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Entitled
DATA ARE NOT REAL!

Presented by
ROMAIN BRAULT *

About
**LARGE-SCALE LEARNING ON STRUCTURED INPUT-OUTPUT
DATA WITH OPERATOR-VALUED KERNELS .**

Under supervision of
Professor (Prof.) FLORENCE D'ALCHÉ-BUC †



M. John DOE	UEVE	examinator,
M. John DOE	UEVE	director,
M. John DOE	UEVE	examinator,
M. John DOE	UEVE	examinator,
M. John DOE	UEVE	reporter,

accepted on the recommendation of M. John DOE (UEVE) and M. John DOE (UEVE).

Computer Science
IBISC
Université d'Évry val d'Essonne

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* Email: romain.brault@ibisc.fr

† Email: florence.dalche@telecom-paristech.fr

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SUPERVISOR:

Professor (Prof.) Florence d'Alché-Buc

LOCATION:

15, Rue Plumet, 75015 - Paris, France

ABSTRACT

In this thesis we study scalable methods to perform regression with *Operator-Valued Kernels* in order to learn *vector-valued functions*.

When data present structure, or relations between them or their different components, a common approach is to treat the data as a vector living in an appropriate vector space. Thus many problems in machine learning can be cast into learning vector-valued functions. Operator-Valued Kernels *Operator-Valued Kernels* and *vector-valued Reproducing Kernel Hilbert Spaces* provide a theoretical and practical framework to address that issue, naturally extending the well-known framework of scalar-valued kernels.

In the context of scalar-valued function learning, a scalar-valued kernel can be seen as a similarity measure between two data points. A solution of the learning problem has the form of a linear combination of these similarities with respect to unknown weights to be found in order to have the best “fit” of the data. When dealing with Operator-Valued Kernels, the evaluation of the kernel is no longer a scalar similarity, but a function acting on vectors. A solution is then a linear combination of operators with respect to vector weights.

Although Operator-Valued Kernels generalize strictly scalar-valued kernels, large scale applications are usually not affordable with these tools that require an important computational power along with a large memory capacity. In this thesis, we propose and study scalable methods to perform regression with *Operator-Valued Kernels*. To achieve this goal, we extend Random Fourier Features, an approximation technique originally introduced for scalar-valued kernels, to *Operator-Valued Kernels*. The idea is to take advantage of an approximated operator-valued feature map in order to come up with a linear model in a finite dimensional space.

This thesis is structured as follows. First we develop a general framework devoted to the approximation of shift-invariant Mercer kernels on Locally Compact Abelian groups and study their properties along with the complexity of the algorithms based on them. Second we show theoretical guarantees by bounding the error due to the approximation, with high probability. Third, we study various applications of Operator Random Fourier Features to different tasks of Machine learning such as multi-class classification, multi-task learning, time series modeling, functional regression and anomaly detection. We also compare the proposed framework with other state of the art methods. Fourth, we conclude by drawing short-term and mid-term perspectives of this work.

PUBLICATIONS

Some ideas and figures have appeared previously in the following publications:

Put your publications from the thesis here. The packages `multibib` or `bibtopic` etc. can be used to handle multiple different bibliographies in your document.

*We have seen that computer programming is an art,
because it applies accumulated knowledge to the world,
because it requires skill and ingenuity, and especially
because it produces objects of beauty.*

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¹ Members of GuIT (Gruppo Italiano Utilizzatori di T_EX e L^AT_EX)

CONTENTS

I	INTRODUCTION	I
I	MOTIVATIONS	3
2	BACKGROUND	5
2.1	Notations	6
2.2	About statistical learning	6
2.3	On large-scale learning	6
2.4	History and state of the art of large scale learning with kernels	6
2.4.1	Introduction to kernel methods	6
2.4.2	Quadratic programing, subsampling	6
2.4.3	Gradient descents	6
2.4.4	Mercer Theorem, Nyström method and feature maps	6
2.4.5	Recent extensions	6
2.5	Elements of abstract harmonic analysis	6
2.5.1	Locally compact Abelian groups	6
2.5.2	The Haar measure	6
2.5.3	Even and odd functions	8
2.5.4	Characters	9
2.5.5	The Fourier transform	11
2.5.6	Representation of Groups	11
2.6	On operator-valued kernels	12
2.6.1	Definitions and properties	12
2.6.2	Shift-Invariant operator-valued kernels	18
2.6.3	Examples of operator-valued kernels	19
II	CONTRIBUTIONS	21
3	OPERATOR-VALUED RANDOM FOURIER FEATURES	23
3.1	Motivations	24
3.2	Construction	25
3.2.1	Theoretical study	25
3.2.2	Sufficient conditions of existence	26
3.2.3	Examples of spectral decomposition	31
3.2.4	Functional Fourier feature map	36
3.2.5	Building Operator-valued Random Fourier Fea- tures	37
3.2.6	Examples of Operator Random Fourier Feature maps	42
3.2.7	Regularization property	45
3.2.8	Operator Random Feature engineering	47
3.3	Learning with ORFF	48
3.3.1	Warm-up: supervised regression	48

3.3.2	Semi-supervised regression	49
3.4	Solution of the empirical risk minimization	56
3.4.1	Gradient methods	56
3.4.2	Efficient linear operators	62
3.4.3	Decomposable kernel	62
3.4.4	Curl-free kernel	62
3.4.5	Divergence-free kernel	62
3.5	Conclusions	63
4	CONSISTENCY AND GENERALIZATION	65
4.1	Consistency of the ORFF estimator	66
4.1.1	Scalar random Fourier Features and decomposable kernel	67
4.1.2	Uniform convergence of ORFF approximation on LCA groups	68
4.1.3	Variance of the ORFF approximation	70
4.1.4	Application on decomposable, curl-free and divergence-free OVKs	71
5	APPLICATIONS	73
5.1	Introduction	74
5.2	Time series modelling	74
5.3	Functional data analysis	74
5.4	Neural networks, deep learning	74
5.5	Conclusions	74
III	FINAL WORDS AND PERSPECTIVES	75
6	CONCLUSIONS	77
IV	APPENDIX	79
A	OPERATOR-VALUED FUNCTIONS AND INTEGRATION	81
B	PROOFS OF THEOREMS	83
B.1	Proof of	83
	BIBLIOGRAPHY	85

LIST OF FIGURES

Figure 1	Different outcomes of a Gaussian kernel approximation	39
Figure 2	Relationships between feature-maps.	43
Figure 3	ORFF Representer theorem	61
Figure 4	Error reconstructing the target operator-valued kernel k with ORFF approximation \tilde{K} for the decomposable, curl-free and divergence-free kernel.	66

LIST OF TABLES

Table 1	Mathematical symbols used throughout the paper and their signification.	7
Table 2	Classification of Fourier transforms in terms of their domain and transform domain.	10

LISTINGS

ACRONYMS

MSE	Mean Squared Error.
OVK	Operator-Valued Kernel.
RFF	Random Fourier Feature.
ORFF	Operator-valued Random Fourier Feature.
POVM	Positive Operator-Valued Measure.
RKHS	Reproducing Kernel Hilbert Space.
\mathcal{Y} -RKHS	\mathcal{Y} -Reproducing Kernel Hilbert Space.
LCA	Locally Compact Abelian.
FT	Fourier transform.
IFT	Inverse Fourier transform.

Part I

INTRODUCTION

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MOTIVATIONS



BACKGROUND

2.1 NOTATIONS

The euclidean inner product in \mathbb{R}^d is denoted $\langle \cdot, \cdot \rangle$ and the euclidean norm is denoted $\|\cdot\|$. The unit pure imaginary number $\sqrt{-1}$ is denoted i . $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ -algebra on \mathbb{R}^d . If \mathcal{X} and \mathcal{Y} are two vector spaces, we denote by $\mathcal{F}(\mathcal{X}; \mathcal{Y})$ the vector space of functions $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $\mathcal{C}(\mathcal{X}; \mathcal{Y}) \subset \mathcal{F}(\mathcal{X}; \mathcal{Y})$ the subspace of continuous functions. If \mathcal{H} is an Hilbert space we denote its scalar product by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and its norm by $\|\cdot\|_{\mathcal{H}}$. We set $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}; \mathcal{H})$ to be the space of linear operators from \mathcal{H} to itself. If $W \in \mathcal{L}(\mathcal{H})$, $\text{Ker } W$ denotes the nullspace, $\text{Im } W$ the image and $W^* \in \mathcal{L}(\mathcal{H})$ the adjoint operator (transpose when W is a real matrix). All these notations are summarized in table 1.

2.2 ABOUT STATISTICAL LEARNING

2.3 ON LARGE-SCALE LEARNING

2.4 HISTORY AND STATE OF THE ART OF LARGE SCALE LEARNING WITH KERNELS

2.4.1 *Introduction to kernel methods*2.4.2 *Quadratic programing, subsampling*2.4.3 *Gradient descents*2.4.4 *Mercer Theorem, Nyström method and feature maps*2.4.5 *Recent extensions*

2.5 ELEMENTS OF ABSTRACT HARMONIC ANALYSIS

2.5.1 *Locally compact Abelian groups*

Definition 2.1 *Locally Compact Abelian group. A group \mathcal{X} endowed with a binary operation \star is said to be Locally Compact Abelian if \mathcal{X} is a topological commutative group w. r. t. \star for which every point has a compact neighborhood and is Hausdorff.*

2.5.2 *The Haar measure*

Definition 2.2 (The Haar measure). *There is, up to a positive multiplicative constant, a unique countably additive, nontrivial measure **Haar** on the Borel subsets of \mathcal{X} satisfying the following properties¹:*

¹ For more details and constructive proofs see [1, 14, 17].

Table 1: Mathematical symbols used throughout the paper and their significance.

Symbol	Meaning
i	Unit pure imaginary number $\sqrt{-1}$.
e	Euler constant.
$\langle \cdot, \cdot \rangle$	Euclidean inner product.
$\ \cdot\ $	Euclidean norm.
\mathcal{X}	Input space.
$\widehat{\mathcal{X}}$	The Pontryagin dual of \mathcal{X} .
\mathcal{Y}	Output space (Hilbert space).
\mathcal{H}	Feature space (Hilbert space).
$\langle \cdot, \cdot \rangle_{\mathcal{Y}}$	The canonical inner product of the Hilbert space \mathcal{Y} .
$\ \cdot\ _{\mathcal{Y}}$	The canonical norm induced by the inner product of the Hilbert space \mathcal{Y} .
$\mathcal{F}(\mathcal{X}; \mathcal{Y})$	Vector space of function from \mathcal{X} to \mathcal{Y} .
$\mathcal{C}(\mathcal{X}; \mathcal{Y})$	The vector subspace of \mathcal{F} of continuous function from \mathcal{X} to \mathcal{Y} .
$\mathcal{L}(\mathcal{H}; \mathcal{Y})$	The set of bounded linear operator from a Hilbert space \mathcal{H} to a Hilbert space \mathcal{Y} .
$\mathcal{L}(\mathcal{Y})$	The set of bounded linear operator from a Hilbert space \mathcal{H} to itself.
$\mathcal{L}_+(\mathcal{Y})$	The set of non-negative bounded linear operator from a Hilbert space \mathcal{H} to itself.
$\mathcal{B}(\mathcal{X})$	Borel σ -algebra on \mathcal{X} .
$\mu(\mathcal{X})$	A scalar positive measure of \mathcal{X} .
$L^p(\mathcal{X}, \mu)$	The Banach space of $ \cdot ^p$ -integrable function from $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mu)$ to \mathbb{C} .
$L^p(\mathcal{X}, \mu; \mathcal{Y})$	The Banach space of $\ \cdot\ _{\mathcal{Y}^p}$ (Bochner)-integrable function from $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mu)$ to \mathcal{Y} .

1. The measure **Haar** is translation-invariant.

$$\mathbf{Haar}(x \star E) = \mathbf{Haar}(E),$$

for every x in \mathcal{X} and Borel set $E \in \mathcal{B}(\mathcal{X})$.

2. The measure **Haar** is finite on every compact set.

$$\mathbf{Haar}(K) < \infty$$

for any compact set $K \in \mathcal{B}(\mathcal{X})$.

3. The measure **Haar** is outer regular on any Borel sets E .

$$\mathbf{Haar}(E) = \inf \{ \mathbf{Haar}(U) \mid E \subseteq U \}$$

For any open set U .

4. The measure **Haar** is inner regular on open sets E .

$$\mathbf{Haar}(E) = \sup \{ \mathbf{Haar}(K) \mid K \subseteq E \},$$

for any compact set K .

² If \mathcal{X} was not supposed to be Abelian, we should have defined a left Haar measure and a right Haar measure. In our case both measures are the same, so we refer to both of them as Haar measure

Such a measure on G is called a Haar measure². An immediate consequence of the invariance is that for any $c \in \mathcal{X}$,

$$\int_{\mathcal{X}} f(s \star x) d\mathbf{Haar}(x) = \int_{\mathcal{X}} f(x) d\mathbf{Haar}(x).$$

It can be shown that $\mathbf{Haar}(U) > 0$ for every non-empty open subset U . In particular, if \mathcal{X} is compact then $\mathbf{Haar}(\mathcal{X})$ is finite and positive, so we can uniquely specify a left Haar measure on \mathcal{X} by adding the normalization condition $\mathbf{Haar}(\mathcal{X}) = 1$. Another useful property is that given a continuous function $f \in \mathcal{C}(\mathcal{X})$,

$$\int_{\mathcal{X}} f(x) d\mathbf{Haar}(x) = \int_{\mathcal{X}} f(x^{-1}) d\mathbf{Haar}(x).$$

We call measured space the space $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbf{Haar})$ the space \mathcal{X} endowed with its Borel sigma algebra and some measure **Haar**. If $\mathbf{Haar}(\mathcal{X}) = 1$ then the space $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbf{Haar})$ is called a probability space.

2.5.3 Even and odd functions

We say that a function $f : \mathcal{X} \rightarrow \mathcal{C}$ is even if for all $x \in \mathcal{X}$, $f(x) = f(x^{-1})$ and odd if $f(x) = -f(x^{-1})$. The definition can be extended to operator-valued functions.

Definition 2.3 (Even and odd function on a LCA group). Let \mathcal{X} be a measured LCA group. A function $f : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y})$ is (weakly) even if for all $x \in \mathcal{X}$ and all $y, y' \in \mathcal{Y}$

$$\langle y, f(x^{-1}) y' \rangle = \langle y, f(x) y' \rangle \quad (2.1)$$

and (weakly) odd if

$$\langle y, f(x^{-1}) y' \rangle = -\langle y, f(x) y' \rangle \quad (2.2)$$

It is easy to check that if f is odd then $\int_{\mathcal{X}} \langle y, f(x) y' \rangle d\mathbf{Haar}(x) = 0$.

Proof

$$\begin{aligned}
\int_{\mathcal{X}} \langle y, f(x)y' \rangle d\mathbf{Haar}(x) &= \frac{1}{2} \left(\int_{\mathcal{X}} \langle y, f(x^{-1})y' \rangle d\mathbf{Haar}(x) \right. \\
&\quad \left. + \int_{\mathcal{X}} \langle y, f(x)y' \rangle d\mathbf{Haar}(x) \right) \\
&= \frac{1}{2} \left(- \int_{\mathcal{X}} \langle y, f(x)y' \rangle d\mathbf{Haar}(x) \right. \\
&\quad \left. + \int_{\mathcal{X}} \langle y, f(x)y' \rangle d\mathbf{Haar}(x) \right) \\
&= 0.
\end{aligned}$$

Besides the product of an even and an odd function is odd. Indeed for all $f, g \in \mathcal{F}(\mathcal{X}; \mathcal{L}(\mathcal{Y}))$, where f is even and g odd. Define $h(x) = \langle y, f(x)g(x)y' \rangle$. Then we have

$$\begin{aligned}
h(x^{-1}) &= \langle y, f(x^{-1})g(x^{-1})y' \rangle \\
&= \langle y, f(x)(-g(x))y' \rangle \\
&= -h(x).
\end{aligned} \tag{2.3}$$

2.5.4 Characters

Locally Compact Abelian (LCA) groups are central to the general definition of Fourier Transform which is related to the concept of Pontryagin duality [17]. Let (\mathcal{X}, \star) be a LCA group with e its neutral element and the notation, x^{-1} , for the inverse of $x \in \mathcal{X}$. A *character* is a complex continuous homomorphism $\omega : \mathcal{X} \rightarrow \mathbb{U}$ from \mathcal{X} to the set of complex numbers of unit module \mathbb{U} . The set of all characters of \mathcal{X} forms the Pontryagin *dual group* $\widehat{\mathcal{X}}$. The dual group of an LCA group is an LCA group so that we can endow $\widehat{\widehat{\mathcal{X}}}$ with a “dual” Haar measure noted $\widehat{\mathbf{Haar}}$. Then the dual group operation is defined by

$$(\omega_1 \star \omega_2)(x) = \omega_1(x)\omega_2(x) \in \mathbb{U}.$$

The Pontryagin duality theorem states that $\widehat{\widehat{\mathcal{X}}} \cong \mathcal{X}$. I. e. there is a canonical isomorphism between any LCA group and its double dual. To emphasize this duality the following notation is usually adopted

$$\omega(x) = (x, \omega) = (\omega, x) = x(\omega), \tag{2.4}$$

where $x \in \mathcal{X} \cong \widehat{\widehat{\mathcal{X}}}$ and $\omega \in \widehat{\mathcal{X}}$. Another important property involves the complex conjugate of the pairing which is defined as

$$\overline{(x, \omega)} = (x^{-1}, \omega) = (x, \omega^{-1}). \tag{2.5}$$

Table 2: Classification of Fourier transforms in terms of their domain and transform domain.

$\mathcal{X} =$	$\widehat{\mathcal{X}} \cong$	Operation	Pairing
\mathbb{R}^d	\mathbb{R}^d	$+$	$(x, \omega) = \exp(i\langle x, \omega \rangle)$
$\mathbb{R}_{*,+}^d$	\mathbb{R}^d	\cdot	$(x, \omega) = \exp(i\langle \log(x), \omega \rangle)$
$(-c; +\infty)^d$	\mathbb{R}^d	\odot	$(x, \omega) = \exp(i\langle \log(x+c), \omega \rangle)$

We notice that for any pairing depending of ω , there exists a function $h_\omega : \mathcal{X} \rightarrow \mathbb{R}$ such that $(x, \omega) = \exp(ih_\omega(x))$ since any pairing maps into \mathbb{U} . Moreover,

$$\begin{aligned}
 (x \star z^{-1}, \omega) &= \omega(x)\omega(z^{-1}) \\
 &= \exp(+ih_\omega(x)) \exp(+ih_\omega(z^{-1})) \\
 &= \exp(+ih_\omega(x)) \exp(-ih_\omega(z)).
 \end{aligned}$$

Example 2.1 $\widehat{\mathbb{R}} \cong \mathbb{R}$ with the duality pairing $(x, \omega) = \exp(ix\omega)$ for all $x \in \mathbb{R}$ and all $\omega \in \mathbb{R}$.

Proof If $\omega \in \widehat{\mathbb{R}}$ then $\omega(0) = 1$ since ω is an homeomorphism from \mathbb{R} to \mathbb{U} . Therefore there exists $a > 0$ such that $\int_0^a \omega(t) d\text{Leb}(t) \neq 0$. Setting $A\omega = \int_0^a \omega(t) d\text{Leb}(t)$ we have

$$(A\omega)(x) = \int_0^a \omega(x+t) d\text{Leb}(t) = \int_x^{a+x} \omega(t) d\text{Leb}(t).$$

so ω is differentiable and

$$\omega'(x) = A^{-1}(\omega(a+x) - \omega(x)) = c\omega(x) \quad \text{where} \quad c = A^{-1}(\omega(a) - 1).$$

It follow that $\omega(x) = e^{cx}$, and since $|\omega| = 1$, one can take $c = i\xi$ for some $\xi \in \mathbb{R}$. Hence we can identify ω with ξ , and $\widehat{\mathbb{R}}$ with \mathbb{R} since ξ uniquely determines ω . With mild abuse of notation we identify $\omega = \xi$. \square

Table 2 provide an explicit list of pairings for various groups based on \mathbb{R}^d or its subsets. We especially mention the duality pairing associated to the skewed multiplicative [LCA](#) product group $\mathcal{X} = (-c_k; +\infty)_{k=1}^d$ endowed with the group operation \odot defined component-wise for all $x, z \in \mathcal{X}$ as follow.

$$x \odot z := ((x_k + c)(z_k + c) - c)_{k=1}^d.$$

Then the duality pairing is defined by

$$(x, \omega) = \exp\left(i \sum_{k=1}^d \omega_k \log(x_k + c_k)\right).$$

Hence $h_\omega(x) = \sum_{k=1}^d \omega_k \log(x_k + c_k)$. This group together with the operation \odot has been proposed by [\[23\]](#) to handle histograms features especially useful in image recognition applications.

2.5.5 The Fourier transform

For a function with values in a separable Hilbert space $f \in L^1(\mathcal{X}, \mathbf{Haar}; \mathcal{Y})$, we denote $\mathcal{F}[f]$ its Fourier transform (FT) which is defined by

$$\forall \omega \in \widehat{\mathcal{X}}, \quad \mathcal{F}[f](\omega) = \int_{\mathcal{X}} \overline{\langle x, \omega \rangle} f(x) d\mathbf{Haar}(x).$$

For a measure \mathbf{Haar} defined on \mathcal{X} , there exists a unique suitably normalized measure $\widehat{\mathbf{Haar}}$ on $\widehat{\mathcal{X}}$ such that for all $f \in L^1(\mathcal{X}, \mathbf{Haar}; \mathcal{Y})$ and if $\mathcal{F}[f] \in L^1(\widehat{\mathcal{X}}, \widehat{\mathbf{Haar}}; \mathcal{Y})$ we have

$$\forall x \in \mathcal{X}, \quad f(x) = \int_{\widehat{\mathcal{X}}} \mathcal{F}[f](\omega) \langle x, \omega \rangle d\widehat{\mathbf{Haar}}(\omega). \quad (2.6)$$

Moreover if $\widehat{\mathbf{Haar}}$ is normalized, \mathcal{F} extends to a unitary operator from $L^2(\mathcal{X}, \mathbf{Haar}; \mathcal{Y})$ onto $L^2(\widehat{\mathcal{X}}, \widehat{\mathbf{Haar}}; \mathcal{Y})$. Then the Inverse Fourier transform (IFT) of a function $g \in L^1(\widehat{\mathcal{X}}, \widehat{\mathbf{Haar}}; \mathcal{Y})$ (where $\widehat{\mathbf{Haar}}$ is suitably normalized w.r.t. \mathbf{Haar}) is noted $\mathcal{F}^{-1}[g]$ defined by

$$\forall x \in \mathcal{X}, \quad \mathcal{F}^{-1}[g](x) = \int_{\widehat{\mathcal{X}}} \langle x, \omega \rangle g(\omega) d\widehat{\mathbf{Haar}}(\omega),$$

Equation (2.5) gives some examples of real Abelian groups with their associated dual and pairing. The interested reader can refer to Folland [17] for a more detailed construction of LCA, Pontryagin duality and Fourier transforms on LCA. For the familiar case of a scalar-valued function f on the LCA group $(\mathbb{R}^d, +)$, we have for all $\omega \in \widehat{\mathcal{X}}$

$$\begin{aligned} \mathcal{F}[f](\omega) &= \int_{\widehat{\mathcal{X}}} \overline{\langle x, \omega \rangle} f(x) d\mathbf{Haar}(x) \\ &= \int_{\mathbb{R}^d} e^{-i\langle x, \omega \rangle} f(x) d\mathbf{Leb}(x), \end{aligned} \quad (2.7)$$

the Haar measure being here the Lebesgue measure.

