

DATA ARE NOT REAL!

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Large-scale learning on structured input-output data with operator-valued kernels

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## ABSTRACT

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Short summary of the contents...a great guide by Kent Beck how to write good abstracts can be found here:

<https://plg.uwaterloo.ca/~migod/research/beck00PSLA.html>



## PUBLICATIONS

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Some ideas and figures have appeared previously in the following publications:

Put your publications from the thesis here. The packages `multibib` or `bibtopic` etc. can be used to handle multiple different bibliographies in your document.





*We have seen that computer programming is an art,  
because it applies accumulated knowledge to the world,  
because it requires skill and ingenuity, and especially  
because it produces objects of beauty.*

## ACKNOWLEDGEMENTS

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Put your acknowledgements here.

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<sup>1</sup> Members of GuIT (Gruppo Italiano Utilizzatori di  $\text{\TeX}$  e  $\text{\LaTeX}$ )



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## LISTINGS

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## ACRONYMS

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OVK Operator-Valued Kernel.

ORFF Operator-valued Random Fourier Feature.

RKHS Reproducing Kernel Hilbert Space.

vv-RKHS vector-valued Reproducing Kernel Hilbert Space.

LCA Locally Compact Abelian.

FT Fourier transform.

IFT inverse Fourier transform.



## Part I

### INTRODUCTION

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## MOTIVATIONS

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## BACKGROUND

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## 2.1 NOTATIONS

The euclidean inner product in  $\mathbb{R}^d$  is denoted  $\langle \cdot, \cdot \rangle$  and the euclidean norm is denoted  $\|\cdot\|$ . The unit pure imaginary number  $\sqrt{-1}$  is denoted  $i$ .  $\mathcal{B}(\mathbb{R}^d)$  is the Borel  $\sigma$ -algebra on  $\mathbb{R}^d$ . If  $\mathcal{X}$  and  $\mathcal{Y}$  are two vector spaces, we denote by  $\mathcal{F}(\mathcal{X}; \mathcal{Y})$  the vector space of functions  $f : \mathcal{X} \rightarrow \mathcal{Y}$  and  $\mathcal{C}(\mathcal{X}; \mathcal{Y}) \subset \mathcal{F}(\mathcal{X}; \mathcal{Y})$  the subspace of continuous functions. If  $\mathcal{H}$  is an Hilbert space we denote its scalar product by  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  and its norm by  $\|\cdot\|_{\mathcal{H}}$ . We set  $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}; \mathcal{H})$  to be the space of linear operators from  $\mathcal{H}$  to itself. If  $W \in \mathcal{L}(\mathcal{H})$ ,  $\text{Ker } W$  denotes the nullspace,  $\text{Im } W$  the image and  $W^* \in \mathcal{L}(\mathcal{H})$  the adjoint operator (transpose when  $W$  is a real matrix). All these notations are summarized in table 1.

## 2.2 ABOUT STATISTICAL LEARNING

## 2.3 ON LARGE-SCALE LEARNING

## 2.4 ELEMENTS OF ABSTRACT HARMONIC ANALYSIS

### 2.4.1 Locally compact Abelian groups

**Definition 1.** *Locally Compact Abelian group. A group  $(\mathcal{X}, \star)$  is said to be Locally Compact Abelian if it is a topological commutative group  $\mathcal{X}$  for which every point has a compact neighborhood and is Hausdorff.*

Locally Compact Abelian (LCA) groups are central to the general definition of Fourier Transform which is related to the concept of Pontryagin duality [8]. Let  $(\mathcal{X}, \star)$  be a LCA group with  $e$  its neutral element and the notation,  $x^{-1}$ , for the inverse of  $x \in \mathcal{X}$ . A *character* is a complex continuous homomorphism  $\omega : \mathcal{X} \rightarrow \mathbb{U}$  from  $\mathcal{X}$  to the set of complex numbers of unit module  $\mathbb{U}$ . The set of all characters of  $\mathcal{X}$  forms the Pontryagin *dual group*  $\hat{\mathcal{X}}$ . The dual group of an LCA group is an LCA group and the dual group operation is defined by

$$(\omega_1 \star \omega_2)(x) = \omega_1(x)\omega_2(x) \in \mathbb{U}.$$

The Pontryagin duality theorem states that  $\hat{\hat{\mathcal{X}}} \cong \mathcal{X}$ . I.e. there is a canonical isomorphism between any LCA group and its double dual. To emphasize this duality the following notation is usually adopted:  $\omega(x) = (x, \omega) = (\omega, x)$ , where  $x \in \mathcal{X}$ ,  $\omega \in \hat{\mathcal{X}}$ . Another important property involves the complex conjugate of the pairing which is defined as  $\overline{(x, \omega)} = (x^{-1}, \omega)$ .

Table 1: Mathematical symbols used throughout the paper and their signification.

Symbol	Meaning
$i$	Unit pure imaginary number $\sqrt{-1}$ .
$e$	Euler constant.
$\langle \cdot, \cdot \rangle$	Euclidean inner product.
$\ \cdot\ $	Euclidean norm.
$\mathcal{X}$	Input space $(\cdot)$ .
$\hat{\mathcal{X}}$	The Pontryagin dual of $\mathcal{X}$ .
$\mathcal{Y}$	Output space (Hilbert space).
$\mathcal{H}$	Feature space (Hilbert space).
$\langle \cdot, \cdot \rangle_{\mathcal{Y}}$	The canonical inner product of the Hilbert space $\mathcal{Y}$ .
$\ \cdot\ _{\mathcal{Y}}$	The canonical norm induced by the inner product of the Hilbert space $\mathcal{Y}$ .
$\mathcal{F}(\mathcal{X}; \mathcal{Y})$	Vector space of function from $\mathcal{X}$ to $\mathcal{Y}$ .
$\mathcal{C}(\mathcal{X}; \mathcal{Y})$	The vector subspace of $\mathcal{F}$ of continuous function from $\mathcal{X}$ to $\mathcal{Y}$ .
$\mathcal{L}(\mathcal{H}; \mathcal{Y})$	The set of bounded linear operator from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{Y}$ .
$\mathcal{L}(\mathcal{Y})$	The set of bounded linear operator from a Hilbert space $\mathcal{H}$ to itself.
$\mathcal{L}_+(\mathcal{Y})$	The set of non-negative bounded linear operator from a Hilbert space $\mathcal{H}$ to itself.
$\mathcal{B}(\mathcal{X})$	Borel $\sigma$ -algebra on $\mathcal{X}$ .
$\mu(\mathcal{X})$	A scalar positive measure of $\mathcal{X}$ .
$p_{\mu}(x)$	The Radon-Nikodym derivative of $\mu$ w.r.t. the Lebesgue measure.
$dx, d\omega$	The canonical Haar measure of the LCA group $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ . (resp. $(\hat{\mathcal{X}}, \mathcal{B}(\hat{\mathcal{X}}))$ ).
$L^p(\mathcal{X}, dx)$	The Banach space of $ \cdot ^p$ -integrable function from $(\mathcal{X}, \mathcal{B}(\mathcal{X}), dx)$ to $\mathbb{C}$ .
$L^p(\mathcal{X}, dx; \mathcal{Y})$	The Banach space of $\ \cdot\ _{\mathcal{Y}^p}$ (Bochner)-integrable function from $(\mathcal{X}, \mathcal{B}(\mathcal{X}), dx)$ to $\mathcal{Y}$ .

