non negative :- whole number (Rank can be +ve integer or 0)

rank cannot be 1/2 not an integer

MODULE-4

MATRICES

Elementary Row transformation:-

$R_1 \rightarrow R_1 \leftrightarrow R_2$

$R_1 \rightarrow R_1 \pm kR_2$

$R_1 \rightarrow \pm kR_1$; $k \neq 0$

$2R_1 \pm 3R_2$ Incorrect (not elementary row transformation)

$R_1 \pm \frac{3}{2} R_2$ ✓ (correct)

$R_1 \times R_2$ (Incorrect)

$R_1 \times 1/2$ (correct)

Definition :- Rank of a matrix

means it can be square matrix as well. Since square matrix is special case of rectangular matrix

If A is a rectangular matrix of order $m \times n$ then ~~an intege~~ a non-negative integer $\boxed{r}$ is said to be a rank of a matrix if

(i) There exists atleast one non-zero minor of order $\boxed{r}$

(ii) Every minor of order greater than r is equal to zero.

Eg :- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ → has 4 minors of order 1

$|1|, |2|, |3|, |4|$

→ has 1 minor of order 2 → $\cancel{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}$

$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2 \neq 0$, hence rank cannot be 1

$\Rightarrow$ rank of matrix is $\boxed{2}$

for $r = 2$, (ii) condn does not exist. bcoz we cannot generate 3x3 matrix from 2x2.

Eg $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \underset{\neq 0}{|1|}, \underset{= 0}{|0|}, \underset{\neq 0}{|1|}, \underset{= 0}{|0|},$ → (i) condⁿ is satisfied → for $r = 1$

$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$ (ii) condⁿ is not satisfied

$\Rightarrow$ rank 2 $\therefore$ rank $= 2$

Eg $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow |1| \neq 0$

$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$ $\Rightarrow$ rank $= 1$

eg $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \rightarrow \begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix} = 0$

$|2| \neq 0$

$\Rightarrow$ rank $= 1$

$| \ | \neq 0$ $\Rightarrow$ non singular matrix
determinant
$\therefore$ rank will be same as its order

$| \ | = 0$ $\Rightarrow$ singular matrix
$\therefore$ rank will be smaller than that of its order.

Note :- 1) Rank of a matrix A is denoted by $\rho(A)$ rho
2) If A is non singular i.e. $|A| \neq 0$
then $\rho(A) =$ order of A
3) If A is singular i.e. $|A| = 0$
then $\rho(A) <$ order of A
4) If A is null matrix then
$\rho(A) = 0$

(If Matrix :- 4×5 order then max rank can be 4
bcoz we cannot form 5×5 matrix from 4×5)

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* Methods of finding rank

1) Row echelon form
2) Normal form
3) PAQ normal form

I) Row echelon form of a matrix

A matrix is said to be in row-echelon form if
(i) There are some zero rows appearing at the bottom of the matrix.
&/or (ii) The number of zero elements before a non zero element in a row is less than next such row

If either of the 2 condn is true, then it is row echelon form. Both condn must not be true.

Eg
0 n
0 0 n
0 0 0 n

Number of zeroes should be in ascending order (before a non zero number)

where n → non zero number

(If any one condn is true then → Row echelon)

In row echelon form number of non-zero rows determine rank of matrix.

Eg $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ → (1st condn is applied)
rank = 1

Eg $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ → Here both condn are applied
Here there is no scope for non zero term in 2nd row
Still 2nd condn is applied

rank = 1
↳ rank will be 1 bcoz 1st row is not entirely zero

ONLY $\rho(A) = 0$ where A = null matrix

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$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \Rightarrow rank = 3$$

All these three eg are triangular matrix
To be specific:- upper triangular matrix
∴ All upper △ matrix are in Row echelon form

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix} \rightarrow$$ Not in row echelon form (Lower triangular)

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$ rank = 2

1^{st} row me non zero term ke baad 0 aa raha hai ∴ we will not consider it

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow rank = 1$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 3 & 0 \end{bmatrix}$$ → not in row echelon form → 0 should be in ascending order in all the rows

(*) To reduce any matrix into row echelon

$$\begin{bmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & \boxed{a_{22}} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

using $\boxed{a_{11}}$ reduce below elements into zero

Next using $\boxed{a_{22}}$ reduce below elements into zero and so on. (elements below a_{22} in same coloumn)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

1. $R_i \rightarrow R_i \leftrightarrow R_j$
2. $R_i \rightarrow R_i \pm kR_j$
3. $R_i \rightarrow \pm kR_i$

Q1) Evaluate the rank of following matrices by reducing them in to ~~row~~ row echelon form.

(a) $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

→ Given matrix is already in row echelon form (bcoz of both cond^n)

2nd row → 1 zero
3rd row → 3 zero

$1 < 3$ → ascending order ∴ 2nd cond^n is applied

$\rho(A) =$ no. of non zero rows $= 2$

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

$\rightarrow A = \begin{bmatrix} \boxed{1} & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

By $R_2 \rightarrow R_2 - 2R_1$
$R_3 \rightarrow R_3 - 3R_1$

$\sim A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (1^{st} condn. is applied)

which is in row echelon form

$\rho(A) = 1$

(c) $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

$\rightarrow A = \begin{bmatrix} \boxed{1} & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

By $R_2 \rightarrow R_2 + 2R_1$
$R_3 \rightarrow R_3 - R_1$

jis row me change kar rahe ho,
uska coeff 1 rahega

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & \boxed{3} & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

by $R_3 \to R_3 + \frac{2}{3} R_2$, $R_4 \to R_4 - \frac{1}{3} R_2$

coeff should be 1

You cannot write $3R_3 + 2R_2$

although you'll get same ans, but it is not elementary transformation.

$$** \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is is row-echelon

$\rho(A) = 2$

II] Normal form of a matrix -

A rectangular matrix of order $m \times n$ can be reduced to $\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]_{m\times n}$ by using finite sequence of elementary row and coloumn transformation is known as Normal or first canonical form.

this 0 is not an element
It is zero matrix

where $I_r \to$ identity matrix of order $\boxed{r}$

then rank of matrix is $\boxed{r}$

eg $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} I_1 & 0 \\ \hline 0 & 0 \end{array}\right] \Rightarrow$ rank $= 1$

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Eg $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{rank} = 2$

Divide in such a way that we get identity matrix on top left

Eg $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [I_3] \rightarrow \text{rank} = 3$

→ There is no scope of $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ – elements

Eg $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] = [I_2 \;\; 0]$

→ rank = 2

$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ There is no scope of – elements

Eg $\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right] = \begin{bmatrix} I_2 \\ 0 \end{bmatrix} \rightarrow \text{rank} = 2$

11/01/22

Q Evaluate the rank of the following matrices by reducing them to Normal form.

