Tutorial 3: Dimensionality Reduction

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Feb. 6, 2014

Outline

1. Generalized Rayleigh Quotient
2. Lagrange Optimization
3. Exercises

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1. Generalized Rayleigh Quotient

Fisher Criterion

$$J(\mathbf{w}) = \frac{\mathbf{w}^t \mathbf{S}_B \mathbf{w}}{\mathbf{w}^t \mathbf{S}_W \mathbf{w}} \tag{1}$$

$J(\mathbf{w})$ is the generalized Rayleigh quotient. A vector $\mathbf{w}$ that maximizes $J(\cdot)$ must satisfy

$$\mathbf{S}_B \mathbf{w} = \lambda \mathbf{S}_W \mathbf{w} \tag{2}$$

for some constant λ

1. ML estimate: ^ of multivariate Gaussian

Multivariate normal density

$$p(\mathbf{x} \mid \, , \,) = \frac{1}{(2\pi)^{d/2}||^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-)^{t\,-1}(\mathbf{x}-) \right] \qquad (3)$$

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Draw $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ independently from $p(\mathbf{x} \mid \ , \ )$, and the joint density (likelihood) is:

$$p(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n \mid ,) = \tag{4}$$

$$\frac{1}{(2\pi)^{nd/2}||^{n/2}} \exp\left[-\frac{1}{2}\sum_{k=1}^{n}(\mathbf{x}_k-)^{t-1}(\mathbf{x}_k-) \right] \tag{5}$$

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Log-likelihood $l(\ , \ )$ is

$$l(,) = -\frac{nd}{2} \ln(2\pi) - \frac{n}{2} \ln |\,| - \frac{1}{2} \sum_{k=1}^{n} (\mathbf{x}_k -)^{t \ -1} (\mathbf{x}_k -) \tag{6}$$

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(9)

(10)

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$$l(,) = -\frac{nd}{2}\ln(2\pi) + \frac{n}{2}\ln\mathbf{A} - \frac{1}{2}\sum_{k=1}^{n}(\mathbf{x}_k -)^{t}\mathbf{A}(\mathbf{x}_k -) \tag{8}$$

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$$\frac{\partial l(,)}{\partial \mathbf{A}} = \frac{n}{2}\mathbf{A}^{-1} - \frac{1}{2}\sum_{k=1}^{n}(\mathbf{x}_k -)(\mathbf{x}_k -)^{t} = 0 \tag{9}$$

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$$\hat{} = \frac{1}{n}\sum_{k=1}^{n}(\mathbf{x}_k - \hat{})(\mathbf{x}_k - \hat{})^{t}, \quad (\hat{} = \frac{1}{n}\sum_{k=1}^{n}\mathbf{x}_k) \tag{10}$$

Outline

1. ML estimate: $\hat{}$ of multivariate Gaussian
2. ML estimate bias: $\hat{\sigma}^2$ of univariate Gaussian
3. Bayesian estimate: Univariate Gaussian
   - brief review
   - posteriori $p(\mu|)$
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An elementary unbiased estimator for σ^2 is given by $\frac{1}{n-1}\sum_{k=1}^{n}(x_k - \hat{\mu})$.

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$$(20)$$

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(23)

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(23)

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(23)

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3.2 Bayesian estimate of univariate Gaussian: posteriori $p(\mu|)$

Bayesian estimate

$P(w_i|\mathbf{x},) \Rightarrow p(\mathbf{x}|w_i,) \Rightarrow p(\mathbf{x}|) \Rightarrow p(\mathbf{x}|\theta)\ \&\ p(\theta|)$

(25)

3.2 Bayesian estimate of univariate Gaussian: posteriori $p(\mu|)$

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(25)

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Posteriori $p(\mu|)$ of univariate Gaussian

$$p(\mu|) = \frac{p(|\mu)p(\mu)}{\int p(|\mu)p(\mu)d\mu} \tag{24}$$

$$= \alpha \prod_{k=1}^{n} p(x_k|\mu)p(\mu) \tag{25}$$

3.2 Bayesian estimate of univariate Gaussian: posteriori $p(\mu|)$

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Posteriori $p(\mu|)$ of univariate Gaussian

$$p(\mu|) = \frac{p(|\mu)p(\mu)}{\int p(|\mu)p(\mu)d\mu} \tag{24}$$

$$= \alpha \prod_{k=1}^{n} p(x_k|\mu)p(\mu) \tag{25}$$

where $p(x_k|\mu) \sim \mathcal{N}(\mu, \sigma^2)$, and $p(\mu) \sim \mathcal{N}(\mu_0, \sigma_0^2)$.

