

DATA ARE NOT REAL!

ROMAIN BRAULT^{*}

UNDER SUPERVISION OF:
Professor (Prof.) Florence d'Alché-Buc[†]



Large-scale learning on structured input-output data with operator-valued kernels

Engineer (Eng.)
Computer Science
IBISC
Université d'Évry val d'Essonne

Septembre 2016 – version 0.1

^{*} Email: romain.brault@ibisc.fr

[†] Email: florence.dalche@telecom-paristech.fr

Romain Brault: *Data are not real!*, Large-scale learning on structured input-output data with operator-valued kernels, © Septembre 2016

SUPERVISOR:

Professor (Prof.) Florence d'Alché-Buc

LOCATION:

15, Rue Plumet, 75015 - Paris, France

ABSTRACT

Short summary of the contents... a great guide by Kent Beck how to write good abstracts can be found here:

<https://plg.uwaterloo.ca/~migod/research/beck00PSLA.html>

PUBLICATIONS

Some ideas and figures have appeared previously in the following publications:

Put your publications from the thesis here. The packages `multibib` or `bibtopic` etc. can be used to handle multiple different bibliographies in your document.

*We have seen that computer programming is an art,
because it applies accumulated knowledge to the world,
because it requires skill and ingenuity, and especially
because it produces objects of beauty.*

ACKNOWLEDGEMENTS

Put your acknowledgements here.

Many thanks to everybody who already sent me a postcard!

Regarding the typography and other help, many thanks go to Marco Kuhlmann, Philipp Lehman, Lothar Schlesier, Jim Young, Lorenzo Pantieri and Enrico Gregorio¹, Jörg Sommer, Joachim Köstler, Daniel Gottschlag, Denis Aydin, Paride Legovini, Steffen Prochnow, Nicolas Repp, Hinrich Harms, Roland Winkler, and the whole L^AT_EX-community for support, ideas and some great software.

Regarding LyX: The LyX port was initially done by *Nicholas Mariette* in March 2009 and continued by *Ivo Pletikosić* in 2011. Thank you very much for your work and the contributions to the original style.

¹ Members of GuIT (Gruppo Italiano Utilizzatori di T_EX e L^AT_EX)

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LISTINGS

ACRONYMS

OVK Operator-Valued Kernel.

ORFF Operator-valued Random Fourier Feature.

RKHS Reproducing Kernel Hilbert Space.

vv-RKHS vector-valued Reproducing Kernel Hilbert Space.

LCA Locally Compact Abelian.

FT Fourier transform.

IFT inverse Fourier transform.

Part I

INTRODUCTION

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MOTIVATIONS

BACKGROUND

2.1 NOTATIONS

The euclidean inner product in \mathbb{R}^d is denoted $\langle \cdot, \cdot \rangle$ and the euclidean norm is denoted $\|\cdot\|$. The unit pure imaginary number $\sqrt{-1}$ is denoted i . $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ -algebra on \mathbb{R}^d . If \mathcal{X} and \mathcal{Y} are two vector spaces, we denote by $\mathcal{F}(\mathcal{X}; \mathcal{Y})$ the vector space of functions $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $\mathcal{C}(\mathcal{X}; \mathcal{Y}) \subset \mathcal{F}(\mathcal{X}; \mathcal{Y})$ the subspace of continuous functions. If \mathcal{H} is an Hilbert space we denote its scalar product by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and its norm by $\|\cdot\|_{\mathcal{H}}$. We set $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}; \mathcal{H})$ to be the space of linear operators from \mathcal{H} to itself. If $W \in \mathcal{L}(\mathcal{H})$, $\text{Ker } W$ denotes the nullspace, $\text{Im } W$ the image and $W^* \in \mathcal{L}(\mathcal{H})$ the adjoint operator (transpose when W is a real matrix). All these notations are summarized in table 1.

2.2 ABOUT STATISTICAL LEARNING

2.3 ON LARGE-SCALE LEARNING

2.4 ELEMENTS OF ABSTRACT HARMONIC ANALYSIS

2.4.1 Locally compact Abelian groups

Definition 1. *Locally Compact Abelian group. A group (\mathcal{X}, \star) is said to be Locally Compact Abelian if it is a topological commutative group \mathcal{X} for which every point has a compact neighborhood and is Hausdorff.*

Locally Compact Abelian (LCA) groups are central to the general definition of Fourier Transform which is related to the concept of Pontryagin duality [8]. Let (\mathcal{X}, \star) be a LCA group with e its neutral element and the notation, x^{-1} , for the inverse of $x \in \mathcal{X}$. A *character* is a complex continuous homomorphism $\omega : \mathcal{X} \rightarrow \mathbb{U}$ from \mathcal{X} to the set of complex numbers of unit module \mathbb{U} . The set of all characters of \mathcal{X} forms the Pontryagin *dual group* $\hat{\mathcal{X}}$. The dual group of an LCA group is an LCA group and the dual group operation is defined by

$$(\omega_1 \star \omega_2)(x) = \omega_1(x)\omega_2(x) \in \mathbb{U}.$$

The Pontryagin duality theorem states that $\hat{\hat{\mathcal{X}}} \cong \mathcal{X}$. I.e. there is a canonical isomorphism between any LCA group and its double dual. To emphasize this duality the following notation is usually adopted: $\omega(x) = (x, \omega) = (\omega, x)$, where $x \in \mathcal{X}$, $\omega \in \hat{\mathcal{X}}$. Another important property involves the complex conjugate of the pairing which is defined as $\overline{(x, \omega)} = (x^{-1}, \omega)$.