(i) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

Maximum rank can be 3
Minimum rank can be 1
(Here rank cannot be 0 bcoz it is not a zero matrix)

→ $A = \begin{bmatrix} \boxed{1} & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

This element should be 1

for normal form

* Make sure 1st row, 1st coloumn element is 1. If not make it 1

By $R_2 - R_1$ and $R_3 - 3R_1$

$\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$ $\left[\begin{array}{c|c} I_r & 0 \end{array}\right]$ $\left[\begin{array}{c} I_r \\ \hline 0 \end{array}\right]$ $[I_r]$

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□ → which has to be used ○ → has to be made zero

$$\sim \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix}$$

Make adjacent elements 0 (adjacent to 1st row 1st coloumn elements all elements should be 0)

By $C_2 - C_1$ and $C_3 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix}$$

Now apply the process that we applied for 1st row 1st coloumn element on 2nd row 2nd coloumn element

By $R_2 \times \left(-\frac{1}{2}\right)$

→ Don't write $\frac{R_2}{2}$ (Division : not allowed in matrices

Only add, sub & mul is allowed

∴ Write it as $R_2 \times \left(-\frac{1}{2}\right)$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & -2 & -2 \end{bmatrix}$$

By $R_3 + 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Make adjacent element zero

By $C_3 - C_2$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right]$$ which is in normal form

$\Rightarrow$ rank of A is 2

(ii) $$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 4 & 3 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

You can interchange 1st and 2nd row or 1st and 2nd coloumn

By $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} \boxed{1} & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

By $R_3 - 3R_1$ & $R_4 - R_1$

$$\sim \begin{bmatrix} \boxed{1} & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -12 & -7 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

By $C_3 - 4C_1$ & $C_4 - 3C_1$

matrix
~ reduces to
sign

□ elements are called pivot element

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & -3 & -1 \\ 0 & 1 & -12 & -7 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

By $R_3 - R_2$ & $R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & -3 & -1 \\ 0 & 0 & -9 & -6 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

By $C_3 + 3C_2$ & $C_4 + C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -9 & -6 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

We'll try to avoid fraction. If you want you can go ahead with fraction

∴ Instead of multiplying R_3 with $-\frac{1}{9}$, we can multiply with $-\frac{1}{3}$

∴ $R_3 \rightarrow 0 \; 0 \; 3 \; 2$

then $C_3 - C_4$

$0 \; 0 \; 1 \; 2$

∵ We are towards the end of the sum, fraction won't become difficult.

∴ You can multiply with $-\frac{1}{9}$

By $R_3 \times -\frac{1}{9}$ or/ you can use $C_3 \times -\frac{1}{9}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 2/3 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

By $R_4 + 3R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By $C_4 - 2/3\, C_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} I_3 & 0 \\ \hline 0 & 0 \end{array}\right]$$

$\Rightarrow$ rank $= 3$ //

(iii) $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & -1 & 2 & 2 \\ 4 & 1 & 0 & -1 \\ 9 & 1 & 5 & 6 \end{bmatrix}$

$\rightarrow$ By $C_1 \leftrightarrow C_2$

$$\sim \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ -1 & 3 & 2 & 2 \\ 1 & 4 & 0 & -1 \\ 1 & 9 & 5 & 6 \end{bmatrix}$$

By $R_2 + R_1$, $R_3 - R_1$ and $R_4 - R_1$.

$$\sim \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 5 & 5 & 6 \\ 0 & 2 & -3 & -5 \\ 0 & 7 & 2 & 2 \end{bmatrix}$$

By $C_2 - 2C_1$, $C_3 - 3C_1$ and $C_4 - 4C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 5 & 6 \\ 0 & 2 & -3 & -5 \\ 0 & 7 & 2 & 2 \end{bmatrix}$$

Instead of $\underbrace{R_2 \times \frac{1}{5}}$, we can do $R_2 - 2R_3$

This will generate fractions.

By $R_2 - 2R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 11 & 16 \\ 0 & 2 & -3 & -5 \\ 0 & 7 & 2 & 2 \end{bmatrix}$$

By $R_3 - 2R_2$ and $R_4 - 7R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 11 & 16 \\ 0 & 0 & -25 & -37 \\ 0 & 0 & -75 & -110 \end{bmatrix}$$

By $C_3 - 11C_2$ and $C_4 - 16C_2$

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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -25 & -37 \\ 0 & 0 & -75 & -110 \end{bmatrix}$$

By $c_3 \times \left(\frac{-1}{25}\right)$ [We'll use column bcoz 75 = 25×3

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & -37 \\ 0 & 0 & 3 & -110 \end{bmatrix}$$

By $R_4 - 3R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & -37 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $c_4 + 37c_3$.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$= [I_4]$

which is in normal form

$\therefore$ rank = 4 //

$I \times A = A$ } any Matrix multiplied by
$A \times I = A$ } Identity matrix gives matrix itself

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Imp
III

PAQ normal form

If A is rectangular matrix of order $m \times n$

then $\boxed{A = \underbrace{I_m}_{\text{prefactor}} \cdot A \cdot \underbrace{I_n}_{\text{post factor}}}$

~~Now reduce A on LHS into normal form (by means of finite number of elementary row and coloumn transformation) by~~

Now reduce A on LHS into normal form by finite number of elementary row and coloumn transformations

On RHS prefactor gets affected by row transformation and post factor gets affected by column transformation.

i.e. $A = I_m \cdot A \cdot I_n$ $\quad$ $A = I_m \cdot A \cdot I_n$

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = P \quad A \quad Q$$

non singular ⇓ $|A| \neq 0$ ⇓ A^{-1} exists

$|A| = 0$ (A^{-1} exists)

** where P, Q are non singular matrices which are completely determined by elementary row / coloumn transformation. → everyone's way of solving will be different ∴ P & Q will be different for everyone but rank will be same

Note:- If A is square matrix of order $\boxed{n}$ and rank of A is $\boxed{n}$ (like in previous eg iii) then PAQ normal form gives

$$[I_n] = PAQ$$

$$\text{i.e. } I = PAQ$$

P & Q are non singular ⇒ Inverse exists

Multiply P^{-1} Q^{-1} on both sides inverse of $(a^{-1}b^{-1}) = (b^{-1})^{-1}(a^{-1})^{-1}$

NOT $(a^{-1})^{-1}(b^{-1})^{-1}$

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$\Rightarrow P^{-1} I Q^{-1} = (P^{-1}P) A (Q Q^{-1})$

$P^{-1}Q^{-1} = IAI = A$

$\Rightarrow A^{-1} = (P^{-1}Q^{-1})^{-1} = (Q^{-1})^{-1}(P^{-1})^{-1}$

$$\boxed{A^{-1} = QP}$$

1) Find A^{-1} and $\rho(A)$ after converting to PAQ form where $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Here if rank is 3, then A^{-1} will exist

$\rightarrow$ Write $A = I_3 \cdot A \cdot I_3$ because A is a square matrix of 3×3 (order 3)

$$\begin{bmatrix} \boxed{1} & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

don't write matrix A again on RHS

Whatever will be rank of A PAQ will have same rank

By $R_2 - R_1$ and $R_3 - R_1$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $C_2 - 3C_1$ & $C_3 - 3C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$[I_3] = P\ A\ Q$ which is in normal form

where $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ & $Q = \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are non-singular matrices

(Extra information: $|P|$ & $|Q| = 1$

why? $P \to$ lower triangular matrix

$Q \to$ upper triangular matrix

determinant of Δ matrix = product of diagonal elements)

$\therefore \rho(A) = 3 = \rho(PAQ)$

Now $I = PAQ$

$P^{-1} I Q^{-1} = A$

$A = P^{-1} Q^{-1}$

$\therefore A^{-1} = (P^{-1} Q^{-1})^{-1} = QP$

$$A^{-1} = \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$ //

In Normal form 1st row 1st coloumn element → 1
Row echelon " " " " → nonzero

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Q2) Find non-singular matrices P and Q such that PAQ is in normal form. Hence find $\rho(A)$ and $\rho(PAQ)$ for $A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 1 & 4 & -5 \\ -1 & -5 & -5 & 7 \end{bmatrix}$

