DATA ARE NOT REAL!

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UNDER SUPERVISION OF:
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Large-scale learning on structured input-output data with operator-valued kernels

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ABSTRACT

Short summary of the contents... a great guide by Kent Beck how to write good abstracts can be found here:

https://plg.uwaterloo.ca/~migod/research/beckOOPSLA.html

PUBLICATIONS

Some ideas and figures have appeared previously in the following publications:

Put your publications from the thesis here. The packages `multibib` or `bibtopic` etc. can be used to handle multiple different bibliographies in your document.

We have seen that computer programming is an art,
because it applies accumulated knowledge to the world,
because it requires skill and ingenuity, and especially
because it produces objects of beauty.

ACKNOWLEDGEMENTS

Put your acknowledgements here.

Many thanks to everybody who already sent me a postcard!

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1 Members of GuIT (Gruppo Italiano Utilizzatori di TeX e LaTeX)

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ACRONYMS

OVK Operator-Valued Kernel

ORFF Operator-valued Random Fourier Feature

RKHS Reproducing Kernel Hilbert Space

vv-RKHS vector-valued Reproducing Kernel Hilbert Space

LCA Locally Compact Abelian

Part I

INTRODUCTION

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1 MOTIVATIONS

2 BACKGROUND

2.1 NOTATIONS

The euclidean inner product in $\mathbb{R}^d$ is denoted $\langle \cdot, \cdot \rangle$ and the euclidean norm is denoted $\|\cdot\|$. The unit pure imaginary number $\sqrt{-1}$ is denoted i. $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra on $\mathbb{R}^d$. If $\mathcal{X}$ and $\mathcal{Y}$ are two vector spaces, we denote by $\mathcal{F}(\mathcal{X};\mathcal{Y})$ the vector space of functions $f : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{C}(\mathcal{X};\mathcal{Y}) \subset \mathcal{F}(\mathcal{X};\mathcal{Y})$ the subspace of continuous functions. If $\mathcal{H}$ is an Hilbert space we denote its scalar product by $\langle .,. \rangle_{\mathcal{H}}$ and its norm by $\|.\|_{\mathcal{H}}$. We set $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H};\mathcal{H})$ to be the space of linear operators from $\mathcal{H}$ to itself. If $W \in \mathcal{L}(\mathcal{H})$, Ker W denotes the nullspace, Im W the image and $W^* \in \mathcal{L}(\mathcal{H})$ the adjoint operator (transpose when W is a real matrix). All these notations are summarized in table 1.

2.2 ABOUT STATISTICAL LEARNING

2.3 ON LARGE-SCALE LEARNING

2.4 ELEMENTS OF ABSTRACT HARMONIC ANALYSIS

2.4.1 *Locally compact Abelian groups*

Definition 1. *Locally Compact Abelian group. A group $(\mathcal{X}, \star)$ is said to be Locally Compact Abelian if it is a topological* commutative *group $\mathcal{X}$ for which every point has a compact neighborhood and is Hausdorff.*

Locally Compact Abelian (LCA) groups are central to the general definition of Fourier Transform which is related to the concept of Pontryagin duality [8]. Let $(\mathcal{X}, \star)$ be a LCA group with e its neutral element and the notation, x^{-1}, for the inverse of $x \in \mathcal{X}$. A *character* is a complex continuous homomorphism $\omega : \mathcal{X} \to \mathbb{U}$ from $\mathcal{X}$ to the set of complex numbers of unit module $\mathbb{U}$. The set of all characters of $\mathcal{X}$ forms the Pontryagin *dual group* $\hat{\mathcal{X}}$. The dual group of an LCA group is an LCA group and the dual group operation is defined by

$$(\omega_1 \star \omega_2)(x) = \omega_1(x)\omega_2(x) \in \mathbb{U}.$$

The Pontryagin duality theorem states that $\hat{\hat{\mathcal{X}}} \cong \mathcal{X}$. I.e. there is a canonical isomorphism between any LCA group and its double dual. To emphasize this duality the following notation is usually adopted: $\omega(x) = (x, \omega) = (\omega, x)$, where $x \in \mathcal{X}$, $\omega \in \hat{\mathcal{X}}$. Another important property involves the complex conjugate of the pairing which is defined as $\overline{(x, \omega)} = (x^{-1}, \omega)$.

Table 1: Mathematical symbols used throughout the parper and their signification.