We notice that for any pairing depending of  $\omega$ , there exists a function  $h_{\omega} : \mathcal{X} \rightarrow \mathbb{R}$  such that:  $(x, \omega) = \exp(-ih_{\omega}(x))$  since any pairing maps into  $\mathbb{U}$ . Moreover,

$$\begin{aligned} (x \star z^{-1}, \omega) &= \omega(x)\omega(z^{-1}) = \exp(-ih_{\omega}(x)) \exp(-ih_{\omega}(z^{-1})) \\ &= \exp(-ih_{\omega}(x)) \exp(+ih_{\omega}(z)). \end{aligned}$$

Table 2: Classification of Fourier transforms in terms of their domain and transform domain.

$\mathcal{X}$	$\hat{\mathcal{X}}$	Operation	Pairing
$\mathbb{R}^d$	$\mathbb{R}^d$	$+$	$(x, \omega) = \exp(i\langle x, \omega \rangle)$
$\mathbb{R}_{*,+}^d$	$\mathbb{R}^d$	$\cdot$	$(x, \omega) = \exp(i\langle \log(x), \omega \rangle)$
$(-c; +\infty)^d$	$\mathbb{R}^d$	$\odot$	$(x, \omega) = \exp(i\langle \log(x+c), \omega \rangle)$

Table 2 provide an explicit list of pairings for various groups based on  $\mathbb{R}^d$  or its subsets. We especially mention the duality pairing associated to the skewed multiplicative LCA group  $\mathcal{X} = ((-c; +\infty)^d, \odot)$  ( $x_k+c)(z_k+c) - c$ , Hence  $h_\omega(x) = \sum_{k=1}^d \omega_k \log(x_k + c)$ ). This group together with the operation  $\odot$  has been proposed by [10] to handle histograms features especially useful in image recognition applications.

#### 2.4.2 The Fourier transform

For a function with values in a separable Hilbert space  $f \in L^1(\mathcal{X}, dx; \mathcal{Y})$ , where  $dx$  is the Haar measure on  $\mathcal{X}$ , we denote  $\mathcal{F}[f]$  its Fourier transform (FT) which is defined by

$$\forall \omega \in \hat{\mathcal{X}}, \quad \mathcal{F}[f](\omega) = \int_{\hat{\mathcal{X}}} \overline{(x, \omega)} f(x) dx.$$

For a measure defined on  $\mathcal{X}$ , there exists a unique suitably normalized measure  $d\omega$  on  $\hat{\mathcal{X}}$  such that  $\forall f \in L^1(\mathcal{X}, dx; \mathcal{Y})$  and if  $\mathcal{F}[f] \in L^1(\hat{\mathcal{X}}, d\omega, \mathcal{Y})$  we have

$$\forall x \in \mathcal{X}, \quad f(x) = \int_{\hat{\mathcal{X}}} \mathcal{F}[f](\omega) (x, \omega) d\omega. \quad (1)$$

Moreover if  $d\omega$  is normalized,  $\mathcal{F}$  extends to a unitary operator from  $L^2(\mathcal{X}, dx, \mathcal{Y})$  onto  $L^2(\hat{\mathcal{X}}, d\omega, \mathcal{Y})$  Then the inverse Fourier transform (IFT) of a function  $g \in L^1(\hat{\mathcal{X}}, d\omega, \mathcal{Y})$  (where  $d\omega$  is a Haar measure on  $\hat{\mathcal{X}}$  suitably normalized w. r. t. the Haar measure  $dx$ ) is noted  $\mathcal{F}^{-1}[g]$  defined by

$$\forall x \in \mathcal{X}, \quad \mathcal{F}^{-1}[g](x) = \int_{\hat{\mathcal{X}}} (x, \omega) g(\omega) d\omega,$$

Equation (1) gives some examples of real Abelian groups with their associated dual and pairing. The interested reader can refer to Folland [8] for a more detailed construction of LCA, Pontryagin duality and Fourier transforms on LCA. For the familiar case of a scalar-valued function  $f$  on the LCA group  $(\mathbb{R}^d, +)$ , we have:

$$\forall \omega \in \hat{\mathcal{X}}, \quad \mathcal{F}[f](\omega) = \int_{\mathbb{R}^d} e^{-i\langle \omega, x \rangle} f(x) dx, \quad (2)$$

the Haar measure being here the Lebesgue measure.



## 2.5 ON OPERATOR-VALUED KERNELS

We now introduce the theory of vector-valued Reproducing Kernel Hilbert Space (**vv-RKHS**) that provides a flexible framework to study and learn vector-valued functions.

## 2.5.1 Definitions and properties

An operator-valued kernel is defined here as a  $\mathcal{Y}$ -reproducing kernel Carmeli et al. [5].

**Definition 2.** Given  $\mathcal{X}$ , a Polish space and  $\mathcal{Y}$ , a Hilbert Space, a map  $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y})$  is called a  $\mathcal{Y}$ -reproducing kernel if

$$\sum_{i,j=1}^N \langle K(x_i, x_j) y_j, y_i \rangle_{\mathcal{Y}} \geq 0,$$

for all  $x_1, \dots, x_N$  in  $\mathcal{X}$ , all  $y_1, \dots, y_N$  in  $\mathcal{Y}$  and  $N \geq 1$ . Given  $x \in \mathcal{X}$ ,  $K_x : \mathcal{Y} \rightarrow \mathcal{F}(\mathcal{X}; \mathcal{Y})$  denotes the linear operator whose action on a vector  $y$  is the function  $K_x y \in \mathcal{F}(\mathcal{X}; \mathcal{Y})$  defined by  $(K_x y)(z) = K(z, x)y$ , for all  $z \in \mathcal{X}$ .

