. $\boxed{x = 4^{1/5}\, e^{i(2k+1)\pi/5}}$, $k = 0, 1, 2, 3, 4$

gives 5 distinct roots of $x^5 = (1+i)^4$

$\therefore$ The continued product of $(1+i)^{4/5}$ = The continued product of roots of $x^5 = (1+i)^4$

by substituting k=0

$= 4^{1/5}e^{i\pi/5}, 4^{1/5}e^{i3\pi/5}, 4^{1/5}e^{i5\pi/5}, 4^{1/5}e^{i7\pi/5}, 4^{1/5}e^{i9\pi/5}$

$= \cancel{(4^{1/5})^5 \cdot e^{i\pi/5+}}$ This is multiplied

$$= \left(4^{1/5}\right)^5 \cdot e^{i\pi/5 + i3\pi/5 + i5\pi/5 + i7\pi/5 + i9\pi/5}$$

$$= \left(4^{1/5}\right)^5 e^{\left(i\pi/5 + i3\pi/5 + i5\pi/5 + i7\pi/5 + i9\pi/5\right)}$$

$$= 4 e^{i\pi/5(1+3+5+7+9)}$$

$$= 4e^{i25\pi/5} = 4e^{i5\pi}$$

$$= 4\left(\cos 5\pi + i\sin 5\pi\right)$$

$$= 4\left(-1+i0\right)$$

$= -4$ → This is the continued product

* HYPERBOLIC FUNCTIONS :

For x real or complex

(i) $\boxed{\sinh x = \frac{e^x - e^{-x}}{2}}$ → true for x = real and x = complex

(ii) $\boxed{\cosh x = \frac{e^x + e^{-x}}{2}}$

(iii) $\tanh x = \frac{\sinh x}{\cosh x} = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$

(iv) $\coth x = \frac{1}{\tanh x} = \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)$

(v) $\operatorname{sech} x = \frac{1}{\cosh x}$

(vi) $\text{cosech}\,x = \frac{1}{\sinh x}$

Note: 1) $\sinh(0) = 0$ $\left(\frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0\right)$

2) $\cosh(0) = 1$

3) $\cosh(x) \neq 0$, always (cos h(x) is never equal to 0 even for $x = \infty$)

4) $\frac{d}{dx}(\sinh x) = \cosh x$

5) $\frac{d}{dx}(\cosh x) = \sinh x$

6) Euler's expressions:

(i) $e^{i\theta} = \cos\theta + i\sin\theta$

(ii) $e^{-i\theta} = \cos\theta - i\sin\theta$

(i) + (ii) $\Rightarrow \cos\theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)$ → exponents are purely imaginary

(i) − (ii) $\Rightarrow \sin\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)$

* Relations between trigonometric & Hyperbolic functions

→ Trigo to Hyperbolic

*** 1) $\boxed{\sin(i\theta) = i\sinh\theta}$

*** 2) $\boxed{\cos(i\theta) = \cosh\theta}$

Trigonometric Hyperbolic

From these 2 relations, rest of the 4 relation can be derived

3) $\tan(i\theta) = i\tanh\theta$

4) $\cot(i\theta) = -i\coth\theta$ $\left[\because \frac{1}{i} = \frac{1}{i} \times \frac{-i}{-i} = \frac{-i}{-(i)^2} = \frac{-i}{-(-1)} = -i\right]$

$\frac{1}{i} = \frac{1}{i} \times \frac{-i}{-i} = \frac{-i}{-(i)^2} = \frac{-i}{-(-1)} = -i$ rationalize

5) $\sec(i\theta) = \text{sech}\,\theta$

6) $\text{cosec}(i\theta) = -i\,\text{cosech}\,\theta$

→ Hyperbolic to Trigo

1) $\boxed{\sinh(i\theta) = i\sin\theta}$ (Replace sinh & sin in $x \to i$)

2) $\boxed{\cosh(i\theta) = \cos\theta}$ → Derive remaining 4 from these 2

3) $\tanh(i\theta) = i\tan\theta$

4) $\coth(i\theta) = -i\cot\theta$

5) ~~cosec~~ $\text{sech}(i\theta) = \sec\theta$

6) $\text{cosech}(i\theta) = -i\,\text{cosec}\,\theta$

* Hyperbolic Identities

1) ~~$\cosh^2 x +$~~ $\boxed{\cosh^2 x - \sinh^2 x = 1}$

Proof

→ We have $\cos^2 x + \sin^2 x = 1$

replace x by ix

$\Rightarrow \underbrace{\{\cos(ix)\}^2 + \{\sin(ix)\}^2}_{\text{trig to hyperbolic}} = 1$