Table 1: Mathematical symbols used throughout the paper and their signification.

Symbol	Meaning
i	Unit pure imaginary number $\sqrt{-1}$.
e	Euler constant.
$\langle \cdot, \cdot \rangle$	Euclidean inner product.
$\ \cdot\ $	Euclidean norm.
\mathcal{X}	Input space $()$.
$\hat{\mathcal{X}}$	The Pontryagin dual of \mathcal{X} .
\mathcal{Y}	Output space (Hilbert space).
\mathcal{H}	Feature space (Hilbert space).
$\langle \cdot, \cdot \rangle_{\mathcal{Y}}$	The canonical inner product of the Hilbert space \mathcal{Y} .
$\ \cdot\ _{\mathcal{Y}}$	The canonical norm induced by the inner product of the Hilbert space \mathcal{Y} .
$\mathcal{F}(\mathcal{X}; \mathcal{Y})$	Vector space of function from \mathcal{X} to \mathcal{Y} .
$\mathcal{C}(\mathcal{X}; \mathcal{Y})$	The vector subspace of \mathcal{F} of continuous function from \mathcal{X} to \mathcal{Y} .
$\mathcal{L}(\mathcal{H}; \mathcal{Y})$	The set of bounded linear operator from a Hilbert space \mathcal{H} to a Hilbert space \mathcal{Y} .
$\mathcal{L}(\mathcal{Y})$	The set of bounded linear operator from a Hilbert space \mathcal{H} to itself.
$\mathcal{L}_+(\mathcal{Y})$	The set of non-negative bounded linear operator from a Hilbert space \mathcal{H} to itself.
$\mathcal{B}(\mathcal{X})$	Borel σ -algebra on \mathcal{X} .
$\mu(\mathcal{X})$	A scalar positive measure of \mathcal{X} .
$p_{\mu}(x)$	The Radon-Nikodym derivative of μ w.r.t. the Lebesgue measure.
$dx, d\omega$	The canonical Haar measure of the LCA group $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. (resp. $(\hat{\mathcal{X}}, \mathcal{B}(\hat{\mathcal{X}}))$).
$L^p(\mathcal{X}, dx)$	The Banach space of $ \cdot ^p$ -integrable function from $(\mathcal{X}, \mathcal{B}(\mathcal{X}), dx)$ to \mathbb{C} .
$L^p(\mathcal{X}, dx; \mathcal{Y})$	The Banach space of $\ \cdot\ _{\mathcal{Y}}$ (Bochner)-integrable function from $(\mathcal{X}, \mathcal{B}(\mathcal{X}), dx)$ to \mathcal{Y} .

We notice that for any pairing depending of ω , there exists a function $h_{\omega} : \mathcal{X} \rightarrow \mathbb{R}$ such that: $(x, \omega) = \exp(-ih_{\omega}(x))$ since any pairing maps into \mathbb{U} . Moreover,

$$\begin{aligned} (x \star z^{-1}, \omega) &= \omega(x)\omega(z^{-1}) = \exp(-ih_{\omega}(x)) \exp(-ih_{\omega}(z^{-1})) \\ &= \exp(-ih_{\omega}(x)) \exp(+ih_{\omega}(z)). \end{aligned}$$

Table 2: Classification of Fourier transforms in terms of their domain and transform domain.

\mathcal{X}	$\hat{\mathcal{X}}$	Operation	Pairing
\mathbb{R}^d	\mathbb{R}^d	$+$	$(x, \omega) = \exp(i \langle x, \omega \rangle)$
$\mathbb{R}_{*,+}^d$	\mathbb{R}^d	\cdot	$(x, \omega) = \exp(i \langle \log(x), \omega \rangle)$
$(-c; +\infty)^d$	\mathbb{R}^d	\odot	$(x, \omega) = \exp(i \langle \log(x+c), \omega \rangle)$

Table 2 provide an explicit list of pairings for various groups based on \mathbb{R}^d or its subsets. We especially mention the duality pairing associated to the skewed multiplicative LCA group $\mathcal{X} = ((-c; +\infty)^d, \odot)$ $(x_k+c)(z_k+c) - c$, Hence $h_\omega(x) = \sum_{k=1}^d \omega_k \log(x_k + c)$. This group together with the operation \odot has been proposed by [10] to handle histograms features especially useful in image recognition applications.