Additionally, given a  $\mathcal{Y}$ -reproducing kernel  $K$ , there is a unique Hilbert space  $\mathcal{H}_K \subset \mathcal{F}(\mathcal{X}; \mathcal{Y})$  satisfying  $K_x \in \mathcal{L}(\mathcal{Y}; \mathcal{H}_K)$ , for all  $x \in \mathcal{X}$  and  $\forall x \in \mathcal{X}, \forall f \in \mathcal{H}_K$ ,  $f(x) = K_x^* f$ , where  $K_x^* : \mathcal{H}_K \rightarrow \mathcal{Y}$  is the adjoint of  $K_x$ . The space  $\mathcal{H}_K$  is called the *vector-valued Reproducing Kernel Hilbert Space* associated with  $K$ . The corresponding product and norm are denoted by  $\langle \cdot, \cdot \rangle_K$  and  $\|\cdot\|_K$ , respectively. As a consequence [5] we have:

$$K(x, z) = K_x^* K_z \quad \forall x, z \in \mathcal{X}, \\ \mathcal{H}_K = \overline{\text{span}} \{ K_x y \mid \forall x \in \mathcal{X}, \forall y \in \mathcal{Y} \}.$$

Another way to describe functions of  $\mathcal{H}_K$  consists in using a suitable feature map.

**Proposition 3** (Feature Operator Carmeli et al. [5]). Let  $\mathcal{H}$  be a Hilbert space and  $\phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y}; \mathcal{H})$ , with  $\phi_x := \phi(x)$ . Then the operator  $W : \mathcal{H} \rightarrow \mathcal{F}(\mathcal{X}; \mathcal{Y})$  defined for all  $g \in \mathcal{H}$ , and for all  $x \in \mathcal{X}$  by  $(Wg)(x) = \phi_x^* g$  is a partial isometry from  $\mathcal{H}$  onto the **vv-RKHS**  $\mathcal{H}_K$  with reproducing kernel

$$K(x, z) = \phi_x^* \phi_z, \quad \forall x, z \in \mathcal{X}.$$

$W^* W$  is the orthogonal projection onto

$$\text{Ker } W^\perp = \overline{\text{span}} \{ \phi_x y \mid \forall x \in \mathcal{X}, \forall y \in \mathcal{Y} \}.$$

Then  $\|f\|_K = \inf \{ \|g\|_{\mathcal{H}} \mid \forall g \in \mathcal{H}, Wg = f \}$ .

We call  $\phi$  a *feature map*,  $W$  a *feature operator* and  $\mathcal{H}$  a *feature space*.

### 2.5.2 Examples of operator-valued kernels

Operator-valued kernels have been first introduced in Machine Learning to solve multi-task regression problems. Multi-task regression is encountered in many fields such as structured classification when classes belong to a hierarchy for instance. Instead of solving independently  $p$  single output regression task, one would like to take advantage of the relationships between output variables when learning and making a decision.

**Definition 4** (Decomposable kernel). *Let  $A$  be a positive semi-definite operator of  $\mathcal{L}(\mathcal{Y})$ .  $K$  is said to be a  $\mathcal{Y}$ -Mercer decomposable kernel<sup>1</sup> if for all  $(x, z) \in \mathcal{X}^2$ ,*

$$K(x, z) = k(x, z)A,$$

where  $k$  is a scalar Mercer kernel.

When  $\mathcal{Y} = \mathbb{R}^p$ , the matrix  $A$  is interpreted as encoding the relationships between the outputs coordinates. If a graph coding for the proximity between tasks is known, then it is shown in Álvarez, Rosasco, and Lawrence [1], Baldassarre et al. [2], and Evgeniou, Micchelli, and Pontil [7] that  $A$  can be chosen equal to the pseudo inverse  $L^\dagger$  of the graph Laplacian such that the norm in  $\mathcal{H}_K$  is a graph-regularizing penalty for the outputs (tasks). When no prior knowledge is available,  $A$  can be set to the empirical covariance of the output training data or learned with one of the algorithms proposed in the literature [6, 12, 15]. Another interesting property of the decomposable kernel is its universality (a kernel which may approximate an arbitrary continuous target function uniformly on any compact subset of the input space). A reproducing kernel  $K$  is said *universal* if the associated [vv-RKHS](#)  $\mathcal{H}_K$  is dense in the space  $\mathcal{C}(\mathcal{X}, \mathcal{Y})$ . The conditions for a kernel to be universal have been discussed in Caponnetto et al. [4] and Carmeli et al. [5]. In particular they show that a decomposable kernel is universal provided that the scalar kernel  $k$  is universal and the operator  $A$  is injective.

Curl-free and divergence-free kernels provide an interesting application of operator-valued kernels [3, 13, 14] to *vector field* learning, for which input and output spaces have the same dimensions ( $d = p$ ). Applications cover shape deformation analysis [14] and magnetic fields approximations [16]. These kernels discussed in [9] allow encoding input-dependent similarities between vector-fields.

**Definition 5** (Curl-free and Div-free kernel). *Assume  $\mathcal{X} = (\mathbb{R}^d, +)$  and  $\mathcal{Y} = \mathbb{R}^p$  with  $d = p$ . The divergence-free kernel is defined as*

$$K^{\text{div}}(x, z) = K_0^{\text{div}}(\delta) = (\nabla \nabla^T - \text{I})k_0(\delta)$$

<sup>1</sup> Some authors also refer to as separable kernels.

and the curl-free kernel as

$$K^{\text{curl}}(x, z) = K_0^{\text{curl}}(\delta) = -\nabla \nabla^T k_0(\delta),$$

where  $\nabla \nabla^T$  is the Hessian operator and  $\delta$  is the Laplacian operator.

Although taken separately these kernels are not universal, a convex combination of the curl-free and divergence-free kernels allows to learn any vector field that satisfies the Helmholtz decomposition theorem [3, 13].



