Complex Variable - Differentiation

Module $\boxed{6}$

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$Z = x + iy$ is complex number $\Rightarrow$ x & y are constant (real)

$Z = x + iy$ is Complex variable $\Rightarrow$ x & y are variable (real)

Let $\boxed{w = f(z)}$ becomes function of complex variable $z = x + iy$

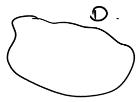
Then $w = f(z) = u + iv$ where u & v are functions of x & y.

eg for $z = x + iy$ Then $w = f(z) = e^{z} = e^{x+iy} = e^{x} \cdot e^{iy} = e^{x}(\cos y + i \sin y)$

$= \underbrace{e^{x}\cos y}_{u} + i \underbrace{e^{x}\sin y}_{v}$

Defn :- Analytic function :-

$w = f(z) = u + iv$ is said to be analytic function in domain D if it is defined and differentiable at every point of domain D.



D.

§ Necessary & sufficient conditions for analytic function :-

Suppose $w = f(z) = u(x,y) + iv(x,y)$ is a function of complex variable z defined on domain D. Then The necessary and sufficient conditions for $f(z) = u + iv$ to be analytic in domain D are

✓ (i) $\boxed{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}}$ & $\boxed{\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}}$ for every pt (x,y) of domain D

✓ (ii) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are continuous function of x, y at every pt of domain D

§ Note :- a) Condition (i) is just a necessary condition for $f(z)$ to be analytic and is also known as Cauchy-Riemann Equations briefly known as C-R equations.

b) Condition (ii) is sufficient condition for $f(z)$ be analytic.

c) If $f(z) = u + iv$ satisfies C-R eqns then it is not necessary that $f(z)$ is analytic. but If $f(z) = u + iv$ is analytic then it must satisfy C-R eqns.

eg :- 1) check whether $f(z) = e^{z}$ satisfies C-R eqn or not.

$\rightarrow$ $f(z) = e^{z} = e^{x+iy} = e^{x}e^{iy} = e^{x}(\cos y + i \sin y) = e^{x}\cos y + i e^{x}\sin y$

$\Rightarrow$ $u = e^{x}\cos y$ & $v = e^{x}\sin y$

$\therefore$ $\dfrac{\partial u}{\partial x} = e^{x}\cos y$; $\dfrac{\partial u}{\partial y} = e^{x}(-\sin y) = -e^{x}\sin y$

$\therefore \frac{\partial u}{\partial x} = e^{x}\cos y$; $\frac{\partial u}{\partial y} = e^{x}(-\sin y) = -e^{x}\sin y$

& $\frac{\partial v}{\partial x} = e^{x}\sin y$; $\frac{\partial v}{\partial y} = e^{x}\cos y$

$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ & $\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$ $\Rightarrow$ $f(z) = e^{z}$ satisfies C-R eqⁿs.

eg 2) Check whether $f(z) = x^2 + iy^3$ analytic or not.

$\rightarrow$ Here $u = x^2$ & $v = y^3$

$\therefore \frac{\partial u}{\partial x} = 2x$; $\frac{\partial u}{\partial y} = 0$ & $\frac{\partial v}{\partial x} = 0$; $\frac{\partial v}{\partial y} = 3y^2$

$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ $\Rightarrow$ $f(z)$ does not satisfies C-R eqⁿs.

$\Rightarrow f(z) = x^2 + iy^3$ is not analytic

eg 3) Check $f(z) = \sinh(z)$ analytic or not ?