3.2 Bayesian estimate of univariate Gaussian: posteriori $p(\mu|)$

Posteriori $p(\mu|)$ of univariate Gaussian

$$p(\mu|) = \alpha \prod_{k=1}^{n} p(x_k|\mu)p(\mu) \tag{26}$$

(32)

3.2 Bayesian estimate of univariate Gaussian: posteriori $p(\mu|)$

Posteriori $p(\mu|)$ of univariate Gaussian

$$p(\mu|) = \alpha \prod_{k=1}^{n} p(x_k|\mu)p(\mu) \tag{26}$$

$$= \alpha \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\Big[-\frac{1}{2}\Big(\frac{x_k-\mu}{\sigma}\Big)^2\Big] \frac{1}{\sqrt{2\pi}\sigma_0} \exp\Big[-\frac{1}{2}\Big(\frac{\mu-\mu_0}{\sigma_0}\Big)^2\Big] \tag{27}$$

(32)

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$$= \alpha' \exp\Big[-\frac{1}{2}\Big(\sum_{k=1}^{n}(\frac{\mu-x_k}{\sigma})^2 + (\frac{\mu-\mu_0}{\sigma_0})^2\Big)\Big] \tag{28}$$

(32)

3.2 Bayesian estimate of univariate Gaussian: posteriori $p(\mu|)$

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$$= \alpha' \exp\Big[-\frac{1}{2}\Big(\sum_{k=1}^{n}(\frac{\mu-x_k}{\sigma})^2 + (\frac{\mu-\mu_0}{\sigma_0})^2\Big)\Big] \tag{28}$$

$$= \alpha'' \exp\Big[-\frac{1}{2}\Big[(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2})\mu^2 - 2(\frac{1}{\sigma^2}\sum_{k=1}^{n} x_k + \frac{\mu_0}{\sigma_0^2})\mu\Big]\Big] \tag{29}$$

$$\tag{32}$$

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Posteriori $p(\mu|)$ of univariate Gaussian

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$$\therefore \quad p(\mu|) \sim \mathcal{N}(\mu_n, \sigma_n^2),\ i.e.\ p(\mu|) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\Big[-\frac{1}{2}\Big(\frac{\mu-\mu_n}{\sigma_n}\Big)^2\Big] \tag{30}$$

(32)

3.2 Bayesian estimate of univariate Gaussian: posteriori $p(\mu|)$

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$$= \alpha'' \exp\Big[-\frac{1}{2}\Big[(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2})\mu^2 - 2(\frac{1}{\sigma^2}\sum_{k=1}^{n} x_k + \frac{\mu_0}{\sigma_0^2})\mu\Big]\Big] \tag{29}$$

$$\therefore \quad p(\mu|) \sim \mathcal{N}(\mu_n, \sigma_n^2), \ \ i.e. \ p(\mu|) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\Big[-\frac{1}{2}\Big(\frac{\mu-\mu_n}{\sigma_n}\Big)^2\Big] \tag{30}$$

$$\Longrightarrow \frac{1}{\sigma_n^2} = \frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}, \ \ \frac{\mu_n}{\sigma_n^2} = \frac{1}{\sigma^2}\sum_{k=1}^{n} x_k + \frac{\mu_0}{\sigma_0^2} = \frac{n}{\sigma^2}\hat{\mu}_n + \frac{\mu_0}{\sigma_0^2} \tag{31}$$

(32)