2.5.6 Representation of Groups

Representations of groups are convenient tools that allows group-theoretic problems to be replaced by linear algebra problems. Let $GL(\mathcal{H})$ be the group of continuous isomorphism of \mathcal{H} , a Hilbert space, onto itself. A representation π of a LCA group \mathcal{X} in \mathcal{H} is an homomorphism π :

$$\pi : \mathcal{X} \rightarrow GL(\mathcal{H})$$

for which all the maps $\mathcal{X} \rightarrow \mathcal{H}$ defined for all $v \in \mathcal{H}$ as $x \mapsto \pi(x)v$, are continuous. The space \mathcal{H} in which the representation takes place is called the representation space of π . A representation π of a group \mathcal{X} in a vector space \mathcal{H} defines an action defined for all $x \in \mathcal{X}$ by

$$\pi_x : \begin{cases} \mathcal{H} & \rightarrow \mathcal{H} \\ v & \mapsto \pi(x)v. \end{cases}$$

If for all $x \in \mathcal{X}$, $\pi(x)$ is a unitary operator, then the group representation π is said to be unitary (i. e. $\forall x \in \mathcal{X}$, $\pi(x)$ is isometric and surjective). Thus π is unitary when for all $x \in \mathcal{X}$

$$\pi(x)^* = \pi(x)^{-1} = \pi(x^{-1}).$$

The representation π of \mathcal{X} in \mathcal{H} is said to be irreducible when $\mathcal{H} \neq \{0\}$ and are the only two stable invariant subspaces under all operators $\pi(x)$ for all $x \in \mathcal{X}$. I. e. $\forall \mathcal{U} \subset \mathcal{H}$, $\mathcal{U} \neq \{0\}$, $\{\pi(x)v \mid \forall x \in \mathcal{X}, \forall v \in \mathcal{U}\} \neq \mathcal{U}$.

To study LCA groups we also introduce the left regular representation of \mathcal{X} acting on a Hilbert space of function $\mathcal{H} \subset \mathcal{F}(\mathcal{X}; \mathcal{Y})$. For all $x, z \in \mathcal{X}$ and for all $f \in \mathcal{H}$,

$$(\lambda_z f)(x) := f(z^{-1} \star x).$$

The representation λ of \mathcal{X} defines an action λ_x on \mathcal{H} which is the translation of $f(x)$ by z^{-1} . With this definition one has for all $x, z \in \mathcal{X}$, $\lambda_x \lambda_z = \lambda_{x^{-1} \star z}$. Such representations are faithful, that is $\lambda_x = 1 \iff x = e$.

2.6 ON OPERATOR-VALUED KERNELS

We now introduce the theory of \mathcal{Y} -Reproducing Kernel Hilbert Space (\mathcal{Y} -RKHS) that provides a flexible framework to study and learn vector-valued functions.

2.6.1 Definitions and properties

A \mathcal{Y} -Reproducing Kernel Hilbert Space (\mathcal{Y} -RKHS) is a functional vector space defined as follow.

Definition 2.4 (\mathcal{Y} -Reproducing Kernel Hilbert Space [12, 26]). Let \mathcal{Y} be a (real or complex) Hilbert space. A \mathcal{Y} -Reproducing Kernel Hilbert Space on \mathcal{X} , a locally compact second countable topological space is a Hilbert space \mathcal{H} such that

1. the elements of \mathcal{H} are functions from \mathcal{X} to \mathcal{Y} (i. e. $\mathcal{H} \subset \mathcal{F}(\mathcal{X}, \mathcal{Y})$);
2. for all $x \in \mathcal{X}$, there exists a positive constant C_x such that for all $f \in \mathcal{H}$

$$\|f(x)\|_{\mathcal{Y}} \leq C_x \|f\|_{\mathcal{H}}. \quad (2.8)$$

An operator-valued kernel is defined here as a Operator-Valued Kernel of positive type Carmeli et al. [13].

Definition 2.5 (Operator-Valued Kernel of positive type acting on a complex space). Given \mathcal{X} a locally compact second countable topological space and \mathcal{Y} a complex Hilbert Space, a map $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y})$ is called an Operator-Valued Kernel of positive type kernel if

$$\sum_{i,j=1}^N \langle K(x_i, x_j) y_j, y_i \rangle_{\mathcal{Y}} \geq 0, \quad (2.9)$$

for all $N \in \mathbb{N}$ and for all sequences of points $(x_i)_{i=1}^N$ in \mathcal{X}^N and all sequences of points $(y_i)_{i=1}^N$ in \mathcal{Y}^N .

if \mathcal{Y} is a complex Hilbert space, an Operator-Valued Kernel of positive type is always Hermitian, i. e. $K(x, z) = K(z, x)^*$. This gives rise to the following definition of Operator-Valued Kernel of positive type acting on a real Hilbert space.

Definition 2.6 (Operator-Valued Kernel of positive type acting on a real Hilbert space). Given \mathcal{X} a locally compact second countable topological space and \mathcal{Y} a real Hilbert Space, a map $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y})$ is called an Operator-Valued Kernel of positive type kernel if

$$K(x, z) = K(z, x) \quad (2.10)$$

and

$$\sum_{i,j=1}^N \langle K(x_i, x_j) y_j, y_i \rangle_{\mathcal{Y}} \geq 0, \quad (2.11)$$

for all $N \in \mathbb{N}$ and for all sequences of points $(x_i)_{i=1}^N$ in \mathcal{X}^N , and all sequences of points $(y_i)_{i=1}^N$ in \mathcal{Y}^N .

As in the scalar case any \mathcal{Y} -Reproducing Kernel Hilbert Space defines a unique Operator-Valued Kernel of positive type and conversely an Operator-Valued Kernel of positive type defines a unique \mathcal{Y} -Reproducing Kernel Hilbert Space.

Proposition 2.1 Given a \mathcal{Y} -Reproducing Kernel Hilbert Space there is a unique Operator-Valued Kernel $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y})$ of positive type.

Proof Given $x \in \mathcal{X}$, eq. (2.8) ensure that the evaluation map at x defined as

$$ev_x : \begin{cases} \mathcal{H} \rightarrow \mathcal{Y} \\ f \mapsto f(x) \end{cases}$$

is a bounded operator and the Operator-Valued Kernel K associated to \mathcal{H} is defined as

$$K : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y}) \quad K(x, z) = ev_x ev_z^*.$$

Since for all $(x_i)_{i=1}^N$ in \mathcal{X}^N and all $(y_i)_{i=1}^N$ in \mathcal{Y}^N ,

$$\begin{aligned} \sum_{i,j=1}^N \langle K(x_i, x_j) y_j, y_i \rangle_{\mathcal{Y}} &= \sum_{i,j=1}^N \langle ev_{x_j}^* y_j, ev_{x_i}^* y_i \rangle_{\mathcal{Y}} \\ &= \left\langle \sum_{i=1}^N ev_{x_i}^* y_i, \sum_{i=1}^N ev_{x_i}^* y_i \right\rangle_{\mathcal{Y}} \\ &= \left\| \sum_{i=1}^N ev_{x_i}^* y_i \right\|_{\mathcal{Y}}^2 \geq 0, \end{aligned}$$

the map K is of positive type. \square

Given $x \in \mathcal{X}$, $K_x : \mathcal{Y} \rightarrow \mathcal{F}(\mathcal{X}; \mathcal{Y})$ denotes the linear operator whose action on a vector y is the function $K_x y \in \mathcal{F}(\mathcal{X}; \mathcal{Y})$ defined for all $z \in \mathcal{X}$ by $K_x = \text{ev}_x^*$. As a consequence we have that

$$K(x, z)y = \text{ev}_x \text{ev}_z^* y = K_x^* K_z y = (K_z y)(x). \quad (2.12)$$

Some direct consequences follows from the definition.

1. The kernel reproduces the value of a function $f \in \mathcal{H}$ at a point $x \in \mathcal{X}$ since for all $y \in \mathcal{Y}$ and $x \in \mathcal{X}$, $\text{ev}_x^* y = K_x y = K(\cdot, x)y$ so that

$$\langle f(x), y \rangle_y = \langle f, K(\cdot, x)y \rangle_{\mathcal{H}} = \langle K_x^* f, y \rangle_y. \quad (2.13)$$

2. The set $\{K_x y \mid \forall x \in \mathcal{X}, \forall y \in \mathcal{Y}\}$ is total in \mathcal{H} . Namely,

$$\left(\bigcup_{x \in \mathcal{X}} \text{Im } K_x \right)^\perp = \{0\}.$$

If $f \in (\bigcup_{x \in \mathcal{X}} \text{Im } K_x)^\perp$, then for all $x \in \mathcal{X}$, $f \in (\text{Im } K_x)^\perp = \text{Ker } K_x^*$, hence $f(x) = 0$ for all $x \in \mathcal{X}$ that is $f = 0$.

3. Finally for all $x \in \mathcal{X}$ and all $f \in \mathcal{H}$, $\|f(x)\|_y \leq \sqrt{\|K(x, x)\|_{\mathcal{Y}, \mathcal{Y}}} \|f\|_{\mathcal{H}}$. This come from the fact that $\|K_x\|_{\mathcal{Y}, \mathcal{H}} = \|K_x^*\|_{\mathcal{H}, \mathcal{Y}} = \sqrt{\|K(x, x)\|_{\mathcal{Y}, \mathcal{Y}}}$ and the operator norm is sub-multiplicative.

Additionally given an Operator-Valued Kernel of positive type, it defines a unique [y-RKHS](#).

Proposition 2.2 *Given an Operator-Valued Kernel of positive type there $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y})$, there is a unique \mathcal{Y} -Reproducing Kernel Hilbert Space \mathcal{H} on \mathcal{X} with reproducing kernel K .*

Proof Let $K_{x,y} = K(\cdot, x)y \in \mathcal{F}(\mathcal{X}; \mathcal{Y})$ and let

$$\mathcal{H}_0 = \text{span}\{K_{x,y} \mid \forall x \in \mathcal{X}, \forall y \in \mathcal{Y}\} \subset \mathcal{F}(\mathcal{X}; \mathcal{Y}).$$

If $f = \sum_{i=1}^N c_i K_{x_i, y_i}$ and $g = \sum_{i=1}^N d_i K_{z_i, y'_i}$ are elements of \mathcal{H}_0 we have that

$$\sum_{j=1}^N \overline{d_j} \langle f(z_j), y'_j \rangle_y = \sum_{i,j=1}^N c_i \overline{d_j} \langle K(z_j, x_i) y_i, y'_j \rangle_y = \sum_{i=1}^N c_i \langle y_i, g(x_i) \rangle_y,$$

so that the sesquilinear form on $\mathcal{H}_0 \times \mathcal{H}_0$

$$\langle f, g \rangle_{\mathcal{H}_0} = \sum_{i,j=1}^N c_i \overline{d_j} \langle K(z_j, x_i) y_i, y'_j \rangle_y$$

is well defined. Since K is an Operator-Valued Kernel of positive type, we have for all $f \in \mathcal{H}_0$ that $\langle f, f \rangle \geq 0$. Moreover the sesquilinear form is positive, so that it

is also Hermitian. Choosing $g = K_{x,y}$ in the above definition yields for all $x \in \mathcal{X}$, all $f \in \mathcal{H}_0$ and all $y \in \mathcal{Y}$

$$\langle f, K_{x,y} \rangle_{\mathcal{H}_0} = \langle f(x), y \rangle_{\mathcal{Y}}.$$

Besides if $f \in \mathcal{H}_0$ for all unitary vector $y \in \mathcal{Y}$, by the Cauchy-Schwartz inequality we have

$$\begin{aligned} |\langle f(x), y \rangle_{\mathcal{Y}}| &= |\langle f, K_{x,y} \rangle_{\mathcal{H}_0}| \leq \sqrt{\langle f, f \rangle_{\mathcal{H}_0}} \sqrt{\langle K_{x,y}, K_{x,y} \rangle_{\mathcal{H}_0}} \\ &= \sqrt{\langle f, f \rangle_{\mathcal{H}_0}} \sqrt{\langle K(x, x)y, y \rangle_{\mathcal{Y}}} \leq \sqrt{\langle f, f \rangle_{\mathcal{H}_0}} \sqrt{\|K(x, x)\|_{\mathcal{Y}, \mathcal{Y}}}, \end{aligned}$$

which implies that

$$\|f(x)\|_{\mathcal{Y}} \leq \|f\|_{\mathcal{H}_0} \sqrt{\|K(x, x)\|_{\mathcal{Y}, \mathcal{Y}}}$$

Therefore if $\langle f, f \rangle_{\mathcal{H}_0} = 0$ then $f = 0$. Eventually we deduce that $\langle \cdot, \cdot \rangle_{\mathcal{H}_0}$ is a scalar product on \mathcal{H}_0 . Hence \mathcal{H}_0 is a pre-Hilbert space. To make it a (complete) Hilbert space we need to take the completion of this space. Let \mathcal{H} be the completion of \mathcal{H}_0 . Moreover let $K_x : \mathcal{Y} \rightarrow \mathcal{H}$ where $K_x y = K_{x,y}$. By construction K_x is bounded and let $W : \mathcal{H} \rightarrow \mathcal{F}(\mathcal{X}; \mathcal{Y})$ where $(Wf)(x) = K_x^* f$. The operator W is injective. Indeed if $Wf = 0$ then for all $x \in \mathcal{X}$, $f \in \text{Ker } K_x^* = (\text{Im } f)^\perp$. since the set $\cup_{x \in \mathcal{X}} \text{Im } K_x = \{K_x y \mid \forall x \in \mathcal{X}, \forall y \in \mathcal{Y}\}$ generates by definition \mathcal{H}_0 , we have $f = 0$. Besides, as W is injective, we have for all $f_1, f_2 \in \mathcal{H}_0$ $(Wf_1)(x) = (Wf_2)(x) \implies f_1(x) = f_2(x)$ pointwise in \mathcal{H} so that we can identify \mathcal{H} with a subspace of $\mathcal{F}(\mathcal{X}; \mathcal{Y})$. Hence $K_x^* f = (Wf)(x) = f(x) = ev_x f$, showing that \mathcal{H} is a \mathcal{Y} -Reproducing Kernel Hilbert Space with reproducing kernel

$$K_{\mathcal{H}}(x, z)y = (ev_z^* y)(x) = K(x, z)y.$$

The uniqueness of \mathcal{H} comes from the uniqueness of the completion of \mathcal{H}_0 up to an isometry. \square

The above theorem holds also if \mathcal{Y} is a real vector space provided we add the assumption that K is symmetric, i. e. $K(x, z) = K(z, x)$.

Since an Operator-Valued Kernel of positive type defines a unique \mathcal{Y} -RKHS and conversely a \mathcal{Y} -RKHS defines a unique Operator-Valued Kernel, we denote the Hilbert space \mathcal{H} endowed with the scalar product $\langle \cdot, \cdot \rangle$ respectively \mathcal{H}_K and $\langle \cdot, \cdot \rangle_K$. From now we refer to Operator-Valued Kernel of positive type as Operator-Valued Kernel whether they act on complex or real Hilbert spaces. As a consequence, given K an Operator-Valued Kernel of positive type, define $K_x = K(\cdot, x)$ we have

$$K(x, z) = K_x^* K_z \quad \forall x, z \in \mathcal{X}, \quad (2.14a)$$

$$\mathcal{H}_K = \overline{\text{span}} \{ K_x y \mid \forall x \in \mathcal{X}, \forall y \in \mathcal{Y} \}. \quad (2.14b)$$

Another way to describe functions of \mathcal{H}_K consists in using a suitable feature map.

Proposition 2.3 (Feature Operator [13]). *Let \mathcal{H} be any Hilbert space and $\Phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y}; \mathcal{H})$, with $\Phi_x := \Phi(x)$. Then the operator $W : \mathcal{H} \rightarrow \mathcal{F}(\mathcal{X}; \mathcal{Y})$ defined for all $g \in \mathcal{H}$, and for all $x \in \mathcal{X}$ by $(Wg)(x) = \Phi_x^* g$ is a partial isometry from \mathcal{H} onto the \mathcal{Y} -RKHS \mathcal{H}_K with reproducing kernel*

$$K(x, z) = \Phi_x^* \Phi_z, \quad \forall x, z \in \mathcal{X}.$$

W^*W is the orthogonal projection onto

$$(\text{Ker } W)^\perp = \overline{\text{span}} \{ \Phi_x y \mid \forall x \in \mathcal{X}, \forall y \in \mathcal{Y} \}.$$

Then

$$\|f\|_K = \inf \{ \|g\|_{\mathcal{H}} \mid \forall g \in \mathcal{H}, Wg = f \}. \quad (2.15)$$

Proof The operator $(Wg)(x) = \Phi(x)^* g$ ensure that the nullspace of W is $\mathcal{N} = \text{Ker } W = \bigcap_{x \in \mathcal{X}} \text{Ker } \Phi(x)^*$. Since $\Phi(x)$ is bounded, $\Phi(x)$ is a continuous operator, thus for all $x \in \mathcal{X}$, $\text{Ker } \Phi(x)^*$ is closed so that \mathcal{N} is closed. Moreover,

$$\mathcal{N} = \text{Ker } W = \bigcap_{x \in \mathcal{X}} \text{Ker } \Phi(x)^* = \bigcap_{x \in \mathcal{X}} (\text{Im } \Phi(x))^\perp = \left(\bigcup_{x \in \mathcal{X}} \text{Im } \Phi(x) \right)^\perp$$

So that $\mathcal{N}^\perp = \bigcup_{x \in \mathcal{X}} \text{Im } \Phi(x)$ and the restriction of W to \mathcal{N}^\perp is injective.

Let $\mathcal{H}_K = \text{Im } W$ be a vector space. Define the unique inner product on \mathcal{H}_K such that W becomes a partial isometry from \mathcal{H} onto \mathcal{H}_K . We call this new partial isometry (again) W . We show that \mathcal{H}_K is a \mathcal{Y} -Reproducing Kernel Hilbert Space. Since W^*W is a projection on \mathcal{N}^\perp , given $f \in \mathcal{H}_K$, where $f = Wg$ and $g \in \mathcal{N}^\perp$ we have for all $x \in \mathcal{X}$

$$f(x) = (Wg)(x) = \Phi(x)^* g = \Phi(x)^* W^* Wg = (W\Phi(x))^* f.$$

Since $\text{Ker } W$ is closed, W is bounded, and $\Phi(x)$ is bounded by definition so that the evaluation map

$$ev_x = (W\Phi(x))^*$$

is bounded so continuous. Then the reproducing kernel is given for all $x, z \in \mathcal{X}$ by

$$K(x, z) = ev_x ev_z^* = (W\Phi(x))^* (W\Phi(z)) = \Phi(x)^* W^* W\Phi(z) = \Phi(x)^* \Phi(z),$$

Since W^*W is the identity on $\text{Im } \Phi(z)$. Hence \mathcal{H}_K is a \mathcal{Y} -RKHS (see proof of proposition 2.1). \square

We call Φ a *feature map*, W a *feature operator* and \mathcal{H} a *feature space*. Since W is an isometry from $(\text{Ker } W)^\perp$ onto \mathcal{H}_K , the map W allows us to identify \mathcal{H}_K with the closed subspace $(\text{Ker } W)^\perp$ of \mathcal{H} . Notice that W is a partial isometry, meaning that there can exist multiple functions $g \in \mathcal{H}$, the re-description space, such that $Wg = f$ where f is a function of the \mathcal{Y} -RKHS \mathcal{H}_K . However eq. (2.15) shows that there is a unique function $g \in \mathcal{H}$ such that $Wg = f$, and $\|g\|_{\mathcal{H}} = \|f\|_{\mathcal{H}_K}$. Among all functions $g \in \mathcal{H}$ such that $Wg = f$, the only one making the norm in the \mathcal{Y} -RKHS and the re-description space is the one with minimal norm.

In this work we mainly focus on the class kernels inducing a \mathcal{Y} -RKHS of continuous functions. Such kernels are named \mathcal{Y} -Mercer kernels.

Definition 2.7 (\mathcal{Y} -Mercer kernel). A reproducing kernel $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y})$ is called \mathcal{Y} -Mercer kernel if \mathcal{H}_K is a subspace of $\mathcal{C}(\mathcal{X}; \mathcal{Y})$.

The following proposition characterize \mathcal{Y} -Mercer kernel in terms of the properties of a kernels rather than properties of the \mathcal{Y} -RKHS.

Proposition 2.4 (Characterization of \mathcal{Y} -Mercer kernel [13]). Let K be a reproducing kernel. The kernel K is Mercer if and only if the function $x \mapsto \|K(x, x)\|$ is locally bounded and for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$, $K_x y \in \mathcal{C}(\mathcal{X}; \mathcal{Y})$.

Proof If $\mathcal{H}_K \subset \mathcal{C}(\mathcal{X}; \mathcal{Y})$, then for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$, $K_x y$ is an element of $\mathcal{C}(\mathcal{X}; \mathcal{Y})$ (see eq. (2.14b)). In addition for all $f \in \mathcal{H}_K$, $\|K_x^* f\| = \|f(x)\| \leq \|f\|_\infty$. Hence there exist a constant $M \in \mathbb{R}_+$ such that for all $x \in \mathcal{X}$, $\|K_x\| \leq M$. Therefore from eq. (2.14a), for all $x \in \mathcal{X}$, $\|K(x, x)\| = \|K_x^*\|^2 \leq M^2$. Conversely assume that the function $x \mapsto \|K(x, x)\|$ is locally bounded and $K_x y \in \mathcal{C}(\mathcal{X}; \mathcal{Y})$. For all $f \in \mathcal{H}_K$ and all $x \in \mathcal{X}$,

$$\|f(x)\| = \|f\|_K \sqrt{\|K(x, x)\|} \leq M \|f\|_K.$$

Thus convergence in \mathcal{H}_K implies uniform convergence. Since by assumption

$$\{K_x y \mid \forall x \in \mathcal{X}, \forall y \in \mathcal{Y}\} \subset \mathcal{C}(\mathcal{X}; \mathcal{Y}),$$

then the \mathcal{Y} -Reproducing Kernel Hilbert Space

$$\begin{aligned} \mathcal{H}_K &= \overline{\text{span}} \{K_x y \mid \forall x \in \mathcal{X}, \forall y \in \mathcal{Y}\} \\ &\subset \mathcal{C} \end{aligned}$$

is also a subset of $\mathcal{C}(\mathcal{X}; \mathcal{Y})$ by the uniform convergence theorem. \square

Lemma 2.1 Let \mathcal{H}_K be a \mathcal{Y} -Reproducing Kernel Hilbert Space of continuous function $f : \mathcal{X} \rightarrow \mathcal{Y}$. If \mathcal{X} and \mathcal{Y} are separable then \mathcal{H}_K is separable.

