Tutorial 3: Dimensionality Reduction

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Feb. 6, 2014

Outline

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1. Lagrange Optimization

Constrained optimization problem:

$$\max_{\mathbf{x}} \quad f(\mathbf{x}), \tag{1}$$

$$s.t. \quad g(\mathbf{x}) = 0. \tag{2}$$

The solution can often be found by Lagrangian method. The Lagrangian is defined as:

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x}). \tag{3}$$

1. Lagrange Optimization

Lagrangian Sufficiency Theorem: Suppose there exist $\mathbf{x}^* \in \mathbf{X}$ and λ^*, such that $\mathbf{x}^*$ maximize $L(\mathbf{x}, \lambda^*)$ over all $\mathbf{x} \in \mathbf{X}$, and $g(\mathbf{x}^*) = 0$. Then $\mathbf{x}^*$ solves the optimization problem.

Proof.

$$\begin{aligned}
& \max_{\mathbf{x}\in\mathbf{X},\ g(\mathbf{x})=0} f(\mathbf{x}) && (4)\\
= & \max_{\mathbf{x}\in\mathbf{X},\ g(\mathbf{x})=0} [f(\mathbf{x}) + \lambda^* g(\mathbf{x})] && (5)\\
\le & \max_{\mathbf{x}\in\mathbf{X}} [f(\mathbf{x}) + \lambda^* g(\mathbf{x})] && (6)\\
= & [f(\mathbf{x}^*) + \lambda^* g(\mathbf{x}^*)] && (7)\\
= & f(\mathbf{x}^*) && (8)
\end{aligned}$$

Solve Lagrange Optimization: solve the unconstrained problem by taking the derivative w.r.t. $\mathbf{x}$ and λ:

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \mathbf{x}} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} + \lambda \frac{\partial g(\mathbf{x})}{\partial \mathbf{x}} = 0 \tag{9}$$

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \lambda} = 0 \tag{10}$$

$$g(\mathbf{x}) = 0 \tag{11}$$

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2. Generalized Rayleigh Quotient

Fisher Criterion

$$J(\mathbf{w}) = \frac{\mathbf{w}^t \mathbf{S}_B \mathbf{w}}{\mathbf{w}^t \mathbf{S}_W \mathbf{w}} \tag{12}$$

$J(\mathbf{w})$ is the generalized Rayleigh quotient. A vector $\mathbf{w}$ that maximizes $J(\cdot)$ must satisfy

$$\mathbf{S}_B \mathbf{w} = \lambda \mathbf{S}_W \mathbf{w} \tag{13}$$

for some constant λ.

2. Generalized Rayleigh Quotient

Maximizing $J(\mathbf{w})$ is equivalent to

$$\max_{\mathbf{w}} \quad \mathbf{w}^t \mathbf{S}_B \mathbf{w} \tag{14}$$

$$s.t. \quad \mathbf{w}^t \mathbf{S}_W \mathbf{w} = K \tag{15}$$

which can be solved using Lagrange multipliers.

2. Generalized Rayleigh Quotient

Define the Lagrangian:

$$L = \mathbf{w}^t \mathbf{S}_B \mathbf{w} - \lambda(\mathbf{w}^t \mathbf{S}_W \mathbf{w} - K) \tag{16}$$

Maximize with respect to $\mathbf{w}$:

$$\nabla_{\mathbf{w}} L = 2(\mathbf{S}_B - \lambda \mathbf{S}_W)\mathbf{w} = 0 \tag{17}$$

To obtain the solution:

$$\mathbf{S}_B \mathbf{w} = \lambda \mathbf{S}_W \mathbf{w} \tag{18}$$

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