→ $A = I_3 \cdot A \cdot I_4$

$$\begin{bmatrix} \boxed{1} & 2 & 3 & -4 \\ 2 & 1 & 4 & -5 \\ -1 & -5 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - 2R_1$ and $R_3 + R_1$

$$\begin{bmatrix} \boxed{1} & 2 & 3 & -4 \\ 0 & -3 & -2 & 3 \\ 0 & -3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_2 - 2C_1$, $C_3 - 3C_1$, $C_4 + 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & 3 \\ 0 & -3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_2 \times \frac{-1}{3}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & -2 & 3 \\ 0 & 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & -3 & 4 \\ 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & -3 & 4 \\ 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_3 + 2C_2$, $C_4 - 3C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & -5/3 & 2 \\ 0 & -1/3 & -2/3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

$$\rho(A) = 2 = \rho(PAQ)$$

$x + 2y = 0$, $x - 3y = 0$ } Homogenous equation ⇓ constant = 0

$x + y = 1$, $x - 2y = 3$ } Non-Homogenous

⇓ Easier to solve

Homogenous equation will always have 1 solⁿ i.e (0,0)
(which means $x = 0$, $y = 0$ will always satisfy a homogenous equation)

System of Equation:- upto 5 marks

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* System of Linear algebraic equations:-

Consider a system of $\boxed{m}$ equations in $\boxed{n}$ unknowns/ variables

$$\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots\cdots &+ a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots\cdots &+ a_{2n}x_n = b_2 \\
\vdots \qquad\qquad & \qquad\qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots &+ a_{mn}x_n = b_m
\end{aligned}\right\} \leftarrow (*)$$

In matrix form it is written as $\boxed{AX = B}$

where $A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix}_{m \times n}$ → coefficient matrix

no. of columns of 1st matrix should be equal to no. of rows of 2nd matrix for multiplication to exist

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$ → column matrix of unknowns/variables

$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$ → column matrix of rhs constants

$[A \,\vdots\, B]$ = augmented matrix

$n \rightarrow$ no. of unknown

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case (i) : If $\boxed{\rho(A) = \rho(A \vdots B)}$

then system is consistent

$\Downarrow$

i.e. It has a solution

Solution are of 2 types < finite / infinite

Solution is finite or infinite depends on rank of matrix

Further (a) $\rho(A) = n$ = no. of variables/unknown

$\Rightarrow$ unique solution.

(b) $\rho(A) < n$ = no. of unknowns

$\Rightarrow$ infinite solution

$\rightarrow$ Rank of coeff matrix should be equal to rank of augmented matrix.

Case (ii) If $\boxed{\rho(A) \neq \rho(A \vdots B)}$

then system is inconsistent

i.e. It has no sol^n

Don't waste time by finding rank of A (coeff matrix) & A;B (augmented matrix) separately

Because A;B also includes A matrix

$\therefore$ just find rank of augmented matrix

$\Rightarrow$ Discuss the consistency of the system and solve them if consistent.

(i) $x + 2y + 3z = 10$
$x + y + z = 5$
$x + 2y + 3z = 8$

$\rightarrow$ In matrix form $AX = B$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 8 \end{bmatrix}$$

$\therefore$ Augmented matrix $[A \vdots B] = \left[\begin{array}{ccc:c} \boxed{1} & 2 & 3 & 10 \\ 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 8 \end{array}\right]$

Should be non zero for row echelon form

indicates augmented

You can make it dotted or | straight line

By $R_2 - R_1$ & $R_3 - R_1$

row echelon form
0 before non zero terms are in ascending order
1st row :- zero 0
2nd row :- one 0
3rd row :- three 0

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 10 \\ 0 & -1 & -2 & -5 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

which is in row echelon form

for rank of A last column ko hide karo

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rightarrow$ for A :- no. of non zero rows = 2

$\therefore$ rank = 2

augmented matrix → consider full matrix
All 3 rows are non zero
⇒ rank = 3

$\therefore \rho(A) = 2 \quad \& \quad \rho(A \vdots B) = 3$

$\Rightarrow \rho(A) \neq \rho(A \vdots B)$

⇒ system is inconsistent
i.e. It has no solution

(Orally :- $x + 2y + 3z = 10$
$x + 2y + 3z = 8$ ⇒ LHS same But RHS different
This cannot exist)

(ii) $x + y + z = 6$
$x + 2y + 3z = 14$
$2x + 4y + 7z = 30$

→ In matrix form $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

Aug matrix $\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 2 & 4 & 7 & 30 \end{array}\right]$

By $R_2 - R_1$ & $R_3 - 2R_1$

In 3x3 matrix
If rank of A is 3, then rank of A|B must be 3, it cannot be less than 3

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$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & \boxed{1} & 2 & 8 \\ 0 & 2 & 5 & 18 \end{array}\right]$$

By $R_3 - 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Even if this matrix was

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow$ rank of $A:B = 3$

which is in row echelon form

$\rho(A) = 3$ and $\rho(A;B) = 3$

$\therefore \rho(A) = \rho(A|B)$

$\Rightarrow$ system is consistent

i.e. It has a solution

Further $\rho(A) = 3 =$ no. of unknowns

$\Rightarrow$ It has unique/finite solution

**** To find Sol^n we rewrite eq^{n}s from row echelon form

$$\left.\begin{aligned} x + y + z &= 6 \\ y + 2z &= 8 \\ z &= 2 \end{aligned}\right\} \Rightarrow \boxed{z = 2}$$

$y + 2z = 8 \Rightarrow y + 2(2) = 8 \Rightarrow \boxed{y = 4}$

$x + y + z = 6 \Rightarrow x + 4 + 2 = 6 \Rightarrow \boxed{x = 0}$

Discuss the consistency of the system and solve them if consistent.

1) $x - 2y + z - w = 2$
$x + 2y + 4w = 1$
$4x - z + 3w = -1$

→ In matrix form $AX = B$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 2 & 0 & 4 \\ 4 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

We'll get a unique sol^n only when rank = no. of variables = 4

→ 3×4 matrix ∴ maximum rank = 3

∴ this cannot give a unique solution

Aug matrix $[A|B] = \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 2 \\ 1 & 2 & 0 & 4 & 1 \\ 4 & 0 & -1 & 3 & -1 \end{array}\right]$

By $R_2 - R_1$ and $R_3 - 4R_1$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 2 \\ 0 & 4 & -1 & 5 & -1 \\ 0 & 8 & -5 & 7 & -9 \end{array}\right]$$

By $R_3 - 2R_2$

$$\begin{bmatrix} 1 & -2 & 1 & -1 & 2 \\ 0 & 4 & -1 & 5 & -1 \\ 0 & 0 & -3 & -3 & -7 \end{bmatrix}$$

which is in row echelon form.

$\cancel{\rho(A) = 3 =}$ $\rho(A) = 3 = \rho(A|B)$

$\Rightarrow$ system is consistent

i.e. It has a solution

Further $\rho(A) = 3 < 4 =$ no. of variables

$\Rightarrow$ Infinitely many solution

To find the infinitely many solution, whatever is the gap b/w rank & no. of variables we'll introduce that many parameters

To find solution we use $(4-3) = 1$ no. of parameter in rewriting eqn's from row ~~of~~ echelon form.

if gap would have been 2 (last eqn last 2 variables) then set $z = t_1$, $w = t_2$

$$x - 2y + z - w = 2$$
$$4y - z + 5w = -1$$
$$-3z - 3w = -7$$

Parameter needs to be introduced in the last eqn and last variable

$\Rightarrow$ last eqn, preferably ~~last variable w~~

You can consider z as t But prefer variable of last eqn i.e. w last

Set $\boxed{w = t}$ where t - parameter

$$-3z - 3w = -7$$
$$\Rightarrow -3z - 3t = -7$$

i.e. $\cancel{\frac{7-3t}{3} = z}$ $\boxed{z = \frac{7-3t}{3}}$

Now $4y - z + 5w = -1$

$$4y - \left(\frac{7-3t}{3}\right) + 5t = -1$$

$$4y = \frac{7-3t}{3} - 5t - 1$$

$$4y = \frac{7-3t-15t-3}{3}$$

$$4y = \frac{4-18t}{3}$$

$$\boxed{y = \frac{2-9t}{6}}$$

Now $x - 2y + z - w = 2$

$$\Rightarrow x - 2\left(\frac{2-9t}{6}\right) + \left(\frac{7-3t}{3}\right) - t = 2$$

$$x = 2 + \left(\frac{2-9t}{3}\right) - \left(\frac{7-3t}{3}\right) + t$$

$$x = \frac{6+2-9t-7+3t+3t}{3}$$

$$x = \frac{-3t+1}{3} \Rightarrow \boxed{x = \frac{1-3t}{3}}$$

This is known as parametric solution

t can take any value.