2.4.2 The Fourier transform

For a function with values in a separable Hilbert space $f \in L^1(\mathcal{X}, dx; \mathcal{Y})$, where dx is the Haar measure on \mathcal{X} , we denote $\mathcal{F}[f]$ its Fourier transform (FT) which is defined by

$$\forall \omega \in \hat{\mathcal{X}}, \quad \mathcal{F}[f](\omega) = \int_{\hat{\mathcal{X}}} \overline{(x, \omega)} f(x) dx.$$

For a measure defined on \mathcal{X} , there exists a unique suitably normalized measure $d\omega$ on $\hat{\mathcal{X}}$ such that $\forall f \in L^1(\mathcal{X}, dx; \mathcal{Y})$ and if $\mathcal{F}[f] \in L^1(\hat{\mathcal{X}}, d\omega, \mathcal{Y})$ we have

$$\forall x \in \mathcal{X}, \quad f(x) = \int_{\hat{\mathcal{X}}} \mathcal{F}[f](\omega) (x, \omega) d\omega. \quad (1)$$

Moreover if $d\omega$ is normalized, \mathcal{F} extends to a unitary operator from $L^2(\mathcal{X}, dx, \mathcal{Y})$ onto $L^2(\hat{\mathcal{X}}, d\omega, \mathcal{Y})$ Then the inverse Fourier transform (IFT) of a function $g \in L^1(\hat{\mathcal{X}}, d\omega, \mathcal{Y})$ (where $d\omega$ is a Haar measure on $\hat{\mathcal{X}}$ suitably normalize w.r.t. the Haar measure dx) is noted $\mathcal{F}^{-1}[g]$ defined by

$$\forall x \in \mathcal{X}, \quad \mathcal{F}^{-1}[g](x) = \int_{\hat{\mathcal{X}}} (x, \omega) g(\omega) d\omega,$$

Section 2.4.1 gives some examples of real Abelian groups with their associated dual and pairing. The interested reader can refer to Folland [8] for a more detailed construction of LCA, Pontryagin duality and Fourier transforms on LCA. For the familiar case of a scalar-valued function f on the LCA group $(\mathbb{R}^d, +)$, we have:

$$\forall \omega \in \hat{\mathcal{X}}, \quad \mathcal{F}[f](\omega) = \int_{\mathbb{R}^d} e^{-i \langle \omega, x-z \rangle} f(x) dx, \quad (2)$$

the Haar measure being here the Lebesgue measure.

2.5 ON OPERATOR-VALUED KERNELS

We now introduce the theory of vector-valued Reproducing Kernel Hilbert Space (**vv-RKHS**) that provides a flexible framework to study and learn vector-valued functions.

2.5.1 Definitions and properties

An operator-valued kernel is defined here as a \mathcal{Y} -reproducing kernel Carmeli et al. [5].

Definition 2. Given \mathcal{X} , a Polish space and \mathcal{Y} , a Hilbert Space, a map $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y})$ is called a \mathcal{Y} -reproducing kernel if

$$\sum_{i,j=1}^N \langle K(x_i, x_j) y_j, y_i \rangle_{\mathcal{Y}} \geq 0,$$

for all x_1, \dots, x_N in \mathcal{X} , all y_1, \dots, y_N in \mathcal{Y} and $N \geq 1$. Given $x \in \mathcal{X}$, $K_x : \mathcal{Y} \rightarrow \mathcal{F}(\mathcal{X}; \mathcal{Y})$ denotes the linear operator whose action on a vector y is the function $K_x y \in \mathcal{F}(\mathcal{X}; \mathcal{Y})$ defined by $(K_x y)(z) = K(z, x)y$, for all $z \in \mathcal{X}$.

Additionally, given a \mathcal{Y} -reproducing kernel K , there is a unique Hilbert space $\mathcal{H}_K \subset \mathcal{F}(\mathcal{X}; \mathcal{Y})$ satisfying $K_x \in \mathcal{L}(\mathcal{Y}; \mathcal{H}_K)$, for all $x \in \mathcal{X}$ and $\forall x \in \mathcal{X}, \forall f \in \mathcal{H}_K, f(x) = K_x^* f$, where $K_x^* : \mathcal{H}_K \rightarrow \mathcal{Y}$ is the adjoint of K_x . The space \mathcal{H}_K is called the *vector-valued Reproducing Kernel Hilbert Space* associated with K . The corresponding product and norm are denoted by $\langle \cdot, \cdot \rangle_K$ and $\|\cdot\|_K$, respectively. As a consequence [5] we have:

$$\begin{aligned} K(x, z) &= K_x^* K_z \quad \forall x, z \in \mathcal{X}, \\ \mathcal{H}_K &= \overline{\text{span}} \{ K_x y \mid \forall x \in \mathcal{X}, \forall y \in \mathcal{Y} \}. \end{aligned}$$

Another way to describe functions of \mathcal{H}_K consists in using a suitable feature map.

Proposition 3 (Feature Operator Carmeli et al. [5]). *Let \mathcal{H} be a Hilbert space and $\phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y}; \mathcal{H})$, with $\phi_x := \phi(x)$. Then the operator $W : \mathcal{H} \rightarrow \mathcal{F}(\mathcal{X}; \mathcal{Y})$ defined for all $g \in \mathcal{H}$, and for all $x \in \mathcal{X}$ by $(Wg)(x) = \phi_x^* g$ is a partial isometry from \mathcal{H} onto the **vv-RKHS** \mathcal{H}_K with reproducing kernel*

$$K(x, z) = \phi_x^* \phi_z, \quad \forall x, z \in \mathcal{X}.$$

W^*W is the orthogonal projection onto

$$\text{Ker } W^\perp = \overline{\text{span}} \{ \phi_x y \mid \forall x \in \mathcal{X}, \forall y \in \mathcal{Y} \}.$$

Then $\|f\|_K = \inf \{ \|g\|_{\mathcal{H}} \mid \forall g \in \mathcal{H}, Wg = f \}$.