Proof The separability of \mathcal{X} assure that there exist a countable dense subset $\mathcal{X}_0 \subseteq \mathcal{X}$. Since \mathcal{Y} is separable,

$$\mathcal{S} = \bigcup_{x \in \mathcal{X}_0} \text{Im } K_x = \{K_x y \mid \forall x \in \mathcal{X}_0, \forall y \in \mathcal{Y}\} \subset \mathcal{H}_K$$

is separable too. We show that \mathcal{S} is total in \mathcal{H}_K so that \mathcal{H}_K is separable. If for all $x \in \mathcal{X}_0$, $f \in \mathcal{S}^\perp$, then $f \in \text{Ker } K_x^*$. Namely $f(x) = \text{ev}_x f = 0$. Since f is continuous and \mathcal{X}_0 is dense in \mathcal{X} , for all $x \in \mathcal{X}$, $f(x) = 0$ so $f = 0$. \square

Proposition 2.5 (Countable orthonormal basis for \mathcal{Y} -Mercer kernel [12]). Let $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y})$ be a reproducing kernel where \mathcal{X} and \mathcal{Y} are separable spaces. If K is a \mathcal{Y} -Mercer kernel then \mathcal{H}_K has a countable orthonormal basis.

Proof From proposition 2.4 K is a \mathcal{Y} -Mercer kernel if and only if $\mathcal{H}_K \subset \mathcal{C}(\mathcal{X}; \mathcal{Y})$. Applying lemma 2.1 of [12], we have that \mathcal{H}_K is separable. Thus since \mathcal{H}_K is also a Hilbert space it has a countable orthonormal basis. \square

An important consequence is that if K is a \mathcal{Y} -Mercer and \mathcal{X} and \mathcal{Y} are separable then \mathcal{H}_K and any resdescription is isometrically isomorphic to ℓ^2 .

2.6.2 Shift-Invariant operator-valued kernels

The main subjects of interest of chapter 3 are shift-invariant Operator-Valued Kernel. When referring to a shift-invariant **OVK** $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y})$ we assume that \mathcal{X} is a locally compact second countable topological group with identity e .

Definition 2.8 (Shift-invariant OVK). *A reproducing Operator-Valued Kernel $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y})$ is called shift-invariant³ if for all $x, z, t \in \mathcal{X}$,*

$$K(x \star t, z \star t) = K(x, z). \quad (2.16)$$

A shift-invariant kernel can be characterize by a function of one variable K_e called the signature of K . Here e denotes the neutral element of the **LCA** group \mathcal{X} endowed with the operator \star .

We recall the definition of left regular representation of \mathcal{X} acting on \mathcal{H}_K which is useful to study **LCA** groups. For all $x, z \in \mathcal{X}$ and for all $f \in \mathcal{H}_K$,

$$(\lambda_z f)(x)G := f(z^{-1} \star x).$$

A group representation λ_z describe the group by making it act on a vector space (here \mathcal{H}_K) in a linear manner. In other word, the group representation let us see a group as a linear operator which are well studied mathematical object.

Proposition 2.6 (Kernel signature [13]). *Let $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y})$ be a reproducing kernel. The following conditions are equivalents.*

1. K is a shift-invariant of positive type.
2. There is a function $K_e : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y})$ of completely positive type such that $K(x, z) = K_e(z^{-1} \star x)$.

If one of the above conditions is satisfied, then the representation λ leaves invariant \mathcal{H}_K , its action on \mathcal{H}_K is unitary and

$$K(x, z) = K_e^* \lambda_{x^{-1} \star z} K_e \quad \forall (x, z) \in \mathcal{X}^2. \quad (2.17a)$$

$$\|K(x, x)\| = \|K_e(e)\| \quad \forall x \in \mathcal{X} \quad (2.17b)$$

Proof Assume eq. (2.17) holds true. Given $x, z \in \mathcal{X}$, eq. (2.12) and eq. (2.16) yields

$$K_e(z^{-1} \star x) = K(z^{-1} \star x, e) = K(x, z).$$

Since K is a reproducing kernel, K_e is of completely positive type, so that proposition 2.6 item 2 holds true. Besides if proposition 2.6 item 2 holds true obviously the definition of a reproducing kernel (definition 2.5) is fulfilled so that holds true.

³ Also referred to as translation-invariant **OVK**.

Suppose that K is a shift-invariant reproducing kernel. Given $t \in \mathcal{X}$ and $y \in \mathcal{Y}$, for all $x, z \in \mathcal{X}$,

$$(\lambda_x K_t y)(z) = (K_t y)(x^{-1} \star z) = K(x^{-1} \star z, t) = K(z, x \star t) = (K_{x \star t} y)z,$$

that is $\lambda_x K_t = K_{x \star t}$. Besides for all $y, y' \in \mathcal{Y}$ and all $x, z, t, t' \in \mathcal{X}$,

$$\begin{aligned} \langle \lambda_x K_t y, \lambda_x K_{t'} y' \rangle_K &= \langle K_{x \star t} y, K_{x \star t'} y' \rangle_K = \langle K(x \star t', x \star t) y, y' \rangle \\ &= \langle K(t', t) y, y' \rangle = \langle K_t y, K_{t'} y' \rangle_K \end{aligned}$$

This means that λ leaves the set $\{K_x y \mid \forall x \in \mathcal{X}, \forall y \in \mathcal{Y}\}$ invariant. Since

$$\{K_x y \mid \forall x \in \mathcal{X}, \forall y \in \mathcal{Y}\}$$

is total in \mathcal{H}_K (see eq. (2.14b)), λ is surjective and because it also leaves the inner product invariant, the first two claims follow. \square

The notation K_e for the function of completely positive type associated with the reproducing kernel K is consistent with the definition given by eq. (2.12) since for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$

$$(K_e y)(x) = K_e(x) y.$$

2.6.3 Examples of operator-valued kernels

Operator-valued kernels have been first introduced in Machine Learning to solve multi-task regression problems. Multi-task regression is encountered in many fields such as structured classification when classes belong to a hierarchy for instance. Instead of solving independently p single output regression task, one would like to take advantage of the relationships between output variables when learning and making a decision.

Definition 2.9 (Decomposable kernel). Let Γ be a non-negative operator of $\mathcal{L}_+(\mathcal{Y})$. K is said to be a \mathcal{Y} -Mercer decomposable kernel⁴ if for all $(x, z) \in \mathcal{X}^2$,

$$K(x, z) := k(x, z) \Gamma,$$

where k is a scalar Mercer kernel.

When $\mathcal{Y} = \mathbb{R}^p$, the matrix Γ is interpreted as encoding the relationships between the outputs coordinates. If a graph coding for the proximity between tasks is known, then it is shown in Álvarez, Rosasco, and Lawrence [2], Baldassarre et al. [3], and Evgeniou, Micchelli, and Pontil [16] that Γ can be chosen equal to the pseudo inverse L^\dagger of the graph Laplacian such that the norm in \mathcal{H}_K is a graph-regularizing penalty for the outputs (tasks). When no prior knowledge is available, Γ can be set to the empirical covariance of the output training data or learned with one of the algorithms proposed in the literature [15, 24, 36]. Another interesting property of the decomposable kernel is its universality (a kernel which may approximate an arbitrary continuous target function uniformly on any compact subset

⁴ Some authors also refer to as separable kernels.

of the input space). A reproducing kernel K is said *universal* if the associated \mathcal{Y} -RKHS \mathcal{H}_K is dense in the space $\mathcal{C}(\mathcal{X}, \mathcal{Y})$. The conditions for a kernel to be universal have been discussed in Caponnetto et al. [11] and Carmeli et al. [13]. In particular they show that a decomposable kernel is universal provided that the scalar kernel k is universal and the operator Γ is injective.

Proposition 2.7 (Kernels and Regularizers [2]). *Let $K(x, z) := k(x, z)\Gamma$ for all $x, z \in \mathcal{X}$ be a decomposable kernel where Γ is a matrix of size $p \times p$. Then for all $f \in \mathcal{H}_K$,*

$$\|f\|_K = \sum_{i,j=1}^p \Gamma_{ij}^\dagger \langle f_i, f_j \rangle_k \quad (2.18)$$

where $f_i = \langle f, e_i \rangle$ (resp $f_j = \langle f, e_j \rangle$), denotes the i -th (resp j -th) component of f .

Curl-free and divergence-free kernels provide an interesting application of operator-valued kernels [4, 25, 27] to *vector field* learning, for which input and output spaces have the same dimensions ($d = p$). Applications cover shape deformation analysis [27] and magnetic fields approximations [45]. These kernels discussed in [18] allow encoding input-dependent similarities between vector-fields.

Definition 2.10 (Curl-free and Div-free kernel). *Assume $\mathcal{X} = (\mathbb{R}^d, +)$ and $\mathcal{Y} = \mathbb{R}^p$ with $d = p$. The divergence-free kernel is defined as*

$$K^{\text{div}}(x, z) = K_0^{\text{div}}(\delta) = (\nabla \nabla^\top - \Delta I)k_0(\delta)$$

and the curl-free kernel as

$$K^{\text{curl}}(x, z) = K_0^{\text{curl}}(\delta) = -\nabla \nabla^\top k_0(\delta),$$

where ∇ is the gradient operator⁵, $\nabla \nabla^\top$ is the Hessian operator and Δ is the Laplacian operator.

Although taken separately these kernels are not universal, a convex combination of the curl-free and divergence-free kernels allows to learn any vector field that satisfies the Helmholtz decomposition theorem [4, 25].

⁵ See section 3.4.1 for the formal definition of the operator ∇ .

Part II

CONTRIBUTIONS

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OPERATOR-VALUED RANDOM FOURIER FEATURES

3.1 MOTIVATIONS

Random Fourier Features have been proved useful to implement efficiently kernel methods in the scalar case, allowing to learn a linear model based on an approximated feature map. In this work, we are interested to construct approximated operator-valued feature maps to learn vector-valued functions. With an explicit (approximated) feature map, one converts the problem of learning a function f in the vector-valued Reproducing Kernel Hilbert Space \mathcal{H}_K into the learning of a linear model \tilde{f} defined by:

$$\tilde{f}(x) = \tilde{\Phi}(x)^* \theta,$$

where $\Phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{Y})$ and $\theta \in \mathcal{H}$. The methodology we propose works for operator-valued kernels defined on any Locally Compact Abelian (LCA) group, noted (\mathcal{X}, \star) , for some operation noted \star . This allows us to use the general context of Pontryagin duality for Fourier transform of functions on LCA groups. Building upon a generalization of Bochner's theorem for operator-valued measures, an operator-valued kernel is seen as the *Fourier transform* of an operator-valued positive measure. From that result, we extend the principle of Random Fourier Feature for scalar-valued kernels and derive a general methodology to build Operator Random Fourier Feature when operator-valued kernels are shift-invariant according to the chosen group operation. Elements of this chapter has been developped in Brault, Heinonen, and Buc [7].

3.2 CONSTRUCTION

We present a construction of Operator-valued Random Fourier Feature (ORFF) such that $f : \mathcal{X} \mapsto \tilde{\Phi}(x)^* \theta$ is a continuous function that maps an arbitrary LCA group \mathcal{X} as input space to an arbitrary output Hilbert space \mathcal{Y} . First we define a functional *Fourier feature map*, and then propose a Monte-Carlo sampling from this feature map to construct an approximation of a shift-invariant \mathcal{Y} -Mercer kernel. Then, we prove the convergence of the kernel approximation $\tilde{K}(x, z) = \tilde{\Phi}(x)^* \tilde{\Phi}(z)$ with high probability on compact subsets of the LCA \mathcal{X} , when \mathcal{Y} is *finite dimensional*. Eventually we conclude with some numerical experiments.

3.2.1 Theoretical study

The following proposition of Carmeli et al. [13] and Zhang, Xu, and Zhang [47] extends Bochner's theorem to any shift-invariant \mathcal{Y} -Mercer kernel.

Proposition 3.1 (Operator-valued Bochner's theorem [32, 47]).

If a continuous function K from $\mathcal{X} \times \mathcal{X}$ to \mathcal{Y} is a shift-invariant \mathcal{Y} -Mercer kernel on \mathcal{X} , then there exists a unique positive projection-valued measure $\hat{Q} : \mathcal{B}(\mathcal{X}) \rightarrow \mathcal{L}_+(\mathcal{Y})$ such that for all $x, z \in \mathcal{X}$,

$$K(x, z) = \int_{\hat{\mathcal{X}}} \overline{(x \star z^{-1}, \omega)} d\hat{Q}(\omega), \quad (3.1)$$

where \hat{Q} belongs to the set of all the projection-valued measures of bounded variation on the σ -algebra of Borel subsets of $\hat{\mathcal{X}}$. Conversely, from any positive operator-valued measure M , a shift-invariant kernel K can be defined by proposition 3.1.

Although this theorem is central to the spectral decomposition of shift-invariant \mathcal{Y} -Mercer OVK, the following results proved by Carmeli et al. [13] provides insights about this decomposition that are more relevant in practice. It first gives the necessary conditions to build shift-invariant \mathcal{Y} -Mercer kernel with a pair $(A, \hat{\mu})$ where A is an operator-valued function on $\hat{\mathcal{X}}$ and $\hat{\mu}$ is a real-valued positive measure on $\hat{\mathcal{X}}$. Note that obviously such a pair is not unique and the choice of this paper may have an impact on theoretical properties as well as practical computations. Secondly it also states that any OVK have such a spectral decomposition when \mathcal{Y} is finite dimensional or \mathcal{X} .

Proposition 3.2 (Carmeli et al. [13]). *Let $\hat{\mu}$ be a positive measure on $\mathcal{B}(\hat{\mathcal{X}})$ and $A : \hat{\mathcal{X}} \rightarrow \mathcal{L}(\mathcal{Y})$ such that $\langle A(\cdot)y, y' \rangle \in L^1(\mathcal{X}, \hat{\mu})$ for all $y, y' \in \mathcal{Y}$ and $A(\omega) \succcurlyeq 0$ for $\hat{\mu}$ -almost all $\omega \in \hat{\mathcal{X}}$. Then, for all $\delta \in \mathcal{X}$,*

$$K_e(\delta) = \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} A(\omega) d\hat{\mu}(\omega) \quad (3.2)$$

is the kernel signature of a shift-invariant \mathcal{Y} -Mercer kernel K such that $K(x, z) = K_e(x \star z^{-1})$. The \mathcal{Y} -RKHS \mathcal{H}_K is embed in $L^2(\hat{\mathcal{X}}, \hat{\mu}; \mathcal{Y})$ by mean of the feature operator

$$(Wg)(x) = \int_{\hat{\mathcal{X}}} \overline{(x, \omega)} B(\omega) g(\omega) d\hat{\mu}(\omega), \quad (3.3)$$

Where $B(\omega)B(\omega)^* = A(\omega)$ and both integral converges in the weak sense. If \mathcal{Y} is finite dimensional or \mathcal{X} is compact, any shift-invariant kernel is of the above form for some pair $(A, \hat{\mu})$.

When $p = 1$ one can always assume A is reduced to the scalar 1, $\hat{\mu}$ is still a bounded positive measure and we retrieve the Bochner theorem applied to the scalar case (??).

Proposition 3.2 shows that a given pair $(A, \hat{\mu})$ characterize an [OVK](#). Namely given a measure $\hat{\mu}$ and a function A such that $\langle A(\cdot)y, y' \rangle \in L^1(\mathcal{X}, \hat{\mu})$ for all $y, y' \in \mathcal{Y}$ and $A(\omega) \succcurlyeq 0$ for $\hat{\mu}$ -almost all ω , it gives rise to an [OVK](#). Since $(A, \hat{\mu})$ determine a unique kernel we can write $\mathcal{H}_{(A, \hat{\mu})} \Rightarrow \mathcal{H}_K$ where K is defined as in eq. (3.2). However the converse is to true: Given a \mathcal{Y} -Mercer shift invariant Operator-Valued Kernel, there exist infinitely many pairs $(A, \hat{\mu})$ that characterize an [OVK](#).

The main difference between proposition 3.1 and proposition 3.2 is that the first one characterize an [OVK](#) by a unique Positive Operator-Valued Measure ([POVM](#)), while the second one shows that the [POVM](#) that uniquely characterize a \mathcal{Y} -Mercer [OVK](#) has an operator-valued density with respect to a *scalar* measure $\hat{\mu}$; and that this operator-valued density is not unique.

Finally proposition 3.2 does not provide any *constructive* way to obtain the pair $(A, \hat{\mu})$ that characterize an [OVK](#). The following section 3.2.2 is based on an other proposition of Carmeli, De Vito, and Toigo and show that if the kernel signature $K_e(\delta)$ of an [OVK](#) is in L^1 then it is possible to construct *explicitly* a pair $(C, \widehat{\text{Haar}})$ from it. Additionally, we show that we can always extract a scalar-valued *probability* density function from C such that we obtain a pair $(A, \text{Pr}_{\hat{\mu}, \rho})$ where $\text{Pr}_{\hat{\mu}, \rho}$ is a *probability* distribution absolutely continuous with respect to $\hat{\mu}$ and with associated probability density function (p. d. f.) ρ . Thus for all $\mathcal{Z} \subset \mathcal{B}(\widehat{\mathcal{X}})$,

$$\text{Pr}_{\hat{\mu}, \rho}(\mathcal{Z}) = \int_{\mathcal{Z}} \rho(\omega) d\hat{\mu}(\omega).$$

When the reference measure $\hat{\mu}$ is the Lebesgue measure, we note $\text{Pr}_{\hat{\mu}, \rho} = \text{Pr}_{\rho}$.

3.2.2 Sufficient conditions of existence

While proposition 3.2 gives some insights on how to build an approximation of a \mathcal{Y} -Mercer kernel, we need a theorem that provides an explicit construction of the pair $(A, \text{Pr}_{\hat{\mu}, \rho})$ from the kernel signature K_e . Proposition 14 in Carmeli et al. [13] gives the solution, and also provide a sufficient condition for proposition 3.2 to apply.

Proposition 3.3 (Carmeli et al. [13]). *Let K be a shift-invariant \mathcal{Y} -Mercer kernel. Suppose that for all $z \in \mathcal{X}$ and for all $y, y' \in \mathcal{Y}$, $\langle K_e(\cdot)y, y' \rangle \in L^1(\mathcal{X}, \mathbf{Haar})$ where \mathcal{X} is endowed with the group law \star . For all $\omega \in \widehat{\mathcal{X}}$ and for all y, y' in \mathcal{Y} , let*

$$\begin{aligned} \langle y', C(\omega)y \rangle &= \int_{\mathcal{X}} (\delta, \omega) \langle y', K_e(\delta)y \rangle d\delta \\ &= \mathcal{F}^{-1} [\langle y', K_e(\cdot)y \rangle] (\omega). \end{aligned} \quad (3.4)$$

Then

1. $C(\omega)$ is a bounded non-negative operator for all $\omega \in \widehat{\mathcal{X}}$,
2. $\langle y, C(\cdot)y' \rangle \in L^1(\widehat{\mathcal{X}}, \widehat{\mathbf{Haar}})$ for all $y, y' \in \mathcal{X}$,
3. for all $\delta \in \mathcal{X}$ and for all y, y' in \mathcal{Y} ,

$$\begin{aligned} \langle y', K_e(\delta)y \rangle &= \int_{\widehat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y', C(\omega)y \rangle d\widehat{\mathbf{Haar}}(\omega) \\ &= \mathcal{F} [\langle y', C(\cdot)y \rangle] (\delta). \end{aligned}$$

There have been a lot of confusion in the literature whether a kernel is the Fourier transform or Inverse Fourier transform of a measure. However lemma 3.1 clarify the relation between the Fourier transform and Inverse Fourier transform for a translation invariant Operator-Valued Kernel. Notice that in the real scalar case the Fourier transform and Inverse Fourier transform of a shift-invariant kernel are the same, while the difference is significant for [OVK](#).

The following lemma is a direct consequence of the definition of $C(\omega)$ as the Fourier transform of the adjoint of K_e and also helps simplifying the definition of [ORFF](#).

Lemma 3.1 *Let K_e be the signature of a shift-invariant \mathcal{Y} -Mercer kernel and let $\langle y', C(\cdot)y \rangle = \mathcal{F}^{-1} [\langle y', K_e(\cdot)y \rangle]$ for all $y, y' \in \mathcal{Y}$. Then*

1. $C(\omega)$ is self-adjoint and C is even.
2. $\mathcal{F}^{-1} [\langle y', K_e(\cdot)y \rangle] = \mathcal{F} [\langle y', K_e(\cdot)y \rangle]$.
3. $K_e(\delta)$ is self-adjoint and K_e is even.

Proof For any function f on (\mathcal{X}, \star) define the flip operator \mathcal{R} by

$$\mathcal{R}f(x) := f(x^{-1}).$$

For any shift invariant \mathcal{Y} -Mercer kernel and for all $\delta \in \mathcal{X}$, $K_e(\delta) = K_e(\delta^{-1})^*$. Indeed,

$$\begin{aligned} \mathcal{R}\langle y, K_e(x \star z^{-1})y' \rangle &= \langle y, K_e((x \star z^{-1})^{-1})y' \rangle \\ &= \langle y, K_e(x^{-1} \star z)y' \rangle \\ &= \langle y, K_e(x \star z^{-1})^*y' \rangle. \end{aligned}$$

Proposition 3.3 item 1: taking the Fourier transform yields,

$$\mathcal{F}^{-1} [\langle y', K_e(\cdot)y \rangle] = \mathcal{F}^{-1} \mathcal{R} [\langle y', K_e(\cdot)y \rangle] = \mathcal{R} \langle y', C(\cdot)y \rangle.$$

Hence $C(\omega) = C(\omega^{-1})^*$. Suppose that \mathcal{Y} is a complex Hilbert space. Since for all $\omega \in \hat{\mathcal{X}}$, $C(\omega)$ is bounded and non-negative so $C(\omega)$ is self-adjoint. Besides we have $C(\omega) = C(\omega^{-1})^*$ to C must be pair. Suppose that \mathcal{Y} is a real Hilbert space. Then we have the additional hypothesis that $K_e(\delta) = K_e(\delta)^*$. Taking the Fourier transform yields that $C(\omega) = C(\omega)^*$. Since for any shift invariant \mathcal{Y} -Mercer kernel $C(\omega) = C(\omega^{-1})^*$ we also conclude that $C(\omega^{-1}) = C(\omega)$.

Proposition 3.3 item 2: simply, for all $y, y' \in \mathcal{Y}$, $\langle y, C(\omega^{-1})y' \rangle = \langle y', C(\omega)y \rangle$ thus $\mathcal{F}^{-1} [\langle y', C(\cdot)y \rangle] = \mathcal{F} \mathcal{R} [\langle y', C(\cdot)y \rangle] = \mathcal{F} [\langle y', C(\cdot)y \rangle]$.

Proposition 3.3 item 3: from proposition 3.3 item 2 we have $\mathcal{F}^{-1} [\langle y', K_e(\cdot)y \rangle] = \mathcal{F}^{-1} \mathcal{R} \langle y', K_e(\cdot)y \rangle$. By injectivity of the Fourier transform, K_e is even. Since $K_e(\delta) = K_e(\delta^{-1})^$, we must have $K_e(\delta) = K_e(\delta)^*$. \square*

While proposition 3.3 gives an explicit form of the operator $C(\omega)$ defined as the Fourier transform of the kernel K , it is not really convenient to work with the Haar measure $\widehat{\mathbf{Haar}}$ on $\mathcal{B}(\hat{\mathcal{X}})$. However it is easily possible to turn $\widehat{\mathbf{Haar}}$ into a probability measure to allow efficient integration over an infinite domain.

The following proposition allows to build a spectral decomposition of a shift-invariant \mathcal{Y} -Mercer kernel on a LCA group \mathcal{X} endowed with the group law \star with respect to a scalar probability measure, by extracting a scalar probability density function from C .