Symbol	Meaning
i	Unit pure imaginary number $\sqrt{-1}$.
e	Euler constant.
$\langle \cdot, \cdot \rangle$	Euclidean inner product.
$\lVert \cdot \rVert$	Euclidean norm.
$\mathcal{X}$	Input space ().
$\hat{\mathcal{X}}$	The Pontryagin dual of $\mathcal{X}$.
$\mathcal{Y}$	Output space (Hilbert space).
$\mathcal{H}$	Feature space (Hilbert space).
$\langle \cdot, \cdot \rangle_{\mathcal{Y}}$	The canonical inner product of the Hilbert space $\mathcal{Y}$.
$\lVert \cdot \rVert_{\mathcal{Y}}$	The canonical norm induced by the inner product of the Hilbert space $\mathcal{Y}$.
$\mathcal{F}(\mathcal{X}; \mathcal{Y})$	Vector space of function from $\mathcal{X}$ to $\mathcal{Y}$.
$\mathcal{C}(\mathcal{X}; \mathcal{Y})$	The vector subspace of $\mathcal{F}$ of continuous function from $\mathcal{X}$ to $\mathcal{Y}$.
$\mathcal{L}(\mathcal{H}; \mathcal{Y})$	The set of bounded linear operator from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{Y}$.
$\mathcal{L}(\mathcal{Y})$	The set of bounded linear operator from a Hilbert space $\mathcal{H}$ to itself.
$\mathcal{B}(\mathcal{X})$	Borel σ-algebra on $\mathcal{X}$.
$\mu(\mathcal{X})$	A scalar positive measure of $\mathcal{X}$.
$p_{\mu}(x)$	The Radon-Nikodym derivative of μ w. r. t. the Lebesgue measure.
dx, $d\omega$	The canonical Haar measure of the LCA group $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. (resp. $(\hat{\mathcal{X}}, \mathcal{B}(\hat{\mathcal{X}}))$.
$L^p(\mathcal{X}, dx)$	The Banach space of $\lvert \cdot \rvert^p$-integrable function from $(\mathcal{X}, \mathcal{B}(\mathcal{X}, dx))$ to $\mathbb{C}$.
$L^p(\mathcal{X}, dx; \mathcal{Y})$	The Banach space of $\lVert \cdot \rVert_{\mathcal{Y}}^p$ (Bochner)-integrable function from $(\mathcal{X}, \mathcal{B}(\mathcal{X}), dx)$ to $\mathcal{Y}$.

We notice that for any pairing depending of ω, there exists a function $h_\omega : \mathcal{X} \to \mathbb{R}$ such that: $(x, \omega) = \exp(-ih_\omega(x))$ since any pairing maps into $\mathbb{U}$. Moreover,

$$\begin{aligned}(x \star z^{-1}, \omega) = \omega(x)\omega(z^{-1}) &= \exp(-ih_\omega(x)) \exp(-ih_\omega(z^{-1})) \\ &= \exp(-ih_\omega(x)) \exp(+ih_\omega(z)).\end{aligned}$$

Table 2 provide an explicit list of pairings for various groups based on $\mathbb{R}^d$ or its subsets. We especially mention the duality pairing asso-

Table 2: Classification of Fourier transforms in terms of their domain and transform domain.

$\mathcal{X}$	$\hat{\mathcal{X}}$	Operation	Pairing
$\mathbb{R}^d$	$\mathbb{R}^d$	$+$	$(x, \omega) = \exp\left(i\langle x, \omega\rangle\right)$
$\mathbb{R}^d_{*,+}$	$\mathbb{R}^d$	$\cdot$	$(x, \omega) = \exp\left(i\langle \log(x), \omega\rangle\right)$
$(-c; +\infty)^d$	$\mathbb{R}^d$	$\odot$	$(x, \omega) = \exp\left(i\langle \log(x+c), \omega\rangle\right)$

ciated to the skewed multiplicative LCA group $\mathcal{X} = ((-c; +\infty)^d, \odot)$ Hence $h_\omega(x) = \sum_{k=1}^d \omega_k \log(x_k + c)$. This group together with the operation $\odot$ has been proposed by [10] to handle histograms features especially useful in image recognition applications.