$\rightarrow$

$$
\begin{aligned}
f(z) &= \sinh(z) \\
&= \sinh(x+iy) \\
&= \sinh x \cdot \cosh(iy) + \cosh x \cdot \sinh(iy) \quad \text{... using } \sinh(A+B) \text{ identity} \\
&= \sinh x \cos y + \cosh x \cdot i\sin y \\
&= (\sinh x \cdot \cos y) + i(\cosh x \sin y) \\
&= u + i\,v
\end{aligned}
$$

Now $u = \sinh x \cos y \Rightarrow u_x = \cosh x \cdot \cos y$; $u_y = -\sinh x \sin y$

$v = \cosh x \cdot \sin y \Rightarrow v_x = \sinh x \cdot \sin y$; $v_y = \cosh x \cos y$

$\therefore \boxed{u_x = v_y}$ & $\boxed{u_y = -v_x}$ $\Rightarrow$ $f(z)$ satisfies C-R eqⁿs. ✓

Further u_x, u_y, v_x, v_y are continuous function of x, y being differentiable. ✓

$\therefore$ By Necessary & sufficient condⁿ $f(z) = \sinh z$ is analytic.

eg 4) Is $f(z) = \frac{z}{\bar{z}}$ analytic function ?

$\rightarrow$ $f(z) = \frac{z}{\bar{z}} = \frac{x+iy}{\overline{x+iy}} = \frac{x+iy}{x-iy} \times \frac{(x+iy)}{(x+iy)} = \frac{(x+iy)^2}{x^2-(iy)^2} = \frac{x^2+i^2y^2+2xiy}{x^2+y^2}$

$\therefore f(z) = \frac{(x^2-y^2)+i(2xy)}{(x^2+y^2)} = \left(\frac{x^2-y^2}{x^2+y^2}\right) + i\left(\frac{2xy}{x^2+y^2}\right) = u + iv$

$\therefore u = \frac{x^2-y^2}{x^2+y^2}$ and $v = \frac{2xy}{x^2+y^2}$

$$\therefore\ u_x = \frac{(x^2+y^2)\,2x - (x^2-y^2)\,2x}{(x^2+y^2)^2} = \frac{2x^3+2xy^2-2x^3+2xy^2}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}$$

$$\therefore\ u_y = \frac{(x^2+y^2)(-2y) - (x^2-y^2)\,2y}{(x^2+y^2)^2} = \frac{-2x^2y-2y^3-2x^2y+2y^3}{(x^2+y^2)^2} = \frac{-4x^2y}{(x^2+y^2)^2}$$

lly $$v_x = \frac{(x^2+y^2)\,2y - 2xy(2x)}{(x^2+y^2)^2} = \frac{2x^2y+2y^3-4x^2y}{(x^2+y^2)^2} = \frac{2y^3-2x^2y}{(x^2+y^2)^2}$$

$$v_y = \frac{(x^2+y^2)\,2x - 2xy(2y)}{(x^2+y^2)^2} = \frac{2x^3+2xy^2-4xy^2}{(x^2+y^2)^2} = \frac{2x^3-2xy^2}{(x^2+y^2)^2}$$

$\therefore\ u_x \neq v_y$ & $u_y \neq -v_x$ $\Rightarrow$ C-R eqn does not holds

$\Rightarrow f(z) = \frac{z}{\bar{z}}$ is not analytic.

eg 5) Determine constants a, b, c, d such that $f(z) = x^2+axy+by^2+i(cx^2+dxy+y^2)$ is analytic.

$\rightarrow$ Given $f(z) = (x^2+axy+by^2) + i\,(cx^2+dxy+y^2)$ is analytic

$\Rightarrow f(z) = u+iv$ satisfies C-R eqns ie $\boxed{u_x = v_y}$ & $\boxed{u_y = -v_x}$ — (*)

where $u = x^2+axy+by^2$ and $v = cx^2+dxy+y^2$

$\therefore\ u_x = 2x+ay$ and $v_x = 2cx+dy$

$u_y = ax+2by$ $\quad v_y = dx+2y$

$\therefore$ by eqn (*) $\quad u_x = v_y \Rightarrow \boxed{2x+ay = dx+2y} \Rightarrow d=2$ & $a=2$