Trigonometric functions are also called circular functions

$(i)^2 = -1 \quad (-i) = -1$

$\{\cosh x\}^2 + \{i \sinh x\}^2 = 1$

$\therefore$ By relation between hyperbolic and trigonometric

$\cosh^2 x - \sinh^2 x = 1$ //

2) $1 - \tanh^2 x = \operatorname{sech}^2 x$

(In trigo to hyperbolic relations except for cos & sec, rest all have i or $-i$
and square of i & $-i = -1$
$\therefore$ wherever $+ \longrightarrow -$
for these functions (except cos & sec))

3) $\boxed{1 - \coth^2 x = - \operatorname{cosech}^2 x}$

4) $\sinh(2x) = 2 \sinh x \cosh x$

$= \dfrac{2\tanh x}{1 - \tanh^2 x}$

$\sin 2x = 2 \sin x \cos x$
$\sin 2x = \dfrac{2\tan x}{1 + \tan^2 x}$

5) $\cosh(2x) = 2\cosh^2 x - 1$

$= 1 + 2\sinh^2 x$

$= \cosh^2 x + \sinh^2 x$

$= \dfrac{1 + \tanh^2 x}{1 - \tanh^2 x}$

$\cos 2x = 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$
$= \cos^2 x - \sin^2 x$
$= \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

6) $\tanh(2x) = \dfrac{2\tanh x}{1 + \tanh^2 x}$

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1) If $\tanh(x) = \frac{2}{3}$ find value of x and $\cosh(2x)$

$\Rightarrow \tanh(x) = \frac{2}{3}$

By definition of $\tanh(x)$

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{2}{3}$$

$$\therefore 3(e^x - e^{-x}) = 2(e^x + e^{-x})$$

$$3e^x - 3e^{-x} = 2e^x + 2e^{-x}$$

$$1e^x - 5e^{-x} = 0$$

$$e^x - \frac{5}{e^x} = 0$$

If we solve this as quadratic we'll get 2 ans. $(e^x)^2 = 5$

$e^x = \pm 5 \Rightarrow x = \ln 5$ & $x = \ln(-5)$ not defined

$$\frac{(e^x)^2 - 5}{e^x} = 0$$

$\therefore$ At the end $x = \ln 5$

(here we solved using indices)

e^x is non zero quantity $\therefore$ we can multiply it with zero

$$(e^x)^2 - 5 = 0$$

$\rightarrow$ This is a quadratic equation in e^x

$$\therefore e^{2x} = 5$$

$$\cancel{2x = \log 5} \quad 2x = \ln(5)$$

$$\boxed{x = \frac{1}{2}\ln(5)}$$

($\ln$) stands for log with base e

log stands for log with base 10

$$\text{or} \quad x = \ln(5)^{1/2}$$

$$x = \ln(\sqrt{5}) \; //$$

By hyperbolic identity

we have $$\cosh(2x) = \frac{1 + \tanh^2(x)}{1 - \tanh^2(x)}$$

$$= \frac{1 + (2/3)^2}{1 - (2/3)^2}$$

$$= \frac{1+4/9}{1-4/9}$$

$$= \frac{13/9}{5/9}$$

$$\therefore \cosh(2x) = \frac{13}{5} \,//$$

Q2) Prove that $(\cosh x - \sinh x)^n = \cosh(nx) - \sinh(nx)$

This formula is similar to D'M theorem
But it is NOT D'M theorem

D'M	this formula
i present	No i
+ sign	− sign
trigo function	hyperbolic function

$\Rightarrow$ LHS $= (\cosh x - \sinh x)^n$

$$= \left\{ \left(\frac{e^x + e^{-x}}{2} \right) - \left(\frac{e^x - e^{-x}}{2} \right) \right\}^n \quad \text{... By definition of sinh and cosh}$$

$$= \left\{ \frac{\cancel{e^x} + e^{-x} - \cancel{e^x} + e^{-x}}{2} \right\}^n$$

$$= \left\{ \frac{\cancel{2} e^{-x}}{\cancel{2}} \right\}^n$$

$$= (e^{-x})^n = e^{-nx} \quad \text{——— (1)}$$

$$RHS = \cosh(nx) - \sinh(nx)$$

$$= \left(\frac{e^{nx} + e^{-nx}}{2}\right) - \left(\frac{e^{nx} - e^{-nx}}{2}\right) \quad \text{... by def}^n \text{ of hyperbolic}$$

$$= \frac{\cancel{e^{nx}} + e^{-nx} - \cancel{e^{nx}} + e^{-nx}}{2}$$

$$= \frac{\cancel{2} e^{-nx}}{\cancel{2}}$$

$$= e^{-nx} \quad \text{———} \quad (2)$$

$\therefore$ (1) and (2) $\Rightarrow$ LHS = RHS //

3) Solve $\sinh x - \cosh x - 5 = 0$ for real values of x

$\rightarrow$ $\sinh x - \cosh x - 5 = 0$

$$\left(\frac{e^{x} - e^{-x}}{2}\right) - \left(\frac{e^{x} + e^{-x}}{2}\right) - 5 = 0$$

$$\frac{\cancel{e^{x}} - e^{-x} - \cancel{e^{x}} - e^{-x}}{2} = 5$$

$$-2e^{-x} = 10$$

i.e. $-e^{-x} = 5$

Don't write it as $\frac{1}{e^x}$ bcoz then

$\frac{-1}{5} = e^{x}$ (exponention can never be -ve)