Proposition 3.4 (Shift-invariant \mathcal{Y} -Mercer kernel spectral decomposition). *Let K_e be the signature of a shift-invariant \mathcal{Y} -Mercer kernel. If for all $y, y' \in \mathcal{Y}$, $\langle K_e(\cdot)y, y' \rangle \in L^1(\mathcal{X}, \widehat{\mathbf{Haar}})$ then there exists a positive probability measure $\mathbf{Pr}_{\widehat{\mathbf{Haar}}, \rho}$ and an operator-valued function A such that for all $y, y' \in \mathcal{Y}$,*

$$\langle y', K_e(\delta)y \rangle = \mathbf{E}_{\widehat{\mathbf{Haar}}, \rho} \left[\overline{(\delta, \omega)} \langle y', A(\omega)y \rangle \right], \quad (3.5)$$

with

$$\langle y', A(\omega)y \rangle \rho(\omega) = \mathcal{F} [\langle y', K_e(\cdot)y \rangle] (\omega). \quad (3.6)$$

Moreover

1. for all $y, y' \in \mathcal{Y}$, $\langle A(\cdot)y, y' \rangle \in L^1(\hat{\mathcal{X}}, \mathbf{Pr}_{\widehat{\mathbf{Haar}}, \rho})$,
2. $A(\omega)$ is non-negative for $\mathbf{Pr}_{\widehat{\mathbf{Haar}}, \rho}$ -almost all $\omega \in \hat{\mathcal{X}}$,
3. $A(\cdot)$ and $\rho(\cdot)$ are even functions.

Proof This is a simple consequence of proposition 3.3 and lemma 3.1. By taking $\langle y', C(\omega)y \rangle = \mathcal{F}^{-1}[\langle y', K_e(\cdot)y \rangle](\omega) = \mathcal{F}[\langle y', K_e(\cdot)y \rangle](\omega)$ we can write the following equality concerning the *OVK* signature K_e .

$$\begin{aligned} \langle y', K_e(\delta)y \rangle(\omega) &= \int_{\widehat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y', C(\omega)y \rangle d\widehat{\mathbf{Haar}}(\omega) \\ &= \int_{\widehat{\mathcal{X}}} \overline{(\delta, \omega)} \left\langle y', \frac{1}{\rho(\omega)} C(\omega)y \right\rangle \rho(\omega) d\widehat{\mathbf{Haar}}(\omega). \end{aligned}$$

It is always possible to choose $\rho(\omega)$ such that $\int_{\widehat{\mathcal{X}}} \rho(\omega) d\widehat{\mathbf{Haar}}(\omega) = 1$. For instance choose

$$\rho(\omega) = \frac{\|C(\omega)\|_{\mathcal{Y}, \mathcal{Y}}}{\int_{\widehat{\mathcal{X}}} \|C(\omega)\|_{\mathcal{Y}, \mathcal{Y}} d\widehat{\mathbf{Haar}}(\omega)}$$

Since for all $y, y' \in \mathcal{Y}$, $\langle y', C(\cdot)y \rangle \in L^1(\widehat{\mathcal{X}}, \widehat{\mathbf{Haar}})$ and \mathcal{Y} is a separable Hilbert space, by *Pettis measurability theorem*, $\int_{\widehat{\mathcal{X}}} \|C(\omega)\|_{\mathcal{Y}, \mathcal{Y}} d\widehat{\mathbf{Haar}}(\omega)$ is finite and so is $\|C(\omega)\|_{\mathcal{Y}, \mathcal{Y}}$ for all $\omega \in \widehat{\mathcal{X}}$. Therefore $\rho(\omega)$ is the density of a probability measure $\mathbf{Pr}_{\widehat{\mathbf{Haar}}, \rho}$, i. e. conclude by taking

$$\mathbf{Pr}_{\widehat{\mathbf{Haar}}, \rho}(\mathcal{Z}) = \int_{\mathcal{Z}} \rho(\omega) d\widehat{\mathbf{Haar}}(\omega),$$

for all $\mathcal{Z} \in \mathcal{B}(\widehat{\mathcal{X}})$. □

In the case where $\mathcal{Y} = \mathbb{R}^p$, we rewrite eq. (3.6) coefficient-wise by choosing an orthonormal basis $\{e_j\}_{j \in \mathbb{N}_p}$ of \mathbb{R}^p .

$$A(\omega)_{ij} \rho(\omega) = \mathcal{F}[K_e(\cdot)_{ij}](\omega). \quad (3.7)$$

It follows that for all i and j in \mathbb{N}_p ,

$$K_e(x \star z^{-1})_{ij} = \mathcal{F}[A(\cdot)_{ij} \rho(\cdot)](x \star z^{-1}) \quad (3.8)$$

Remark 3.1 Note that although the Fourier transform of K_e yields a unique operator-valued function $C(\cdot)$, the decomposition of $C(\cdot)$ into $A(\cdot)\rho(\cdot)$ is again not unique. The choice of the decomposition may be justified by the computational cost or by the nature of the constants involved in the uniform convergence of the estimator.

Another difficulty arise from the fact that the quantity $\sup_{\omega \in \widehat{\mathcal{X}}} \|A(\omega)\|_{\mathcal{Y}, \mathcal{Y}}$ obtained in proposition 3.4 might not be bounded. The unboundedness of $\|A(\cdot)\|_{\mathcal{Y}, \mathcal{Y}}$ forbid the use of the most simple concentration inequalities – which require the boundedness of the random variable to be controlled. Therefore in the context of Operator-Valued Kernel concentration inequalities for unbounded random operators should be used. However, as pointed out by Minh [29], under some condition on the trace of $K_e(\delta)$, it is possible to turn $A(\cdot)$ into a bounded random operator for all ω in $\widehat{\mathcal{X}}$. The idea is to define a sum measure $\rho = \sum_{j \in \mathbb{N}} \rho_{e_j}$, which gives rise to bounded operator $A(\omega)$ and is independent of the $\{e_j\}_{j \in \mathbb{N}}$ base, instead of constructing a

measure from the operator norm as in proposition 3.4. Additionally with such construction the measure associated to $A(\cdot)$ is *independent* from the basis of \mathcal{Y} . In this proof we relax the assumptions of Minh [29] which requires $\int_{\widehat{\mathcal{X}}} |\text{Tr } K_e(\delta)| d\widehat{\text{Haar}}(\delta)$ to be well defined. We only require $\text{Tr } K_e(e)$ to be well defined.

Proposition 3.5 (Bounded shift-invariant \mathcal{Y} -Mercer kernel spectral decomposition). *Let K_e be the signature of a shift-invariant \mathcal{Y} -Mercer kernel. If for all y and y' in \mathcal{Y} , $\langle K_e(\cdot)y, y' \rangle \in L^1(\widehat{\mathcal{X}}, \widehat{\text{Haar}})$ and $\text{Tr } K_e(e) \in \mathbb{R}$, then*

$$\langle y', K_e(\delta)y \rangle = \mathbf{E}_{\widehat{\text{Haar}}, \rho_{\text{Tr}}} \left[\overline{(\delta, \omega)} \langle y, A_{\text{Tr}}(\omega)y' \rangle \right]. \quad (3.9)$$

with

$$\langle y', C(\cdot)y \rangle = \mathcal{F} [\langle y', K_e(\cdot)y \rangle] \quad (3.10a)$$

$$c_{\text{Tr}} = \text{Tr} [K_e(e)] \quad (3.10b)$$

$$A_{\text{Tr}}(\omega) = c_{\text{Tr}}^{-1} \text{Tr} [C(\omega)]^{-1} C(\omega) \quad (3.10c)$$

$$\rho_{\text{Tr}}(\omega) = c_{\text{Tr}}^{-1} \text{Tr} [C(\omega)]. \quad (3.10d)$$

Moreover

1. For all $y, y' \in \mathcal{Y}$, $\langle y, A_{\text{Tr}}(\cdot)y' \rangle \in L^1(\widehat{\mathcal{X}}, \widehat{\text{Pr}}_{\widehat{\text{Haar}}, \rho_{\text{Tr}}})$.
2. $A_{\text{Tr}}(\omega)$ is non-negative for all $\omega \in \widehat{\mathcal{X}}$,
3. $\sup_{\omega \in \widehat{\mathcal{X}}} \|A_{\text{Tr}}(\omega)\|_{\mathcal{Y}, \mathcal{Y}} \leq c_{\text{Tr}}$,
4. $A_{\text{Tr}}(\cdot)$ and ρ_{Tr} are even functions.

Proof Let $\{e_j\}_{j \in \mathbb{N}}$ be an orthonormal basis of \mathcal{Y} . Notice that

$$\begin{aligned} \int_{\widehat{\mathcal{X}}} \langle e_j, C(\omega)e_j \rangle d\widehat{\text{Haar}}(\omega) &= \int_{\widehat{\mathcal{X}}} \underbrace{\overline{(e, \omega)}}_{=1} \langle e_j, C(\omega)e_j \rangle d\widehat{\text{Haar}}(\omega) \\ &= \langle e_j, K_e(e)e_j \rangle. \end{aligned}$$

Since $C(\omega)$ is non-negative, all the $\langle e_j, C(\omega)e_j \rangle$. Thus using the monotone convergence theorem,

$$\begin{aligned} \int_{\widehat{\mathcal{X}}} \text{Tr} [C(\omega)] d\widehat{\text{Haar}}(\omega) &= \int_{\widehat{\mathcal{X}}} \sum_{j \in \mathbb{N}} \langle e_j, C(\omega)e_j \rangle d\widehat{\text{Haar}}(\omega) \\ &= \sum_{k \in \mathbb{N}} \langle e_k, K_e(e)e_k \rangle \\ &= \text{Tr} [K_e(e)] = c_{\text{Tr}} < \infty. \end{aligned}$$

Let $A_{\text{Tr}}(\omega)$ and $\rho_{\text{Tr}}(\omega)$ be defined respectively as in eq. (3.10c) and eq. (3.10d). By definition, $\int_{\widehat{\mathcal{X}}} \rho_{\text{Tr}}(\omega) d\widehat{\text{Haar}}(\omega) = 1$ and $A_{\text{Tr}}(\omega)\rho_{\text{Tr}}(\omega) = C(\omega)$. Now it remains to check the finiteness of $\text{Tr} [C(\omega)]$ for all $\omega \in \widehat{\mathcal{X}}$. Since for all $\omega \in \widehat{\mathcal{X}}$, $\text{Tr} [C(\omega)] \geq 0$,

$$\text{Tr} [C(\omega)] \leq \int_{\widehat{\mathcal{X}}} \text{Tr} [C(\omega)] d\widehat{\text{Haar}}(\omega) = \text{Tr} [K_e(e)] < \infty.$$

Since $\text{Tr}[C(\omega)]$ is positive and its integral is finite, ρ_{Tr} is a probability density function. The Schatten norms $\|\cdot\|_p$ verifies $\text{Tr}[\|\cdot\|] = \|\cdot\|_1 \geq \|\cdot\|_p \geq \|\cdot\|_q \geq \|\cdot\|_{\mathcal{Y}, \mathcal{Y}} = \|\cdot\|_\infty$ for all $p, q \in \mathbb{N}$ such that $1 \leq p \leq q$. Therefore since for all $\omega \in \mathcal{X}$, $C(\omega)$ is non-negative,

$$\begin{aligned} \|A_{\text{Tr}}(\omega)\|_{\mathcal{Y}, \mathcal{Y}} &= c_{\text{Tr}} \text{Tr}[C(\omega)]^{-1} \|C(\omega)\|_\infty \\ &\leq c_{\text{Tr}} \text{Tr}[C(\omega)]^{-1} \|C(\omega)\|_1 \\ &= c_{\text{Tr}} \text{Tr}[C(\omega)]^{-1} \text{Tr}[|C(\omega)|] \\ &= c_{\text{Tr}} \text{Tr}[C(\omega)]^{-1} \text{Tr}[C(\omega)] \\ &\leq c_{\text{Tr}} < \infty. \end{aligned}$$

Thus $\sup_{\omega \in \widehat{\mathcal{X}}} \|A(\omega)\|_{\mathcal{Y}, \mathcal{Y}} \leq c_{\text{Tr}} < \infty$. As C is an even function, so are A_{Tr} and ρ_{Tr} . Eventually $\langle y', C(\cdot)y \rangle$ is in $L^1(\widehat{\mathcal{X}}, \widehat{\text{Haar}})$, thus $\langle y, A_{\text{Tr}}(\cdot)\rho_{\text{Tr}}(\cdot)y' \rangle$ is in $L^1(\widehat{\mathcal{X}}, \widehat{\text{Haar}})$, hence $\langle y, A_{\text{Tr}}(\cdot)y' \rangle \in L^1(\widehat{\mathcal{X}}, \text{Pr}_{\widehat{\text{Haar}}, \rho_{\text{Tr}}})$. Since the trace is independent of the basis of \mathcal{Y} , so is ρ_{Tr} . \square

If \mathcal{Y} is finite dimensional then $\text{Tr}[K_e(e)]$ is well defined hence proposition 3.5 is valid as long as $K_e(\cdot)_{ij} \in L^1(\mathcal{X}, \text{Haar})$ for all $i, j \in \mathbb{N}_p$, where p is the dimension of \mathcal{Y} .

3.2.3 Examples of spectral decomposition

In this section we give exemple of spectral decomposition of various \mathcal{Y} -Mercer kernel, based on proposition 3.4 and proposition 3.5.

3.2.3.1 Gaussian decomposable kernel

Recall that a decomposable \mathbb{R}^p -Mercer has the form $K(x, z) = k(x, z)\Gamma$, where $k(x, z)$ is a scalar Mercer kernel and $\Gamma \in \mathcal{L}(\mathbb{R}^p)$ is a non-negative operator. Let $K_e^{\text{dec, gauss}}(\cdot) = k_e^{\text{gauss}}(\cdot)\Gamma$ be the Gaussian decomposable kernel where K_e and k_e are respectively the signature of K and k on the additive group $\mathcal{X} = (\mathbb{R}^d, +)$ - i. e. $\delta = x - z$ and $e = 0$. The scalar Gaussian kernel reads for all $\delta \in \mathbb{R}^d$

$$k_0^{\text{gauss}}(\delta) = \exp\left(-\frac{1}{2\sigma^2} \|\delta\|_2^2\right)$$

where $\sigma \in \mathbb{R}_+$ is an hyperparameter corresponding to the bandwith of the kernel. The Pontryagin dual group of $\mathcal{X} = (\mathbb{R}^d, +)$ is $\widehat{\mathcal{X}} \cong (\mathbb{R}^d, +)$ with the pairing

$$(\delta, \omega) = \exp(i\langle \delta, \omega \rangle)$$

where δ and $\omega \in \mathbb{R}^d$. In this case the Haar measures on \mathcal{X} and $\widehat{\mathcal{X}}$ are in both case the Lebesgue measure. However in order to have the property that $\mathcal{F}^{-1}[\mathcal{F}[f]] = f$ and $\mathcal{F}^{-1}[f] = \mathcal{R}\mathcal{F}[f]$ one must normalize both measures by $\sqrt{2\pi}^{-d}$, i. e. for all $\mathcal{Z} \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned} \sqrt{2\pi}^d \text{Haar}(\mathcal{Z}) &= \text{Leb}(\mathcal{Z}) \text{ and} \\ \sqrt{2\pi}^d \widehat{\text{Haar}}(\mathcal{Z}) &= \text{Leb}(\mathcal{Z}). \end{aligned}$$

Then the Fourier transform on $(\mathbb{R}^d, +)$ is

$$\begin{aligned}\mathcal{F}[f](\omega) &= \int_{\mathbb{R}^d} \exp(-i\langle \delta, \omega \rangle) f(x) d\mathbf{Haar}(\delta) \\ &= \int_{\mathbb{R}^d} \exp(-i\langle \delta, \omega \rangle) f(x) \frac{d\mathbf{Leb}(\delta)}{\sqrt{2\pi}^d}.\end{aligned}$$

Since $k_0^{\text{gauss}} \in L^1$ and Γ is bounded, it is possible to apply proposition 3.4, and obtain for all $i, j \in \mathbb{N}_p$,

$$\begin{aligned}C^{\text{dec,gauss}}(\omega)_{ij} &= \mathcal{F}\left[K_0^{\text{dec,gauss}}(\cdot)_{ij}\right](\omega) \\ &= \mathcal{F}\left[k_0^{\text{gauss}}\right](\omega)\Gamma_{ij} \\ &= \int_{\mathbb{R}^d} \exp(-i\langle \omega, x \rangle) \exp\left(-\frac{\|\delta\|_2^2}{2\sigma^2}\right) \frac{d\mathbf{Leb}(\delta)}{\sqrt{2\pi}^d} \Gamma_{ij} \\ &= \frac{1}{\sqrt{2\pi}^{\frac{1}{\sigma^2}}^d} \exp\left(-\frac{\sigma^2}{2}\|\omega\|_2^2\right) \sqrt{2\pi}^d \Gamma_{ij}.\end{aligned}$$

Hence

$$C^{\text{dec,gauss}}(\omega) = \underbrace{\frac{1}{\sqrt{2\pi}^{\frac{1}{\sigma^2}}^d} \exp\left(-\frac{\sigma^2}{2}\|\omega\|_2^2\right) \sqrt{2\pi}^d}_{\rho(\cdot) = \mathcal{N}(0, \sigma^{-2}I_d)\sqrt{2\pi}^d} \underbrace{\Gamma}_{A(\cdot) = \Gamma}$$

Therefore the canonical decomposition of $C^{\text{dec,gauss}}$ is $A^{\text{dec,gauss}}(\omega) = \Gamma$ and $\rho^{\text{dec,gauss}} = \mathcal{N}(0, \sigma^{-2}I_d)\sqrt{2\pi}^d$, where \mathcal{N} is the Gaussian probability distribution. Note that this decomposition is done with respect to the *normalized* Lebesgue measure $\widehat{\mathbf{Haar}}$, meaning that for all $z \in \mathcal{B}(\widehat{\mathcal{X}})$,

$$\begin{aligned}\mathbf{Pr}_{\widehat{\mathbf{Haar}}, \mathcal{N}(0, \sigma^{-2}I_d)\sqrt{2\pi}^d}(z) &= \int_{\widehat{\mathcal{X}}} \mathcal{N}(0, \sigma^{-2}I_d)\sqrt{2\pi}^d d\widehat{\mathbf{Haar}}(\omega) \\ &= \int_{\widehat{\mathcal{X}}} \mathcal{N}(0, \sigma^{-2}I_d) d\mathbf{Leb}(\omega) \\ &= \mathbf{Pr}_{\mathcal{N}(0, \sigma^{-2}I_d)}(z)\end{aligned}$$

Thus, the same decomposition with respect to the usual –non-normalized– Lebesgue measure \mathbf{Leb} yields

$$A^{\text{dec,gauss}}(\cdot) = \Gamma \tag{3.11a}$$

$$\rho^{\text{dec,gauss}} = \mathcal{N}(0, \sigma^{-2}I_d) \tag{3.11b}$$

If Γ is a trace class operator, applying proposition 3.5 yields the same decomposition since $\mathbf{Tr}\left[K_0^{\text{dec,gauss}}(0)\right] = \mathbf{Tr}[\Gamma]$ and $\mathbf{Tr}\left[C^{\text{dec,gauss}}(\cdot)\right] = \mathcal{N}(0, \sigma^{-2}I_d)\sqrt{2\pi}^d \mathbf{Tr}[\Gamma]$.

3.2.3.2 Skewed- χ^2 decomposable kernel

The skewed- χ^2 scalar kernel is defined on the LCA product group $\mathcal{X} = ((-c_k; +\infty)_{k=1}^d, \odot)$, with $c_k \in \mathbb{R}_+$. Let $(e_k)_{k=1}^d$ be the standard basis of \mathcal{X} and $\iota_k : x \mapsto \langle x, e_k \rangle$. The operator $\odot : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ is defined by

$$x \odot z = ((x_k + c_k)(z_k + c_k) - c_k)_{k=1}^d.$$

The identity element e is $(1 - c_k)_{k=1}^d$ since $(1 - c) \odot x = x$. Thus the inverse element x^{-1} is $((x_k + c_k)^{-1} - c_k)_{k=1}^d$. The skewed- χ^2 scalar kernel reads

$$k_{1-c}^{\text{skewed}}(\delta) = \prod_{k=1}^d \frac{2}{\sqrt{\delta_k + c_k} + \sqrt{\frac{1}{\delta_k + c_k}}}. \quad (3.12)$$

The dual of \mathcal{X} is $\widehat{\mathcal{X}} \cong \mathbb{R}$ with the pairing

$$(\delta, \omega) = \prod_{k=1}^d \exp(i \langle \log(\delta_k + c_k), \omega_k \rangle).$$

The Haar measure are defined for all $z \in \mathcal{B}((-c; +\infty)^d)$ and all $\widehat{z} \in \mathcal{B}(\mathbb{R}^d)$ by

$$\begin{aligned} \sqrt{2\pi}^d \mathbf{Haar}(z) &= \int_{\mathcal{Z}} \prod_{k=1}^d \frac{1}{z_k + c_k} d\mathbf{Leb}(z) \\ \sqrt{2\pi}^d \widehat{\mathbf{Haar}}(\widehat{z}) &= \mathbf{Leb}(\widehat{z}). \end{aligned}$$

Thus the Fourier transform is

$$\mathcal{F}[f](\omega) = \int_{(-c; +\infty)^d} \prod_{k=1}^d \frac{\exp(-i \langle \log(\delta_k + c_k), \omega_k \rangle)}{t_k + c_k} f(\delta) \frac{d\mathbf{Leb}(\delta)}{\sqrt{2\pi}^d}.$$

Then, applying Fubini's theorem over product space,

$$\begin{aligned} \mathcal{F}[k_0^{\text{skewed}}](\omega) &= \prod_{k=1}^d \int_{-c_k}^{\infty} \frac{2 \exp(-i \langle \log(\delta_k + c_k), \omega_k \rangle)}{(t_k + c_k) \left(\sqrt{\delta_k + c_k} + \sqrt{\frac{1}{\delta_k + c_k}} \right)} \frac{d\mathbf{Leb}(\delta_k)}{\sqrt{2\pi}^d} \\ &= \prod_{k=1}^d \int_{-c_k}^{\infty} \frac{2 \exp(-i \langle \delta_k, \omega_k \rangle)}{\exp(\frac{1}{2} \delta_k) + \exp(-\frac{1}{2} \delta_k)} \frac{d\mathbf{Leb}(\delta_k)}{\sqrt{2\pi}^d} \\ &= \sqrt{2\pi}^d \prod_{k=1}^d \text{sech}(\pi \omega_k) \end{aligned}$$

Since $k_{1-c}^{\text{skewed}} \in L^1$ and Γ is bounded, it is possible to apply proposition 3.4, and obtain for all $i, j \in \mathbb{N}_p$,

$$\begin{aligned} C^{\text{dec,skewed}}(\omega) &= \mathcal{F}[k_{1-c}^{\text{skewed}}](\omega) \Gamma \\ &= \underbrace{\sqrt{2\pi}^d \prod_{k=1}^d \text{sech}(\pi \omega_k)}_{\rho(\cdot) = \mathcal{S}(0, 2^{-1})^d \sqrt{2\pi}^d} \underbrace{\Gamma}_{A(\cdot)} \end{aligned}$$