We call ϕ a *feature map*, W a *feature operator* and \mathcal{H} a *feature space*.

2.5.2 Examples of operator-valued kernels

Operator-valued kernels have been first introduced in Machine Learning to solve multi-task regression problems. Multi-task regression is encountered in many fields such as structured classification when classes belong to a hierarchy for instance. Instead of solving independently p single output regression task, one would like to take advantage of the relationships between output variables when learning and making a decision.

Definition 4 (Decomposable kernel). *Let A be a positive semi-definite operator of $\mathcal{L}(\mathcal{Y})$. K is said to be a \mathcal{Y} -Mercer decomposable kernel¹ if for all $(x, z) \in \mathcal{X}^2$,*

$$K(x, z) = k(x, z)A,$$

where k is a scalar Mercer kernel.

When $\mathcal{Y} = \mathbb{R}^p$, the matrix A is interpreted as encoding the relationships between the outputs coordinates. If a graph coding for the proximity between tasks is known, then it is shown in Álvarez, Rosasco, and Lawrence [1], Baldassarre et al. [2], and Evgeniou, Micchelli, and Pontil [7] that A can be chosen equal to the pseudo inverse L^\dagger of the graph Laplacian such that the norm in \mathcal{H}_K is a graph-regularizing penalty for the outputs (tasks). When no prior knowledge is available, A can be set to the empirical covariance of the output training data or learned with one of the algorithms proposed in the literature [6, 11, 14]. Another interesting property of the decomposable kernel is its universality (a kernel which may approximate an arbitrary continuous target function uniformly on any compact subset of the input space). A reproducing kernel K is said *universal* if the associated [vv-RKHS](#) \mathcal{H}_K is dense in the space $\mathcal{C}(\mathcal{X}, \mathcal{Y})$. The conditions for a kernel to be universal have been discussed in Caponnetto et al. [4] and Carmeli et al. [5]. In particular they show that a decomposable kernel is universal provided that the scalar kernel k is universal and the operator A is injective.

Curl-free and divergence-free kernels provide an interesting application of operator-valued kernels [3, 12, 13] to *vector field* learning, for which input and output spaces have the same dimensions ($d = p$). Applications cover shape deformation analysis [13] and magnetic fields approximations [15]. These kernels discussed in [9] allow encoding input-dependent similarities between vector-fields.

Definition 5 (Curl-free and Div-free kernel). *Assume $\mathcal{X} = (\mathbb{R}^d, +)$ and $\mathcal{Y} = \mathbb{R}^p$ with $d = p$. The divergence-free kernel is defined as*

$$K^{\text{div}}(x, z) = K_0^{\text{div}}(\delta) = (\nabla \nabla^T - \mathbb{I})k_0(\delta)$$

¹ Some authors also refer to as separable kernels.

and the curl-free kernel as

$$K^{\text{curl}}(x, z) = K_0^{\text{curl}}(\delta) = -\nabla \nabla^T k_0(\delta),$$

where $\nabla \nabla^T$ is the Hessian operator and δ is the Laplacian operator.

Although taken separately these kernels are not universal, a convex combination of the curl-free and divergence-free kernels allows to learn any vector field that satisfies the Helmholtz decomposition theorem [3, 12].

Part II

CONTRIBUTIONS

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OPERATOR-VALUED RANDOM FOURIER FEATURES

3.1 MOTIVATIONS

Random Fourier Features have been proved useful to implement efficiently kernel methods in the scalar case, allowing to learn a linear model based on an approximated feature map. In this work, we are interested to construct approximated operator-valued feature maps to learn vector-valued functions. With an explicit (approximated) feature map, one converts the problem of learning a function f in the vector-valued Reproducing Kernel Hilbert Space \mathcal{H}_K into the learning of a linear model \tilde{f} defined by:

$$\tilde{f}(x) = \tilde{\phi}(x)^* \theta,$$

where $\phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{Y})$ and $\theta \in \mathcal{H}$. The methodology we propose works for operator-valued kernels defined on any Locally Compact Abelian (LCA) group, noted (\mathcal{X}, \star) , for some operation noted \star . This allows us to use the general context of Pontryagin duality for Fourier transform of functions on LCA groups. Building upon a generalization of Bochner's theorem for operator-valued measures, an operator-valued kernel is seen as the *Fourier transform* of an operator-valued positive measure. From that result, we extend the principle of Random Fourier Feature for scalar-valued kernels and derive a general methodology to build Operator Random Fourier Feature when operator-valued kernels are shift-invariant according to the chosen group operation.