Hence infinite solution

$t=0$, soln $\left(\frac{1}{3}, \frac{1}{3}, \frac{7}{3}, 0\right)$

$t=1$, soln $\left(\frac{-2}{3}, \frac{-7}{6}, \frac{4}{3}, 1\right)$

t can be +ve, -ve, fraction rational or irrational

* But t cannot be complex *

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Q2) Investigate for what values of λ and μ the system of equations $2x+3y+5z=9$; $7x+3y-2z=8$; $2x+3y+\lambda z=\mu$ have (i) no solution
(ii) unique solution
(iii) many solution (many means infinitely many solution)

→ In matrix form $AX=B$

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

Aug matrix
$$\left[\begin{array}{ccc|c} \boxed{2} & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right]$$

By $R_2 - \frac{7}{2}R_1$ and $R_3 - R_1$

Do not write in decimal

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 0 & -\frac{15}{2} & -\frac{39}{2} & -\frac{47}{2} \\ 0 & 0 & \lambda-5 & \mu-9 \end{array}\right]$$

which is in row echelon

We cannot say rank of $A=3$

if $\lambda=5$ ∴ rank of $A=2$

λ other than 5 ∴ rank $=3$

rank of $(A|B)$ is dependent on both λ & μ

rank of (A|B) $\geq$ rank of A

We'll consider each situation

no soln :- rank of A $\neq$ rank of A|B (should not be equal)

$\Rightarrow$ max rank of (A|B) = 3

$\Rightarrow$ rank of A must be 2

$\therefore$ for that $\lambda = 5$

case (i) when $\boxed{\lambda = 5}$ and $\boxed{\mu \neq 9}$

$\lambda = 5 \Rightarrow (\lambda - 5) = 0 \Rightarrow \rho(A) = 2$

$\mu \neq 9 \Rightarrow (\mu - 9) \neq 0 \Rightarrow \rho(A|B) = 3$

$\therefore \rho(A) \neq \rho(A|B) \Rightarrow$ no solution

case (ii) when $\boxed{\lambda \neq 5}$ and μ can have any value

$\lambda \neq 5 \Rightarrow (\lambda - 5) \neq 0 \Rightarrow \rho(A) = 3 = \rho(A|B)$

= no. of variables

if $\lambda - 5 \neq 0 \Rightarrow$ 3rd row me we got a non zero element

$\Rightarrow$ unique solution

$\therefore$ 3rd row will be non zero

Case (iii) We'll have infinitely many soln when $\rho(A)$ & $\rho(A|B)$ is less than 3

$\Rightarrow$ rank can either be 1 or 0

rank cannot be 1

bcoz $\frac{-15}{2}$, $\frac{-39}{2}$ } $\rightarrow$ these elements cannot be zero

when $\boxed{\lambda = 5}$ and $\boxed{\mu = 9}$

$\lambda = 5 \Rightarrow (\lambda - 5) = 0 \Rightarrow \rho(A) = 2$

$\mu = 9 \Rightarrow (\mu - 9) = 0 \Rightarrow \rho(A|B) = 2$

$\rho(A) = \rho(A|B) = 2 < 3$ = no. of variables

$\Rightarrow$ Infinitely many solution. //

Extra:- we cannot find solⁿ in case (i)

We cannot find solⁿ in case (ii) also

bcoz $\lambda \neq 5 \Rightarrow \lambda$ can take any value except for 5

We can only define value in case (iii)

Q3 Determine λ so that $x+y+z=1$; $x+2y+4z=\lambda$; $x+4y+10z=\lambda^2$ have a solⁿ and solve them in each case completely. $\Rightarrow \rho(A) = \rho(A|B)$

Aug matrix [A][B]

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{array}\right]$$

By $R_2 - R_1$ and $R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \boxed{1} & 3 & \lambda-1 \\ 0 & 3 & 9 & \lambda^2-1 \end{array}\right]$$

By $R_3 - 3R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 0 & 0 & \lambda^2-3\lambda+2 \end{array}\right]$$

which is in row echelon form

$\rho(A) = 2$

To have a sol^n $\rho(A|B)$ should be 2

$$\Rightarrow \lambda^2 - 3\lambda + 2 = 0$$

$$\Rightarrow \cancel{\lambda = 2} \quad \boxed{\lambda = 2, 1}$$

rank $\neq$ no. of variables
$\Rightarrow$ infinitely many sol^n

Case (i) when $\boxed{\lambda = 1}$

Echelon form $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

2 is rank of $(A|B)$

$$\rho(A) = \rho(A|B) = 2 < 3 = \text{no. of variables}$$

$\Rightarrow$ many sol^n — 3 is no. of variables

Rewriting eq^n $\boxed{\begin{array}{l} x + y + z = 1 \\ \quad y + 3z = 0 \end{array}}$

put $\boxed{z = t}$, $t \to$ parameter

$$y + 3z = 0 \Rightarrow \boxed{y = -3t}$$

$$x + y + z = 1 \Rightarrow x = 1 + 3t - t$$

$$\boxed{x = 1 + 2t}$$

Case (ii) when $\boxed{\lambda = 2}$

MODULE-5

CALCULUS-II

* Gamma function

For $n>0$, the definite integral $\int_0^{\infty} e^{-x} x^{n-1}\, dx$

is called Gamma function and is denoted ~~by~~ by $\sqrt{n}$

i.e. $\boxed{\sqrt{n} = \int_0^{\infty} e^{-x} x^{n-1}\, dx}$ if we replace x with any other it will still be same

we won't use definitions we will use properties

Eg $\sqrt{2} = \int_0^{\infty} e^{-x} x^{2-1}\, dx$

$$= \int_0^{\infty} e^{-x}\, x\, dx$$

LIATE rule, Algebraic comes first then integration

$x \to u,\ e^{-x} \to v$

$$= \left\{ x \int e^{-x} dx - \int \int e^{-x} dx \frac{d}{dx}(x)\, dx \right\}_0^{\infty}$$

$$= \left\{ x \left(\frac{e^{-x}}{-1}\right) - \int \frac{e^{-x}}{-1} \cdot 1\, dx \right\}_0^{\infty}$$

$e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} \Rightarrow \to 0$

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$$\Gamma 2 = \left\{ -xe^{-x} - e^{-x} \right\}_0^{\infty}$$

$$= (0-0) - (0 - e^{0}) \quad \because e^{-\infty} = 0$$

$$= 0 - (0-1)$$

$$\Gamma 2 = 1$$

Note 1) $\Gamma n = \int_0^{\infty} e^{-t}\, t^{n-1}\, dt$ yaha jo bhi rahega idhar +1 kar lo

2) $\Gamma n+1 = \int_0^{\infty} e^{-t}\, t^{n}\, dt$

Properties of Gamma function

1) $\Gamma 1 = 1$

2) $\Gamma n+1 = n\Gamma n$; for n - positive fraction

3) $\Gamma n+1 = n!$; for n - positive integer

means natural numbers
0 won't be included

4) $\Gamma n = \frac{\Gamma n+1}{n}$; for n - negative fraction

4th property comes from 2nd property just by rearranging terms

In 2) $\Gamma n+1 = n\Gamma n \Rightarrow \Gamma n = \frac{\Gamma n+1}{n}$

But the terms on LHS are different

5) $\Gamma n \, \Gamma(1-n) = \dfrac{\pi}{\sin(n\pi)}$

6) $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$

(Gamma is not defined for negative integer.
But it is defined for negative fraction)

$\Downarrow$

negative fraction can be converted into positive by using properties.