We cannot take log directly

log (-ve) not defined

eliminate -ve sign $\therefore$ square both sides on squaring we get

$$e^{-2x} = 25$$
$$\Rightarrow -2x = \ln(25)$$

$$x = -\frac{1}{2}\ln(25) = -\ln(25)^{1/2} = -\ln\sqrt{25}$$
$$= -\ln(5) \text{ or } \ln\left(\frac{1}{5}\right)$$

4) Prove that $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{(1+\sinh^2 x)}}} = -\sinh^2(x)$

Avoid doing this example by definition. It will become lengthy. ∴ Solve using identity. We get this hint by seeing $(1+\sinh^2 x)$ which is equal to $\cosh^2 x$

$$\rightarrow LHS = \frac{1}{1-\dfrac{1}{1-\dfrac{1}{(1+\sinh^2 x)}}} = \frac{1}{1-\dfrac{1}{1-\left\{\dfrac{1}{(\cosh^2 x)}\right\}}}$$

.... using $\cosh^2 x - \sinh^2 x = 1$

$$= \frac{1}{1-\dfrac{1}{(1-\operatorname{sech}^2 x)}} = \frac{1}{1-\left\{\dfrac{1}{(\tanh^2 x)}\right\}}$$

...using $1-\tanh^2 x = \operatorname{sech}^2 x$

$$= \frac{1}{1-\coth^2 x} = \frac{1}{-\operatorname{cosech}^2 x}$$ ... using $1-\coth^2 x = -\operatorname{cosech}^2 x$

$$= -\sinh^2 x$$
$$= RHS$$

trigo hyperbolic

5) If $\tan\left(\frac{x}{2}\right) = \tanh\left(\frac{u}{2}\right)$ then Prove that

(i) $\sinh u = \tan x$ (ii) $\cosh u = \sec x$

(iii) $u = \log\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$

→ By Hyperbolic identity

$$\boxed{\sinh(2u) = \cancel{\frac{2\tanh u}{1-\tan^2}} \quad \frac{2\tanh(u)}{1-\tanh^2(u)}}$$

$$\Rightarrow \sinh(u) = \frac{2\tanh(u/2)}{1-\tanh^2(u/2)} \qquad \left(\tanh\left(\frac{u}{2}\right) = \tan\frac{x}{2}\right)$$

$$= \frac{2\tan(x/2)}{1-\tan^2(x/2)} \quad \text{—— (1) ... By data}$$

$$\Rightarrow \tan(2x) = \frac{2\tan x}{1-\tan^2 x}$$

$$\Rightarrow \tan x = \frac{2\tan(x/2)}{1-\tan^2(x/2)} \quad \text{—— (2)}$$

By (1) and (2) $\Rightarrow \sinh(u) = \tan x$ //

similarly

$||^y$ we have $\boxed{\cosh(2u) = \frac{1+\tanh^2 u}{1-\tanh^2 u}}$

$$\Rightarrow \cosh(u) = \frac{1+\tanh^2(u/2)}{1-\tanh^2(u/2)}$$

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$$\cosh(u) = \frac{1+\tan^2(x/2)}{1-\tan^2(x/2)} \quad \text{--- (3)}$$

Now :- $\cos(2x) = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

$$\Rightarrow \cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$$

$$\Rightarrow \sec(x) = \frac{1+\tan^2(x/2)}{1-\tan^2(x/2)} \quad \text{--- (4)}$$

By (3) and (4) $\Rightarrow \cosh u = \sec x$

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ii) $\rightarrow$ To prove $u = \log\left[\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)\right]$

enough to prove $e^u = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$

i.e. $e^u = \dfrac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1-\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{x}{2}\right)}$

i.e. $e^u = \dfrac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}$

We have $\tan\left(\frac{x}{2}\right) = \tanh\left(\frac{u}{2}\right)$

$$\tan\left(\frac{x}{2}\right) = \frac{e^{u/2}-e^{-u/2}}{e^{u/2}+e^{-u/2}}$$

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$\cancel{\tan\left(\frac{x}{2}\right)}$ i.e. $\dfrac{e^{u/2}-e^{-u/2}}{e^{u/2}+e^{-u/2}} = \dfrac{\tan\left(\frac{x}{2}\right)}{1}$

use componendo - dividendo to get x.x

By componendo and dividendo rule,

$$\frac{(e^{u/2}-\cancel{e^{-u/2}})+(e^{u/2}+\cancel{e^{-u/2}})}{(\cancel{e^{u/2}}-e^{-u/2})-(\cancel{e^{u/2}}+e^{-u/2})} = \frac{\tan(x/2)+1}{\tan(x/2)-1}$$

$$\frac{\cancel{2}e^{u/2}}{-\cancel{2}e^{-u/2}} = \frac{[1+\tan(x/2)]}{\cancel{-}[1-\tan(x/2)]}$$

$$e^{u/2}\cdot e^{u/2} = \frac{1+\tan(x/2)}{1-\tan(x/2)}$$

$$\Rightarrow \quad e^{u} = \frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)} \; //$$