3.2 CONSTRUCTION

We present a construction of Operator-valued Random Fourier Feature (ORFF) such that $f : x \mapsto \tilde{\phi}(x)^* \theta$ is a continuous function that maps an arbitrary LCA group \mathcal{X} as input space to an arbitrary output Hilbert space \mathcal{Y} . First we define a functional *Fourier feature map*, and then propose a Monte-Carlo sampling from this feature map to construct an approximation of a shift-invariant \mathcal{Y} -Mercer kernel. Then, we prove the convergence of the kernel approximation $\tilde{K}(x, z) = \tilde{\phi}(x)^* \tilde{\phi}(z)$ with high probability on *compact* subsets of the LCA \mathcal{X} , when \mathcal{Y} is *finite dimensional*. Eventually we conclude with some numerical experiments.

3.2.1 Theoretical study

The following proposition of Carmeli et al. [5] and Zhang, Xu, and Zhang [16] extends Bochner's theorem to any shift-invariant \mathcal{Y} -Mercer kernel.

Proposition 6 (Operator-valued Bochner's theorem [16]). *If a continuous function K from $\mathcal{X} \times \mathcal{X}$ to \mathcal{Y} is a shift-invariant \mathcal{Y} -Mercer kernel on \mathcal{X} ,*

then there exists a unique positive operator-valued measure $M : \mathcal{B}(\mathcal{X}) \rightarrow \mathcal{L}_+(\mathcal{Y})$ such that for all $x, z \in \mathcal{X}$,

$$K(x, z) = \int_{\hat{\mathcal{X}}} \overline{(x \star z^{-1}, \omega)} dM(\omega), \quad (3)$$

where M belongs to the set of all the $\mathcal{L}_+(\mathcal{Y})$ -valued measures of bounded variation on the σ -algebra of Borel subsets of $\hat{\mathcal{X}}$. Conversely, from any positive operator-valued measure M , a shift-invariant kernel K can be defined by theorem 6.

Although this theorem is central to the spectral decomposition of shift-invariant \mathcal{Y} -Mercer [OVK](#), the following results proved by Carmeli et al. [5] provides insights about this decomposition that are more relevant in practise. It first shows how to build shift-invariant \mathcal{Y} -Mercer kernel but more importantly, also states that any operator-valued spectral decomposition of such [OVKs](#) when \mathcal{Y} is finite dimensional or \mathcal{X} is compact can be written using a pair (A, μ) where A is an operator-valued function on $\hat{\mathcal{X}}$ and μ is a real-valued positive measure on $\hat{\mathcal{X}}$. Note that obviously such a pair is not unique and the choice of this paper may have an impact on theoretical properties as well as practical computations.

Proposition 7 (Carmeli et al. [5]). *Let μ be a positive measure on $\mathcal{B}(\hat{\mathcal{X}})$ and $A : \hat{\mathcal{X}} \rightarrow \mathcal{L}(\mathcal{Y})$ such that $\langle A(\cdot)y, y' \rangle \in L^1(\mathcal{X}, d\mu)$ for all $y, y' \in \mathcal{Y}$ and $A(\omega) \succcurlyeq 0$ for μ -almost all ω . Then, for all $\delta \in \mathcal{X}$ and for all $y, y' \in \mathcal{Y}$,*

$$K_e(\delta) = \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} A(\omega) d\mu(\omega) \quad (4)$$

is the kernel signature of a shift-invariant \mathcal{Y} -Mercer kernel K such that $K(x, z) = K_e(x \star z^{-1})$. The [vv-RKHS](#) \mathcal{H}_K is embed in $L^2(\hat{\mathcal{X}}, d\mu; \mathcal{Y}')$ by mean of the feature operator

$$(Wg)(x) = \int_{\hat{\mathcal{X}}} \overline{(x, \omega)} B(\omega) g(\omega) d\mu(\omega), \quad (5)$$

Where $B(\omega)B(\omega)^* = A(\omega)$ and both integral converges in the weak sense. If \mathcal{Y} is finite dimensional or \mathcal{X} is compact, any shift-invariant kernel is of the above form for some pair $(A(\omega), \mu(\omega))$.

This theorem is more interesting than theorem 6 in the sense that it shows that we are certain of the existence of a *scalar* measure μ and a positive operator $A(\omega)$, provided that \mathcal{X} is compact or \mathcal{Y} is finite dimensional. When $p = 1$ one can always assume A is reduced to the scalar 1 , μ is still a bounded positive measure and we retrieve the Bochner theorem applied to the scalar case (??).

3.2.2 Functional Fourier feature map

Let us introduce a functional feature map, we call here *Fourier Feature map*, defined by the following proposition as a direct consequence of theorem 7.

Proposition 8 (Fourier feature map). *If there exist an operator-valued function $B : \hat{\mathcal{X}} \rightarrow \mathcal{L}(\mathcal{Y}, \mathcal{Y}')$ such that for all $y, y' \in \mathcal{Y}$, $\langle y, B(\omega)B(\omega)^*y' \rangle = \langle y, A(\omega)y' \rangle$ μ -almost everywhere and $\langle y, A(\omega)y' \rangle \in L^1(\hat{\mathcal{X}}, d\mu)$ then the operator ϕ_x defined for all y in \mathcal{Y} by*

$$(\phi_x y)(\omega) = (x, \omega)B(\omega)^*y, \quad (6)$$

² I.e. it satisfies for all $x, z \in \mathcal{X}$, $\phi_x^* \phi_z = K(x, z)$ where K is an operator-valued kernel.

is a feature map² of some shift-invariant kernel K .