$u_y = -v_x \Rightarrow ax+2by = -(2cx+dy)$

$\Rightarrow 2x+2by = -2cx-2y$ $\quad \cdots\ \because a=2$ & $d=2$

$\Rightarrow b=-1$ & $c=-1$ $\quad \cdots$ by term by term comparision

$\therefore\ a=2;\ b=-1;\ c=-1;\ d=2$

Note:- If $f(z) = u+iv$ is analytic then $f'(z)$ exist

(Analytic $\Rightarrow$ define & differentiable)

$\therefore f'(z) = u_x + i\,v_x$

$= u_y + i\,v_y$

$= u_x - i\,u_y$ $\quad \cdots$ by C-R eqn $\boxed{u_y = -v_x}$

$= -v_x + i\,v_u$ $\quad \cdots$ by C-R eqn $\boxed{u_y = -v_x}$

$= u_x - i\,u_y$ .... by C-R eq^n $\boxed{u_y = -v_x}$

$= -v_x + i\,v_y$ .... by C-R eq^n $\boxed{u_y = -v_x}$

eg 1) Construct an analytic function whose real part is $3x^2y + 2x^2 - y^3 - 2y^2$

→ Given $u = 3x^2y + 2x^2 - y^3 - 2y^2$ ✓

Suppose $f(z) = u + iv$ is the required analytic f^n.

$\therefore f'(z) = u_x + i\,v_x$

$= u_x + i(-u_y)$ .... by C-R eq^n $\boxed{u_y = -v_x}$ as $f(z) = u + iv$ analytic

$\boxed{f'(z) = u_x - i\,u_y}$

By Milne-Thompson method: $\boxed{f'(z) = u_x(z,0) - i\,u_y(z,0)}$ —— (1)

where $u_x = 6xy + 4x \Rightarrow u_x(z,0) = 6z(0) + 4z = 4z$

$u_y = 3x^2 - 3y^2 - 4y \Rightarrow u_y(z,0) = 3z^2 - 3(0^2) - 4(0) = 3z^2$

$\therefore$ eq^n (1) becomes $f'(z) = 4z - i\,3z^2$

$\therefore f(z) = \int (4z - i3z^2)\,dz + c$

$f(z) = 4\frac{z^2}{2} - i3\frac{z^3}{3} + c$

$f(z) = 2z^2 - iz^3 + c$ is the required analytic f^n.

eg 2) Construct an analytic function whose imaginary part is $e^x(x\sin y + y\cos y)$

→ Given $v = e^x(x\sin y + y\cos y)$

let $f(z) = u + iv$ be the required analytic function.

$\therefore f'(z) = u_x + i\,v_x$

$= v_y + i\,v_x$ ... by C-R eq^{n}s $\boxed{u_x = v_y}$ as $f(z)$ analytic

$\therefore \boxed{f'(z) = v_y + i\,v_x}$

$\therefore$ By Milne-Thompson method: $\boxed{f'(z) = v_y(z,0) + i\,v_x(z,0)}$ —— (1)

Now $v_x = e^x(x\sin y + y\cos y) + e^x\sin y \Rightarrow v_x(z,0) = e^z(0+0) + e^z(0) = 0$

$v_y = e^x(x\cos y + \cos y - y\sin y) \Rightarrow v_y(z,0) = e^z(z + 1 - 0) = e^z(z+1)$

$\therefore$ eq^n (1) becomes: $f'(z) = e^z(z+1) + i\,0 = e^z(z+1)$

$\therefore f(z) = \int e^z(z+1)\,dz + c$

$$= (z+1)\int e^z\,dz - \int\int e^z\,dz \cdot \frac{d}{dz}(z+1)\,dz + C$$

$$= (z+1)\,e^z - \int e^z \cdot (1)\,dz + C$$

$$= (z+1)\,e^z - e^z + C$$

$$= z e^z + e^z - e^z + C$$

$\therefore f(z) = z e^z + C$ is the required analytic function.