2.4.2 *The Fourier transform*

For a function with values in a separable Hilbert space $f \in L^1(\mathcal{X}, dx; \mathcal{Y})$, where dx is the Haar measure on $\mathcal{X}$, we denote $\mathcal{F}[f]$ its Fourier transform which is defined by

$$\forall \omega \in \hat{\mathcal{X}}, \ \ \mathcal{F}[f](\omega) = \int_{\hat{\mathcal{X}}} \overline{(x, \omega)} f(x) dx.$$

For a measure defined on $\mathcal{X}$, there exists a unique suitably normalized measure $d\omega$ on $\hat{\mathcal{X}}$ such that $\forall f \in L^1(\mathcal{X}, dx; \mathcal{Y})$ and if $\mathcal{F}[f] \in L^1(\hat{\mathcal{X}}, d\omega, \mathcal{Y})$ we have

$$\forall x \in \mathcal{X}, \ \ f(x) = \int_{\hat{\mathcal{X}}} \mathcal{F}[f](\omega)(x, \omega) d\omega. \tag{1}$$

Moreover if $d\omega$ is normalized, $\mathcal{F}$ extends to a unitary operator from $L^2(\mathcal{X}, dx, \mathcal{Y})$ onto $L^2(\hat{\mathcal{X}}, d\omega, \mathcal{Y})$ Then the inverse Fourier transform of a function $g \in L^1(\hat{\mathcal{X}}, d\omega, \mathcal{Y})$ (where $d\omega$ is a Haar measure on $\hat{\mathcal{X}}$ suitably normalize w.r.t. the Haar measure dx) is noted $\mathcal{F}^{-1}[g]$ defined by

$$\forall x \in \mathcal{X}, \ \ \mathcal{F}^{-1}[g](x) = \int_{\hat{\mathcal{X}}} (x, \omega) g(\omega) d\omega,$$

For the familiar case of a scalar-valued function f on the LCA group $(\mathbb{R}^d, +)$, we have:

$$\forall \omega \in \hat{\mathcal{X}}, \ \ \mathcal{F}[f](\omega) = \int_{\mathbb{R}^d} e^{-i\langle \omega, x - z\rangle} f(x) dx, \tag{2}$$

the Haar measure being here the Lebesgue measure.

2.5 ON OPERATOR-VALUED KERNELS

We now introduce the theory of vector-valued Reproducing Kernel Hilbert Space (vv-RKHS) that provides a flexible framework to study and learn vector-valued functions.

2.5.1 *Definitions and properties*

An operator-valued kernel is defined here as a $\mathcal{Y}$-reproducing kernel Carmeli et al. [5].

Definition 2. *Given $\mathcal{X}$, a Polish space and $\mathcal{Y}$, a Hilbert Space, a map $K : \mathcal{X} \times \mathcal{X} \to \mathcal{L}(\mathcal{Y})$ is called a $\mathcal{Y}$-reproducing kernel if*

$$\sum_{i,j=1}^{N} \langle K(x_i, x_j) y_j, y_i \rangle_{\mathcal{Y}} \geqslant 0,$$

for all $x_1, \dots, x_N$ in $\mathcal{X}$, all $y_1, \dots, y_N$ in $\mathcal{Y}$ and $N \geqslant 1$. Given $x \in \mathcal{X}$, $K_x : \mathcal{Y} \to \mathcal{F}(\mathcal{X}; \mathcal{Y})$ denotes the linear operator whose action on a vector y is the function $K_x y \in \mathcal{F}(\mathcal{X}; \mathcal{Y})$ defined by $(K_x y)(z) = K(z, x) y$, for all $z \in \mathcal{X}$.

Additionally, given a $\mathcal{Y}$-reproducing kernel K, there is a unique Hilbert space $\mathcal{H}_K \subset \mathcal{F}(\mathcal{X}; \mathcal{Y})$ satisfying $K_x \in \mathcal{L}(\mathcal{Y}; \mathcal{H}_K)$, for all $x \in \mathcal{X}$ and $\forall x \in \mathcal{X}, \forall f \in \mathcal{H}_K,\ f(x) = K_x^* f$, where $K_x^* : \mathcal{H}_K \to \mathcal{Y}$ is the adjoint of K_x. The space $\mathcal{H}_K$ is called the *vector-valued Reproducing Kernel Hilbert Space* associated with K. The corresponding product and norm are denoted by $\langle ., . \rangle_K$ and $\|.\|_K$, respectively. As a consequence [5] we have:

$$K(x, z) = K_x^* K_z \quad \forall x, z \in \mathcal{X}$$
$$\mathcal{H}_K = \overline{\text{span}}\ \{K_x y \mid \forall x \in \mathcal{X},\ \forall y \in \mathcal{Y}\}$$

Another way to describe functions of $\mathcal{H}_K$ consists in using a suitable feature map.