Proof. For all $y, y' \in \mathcal{Y}$ and $x, z \in \mathcal{X}$,

$$\begin{aligned} \langle y, \phi_x^* \phi_z y' \rangle_y &= \langle \phi_x y, \phi_z y' \rangle_{L^2(\hat{\mathcal{X}}, \mu, \mathcal{Y}')} \\ &= \int_{\hat{\mathcal{X}}} \overline{(x, \omega)} \langle y, B(\omega)(z, \omega)B(\omega)^*y' \rangle d\mu(\omega) \\ &= \int_{\hat{\mathcal{X}}} \overline{(x \star z^{-1}, \omega)} \langle y B(\omega)B(\omega)^*y' \rangle d\mu(\omega) \\ &= \int_{\hat{\mathcal{X}}} \overline{(x \star z^{-1}, \omega)} \langle y, A(\omega)y' \rangle d\mu(\omega), \end{aligned}$$

which defines a \mathcal{Y} -Mercer according to theorem 7 of Carmeli et al. [5]. With this notation notice that $\phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y}; L^2(\hat{\mathcal{X}}, \mu; \mathcal{Y}'))$ such that $\phi_x \in \mathcal{L}(\mathcal{Y}; L^2(\hat{\mathcal{X}}, \mu; \mathcal{Y}'))$ where $\phi_x := \phi(x)$. \square

3.2.3 Sufficient conditions of existence

While theorem 7 gives some insights on how to build an approximation of a \mathcal{Y} -Mercer kernel, we need a theorem that provides an explicit construction of the pair $A(\omega), \mu(\omega)$ from the kernel signature. Proposition 14 in Carmeli et al. [5] gives the solution, and also provide a sufficient condition for theorem 7 to apply.

Proposition 9 (Carmeli et al. [5]). *Let K be a shift-invariant \mathcal{Y} -Mercer kernel. Suppose that $\forall z \in \mathcal{X}$ and $\forall y, y' \in \mathcal{Y}$, $\langle K_e(\cdot)y, y' \rangle \in L^1(\mathcal{X}, dx)$ where dx denotes the Haar measure on (\mathcal{X}, \star) . Define C such that for all $\omega \in \hat{\mathcal{X}}$ and for all y, y' in \mathcal{Y} ,*

$$\begin{aligned} \langle y, C(\omega)y' \rangle &= \int_{\mathcal{X}} (\delta, \omega) \langle y, K_e(\delta)y' \rangle d\delta \\ &= \mathcal{F}^{-1} [\langle y, K_e(\cdot)y' \rangle] (\omega) \end{aligned} \quad (7)$$

Then

- i) $C(\omega)$ is a bounded non-negative operator for all $\omega \in \hat{\mathcal{X}}$,
- ii) $\langle y, C(\cdot)y' \rangle \in L^1(\hat{\mathcal{X}}, d\omega)$ for all $y, y' \in \mathcal{Y}$,
- iii) for all $\delta \in \mathcal{X}$ and for all y, y' in \mathcal{Y} ,

$$\langle y, K_e(\delta)y' \rangle = \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y, C(\omega)y' \rangle d\omega.$$

Gathering the two propositions, we present now the following property that allows to build a spectral decomposition of a shift-invariant \mathcal{Y} -Mercer kernel on a LCA group (\mathcal{X}, \star) .

Proposition 10 (Sufficient condition for shift-invariant \mathcal{Y} -Mercer kernel spectral decomposition). *Let K_e be the signature of a shift-invariant \mathcal{Y} -Mercer kernel on (\mathcal{X}, \star) .*

If for all $y, y' \in \mathcal{Y}$, $\langle K_e(\cdot)y, y' \rangle \in L^1(\mathcal{X}, dx)$, then there exists a positive measure μ with density p_μ on $\mathcal{B}(\hat{\mathcal{X}})$ and $A : \hat{\mathcal{X}} \rightarrow \mathcal{L}_+(\mathcal{Y})$ an operator-valued functions such that for all $y, y' \in \mathcal{Y}$ $\langle A(\cdot)y, y' \rangle \in L^1(\mathcal{X}, d\mu)$ and for all y, y' in \mathcal{Y} ,

$$\langle y, K_e(\delta)y' \rangle = \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y, A(\omega)y' \rangle p_\mu(\omega) d\omega.$$

where $\langle y, A(\omega)y' \rangle p_\mu(\omega) = \mathcal{F}^{-1} [\langle y, K_e(x \star z^{-1})y' \rangle]$.