## Part II

### CONTRIBUTIONS

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### 3.1 MOTIVATIONS

Random Fourier Features have been proved useful to implement efficiently kernel methods in the scalar case, allowing to learn a linear model based on an approximated feature map. In this work, we are interested to construct approximated operator-valued feature maps to learn vector-valued functions. With an explicit (approximated) feature map, one converts the problem of learning a function  $f$  in the vector-valued Reproducing Kernel Hilbert Space  $\mathcal{H}_K$  into the learning of a linear model  $\tilde{f}$  defined by:

$$\tilde{f}(x) = \tilde{\phi}(x)^* \theta,$$

where  $\phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{Y})$  and  $\theta \in \mathcal{H}$ . The methodology we propose works for operator-valued kernels defined on any Locally Compact Abelian (LCA) group, noted  $(\mathcal{X}, \star)$ , for some operation noted  $\star$ . This allows us to use the general context of Pontryagin duality for Fourier transform of functions on LCA groups. Building upon a generalization of Bochner's theorem for operator-valued measures, an operator-valued kernel is seen as the *Fourier transform* of an operator-valued positive measure. From that result, we extend the principle of Random Fourier Feature for scalar-valued kernels and derive a general methodology to build Operator Random Fourier Feature when operator-valued kernels are shift-invariant according to the chosen group operation.

### 3.2 CONSTRUCTION

We present a construction of Operator-valued Random Fourier Feature (ORFF) such that  $f : x \mapsto \tilde{\phi}(x)^* \theta$  is a continuous function that maps an arbitrary LCA group  $\mathcal{X}$  as input space to an arbitrary output Hilbert space  $\mathcal{Y}$ . First we define a functional *Fourier feature map*, and then propose a Monte-Carlo sampling from this feature map to construct an approximation of a shift-invariant  $\mathcal{Y}$ -Mercer kernel. Then, we prove the convergence of the kernel approximation  $\tilde{K}(x, z) = \tilde{\phi}(x)^* \tilde{\phi}(z)$  with high probability on compact subsets of the LCA  $\mathcal{X}$ , when  $\mathcal{Y}$  is *finite dimensional*. Eventually we conclude with some numerical experiments.

#### 3.2.1 Theoretical study

The following proposition of Carmeli et al. [5] and Zhang, Xu, and Zhang [17] extends Bochner's theorem to any shift-invariant  $\mathcal{Y}$ -Mercer kernel.

**Proposition 6** (Operator-valued Bochner's theorem [17]). *If a continuous function  $K$  from  $\mathcal{X} \times \mathcal{X}$  to  $\mathcal{Y}$  is a shift-invariant  $\mathcal{Y}$ -Mercer kernel*



on  $\mathcal{X}$ , then there exists a unique positive operator-valued measure  $M : \mathcal{B}(\mathcal{X}) \rightarrow \mathcal{L}_+(\mathcal{Y})$  such that for all  $x, z \in \mathcal{X}$ ,

$$K(x, z) = \int_{\hat{\mathcal{X}}} \overline{(x \star z^{-1}, \omega)} dM(\omega), \quad (3)$$

where  $M$  belongs to the set of all the  $\mathcal{L}_+(\mathcal{Y})$ -valued measures of bounded variation on the  $\sigma$ -algebra of Borel subsets of  $\hat{\mathcal{X}}$ . Conversely, from any positive operator-valued measure  $M$ , a shift-invariant kernel  $K$  can be defined by proposition 6.

Although this theorem is central to the spectral decomposition of shift-invariant  $\mathcal{Y}$ -Mercer [ovk](#), the following results proved by Carmeli et al. [5] provides insights about this decomposition that are more relevant in practise. It first shows how to build shift-invariant  $\mathcal{Y}$ -Mercer kernel but more importantly, also states that any operator-valued spectral decomposition of such [ovks](#) when  $\mathcal{Y}$  is finite dimensional or  $\mathcal{X}$  is compact can be written using a pair  $(A, \mu)$  where  $A$  is an operator-valued function on  $\hat{\mathcal{X}}$  and  $\mu$  is a real-valued positive measure on  $\hat{\mathcal{X}}$ . Note that obviously such a pair is not unique and the choice of this paper may have an impact on theoretical properties as well as practical computations.

**Proposition 7** (Carmeli et al. [5]). *Let  $\mu$  be a positive measure on  $\mathcal{B}(\hat{\mathcal{X}})$  and  $A : \hat{\mathcal{X}} \rightarrow \mathcal{L}(\mathcal{Y})$  such that  $\langle A(\cdot)y, y' \rangle \in L^1(\mathcal{X}, d\mu)$  for all  $y, y' \in \mathcal{Y}$  and  $A(\omega) \succcurlyeq 0$  for  $\mu$ -almost all  $\omega$ . Then, for all  $\delta \in \mathcal{X}$  and for all  $y, y' \in \mathcal{Y}$ ,*

$$K_e(\delta) = \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} A(\omega) d\mu(\omega) \quad (4)$$

*is the kernel signature of a shift-invariant  $\mathcal{Y}$ -Mercer kernel  $K$  such that  $K(x, z) = K_e(x \star z^{-1})$ . The [vw-RKHS](#)  $\mathcal{H}_K$  is embed in  $L^2(\hat{\mathcal{X}}, d\mu; \mathcal{Y}')$  by mean of the feature operator*

$$(Wg)(x) = \int_{\hat{\mathcal{X}}} \overline{(x, \omega)} B(\omega) g(\omega) d\mu(\omega), \quad (5)$$

*Where  $B(\omega)B(\omega)^* = A(\omega)$  and both integral converges in the weak sense. If  $\mathcal{Y}$  is finite dimensional or  $\mathcal{X}$  is compact, any shift-invariant kernel is of the above form for some pair  $(A(\omega), \mu(\omega))$ .*

This theorem is more interesting than proposition 6 in the sense that it shows that we are certain of the existence of a scalar measure  $\mu$  and a positive operator  $A(\omega)$ , provided that  $\mathcal{X}$  is compact or  $\mathcal{Y}$  is finite dimensional. When  $p = 1$  one can always assume  $A$  is reduced to the scalar  $1$ ,  $\mu$  is still a bounded positive measure and we retrieve the Bochner theorem applied to the scalar case ([??](#)).

### 3.2.2 Functional Fourier feature map

Let us introduce a functional feature map, we call here *Fourier Feature map*, defined by the following proposition as a direct consequence of proposition 7.