Proposition 3 (Feature Operator Carmeli et al. [5])**.** *Let $\mathcal{H}$ be a Hilbert space and $\Phi : \mathcal{X} \to \mathcal{L}(\mathcal{Y}; \mathcal{H})$, with $\Phi_x := \Phi(x)$. Then the operator $W : \mathcal{H} \to \mathcal{F}(\mathcal{X}; \mathcal{Y})$ defined for all $g \in \mathcal{H}$, and for all $x \in \mathcal{X}$ by $(Wg)(x) = \Phi_x^* g$ is a partial isometry from $\mathcal{H}$ onto the vv-RKHS $\mathcal{H}_K$ with reproducing kernel*

$$K(x, z) = \Phi_x^* \Phi_z, \quad \forall x, z \in \mathcal{X}.$$

$W^* W$ is the orthogonal projection onto

$$\textit{Ker}\ W^{\perp} = \overline{\text{span}}\ \{\Phi_x y \mid \forall x \in \mathcal{X},\ \forall y \in \mathcal{Y}\}.$$

Then $\|f\|_K = \inf\{\|g\|_{\mathcal{H}} \mid \forall g \in \mathcal{H},\ Wg = f\}$.

We call Φ a *feature map*, W a *feature operator* and $\mathcal{H}$ a *feature space*.

2.5.2 *Examples of operator-valued kernels*

Operator-valued kernels have been first introduced in Machine Learning to solve multi-task regression problems. Multi-task regression is encountered in many fields such as structured classification when classes belong to a hierarchy for instance. Instead of solving independently p single output regression task, one would like to take advantage of the relationships between output variables when learning and making a decision.

Definition 4 (Decomposable kernel)**.** *Let A be a positive semi-definite operator of $\mathcal{L}(\mathcal{Y})$. K is said to be a $\mathcal{Y}$-Mercer* decomposable *kernel if for all $(x, z) \in \mathcal{X}^2$,*

Some authors also refer to as separable *kernels.*

$$K(x, z) = k(x, z)A,$$

where k is a scalar *Mercer kernel.*

When $\mathcal{Y} = \mathbb{R}^p$, the matrix A is interpreted as encoding the relationships between the outputs coordinates. If a graph coding for the proximity between tasks is known, then it is shown in Álvarez, Rosasco, and Lawrence [1], Baldassarre et al. [2], and Evgeniou, Micchelli, and Pontil [7] that A can be chosen equal to the pseudo inverse $L^\dagger$ of the graph Laplacian such that the norm in $\mathcal{H}_K$ is a graph-regularizing penalty for the outputs (tasks). When no prior knowledge is available, A can be set to the empirical covariance of the output training data or learned with one of the algorithms proposed in the literature [6, 11, 14]. Another interesting property of the decomposable kernel is its universality (a kernel which may approximate an arbitrary continuous target function uniformly on any compact subset of the input space). A reproducing kernel K is said *universal* if the associated vv-RKHS $\mathcal{H}_K$ is dense in the space $\mathcal{C}(\mathcal{X}, \mathcal{Y})$. The conditions for a kernel to be universal have been discussed in Caponnetto et al. [4] and Carmeli et al. [5]. In particular they show that a decomposable kernel is universal provided that the scalar kernel k is universal and the operator A is injective.

Curl-free and divergence-free kernels provide an interesting application of operator-valued kernels [3, 12, 13] to *vector field* learning, for which input and output spaces have the same dimensions ($d = p$). Applications cover shape deformation analysis [13] and magnetic fields approximations [15]. These kernels discussed in [9] allow encoding input-dependent similarities between vector-fields.

Definition 5 (Curl-free and Div-free kernel)**.** *Assume $\mathcal{X} = (\mathbb{R}^d, +)$ and $\mathcal{Y} = \mathbb{R}^p$ with $d = p$. The* divergence-free *kernel is defined as*

$$K^{div}(x, z) = K_0^{div}(\delta) = (\nabla\nabla^T - \Delta I)k_0(\delta)$$

and the curl-free *kernel as*

$$K^{curl}(x, z) = K_0^{curl}(\delta) = -\nabla\nabla^T k_0(\delta),$$

where $\nabla\nabla^T$ is the Hessian operator and Δ is the Laplacian operator.