Definition:- # Inverse Hyperbolic functions:

1) $\sinh^{-1}(x) = \log\left(x + \sqrt{x^2+1}\right)$

this was the answer of one of the standard integration

$\int \frac{1}{\sqrt{1+x^2}} = \log\left(x+\sqrt{x^2+1}\right)$

Proof:- Let $\boxed{\sinh^{-1}(x) = y}$ —— ①

i.e. $x = \sinh(y)$

$\Rightarrow x + \sqrt{x^2+1} = \sinh(y) + \sqrt{\sinh^2 y + 1}$

$= \sinh(y) + \sqrt{\cosh^2 y}$ ... ~~cosh~~

$\because \cosh^2 y - \sinh^2 y = 1$

$= \sinh y + \cosh y$

$= \left(\frac{e^y - e^{-y}}{2}\right) + \left(\frac{e^y + e^{-y}}{2}\right) = \frac{2e^y}{2} = e^y$

$\Rightarrow x + \sqrt{x^2+1} = e^y$

$\log\left(x + \sqrt{x^2+1}\right) = y$ —— ②

$\therefore$ By ① and ②

$\sinh^{-1}(x) = \log\left(x + \sqrt{x^2+1}\right)$

2) $\cosh^{-1}(x) = \log\left(x + \sqrt{x^2-1}\right)$

(this is also the ans of standard integration)

3) $\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

(sech, coth, cosech can be converted into these 3 forms)

Eg i)

(i)

$\text{sech}^{-1}(y) \neq \frac{1}{\cosh^{-1}(y)}$ $\quad \text{sech}\,y = \frac{1}{\cosh y}$

eg i) Prove that (i) $\tanh^{-1}(x) = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

(ii) $\text{sech}^{-1}(\sin\theta) = \log\left[\cot\left(\frac{\theta}{2}\right)\right]$

→ (i)

Solving this using definition of inverse hyperbolic will become complicated

Although this is a question of inverse, solving RHS using definition will become lengthy

(i) Let $\tanh^{-1}(x) = y$ — ①

i.e. $x = \tanh(y)$

$$\frac{x}{\sqrt{1-x^2}} = \frac{\tanh(y)}{\sqrt{1-\tanh^2 y}}$$

$$\frac{x}{\sqrt{1-x^2}} = \frac{\tanh(y)}{\text{sech}(y)} \quad \because 1-\tanh^2 y = \text{sech}^2 y$$

$$\frac{x}{\sqrt{1-x^2}} = \frac{\sinh y/\cancel{\cosh y}}{1/\cancel{\cosh y}} = \sinh y$$

$$\Rightarrow \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = y \quad \text{— ②}$$

By ① and ②

$$\tanh^{-1}(x) = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

//

(ii) Let $sech^{-1}(\sin\theta) = y$ —— (1)

i.e. $\sin\theta = sech(y)$

$\Rightarrow cosec\,\theta = \cosh(y)$ → You could have written $y = \cosh^{-1}(cosec\theta)$ in this step as well

$\Rightarrow cosec^2\theta = \cosh^2(y)$

$\Rightarrow cosec^2\theta = 1 + \sinh^2(y)$

i.e. $-\sinh^2 y = 1 - cosec^2\theta$

$-\sinh^2 y = -\cot^2\theta$

$\sinh y = \cot\theta$

$\Rightarrow y = \sinh^{-1}(\cot\theta)$

$\therefore y = \log\left(\cot\theta + \sqrt{\cot^2\theta + 1}\right)$ ... by definition of $\sinh^{-1}$

$y = \log(\cot\theta + cosec\theta)$

$y = \log\left(\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta}\right)$

$y = \log\left(\frac{1+\cos\theta}{\sin\theta}\right)$

$y = \log\left[\frac{\cancel{2}\cos^{\cancel{2}}\left(\frac{\theta}{2}\right)}{\cancel{2}\sin\left(\frac{\theta}{2}\right)\cancel{\cos\left(\frac{\theta}{2}\right)}}\right]$

$y = \log\left[\cot\left(\frac{\theta}{2}\right)\right]$ —— (2)

By (1) and (2)