Hence the decomposition with respect to the usual –non-normalized– Lebesgue measure Leb yields

$$A^{\text{dec,skewed}}(\cdot) = \Gamma \quad (3.13a)$$

$$\rho^{\text{dec,skewed}} = \mathcal{S}(0, 2^{-1})^d \quad (3.13b)$$

3.2.3.3 Curl-free Gaussian kernel

The curl-free Gaussian kernel is defined as $K_0^{\text{curl,gauss}} = -\nabla\nabla^T k_0^{\text{gauss}}$. Here $\mathcal{X} = (\mathbb{R}^d, +)$ so the setting is the same than section 3.2.3.1.

$$\begin{aligned} C^{\text{curl,gauss}}(\omega)_{ij} &= \mathcal{F} \left[K_{1-c}^{\text{curl,gauss}}(\cdot)_{ij} \right] (\omega) \\ &= \mathcal{F} \left[-\frac{d^2}{d\delta_i d\delta_j} k_0^{\text{gauss}} \right] (\omega) \\ &= -(i\omega_i)(i\omega_j) \mathcal{F} [k_0^{\text{gauss}}] (\omega) \\ &= \omega_i \omega_j \mathcal{F} [k_0^{\text{gauss}}] (\omega) \\ &= \sqrt{2\pi} \frac{1}{\sigma^2} \exp \left(-\frac{\sigma^2}{2} \|\omega\|_2^2 \right) \sqrt{2\pi}^d \omega_i \omega_j. \end{aligned}$$

Hence

$$C^{\text{curl,gauss}}(\omega) = \underbrace{\frac{1}{\sqrt{2\pi} \frac{1}{\sigma^2}} \exp \left(-\frac{\sigma^2}{2} \|\omega\|_2^2 \right) \sqrt{2\pi}^d}_{\mu(\cdot) = \mathcal{N}(0, \sigma^{-2} I_d) \sqrt{2\pi}^d} \underbrace{\omega \omega^T}_{A(\omega) = \omega \omega^T}.$$

Here a canonical decomposition is $A^{\text{curl,gauss}}(\omega) = \omega \omega^T$ for all $\omega \in \mathbb{R}^d$ and $\mu^{\text{curl,gauss}} = \mathcal{N}(0, \sigma^{-2} I_d) \sqrt{2\pi}^d$ with respect to the normalized Lebesgue measure $d\omega$. Again the decomposition with respect to the usual –non-normalized– Lebesgue measure is for all $\omega \in \mathbb{R}^d$

$$A^{\text{curl,gauss}}(\omega) = \omega \omega^T \quad (3.14a)$$

$$\mu^{\text{curl,gauss}} = \mathcal{N}(0, \sigma^{-2} I_d) \quad (3.14b)$$

Notice that in this case $\|A^{\text{curl,gauss}}(\cdot)\|_{\mathbb{R}^d, \mathbb{R}^d}$ is not bounded. However applying proposition 3.5 yields a different decomposition where the quantity $\|A_{\text{Tr}}^{\text{curl,gauss}}(\cdot)\|_{\mathbb{R}^d, \mathbb{R}^d}$ is bounded. First we have for all $\delta \in \mathbb{R}^d$ and for all $i, j \in \mathbb{N}_d$

$$\frac{d^2}{d\delta_i d\delta_j} k_0^{\text{gauss}}(\delta) = \frac{\exp \left(-\frac{1}{2\sigma^2} \|\delta\|_2^2 \right)}{\sigma^2} \begin{cases} \frac{\delta_i \delta_j}{\sigma^2} & \text{if } i \neq j \\ \left(1 - \frac{\delta_i \delta_j}{\sigma^2} \right) & \text{otherwise.} \end{cases}$$

Hence

$$-\nabla\nabla^T k_0^{\text{gauss}}(\delta) = \left(I_d - \frac{\delta \delta^T}{\sigma^2} \right) \frac{\exp \left(-\frac{1}{2\sigma^2} \|\delta\|_2^2 \right)}{\sigma^2}$$

Thus $\text{Tr} [K_0^{\text{curl,gauss}}(0)] = \text{Tr} [\nabla \nabla^T k_0^{\text{gauss}}(0)] = d\sigma^{-2}$ and $\text{Tr} [C(\omega)] = \|\omega\|_2^2 \mathcal{N}(0, \sigma^{-2} I_d) \sqrt{2\pi}^d$. Apply proposition 3.5 to obtain the decomposition $A_{\text{Tr}}^{\text{curl,gauss}}(\omega) = \omega \omega^T \|\omega\|_2^{-2}$ and the measure $\mu_{\text{Tr}}^{\text{curl,gauss}}(\omega) = \sigma^2 d^{-1} \|\omega\|_2^2 \mathcal{N}(0, \sigma^{-2}) \sqrt{2\pi}^d$ for all $\omega \in \mathbb{R}^d$, with respect to the normalized Lebesgue measure. Therefore the decomposition with respect to the usual non-normalized Lebesgue measure is

$$A_{\text{Tr}}^{\text{curl,gauss}}(\omega) = \frac{\omega \omega^T}{\|\omega\|_2^2} \quad (3.15a)$$

$$\mu_{\text{Tr}}^{\text{curl,gauss}}(\omega) = \frac{\sigma^2}{d} \|\omega\|_2^2 \mathcal{N}(0, \sigma^{-2})(\omega) \quad (3.15b)$$

This example also illustrate that there exist many decomposition of $C(\omega)$ into $(A(\omega), \mu(\omega))$.

3.2.3.4 Divergence-free kernel

The divergence-free Gaussian kernel is defined as $K_0^{\text{div,gauss}} = (\nabla \nabla^T - \Delta) k_0^{\text{gauss}}$ on the group $\mathcal{X} = (\mathbb{R}^d, +)$. The setting is the same than section 3.2.3.1. Hence

$$\begin{aligned} C^{\text{div,gauss}}(\omega)_{ij} &= \mathcal{F} [K_0^{\text{div,gauss}}(\cdot)_{ij}] (\omega) \\ &= \mathcal{F} \left[\frac{d^2}{d\delta_i d\delta_j} k_0^{\text{gauss}} - \delta_{i=j} \sum_{k=1}^d \frac{d^2}{d\delta_k d\delta_k} k_0^{\text{gauss}} \right] (\omega) \\ &= \left(-(i\omega_i)(i\omega_j) - \delta_{i=j} \sum_{k=1}^d (i\omega_k)^2 \right) \mathcal{F} [k_0^{\text{gauss}}] \\ &= \left(\delta_{i=j} \sum_{k=1}^d \omega_k^2 - \omega_i \omega_j \right) \mathcal{F} [k_0^{\text{gauss}}] (\omega). \end{aligned}$$

Hence

$$C^{\text{div,gauss}}(\omega) = \underbrace{\frac{1}{\sqrt{2\pi}^{\frac{1}{\sigma^2}} d} \exp\left(-\frac{\sigma^2}{2} \|\omega\|_2^2\right)}_{\rho(\cdot) = \mathcal{N}(0, \sigma^{-2} I_d) \sqrt{2\pi}^d} \underbrace{\sqrt{2\pi}^d (I_d \|\omega\|_2^2 - \omega \omega^T)}_{A(\omega) = I_d \|\omega\|_2^2 - \omega \omega^T}.$$

Thus the canonical decomposition with respect to the normalized Lebesgue measure is $A^{\text{div,gauss}}(\omega) = I_d \|\omega\|_2^2 - \omega \omega^T$ and the measure $\rho^{\text{div,gauss}} = \mathcal{N}(0, \sigma^{-2} I_d) \sqrt{2\pi}^d$. The canonical decomposition with respect to the usual Lebesgue measure is

$$A^{\text{div,gauss}}(\omega) = I_d \|\omega\|_2^2 - \omega \omega^T \quad (3.16a)$$

$$\rho^{\text{div,gauss}} = \mathcal{N}(0, \sigma^{-2} I_d). \quad (3.16b)$$

To obtain the bounded decomposition, again, apply proposition 3.5. For all $\delta \in \mathbb{R}^d$,

$$\sum_{k=1}^d \frac{d^2}{d\delta_k d\delta_k} k_0^{\text{gauss}}(\delta) = \left(d - \frac{\|\delta\|_2^2}{\sigma^2} \right) \frac{\exp\left(-\frac{1}{2\sigma^2} \|\delta\|_2^2\right)}{\sigma^2}.$$

Thus overall,

$$K_0^{\text{div,gauss}}(\delta) = \left(\frac{\delta\delta^T}{\sigma^2} + \left((d-1) - \frac{\|\delta\|_2^2}{\sigma^2} \right) I_d \right) \frac{\exp\left(-\frac{1}{2\sigma^2}\|\delta\|_2^2\right)}{\sigma^2},$$

Eventually $\text{Tr} [K_0^{\text{div,gauss}}(0)] = \text{Tr} [(\nabla\nabla^T - \Delta)k_0^{\text{gauss}}(0)] = d(d-1)\sigma^{-2}$ and $\text{Tr} [C(\omega)] = (d-1)\|\omega\|_2^2 \mathcal{N}(0, \sigma^2 I_d) \sqrt{2\pi}^d$. As a result the decomposition with respect to the normalized Lebesgue measure is $A_{\text{Tr}}^{\text{div,gauss}}(\omega) = (I_d - \omega\omega^T \|\omega\|_2^{-2})$ and $\rho_{\text{Tr}}^{\text{div,gauss}}(\omega) = d^{-1} \sigma^2 \|\omega\|_2^2 \mathcal{N}(0, \sigma^2 I_d) \sqrt{2\pi}^d$. The decomposition with respect to the normalized Lebesgue measure being

$$A_{\text{Tr}}^{\text{div,gauss}}(\omega) = I_d - \frac{\omega\omega^T}{\|\omega\|_2^2} \quad (3.17a)$$

$$\rho_{\text{Tr}}^{\text{div,gauss}} = \frac{\sigma^2}{d} \|\omega\|_2^2 \mathcal{N}(0, \sigma^{-2} I_d). \quad (3.17b)$$

3.2.4 Functional Fourier feature map

Let us introduce a functional feature map, we call here *Fourier Feature map*, defined by the following proposition as a direct consequence of proposition 3.2.

Proposition 3.6 (Functional Fourier feature map). *Let \mathcal{Y} and \mathcal{Y}' be two Hilbert spaces. If there exist an operator-valued function $B : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y}, \mathcal{Y}')$ such that for all $y, y' \in \mathcal{Y}$,*

$$\langle y, B(\omega)B(\omega)^*y' \rangle_{\mathcal{Y}} = \langle y', A(\omega)y \rangle_{\mathcal{Y}}$$

$\hat{\mu}$ -almost everywhere and $\langle y', A(\cdot)y \rangle \in L^1(\hat{\mathcal{X}}, \hat{\mu})$ then the operator Φ_x defined for all y in \mathcal{Y} by

$$(\Phi_x y)(\omega) = (x, \omega)B(\omega)^*y, \quad (3.18)$$

is a feature map⁶ of some shift-invariant \mathcal{Y} -Mercer kernel K .

Proof For all $y, y' \in \mathcal{Y}$ and $x, z \in \mathcal{X}$,

$$\begin{aligned} \langle y, \Phi_x^* \Phi_z y' \rangle_{\mathcal{Y}} &= \langle \Phi_x y, \Phi_z y' \rangle_{L^2(\hat{\mathcal{X}}, \hat{\mu}; \mathcal{Y})} \\ &= \int_{\hat{\mathcal{X}}} \overline{(x, \omega)} \langle y, B(\omega)(z, \omega)B(\omega)^*y' \rangle d\hat{\mu}(\omega) \\ &= \int_{\hat{\mathcal{X}}} \overline{(x \star z^{-1}, \omega)} \langle y B(\omega)B(\omega)^*y' \rangle d\hat{\mu}(\omega) \\ &= \int_{\hat{\mathcal{X}}} \overline{(x \star z^{-1}, \omega)} \langle y, A(\omega)y' \rangle d\hat{\mu}(\omega), \end{aligned}$$

which defines a \mathcal{Y} -Mercer according to proposition 3.2 of Carmeli et al. [13]. \square

With this notation notice that $\Phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y}; L^2(\hat{\mathcal{X}}, \hat{\mu}; \mathcal{Y}'))$ such that $\Phi_x \in \mathcal{L}(\mathcal{Y}; L^2(\hat{\mathcal{X}}, \hat{\mu}; \mathcal{Y}'))$ where $\Phi_x := \Phi(x)$.

⁶ I. e. it satisfies for all $x, z \in \mathcal{X}$, $\Phi_x^* \Phi_z = K(x, z)$ where K is a \mathcal{Y} -Mercer OVK.

3.2.5 Building Operator-valued Random Fourier Features

As shown in propositions 3.4 and 3.5 it is always possible to find a pair $(A, \widehat{\mathbf{Pr}}_{\text{Haar}, \rho})$ from a shift invariant \mathcal{Y} -Mercer Operator-Valued Kernel K_e such that $\widehat{\mathbf{Pr}}_{\text{Haar}, \rho}$ is a probability measure -i.e. $\int_{\widehat{\mathcal{X}}} \rho d\widehat{\mathbf{Haar}} = 1$ where ρ is the density of $\widehat{\mathbf{Pr}}_{\text{Haar}, \rho}$ - and $K_e(\delta) = \mathbb{E}_\rho[\overline{(\delta, \omega)} A(\omega)]$. In order to obtain an approximation of K from a decomposition $(A, \widehat{\mathbf{Pr}}_{\text{Haar}, \rho})$ we turn our attention to a Monte-Carlo estimation of the expectations eq. (3.9) and eq. (3.5) characterizing a \mathcal{Y} -Mercer shift-invariant Operator-Valued Kernel.

However, for efficient computations as motivated in the introduction, we are interested in finding an approximated *feature map* instead of a kernel approximation. Indeed, an approximated feature map will allow to build linear models in regression tasks. The idea is to start from the Monte-Carlo approximation of the expectation and provide a systematic decomposition of the Monte-Carlo sample mean into an approximate feature map. The following proposition provides the general form of an Operator-valued Random Fourier Feature.

Proposition 3.7 (ORFF). *Let \mathcal{Y} and \mathcal{Y}' be two Hilbert spaces. If one can find $B : \widehat{\mathcal{X}} \rightarrow \mathcal{L}(\mathcal{Y}, \mathcal{Y}')$ and a probability measure $\widehat{\mathbf{Pr}}_{\text{Haar}, \rho}$ on $\mathcal{B}(\widehat{\mathcal{X}})$, such that for all $y \in \mathcal{Y}$ and all $y' \in \mathcal{Y}'$, $\langle y, B(\cdot)y' \rangle \in L^2(\widehat{\mathcal{X}}, \widehat{\mathbf{Pr}}_{\text{Haar}, \rho})$, then the operator-valued function given for all $y \in \mathcal{Y}$ by*

$$\widetilde{\Phi}(x)y = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D (x, \omega_j) B(\omega_j)^* y, \quad \omega_j \sim \widehat{\mathbf{Pr}}_{\text{Haar}, \rho} \text{ i.i.d.}, \quad (3.19)$$

is an approximated feature map⁷ of a shift-invariant \mathcal{Y} -Mercer Operator-Valued Kernel.

Proof Let $(\omega_j)_{j=1}^D$ be a sequence of $D \in \mathbb{N}$ i.i.d. random vectors, each of them following the law $\widehat{\mathbf{Pr}}_{\text{Haar}, \rho}$. For all $x, z \in \widehat{\mathcal{X}}$ and all $y, y' \in \mathcal{Y}$,

$$\begin{aligned} \langle \widetilde{\Phi}(x)y', \widetilde{\Phi}(z)y \rangle &= \frac{1}{D} \left\langle \bigoplus_{j=1}^D (x, \omega_j) B(\omega_j)^* y', \bigoplus_{j=1}^D (z, \omega_j) B(\omega_j)^* y \right\rangle \\ &= \frac{1}{D} \sum_{j=1}^D \langle y', \overline{(x, \omega_j)} B(\omega_j) (z, \omega_j) B(\omega_j)^* y \rangle_y \\ &= \left\langle y', \left(\frac{1}{D} \sum_{j=1}^D \overline{(x \star z^{-1}, \omega_j)} A(\omega_j) \right) y \right\rangle_y, \end{aligned}$$

where $A(\omega) = B(\omega)B(\omega)^*$. By assumption $\langle y, A(\cdot)y' \rangle \in L^1(\widehat{\mathcal{X}}, \widehat{\mathbf{Pr}}_{\text{Haar}, \rho})$ and ω_j are i.i.d.. Hence from the strong law of large numbers and proposition 3.2 with $\widehat{\mu} = \widehat{\mathbf{Pr}}_{\text{Haar}, \rho}$,

$$\frac{1}{D} \sum_{j=1}^D \overline{(x \star z^{-1}, \omega_j)} A(\omega_j) \xrightarrow[D \rightarrow \infty]{a.s.} \mathbb{E}_\rho[\overline{(x \star z^{-1}, \omega)} A(\omega)] = K_e(x \star z^{-1})$$

⁷ I.e. it satisfies $\widetilde{\Phi}(x)^* \widetilde{\Phi}(z) \xrightarrow[D \rightarrow \infty]{a.s.} K(x, z)$ where K is a \mathcal{Y} -Mercer OVK.

where the integral converges in the weak operator topology. \square

Remark 3.2 The approximate feature map proposed in proposition 3.7 has direct link with the functional Fourier feature map defined in proposition 3.6 since we have for all $y \in \mathcal{Y}$

$$\begin{aligned}\tilde{\Phi}(x)y &= \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D (x, \omega_j) B(\omega_j)^* y, \quad \omega_j \sim \widehat{\mathbf{Pr}}_{\text{Haar}, \rho} \text{ i. i. d.} \\ &= \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D (\Phi_x y)(\omega_j).\end{aligned}\tag{3.20}$$

Therefore $\tilde{\Phi}(x)$ can be seen as an “operator-valued vector” corresponding the “stacking” of D i. i. d. operator-valued random variable $(x, \omega_j) B(\omega_j)^*$. Note that we consider ω to be a *variable* in $\hat{\mathcal{X}}$ while ω_j are $\hat{\mathcal{X}}$ -valued *random variables*. I. e. ω_j are $(\hat{\mathcal{X}}, \mathcal{B}(\hat{\mathcal{X}}))$ -valued measurable function and $\widehat{\mathbf{Pr}}_{\text{Haar}, \rho}$ is the distribution of each ω_j . Let

$$\tilde{\mathcal{H}} = \bigoplus_{j=1}^D \mathcal{Y}',$$

be a Hilbert space. Let $\omega \in \hat{\mathcal{X}}^D$ be a sequence where $\omega = (\omega_j)_{j=1}^D$ (i. e. an outcome of the random sequence $(\omega_j)_{j=1}^D$, $\omega_j \sim \widehat{\mathbf{Pr}}_{\text{Haar}, \rho}$). Notice that for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, $\tilde{\Phi}(x)y \in \tilde{\mathcal{H}}$ as we can take $g(\omega_j) = (x, \omega_j) B(\omega_j)^* y$. Thus we have for all $y, y' \in \mathcal{Y}$ and all $x, z \in \mathcal{X}$

$$\langle y', \tilde{K}(x, z)y \rangle_{\mathcal{Y}} = \langle y', \tilde{\Phi}(x)^* \tilde{\Phi}(z)y \rangle_{\mathcal{Y}} = \langle \tilde{\Phi}(x)y', \tilde{\Phi}(z)y \rangle_{\tilde{\mathcal{H}}}.$$

Thus $\tilde{K}(x, z) = \tilde{\Phi}(x)^* \tilde{\Phi}(z)$ is a proper random shift-invariant Operator-Valued Kernel as shown in the following proposition.

Proposition 3.8 Let $\omega \in \hat{\mathcal{X}}^D$. If for all $y, y' \in \mathcal{Y}$

$$\begin{aligned}\langle y', \tilde{K}_e(x \star z^{-1})y \rangle_{\mathcal{Y}} &= \langle \tilde{\Phi}(x)y', \tilde{\Phi}(z)y \rangle_{\tilde{\mathcal{H}}} \\ &= \left\langle y', \frac{1}{D} \sum_{j=1}^D \overline{(x \star z^{-1}, \omega_j)} B(\omega_j) B(\omega_j)^* y \right\rangle_{\mathcal{Y}},\end{aligned}$$

for all $x, z \in \mathcal{X}$, then \tilde{K} is a shift-invariant Operator-Valued Kernel.

Proof Apply proposition 2.3 to $\tilde{\Phi}$ considering the Hilbert space $\tilde{\mathcal{H}}$ to show that \tilde{K} is an *OVK*. Then proposition 2.6 shows that \tilde{K} is shift-invariant since $\tilde{K}(x, z) = \tilde{K}_e(x, \star z^{-1})$. \square

We stress out that if $\omega = (\omega_j)_{j=1}^D \sim \widehat{\mathbf{Pr}}_{\text{Haar}, \rho}$ i. i. d. is a *random sequence* then

$$\tilde{K}_e(x \star z^{-1}) = \tilde{\Phi}(x)^* \tilde{\Phi}(z)$$

is not *sensu stricto* an Operator-Valued Kernel since it is a random variable (and $\tilde{\mathcal{H}}$ is no longer a Hilbert space, since its inner product is a then

random variable and not a scalar). However this is not a problem since any outcome of the random sequence ω gives birth to a (different) Operator-Valued Kernel, and $\mathbb{E}_{\text{Haar}, \rho} \tilde{K}$ is an [OVK](#). This illustrated by fig. 1 where we represented the same function for different realization of $\tilde{K} \approx K$. We generated 250 points equally separated on the segment $(-1; 1)$. We computed

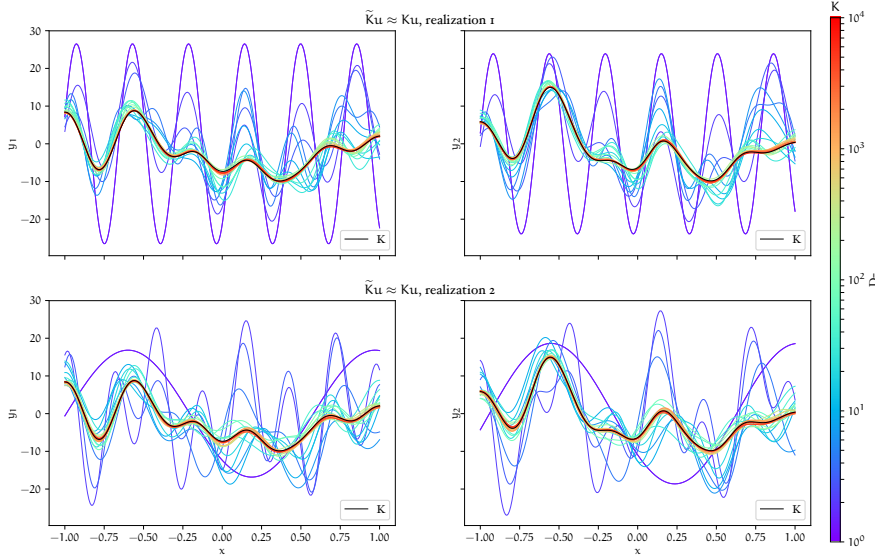


Figure 1: Different outcomes of a Gaussian kernel approximation. Top row and bottom row correspond to two different outcomes of \tilde{K} , which are *different* Operator-Valued Kernel. However when D tends to infinity, the different outcomes fo \tilde{K} yield the samel [OVK](#).

the Gram Matrix of the Gaussian decomposable kernel

$$K(x, z)_{ij} = \exp \left(-\frac{1}{2(0.1)^2(x_i - x_j)^2} \right) \Gamma, \quad \text{for } i, j \in \mathbb{N}_{250}.$$

We computed a reference function (black line) defined as $(y_1, y_2)^T = f(x_i) = \sum_{j=1}^{250} K(x_i, x_j) u_j$ where $u_j \sim \mathcal{N}(0, 1)$ i. i. d.. We took $\Gamma = .5I_2 + .5I_2$ such that the outputs y_1 and y_2 share some similarities. Then we computed an approximate kernel matrix $\tilde{K} \approx K$ for 25 increasing values of D ranging from 1 to 10^4 . The two graphs on the top row shows that the more the number of features increase the closer the model $\tilde{f}(x_i) = \sum_{j=1}^{250} \tilde{K}(x_i, x_j) u_j$ is to f . The bottom row shows the same experiment but for a different realization of \tilde{K} . When D is small the curves of the bottom and top rows are very dissimilar –and sine wave like– while they both converge to f when D increase.