**Proposition 8** (Fourier feature map). *If there exist an operator-valued function  $B : \hat{\mathcal{X}} \rightarrow \mathcal{L}(\mathcal{Y}, \mathcal{Y}')$  such that for all  $y, y' \in \mathcal{Y}$ ,  $\langle y, B(\omega)B(\omega)^*y' \rangle = \langle y, A(\omega)y' \rangle$   $\mu$ -almost everywhere and  $\langle y, A(\omega)y' \rangle \in L^1(\hat{\mathcal{X}}, d\mu)$  then the operator  $\phi_x$  defined for all  $y$  in  $\mathcal{Y}$  by*

$$(\phi_x y)(\omega) = (x, \omega)B(\omega)^*y, \quad (6)$$

*is a feature map<sup>2</sup> of some shift-invariant kernel  $K$ .*

*Proof.* For all  $y, y' \in \mathcal{Y}$  and  $x, z \in \mathcal{X}$ ,

$$\begin{aligned} \langle y, \phi_x^* \phi_z y' \rangle &= \langle \phi_x y, \phi_z y' \rangle_{L^2(\hat{\mathcal{X}}, \mu, \mathcal{Y}')} \\ &= \int_{\hat{\mathcal{X}}} \overline{(x, \omega)} \langle y, B(\omega)(z, \omega)B(\omega)^*y' \rangle d\mu(\omega) \\ &= \int_{\hat{\mathcal{X}}} \overline{(x \star z^{-1}, \omega)} \langle y B(\omega)B(\omega)^*y' \rangle d\mu(\omega) \\ &= \int_{\hat{\mathcal{X}}} \overline{(x \star z^{-1}, \omega)} \langle y, A(\omega)y' \rangle d\mu(\omega), \end{aligned}$$

which defines a  $\mathcal{Y}$ -Mercer according to proposition 7 of Carmeli et al. [5].  $\square$

With this notation notice that  $\phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y}; L^2(\hat{\mathcal{X}}, \mu; \mathcal{Y}'))$  such that  $\phi_x \in \mathcal{L}(\mathcal{Y}; L^2(\hat{\mathcal{X}}, \mu; \mathcal{Y}'))$  where  $\phi_x := \phi(x)$ .

### 3.2.3 Sufficient conditions of existence

While proposition 7 gives some insights on how to build an approximation of a  $\mathcal{Y}$ -Mercer kernel, we need a theorem that provides an explicit construction of the pair  $A(\omega), \mu(\omega)$  from the kernel signature. Proposition 14 in Carmeli et al. [5] gives the solution, and also provide a sufficient condition for proposition 7 to apply.

**Proposition 9** (Carmeli et al. [5]). *Let  $K$  be a shift-invariant  $\mathcal{Y}$ -Mercer kernel. Suppose that  $\forall z \in \mathcal{X}$  and  $\forall y, y' \in \mathcal{Y}$ ,  $\langle K_e(\cdot)y, y' \rangle \in L^1(\mathcal{X}, dx)$  where  $dx$  denotes the Haar measure on  $(\mathcal{X}, \star)$ . Define  $C$  such that for all  $\omega \in \hat{\mathcal{X}}$  and for all  $y, y'$  in  $\mathcal{Y}$ ,*

$$\begin{aligned} \langle y, C(\omega)y' \rangle &= \int_{\mathcal{X}} (\delta, \omega) \langle y, K_e(\delta)y' \rangle d\delta \\ &= \mathcal{F}^{-1} [\langle y, K_e(\cdot)y' \rangle] (\omega) \end{aligned} \quad (7)$$

*Then*

- i)  $C(\omega)$  is a bounded non-negative operator for all  $\omega \in \hat{\mathcal{X}}$ ,
- ii)  $\langle y, C(\cdot)y' \rangle \in L^1(\hat{\mathcal{X}}, d\omega)$  for all  $y, y' \in \mathcal{X}$ ,
- iii) for all  $\delta \in \mathcal{X}$  and for all  $y, y'$  in  $\mathcal{Y}$ ,

$$\langle y, K_e(\delta)y' \rangle = \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y, C(\omega)y' \rangle d\omega.$$

<sup>2</sup> I. e. it satisfies for all  $x, z \in \mathcal{X}$ ,  $\phi_x^* \phi_z = K(x, z)$  where  $K$  is a  $\mathcal{Y}$ -Mercer OVK.

Gathering the two propositions, we present now the following property that allows to build a spectral decomposition of a shift-invariant  $\mathcal{Y}$ -Mercer kernel on a LCA group  $(\mathcal{X}, \star)$ .

**Proposition 10** (Sufficient condition for shift-invariant  $\mathcal{Y}$ -Mercer kernel spectral decomposition). *Let  $K_e$  be the signature of a shift-invariant  $\mathcal{Y}$ -Mercer kernel on  $(\mathcal{X}, \star)$ .*

*If for all  $y, y' \in \mathcal{Y}$ ,  $\langle K_e(\cdot)y, y' \rangle \in L^1(\mathcal{X}, dx)$ , then there exists a positive measure  $\mu$  with density  $p_\mu$  on  $\mathcal{B}(\hat{\mathcal{X}})$  and  $A : \hat{\mathcal{X}} \rightarrow \mathcal{L}_+(\mathcal{Y})$  an operator-valued functions such that for all  $y, y' \in \mathcal{Y}$   $\langle A(\cdot)y, y' \rangle \in L^1(\mathcal{X}, d\mu)$  and for all  $y, y'$  in  $\mathcal{Y}$ ,*

$$\langle y, K_e(\delta)y' \rangle = \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y, A(\omega)y' \rangle p_\mu(\omega) d\omega.$$

where  $\langle y, A(\omega)y' \rangle p_\mu(\omega) = \mathcal{F}^{-1} [\langle y, K_e(x \star z^{-1})y' \rangle]$ .