3.2 Bayesian estimate of univariate Gaussian: posteriori $p(\mu|)$

Posteriori $p(\mu|)$ of univariate Gaussian

$$p(\mu|) = \alpha \prod_{k=1}^{n} p(x_k|\mu)p(\mu) \tag{26}$$

$$= \alpha \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\Big[-\frac{1}{2}\Big(\frac{x_k-\mu}{\sigma}\Big)^2\Big] \frac{1}{\sqrt{2\pi}\sigma_0} \exp\Big[-\frac{1}{2}\Big(\frac{\mu-\mu_0}{\sigma_0}\Big)^2\Big] \tag{27}$$

$$= \alpha' \exp\Big[-\frac{1}{2}\Big(\sum_{k=1}^{n}(\frac{\mu-x_k}{\sigma})^2 + (\frac{\mu-\mu_0}{\sigma_0})^2\Big)\Big] \tag{28}$$

$$= \alpha'' \exp\Big[-\frac{1}{2}\Big[(\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2})\mu^2 - 2(\frac{1}{\sigma^2}\sum_{k=1}^{n} x_k + \frac{\mu_0}{\sigma_0^2})\mu\Big]\Big] \tag{29}$$

$$\therefore \quad p(\mu|) \sim \mathcal{N}(\mu_n, \sigma_n^2), \ i.e.\ p(\mu|) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\Big[-\frac{1}{2}\Big(\frac{\mu-\mu_n}{\sigma_n}\Big)^2\Big] \tag{30}$$

$$\Longrightarrow \frac{1}{\sigma_n^2} = \frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}, \ \frac{\mu_n}{\sigma_n^2} = \frac{1}{\sigma^2}\sum_{k=1}^{n} x_k + \frac{\mu_0}{\sigma_0^2} = \frac{n}{\sigma^2}\hat{\mu}_n + \frac{\mu_0}{\sigma_0^2} \tag{31}$$

$$\Longrightarrow \mu_n = \Big(\frac{n\sigma_0^2}{n\sigma_0^2+\sigma^2}\Big)\hat{\mu}_n + \frac{\sigma^2}{\sigma_0^2+\sigma^2}\mu_0, \ \sigma_n^2 = \frac{\sigma_0^2\sigma^2}{n\sigma_0^2+\sigma^2} \tag{32}$$

Outline

1. ML estimate: $\hat{}$ of multivariate Gaussian
2. ML estimate bias: $\hat{\sigma}^2$ of univariate Gaussian
3. Bayesian estimate: Univariate Gaussian
   - brief review
   - posteriori $p(\mu|)$
   - conditional probability density $p(x|w_{i,i})$

Outline

1. ML estimate: $\hat{}$ of multivariate Gaussian
2. ML estimate bias: $\hat{\sigma}^2$ of univariate Gaussian
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   - brief review
   - posteriori $p(\mu|)$
   - conditional probability density $p(x|w_i,_i\,)$

3.3 Bayesian estimate of univariate Gaussian: conditional probability density $p(x|w_{i,i})$

Conditional probability density $p(x|w_{i,i})$

$$p(x|w_{i,i}) \tag{33}$$

$$\sim p(x|) \tag{34}$$

(38)

3.3 Bayesian estimate of univariate Gaussian: conditional probability density $p(x|w_{i,i})$

Conditional probability density $p(x|w_{i,i})$

$$p(x|w_{i,i}) \quad (33)$$

$$\sim p(x|) \quad (34)$$

$$= \int p(x|\mu)p(\mu|)d\mu \quad (35)$$

(38)

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$$= \int \frac{1}{\sqrt{2\pi}\sigma} \exp\Big[-\frac{1}{2}\Big(\frac{x-\mu}{\sigma}\Big)^2\Big] \frac{1}{\sqrt{2\pi}\sigma_n} \exp\Big[-\frac{1}{2}\Big(\frac{\mu-\mu_n}{\sigma_n}\Big)^2\Big] d\mu$$

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$$= \frac{1}{2\pi\sigma\sigma_n} \exp\left[-\frac{1}{2}\frac{(x-\mu_n)^2}{\sigma^2+\sigma_n^2}\right] f(\sigma,\sigma_n) \tag{36}$$

$$\tag{38}$$

3.3 Bayesian estimate of univariate Gaussian: conditional probability density $p(x|w_{i,i})$

Conditional probability density $p(x|w_{i,i})$

$$p(x|w_{i,i}) \tag{33}$$
$$\sim p(x|) \tag{34}$$
$$= \int p(x|\mu)p(\mu|)d\mu \tag{35}$$
$$= \int \frac{1}{\sqrt{2\pi}\sigma} \exp\Big[-\frac{1}{2}\Big(\frac{x-\mu}{\sigma}\Big)^2\Big] \frac{1}{\sqrt{2\pi}\sigma_n} \exp\Big[-\frac{1}{2}\Big(\frac{\mu-\mu_n}{\sigma_n}\Big)^2\Big] d\mu$$
$$= \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}\frac{(x-\mu_n)^2}{\sigma^2+\sigma_n^2}\Big] f(\sigma,\sigma_n) \tag{36}$$
$$\therefore \quad p(x|) \sim \mathcal{N}(\mu_n, \sigma^2+\sigma_n^2) \tag{37}$$
$$\tag{38}$$