Although taken separately these kernels are not universal, a convex combination of the curl-free and divergence-free kernels allows to learn any vector field that satisfies the Helmholtz decomposition theorem [3, 12].

Part II

CONTRIBUTIONS

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3 OPERATOR-VALUED RANDOM FOURIER FEATURES

3.1 MOTIVATIONS

Random Fourier Features have been proved useful to implement efficiently kernel methods in the scalar case, allowing to learn a linear model based on an approximated feature map. In this work, we are interested to construct approximated operator-valued feature maps to learn vector-valued functions. With an explicit (approximated) feature map, one converts the problem of learning a function f in the vector-valued Reproducing Kernel Hilbert Space $\mathcal{H}_K$ into the learning of a linear model $\tilde{f}$ defined by:

$$\tilde{f}(x) = \tilde{\Phi}(x)^*\theta,$$

where $\Phi : \mathcal{X} \to \mathcal{L}(\mathcal{H}, \mathcal{Y})$ and $\theta \in \mathcal{H}$. The methodology we propose works for operator-valued kernels defined on any Locally Compact Abelian (LCA) group, noted $(\mathcal{X}, \star)$, for some operation noted $\star$. This allows us to use the general context of Pontryagin duality for Fourier transforms of functions on LCA groups. Building upon a generalization of Bochner's theorem for operator-valued measures, an operator-valued kernel is seen as the *Fourier transform* of an operator-valued positive measure. From that result, we extend the principle of Random Fourier Feature for scalar-valued kernels and derive a general methodology to build Operator Random Fourier Feature when operator-valued kernels are shift-invariant according to the chosen group operation.

3.2 CONSTRUCTION

We present a construction of Operator-valued Random Fourier Feature (ORFF) such that $f : x \mapsto \tilde{\Phi}(x)^*\theta$ is a continuous function that maps an arbitrary LCA group $\mathcal{X}$ as input space to an arbitrary output Hilbert space $\mathcal{Y}$. First we define a functional *Fourier feature map*, and then propose a Monte-Carlo sampling from this feature map to construct an approximation of a shift-invariant $\mathcal{Y}$-Mercer kernel. Then, we prove the convergence of the kernel approximation $\tilde{K}(x, z) = \tilde{\Phi}(x)^*\tilde{\Phi}(z)$ with high probability on *compact* subsets of the LCA $\mathcal{X}$, when $\mathcal{Y}$ is *finite dimensional*. Eventually we conclude with some numerical experiments.

The following proposition of Carmeli et al. [5] and Zhang, Xu, and Zhang [16] extends Bochner's theorem to any shift-invariant $\mathcal{Y}$-Mercer kernel.

Proposition 6 (Operator-valued Bochner's theorem [16])**.** *If a continuous function* K *from* $\mathcal{X} \times \mathcal{X}$ *to* $\mathcal{Y}$ *is a shift-invariant* $\mathcal{Y}$*-Mercer kernel on* $\mathcal{X}$*,* *then there exists a unique positive operator-valued measure* $M : \mathcal{B}(\mathcal{X}) \to \mathcal{L}_+(\mathcal{Y})$ *such that:*

$$\forall x, z \in \mathcal{X}, K(x, z) = \int_{\hat{\mathcal{X}}} \overline{(x \star z^{-1}, \omega)} dM(\omega), \tag{3}$$

where M *belongs to the set of all the* $\mathcal{L}_+(\mathcal{Y})$*-valued measures of bounded variation on the* σ*-algebra of Borel subsets of* $\hat{\mathcal{X}}$*. Conversely, from any positive operator-valued measure* M*, a shift-invariant kernel* K *can be defined by eq.* (3)*.*

Although this theorem is central to the spectral decomposition of shift-invariant $\mathcal{Y}$-Mercer OVK, the following results proved by Carmeli et al. [5] provides insights about this decomposition that are more relevant in practise. It first shows how to build shift-invariant $\mathcal{Y}$-Mercer kernel but more importantly, also states that any operator-valued spectral decomposition of such OVKs when $\mathcal{Y}$ is finite dimensional or $\mathcal{X}$ is compact can be written using a pair (A, μ) where A is an operator-valued function on $\hat{\mathcal{X}}$ and μ is a real-valued positive measure on $\hat{\mathcal{X}}$. Note that obviously such a pair is not unique ad the choice of this paper may have an impact on theoretical properties as well as practical computations.