In the same way we defined an [ORFF](#), we can define an approximate feature operator \tilde{W} which maps $\tilde{\mathcal{H}}$ onto $\mathcal{H}_{\tilde{K}}$, where

$$\tilde{K}(x, z) = \tilde{\Phi}(x) * \tilde{\Phi}(z)$$

for all $x, z \in \mathcal{X}$.

Definition 3.1 (Random Fourier feature operator). Let $\omega = (\omega_j)_{j=1}^D \in \widehat{\mathcal{X}}^D$ and let

$$\widetilde{\mathcal{K}}_e = \frac{1}{D} \sum_{j=1}^D \overline{(\cdot, \omega_j)} B(\omega_j) B(\omega_j)^*.$$

We call random Fourier feature operator the linear application $\widetilde{W} : \mathcal{H} \rightarrow \mathcal{H}_{\widetilde{\mathcal{K}}}$ defined as

$$(\widetilde{W}\theta)(x) := \widetilde{\Phi}(x)^* \theta = \frac{1}{D} \sum_{j=1}^D \overline{(x, \omega_j)} B(\omega_j) g(\omega_j)$$

where $\theta = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D g(\omega_j) \in \widetilde{\mathcal{H}}$. Then,

$$\left(\text{Ker } \widetilde{W} \right)^\perp = \overline{\text{span}} \left\{ \widetilde{\Phi}(x)y \mid \forall x \in \mathcal{X}, \forall y \in \mathcal{Y} \right\} \subseteq \widetilde{\mathcal{H}}.$$

The random Fourier feature operator is useful to show the relations between the random Fourier feature map with the functional feature map defined in proposition 3.6. The relationship between the generic feature map (defined for all Operator-Valued Kernel) the functional feature map (defining a shift-invariant \mathcal{Y} -Mercer Operator-Valued Kernel) and the random Fourier feature map is presented in fig. 2.

Proposition 3.9 For any $g \in \mathcal{H} = L^2(\widehat{\mathcal{X}}, \widehat{\text{Pr}}_{\text{Haar}, \rho}; \mathcal{Y}')$, let

$$\theta := \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D g(\omega_j), \quad \omega_j \sim \widehat{\text{Pr}}_{\text{Haar}, \rho} \text{ i. i. d..}$$

Then

1. $(\widetilde{W}\theta)(x) = \widetilde{\Phi}(x)^* \theta \xrightarrow[D \rightarrow \infty]{a.s.} \Phi_x^* g = (Wg)(x),$
2. $\|\theta\|_{\widetilde{\mathcal{H}}}^2 \xrightarrow[D \rightarrow \infty]{a.s.} \|g\|_{\mathcal{H}}^2,$

Proof (of proposition 3.9 item 1) since $(\omega_j)_{j=1}^D$ are i. i. d. random vectors, for all $y \in \mathcal{Y}$ and for all $y' \in \mathcal{Y}'$, $\langle y, B(\cdot)y' \rangle \in L^2(\widehat{\mathcal{X}}, \widehat{\text{Pr}}_{\text{Haar}, \rho})$ and $g \in L^2(\widehat{\mathcal{X}}, \widehat{\text{Pr}}_{\text{Haar}, \rho}; \mathcal{Y}')$, from the strong law of large numbers

$$\begin{aligned} (\widetilde{W}\theta)(x) &= \widetilde{\Phi}(x)^* \theta \\ &= \frac{1}{D} \sum_{j=1}^D \overline{(x, \omega_j)} B(\omega_j) g(\omega_j), \quad \omega_j \sim \widehat{\text{Pr}}_{\text{Haar}, \rho} \text{ i. i. d.} \\ &\xrightarrow[D \rightarrow \infty]{a.s.} \int_{\widehat{\mathcal{X}}} \overline{(x, \omega)} B(\omega) g(\omega) d\widehat{\text{Pr}}_{\text{Haar}, \rho}(\omega) \\ &= (Wg)(x) := \Phi_x^* g. \quad \square \end{aligned}$$

Proof (of proposition 3.9 item 2) again, since $(\omega_j)_{j=1}^D$ are i. i. d. random vectors and $g \in L^2(\hat{\mathcal{X}}, \widehat{\mathbf{Pr}}_{\mathbf{Haar}, \rho}; \mathcal{Y}')$, from the strong law of large numbers

$$\begin{aligned} \|\theta\|_{\mathcal{H}}^2 &= \frac{1}{D} \sum_{j=1}^D \|g(\omega_j)\|_{\mathcal{Y}'}^2, \quad \omega_j \sim \widehat{\mathbf{Pr}}_{\mathbf{Haar}, \rho} \text{ i. i. d.} \\ &\xrightarrow[D \rightarrow \infty]{a. s.} \int_{\hat{\mathcal{X}}} \|g(\omega)\|_{\mathcal{Y}'}^2 d\widehat{\mathbf{Pr}}_{\mathbf{Haar}, \rho}(\omega) \\ &= \|g\|_{L^2(\hat{\mathcal{X}}, \widehat{\mathbf{Pr}}_{\mathbf{Haar}, \rho}; \mathcal{Y}')}^2. \quad \square \end{aligned}$$

We write $\tilde{\Phi}(x)^* \tilde{\Phi}(x) \approx K(x, z)$ when $\tilde{\Phi}(x)^* \tilde{\Phi}(x) \xrightarrow{a. s.} K(x, z)$ in the weak operator topology when D tends to infinity. With mild abuse of notation we say that $\tilde{\Phi}(x)$ is an approximate feature map of Φ_x i. e. $\tilde{\Phi}(x) \approx \Phi_x$, when for all $y', y \in \mathcal{Y}$,

$$\begin{aligned} \langle y, K(x, z)y' \rangle_y &= \langle \Phi_x y, \Phi_z y' \rangle_{\mathcal{L}(\hat{\mathcal{X}}, \widehat{\mathbf{Pr}}_{\mathbf{Haar}, \rho}; \mathcal{Y}')} \\ &\approx \langle \tilde{\Phi}(x)y, \tilde{\Phi}(z)y' \rangle_{\mathcal{H}} := \langle y, \tilde{K}(x, z)y' \rangle_y \end{aligned}$$

where Φ_x is defined in the sense of proposition 3.6. Then corollary 3.1 exhibit a construction of an ORFF directly from an OVK.

Corollary 3.1 *If $K(x, z)$ is a shift-invariant \mathcal{Y} -Mercer kernel such that for all $y, y' \in \mathcal{Y}$, $\langle y', K_e(\cdot)y \rangle \in L^1(\mathcal{X}, \mathbf{Haar})$. Then*

$$\tilde{\Phi}(x)y = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D (x, \omega_j) B(\omega_j)^* y, \quad \omega_j \sim \widehat{\mathbf{Pr}}_{\mathbf{Haar}, \rho} \text{ i. i. d.}, \quad (3.21)$$

where $\langle y, B(\omega)B(\omega)^* y' \rangle_{\rho(\omega)} = \mathcal{F}[\langle y', K_e(\cdot)y \rangle](\omega)$, is an approximated feature map of K .

Proof Find $(A, \widehat{\mathbf{Pr}}_{\mathbf{Haar}, \rho})$ from proposition 3.4, find a decomposition of $A(\omega) = B(\omega)B(\omega)^*$ for $\widehat{\mathbf{Pr}}_{\mathbf{Haar}, \rho}$ -almost all ω and apply proposition 3.7. \square

Remark 3.3 *We find a decomposition such that for all $j = 1, \dots, D$, $A(\omega_j) = B(\omega_j)B(\omega_j)^*$ either by exhibiting an analytic closed-form or using a numerical decomposition.*

Corollary 3.1 allows us to define algorithm 1 for constructing ORFF from an operator valued kernel.

Algorithm 1: Construction of ORFF from OVK

Input : $K(x, z) = K_e(\delta)$ a \mathcal{Y} -shift-invariant Mercer kernel such that $\forall y, y' \in \mathcal{Y}, \langle y', K_e(\cdot)y \rangle \in L^1(\mathbb{R}^d, \mathbf{Haar})$ and D the number of features.

Output: A random feature $\tilde{\Phi}(x)$ such that $\tilde{\Phi}(x)^* \tilde{\Phi}(z) \approx K(x, z)$

- 1 Define the pairing (x, ω) from the LCA group (\mathcal{X}, \star) ;
- 2 Find a decomposition $(B(\omega), \widehat{\mathbf{Pr}}_{\mathbf{Haar}, \rho})$ such that $B(\omega)B(\omega)^* \rho(\omega) = \mathcal{F}^{-1}[K_e](\omega)$;
- 3 Draw D random vectors $(\omega_j)_{j=1}^D$ i. i. d. from the probability law $\widehat{\mathbf{Pr}}_{\mathbf{Haar}, \rho}$;
- 4 **return** $\begin{cases} \tilde{\Phi}(x) & : y \mapsto \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D (x, \omega_j) B(\omega_j)^* y \\ \tilde{\Phi}(x)^* & : \theta \mapsto \frac{1}{\sqrt{D}} \sum_{j=1}^D (x, \omega_j) B(\omega_j) \theta \end{cases}$;

When $\lambda_M = 0$ or there is no

3.2.6 Examples of Operator Random Fourier Feature maps

We now give two examples of operator-valued random Fourier feature map when. First we introduce the general form of an approximated feature map for a matrix-valued kernel on the additive group $(\mathbb{R}^d, +)$.

Example 3.1 (Matrix-valued kernel on the additive group). *In the following, $K(x, z) = K_0(x - z)$ is a \mathcal{Y} -Mercer matrix-valued kernel on $\mathcal{X} = \mathbb{R}^d$ invariant w.r.t. the group operation $+$. Then the function $\tilde{\Phi}$ defined as follow is an Operator-valued Random Fourier Feature of K_0 .*

$$\tilde{\Phi}(x)y = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D \begin{pmatrix} \cos \langle x, \omega_j \rangle B(\omega_j)^* y \\ \sin \langle x, \omega_j \rangle B(\omega_j)^* y \end{pmatrix}, \quad \omega_j \sim \widehat{\mathbf{Pr}}_{\mathbf{Haar}, \rho} \text{ i. i. d.}$$

for all $y \in \mathcal{Y}$.

Proof *The (Pontryagin) dual of $\mathcal{X} = \mathbb{R}^d$ is $\widehat{\mathcal{X}} \cong \mathbb{R}^d$, and the duality pairing is $\langle x - z, \omega \rangle = \exp(i \langle x - z, \omega \rangle)$. The kernel approximation yields:*

$$\begin{aligned} \tilde{K}(x, z) &= \tilde{\Phi}(x)^* \tilde{\Phi}(z) \\ &= \frac{1}{D} \sum_{j=1}^D \begin{pmatrix} \cos \langle x, \omega_j \rangle & \sin \langle x, \omega_j \rangle \end{pmatrix} \begin{pmatrix} \cos \langle z, \omega_j \rangle \\ \sin \langle z, \omega_j \rangle \end{pmatrix} A(\omega_j) \\ &= \frac{1}{D} \sum_{j=1}^D \cos \langle x - z, \omega_j \rangle A(\omega_j) \\ &\xrightarrow[D \rightarrow \infty]{a.s.} \mathbf{E}_\rho [\cos \langle x - z, \omega \rangle A(\omega)] \end{aligned}$$

in the weak operator topology. Since for all $x \in \mathcal{X}$, $\sin \langle x, \cdot \rangle$ is an odd function and $A(\cdot)\rho(\cdot)$ is even,

$$\mathbf{E}_\rho [\cos \langle x - z, \omega \rangle A(\omega)] = \mathbf{E}_\rho [\exp(-i \langle x - z, \omega \rangle) A(\omega)] = K(x, z).$$

$$\begin{array}{c}
\Phi_x^* \Phi_z = K(x, z) \quad = \quad K_e \left(x \star z^{-1} \right) \quad \approx \quad \tilde{K}_e \left(x \star z^{-1} \right) = \tilde{\Phi}(x)^* \tilde{\Phi}(x) \\
\hline
\begin{array}{ccc}
\begin{array}{c} \Phi_x \in \mathcal{L}(y; \mathcal{H}) \xrightarrow{\Phi_x^* g} y \\ \uparrow \Phi \\ x \in \mathcal{X} \end{array} & \begin{array}{c} \Phi_x \in \mathcal{L}\left(y; L^2\left(\widehat{\mathcal{X}}, \widehat{\mathbf{Pr}}_{\widehat{\mathbf{Haar}}, \rho}; y'\right)\right) \xrightarrow{\Phi_x^* g} y \\ \uparrow \Phi \\ x \in \mathcal{X} \end{array} & \begin{array}{c} \tilde{\Phi}(x) \in \mathcal{L}(y; \tilde{\mathcal{H}}) \xrightarrow{\tilde{\Phi}(x)^* \theta} y \\ \uparrow \tilde{\Phi} \\ x \in \mathcal{X} \end{array} \\
\text{Fourier,} & \text{Monte-Carlo,} & \\
\Phi_x(\omega)y = (x, \omega)B(\omega)^*y. & \tilde{\Phi}(x)y = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D (\Phi_x y)(\omega_j), & \\
& \omega_j \sim \widehat{\mathbf{Pr}}_{\widehat{\mathbf{Haar}}, \rho} \text{ i.i.d..} &
\end{array}
\end{array}$$

Figure 2: Relationships between feature-maps. $\tilde{\mathcal{H}} = \bigoplus_{j=1}^D y'$.

Hence $\tilde{K}(x, z) \xrightarrow[D \rightarrow \infty]{a.s.} K(x, z)$. \square

In particular we deduce the following features maps for the kernels proposed in section 3.2.3.

- For the decomposable gaussian kernel $K_0^{\text{dec,gauss}}(\delta) = k_0^{\text{gauss}}(\delta)\Gamma$ for all $\delta \in \mathbb{R}^d$, let $BB^* = \Gamma$. A bounded and unbounded ORFF map is

$$\begin{aligned}\tilde{\Phi}(x)y &= \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D \begin{pmatrix} \cos \langle x, \omega_j \rangle B^* y \\ \sin \langle x, \omega_j \rangle B^* y \end{pmatrix} \\ &= (\tilde{\Phi}(x) \otimes B^*)y,\end{aligned}$$

where $\omega_j \sim \text{Pr}_{\mathcal{N}(0, \sigma^{-2}I_d)}$ i. i. d. and $\tilde{\Phi}(x) = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D \begin{pmatrix} \cos \langle x, \omega_j \rangle \\ \sin \langle x, \omega_j \rangle \end{pmatrix}$ is a scalar RFF map [33].

- For the curl-free gaussian kernel, $K_0^{\text{curl,gauss}} = -\nabla \nabla^T k_0^{\text{gauss}}$ an unbounded ORFF map is

$$\tilde{\Phi}(x)y = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D \begin{pmatrix} \cos \langle x, \omega_j \rangle \omega_j^T y \\ \sin \langle x, \omega_j \rangle \omega_j^T y \end{pmatrix},$$

$\omega_j \sim \text{Pr}_{\mathcal{N}(0, \sigma^{-2}I_d)}$ i. i. d. and a bounded ORFF map is

$$\tilde{\Phi}(x)y = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D \begin{pmatrix} \cos \langle x, \omega_j \rangle \frac{\omega_j^T}{\|\omega_j\|} y \\ \sin \langle x, \omega_j \rangle \frac{\omega_j^T}{\|\omega_j\|} y \end{pmatrix}, \quad \omega_j \sim \text{Pr}_\rho \text{ i. i. d..}$$

where $\rho(\omega) = \frac{\sigma^2 \|\omega\|^2}{d} \mathcal{N}(0, \sigma^{-2}I_d)(\omega)$ for all $\omega \in \mathbb{R}^d$.

- For the divergence-free gaussian kernel $K_0^{\text{div,gauss}}(x, z) = (\nabla \nabla^T - \Delta I_d)k_0^{\text{gauss}}(x, z)$ an unbounded ORFF map is

$$\tilde{\Phi}(x)y = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D \begin{pmatrix} \cos \langle x, \omega_j \rangle B(\omega_j)^T y \\ \sin \langle x, \omega_j \rangle B(\omega_j)^T y \end{pmatrix}, \quad \omega_j \sim \text{Pr}_\rho \text{ i. i. d..}$$

where $B(\omega) = \left(I_d - \frac{\omega \omega^T}{\|\omega\|^2}\right)$ and $\rho = \mathcal{N}(0, \sigma^{-2}I_d)$ for all $\omega \in \mathbb{R}^d$. A bounded ORFF map is

$$\tilde{\Phi}(x)y = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D \begin{pmatrix} \cos \langle x, \omega_j \rangle B(\omega_j)^T y \\ \sin \langle x, \omega_j \rangle B(\omega_j)^T y \end{pmatrix}, \quad \omega_j \sim \text{Pr}_\rho \text{ i. i. d.,}$$

where $B(\omega) = \left(I_d - \frac{\omega \omega^T}{\|\omega\|^2}\right)$ and $\rho(\omega) = \frac{\sigma^2 \|\omega\|^2}{d} \mathcal{N}(0, \sigma^{-2}I_d)$ for all $\omega \in \mathbb{R}^d$.

The second example extends scalar-valued Random Fourier Features on the skewed multiplicative group –described in ?? and section 3.2.3.2– to the operator-valued case.

Example 3.2 (Matrix-valued kernel on the skewed multiplicative group). *In the following, $K(x, z) = K_{1-c}(x \odot z^{-1})$ is a \mathcal{Y} -Mercer matrix-valued kernel on $\mathcal{X} = (-c; +\infty)^d$ invariant w. r. t. the group operation⁸ \odot . Then the function $\tilde{\Phi}$ defined as follow is an Operator-valued Random Fourier Feature of K_{1-c} .*

⁸ The group operation \odot is defined in section 3.2.3.2.

$$\tilde{\Phi}(x)y = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D \begin{pmatrix} \cos \langle \log(x+c), \omega_j \rangle B(\omega_j)^* y \\ \sin \langle \log(x+c), \omega_j \rangle B(\omega_j)^* y \end{pmatrix},$$

$\omega_j \sim \widehat{\text{Pr}}_{\text{Haar}, \rho}$ iid, for all $y \in \mathcal{Y}$.

Proof The dual of $\mathcal{X} = (-c; +\infty)^d$ is $\hat{\mathcal{X}} \cong \mathbb{R}^d$, and the duality pairing is $(x \odot z^{-1}, \omega) = \exp(i \langle \log(x \odot z^{-1} + c), \omega \rangle)$. Following the proof of example 3.1, we have

$$\tilde{K}(x, z) = \frac{1}{D} \sum_{j=1}^D \cos \left\langle \log \left(\frac{x+c}{z+c} \right), \omega_j \right\rangle A(\omega_j).$$

which converges almost surely to

$$\mathbb{E}_\rho[\exp(-i \langle \log(x \odot z^{-1} + c) \rangle) A(\omega)] = \mathbb{E}_\rho[\overline{(\langle x \odot z^{-1}, \omega \rangle)} A(\omega)] = K(x, z)$$

when D tends to infinity, in the weak operator topology. \square

- For the skewed- χ^2 decomposable kernel defined as $K_{1-c}^{\text{dec}, \text{skewed}}(\delta) = k_{1-c}^{\text{skewed}}(\delta)\Gamma$ for all $\delta \in \mathcal{X}$, let $BB^* = \Gamma$. A bounded –and unbounded–ORFF map is

$$\begin{aligned} \tilde{\Phi}(x)y &= \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D \begin{pmatrix} \cos \langle \log(x+c), \omega_j \rangle B^* y \\ \sin \langle \log(x+c), \omega_j \rangle B^* y \end{pmatrix}, \quad \omega_j \sim \text{Pr}_\rho \text{ i. i. d.} \\ &= (\tilde{\Phi}(x) \otimes B^*)y, \end{aligned}$$

where $\rho = \mathcal{S}(0, 2^{-1})$ and $\tilde{\Phi}(x) = \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D \begin{pmatrix} \cos \langle \log(x+c), \omega_j \rangle \\ \sin \langle \log(x+c), \omega_j \rangle \end{pmatrix}$ is a scalar RFF map [23].

3.2.7 Regularization property

We have shown so far that it is always possible to construct a feature map that allows to approximate a shift-invariant \mathcal{Y} -Mercer kernel. However we could also propose a construction of such map by studying the regularization induced with respect to the Fourier transform of a target function $f \in \mathcal{H}_K$. In other words, what is the norm in $L^2(\hat{\mathcal{X}}, \widehat{\text{Haar}}; \mathcal{Y}')$ induced by $\|\cdot\|_K$?

Proposition 3.10 *Let K be a shift-invariant \mathcal{Y} -Mercer Kernel such that for all y, y' in \mathcal{Y} , $\langle y', K_e(\cdot)y \rangle \in L^1(\mathcal{X}, \mathbf{Haar})$. Then for all $f \in \mathcal{H}_K$*

$$\|f\|_K^2 = \int_{\widehat{\mathcal{X}}} \frac{\langle \mathcal{F}[f](\omega), A(\omega)^\dagger \mathcal{F}[f](\omega) \rangle_{\mathcal{Y}}}{\rho(\omega)} d\widehat{\mathbf{Haar}}(\omega). \quad (3.22)$$

where $\langle y', A(\omega)y \rangle \rho(\omega) := \mathcal{F}[\langle y', K_e(\cdot)y \rangle](\omega)$.