3.3 Bayesian estimate of univariate Gaussian: conditional probability density $p(x|w_{i,i}\ )$

Conditional probability density $p(x|w_{i,i}\ )$

$$
\begin{aligned}
& p(x|w_{i,i}\ ) && (33)\\
& \sim p(x|) && (34)\\
& = \int p(x|\mu)p(\mu|)d\mu && (35)\\
& = \int \frac{1}{\sqrt{2\pi}\sigma} \exp\Big[-\frac{1}{2}\Big(\frac{x-\mu}{\sigma}\Big)^2\Big] \frac{1}{\sqrt{2\pi}\sigma_n} \exp\Big[-\frac{1}{2}\Big(\frac{\mu-\mu_n}{\sigma_n}\Big)^2\Big] d\mu \\
& = \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}\frac{(x-\mu_n)^2}{\sigma^2+\sigma_n^2}\Big] f(\sigma,\sigma_n) && (36)\\
& \therefore \quad p(x|) \sim \mathcal{N}(\mu_n, \sigma^2+\sigma_n^2) && (37)\\
& f(\sigma,\sigma_n) = \int \exp\Big[-\frac{1}{2}\frac{\sigma^2+\sigma_n^2}{\sigma^2\sigma_n^2}\Big(\mu - \frac{\sigma_n^2 x + \sigma^2\mu_n}{\sigma^2+\sigma_n^2}\Big)^2\Big] d\mu && (38)
\end{aligned}
$$

3.3 Bayesian estimate of univariate Gaussian: conditional probability density $p(x|w_{i,i}\ )$

Conditional probability density $p(x|w_{i,i}\ )$

$$p(x|w_{i\ ,i}\ ) \sim p(x|) \tag{39}$$

3.3 Bayesian estimate of univariate Gaussian: conditional probability density $p(x|w_{i,i})$

Conditional probability density $p(x|w_{i,i})$

$$p(x|w_{i,i}) \sim p(x|) \tag{39}$$

$$= \int p(x|\mu)p(\mu|)d\mu \tag{40}$$

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$$p(x|w_{i,i}) \sim p(x|) \tag{39}$$

$$= \int p(x|\mu)p(\mu|)d\mu \tag{40}$$

$$= \int \frac{1}{\sqrt{2\pi}\sigma} \exp\Big[-\frac{1}{2}\Big(\frac{x-\mu}{\sigma}\Big)^2\Big] \frac{1}{\sqrt{2\pi}\sigma_n} \exp\Big[-\frac{1}{2}\Big(\frac{\mu-\mu_n}{\sigma_n}\Big)^2\Big] d\mu \tag{41}$$

3.3 Bayesian estimate of univariate Gaussian: conditional probability density $p(x|w_{i,i})$

Conditional probability density $p(x|w_{i,i})$

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$$= \frac{1}{2\pi\sigma\sigma_n} \int \exp\Big[-\frac{1}{2}\Big(\frac{x^2-2x\mu+\mu^2}{\sigma^2} + \frac{\mu^2-2\mu\mu_n+\mu_n^2}{\sigma_n^2}\Big)\Big] d\mu \tag{42}$$

3.3 Bayesian estimate of univariate Gaussian: conditional probability density $p(x|w_{i,i})$

Conditional probability density $p(x|w_{i,i})$

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$$= \int p(x|\mu)p(\mu|)d\mu \tag{40}$$

$$= \int \frac{1}{\sqrt{2\pi}\sigma} \exp\Big[-\frac{1}{2}\Big(\frac{x-\mu}{\sigma}\Big)^2\Big] \frac{1}{\sqrt{2\pi}\sigma_n} \exp\Big[-\frac{1}{2}\Big(\frac{\mu-\mu_n}{\sigma_n}\Big)^2\Big] d\mu \tag{41}$$

$$= \frac{1}{2\pi\sigma\sigma_n} \int \exp\Big[-\frac{1}{2}\Big(\frac{x^2-2x\mu+\mu^2}{\sigma^2} + \frac{\mu^2-2\mu\mu_n+\mu_n^2}{\sigma_n^2}\Big)\Big] d\mu \tag{42}$$

$$= \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}(\frac{x^2}{\sigma^2} + \frac{\mu_n^2}{\sigma_n^2})\Big] \int \exp\Big[-\frac{1}{2}\Big(\frac{\mu^2-2x\mu}{\sigma^2} + \frac{\mu^2-2\mu_n\mu}{\sigma_n^2}\Big)\Big] d\mu \tag{43}$$