Proposition 7 (Carmeli et al. [5])**.** *Let* μ *be a positive measure on* $\mathcal{B}(\hat{\mathcal{X}})$ *and* $A : \hat{\mathcal{X}} \to \mathcal{L}(\mathcal{Y})$ *such that* $\langle A(.)y, y' \rangle \in L^1(\mathcal{X}, d\mu)$ *for all* $y, y' \in \mathcal{Y}$ *and* $A(\omega) \succcurlyeq 0$ *for* μ*-almost all* ω*. Then, for all* $\delta \in \mathcal{X}$ *and for all* $y, y' \in \mathcal{Y}$*,*

$$\langle y, K_e(\delta) y \rangle_{\mathcal{Y}} = \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y, A(\omega) y' \rangle_{\mathcal{Y}} d\mu(\omega) \tag{4}$$

is the kernel signature of a shift-invariant $\mathcal{Y}$*-Mercer kernel* K *such that* $K(x, z) = K_e(x \star z^{-1})$*. In other terms, each function* $K_e(\cdot)$ *is the Fourier transform of* $A(\cdot)p_\mu(\cdot)$ *where the integral converges in the weak operator topology and* $p_\mu(\omega) = \frac{d\mu}{d\omega}$ *is the Radon-Nikodym derivative (density) of the measure* μ*. If* $\mathcal{Y}$ *is finite dimensional or* $\mathcal{X}$ *is compact, any shift-invariant kernel is of the above form for some pair* $(A(\omega), \mu(\omega))$*.*

This theorem in more interesting than eq. (3) in the sense that it shows that we are certain of the existence of a *scalar* measure μ and a positive operator $A(\omega)$, provided that $\mathcal{X}$ is compact or $\mathcal{Y}$ is finite dimensional. When $p = 1$ one can always assume A is reduced to the scalar 1, μ is still a bounded positive measure and we retrieve the Bochner theorem applied to the scalar case (**??**).

While **??** gives some insights on how to build an approximation of a $\mathcal{Y}$-Mercer kernel, we need a theorem that provides an explicit construction of the pair $A(\omega), \mu(\omega)$ from the kernel signature. Proposition 14 in Carmeli et al. [5] gives the solution, and also provide a sufficient condition for theorem 7 to apply.

Proposition 8 (Carmeli et al. [5]). *Let* K *be a shift-invariant* $\mathcal{Y}$*-Mercer kernel. Suppose that* $\forall z \in \mathcal{X}$ *and* $\forall y, y' \in \mathcal{Y}$, $\langle K_e(.)y, y'\rangle \in L^1(\mathcal{X}, dx)$ *where* dx *denotes the Haar measure on* $(\mathcal{X}, \star)$. *Define* C *such that for all* $\omega \in \hat{\mathcal{X}}$ *and for all* y, y' *in* $\mathcal{Y}$,

$$\begin{aligned} \langle y, C(\omega)y'\rangle &= \int_{\mathcal{X}} (\delta, \omega)\langle y, K_e(\delta)y'\rangle d\delta \\ &= \mathcal{F}^{-1}\left[\langle y, K_e(\cdot)y'\rangle\right](\omega) \end{aligned} \tag{5}$$

Then

i) $C(\omega)$ *is a bounded non-negative operator for all* $\omega \in \hat{\mathcal{X}}$,

ii) $\langle y, C(.)y'\rangle \in L^1(\hat{\mathcal{X}}, d\omega)$ *for all* $y, y' \in \mathcal{X}$,

iii) for all $\delta \in \mathcal{X}$ *and for all* y, y' *in* $\mathcal{Y}$,

$$\langle y, K_e(\delta)y'\rangle = \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)}\langle y, C(\omega)y'\rangle d\omega.$$

Gathering the two propositions, we present now the following property that allows to build a spectral decomposition of a shift-invariant $\mathcal{Y}$-Mercer kernel on a LCA group $(\mathcal{X}, \star)$.