*Proof.* From eq. (4) and eq. (7) we can write the following equality concerning the ovk signature  $K_e$ .

$$\begin{aligned} \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y, A(\omega)y' \rangle d\mu(\omega) &= \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y, C(\omega)y' \rangle d\omega. \\ &= \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y, C'(\omega)p(\omega)y' \rangle d\omega \end{aligned}$$

It is always possible to choose  $p(\omega)$  such that  $\int_{\hat{\mathcal{X}}} p(\omega) d\omega = 1$  then  $p(\omega)$  is the density of a probability measure  $\mu$ . In this case we note  $p(\omega) = p_\mu(\omega)$ . By injectivity of the Fourier transform we have for all  $y, y' \in \mathcal{Y}$ ,

$$\langle y, C(\omega)y' \rangle = \langle y, C'(\omega)y' \rangle p_\mu(\omega) = \mathcal{F}^{-1} [\langle y, K_e(\cdot)y' \rangle] (\omega) \quad (8)$$

Conclude by taking  $A(\omega) = C'(\omega)$  and  $d\mu(\omega) = p_\mu(\omega) d\omega$ .  $\square$

In the case where  $\mathcal{Y} = \mathbb{R}^p$ , we rewrite eq. (8) coefficient-wise by choosing an orthonormal basis  $(e_1, \dots, e_p)$  of  $\mathcal{Y}$ , such that for all  $i, j \in \{1, \dots, p\}$ ,

$$\langle e_i, C(\omega)e_j \rangle = C(\omega)_{ij} = A(\omega)_{ij} p_\mu(\omega) = \mathcal{F}^{-1} [K_e(\delta)_{ij}]. \quad (9)$$

It follows that for all  $i, j \in \{1, \dots, p\}$ ,

$$K_e(x \star z^{-1})_{ij} = \mathcal{F} [A(\cdot)_{ij}] \quad (10)$$

**Remark 11.** Note that although the inverse Fourier transform of  $K_e$  yields a unique operator-valued function  $C(\cdot)$ , the decomposition of  $C(\omega)$  into  $A(\omega)p_\mu(\omega)$  is not unique. The choice of the decomposition may be justified by the computational cost or by the nature of the constants involved in the uniform convergence of the estimator.

### 3.2.4 Regularization property

We have shown so far that it is always possible to construct a feature map that allows to approximate a shift-invariant  $\mathcal{Y}$ -Mercer kernel. However we could also propose a construction of such map by studying the regularization induced with respect to the Fourier transform of a target function  $f \in \mathcal{H}_K$ . In other words, what is the norm in  $L^2(\hat{\mathcal{X}}, d\omega, \mathcal{Y}')$  induced by  $\|\cdot\|_K$ ?

**Proposition 12.** *Let  $K$  be a shift-invariant  $\mathcal{Y}$ -Mercer Kernel such that for all  $y, y'$  in  $\mathcal{Y}$ ,  $\langle y, K_e(\cdot)y' \rangle \in L^1(\mathcal{X}, dx)$  and*

*Let  $\langle y, A(\omega)y' \rangle p_\mu(\omega) := \mathcal{F}^{-1}[\langle y, K_e(\cdot)y' \rangle](\omega)$  and let  $f \in \mathcal{H}_K$ . Then*

$$\|f\|_K^2 = \int_{\hat{\mathcal{X}}} \frac{\langle \mathcal{F}[f](\omega), A(\omega)^\dagger \mathcal{F}[f](\omega) \rangle_{\mathcal{Y}}}{p_\mu(\omega)} d\omega. \quad (11)$$

*Proof.* We first show how the Fourier transform relates to the feature operator. Since  $\mathcal{H}_K$  is embed into  $\mathcal{H} = L^2(\hat{\mathcal{X}}, \mu, \mathcal{Y})$  by mean of the feature operator  $W$ , we have:

$$\begin{aligned} \mathcal{F}[\mathcal{F}^{-1}[f]](x) &= \int_{\hat{\mathcal{X}}} \overline{(x, \omega)} \mathcal{F}^{-1}[f](\omega) d\omega = f(x) \\ (Wg)(x) &= \int_{\hat{\mathcal{X}}} \overline{(x, \omega)} p_\mu(\omega) B(\omega) g(\omega) d\omega = f(x). \end{aligned}$$

By injectivity of the Fourier transform,  $\mathcal{F}^{-1}[f](\omega) = p_\mu(\omega) B(\omega) g(\omega)$   $\mu$ -almost everywhere. From proposition 3 we have

$$\begin{aligned} \|f\|_K^2 &= \inf \left\{ \|g\|_{\mathcal{H}}^2 \mid \forall g \in \mathcal{H}, Wg = f \right\} \\ &= \inf \left\{ \int_{\hat{\mathcal{X}}} \|g\|_{\mathcal{Y}}^2 d\mu \mid \forall g \in \mathcal{H}, \mathcal{F}^{-1}[f] = p_\mu(\cdot) B(\cdot) g(\cdot) \right\}. \end{aligned}$$

The pseudo inverse of the operator  $B(\omega)$  (noted  $B(\omega)^\dagger$ ) is the unique solution of the system  $\mathcal{F}^{-1}[f](\omega) = p_\mu(\omega) B(\omega) g(\omega)$  w. r. t.  $g(\omega)$  with minimal norm. Eventually,

$$\begin{aligned} \|f\|_K^2 &= \int_{\hat{\mathcal{X}}} \frac{\|B(\omega)^\dagger \mathcal{F}^{-1}[f](\omega)\|_{\mathcal{Y}}^2}{p_\mu(\omega)^2} d\mu(\omega) \\ &= \int_{\hat{\mathcal{X}}} \frac{\|B(\omega)^\dagger \mathcal{F}[f](\omega)\|_{\mathcal{Y}}^2}{p_\mu(\omega)^2} d\mu(\omega) \end{aligned} \quad (12)$$

Conclude the proof by taking  $d\mu(\omega) = p_\mu(\omega) d\omega$  and rewriting the integral as an expectation.  $\square$

Note that if  $K(x, z) = k(x, z)$  is a scalar kernel then for all  $\omega$  in  $\hat{\mathcal{X}}$ ,  $A(\omega) = \mathbf{1}$ . Therefore we recover a well known results for kernels that is for any  $f \in \mathcal{H}_K$  we have  $\|f\|_K^2 = \int_{\hat{\mathcal{X}}} \mathcal{F}[k_e](\omega)^{-1} \mathcal{F}[f](\omega)^2 d\omega$ . We also note that the regularization property in  $\mathcal{H}_K$  does not depends (as expected) on the decomposition of  $A(\omega)$  into  $B(\omega)B(\omega)^*$ . Therefore the decomposition should be chosen such that it optimizes the computation cost. For instance if  $A(\omega) \in \mathcal{L}(\mathbb{R}^p)$  has rank  $r$ , one could find an operator  $B(\omega) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^r)$  such that  $A(\omega) = B(\omega)B(\omega)^*$ .