Proof. From theorem 7 and theorem 9 we can write the following equality concerning the OVK signature K_e .

$$\begin{aligned} \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y, A(\omega)y' \rangle d\mu(\omega) &= \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y, C(\omega)y' \rangle d\omega \\ &= \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y, C'(\omega)p(\omega)y' \rangle d\omega \end{aligned}$$

It is always possible to choose $p(\omega)$ such that $\int_{\hat{\mathcal{X}}} p(\omega) d\omega = 1$ then $p(\omega)$ is the density of a probability measure μ . In this case we note $p(\omega) = p_\mu(\omega)$. By injectivity of the Fourier transform we have for all $y, y' \in \mathcal{Y}$,

$$\langle y, C(\omega)y' \rangle = \langle y, C'(\omega)y' \rangle p_\mu(\omega) = \mathcal{F}^{-1} [\langle y, K_e(\cdot)y' \rangle] (\omega) \quad (8)$$

□

In the case where $\mathcal{Y} = \mathbb{R}^p$, we rewrite section 3.2.3 coefficient-wise by choosing an orthonormal basis (e_1, \dots, e_p) of \mathcal{Y} , such that for all $i, j \in \{1, \dots, p\}$,

$$\langle e_i, C(\omega)e_j \rangle = C(\omega)_{ij} = A(\omega)_{ij} p_\mu(\omega) = \mathcal{F}^{-1} [K_e(\delta)_{ij}]. \quad (9)$$

It follows that for all $i, j \in \{1, \dots, p\}$,

$$K_e(x \star z^{-1})_{ij} = \mathcal{F} [A(\cdot)_{ij}] \quad (10)$$

Remark 11. *Note that although the inverse Fourier transform of K_e yields a unique operator-valued function $C(\cdot)$, the decomposition of $C(\omega)$ into $A(\omega)p_\mu(\omega)$ is not unique. The choice of the decomposition may be justified by the computational cost or by the nature of the constants involved in the uniform convergence of the estimator.*

3.2.4 Regularization property

We have shown so far that it is always possible to construct a feature map that allows to approximate a shift-invariant \mathcal{Y} -Mercer kernel. However we could also propose a construction of such map by studying the regularization induced with respect to the Fourier transform of a target function $f \in \mathcal{H}_K$. In other words, what is the norm in $L^2(\hat{\mathcal{X}}, d\omega, \mathcal{Y}')$ induced by $\|\cdot\|_K$?

Proposition 12. *Let K be a shift-invariant \mathcal{Y} -Mercer Kernel such that for all y, y' in \mathcal{Y} , $\langle y, K_e(\cdot)y' \rangle \in L^1(\mathcal{X}, dx)$ and*

$$\text{Let } \langle y, A(\omega)y' \rangle p_\mu(\omega) := \mathcal{F}^{-1}[\langle y, K_e(\cdot)y' \rangle](\omega) \text{ and let } f \in \mathcal{H}_K. \text{ Then}$$

$$\|f\|_K^2 = \int_{\hat{\mathcal{X}}} \frac{\langle \mathcal{F}[f](\omega), A(\omega)^\dagger \mathcal{F}[f](\omega) \rangle_{\mathcal{Y}}}{p_\mu(\omega)} d\omega. \quad (11)$$

Proof. We first show how the Fourier transform relates to the feature operator. Since \mathcal{H}_K is embed into $\mathcal{H} = L^2(\hat{\mathcal{X}}, \mu, \mathcal{Y})$ by mean of the feature operator W , we have:

$$\mathcal{F}[\mathcal{F}^{-1}[f]](x) = \int_{\hat{\mathcal{X}}} \overline{(x, \omega)} \mathcal{F}^{-1}[f](\omega) d\omega = f(x)$$

$$(Wg)(x) = \int_{\hat{\mathcal{X}}} \overline{(x, \omega)} p_\mu(\omega) B(\omega) g(\omega) d\omega = f(x).$$

By injectivity of the Fourier transform, $\mathcal{F}^{-1}[f](\omega) = p_\mu(\omega) B(\omega) g(\omega)$ μ -almost everywhere. From theorem 3 we have

$$\|f\|_K^2 = \inf \left\{ \|g\|_{\mathcal{H}}^2 \mid \forall g \in \mathcal{H}, Wg = f \right\}$$

$$= \inf \left\{ \int_{\hat{\mathcal{X}}} \|g\|_{\mathcal{Y}}^2 d\mu \mid \forall g \in \mathcal{H}, \mathcal{F}^{-1}[f] = p_\mu(\cdot) B(\cdot) g(\cdot) \right\}.$$

The pseudo inverse of the operator $B(\omega)$ (noted $B(\omega)^\dagger$) is the unique solution of the system $\mathcal{F}^{-1}[f](\omega) = p_\mu(\omega) B(\omega) g(\omega)$ w. r. t. $g(\omega)$ with minimal norm. Eventually,

$$\|f\|_K^2 = \int_{\hat{\mathcal{X}}} \frac{\|B(\omega)^\dagger \mathcal{F}^{-1}[f](\omega)\|_{\mathcal{Y}}^2}{p_\mu(\omega)^2} d\mu(\omega)$$

$$= \int_{\hat{\mathcal{X}}} \frac{\|B(\omega)^\dagger \mathcal{F}[f](\omega)\|_{\mathcal{Y}}^2}{p_\mu(\omega)^2} d\mu(\omega) \quad (12)$$

Conclude the proof by taking $d\mu(\omega) = p_\mu(\omega) d\omega$. \square

Note that if $K(x, z) = k(x, z)$ is a scalar kernel then for all ω in $\hat{\mathcal{X}}$, $A(\omega) = 1$. Therefore we recover a well known results for kernels that is for any $f \in \mathcal{H}_K$ we have $\|f\|_K = \int_{\hat{\mathcal{X}}} \mathcal{F}[k_e](\omega)^{-1} \mathcal{F}[f](\omega)^2 d\omega$. We also note that the regularization property in \mathcal{H}_K does not depends (as expected) on the decomposition of $A(\omega)$ into $B(\omega)B(\omega)^*$. Therefore the decomposition should be chosen such that it optimizes the computation cost. For instance if $A(\omega) \in \mathcal{L}(\mathbb{R}^p)$ has rank r , one could find an operator $B(\omega) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^r)$ such that $A(\omega) = B(\omega)B(\omega)^*$.