$sech^{-1}(\sin\theta) = \log\left[\cot\left(\frac{\theta}{2}\right)\right]$ //

Eg 2) If $\cosh x = \sec\theta$ Prove that

(i) $x = \log(\sec\theta + \tan\theta)$

(ii) $\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$

(iii) $\sinh x = \tan\theta$ (iv) $\tanh x = \sin\theta$

(v) $\tanh\left(\frac{x}{2}\right) = \pm\tan\left(\frac{\theta}{2}\right)$

→ (i) $\cosh(x) = \sec\theta$

$x = \cosh^{-1}(\sec\theta)$

$x = \log\left(\sec\theta + \sqrt{\sec^2\theta - 1}\right)$ ... By definition of $\cosh^{-1}$

$x = \log(\sec\theta + \tan\theta)$ ... $\because 1 + \tan^2\theta = \sec^2\theta$

(ii) To prove $\boxed{\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})}$

By (i) $x = \log(\sec\theta + \tan\theta)$

$e^{x} = \sec\theta + \tan\theta$

$e^{-x} = \frac{1}{(\sec\theta + \tan\theta)} \times \frac{\sec\theta - \tan\theta}{(\sec\theta - \tan\theta)}$

$e^{-x} = \frac{\sec\theta - \tan\theta}{\sec^2\theta - \tan^2\theta} = \frac{\sec\theta - \tan\theta}{1}$

$e^{-x} = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$

$e^{-x} = \frac{1 - \sin\theta}{\cos\theta} = \frac{(\cos\theta/2 - \sin\theta/2)^2}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}$

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$$e^{-x} = \frac{\left(\cos\theta/_2 - \sin\theta/_2\right)^2}{\left(\cos\theta/_2 - \sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)}$$

$$e^{-x} = \frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}$$ (Dividing by $\cos\frac{\theta}{2}$)

$$= \frac{1 - \tan\theta/_2}{1 + \tan\theta/_2} = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$\Rightarrow \tan^{-1}(e^{-x}) = \frac{\pi}{4} - \frac{\theta}{2}$$

$$\Rightarrow \frac{\theta}{2} = \frac{\pi}{4} - \tan^{-1}(e^{-x})$$

$$\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x}) \;//$$

$$\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x}) \quad //$$

(iii) $$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$= \frac{(\sec\theta + \tan\theta) - (\sec\theta - \tan\theta)}{2} \text{ by result (i)}$$

$$= \frac{\cancel{2}\tan\theta}{\cancel{2}} = \tan\theta \quad //$$

$(x+1)^8 + (x-1)^8 = 0$ Here degree will be 8 bcoz 1st term x^8 for both will add up

$(x+1)^8 - (x-1)^8 = 0$ degree = 7

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TUTORIAL

PRACTICE QUESTIONS

though power is 7, degree is not 7

10 marks ★★★ 1) Show that all the roots of $(x+1)^7 = (x-1)^7$ are given by $\boxed{\pm i\cot\left(\frac{k\pi}{7}\right)}$ where $k = 1, 2, 3$

+i: 3 roots -i: 3 roots Total: 6 roots

→ $(x+1)^7 = (x-1)^7$ is equation of deg. $\boxed{6}$ as the term x^7 gets cancelled after expansion of $(x+1)^7$ & $(x-1)^7$.
∴ It has 6 distinct roots

first term of $(x+1)^7 = x^7$ 1st term of $(x-1)^7 = x^7$
gets cancelled

2nd term of $(x+1)^7 = x^6$ 2nd term of $(x-1)^7 = -x^6$

∴ x^6 will not get cancelled

Hence degree = 6

Now, $(x+1)^7 = (x-1)^7$ ∴ We want a nonzero term on RHS (It can be complex but should be non zero)

$$\frac{(x+1)^7}{(x-1)^7} = 1 = re^{i\theta} = 1e^{i0}$$

$$\frac{(x+1)^7}{(x-1)^7} = 1e^{i0} \Rightarrow \left(\frac{x+1}{x-1}\right)^7 = e^{i0}$$

→ Don't rationalize. If we rationalize, we'll get x^2 in numerator & denominator. We want only x in LHS, NOT x^2.

$$\left(\frac{x+1}{x-1}\right)^7 = e^{i0} = \cos 0 + i\sin 0$$
$$= \cos(2\pi k + 0) + i\sin(2\pi k + 0)$$

$$\frac{x+1}{x-1} = \left\{\cos(2\pi k) + i\sin(2\pi k)\right\}^{1/7}$$
$$= \cos\left(\frac{2\pi k}{7}\right) + i\sin\left(\frac{2\pi k}{7}\right) \quad \text{... by D'M theorem}$$

where $k = 0, 1, 2, \ldots, 6$

We are getting 7 roots which is not matching with the 1st statement that we have written (stating that eq^n has 6 roots).

$\Rightarrow$ Out of 0, 1, 2, ... 6, one of the root is incorrect.

1, 2, 3 cannot be wrong bcoz we have been asked to prove that.