### 3.2.5 Building Operator-valued Random Fourier Features

Throughout the document, without loss of generality, we assume that  $\int_{\mathcal{X}} d\mu(\omega) = 1$  and thus  $d\mu$  is a probability measure with density  $p_\mu$ . As shown in proposition 8 it is always possible to find a pair  $(A(\omega), d\mu)$  such that  $d\mu$  is a probability measure and  $\mathbf{E}_\mu(\overline{\delta(\cdot, \omega)} A(\omega))$ .

Given a  $\mathcal{Y}$ -Mercer shift-invariant kernel  $K$  on  $\mathcal{X}$ , an approximation of  $K$  can be obtained using a decomposition  $(A, \mu)$  and a plug-in Monte-Carlo estimator instead of the expectation. However, for efficient computations, as motivated in the introduction, we are interested in finding an approximated feature map more than a kernel approximation. Indeed, an approximated feature map will allow to build linear models in regression tasks. The following proposition provides the general form of an Operator-valued Random Fourier Feature.

**Proposition 13.** *If one can find  $B : \hat{\mathcal{X}} \rightarrow \mathcal{L}(\mathcal{Y})$  and a probability measure  $\mu$  on  $\mathcal{B}(\hat{\mathcal{X}})$ , such that for all  $y, y' \in \mathcal{Y}$ ,  $\langle y, B(\cdot)B(\cdot)^* y' \rangle \in L^1(\hat{\mathcal{X}}, d\mu)$ , then the operator-valued function*

$$\tilde{\Phi}(x) = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D (x, \omega_j) B(\omega_j)^*, \quad \omega_j \sim \mu \quad (13)$$

*is an approximated feature map of an Operator-Valued Kernel<sup>3</sup>.*

*Proof.* Let  $\omega_1, \dots, \omega_D$  be  $D$  i.i.d. random vectors following the law  $\mu$ . For all  $x, z \in \mathcal{X}$  and all  $y, y' \in \mathcal{Y}$ ,

$$\begin{aligned} & \langle \tilde{\Phi}(x)y, \tilde{\Phi}(z)y' \rangle \\ &= \left\langle y, \left( \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D (z, \omega_j) B(\omega_j)^* \right)^* \left( \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D (x, \omega_j) B(\omega_j)^* \right) y' \right\rangle \\ &= \frac{1}{D} \sum_{j=1}^D \overline{(x \star z^{-1}, \omega_j)} A(\omega_j), \end{aligned}$$

where  $A(\omega) = B(\omega)B(\omega)^*$ . By assumption  $\langle y, A(\cdot)y' \rangle \in L^1(\hat{\mathcal{X}}, \mu)$  and  $\omega_j$  are i.i.d.. Hence from the strong law of large numbers and proposition 7,

$$\frac{1}{D} \sum_{j=1}^D \overline{(x \star z^{-1}, \omega_j)} A(\omega_j) \xrightarrow[D \rightarrow \infty]{\text{a.s.}} \mathbf{E}_\mu[\overline{(x \star z^{-1}, \omega)} A(\omega)] = K_e(x \star z^{-1})$$

in the weak operator topology.  $\square$

The approximate feature map proposed in proposition 13 has direct link with the functional feature map defined in proposition 8 since we have

$$\tilde{\Phi}(x)y = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D (\phi_x y)(\omega_j). \quad (14)$$

<sup>3</sup> I.e. it satisfies  $\tilde{\Phi}(x)^* \tilde{\Phi}(z) \xrightarrow[D \rightarrow \infty]{\text{a.s.}} K(x, z)$  where  $K$  is a  $\mathcal{Y}$ -Mercer OVK.

Therefore  $\tilde{\phi}(x)$  can be seen as an “operator-valued vector” corresponding the “stacking” of i.i.d. operator-valued realization of  $\phi_x$ , the functional feature map. Therefore, in light of eq. (11), it is possible to define an approximate feature map of an operator-valued kernel.

**Corollary 14.** *If  $K(x, z)$  is a shift-invariant  $\mathcal{Y}$ -Mercer kernel such that for all  $y, y' \in \mathcal{Y}$ ,  $\langle y, K_e(\delta)y' \rangle \in L^1(\mathcal{X}, dx)$ . Then*

$$\tilde{\phi}(x) = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D (x, \omega_j) B(\omega_j)^*, \quad \omega_j \sim \mu, \quad (15)$$

where  $\langle y, B(\omega)B(\omega)^*y' \rangle p_\mu(\omega) = \mathcal{F}^{-1}[\langle y, K_e(\cdot)y' \rangle](\omega)$ , is an approximated feature map of  $K$ .

*Proof.* Find  $(A(\omega), d\mu)$  from proposition 10 and apply proposition 13.  $\square$

We write  $\tilde{\phi}(x)^* \tilde{\phi}(x) \approx K(x, z)$  when  $\tilde{\phi}(x)^* \tilde{\phi}(x) \xrightarrow{\text{a.s.}} K(x, z)$  in the weak operator topology when  $D$  tends to infinity. With mild abuse of notation we say that  $\tilde{\phi}(x)$  is an approximate feature map of  $\phi_x$  i.e.  $\tilde{\phi}(x) \approx \phi_x$ , when for all  $y \in \mathcal{Y}$ ,  $\langle y, K(x, z)y' \rangle = \langle \phi_x y, \phi_z y' \rangle \approx \langle \tilde{\phi}(x)y, \tilde{\phi}(z)y' \rangle := \tilde{K}(x, z)$  where  $\phi_x$  is defined in the sense of proposition 8.

The kernel approximation  $\tilde{K}$  can be seen as the sample mean of the product of functional feature map. Indeed  $\tilde{K} = 1/D \sum_{j=1}^D$ .

**Remark 15.** *We find a decomposition such that for all  $j = 1, \dots, D$ ,  $A(\omega_j) = B(\omega_j)B(\omega_j)^*$  either by exhibiting an analytic closed-form or using a numerical decomposition.*

Corollary 14 allows us to define line 1 for constructing ORFF from an operator valued kernel.

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**Algorithm 1:** Construction of ORFF from OVK

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**Input** :  $K(x, z) = K_e(\delta)$  a  $\mathcal{Y}$ -shift-invariant Mercer kernel such that  $\forall y, y' \in \mathcal{Y}$ ,  $\langle y, K_e(\delta)y' \rangle \in L^1(\mathbb{R}^d, dx)$ .