Proof We first show how the Fourier transform relates to the feature operator. Since \mathcal{H}_K is embed into $\mathcal{H} = L^2(\widehat{\mathcal{X}}, \mathbf{Pr}_{\widehat{\mathbf{Haar}}, \rho}; \mathcal{Y})$ by mean of the feature operator W , we have for all $f \in \mathcal{H}_K$, for all $f \in \mathcal{H}$ and for all $x \in \mathcal{X}$

$$\begin{aligned} \mathcal{F}[\mathcal{F}^{-1}[f]](x) &= \int_{\widehat{\mathcal{X}}} \overline{(x, \omega)} \mathcal{F}^{-1}[f](\omega) d\widehat{\mathbf{Haar}}(\omega) = f(x) \\ (Wg)(x) &= \int_{\widehat{\mathcal{X}}} \overline{(x, \omega)} \rho(\omega) B(\omega) g(\omega) d\widehat{\mathbf{Haar}}(\omega) = f(x). \end{aligned}$$

By injectivity of the Fourier transform, $\mathcal{F}^{-1}[f](\omega) = \rho(\omega) B(\omega) g(\omega)$. From proposition 2.3 we have

$$\begin{aligned} \|f\|_K^2 &= \inf \left\{ \|g\|_{\mathcal{H}}^2 \mid \forall g \in \mathcal{H}, Wg = f \right\} \\ &= \inf \left\{ \int_{\widehat{\mathcal{X}}} \|g\|_{\mathcal{Y}}^2 d\mathbf{Pr}_{\widehat{\mathbf{Haar}}, \rho} \mid \forall g \in \mathcal{H}, \mathcal{F}^{-1}[f] = \rho(\cdot) B(\cdot) g(\cdot) \right\}. \end{aligned}$$

The pseudo inverse of the operator $B(\omega)$ – noted $B(\omega)^\dagger$ – is the unique solution of the system $\mathcal{F}^{-1}[f](\omega) = \rho(\omega) B(\omega) g(\omega)$ w. r. t. $g(\omega)$ with minimal norm⁹. Eventually,

$$\|f\|_K^2 = \int_{\widehat{\mathcal{X}}} \frac{\|B(\omega)^\dagger \mathcal{F}^{-1}[f](\omega)\|_{\mathcal{Y}}^2}{\rho(\omega)^2} d\mathbf{Pr}_{\widehat{\mathbf{Haar}}, \rho}(\omega)$$

Using the fact that $\mathcal{F}^{-1}[\cdot] = \mathcal{F}\mathcal{R}[\cdot]$ and $\mathcal{F}^2[\cdot] = \mathcal{R}[\cdot]$,

$$\begin{aligned} \|f\|_K^2 &= \int_{\widehat{\mathcal{X}}} \frac{\|\mathcal{R}[B(\cdot)^\dagger \rho(\cdot)](\omega) \mathcal{F}[f](\omega)\|_{\mathcal{Y}}^2}{\rho(\omega)^2} d\widehat{\mathbf{Haar}}(\omega) \\ &= \int_{\widehat{\mathcal{X}}} \frac{\|B(\omega)^\dagger \rho(\omega) \mathcal{F}[f](\omega)\|_{\mathcal{Y}}^2}{\rho(\omega)^2} d\widehat{\mathbf{Haar}}(\omega) \\ &= \int_{\widehat{\mathcal{X}}} \frac{\langle B(\omega)^\dagger \mathcal{F}[f](\omega), B(\omega)^\dagger \mathcal{F}[f](\omega) \rangle_{\mathcal{Y}}}{\rho(\omega)} d\widehat{\mathbf{Haar}}(\omega) \\ &= \int_{\widehat{\mathcal{X}}} \frac{\langle \mathcal{F}[f](\omega), A(\omega)^\dagger \mathcal{F}[f](\omega) \rangle_{\mathcal{Y}}}{\rho(\omega)} d\widehat{\mathbf{Haar}}(\omega) \end{aligned}$$

Note that if $K(x, z) = k(x, z)$ is a scalar kernel then for all ω in $\widehat{\mathcal{X}}$, $A(\omega) = 1$. Therefore we recover a well known results for kernels that is for any $f \in \mathcal{H}_K$ we have $\|f\|_K = \int_{\widehat{\mathcal{X}}} \mathcal{F}[k_e](\omega)^{-1} \mathcal{F}[f](\omega)^2 d\widehat{\mathbf{Haar}}(\omega)$ [38, 43, 46]. We also note that the regularization property in \mathcal{H}_K does not depends (as expected) on the decomposition of $A(\omega)$ into $B(\omega)B(\omega)^*$. Therefore the decomposition should be chosen such that it optimizes the computation cost. For instance if $A(\omega) \in \mathcal{L}(\mathbb{R}^p)$ has rank r , one could find an

⁹ Note that since $B(\omega)$ is bounded the pseudo inverse of $B(\omega)$ is well defined for $\widehat{\mathbf{Haar}}$ -almost all ω . However if $B(\omega)$ is infinite dimensional, the pseudo inverse is continuous if and only if $A(\omega)$ has closed range. This is always true if \mathcal{Y} is finite dimensional.

operator $B(\omega) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^r)$ such that $A(\omega) = B(\omega)B(\omega)^*$. Moreover, in light of eq. (3.22) the regularization property of the kernel with respect to the Fourier transform, it is also possible to define an approximate feature map of an Operator-Valued Kernel from its regularization properties in the \mathcal{Y} -RKHS as proposed in algorithm 2.

Algorithm 2: Construction of ORFF

Input :

- The pairing (x, ω) of the LCA group (\mathcal{X}, \star) .
- A probability measure $\Pr_{\widehat{\text{Haar}}, \rho}$ with density ρ w. r. t. the haar measure $\widehat{\text{Haar}}$ on $\widehat{\mathcal{X}}$.
- An operator-valued function $B : \widehat{\mathcal{X}} \rightarrow \mathcal{L}(\mathcal{Y}, \mathcal{Y}')$ such that for all $y, y' \in \mathcal{Y}$, $\langle y', B(\cdot)B(\cdot)^*y \rangle \in L^1(\widehat{\mathcal{X}}, \Pr_{\widehat{\text{Haar}}, \rho})$.
- D the number of features.

Output: A random feature $\tilde{\Phi}(x)$ such that $\tilde{\Phi}(x)^* \tilde{\Phi}(z) \approx K_e(x \star z^{-1})$.

1 Draw D random vectors $(\omega_j)_{j=1}^D$ i. i. d. from the probability law

$\Pr_{\widehat{\text{Haar}}, \rho}$;

2 **return** $\begin{cases} \tilde{\Phi}(x) & : y \mapsto \frac{1}{\sqrt{D}} \bigoplus_{j=1}^D (x, \omega_j) B(\omega_j)^* y \\ \tilde{\Phi}(x)^* & : \theta \mapsto \frac{1}{\sqrt{D}} \sum_{j=1}^D (x, \omega_j) B(\omega_j) \theta \end{cases}$;

3.2.8 Operator Random Feature engineering

As in the scalar case, it is possible to construct. We list some examples in the following.

3.2.8.1 Sum of kernels

Proposition 3.11 (Sum of kernels). *Let I be a countable set and let $(K^i)_{i \in I}$ be a family of \mathcal{Y} -reproducing kernels such that for all $y \in \mathcal{Y}$*

$$\sum_{i \in I} \langle y, K^i(x, x)y \rangle < \infty.$$

Given $x, z \in \mathcal{X}$, the serie $\sum_{i \in I} K^i(x, z)$ converges to a bounded operator $K(x, z)$ in the strong operator topology, and the map $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y})$ defined by

$$K(x, z)y = \sum_{i \in I} K^i(x, z)y$$

is a \mathcal{Y} -reproducing kernel. The corresponding space \mathcal{H}_K is embedded in $\bigoplus_{i \in I} \mathcal{H}_{K^i}$ by means of the feature operator

$$(Wf)(x) = \sum_{i \in I} f_i$$

where $f = \oplus_{i \in I} f_i$, $f_i \in \mathcal{H}_{K^i}$, and the sum converges in norm. Moreover if each K^i is a Mercer kernel—resp. \mathcal{C}_0 -kernel—and $x \mapsto \sum_{i \in I} \|K^i(x, x)\|_{\mathcal{Y}, \mathcal{Y}}$ is locally bounded—resp. bounded—then K is Mercer—resp. \mathcal{C}_0 .

3.3 LEARNING WITH ORFF

We now turn our attention to learning function with an ORFF model that approximate an OVK model.

3.3.1 Warm-up: supervised regression

Let $\mathbf{s} = (x_i, y_i)_{i=1}^N \in (\mathcal{X} \times \mathcal{Y})^N$ be a sequence of training samples. Given a local loss function $L : \mathcal{X} \times \mathcal{F} \times \mathcal{Y} \rightarrow \overline{\mathbb{R}}$ such that L is proper, convex and lower semi-continuous in f , we are interested in finding a *vector-valued function* $f_s : \mathcal{X} \rightarrow \mathcal{Y}$, that lives in a RKHS and minimize a tradeoff between a data fitting term L and a regularization term to prevent from overfitting. Namely finding $f_s \in \mathcal{H}_K$ such that

$$f_s = \arg \min_{f \in \mathcal{H}_K} \frac{1}{N} \sum_{i=1}^N L(x_i, f, y_i) + \frac{\lambda}{2} \|f\|_K^2, \quad (3.23)$$

¹⁰ Tychonov regularization.

where $\lambda \in \mathbb{R}_+$ is a regularization¹⁰ parameter. We call the quantity

$$\mathcal{R}(f) = \frac{1}{N} \sum_{i=1}^N L(x_i, f, y_i), \quad \forall f \in \mathcal{H}_K, \forall \mathbf{s} \in (\mathcal{X} \times \mathcal{Y})^N.$$

The empirical risk of the model $f \in \mathcal{H}_K$. A common choice of data fitting term for regression is $L : (x_i, f, y_i) \mapsto \|f(x_i) - y_i\|_{\mathcal{Y}}^2$. We introduce a corollary from Mazur and Schauder proposed in 1936 (see Górniewicz [19] and Kurdila and Zabarankin [22]) showing that eq. (3.23)—and eq. (3.25)—attains a unique minimizer.

Theorem 3.1 (Mazur–Schauder). *Let \mathcal{H} be a Hilbert space and $J : \mathcal{H} \rightarrow \overline{\mathbb{R}}$ be a proper, convex, lower semi-continuous and coercive function. Then J is bounded from below and attains a minimizer. Moreover if J is strictly convex the minimizer is unique.*

This is easily verified for Ridge regression. Define

$$J_\lambda(f) = \frac{1}{N} \sum_{i=1}^N \|f(x_i) - y_i\|_{\mathcal{Y}}^2 + \frac{\lambda}{2} \|f\|_K^2, \quad (3.24)$$

where $f \in \mathcal{H}_K$ and $\lambda \in \mathbb{R}_{>0}$. J_λ is continuous¹¹ and strictly convex. Additionally J_λ is coercive since $\|f\|_K$ is coercive, $\lambda \in \mathbb{R}_{>0}$, and all the summands of J_λ are positive. Hence for all positive λ , $f_s = \arg \min_{f \in \mathcal{H}_K} J_\lambda(f)$ exists, is unique and attained.

¹¹ Reminder, if $f \in \mathcal{H}_K$, $ev_x : f \mapsto f(x)$ is continuous, see proposition 2.1.

Remark 3.4 We consider the optimization problem proposed in eq. (3.24) where $L : (x_i, f, y_i) \mapsto \|f(x_i) - y_i\|_Y^2$. If given a training sample s , we have

$$\frac{1}{N} \sum_{i=1}^N \|y_i\|_Y^2 \leq \sigma_Y^2,$$

then $\lambda \|f_s\|_K \leq 2\sigma_Y^2$. Indeed, since \mathcal{H}_K is a Hilbert space, $0 \in \mathcal{H}_K$, thus

$$\begin{aligned} \frac{\lambda}{2} \|f_s\|_K^2 &\leq \frac{1}{N} \sum_{i=1}^N L(x_i, f_s, y_i) + \frac{\lambda}{2} \|f_s\|_K^2 \\ &\leq \frac{1}{N} \sum_{i=1}^N L(x_i, 0, y_i) \leq \sigma_Y^2, \quad \text{by optimality of } f_s. \end{aligned}$$

Since for all $x \in \mathcal{X}$, $\|f(x)\|_Y \leq \sqrt{\|K(x, x)\|_{Y,Y}} \|f\|_K$, the maximum value that the solution $\|f_s(x)\|_Y$ of eq. (3.24) can reach is $2\sqrt{\|K(x, x)\|_{Y,Y}} \frac{\sigma_Y^2}{\lambda}$. Thus when solving a Ridge regression problem, given a shift-invariant kernel K_e , one should choose

$$0 < \lambda \leq 2 \frac{\sqrt{\|K_e(e)\|_{Y,Y}} \sigma_Y^2}{C}.$$

with $C \in \mathbb{R}_{>0}$ to have a chance to fit all the y_i with norm $\|y_i\|_Y \leq C$ in the train set.

3.3.2 Semi-supervised regression

Regression in \mathcal{Y} -Reproducing Kernel Hilbert Space has been well studied [2, 10, 20, 26, 28, 31, 35], and a cornerstone of learning in \mathcal{Y} -RKHS is the representer theorem¹², which allows to replace the search of a minimizer in a infinite dimensional \mathcal{Y} -RKHS by a finite number of parameters $(u_i)_{i=1}^N$, $u_i \in \mathcal{Y}$. We present here the very general form of Minh, Bazzani, and Murino [31]. This framework encompasses Vector-valued Manifold Regularization [5, 9, 30] and Co-regularized Multi-view Learning [8, 34, 37, 40].

¹² Sometimes referred to as minimal norm interpolation theorem.

In the following we suppose we are given a cost function $c : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}$, such that $c(f(x), y)$ returns the error of the prediction $f(x)$ w.r. t. the ground truth y . A loss function of a model f with respect to an example $(x, y) \in \mathcal{X} \times \mathcal{Y}$ can be naturally defined from a cost function as $L(x, f, y) = c(f(x), y)$. Conceptually the function c evaluate the quality of the prediction versus its ground truth $y \in \mathcal{Y}$ while the loss function L evaluate the quality of the model f at a training point $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Moreover we suppose that we are given a training sample $\mathbf{u} = (x_i)_{i=N}^{N+U} \in \mathcal{X}^U$ of unlabelled example. We note $\mathbf{z} \in (\mathcal{X} \times \mathcal{Y})^N \times \mathcal{X}^U$ the sequence \mathbf{su} concatenating both labeled (s) and unlabelled (u) training examples.

Theorem 3.2 (Representer [31]). Let K be a \mathcal{U} -Mercer Operator-Valued Kernel and \mathcal{H}_K its corresponding \mathcal{U} -Reproducing Kernel Hilbert space.

Let $V : \mathcal{U} \rightarrow \mathcal{Y}$ be a bounded linear operator and let $c : \mathcal{Y} \times \mathcal{Y} \rightarrow \overline{\mathbb{R}}$ be a cost function such that $L(x, f, y) = c(Vf(x), y)$ is a proper convex lower semi-continuous function in f for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$.

Eventually let $\lambda_K \in \mathbb{R}_{>0}$ and $\lambda_M \in \mathbb{R}_+$ be two regularization hyperparameters and $(M_{ik})_{i,k=1}^{N+U}$ be a sequence of data dependent bounded linear operators in $\mathcal{L}(\mathcal{U})$, such that

$$\sum_{i,j=1}^{N+U} \langle u_i, M_{ik} u_k \rangle \geq 0, \quad \forall (u_i)_{i=1}^{N+U} \in \mathcal{U}^{N+U} \text{ and } M_{ik} = M_{ki}^*.$$

The solution $f_z \in \mathcal{H}_K$ of the regularized optimization problem

$$\begin{aligned} f_z = \arg \min_{f \in \mathcal{H}_K} & \frac{1}{N} \sum_{i=1}^N c(Vf(x_i), y_i) + \frac{\lambda_K}{2} \|f\|_K^2 \\ & + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \langle f(x_i), M_{ik} f(x_k) \rangle_{\mathcal{U}} \end{aligned} \quad (3.25)$$

has the form $f_z = \sum_{j=1}^{N+U} K(\cdot, x_j) u_{z,j}$ where $u_{z,j} \in \mathcal{U}$ and

$$\begin{aligned} u_z = \arg \min_{u \in \bigoplus_{i=1}^{N+U} \mathcal{U}} & \frac{1}{N} \sum_{i=1}^N c \left(V \sum_{k=1}^{N+U} K(x_i, x_k) u_k, y_i \right) \\ & + \frac{\lambda_K}{2} \sum_{k=1}^{N+U} u_i^* K(x_i, x_k) u_k \\ & + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \left\langle \sum_{j=1}^{N+U} K(x_i, x_j) u_j, M_{ik} \sum_{j=1}^{N+U} K(x_k, x_j) u_j \right\rangle_{\mathcal{U}} \end{aligned} \quad (3.26)$$

The first representer theorem was first introduced by Wahba [44] in the case where $\mathcal{Y} = \mathbb{R}$. The extension to an arbitrary Hilbert space \mathcal{Y} has been proved by many authors in different forms [9, 20, 26]. The idea behind the representer theorem is that eventhough we minimize over the whole space \mathcal{H}_K , when $\lambda_K > 0$, the solution of eq. (3.25) falls inevitably into the set $\mathcal{H}_{K,z} = \left\{ \sum_{j=1}^{N+U} K_{x_j} u_j \mid \forall (u_i)_{i=1}^{N+U} \in \mathcal{U}^{N+U} \right\}$. Therefore the result can be expressed as a finite linear combination of basis functions of the form $K(\cdot, x_k)$. Remark that we can perform the kernel expansion of $f_z = \sum_{j=1}^{N+U} K(\cdot, x_j) u_{z,j}$ eventhough $\lambda_K = 0$. However f_z is no longer the solution of eq. (3.25) over the whole space \mathcal{H}_K but a projection on the subspace $\mathcal{H}_{K,z}$. While this is in general not a problem for practical applications, it might raise issues for further theoretical investigations. In particular, it makes it difficult to perform theoretical comparison the “exact” solution of eq. (3.25) with respect to the ORFF approximation solution given in corollary 3.2.

We present here the proof of the generic formulation proposed by Minh, Bazzani, and Murino [31]. In the mean time we clarify some elements of the proof. Indeed the existence of a global minimizer is not trivial and we must invoke the Mazur-Schauder theorem. Moreover the coercivity of the objective function required by the Mazur-Schauder theorem is not obvious when we do not require the cost function to take only positive values. However a corollary of Hahn-Banach theorem linking strong convexity to coercivity gives the solution.

Proof Since $f(x) = K_x^* f$ (see eq. (2.13)), the optimization problem reads

$$\begin{aligned} f_z = \arg \min_{f \in \mathcal{H}_K} & \frac{1}{N} \sum_{i=1}^N c(VK_{x_i}^* f, y_i) + \frac{\lambda_K}{2} \|f\|_K^2 \\ & + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \langle K_{x_i}^* f, M_{ik} K_{x_k}^* f \rangle_{\mathcal{U}} \end{aligned}$$

Let $W_{V,s} : \mathcal{H}_K \rightarrow \bigoplus_{i=1}^N \mathcal{Y}$ be a linear operator defined as

$$W_{V,s} f = \bigoplus_{i=1}^N CK_{x_i}^* f,$$

with $VK_{x_i}^* : \mathcal{H}_K \rightarrow \mathcal{Y}$ and $K_{x_i} V^* : \mathcal{Y} \rightarrow \mathcal{H}_K$. Let $Y = \bigoplus_{i=1}^N y_i \in \mathcal{Y}^N$. We have

$$\langle Y, W_{V,s} f \rangle_{\bigoplus_{i=1}^N \mathcal{Y}} = \sum_{i=1}^N \langle y_i, VK_{x_i}^* f \rangle_{\mathcal{Y}} = \sum_{i=1}^N \langle K_{x_i} V^* y_i, f \rangle_{\mathcal{H}_K}.$$

Thus the adjoint operator $W_{V,s}^* : \bigoplus_{i=1}^N \mathcal{Y} \rightarrow \mathcal{H}_K$ is

$$W_{V,s}^* Y = \sum_{i=1}^N K_{x_i} V^* y_i,$$

and the operator $W_{V,s}^* W_{V,s} : \mathcal{H}_K \rightarrow \mathcal{H}_K$ is

$$W_{V,s}^* W_{V,s} f = \sum_{i=1}^N K_{x_i} V^* VK_{x_i}^* f$$

where $V^* V \in \mathcal{L}(\mathcal{U})$. Let

$$\begin{aligned} J_{\lambda_K}(f) &= \underbrace{\frac{1}{N} \sum_{i=1}^N c(Vf(x_i), y_i)}_{=J_c} + \frac{\lambda_K}{2} \|f\|_K^2 \\ &+ \underbrace{\frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \langle f(x_i), M_{ik} f(x_k) \rangle_{\mathcal{U}}}_{=J_M} \end{aligned}$$

To ensure that J_{λ_K} has a global minimizer we need the following technical lemma (which is a consequence of the Hahn-Banach theorem for lower-semicontinuous functional, see Kurdila and Zabrankin [22]).

Lemma 3.2 *Let J be a proper, convex, lower semi-continuous functional, defined on a Hilbert space \mathcal{H} . If J is strongly convex, then J is coercive.*

Proof Consider the convex function $G(f) := J(f) - \lambda \|f\|^2$, for some $\lambda > 0$. Since J is by assumption proper, lower semi-continuous and strongly convex with parameter λ , G is proper, lower semi-continuous and convex. Thus Hahn-Banach theorem apply, stating that G is bounded by below by an affine functional. I. e. there exists f_0 and $f_1 \in \mathcal{H}$ such that

$$G(f) \geq G(f_0) + \langle f - f_0, f_1 \rangle, \quad \text{for all } f \in \mathcal{H}.$$

Then substitute the definition of G to obtain

$$J(f) \geq J(f_0) + \lambda (\|f\| - \|f_0\|) + \langle f - f_0, f_1 \rangle.$$

By the Cauchy-Schwartz inequality, $\langle f, f_1 \rangle \geq -\|f\| \|f_1\|$, thus

$$J(f) \geq J(f_0) + \lambda (\|f\| - \|f_0\|) - \|f\| \|f_1\| - \langle f_0, f_1 \rangle,$$

which tends to infinity as f tends to infinity. Hence J is coercive \square

for all $f \in \mathcal{H}_K$. Since c is proper, lower semi-continuous and convex by assumption, thus the term J_c is also proper, lower semi-continuous and convex. Moreover the term J_M is always positive for any $f \in \mathcal{H}_K$ and $\frac{\lambda_K}{2} \|f\|_K^2$ is strongly convex. Thus J_{λ_K} is strongly convex. Apply lemma 3.2 to obtain the coercivity of J_{λ_K} , and then theorem 3.1 to show that J_{λ_K} has a unique minimizer and is attained. Then let $\mathcal{H}_{K,z} = \left\{ \sum_{j=1}^{N+U} K_{x_j} u_j \mid \forall (u_i)_{i=1}^{N+U} \in \mathcal{U}^{N+U} \right\}$. For $f \in \mathcal{H}_{K,z}^{\perp}$, the operator $W_{V,s}$ satisfies

$$\langle Y, W_{V,s} f \rangle_{\oplus_{i=1}^N \mathcal{Y}} = \left\langle \underbrace{f}_{\in \mathcal{H}_{K,z}^{\perp}}, \underbrace{\sum_{i=1}^{N+U} K_{x_i} V^* y_i}_{\in \mathcal{H}_{K,z}} \right\rangle_{\mathcal{H}_K} = 0$$

for all sequences $(y_i)_{i=1}^N$, since $V^* y_i \in \mathcal{U}$. Hence,

$$(V f(x_i))_{i=1}^N = 0 \quad (3.27)$$

In the same way,

$$\sum_{i=1}^{N+U} \langle K_{x_i}^* f, u_i \rangle_{\mathcal{U}} = \left\langle \underbrace{f}_{\in \mathcal{H}_{K,z}^{\perp}}, \underbrace{\sum_{j=1}^{N+U} K_{x_j} u_j}_{\in \mathcal{H}_{K,z}} \right\rangle_{\mathcal{H}_K} = 0.$$

for all sequences $(u_i)_{i=1}^{N+U} \in \mathcal{U}^{N+U}$. As a result,

$$(f(x_i))_{i=1}^{N+U} = 0. \quad (3.28)$$

Now for an arbitrary $f \in \mathcal{H}_K$, consider the orthogonal decomposition $f = f^{\perp} + f^{\parallel}$, where $f^{\perp} \in \mathcal{H}_{K,z}^{\perp}$ and $f^{\parallel} \in \mathcal{H}_{K,z}$. Then since $\|f^{\perp} + f^{\parallel}\|_{\mathcal{H}_K}^2 = \|f^{\perp}\|_{\mathcal{H}_K}^2 + \|f^{\parallel}\|_{\mathcal{H}_K}^2$, eq. (3.27) and eq. (3.28) shows that if $\lambda_K \geq 0$, clearly then

$$J_{\lambda_K}(f) = J_{\lambda_K}(f^{\perp} + f^{\parallel}) \geq J_{\lambda_K}(f^{\parallel})$$

The last inequality holds only when $\|f^{\perp}\|_{\mathcal{H}_K} = 0$, that is when $f^{\perp} = 0$. As a result since the minimizer of J_{λ_K} is unique and attained, it must lie in $\mathcal{H}_{K,z}$. \square

The representer theorem show that minimizing a functional in a \mathcal{Y} -RKHS yields a solution which depends on all the points in the training set. Assuming that for all x_i , $x \in \mathcal{X}$ and for all $u_i \in \mathcal{U}$ it takes time $O(P)$, to compute $K(x_i, x)u_i$, making a prediction using the representer theorem take $O(2P)$. Obviously If $\mathcal{Y} = \mathbb{R}^p$, Then $P = O(p^2)$ thus making a prediction cost $O(2p^2)$ operations.

