D.C programming with rank constrained problem

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1 D.C Programming

The D.C programming problems have been widely investigated. It has been proven that D.C programming can only converge to a local optimal stable solution. Now we consider a digital precoder problem:

$$\begin{aligned}
\text{maximize} \quad & \sum_{k=1}^{K} R_k = \sum_{k=1}^{K} \log_2 \left(1 + \frac{|\boldsymbol{h}_k \boldsymbol{f}_k|^2}{\sigma_k^2 + \sum_{i \neq k}^{K} |\boldsymbol{h}_k \boldsymbol{f}_i|^2} \right) && (1) \\
& = \sum_{k=1}^{K} \left[\log_2 \sigma_k^2 + \sum_{k=1}^{K} \left(|\boldsymbol{h}_k \boldsymbol{f}_k|^2 \right) - \log_2 \left(\sigma_k^2 + \sum_{i \neq k}^{K} |\boldsymbol{h}_k \boldsymbol{f}_i|^2 \right) \right] \\
\text{subject to} \quad & \|\boldsymbol{V} \boldsymbol{f}_k\|_F \leq 1; \\
& R_k \geq \lambda_k, && (2)
\end{aligned}$$

where $\boldsymbol{V}$ is the analog precoder and $\boldsymbol{h}_k = \boldsymbol{w}_k^H \boldsymbol{h}_k \boldsymbol{V}$ is the effective channel.

Since $\log_2(\cdot)$ is concave while $|\boldsymbol{h}_k \boldsymbol{f}_k|^2$ is convex, the problem (1) can be transformed as D.C problem by defining:

$$\boldsymbol{F}_k = \boldsymbol{f}_k \boldsymbol{f}_k^H \tag{3}$$

Then the problem (1) can be rewritten as a standard D.C problem:

$$\begin{aligned}
\text{maximize} \quad & \sum_{k=1}^{K} R_k = \sum_{k=1}^{K} \left[\log_2 \left(|\boldsymbol{h}_k \boldsymbol{F}_k \boldsymbol{h}_k^H| \right) - \log_2 \left(\sigma_k^2 + \sum_{i \neq k}^{K} |\boldsymbol{h}_k \boldsymbol{F}_i \boldsymbol{h}_k^H| \right) \right] && (4) \\
\text{subject to} \quad & \|\boldsymbol{V} \boldsymbol{f}_k\|_F \leq 1; \\
& R_k \geq \lambda_k; \\
& \text{Rank}(\boldsymbol{F}_k) = 1.
\end{aligned}$$

By ignoring the rank constraint, the digital precoder can be solved alternatively because (1) the degree freedom of variable matrix $F_k, k = 1, \cdots, K$ will be very large such that it's very hard to be solved; (2) the local optimal solution will be sensitive to initial value as the increment of degree freedom. We firstly give the initial value to $\boldsymbol{F}_k^{(0)}, k = 1, \cdots, K$, then the digital precoder is solved iteratively:

$$\text{maximize} \quad f(\boldsymbol{F}_k) - g(\boldsymbol{F}_k): \quad \text{subject to the constraits in (4)} \tag{5}$$

The solution of problem (5) can be calculated by iterative procedure:

$$\text{maximize} \quad f(\boldsymbol{F}_k) - g(\boldsymbol{F}_k^{(t)}) - \langle \bigtriangledown g(\boldsymbol{F}_k^{(t)}), \boldsymbol{F}_k - \boldsymbol{F}_k^{(t)} \rangle: \quad \text{subject to the constraits in (4)} \tag{6}$$

This convex problem in each iterations can be easily solved by CVX toolbox.

1.1 Convergence Analysis

As function $g(\boldsymbol{F}_{\boldsymbol{k}}^{(t)})$ is concave, its gradient $\bigtriangledown g(\boldsymbol{F}_{\boldsymbol{k}}^{(t)})$ is also its super-gradient, so

$$g(\boldsymbol{F}_{\boldsymbol{k}}^{(t+1)}) \leq g(\boldsymbol{F}_{k}^{(t)}) + \langle \bigtriangledown g(\boldsymbol{F}_{k}^{(t)}), \boldsymbol{F}_{k}^{(t+1)} - \boldsymbol{F}_{k}^{(t)} \rangle \tag{7}$$