Proposition 9 (Spectral decomposition of shift-invariant $\mathcal{Y}$-Mercer kernel). *Let* K_e *be the signature of a shift-invariant* $\mathcal{Y}$*-Mercer kernel on* $(\mathcal{X}, \star)$.
Suppose that for all y, $y' \in \mathcal{Y}$, $\langle K_e(.)y, y'\rangle \in L^1(\mathcal{X}, dx)$. *If* $\mathcal{Y}$ *is of finite dimension or* $\mathcal{X}$ *is compact, then there exists* μ, *a positive measure on* $\mathcal{B}(\hat{\mathcal{X}})$, *and* $A : \hat{\mathcal{X}} \to \mathcal{L}_+(\mathcal{Y})$, *a operator-valued functions such that* $\langle A(.)y, y'\rangle \in L^1(\mathcal{X}, d\mu)$ *for all* $y, y' \in \mathcal{Y}$ *and*

$$\forall (y, y') \in \mathcal{Y}^2, \ \ \langle y, K_e(\delta)y'\rangle = \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)}\langle y, A(\omega)y'\rangle p_\mu(\omega) d\omega.$$

where $\langle y, A(\omega)y'\rangle p_\mu(\omega) = \mathcal{F}^{-1}\left[\langle y, K_e(x \star z^{-1})y'\rangle\right]$.

Proof. From eq. (4) and eq. (5), if $\mathcal{X}$ is compact or $\mathcal{Y}$ is finite dimensional, we can write the following equality concerning the operator-valued kernel signature K_e. For all $\delta \in \mathcal{X}$ and for all y, y' in $\mathcal{Y}$

$$\int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)}\langle y, C(\omega)y'\rangle d\omega = \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)}\langle y, A(\omega)y'\rangle d\mu(\omega).$$

Since both sides of the equation define continuous functions, the following equality holds μ-almost everywhere. For all $\omega \in \hat{\mathcal{X}}$ and for all y, $y' \in \mathcal{Y}$,

$$\langle y, C(\omega)y'\rangle = \langle y, A(\omega)y'\rangle p_\mu(\omega) = \mathcal{F}^{-1}\left[\langle y, K_e(\cdot)y'\rangle\right](\omega), \tag{6}$$

where $p_\mu(\omega) = \frac{d\mu}{d\omega}$ is the Radon-Nikodym derivative of the measure μ, e.g. its density. $\square$

In the case where $\mathcal{Y} = \mathbb{R}^p$, we rewrite eq. (6) coefficient-wisely by choosing the orthonormal basis of $\mathcal{Y}$, $(e_1, \dots, e_p)$, such that for all $i, j \in \{1, \dots, p\}$,

$$\langle e_i, C(\omega) e_j \rangle = C(\omega)_{ij} = A(\omega)_{ij} p_\mu(\omega) = \mathcal{F}^{-1}\left[K_e(\delta)_{ij}\right]. \tag{7}$$

It follows that

$$\forall i, j \in \{1, \dots, p\}, \;\; K_e(x \star z^{-1})_{ij} = \mathcal{F}\left[A(\cdot)_{ij}\right] \tag{8}$$

Remark 10. *Note that although the inverse Fourier transform of K_e yields a unique operator-valued function $C(\cdot)$, the decomposition of $C(\omega)$ into $A(\omega)p_\mu(\omega)$ is not unique. The choice of the decomposition may be justified by the computational cost or by the nature of the constants involved in the uniform convergence of the estimator.*

3.2.1 *Construction of ORFF*

Without loss of generality we assume that $\int_{\mathcal{X}} d\mu(\omega) = 1$ and thus, μ is a probability distribution and p_μ, a probability density. Note that this is always possible through an appropriate rescaling of the kernel.

3.3 UNIFORM BOUND ON THE APPROXIMATION

3.4 LEARNING WITH OPERATOR-VALUED RANDOM-FOURIER FEATURES

3.5 CONSISTENCY AND GENERALIZATION BOUNDS

3.6 CONCLUSIONS

4 CONCURRENT METHODS

4.1 BACKGROUND

4.2 THE NYSTRÖM METHOD

4.3 SUB-SAMPLING THE DATA

4.4 CONCLUSIONS

Part III

FINAL WORDS

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5 CONLUSIONS

Part IV

APPENDIX

A

OPERATOR-VALUED FUNCTIONS AND INTEGRATION

B PROOFS OF THEOREMS

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DECLARATION

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15, Rue Plumet, 75015 - Paris, France, Septembre 2016

Romain Brault

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