Eg 1) $\Gamma\left(\dfrac{3}{2}\right) = \Gamma\left(\dfrac{1}{2}+1\right) = \dfrac{1}{2}\Gamma\left(\dfrac{1}{2}\right)$ ... by property ②

$= \dfrac{1}{2}\sqrt{\pi}$ // ... by property ⑥

2) $\Gamma\left(\dfrac{5}{2}\right) = \Gamma\left(\dfrac{3}{2}+1\right) = \dfrac{3}{2}\Gamma\left(\dfrac{3}{2}\right)$ ... by property ②

$= \dfrac{3}{2} \cdot \dfrac{1}{2}\Gamma\left(\dfrac{1}{2}\right)$

$= \dfrac{3}{4}\sqrt{\pi}$ //

3) Keep subtracting 1

$\Gamma\left(\dfrac{11}{2}\right) = \dfrac{9}{2} \cdot \dfrac{7}{2} \cdot \dfrac{5}{2} \cdot \dfrac{3}{2} \cdot \dfrac{1}{2}\Gamma\left(\dfrac{1}{2}\right)$ ... by prop ②

You can write in single line

$= \dfrac{945}{32}\sqrt{\pi}$ //

Keep the number as it is

N < D :- we cannot reduce the fraction we'll get -ve fraction

N > D :- fraction can be reduced

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4) $\Gamma 5 = \Gamma(4+1) = 4!$ ... by prop ③
$= 24$ //

5) $\Gamma\left(\frac{-1}{2}\right) = \dfrac{\Gamma\left(\frac{-1}{2}+1\right)}{\frac{-1}{2}}$ ... by prop ④

$= -2\,\Gamma\left(\frac{1}{2}\right)$

$= -2\sqrt{\pi}$ //

6) $\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)$

We cannot calculate $\Gamma\left(\frac{2}{3}\right)$ by prop ②

We can take $\frac{2}{3}$ as n or $\frac{1}{3}$ as n

$\Gamma\left(\frac{2}{3}\right) = \Gamma\left(\left(\frac{-1}{3}\right)+1\right)$ But n cannot be -ve in prop ②. n should be positive integer in property ②.

$\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)$ or $\Gamma\left(\frac{2}{3}\right)\Gamma\left(1-\frac{2}{3}\right)$

$= \dfrac{\pi}{\sin\left(\frac{\pi}{3}\right)}$ ... by property ⑤

$= \dfrac{\pi}{\sqrt{3}/2} = \dfrac{2\pi}{\sqrt{3}}$ //

For questions related to gamma:-
2 indications $\int_0^\infty$ and e^{-t}
-ve exponential

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Q1) Prove that $\displaystyle\int_0^\infty \frac{e^{-\sqrt{x}}}{x^{7/4}}\,dx = \frac{8}{3}\sqrt{\pi}$

→ Let $I = \displaystyle\int_0^\infty \frac{e^{-\sqrt{x}}}{x^{7/4}}\,dx$

we want this to be in form of $\displaystyle\int_0^\infty e^{-t}\,t^{n-1}\,dt$

This must be a constant

put $\boxed{\sqrt{x}=t}$ i.e. $x = t^2$

$\Rightarrow dx = \cancel{2d}\; 2t\,dt$

when $x=0$; $t=0$ (when $x=0$, $t=\sqrt{0}=0$
$x=\infty$; $t=\infty$ $x=\infty$, $t=\sqrt{\infty}=\infty$)

we don't want limits to change
But we need to check

$$I = \int_0^\infty \frac{e^{-t}}{(t^2)^{7/4}}\,2t\,dt$$

$\dfrac{t^1}{t^{7/2}} = t^{1-7/2}$

$$= 2\int_0^\infty e^{-t}\cdot t^{1-7/2}\,dt$$

$$= \cancel{2\int_0^\infty e^{-t}/t} = 2\int_0^\infty e^{-t}\,t^{-5/2}\,dt$$

$$= 2\,\Gamma\!\left(\frac{-5}{2}+1\right) \quad \ldots \text{by definition}$$

$$\Gamma(n+1) = \int_0^\infty e^{-t}\,t^n\,dt$$

$$= 2\,\Gamma\left(\frac{-3}{2}\right)$$

$$= 2\,\frac{\Gamma\left(\frac{-3}{2}+1\right)}{\frac{-3}{2}}$$

$$= -\frac{4}{3}\,\Gamma\left(\frac{-1}{2}\right) = -\frac{4}{3}\,\frac{\Gamma\left(\frac{-1}{2}+1\right)}{\frac{-1}{2}}$$

$$= \frac{8}{3}\,\Gamma\left(\frac{1}{2}\right) = \frac{8\sqrt{\pi}}{3} \quad //$$

Q2) Prove that

$$\int_0^1 (x\log x)^4\,dx = \frac{4!}{5^5}.$$

Here we cannot directly guess this is a question of gamma function
bcoz limit is not 0 to ∞
and e^{-t} is not present
But we have log which is opposite of exponential.

Let $I = \int_0^1 (x\log x)^4\,dx$

Our objective is to get negative exponential
$\therefore$ Substitute $\log x$ as $-t$

put $\boxed{\log x = -t}$

i.e. $x = e^{-t}$

$\Rightarrow dx = -e^{-t}\,dt$

when $x = 0$; $t = \infty$

$\because \log(0) = -t$

$\Rightarrow -\infty = -t$

$\log 0 = -t$

$\log 0 = \rightarrow -\infty$ $\Rightarrow -\infty = -t \Rightarrow t = \infty$

$x = 1$; $t = 0$

$\log 1 = -t$

$\Rightarrow 0 = -t \Rightarrow t = 0$

e can inverse
the limits by
applying
ve sign

$$I = \int_{\infty}^{0} \{e^{-t}(-t)\}^4 (-e^{-t})\,dt$$

take this -ve sign out
& inverse limit

$$= -\int_{0}^{\infty} (e^{-t})^4 (-t)^4 e^{-t}\,dt$$

$\because (-t)^4 = t^4$

$$= \int_{0}^{\infty} (e^{-t})^5 \cdot t^4\,dt$$

$$= \int_{0}^{\infty} e^{-5t} \cdot t^4\,dt$$

here we only want variable
nothing with variable

put $\boxed{5t = u}$

$\Rightarrow dt = \frac{1}{5} du$

when $t = 0$, $u = 0$ $\quad (u = 5\times 0 = 0)$

$t = \infty$, $u = \infty$

$$= \int_0^\infty e^{-u}\left(\frac{1}{5}u\right)^4 \frac{1}{5}\, du$$

$$= \frac{1}{5^5}\int_0^\infty e^{-u}\cdot u^{4}\, du$$

$$= \frac{1}{5^5}\,\Gamma(4+1) = \frac{1}{5^5}\, 4!$$

//

Evaluate :

$$\int_0^\infty 7^{-4x^2}\, dx$$

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3) Evaluate:

$$\int_0^{\infty} 7^{-4x^2}\, dx$$

in the form of $a^{-4x^2} \Leftarrow 7^{-4x^2}$ is negative exponential but not with base e