$\Rightarrow$ Out of 0, 4, 5, 6, one root is incorrect

$\therefore$ we'll have to check using trial & error

For $k = 0$; $\dfrac{x+1}{x-1} = \cos 0 + i\sin 0 = 1 + i0$

$$\frac{x+1}{x-1} = 1 \qquad \Rightarrow \boxed{x+1 = x-1} \quad (1 \neq -1)$$

which is not true

$\Rightarrow$ k cannot be zero

$$\therefore \frac{x+1}{x-1} = \cos\left(\frac{2\pi k}{7}\right) + i\sin\left(\frac{2\pi k}{7}\right), \; k = 1, 2, 3, 4, 5, 6$$

This does not represent 6 roots

For an expression to represent roots, LHS should be only x, not an expression in x

$\therefore$ use componendo - dividendo

$$\frac{(x+1)+(x-1)}{(x+1)-(x-1)} = \frac{\cos\left(\frac{2\pi k}{7}\right) + i\sin\left(\frac{2\pi k}{7}\right) + 1}{\cos\left(\frac{2\pi k}{7}\right) + i\sin\left(\frac{2\pi k}{7}\right) - 1}$$

$$\frac{2x}{2} = \frac{(\cos\theta + 1) + i\sin\theta}{(\cos\theta - 1) + i\sin\theta} \quad \text{where } \boxed{\theta = \frac{2\pi k}{7}}$$

$$\therefore x = \frac{2\cos^2\left(\frac{\theta}{2}\right) + i2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{-2\sin^2\left(\frac{\theta}{2}\right) + i2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

we have $\cos\frac{\theta}{2}$ as real in numerator $\therefore$ we'll take $i2\sin(\theta/2)$ to keep ~~cos as real~~

$$x = \frac{2\cos\left(\frac{\theta}{2}\right)\left[\cancel{\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)}\right]}{2i\sin\left(\frac{\theta}{2}\right)\left[\cancel{i\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)}\right]} \quad \ldots\ \because -1 = i^2$$

$i^2 = -1$ $\therefore$ ultimately if we open it will be $-2\sin^2\left(\frac{\theta}{2}\right)$

$$x = \frac{1}{i}\cot\left(\frac{\theta}{2}\right) \Rightarrow x = -i\cot\left(\frac{\theta}{2}\right)$$

$$= -i\cot\left(\frac{\pi k}{7}\right); k = 1,2,3,4,5,6$$

represents 6 ~~diff~~ distinct roots of
$(x+1)^7 = (x-1)^7$

~~We'll have to show that 1st three roots are~~ conjugates of last three roots in order to prove $x = +i\cot\left(\frac{\theta}{2}\right)$ (We have already proved $x = -i\cot\left(\frac{\theta}{2}\right)$)

For $k = 1$; $x = -i\cot\left(\frac{\pi}{7}\right)$

For $k = 2$; $x = -i\cot\left(\frac{2\pi}{7}\right)$

For $k = 3$; $x = -i\cot\left(\frac{3\pi}{7}\right)$

we have to prove this to don't touch these conj   if $N<D$ $\frac{D-N}{D}$; $N>D$, $\frac{mD-N}{D}$ next multiple of Denominator

For $k=4$; $x = -i\cot\left(\frac{4\pi}{7}\right) = -i\cot\left((7-3)\frac{\pi}{7}\right)$

$= -i\cot\left(\pi - \frac{3\pi}{7}\right) = i\cot\left(\frac{3\pi}{7}\right)$

$\boxed{\because \cot(\pi-\theta) = -\cot\theta}$

For $k=5$; $x = -i\cot\left(\frac{5\pi}{7}\right) = -i\cot\left((7-2)\frac{\pi}{7}\right)$

$= -i\cot\left(\pi - \frac{2\pi}{7}\right) = i\cot\left(\frac{2\pi}{7}\right)$

For $k=6$; $x = -i\cot\left(\frac{6\pi}{7}\right) = -i\cot\left((7-1)\frac{\pi}{7}\right)$

$= -i\cot\left(\pi - \frac{\pi}{7}\right) = i\cot\left(\frac{\pi}{7}\right)$

$\Rightarrow$ All the roots of $(x+1)^7 = (x-1)^7$ are given by $\boxed{\pm i\cot\left(\frac{k\pi}{7}\right)}$; $k = 1, 2, 3$ //

2) Find the value of $\tanh(\log\sqrt{5})$

$\rightarrow$ $\tanh(\log\sqrt{5}) = \dfrac{e^{\log\sqrt{5}} - e^{-\log\sqrt{5}}}{e^{\log\sqrt{5}} + e^{-\log\sqrt{5}}}$