3.3 Bayesian estimate of univariate Gaussian: conditional probability density $p(x|w_{i,i})$

Conditional probability density $p(x|w_{i,i})$

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$$= \int p(x|\mu)p(\mu|)d\mu \tag{40}$$

$$= \int \frac{1}{\sqrt{2\pi}\sigma} \exp\Big[-\frac{1}{2}\Big(\frac{x-\mu}{\sigma}\Big)^2\Big] \frac{1}{\sqrt{2\pi}\sigma_n} \exp\Big[-\frac{1}{2}\Big(\frac{\mu-\mu_n}{\sigma_n}\Big)^2\Big] d\mu \tag{41}$$

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$$= \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}(\frac{x^2}{\sigma^2} + \frac{\mu_n^2}{\sigma_n^2})\Big] \int \exp\Big[-\frac{1}{2}\Big(\frac{\sigma_n^2+\sigma^2}{\sigma_n^2\sigma^2}\mu^2 - 2(\frac{x}{\sigma^2} + \frac{\mu_n}{\sigma_n^2})\mu\Big)\Big] d\mu \tag{44}$$

3.3 Bayesian estimate of univariate Gaussian: conditional probability density $p(x|w_{i,i})$

Conditional probability density $p(x|w_{i,i})$

$$
\begin{aligned}
& p(x|w_{i,i}) \sim p(x|) && (39)\\
& = \int p(x|\mu)p(\mu|)d\mu && (40)\\
& = \int \frac{1}{\sqrt{2\pi}\sigma} \exp\Big[-\frac{1}{2}\Big(\frac{x-\mu}{\sigma}\Big)^2\Big] \frac{1}{\sqrt{2\pi}\sigma_n} \exp\Big[-\frac{1}{2}\Big(\frac{\mu-\mu_n}{\sigma_n}\Big)^2\Big] d\mu && (41)\\
& = \frac{1}{2\pi\sigma\sigma_n} \int \exp\Big[-\frac{1}{2}\Big(\frac{x^2-2x\mu+\mu^2}{\sigma^2} + \frac{\mu^2-2\mu\mu_n+\mu_n^2}{\sigma_n^2}\Big)\Big] d\mu && (42)\\
& = \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}(\frac{x^2}{\sigma^2} + \frac{\mu_n^2}{\sigma_n^2})\Big] \int \exp\Big[-\frac{1}{2}\Big(\frac{\mu^2-2x\mu}{\sigma^2} + \frac{\mu^2-2\mu_n\mu}{\sigma_n^2}\Big)\Big] d\mu && (43)\\
& = \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}(\frac{x^2}{\sigma^2} + \frac{\mu_n^2}{\sigma_n^2})\Big] \int \exp\Big[-\frac{1}{2}\Big(\frac{\sigma_n^2+\sigma^2}{\sigma_n^2\sigma^2}\mu^2 - 2(\frac{x}{\sigma^2} + \frac{\mu_n}{\sigma_n^2})\mu\Big)\Big] d\mu && (44)\\
& = \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}(\frac{x^2}{\sigma^2} + \frac{\mu_n^2}{\sigma_n^2})\Big] \int \exp\Big[-\frac{1}{2}\frac{\sigma_n^2+\sigma^2}{\sigma_n^2\sigma^2}\Big(\mu^2 - 2\frac{\sigma_n^2 x+\sigma^2\mu_n}{\sigma_n^2+\sigma^2}\mu\Big)\Big] d\mu
\end{aligned}
$$