In such problems directly substitute entire 7^{-4x^2} as e^{-t}

put $\boxed{7^{-4x^2} = e^{-t}}$

$$\log 7^{-4x^2} = \log e^{-t}$$

$$-4x^2(\log 7) = -t(\log e)$$

$$\cancel{-}4x^2(\log 7) = \cancel{-}t$$

$$x^2 = \frac{t}{4(\log 7)}$$

$$x = \sqrt{\frac{t}{4(\log 7)}}$$

$$\Rightarrow dx = \frac{1}{\sqrt{4\log 7}} \cdot \frac{1}{2\sqrt{t}} \cdot dt$$

$$dx = \frac{1}{4\sqrt{\log 7}} \frac{1}{\sqrt{t}} dt$$

when $x = 0$; $t = 0$

$x = \infty$; $t = \infty$

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$$\therefore I = \int_0^\infty e^{-t} \cdot \frac{1}{4\sqrt{\log 7}} \cdot \frac{1}{\sqrt{t}} \, dt$$

$$= \frac{1}{4\sqrt{\log 7}} \int_0^\infty e^{-t} \, t^{-1/2} \, dt$$

$$= \frac{1}{4\sqrt{\log 7}} \Gamma\left(-\frac{1}{2}+1\right) \quad \because \text{by def}^n \text{ of gamma.}$$

$$= \frac{1}{4\sqrt{\log 7}} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{4\sqrt{\log 7}} \sqrt{\pi} \quad //$$

$\sqrt[3]{x} = t$

$-\sqrt[3]{x} = t$

(1) Say question $\int_0^\infty \frac{x^{3/2}}{e^{-\sqrt[3]{x}}} \, dx$ — 1)

Substitute $\sqrt[3]{x} = t$ or $-\sqrt[3]{x} = t$

because we want e^{-t} in numerator

$\therefore$ if Denominator $= e^{t}$.

then only Numerator will become e^{-t}

(2) Question $\int_0^1 \frac{2x^2 \, dx}{9^{\sqrt{x}}}$ $\rightarrow 9^{\sqrt{x}} = e^{t}$.

Questions will be asked only on ①, ②, ③ forms

$-\frac{1}{2} < x < \frac{1}{2}$

3) $\int_0^1 \frac{x^4}{(\log x)^3}\,dx$ put $\log x = t$

$\Rightarrow x = e^t$

Only then numerator will become e^{-t}

4) P.T.

$$\Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right) = \left(\frac{1}{4}-x^2\right)\pi \sec(\pi x)$$

provided $-1 < 2x < 1$

$\Rightarrow -\frac{1}{2} < x < \frac{1}{2}$

LHS $= \Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right)$

we can either use ② or ③ property

+ve fraction   positive integer

$\because x$ lies b/w $-1/2$ & $1/2$

$\Rightarrow x$ cannot be $1/2$

if x was $= 1/2$ then Γ could have been +ve integer

but $x \neq 1/2$

$\Rightarrow$ we'll use prop ②

Since $-1 < 2x < 1$ ie $-\frac{1}{2} < x < \frac{1}{2}$

$$\Rightarrow \left(\frac{1}{2}+x\right) > 0 \ \& \ \left(\frac{1}{2}-x\right) > 0$$

$\frac{1}{2} - x$

$1 - \frac{1}{2} + x = \frac{1}{2}$

$1/2 + 1 = 3/2$

$$LHS = \Gamma\left(\left(\frac{1}{2} - x\right) + 1\right) \Gamma\left(\left(\frac{1}{2} + x\right) + 1\right)$$

$$= \left(\frac{1}{2} - x\right) \Gamma\left(\frac{1}{2} - x\right) \left(\frac{1}{2} + x\right) \Gamma\left(\frac{1}{2} + x\right)$$

.... by prop (2)

~~LHS~~.

$$= \left(\frac{1}{4} - x^2\right) \Gamma\left(\frac{1}{2} - x\right) \Gamma\left(\frac{1}{2} + x\right)$$

or you can take $\Gamma\left(\frac{1}{2} + x\right)$ as n as $n-1$ as $\Gamma\left(\frac{1}{2} - x\right)$

$n \qquad 1-n$

$$= \left(\frac{1}{4} - x^2\right) \Gamma\left(\frac{1}{2} - x\right) \Gamma\left(1 - \left(\frac{1}{2} - x\right)\right)$$

$$= \left(\frac{1}{4} - x^2\right) \frac{\pi}{\sin\left\{\left(\frac{1}{2} - x\right)\pi\right\}}$$

... by Prop (5)

$$= \left(\frac{1}{4} - x^2\right) \frac{\pi}{\sin\left(\frac{\pi}{2} - \pi x\right)}$$

$$= \left(\frac{1}{4} - x^2\right) \frac{\pi}{\cos(\pi x)}$$

$$= \left(\frac{1}{4} - x^2\right) \pi \cdot \sec(\pi x)$$

$= RHS$ //

Definition:- Beta function

For $m, n > 0$, the definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$

is called Beta function and is denoted by $B(m,n)$

i.e. $\boxed{B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx}$

Eg. $B(2,3) = \int_0^1 x^{2-1}(1-x)^{3-1}\,dx$

$= \int_0^1 x\,(1-x)^2\,dx$

this is not u.v rule

we'll expand $(1-x)^2$

$x(1-2x+x^2) \Rightarrow$ integration of 3 terms

$x-2x^2+x^3$

(*) Properties of Beta function

1. $B(m,n) = B(n,m)$ → Symmetric property

*** 2. $B(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$ → Every beta function can be converted into gamma function

3. $B(m,n) = \int_0^\infty \dfrac{x^{m-1}}{(1+x)^{m+n}}\,dx$

(2nd definition of Beta function)

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(4) $B(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta$

put $\boxed{2m-1=p}$ and $\boxed{2n-1=q}$

$$B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} (\sin\theta)^p (\cos\theta)^q\, d\theta$$

→ trigonometric form of Beta function

Eg1) Evaluate $\int_0^{\pi/2} \sqrt{\tan x}\, dx$

integration of tanx is log secx + tanx
But we don't have a formula to integrate root of tanx

If we take $\sqrt{\tan x} = t$ then $\tan x = t^2$
$x = \tan^{-1} t^2$
we have to check limits
& then calculate dx i.e derivative of $\tan^{-1} t^2$, which will become complicated

→ Let $I = \int_0^{\pi/2} \sqrt{\tan x}\, dx$

$$= \int_0^{\pi/2} \sqrt{\frac{\sin x}{\cos x}}\, dx = \cancel{\int (\sin x)^{1/2} (\cos x)^{-1/2} dx}$$

$$= \int_0^{\pi/2} (\sin x)^{1/2} (\cos x)^{-1/2} dx$$

$$= \frac{1}{2} B\left(\frac{\frac{1}{2}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right) \quad \text{... by prop (4)}$$

$$= \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)} \quad \text{... by prop (2)}$$

we cannot reduce only $\Gamma\frac{3}{4}$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma 1} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

$\Gamma 1 = 1$

we cannot reduce only $\Gamma\frac{1}{4}$ bcoz Num < Den

But product of $\Gamma\frac{1}{4}$ & $\Gamma\frac{3}{4}$ can be reduced by using prop 5 of gamma

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{2} \frac{\pi}{\sin\left(\frac{1}{4}\pi\right)} \quad \text{... using } \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}$$

$$= \frac{1}{2} \frac{\pi}{\frac{1}{\sqrt{2}}} = \frac{\pi}{\sqrt{2}}$$

//

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$$\int_0^{\pi/2} \sqrt{\cot x}\, dx = \int (\sin x)^{-1/2} (\cos x)^{1/2} dx$$

$$= \frac{1}{2} B\left(\frac{1}{4}, \frac{3}{4}\right)$$

But $B(m,n) = B(n,m)$

$\therefore$ Ans of $\int_0^{\pi/2} \sqrt{\cot x}\, dx = \int_0^{\pi/2} \sqrt{\tan x}\, dx = \frac{\pi}{\sqrt{2}}$