... by def^n of $\tanh$

antilog. log will cancel only when log ka sign is +ve

$$= \frac{e^{\log\sqrt{5}} - e^{\log(\sqrt{5})^{-1}}}{e^{\log\sqrt{5}} + e^{\log(\sqrt{5})^{-1}}}$$

$$= \frac{\sqrt{5} - (\sqrt{5})^{-1}}{\sqrt{5} + (\sqrt{5})^{-1}} = \frac{\sqrt{5} - \frac{1}{\sqrt{5}}}{\sqrt{5} + \frac{1}{\sqrt{5}}} = \frac{5-1}{5+1}$$

$$= \frac{2}{3}$$ //

3) Prove that $\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \frac{-i}{2}\log\left(\frac{a}{x}\right)$

we don't want $\tan^{-1}$ in RHS

∴ To eliminate $\tan^{-1}$ in LHS, we'll have to write $\left[i\left(\frac{x-a}{(x+a)}\right)\right]$ in terms of tan

∴ for that we'll have to convert $\left(\frac{x-a}{x+a}\right)$ in terms of $\tanh(\ ) \Rightarrow i\tanh(\ ) = \tan(\ )$

$\tan(i\theta) = i\tanh\theta$

→ $LHS = \tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right]$ ~~we cannot directly~~ substitute $x = e^{y}$ bcoz we have ~~constant term~~

To generate exponential form, take 'a' common from Numerator & denominator)

∴ Divide Num & Den by a

$$= \tan^{-1}\left[i\left(\frac{\frac{x}{a}-1}{x/a+1}\right)\right]$$ put $\boxed{\frac{x}{a} = e^{y}}$

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$$LHS = \tan^{-1}\left[i\left(\frac{e^y-1}{e^y+1}\right)\right]$$

$$= \tan^{-1}\left[i\left(\frac{e^y-1}{e^y+1}\right) \times \frac{e^{-y/2}}{e^{-y/2}}\right]$$

$$= \tan^{-1}\left[i\left(\frac{e^{y/2}-e^{-y/2}}{e^{y/2}+e^{-y/2}}\right)\right]$$

$$= \tan^{-1}\left[i\tanh\left(\frac{y}{2}\right)\right]$$

$$= \cancel{\tan^{-1}}\left[\cancel{\tan}\left(\frac{iy}{2}\right)\right] \ldots \text{ using } \tan(i\theta) = i\tanh\theta$$

$$= \frac{i}{2}y$$

$$= \frac{i}{2}\log\left(\frac{x}{a}\right)$$

$$\frac{x}{a} = e^y$$

$$\Rightarrow y = \log\left(\frac{x}{a}\right)$$

$$= \frac{i}{2}\log\left(\frac{a}{x}\right)^{-1}$$

$$= -\frac{i}{2}\log\left(\frac{a}{x}\right) = RHS \,//$$

$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

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Eg 2 :- continued from pg 139

iv) $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\tan\theta}{\sec\theta}$ ... by result (iii) and data

$= \sin\theta$

v) By data we have

$\cosh x = \sec\theta = \frac{1}{\cos\theta}$

$\therefore \frac{1+\tanh^2\left(\frac{x}{2}\right)}{1-\tanh^2\left(\frac{x}{2}\right)} = \frac{1}{\frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}}$

$\frac{1+\tanh^2\left(\frac{x}{2}\right)}{1-\tanh^2\left(\frac{x}{2}\right)} = \frac{1+\tan^2\left(\frac{\theta}{2}\right)}{1-\tan^2\left(\frac{\theta}{2}\right)}$

On term by term comparison we get $\tanh^2\left(\frac{x}{2}\right) = \tan^2\left(\frac{\theta}{2}\right)$

$\Rightarrow \tanh\left(\frac{x}{2}\right) = \pm\tan\left(\frac{\theta}{2}\right)$ //

$\pm$ bcoz we have taken squareroot

Logarithm of complex Number

By exponential form $z = re^{i\theta}$ where $r = |z|$

$\theta = \arg(z)$ ~~$\theta = \arg(z)$~~

$$\Rightarrow \log z = \log(re^{i\theta})$$

$$= \log r + \log e^{i\theta}$$

$$= \log r + i\theta \log e \qquad (\log e = 1)$$

$$\boxed{\log(z) = \log r + i\theta} \rightarrow \text{polar form}$$

this formula is highly recommend if point is in 1st quad for each quad tan⁻¹( ) ka term will change

i.e. $\boxed{\log(x+iy) = \log\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)} \rightarrow$ cartesion form

or $\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

∴ Don't prefer cartesion form Prefer polar form

Eg1) Find $\log(1+i)$

↳ Usually after seeing this you'll say it is similar to LHS in cartesian form. But prefer polar form

$$\Rightarrow (1+i) = re^{i\theta} = \sqrt{2}e^{i\pi/4}$$

$$\log(1+i) = \log(\sqrt{2}e^{i\pi/4})$$

$$= \log\sqrt{2} + \log e^{i\pi/4}$$

$$= \log\sqrt{2} + i\pi/4$$

2) Find $\log(-1)$

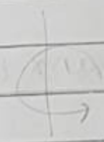
$-1 = re^{i\theta} = 1e^{i\pi}$ (Take log on both sides)