3.2.5 Building Operator-valued Random Fourier Features

Without loss of generality we assume that $\int_{\mathcal{X}} d\mu(\omega) = 1$ and thus, μ is a probability distribution and p_μ , a probability density. Note that this is always possible through an appropriate rescaling of the kernel.

Given a \mathcal{Y} -Mercer shift-invariant kernel K on \mathcal{X} , an approximation of K can be obtained using a decomposition (A, μ) and a plug-in Monte-Carlo estimator instead of the expectation. However, for efficient computations, as motivated in the introduction, we are interested in finding an approximated feature map more than a kernel approximation. Indeed, an approximated feature map will allow to build linear models in regression tasks. The following proposition provides the general form of an Operator Random Fourier Feature map:

Proposition 13. *If one can find $B : \hat{\mathcal{X}} \rightarrow \mathcal{L}(\mathcal{Y})$, such that for all $y, y' \in \mathcal{Y}$, $\langle y, A(\omega)y' \rangle = \langle y, B(\omega)B(\omega)^*y' \rangle \in L^1(\hat{\mathcal{X}}, d\mu)$, then the operator-valued function*

$$\tilde{\Phi}(x) = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D (x, \omega_j) B(\omega_j)^*, \quad \omega_j \sim \mu \quad (13)$$

is an approximated feature map of kernel K

Proof. Let $\omega_1, \dots, \omega_D$ be D i.i.d random vectors following the law μ . For all $(x, z) \in \mathcal{X}^2$,

$$\begin{aligned} \tilde{\Phi}(x)^* \tilde{\Phi}(z) &= \left(\frac{1}{\sqrt{D}} \bigoplus_{j=1}^D \exp(i h_{\omega_j}(x) B(\omega_j)^*) \right)^* \\ &\quad \left(\frac{1}{\sqrt{D}} \bigoplus_{j=1}^D \exp(i h_{\omega_j}(z) B(\omega_j)^*) \right) \\ &= \frac{1}{D} \sum_{j=1}^D \exp(-i(h_{\omega_j}(x) - h_{\omega_j}(z)) A(\omega_j)) \\ &= \frac{1}{D} \sum_{j=1}^D \overline{(x, z)} A(\omega_j) \end{aligned}$$

From the strong law of large numbers, $\frac{1}{D} \sum_{j=1}^D \overline{(x, z)} A(\omega_j)$ converges almost-surely in the weak operator topology to $\mathbf{E}_\mu[(\bar{x} \star z^{-1}, \omega_j) A(\omega)]$ when D tends to infinity. \square

Remark 14. *We find a decomposition such that for all $j = 1, \dots, D$, $A(\omega_j) = B(\omega_j)B(\omega_j)^*$ either by exhibiting an analytic closed-form or using a numerical decomposition.*

This proposition leads to the following construction algorithm.

Algorithm 1: Construction of ORFF

Input : $K(x, z) = K_e(\delta)$ a \mathcal{Y} -shift-invariant Mercer kernel such that $\forall y, y' \in \mathcal{Y}, \langle y, K_e(\delta)y' \rangle \in L^1(\mathbb{R}^d, dx)$.

Output: A random feature $\tilde{\phi}(x)$ such that $\tilde{\phi}(x)^* \tilde{\phi}(z) \approx K(x, z)$

- 1 Define the pairing (x, ω) from the [LCA](#) group (\mathcal{X}, \star) ;
 - 2 Find a decomposition $(B(\omega), p_\mu(\omega))$ such that $B(\omega)B(\omega)^* p_\mu(\omega) = \mathcal{F}^{-1}[K_e](\omega)$;
 - 3 Draw D random vectors $\omega_j, j = 1, \dots, D$ from the probability distribution μ ;
 - 4 **return** $\tilde{\phi}(x) = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D (x, \omega_j) B(\omega_j)^*$;
-

3.3 UNIFORM BOUND ON THE APPROXIMATION

3.4 LEARNING WITH OPERATOR-VALUED RANDOM-FOURIER FEATURES

3.5 CONSISTENCY AND GENERALIZATION BOUNDS

3.6 CONCLUSIONS

4.1 BACKGROUND

4.2 THE NYSTRÖM METHOD

4.3 SUB-SAMPLING THE DATA

4.4 CONCLUSIONS

Part III

FINAL WORDS

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CONCLUSIONS

Part IV

APPENDIX

OPERATOR-VALUED FUNCTIONS AND
INTEGRATION

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DECLARATION

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15, Rue Plumet, 75015 - Paris, France, Septembre 2016

Romain Brault

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