Eg2) Evaluate $\int_0^2 x^4 (8-x^3)^{-1/3} dx$

Let $I = \int_0^2 x^4 (8-x^3)^{-1/3} dx$

According to defn we want this to be in form of $(1-t)$

$m-1 \; (1-t)^{n-1} dt$

$(8-x^3) \Rightarrow 1-t$

$\therefore$ Substitute $x^3 = 8t$

put $\boxed{x^3 = 8t}$

i.e. $x = (8t)^{1/3} = 2t^{1/3}$

$\Rightarrow dx = 2 \cdot \frac{1}{3} t^{1/3 - 1} dt$

we'll keep the power in this form bcoz according to defn we want it in form of $n-1$

when $x = 0$; $t = 0$

$x = 2$; $t = 1$

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$$\therefore I = \int_0^1 (2t^{1/3})^4 (8-8t)^{-1/3} \frac{2}{3} t^{\frac{1}{3}-1} \, dt$$

$$\therefore I = \int_0^1 2^4 t^{4/3} [8(1-t)]^{-1/3} \frac{2}{3} \cdot t^{\frac{1}{3}-1} \, dt$$

$$= \frac{2^5}{3} \int_0^1 t^{5/3-1} \cdot 8^{-1/3} (1-t)^{-1/3} \, dt$$

$$= \frac{32}{3} \cdot \frac{1}{8^{1/3}} \int_0^1 t^{5/3-1} (1-t)^{2/3-1} \, dt$$

$$= \frac{32}{3} \cdot \frac{1}{2} \; B\left(\frac{5}{3}, \frac{2}{3}\right)$$

$$= \frac{16}{3} \frac{\Gamma\frac{5}{3} \Gamma\frac{2}{3}}{\Gamma\left(\frac{5}{3}+\frac{2}{3}\right)}$$

$$= \frac{16}{3} \frac{\Gamma\frac{5}{3} \Gamma\frac{2}{3}}{\Gamma\frac{7}{3}}$$

$\Gamma\frac{5}{3}$ & $\Gamma\frac{7}{3}$ can be reduced further

But $\Gamma\frac{2}{3}$ cannot be reduced bcoz $2 < 3$

$\therefore$ We'll keep $\Gamma\frac{2}{3}$ as it is

$$= \frac{\frac{16}{3}\,\Gamma\left(\frac{2}{3}+1\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{4}{3}+1\right)}$$

$$= \frac{\frac{16}{3}\cdot\frac{2}{\cancel{3}}\,\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\frac{4}{\cancel{3}}\,\Gamma\left(\frac{4}{3}\right)}$$ ...using $\Gamma(n+1) = n\Gamma(n)$

$$= \frac{\frac{16}{3}\cdot\frac{2}{4}\left(\Gamma\left(\frac{2}{3}\right)\right)^2}{\Gamma\left(\frac{1}{3}+1\right)}$$

$$= \frac{\frac{8}{\cancel{3}}\left(\Gamma\left(\frac{2}{3}\right)\right)^2}{\frac{1}{\cancel{3}}\,\Gamma\left(\frac{1}{3}\right)}$$

$$= \frac{8\left(\Gamma\left(\frac{2}{3}\right)\right)^2}{\Gamma\left(\frac{1}{3}\right)}$$ //

Note:- $\Gamma\frac{2}{3}$ & $\Gamma\frac{1}{3}$ cannot be reduced further. ∴ we'll "keep it as it is".

We don't have to focus on limit
We have to focus on factors
~~~ are the factors

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Q3 Show that

$$\int_0^1 \sqrt{1-\sqrt{x}}\,dx \int_0^{1/2} \sqrt{2y-(2y)^2}\,dy = \frac{\pi}{30}$$

We have to convert this into $1-t$
$\Rightarrow$ put $\sqrt{x}=t$

convert into $(1-t)$
put $2y = t$

We'll solve both seperately

$\rightarrow$ Let $I_1 = \int_0^1 \sqrt{1-\sqrt{x}}\,dx$ put $\boxed{\sqrt{x}=t}$

i.e. $x = t^2$

$\Rightarrow dx = 2t\,dt$

when $x=0$; $t=0$
$x=1$; $t=1$

$$= \int_0^1 \sqrt{1-t}\;2t\,dt$$

whatever we have here, we add 1 to it to get m, n

$$\int_0^1 t^{m-1}(1-t)^{n-1}dt = B(m,n)$$

add 1 to get m
add 1 to get n

$$= 2\int_0^1 t^{1}(1-t)^{1/2}\,dt$$

We can further simplify here or later

$$= 2B\left(2, \frac{3}{2}\right) \quad \text{—— (1)}$$

Now $I_2 = \int_0^{1/2} \sqrt{2y-(2y)^2}\,dy$

put $\boxed{2y=t}$ $\Rightarrow dy = \frac{dt}{2}$
~~~

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when $y=0$; $t=0$

$y=\frac{1}{2}$; $t=1$

$$I_2 = \int_0^1 \sqrt{t-t^2}\,\frac{dt}{2}$$

$$= \frac{1}{2}\int_0^1 \sqrt{t(1-t)}\,dt$$

$$= \frac{1}{2}\int_0^1 t^{1/2}(1-t)^{1/2}\,dt$$

$$= \frac{1}{2}B\left(\frac{3}{2},\frac{3}{2}\right) \quad \text{——} \quad (2)$$

$\therefore\ I = I_1 \cdot I_2$

$$= \cancel{2}B\left(2,\frac{3}{2}\right)\cdot\frac{1}{\cancel{2}}B\left(\frac{3}{2},\frac{3}{2}\right)$$

$$= \frac{\Gamma 2\,\Gamma\frac{3}{2}}{\Gamma\left(2+\frac{3}{2}\right)} \cdot \frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)}$$

$$= 1!\,\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} \cdot \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}$$

$$= \frac{\cancel{\Gamma\left(\frac{3}{2}\right)}}{\frac{5}{2}\cdot\frac{3}{2}\,\cancel{\Gamma\left(\frac{3}{2}\right)}} \cdot \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\,\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2!}$$

$$= \frac{1}{\frac{15}{\cancel{4}}} \cdot \frac{\frac{1}{\cancel{4}}\sqrt{\pi}\,\sqrt{\pi}}{2}$$

$$= \frac{\pi}{30} \quad //$$

4) Prove that $\displaystyle\int_0^\infty \frac{x^5}{(2+3x)^{16}}\,dx = \cancel{5!\,9!}\ \frac{5!\,9!}{2^{10}\,3^{6}\,15!}$

In the form of 2nd def^n of Beta function

$\therefore$ we want denominator in the form of $(1+t)$

$\therefore$ put $3x = 2t$

$(2+3x) \to (1+t)$, $3x \to 2t$

$\rightarrow$ put $\boxed{3x = 2t}$

$\Rightarrow dx = \frac{2}{3}\,dt$

When $x = 0$; $t = 0$

$x = \infty$; $t = \infty$

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$$I = \int_0^\infty \frac{\left(\frac{2}{3}t\right)^5}{(2+2t)^{16}} \frac{2}{3} dt$$

$$= \left(\frac{2}{3}\right)^6 \int_0^\infty \frac{t^5}{\{2(1+t)\}^{16}} dt$$

$$= \frac{2^6}{3^6 2^{16}} \int_0^\infty \frac{t^5}{(1+t)^{16}} dt$$

$$= \frac{1}{3^6 2^{10}} \int_0^\infty \frac{t^{6-1}}{(1+t)^{6+10}} dt$$

$$= \frac{1}{3^6 2^{10}} B(6,10) \ldots$$ by 2^{nd} definition of Beta function

$$= \frac{1}{3^6 2^{10}} \frac{\Gamma 6 \, \Gamma 10}{\Gamma 6+10}$$

$$= \frac{1}{3^6 2^{10}} \cdot \frac{5!\,9!}{15!}$$ //

5) P.T. $\int_5^9 \sqrt[4]{(9-x)(x-5)}\, dx = \frac{2\left(\Gamma\frac{1}{4}\right)^2}{3\sqrt{\pi}}$