Then

$$f(\boldsymbol{F}_{\boldsymbol{k}}^{(t+1)}) - g(\boldsymbol{F}_{\boldsymbol{k}}^{(t+1)}) \geq f(\boldsymbol{F}_{\boldsymbol{k}}^{(t+1)}) - \left[g(\boldsymbol{F}_{k}^{(t)}) + \langle \bigtriangledown g(\boldsymbol{F}_{k}^{(t)}), \boldsymbol{F}_{k}^{(t+1)} - \boldsymbol{F}_{k}^{(t)} \rangle \right] \tag{8}$$

and from Equation (6), we know

$$\begin{aligned} & f(\boldsymbol{F}_{k}^{(t+1)}) - g(\boldsymbol{F}_{k}^{(t)}) - \langle \bigtriangledown g(\boldsymbol{F}_{k}^{(t)}), \boldsymbol{F}_{k}^{(t+1)} - \boldsymbol{F}_{k}^{(t)} \rangle \\ \geq \quad & f(\boldsymbol{F}_{k}^{(t)}) - g(\boldsymbol{F}_{k}^{(t)}) - \langle \bigtriangledown g(\boldsymbol{F}_{k}^{(t)}), \boldsymbol{F}_{k}^{(t)} - \boldsymbol{F}_{k}^{(t)} \rangle \\ = \quad & f(\boldsymbol{F}_{k}^{(t)}) - g(\boldsymbol{F}_{k}^{(t)}) \end{aligned} \tag{9}$$

Combining Equation (8) and (9), we have

$$f(\boldsymbol{F}_{\boldsymbol{k}}^{(t)}) - g(\boldsymbol{F}_{\boldsymbol{k}}^{(t)}) \leq f(\boldsymbol{F}_{\boldsymbol{k}}^{(t+1)}) - g(\boldsymbol{F}_{\boldsymbol{k}}^{(t+1)}) \tag{10}$$

2 The rank constrained problem

A general rank constrained problem for SDP variable is formulated as:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{X}) && (11) \\ \text{subject to} \quad & \boldsymbol{X} \succeq 0; \\ & \boldsymbol{x} \in \mathcal{C}, \\ & \text{Rank}(\boldsymbol{X}) \leq r && (12) \end{aligned}$$

where $\mathcal{C}$ is a convex set and $f(\boldsymbol{x})$ is a convex function.

This problem can be solved by an iterative method. At each step t,

$$\begin{aligned} \min_{\boldsymbol{X}_t, e_t} \quad & f(\boldsymbol{X}) + w_t e_t \\ \text{subject to} \quad & \boldsymbol{x}_t \succeq 0; \\ & \boldsymbol{x}_t \in \mathcal{C}, \\ & e_t \boldsymbol{I}_{n-r} - \boldsymbol{U}_{t-1}^T \boldsymbol{X}_t \boldsymbol{U}_{t-1} \succeq \boldsymbol{0}; \\ & e_t \leq e_{t-1} \end{aligned} \tag{13}$$

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Combining the (13) with (6), the F_k can be solved by

$$\min_{\boldsymbol{X}_t, e_t} \quad -f(\boldsymbol{F}_k) + g(\boldsymbol{F}_k^{(t)}) + < \bigtriangledown g(\boldsymbol{F}_k^{(t)}), \boldsymbol{F}_k - \boldsymbol{F}_k^{(t)} > + w_t e_t \tag{14}$$

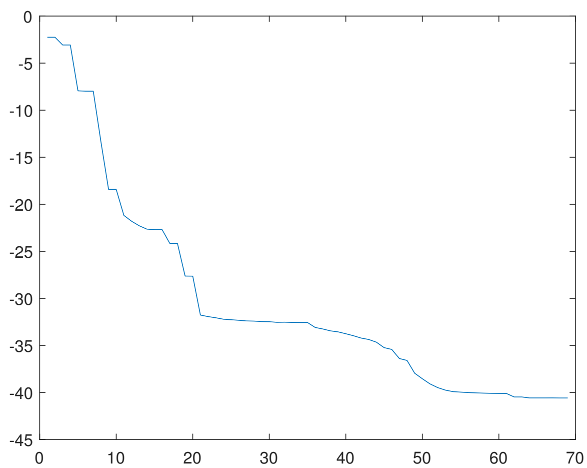

Figure 1: The convergence of proposed algorithm.