Support Vector Machine

$$(P) \quad \min \frac{\|W\|^2}{2}$$

$$\text{s.t.} \quad y_i(w \cdot x_i + b) - 1 \geq 0 \;, \quad i = 1, \cdots, n$$

$$\Leftrightarrow \quad J(w, b) = \begin{cases} \frac{1}{2}\|W\|^2 \;, & y_i(w \cdot x_i + b) - 1 \geq 0 \;, \; i = 1, \cdots, n \\ \infty \;, & \text{otherwise} \end{cases}$$

$$= \frac{\|W\|^2}{2} + \begin{cases} 0 \;, & y_i(w \cdot x_i + b) - 1 \geq 0 \;, \; i = 1, \cdots, n \\ \infty \;, & \text{otherwise} \end{cases}$$

Lagrange Multipliers $\alpha = \{\alpha_i\}$

$$\max_{a_i \geq 0} \alpha_i[1 - y_i(w \cdot x_i + b)] = \begin{cases} 0 \;. & y_i(wx_i + b) - 1 \geq 0 \\ \infty \;. & \text{otherwise} \end{cases}$$

$$J(w, b) = \frac{\|W\|^2}{2} + \sum_{i=1}^{n} \max_{\alpha_i \geq 0} \alpha_i [1 - y_i(w \cdot x + b)]$$

$$(P) \Leftrightarrow \quad \min_{w, b} \left\{ \frac{\|W\|^2}{2} + \sum_{i=1}^{n} \max_{\alpha_i \geq 0} \alpha_i [1 - y_i(w \cdot x_i + b)] \right\}$$

$$= \max_{\alpha_i \geq 0} \min_{w, b} \left\{ \frac{\|W\|^2}{2} + \sum_{i=1}^{n} \alpha_i [1 - y_i(w \cdot x_i + b)] \right\} \qquad (P1)$$

$$\text{Let } J(w, b; \alpha) = \frac{\|W\|^2}{2} + \sum_{i=1}^{n} \alpha_i [1 - y_i(w \cdot x_i + b)]$$

$$\frac{\partial J(w, b; \alpha)}{\partial w} = w - \sum_{i=1}^{n} \alpha_i y_i x_i = 0 \qquad \Rightarrow \quad w^* = \sum_{i=1}^{n} \alpha_i y_i x_i \neq 0$$

$$\frac{\partial J(w, b; \alpha)}{\partial b} = -\sum_{i=1}^{n} \alpha_i y_i = 0$$

$$(P1) \quad = \max_{\substack{\alpha_i \geq 0 \\ \sum_{i=1}^{n} \alpha_i y_i = 0}} \left\{ \frac{\|w^*\|^2}{2} + \sum_{i=1}^{n} \alpha_i [1 - y_i(w^* x_i + b^*)] \right\}$$

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$$= \max_{\substack{\alpha_i \geq 0 \\ \sum_{i=1}^{n} \alpha_i y_i = 0}} \left\{ \frac{\|w^*\|^2}{2} + \sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} \alpha_i y_i x_i \cdot w^* - \underbrace{\sum_{i=1}^{n} \alpha_i y_i b^*}_{=0} \right\}$$

$$= \max_{\substack{\alpha_i \geq 0 \\ \sum_{i=1}^{n} \alpha_i y_i = 0}} \left\{ \frac{\|w^*\|^2}{2} + \sum_{i=1}^{n} \alpha_i - \|w^*\|^2 \right\}$$

$$= \max_{\substack{\alpha_i \geq 0 \\ \sum_{i=1}^{n} \alpha_i y_i = 0}} \left\{ \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j (x_i \cdot x_j) \right\}$$

$$\Rightarrow \max \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} y_i y_j \alpha_i \alpha_j (x_i \cdot x_j)$$

s.t. $\alpha_i \geq 0$

$\sum_{i=1}^{n} \alpha_i y_i = 0$

if $\alpha_i^* > 0 \Rightarrow y_i (w^* \cdot x_i + b^*) = 1$

if $\alpha_i^* = 0 \Rightarrow y_i (w^* \cdot x_i + b^*) > 1$

$$w^* = \sum_{i \in SV} \alpha_i^* y_i x_i$$