We can't predict if original defⁿ or 2nd def is used bcoz of limit
But we can consider the factors. Factors have negative sign in middle which matches somewhat with 1st defⁿ

→ You have to substitute such a value so that when $x = 5$, $t = 0$

write
x t
when $x=5, t=0$
∴ subtract 5 from x
$x-5 = t$

$(x-5) = t$
But $x = 9$; t should be 1
If we substitute 9 here, we'll get 4

⇒ divide by 4 ∴ $\frac{x-5}{4} = t$

⇒ put $\boxed{\frac{(x-5)}{4} = t}$ → $x - 5 = 4t$

i.e. $x = 5 + 4t$
$\Rightarrow dx = 4dt$
when $x = 5$, $t = 0$
$x = 9$; $t = 1$

$$I = \int_0^1 \sqrt[4]{\{9-(5+4t)\}\, 4t}\ .4dt$$

$$= 4\int_0^1 (4-4t)^{1/4}(4t)^{1/4}\, dt$$

$$= 4(4)^{1/4}\int_0^1 \{4(1-t)\}^{1/4}\, t^{1/4}\, dt$$

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$$= 4(4)^{1/4}(4)^{1/4}\left\{\int_0^1 t^{1/4}(1-t)^{1/4}\,dt\right\}$$

$$= 8\int_0^1 t^{5/4-1}(1-t)^{5/4-1}\,dt$$

$$= 8\,B\left(\frac{5}{4},\frac{5}{4}\right)$$

$$= 8\,\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{5}{4}+\frac{5}{4}\right)}$$

$$= 8\cdot\frac{\Gamma\left(\frac{1}{4}+1\right)\Gamma\left(\frac{1}{4}+1\right)}{\Gamma\left(\frac{5}{2}\right)} = \frac{8\cdot\frac{1}{4}\Gamma\left(\frac{1}{4}\right)\cdot\frac{1}{\cancel{4}}\Gamma\left(\frac{1}{4}\right)}{\frac{3}{\cancel{2}}\cdot\frac{1}{\cancel{2}}\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{2\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{3\sqrt{\pi}}$$

//

$t = \frac{x-a}{b-a}$

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H.W. Eg. Evaluate

$$\int_a^b \sqrt[n]{(b-x)(x-a)}\ dx$$

Q6) Show that $\int_0^a \sqrt{\frac{x^3}{a^3-x^3}}\ dx = \frac{a\sqrt{\pi}\ \Gamma\frac{5}{6}}{\Gamma\frac{1}{3}}$

$(a^3-x^3) \rightarrow (1-t)$

put $x^3 = a^3t$ → Never think of substitution for $x =$ consider substitution for entire term.

→ Let

$$I = \int_0^a \sqrt{\frac{x^3}{a^3-x^3}}\ dx$$

put $\boxed{x^3 = a^3t}$

i.e. $x = (a^3t)^{1/3} = at^{1/3}$

$\Rightarrow dx = \frac{a}{3} t^{1/3-1}\ dt$

when $x = 0$ $\quad t = 0$ $\quad \Rightarrow t = \frac{0}{a^3} = 0$

$x = a$ $\quad t = 1$

$$I = \int_0^1 \sqrt{\frac{a^3t}{a^3-a^3t}}\ \frac{a}{3}t^{1/3-1}\ dt$$

constant

$$I = \int_0^1 \sqrt{\frac{\cancel{a^3} t}{\cancel{a^3}(1-t)}} \; \frac{a}{3} t^{1/3-1} \, dt$$

$$= \frac{a}{3} \int_0^1 t^{1/2} (1-t)^{-1/2} \, t^{1/3-1} \, dt$$

$$= \frac{a}{3} \int_0^1 t^{5/6-1} (1-t)^{\frac{1}{2}-1} \, dt$$

$$= \frac{a}{3} B\left(\frac{5}{6}, \frac{1}{2}\right)$$

$$= \frac{a}{3} \frac{\Gamma\frac{5}{6}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{5}{6}+\frac{1}{2}\right)}$$

cannot be reduced further because $5 < 6$

$$= \frac{\frac{a}{3}\Gamma\frac{5}{6}\sqrt{\pi}}{\Gamma\frac{4}{3}}$$

$$= \frac{\frac{a}{\cancel{3}}\Gamma\frac{5}{6}\sqrt{\pi}}{\frac{1}{\cancel{3}}\Gamma\frac{1}{3}} = \frac{a\,\Gamma\frac{5}{6}\sqrt{\pi}}{\Gamma\frac{1}{3}}$$

//

7) Evaluate $\displaystyle\int_0^\infty \frac{\sqrt{x}}{(1+2x+x^2)}\,dx$

limit $\rightarrow 0$ to ∞ $\Rightarrow$ 2nd form

$$\rightarrow I = \int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx$$

$$= \int_0^\infty \frac{x^{3/2-1}}{(1+x)^{3/2+1/2}}\,dx$$

$\rightarrow m = 3/2$ $\Rightarrow m + \square = 2$, $\square = 2-$

$$= B\left(\frac{3}{2}, \frac{1}{2}\right)$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2}\right)}$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \frac{\frac{1}{2}\sqrt{\pi}\sqrt{\pi}}{1!}$$

$$= \frac{\pi}{2}$$

~~Duplif~~ Duplication formula on beta function

For $m > 0$, $\boxed{\Gamma m \, \Gamma\left(m+\frac{1}{2}\right) = \frac{\Gamma 2m \sqrt{\pi}}{2^{2m-1}}}$

Eg i) Prove that $B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{1}{2^{2n}} \frac{\Gamma\left(n+\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(n+1)}$

$\rightarrow$ LHS $= B\left(n+\frac{1}{2}, n+\frac{1}{2}\right)$

$$= \frac{\left(\Gamma\left(n+\frac{1}{2}\right)\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}+n+\frac{1}{2}\right)}$$

we wont one $\Gamma\left(n+\frac{1}{2}\right)$ in RHS
$\therefore$ we'll keep one $\Gamma\left(n+\frac{1}{2}\right)$ as it is and apply duplication formula on one $\Gamma\left(n+\frac{1}{2}\right)$

$$= \frac{\left(\frac{\frac{\Gamma 2n \sqrt{\pi}}{2^{2\cancel{n}-1}}}{\Gamma n}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)}$$

... by Duplication formula

When we are applying Duplication formula, it is for $n > 0$
$\therefore n$ cannot be -ve

we cannot say $\Gamma(2n+1) = 2n!$
bcoz we cannot guarantee that n is positive integer.

$$= \frac{\frac{\Gamma 2n \sqrt{\pi}}{2^{2n-1}\,\Gamma n}\Gamma\left(n+\frac{1}{2}\right)}{(2n)\,\Gamma 2n}$$

... using $\Gamma(n+1) = n\,\Gamma n$

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$$= \frac{\cancel{\Gamma(2n)}\,\sqrt{\pi}}{2^{2n-1}\,\Gamma(n)\cdot(2n)\,\cancel{\Gamma(2n)}}\;\Gamma\!\left(n+\frac{1}{2}\right)$$

2^{2n-1+1}
$= 2^{2n}$

$$= \frac{\sqrt{\pi}}{2^{2n}}\cdot\frac{1}{n\,\Gamma(n)}\;\Gamma\!\left(n+\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2^{2n}}\cdot\frac{1}{\Gamma(n+1)}\;\Gamma\!\left(n+\frac{1}{2}\right)$$