$\Rightarrow \log(-1) = \log(1e^{i\pi}) = \log(e^{i\pi})$

$= i\pi$

($\log(-1)$ doesn't exist for real number
But it DOES exist for complex numbers)

calculator always calculates principle value of argument NOT general value
($-\pi$ to π)
(0 to 2π)



anticlockwise direction me calculate karenge to we'll get general value

Clockwise :- principle value

Note :- 1) $\boxed{\log(z) = \log r + i\theta}$ is called principle value of logarithm

2) $\boxed{\operatorname{Log}(z) = \log(r) + i(2\pi k + \theta)}$

capital L is for general value

is called general value of logarithm
where k is general integer
can be +ve, -ve, 0

3) $\operatorname{Log}(z) = \log(r) + i\theta + i2\pi k$ ← open bracket

$\boxed{\operatorname{Log}(z) = \log(z) + i2\pi k}$ is called relation between general and principle value of logarithm.

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Capital L :- General value
But sometimes capital L is not written
Then if there is any general integer :- implies general value

Eg 1) Find $Log(-1)$

$\rightarrow Log(-1) = log(-1) + i2\pi k$ ... by relation between Log and log

$= i\pi + i2\pi k$

calculated in previous question

$= i\pi(2k+1)$

2) Show that

$$\log\left\{\frac{\cos(x-iy)}{\cos(x+iy)}\right\} = 2i\tan^{-1}(\tan x \tanh y)$$

Here you cannot say that den → 1 quad
You first have to expand it

$$LHS = \log\left\{\frac{\cos(x-iy)}{\cos(x+iy)}\right\}$$

applying $\log\left(\frac{a}{b}\right) = \log a - \log b$ here will make the problem lengthy

Here $(x-iy)$ & $(x+iy)$ are conjugates of each other
∴ When there is a ratio of conjugates write in exponential form.

First expand $\cos(A-B)$ & $\cos(A+B)$

$$= \log\left\{\frac{\cos x\cos(iy) + \sin x\sin(iy)}{\cos x\cos(iy) - \sin x\cdot\sin(iy)}\right\}$$

By relation b/w trigo & hyperbolic functions

$$= \log\left\{\frac{\cos x\cosh y + \sin x\, i\sinh y}{\cos x\cosh y - \sin x\, i\sinh y}\right\}$$

$$= \log\left(\frac{\cos x\cosh y + i\sin x\sinh y}{\cos x\cosh y - i\sin x\sinh y}\right)$$

→ Num → 1^{st} quad
Den → 4^{th} quad

Separation into real and dynamic :- Self Study
(no question will be asked)

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Imp skip **

$$= \log\left(\frac{re^{i\theta}}{re^{-i\theta}}\right) \text{ ... by exponential form}$$

$$= \log\left(e^{i2\theta}\right)$$

$$= i2\theta \quad \text{where } \theta = \arg(\cos x\cosh y + i\sin x\sinh y)$$

θ is arg of numerator

denominator :- is $(-\theta)$

This is in the form of $\theta = \arg(x+iy)$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)$

$$= i\,2\tan^{-1}\left(\frac{\sin x\sinh y}{\cos x\cosh y}\right)$$

$$= 2i\tan^{-1}(\tan x \tanh y) \;//$$

Q1) Show that :-

$$\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2-b^2}$$

$$\rightarrow \text{LHS} = \tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right]$$

$$= \tan\left[i\log\left(\frac{re^{-i\theta}}{re^{i\theta}}\right)\right] \text{ ....by exp. form}$$

Always prefer writing the expression in 1st quadrant in the form of $re^{i\theta}$

$$\therefore = \tan\left[i\log\left(e^{-i2\theta}\right)\right]$$

we have used definition here unknowing $\log r + i\theta$

$\log 1 + (-i2\theta)$

$0 - i2\theta$ by

but it is overlapped property of log

$\log a^b = b\log a$

$$= \tan\left[i(-i2\theta)\right]$$

$$= \tan(2\theta) \quad \text{where } \theta = \arg(a+ib)$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

you cannot substitute value of θ and cancel $\tan^{-1}$ and $\tan$ because it is 2θ not θ

$$\tan\left(2\tan^{-1}\left(\frac{b}{a}\right)\right)$$

$$\Rightarrow \quad = \tan(2\theta) \quad \text{where } \theta = \arg(a+ib)$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$= \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2\left(b/a\right)}{1-\left(b/a\right)^2} = \frac{2b/a}{\frac{a^2-b^2}{a^2}} = \frac{2ab}{a^2-b^2} = RHS$$

We have unknowingly used definition of log in step .. to ..
But it is overlapped by normal property of log
Hence we don't notice the definition