3.3 Bayesian estimate of univariate Gaussian: conditional probability density $p(x|w_{i,i})$

Conditional probability density $p(x|w_{i,i})$

$$p(x|w_{i,i}) \sim p(x|) \tag{39}$$

$$= \int p(x|\mu)p(\mu|)d\mu \tag{40}$$

$$= \int \frac{1}{\sqrt{2\pi}\sigma} \exp\Big[-\frac{1}{2}\Big(\frac{x-\mu}{\sigma}\Big)^2\Big] \frac{1}{\sqrt{2\pi}\sigma_n} \exp\Big[-\frac{1}{2}\Big(\frac{\mu-\mu_n}{\sigma_n}\Big)^2\Big] d\mu \tag{41}$$

$$= \frac{1}{2\pi\sigma\sigma_n} \int \exp\Big[-\frac{1}{2}\Big(\frac{x^2-2x\mu+\mu^2}{\sigma^2} + \frac{\mu^2-2\mu\mu_n+\mu_n^2}{\sigma_n^2}\Big)\Big] d\mu \tag{42}$$

$$= \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}(\frac{x^2}{\sigma^2} + \frac{\mu_n^2}{\sigma_n^2})\Big] \int \exp\Big[-\frac{1}{2}\Big(\frac{\mu^2-2x\mu}{\sigma^2} + \frac{\mu^2-2\mu_n\mu}{\sigma_n^2}\Big)\Big] d\mu \tag{43}$$

$$= \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}(\frac{x^2}{\sigma^2} + \frac{\mu_n^2}{\sigma_n^2})\Big] \int \exp\Big[-\frac{1}{2}\Big(\frac{\sigma_n^2+\sigma^2}{\sigma_n^2\sigma^2}\mu^2 - 2(\frac{x}{\sigma^2} + \frac{\mu_n}{\sigma_n^2})\mu\Big)\Big] d\mu \tag{44}$$

$$= \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}(\frac{x^2}{\sigma^2} + \frac{\mu_n^2}{\sigma_n^2})\Big] \int \exp\Big[-\frac{1}{2}\frac{\sigma_n^2+\sigma^2}{\sigma_n^2\sigma^2}\Big(\mu^2 - 2\frac{\sigma_n^2 x+\sigma^2\mu_n}{\sigma_n^2+\sigma^2}\mu\Big)\Big] d\mu$$

$$= \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}(\frac{x^2}{\sigma^2} + \frac{\mu_n^2}{\sigma_n^2} - \frac{(\sigma_n^2 x+\sigma^2\mu_n)^2}{\sigma_n^2\sigma^2(\sigma_n^2+\sigma^2)})\Big] f(\sigma,\sigma_n) \tag{45}$$

3.3 Bayesian estimate of univariate Gaussian: conditional probability density $p(x|w_{i,i})$

Conditional probability density $p(x|w_{i,i})$

$$p(x|w_{i,i}) \sim p(x|) \quad (39)$$

$$= \int p(x|\mu)p(\mu|)d\mu \quad (40)$$

$$= \int \frac{1}{\sqrt{2\pi}\sigma} \exp\Big[-\frac{1}{2}\Big(\frac{x-\mu}{\sigma}\Big)^2\Big] \frac{1}{\sqrt{2\pi}\sigma_n} \exp\Big[-\frac{1}{2}\Big(\frac{\mu-\mu_n}{\sigma_n}\Big)^2\Big] d\mu \quad (41)$$

$$= \frac{1}{2\pi\sigma\sigma_n} \int \exp\Big[-\frac{1}{2}\Big(\frac{x^2-2x\mu+\mu^2}{\sigma^2} + \frac{\mu^2-2\mu\mu_n+\mu_n^2}{\sigma_n^2}\Big)\Big] d\mu \quad (42)$$

$$= \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}(\frac{x^2}{\sigma^2} + \frac{\mu_n^2}{\sigma_n^2})\Big] \int \exp\Big[-\frac{1}{2}\Big(\frac{\mu^2-2x\mu}{\sigma^2} + \frac{\mu^2-2\mu_n\mu}{\sigma_n^2}\Big)\Big] d\mu \quad (43)$$

$$= \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}(\frac{x^2}{\sigma^2} + \frac{\mu_n^2}{\sigma_n^2})\Big] \int \exp\Big[-\frac{1}{2}\Big(\frac{\sigma_n^2+\sigma^2}{\sigma_n^2\sigma^2}\mu^2 - 2(\frac{x}{\sigma^2} + \frac{\mu_n}{\sigma_n^2})\mu\Big)\Big] d\mu \quad (44)$$