**Output:** A random feature  $\tilde{\phi}(x)$  such that  $\tilde{\phi}(x)^* \tilde{\phi}(z) \approx K(x, z)$

- 1 Define the pairing  $(x, \omega)$  from the LCA group  $(\mathcal{X}, \star)$ ;
  - 2 Find a decomposition  $(B(\omega), p_\mu(\omega))$  such that  $B(\omega)B(\omega)^* p_\mu(\omega) = \mathcal{F}^{-1}[K_e](\omega)$ ;
  - 3 Draw  $D$  random vectors  $\omega_j, j = 1, \dots, D$  from the probability distribution  $\mu$ ;
  - 4 **return**  $\tilde{\phi}(x) = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D (x, \omega_j) B(\omega_j)^*$ ;
-

### 3.2.6 Examples of Operator Random Fourier Feature maps

We now give two examples of operator-valued random Fourier feature map when  $\mathcal{Y} \subset \mathbb{R}^p$ . First we introduce the general form of an approximated feature map for a matrix-valued kernel on the additive group  $(\mathbb{R}^d, +)$ .

**Example 1** (Matrix-valued kernel on the additive group). *In the following,  $K(x, z) = K_0(x - z)$  is a  $\mathbb{R}^p$ -Mercer matrix-valued kernel invariant w.r.t. the group operation  $+$ . An Operator Random Fourier feature map of an  $\mathbb{R}^p$ -Mercer shift-invariant matrix-valued kernel takes the general form:*

$$\tilde{\Phi}(x) = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D \begin{pmatrix} \cos \langle x, \omega_j \rangle B(\omega_j)^* \\ \sin \langle x, \omega_j \rangle B(\omega_j)^* \end{pmatrix}, \quad \omega_j \sim \mu.$$

*Proof.* The (Pontryagin) dual of  $\mathcal{X} = \mathbb{R}^d$  is  $\hat{\mathcal{X}} = \mathbb{R}^d$ , and the duality pairing is  $\langle x - z, \omega \rangle = \exp(i \langle x - z, \omega \rangle)$ . A  $\mathbb{R}^p$ -operator-valued function has a real operator-valued Fourier transform if and only if  $A(\omega)$  is even with respect to  $\omega$ . Taking this point into account, the kernel approximation yields:

$$\begin{aligned} \tilde{K}(x, z) &= \tilde{\Phi}(x)^* \tilde{\Phi}(z) \\ &= \frac{1}{D} \sum_{j=1}^D \cos \langle x, \omega_j \rangle \cos \langle z, \omega_j \rangle A(\omega_j) + \sin \langle x, \omega_j \rangle \sin \langle z, \omega_j \rangle A(\omega_j) \\ &= \frac{1}{D} \sum_{j=1}^D \cos \langle x - z, \omega_j \rangle A(\omega_j) \\ &= \frac{1}{D} \sum_{j=1}^D \exp(-i \langle x - z, \omega_j \rangle) A(\omega_j). \end{aligned}$$

which tends to  $\mathbf{E}_\mu[\exp(-i \langle x - z, \omega \rangle) A(\omega)] = \mathbf{E}_\mu[\overline{\langle x - z, \omega \rangle} A(\omega)] = K(x, z)$  when  $D$  tends to infinity.  $\square$

The second example extends scalar-valued Random Fourier Features on the skewed multiplicative group described in ?? [10]) to the operator-valued case.

**Example 2** (Matrix-valued kernel on the skewed multiplicative group). *In the following, suppose that  $\mathcal{X} = (-c; +\infty)^d$ ,  $\mathcal{Y} = \mathbb{R}^p$  and  $K(x, z) = K_{1-c}(x \odot z^{-1})$  is a  $\mathbb{R}^p$ -Mercer matrix-valued kernel invariant w.r.t. the group operation  $\odot$  defined in ??. The group operation is defined coefficient-wise for all  $k \in \{1, \dots, d\}$  as  $x_k \odot z_k := (x_k + c)(z_k + c) - c$ . As a consequence  $z_k^{-1} = 1/(z_k + c) - c$ . The following function  $\tilde{\Phi}$  is an operator-valued Random Fourier Feature map built following the construction principle:*

$$\tilde{\Phi}(x) = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D \begin{pmatrix} \cos \langle \log(x + c), \omega_j \rangle B(\omega_j)^* \\ \sin \langle \log(x + c), \omega_j \rangle B(\omega_j)^* \end{pmatrix}, \quad \omega_j \sim \mu.$$

*Proof.* The dual of  $\mathcal{X} = (-c; +\infty)^d$  is  $\hat{\mathcal{X}} = \mathbb{R}^d$ , and the duality pairing is  $(x \odot z^{-1}, \omega) = \exp(i\langle \log(x \odot z^{-1} + c), \omega \rangle)$  (see Li, Ionescu, and Sminchisescu [11]). Following the proof of example 1, we have

$$\tilde{K}(x, z) = \frac{1}{D} \sum_{j=1}^D e^{i\langle \log(\frac{x+c}{z+c}), \omega_j \rangle} A(\omega_j).$$

which tends to  $\mathbf{E}_{\mu}[\exp(-i\langle \log(x \odot z^{-1} + c), \omega \rangle) A(\omega)] = \mathbf{E}_{\mu}[\overline{(x \odot z^{-1}, \omega)} A(\omega)] = K(x, z)$  when  $D$  tends to infinity.  $\square$

### 3.3 UNIFORM BOUND ON THE APPROXIMATION

### 3.4 LEARNING WITH OPERATOR-VALUED RANDOM-FOURIER FEATURES

### 3.5 CONSISTENCY AND GENERALIZATION BOUNDS

### 3.6 CONCLUSIONS





#### 4.1 BACKGROUND

#### 4.2 THE NYSTRÖM METHOD

#### 4.3 SUB-SAMPLING THE DATA

#### 4.4 CONCLUSIONS

## Part III

### FINAL WORDS

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## CONCLUSIONS

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## Part IV

### APPENDIX







## OPERATOR-VALUED FUNCTIONS AND INTEGRATION

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## DECLARATION

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Put your declaration here.

*15, Rue Plumet, 75015 - Paris, France, Septembre 2016*

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Romain Brault





## COLOPHON

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