$$= \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}(\frac{x^2}{\sigma^2} + \frac{\mu_n^2}{\sigma_n^2})\Big] \int \exp\Big[-\frac{1}{2}\frac{\sigma_n^2+\sigma^2}{\sigma_n^2\sigma^2}\Big(\mu^2 - 2\frac{\sigma_n^2 x+\sigma^2\mu_n}{\sigma_n^2+\sigma^2}\mu\Big)\Big] d\mu$$

$$= \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}(\frac{x^2}{\sigma^2} + \frac{\mu_n^2}{\sigma_n^2} - \frac{(\sigma_n^2 x+\sigma^2\mu_n)^2}{\sigma_n^2\sigma^2(\sigma_n^2+\sigma^2)})\Big] f(\sigma,\sigma_n) \quad (45)$$

$$f(\sigma,\sigma_n) = \int \exp\Big[-\frac{1}{2}\frac{\sigma_n^2+\sigma^2}{\sigma_n^2\sigma^2}\Big(\mu - \frac{\sigma_n^2 x+\sigma^2\mu_n}{\sigma^2+\sigma_n^2}\Big)^2\Big] d\mu$$

3.3 Bayesian estimate of univariate Gaussian: conditional probability density $p(x|w_{i,i})$

Conditional probability density $p(x|w_{i,i})$

$$p(x|) \tag{46}$$

$$= \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}\Big(\frac{x^2}{\sigma^2} + \frac{\mu_n^2}{\sigma_n^2} - \frac{(\sigma_n^2 x + \sigma^2 \mu_n)^2}{\sigma_n^2\sigma^2(\sigma_n^2+\sigma^2)}\Big)\Big] f(\sigma, \sigma_n) \tag{47}$$

(51)

$$f(\sigma, \sigma_n) = \int \exp\Big[-\frac{1}{2}\frac{\sigma_n^2+\sigma^2}{\sigma_n^2\sigma^2}\Big(\mu - \frac{\sigma_n^2 x + \sigma^2\mu_n}{\sigma^2+\sigma_n^2}\Big)^2\Big] d\mu$$

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$$= \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}\Big(\frac{x^2}{\sigma_n^2 + \sigma^2} - \frac{2x\mu_n}{\sigma_n^2 + \sigma^2} + \frac{\mu_n^2}{\sigma_n^2 + \sigma^2}\Big)\Big] f(\sigma, \sigma_n) \tag{49}$$

(51)

$$f(\sigma, \sigma_n) = \int \exp\Big[-\frac{1}{2}\frac{\sigma_n^2 + \sigma^2}{\sigma_n^2\sigma^2}\Big(\mu - \frac{\sigma_n^2 x + \sigma^2\mu_n}{\sigma^2 + \sigma_n^2}\Big)^2\Big] d\mu$$

3.3 Bayesian estimate of univariate Gaussian: conditional probability density $p(x|w_{i,i})$

Conditional probability density $p(x|w_{i,i})$

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$$= \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}\Big(\frac{x^2}{\sigma^2} + \frac{\mu_n^2}{\sigma_n^2} - \frac{(\sigma_n^2 x + \sigma^2\mu_n)^2}{\sigma_n^2\sigma^2(\sigma_n^2+\sigma^2)}\Big)\Big] f(\sigma, \sigma_n) \tag{47}$$

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$$= \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}\Big(\frac{x^2}{\sigma_n^2+\sigma^2} - \frac{2x\mu_n}{\sigma_n^2+\sigma^2} + \frac{\mu_n^2}{\sigma_n^2+\sigma^2}\Big)\Big] f(\sigma, \sigma_n) \tag{49}$$

$$= \frac{1}{2\pi\sigma\sigma_n} \exp\Big[-\frac{1}{2}\frac{(x-\mu_n)^2}{\sigma^2+\sigma_n^2}\Big] f(\sigma, \sigma_n) \tag{50}$$

$$\tag{51}$$

$$f(\sigma, \sigma_n) = \int \exp\Big[-\frac{1}{2}\frac{\sigma_n^2+\sigma^2}{\sigma_n^2\sigma^2}\Big(\mu - \frac{\sigma_n^2 x + \sigma^2\mu_n}{\sigma^2+\sigma_n^2}\Big)^2\Big] d\mu$$