Instead learning a model f that depends on all the points of the training set, we would like to learn a parametric model of the form $\tilde{f}(x) = \tilde{\Phi}(x)^* \theta$, where θ lives in some redescription space $\tilde{\mathcal{H}}$. We are interested in finding a parameter vector θ_z such that

$$\begin{aligned} \theta_z = \arg \min_{\theta \in \tilde{\mathcal{H}}} & \frac{1}{N} \sum_{i=1}^N c \left(V \tilde{\Phi}(x_i)^* \theta, y_i \right) + \frac{\lambda_K}{2} \|\theta\|_{\tilde{\mathcal{H}}}^2 \\ & + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \langle \theta, \tilde{\Phi}(x_i) M_{ik} \tilde{\Phi}(x_k)^* \theta \rangle_{\tilde{\mathcal{H}}} \end{aligned} \quad (3.29)$$

Corollary 3.2 (ORFF representer). *Let \tilde{K} be an Operator-Valued Kernel such that for all $x, z \in \mathcal{X}$, $\tilde{\Phi}(x)^* \tilde{\Phi}(z) = \tilde{K}(x, z)$ where \tilde{K} is a \mathcal{U} -Mercer OVK and $\mathcal{H}_{\tilde{K}}$ its corresponding \mathcal{U} -Reproducing kernel Hilbert space.*

Let $V : \mathcal{U} \rightarrow \mathcal{Y}$ be a bounded linear operator and let $c : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}$ be a cost function such that $L(x, f, y) = c(Vf(x), y)$ is a proper convex lower semi-continuous function in $\tilde{f} \in \mathcal{H}_{\tilde{K}}$ for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$.

Eventually let $\lambda_K \in \mathbb{R}_{>0}$ and $\lambda_M \in \mathbb{R}_+$ be two regularization hyperparameters and $(M_{ik})_{i,k=1}^{N+U}$ be a sequence of data dependent bounded linear operators in $\mathcal{L}(\mathcal{U})$, such that

$$\sum_{i,j=1}^{N+U} \langle u_i, M_{ik} u_k \rangle \geq 0, \quad \forall (u_i)_{i=1}^{N+U} \in \mathcal{U}^{N+U} \text{ and } M_{ik} = M_{ki}^*.$$

The solution $f_z \in \mathcal{H}_{\tilde{K}}$ of the regularized optimization problem

$$\begin{aligned} \tilde{f}_z = \arg \min_{\tilde{f} \in \mathcal{H}_{\tilde{K}}} & \frac{1}{N} \sum_{i=1}^N c \left(V \tilde{f}(x_i), y_i \right) + \frac{\lambda_K}{2} \|\tilde{f}\|_{\tilde{K}}^2 \\ & + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \langle \tilde{f}(x_i), M_{ik} \tilde{f}(x_k) \rangle_{\mathcal{U}} \end{aligned} \quad (3.30)$$

has the form $\tilde{f}_z = \tilde{\Phi}(\cdot)^ \theta_z$, where $\theta_z \in (\text{Ker } \tilde{W})^\perp$ and*

$$\begin{aligned} \theta_z = \arg \min_{\theta \in \tilde{\mathcal{H}}} & \frac{1}{N} \sum_{i=1}^N c \left(V \tilde{\Phi}(x_i)^* \theta, y_i \right) + \frac{\lambda_K}{2} \|\theta\|_{\tilde{\mathcal{H}}}^2 \\ & + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \langle \theta, \tilde{\Phi}(x_i) M_{ik} \tilde{\Phi}(x_k)^* \theta \rangle_{\tilde{\mathcal{H}}}. \end{aligned} \quad (3.31)$$

Proof Since \tilde{K} is an operator-valued kernel, from theorem 3.2, eq. (3.30) has a solution of the form

$$\begin{aligned}\tilde{f}_z &= \sum_{i=1}^{N+U} \tilde{K}(\cdot, x_i) u_i, \quad u_i \in \mathcal{U}, x_i \in \mathcal{X} \\ &= \sum_{i=1}^N \tilde{\Phi}(\cdot)^* \tilde{\Phi}(x_i) u_i = \tilde{\Phi}(\cdot)^* \underbrace{\left(\sum_{i=1}^{N+U} \tilde{\Phi}(x_i) u_i \right)}_{=\theta \in (\text{Ker } \tilde{W})^\perp \subset \tilde{\mathcal{H}}}.\end{aligned}$$

Let

$$\begin{aligned}\theta_z &= \arg \min_{\theta \in (\text{Ker } \tilde{W})^\perp} \frac{1}{N} \sum_{i=1}^N c \left(V \tilde{\Phi}(x_i)^* \theta, y_i \right) + \frac{\lambda_K}{2} \left\| \tilde{\Phi}(\cdot)^* \theta \right\|_{\tilde{K}}^2 \\ &\quad + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \left\langle \tilde{\Phi}(x_i)^* \theta, M_{ik} \tilde{\Phi}(x_k)^* \theta \right\rangle_{\mathcal{U}}.\end{aligned}$$

Since $\theta \in (\text{Ker } \tilde{W})^\perp$ and W is an isometry from $(\text{Ker } \tilde{W})^\perp \subset \tilde{\mathcal{H}}$ onto $\mathcal{H}_{\tilde{K}}$, we have $\left\| \tilde{\Phi}(\cdot)^* \theta \right\|_{\tilde{K}}^2 = \|\theta\|_{\tilde{\mathcal{H}}}^2$. Hence

$$\begin{aligned}\theta_z &= \arg \min_{\theta \in (\text{Ker } \tilde{W})^\perp} \frac{1}{N} \sum_{i=1}^N c \left(V \tilde{\Phi}(x_i)^* \theta, y_i \right) + \frac{\lambda_K}{2} \|\theta\|_{\tilde{\mathcal{H}}}^2 \\ &\quad + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \left\langle \tilde{\Phi}(x_i)^* \theta, M_{ik} \tilde{\Phi}(x_k)^* \theta \right\rangle_{\mathcal{U}}.\end{aligned}$$

Finding a minimizer θ_z over $(\text{Ker } \tilde{W})^\perp$ is not the same than finding a minimizer over $\tilde{\mathcal{H}}$. Although in both cases Mazur-Schauder's theorem guarantee that the respective minimizers are unique, they might not be the same. Since \tilde{W} is bounded, $\text{Ker } \tilde{W}$ is closed, so that we can perform the decomposition $\tilde{\mathcal{H}} = (\text{Ker } \tilde{W})^\perp \oplus (\text{Ker } \tilde{W})$. Then clearly by linearity of W and the fact that for all $\theta^\parallel \in \text{Ker } \tilde{W}$, $\tilde{W}\theta^\parallel = 0$, if $\lambda > 0$ we have

$$\begin{aligned}\theta_z &= \arg \min_{\theta \in \tilde{\mathcal{H}}} \frac{1}{N} \sum_{i=1}^N c \left(V \tilde{\Phi}(x_i)^* \theta, y_i \right) + \frac{\lambda_K}{2} \|\theta\|_{\tilde{\mathcal{H}}}^2 \\ &\quad + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \left\langle \tilde{\Phi}(x_i)^* \theta, M_{ik} \tilde{\Phi}(x_k)^* \theta \right\rangle_{\mathcal{U}}\end{aligned}$$

Thus

$$\begin{aligned}
\theta_z = & \arg \min_{\substack{\theta^\perp \in (\text{Ker } \widetilde{W})^\perp, \\ \theta^\parallel \in \text{Ker } \widetilde{W}}} \frac{1}{N} \sum_{i=1}^N c \left(v \left(\widetilde{W} \theta^\perp \right) (x) + \underbrace{v \left(\widetilde{W} \theta^\parallel \right) (x)}_{=0 \text{ for all } \theta^\parallel}, y_i \right) \\
& + \frac{\lambda_K}{2} \left\| \theta^\perp \right\|_{\widetilde{\mathcal{H}}}^2 + \underbrace{\frac{\lambda}{2} \left\| \theta^\parallel \right\|_{\widetilde{\mathcal{H}}}^2}_{=0 \text{ only if } \theta^\parallel=0} \\
& + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \left\langle \widetilde{\Phi}(x_i)^* \theta^\perp, M_{ik} \left(\widetilde{W} \theta^\perp \right) (x_k) \right\rangle_{\mathcal{U}} \\
& + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \left\langle \underbrace{\left(\widetilde{W} \theta^\parallel \right) (x_i)}_{=0 \text{ for all } \theta^\parallel}, M_{ik} \left(\widetilde{W} \theta^\perp \right) (x_k) \right\rangle_{\mathcal{U}} \\
& + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \left\langle \left(\widetilde{W} \theta^\perp \right) (x_i), \underbrace{M_{ik} \left(\widetilde{W} \theta^\parallel \right) (x_k)}_{=0 \text{ for all } \theta^\parallel} \right\rangle_{\mathcal{U}} \\
& + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \left\langle \underbrace{\left(\widetilde{W} \theta^\parallel \right) (x_i)}_{=0 \text{ for all } \theta^\parallel}, \underbrace{M_{ik} \left(\widetilde{W} \theta^\parallel \right) (x_k)}_{=0 \text{ for all } \theta^\parallel} \right\rangle_{\mathcal{U}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\theta_z = & \arg \min_{\theta^\perp \in (\text{Ker } \widetilde{W})^\perp} \frac{1}{N} \sum_{i=1}^N c \left(v \left(\widetilde{W} \theta^\perp \right) (x), y_i \right) + \frac{\lambda_K}{2} \left\| \theta^\perp \right\|_{\widetilde{\mathcal{H}}}^2 \\
& + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \left\langle \widetilde{\Phi}(x_i)^* \theta^\perp, M_{ik} \left(\widetilde{W} \theta^\perp \right) (x_k) \right\rangle_{\mathcal{U}}.
\end{aligned}$$

Hence minimizing over $(\text{Ker } \widetilde{W})^\perp$ or $\widetilde{\mathcal{H}}$ is the same when $\lambda_K > 0$. Eventually,

$$\begin{aligned}
\theta_z = & \arg \min_{\theta \in \widetilde{\mathcal{H}}} \frac{1}{N} \sum_{i=1}^N c \left(v \widetilde{\Phi}(x_i)^* \theta, y_i \right) + \frac{\lambda_K}{2} \left\| \theta \right\|_{\widetilde{\mathcal{H}}}^2 \\
& + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \left\langle \widetilde{\Phi}(x_i)^* \theta, M_{ik} \widetilde{\Phi}(x_k)^* \theta \right\rangle_{\mathcal{U}} \\
= & \arg \min_{\theta \in \widetilde{\mathcal{H}}} \frac{1}{N} \sum_{i=1}^N c \left(v \widetilde{\Phi}(x_i)^* \theta, y_i \right) + \frac{\lambda_K}{2} \left\| \theta \right\|_{\widetilde{\mathcal{H}}}^2 \\
& + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \left\langle \theta, \widetilde{\Phi}(x_i) M_{ik} \widetilde{\Phi}(x_k)^* \theta \right\rangle_{\widetilde{\mathcal{H}}}.
\end{aligned}$$

This shows that when $\lambda_K > 0$ the solution of eq. (3.26) with the approximated kernel $K(x, z) \approx \tilde{K}(x, z) = \tilde{\Phi}(x)^* \tilde{\Phi}(z)$ is the same than the solution of eq. (3.31) up to a transformation. Namely, if u_z is the solution of eq. (3.26), θ_z is the solution of eq. (3.31) and $\lambda_K > 0$ we have

$$\theta_z = \sum_{j=1}^{N+U} \tilde{\Phi}(x_j) u_z \in \tilde{\mathcal{H}}.$$

If $\lambda_K = 0$ we can still find a solution u_z of eq. (3.26). By construction of the kernel expansion, we have $u_z \in (\text{Ker } W)^\perp$. However looking at the proof of corollary 3.2 we see that θ_z might *not* belong to $(\text{Ker } W)^\perp$. Then we can compute a residual vector

$$r_z = \theta_z - \sum_{j=1}^{N+U} \tilde{\Phi}(x_j) u_z \in \tilde{\mathcal{H}}.$$

Then if $r_z = 0$, it means that λ_K is large enough for both representer theorem and ORFF representer theorem to apply. If $r_z \neq 0$ but $\tilde{\Phi}(\cdot)^* r_z = 0$ means that both θ_z and $\sum_{j=1}^{N+U} \tilde{\Phi}(x_j) u_z$ are in $(\text{Ker } W)^\perp$, thus the representer theorem fails to find the “true” solution over the whole space \mathcal{H}_K but returns a projection onto $\mathcal{H}_{\tilde{K}, z}$ of the solution. If $r_z \neq 0$ and $\tilde{\Phi}(\cdot)^* r_z \neq 0$ means that θ_z is *not* in $(\text{Ker } W)^\perp$, thus the ORFF representer theorem fails to apply. This remark is illustrated by fig. 3.

3.4 SOLUTION OF THE EMPIRICAL RISK MINIMIZATION

3.4.1 Gradient methods

In order to find a solution to eq. (3.31), we turn our attention to gradient descent methods. In the following we let

$$\begin{aligned} J_{\lambda_K}(\theta) = & \frac{1}{N} \sum_{i=1}^N c\left(V \tilde{\Phi}(x_i)^* \theta, y_i\right) + \frac{\lambda_K}{2} \|\theta\|_{\tilde{\mathcal{H}}}^2 \\ & + \frac{\lambda_M}{2} \sum_{i,k=1}^{N+U} \langle \theta, \tilde{\Phi}(x_i) M_{ik} \tilde{\Phi}(x_k)^* \theta \rangle_{\tilde{\mathcal{H}}}. \end{aligned} \quad (3.32)$$

Since the solution of eq. (3.31) is unique when $\lambda_K > 0$, a sufficient and necessary condition is that the gradient of J_{λ_K} at the minimizer θ_z is zero. We use the Frechet derivative, the strongest notion of derivative in Banach spaces¹⁴ [14, 22] which directly generalizes the notion of gradient to Banach spaces. A function $f : \mathcal{H}_0 \rightarrow \mathcal{H}_1$ is call Frechet differentiable at $\theta_0 \in \mathcal{H}_0$ if there exist a bounded linear operator $A : \mathcal{H}_0 \rightarrow \mathcal{H}_1$ such that

$$\lim_{h \rightarrow 0} \frac{\|f(\theta_0 + h) - f(\theta_0) - Ah\|_{\mathcal{H}_1}}{\|h\|_{\mathcal{H}_0}} = 0$$

¹⁴ Here we view the Hilbert space \mathcal{H} (feature space) as a reflexive Banach space.

We write

$$(D_F f)(\theta_0) = \left. \frac{\partial f(\theta)}{\partial \theta} \right|_{\theta=\theta_0} = A$$

and call it Frechet derivative of f with respect to θ at θ_0 . With mild abuse of notation we write $\frac{\partial f(\theta_0)}{\partial \theta_0} = \left. \frac{\partial f(\theta)}{\partial \theta} \right|_{\theta=\theta_0}$. The Frechet derivative is an unbounded linear operator. The chain rule is valid in this context. Namely if a function $f : \mathcal{H}_0 \rightarrow \mathcal{H}_1$ is Frechet differentiable at θ and $g : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is Frechet differentiable at $f(\theta)$ then $g \circ f$ is Frechet differentiable at θ and

$$\frac{\partial}{\partial \theta}(g \circ f)(\theta) = \frac{\partial g(f(\theta))}{\partial f(\theta)} \circ \frac{\partial f(\theta)}{\partial \theta}.$$

If $f : \mathcal{H} \rightarrow \mathbb{R}$ we define the gradient of f as

$$\nabla_{\theta} f(\theta) = \left(\frac{\partial f(\theta)}{\partial \theta} \right)^*$$

and in the same way, for a function $f : \mathcal{H}_0 \rightarrow \mathcal{H}_1$ the jacobian of f as

$$J_{\theta} f(\theta) = \left(\frac{\partial f(\theta)}{\partial \theta} \right)^*.$$

In this context if $f : \mathcal{H}_0 \rightarrow \mathcal{H}_1$ and $g : \mathcal{H}_1 \rightarrow \mathbb{R}$ the chain rule reads

$$\nabla_{\theta}(g \circ f)(\theta) = J_{\theta} f(\theta) \circ \nabla_{f(\theta)} g(f(\theta)).$$

We consider that \mathcal{U} and \mathcal{U}' are both isometrically isomorphic to some ℓ^2 such that $\nabla_{\theta} \|\theta\|^2 = 2\theta$. By linearity and applying the chaine rule to eq. (3.31) and since $M_{ik}^* = M_{ki}$ for all $i, k \in \mathbb{N}_{N+U}$, we have

$$\begin{aligned} \nabla_{\theta} c(V\tilde{\Phi}(x_i)^* \theta, y_i) &= \tilde{\Phi}(x_i) V^* \left(\left. \frac{\partial}{\partial y} c(y, y_i) \right|_{y=V\tilde{\Phi}(x_i)^* \theta} \right)^*, \\ \nabla_{\theta} \left\langle \tilde{\Phi}(x_i)^* \theta, M_{ik} \tilde{\Phi}(x_k)^* \theta \right\rangle_{\mathcal{U}} &= \tilde{\Phi}(x_i) (M_{ik} + M_{ki}) \tilde{\Phi}(x_k)^* \theta, \\ \nabla_{\theta} \|\theta\|_{\mathcal{H}}^2 &= 2\theta. \end{aligned}$$

Provided that $c(y, y_i)$ is Frechet differentiable w. r. t. y , for all y and $y_i \in \mathcal{Y}$ we have

$$\begin{aligned} \nabla_{\theta} J_{\lambda_K}(\theta) &= \frac{1}{N} \sum_{i=1}^N \tilde{\Phi}(x_i) V^* \left(\left. \frac{\partial}{\partial y} c(y, y_i) \right|_{y=V\tilde{\Phi}(x_i)^* \theta} \right)^* \\ &\quad + \lambda_K \theta + \lambda_M \sum_{i,k=1}^{N+U} \tilde{\Phi}(x_i) M_{ik} \tilde{\Phi}(x_k)^* \theta \end{aligned}$$

Therefore after factorization, considering $\lambda_K > 0$,

$$\begin{aligned} \nabla_{\theta} J_{\lambda_K}(\theta) &= \frac{1}{N} \sum_{i=1}^N \tilde{\Phi}(x_i) V^* \left(\left. \frac{\partial}{\partial y} c(y, y_i) \right|_{y=V\tilde{\Phi}(x_i)^* \theta} \right)^* \\ &\quad + \lambda_K \left(I_{\mathcal{H}} + \frac{\lambda_M}{\lambda_K} \sum_{i,k=1}^{N+U} \tilde{\Phi}(x_i) M_{ik} \tilde{\Phi}(x_k)^* \right) \theta \end{aligned}$$

We note the quantity $\widetilde{\mathbf{M}}_{(\lambda_K, \lambda_M)} = I_{\mathcal{H}} + \frac{\lambda_M}{\lambda_K} \sum_{i,k=1}^{N+U} \widetilde{\Phi}(x_i) M_{ik} \widetilde{\Phi}(x_k)^* \in \bigoplus_{j=1}^D \mathcal{L}(\mathcal{U}')$ so that

$$\nabla_{\theta} J_{\lambda_K}(\theta) = \frac{1}{N} \sum_{i=1}^N \widetilde{\Phi}(x_i) V^* \left(\frac{\partial}{\partial y} c(y, y_i) \Big|_{y=V\widetilde{\Phi}(x_i)^*\theta} \right)^* + \lambda_K \widetilde{\mathbf{M}}_{(\lambda_K, \lambda_M)} \theta$$

Example 3.3 (Naive close form for the squared error cost). Consider the cost function defined for all $y, y' \in \mathcal{Y}$ by $c(y, y') = \|y - y'\|_2^2$. Then

$$\left(\frac{\partial}{\partial y} c(y, y_i) \Big|_{y=V\widetilde{\Phi}(x_i)^*\theta} \right)^* = 2 \left(V\widetilde{\Phi}(x_i)^*\theta - y_i \right).$$

Thus, since the optimal solution θ_z verifies $\nabla_{\theta_z} J_{\lambda_K}(\theta_z) = 0$ we have

$$\frac{1}{N} \sum_{i=1}^N \widetilde{\Phi}(x_i) V^* \left(V\widetilde{\Phi}(x_i)^*\theta_z - y_i \right) + \lambda_K \widetilde{\mathbf{M}}_{(\lambda_K, \lambda_M)} \theta_z = 0.$$

Therefore,

$$\left(\frac{1}{N} \sum_{i=1}^N \widetilde{\Phi}(x_i) V^* V \widetilde{\Phi}(x_i)^* + \lambda_K \widetilde{\mathbf{M}}_{(\lambda_K, \lambda_M)} \right) \theta_z = \frac{1}{N} \sum_{i=1}^N \widetilde{\Phi}(x_i) V^* y_i.$$

Suppose that $\mathcal{Y} \subseteq \mathbb{R}^p$, $V : \mathcal{U} \rightarrow \mathcal{Y}$ where $\mathcal{U} \subseteq \mathbb{R}^u$ and for all $x \in \mathcal{X}$, $\widetilde{\Phi}(x) : \mathbb{R}^r \rightarrow \mathbb{R}^u$. From this we can derive algorithm 3 which return the close form solution of eq. (3.32) for $c(y, y') = \|y - y'\|_2^2$.

COMPLEXITY ANALYSIS Algorithm 3 constitute our first step toward large-scale learning with operator-valued kernels. We can easily compute the time complexity of algorithm 3 when all the operators act on finite dimensional Hilbert spaces. Suppose that $u = \dim(\mathcal{U}) < +\infty$, $u' = \dim(\mathcal{U}') < +\infty$ and for all $x \in \mathcal{X}$, $\widetilde{\Phi}(x) : \mathcal{U}' \rightarrow \mathcal{H}$ where $r = \dim(\mathcal{H}) < +\infty$ is the dimension of the redescription space $\mathcal{H} = \mathbb{R}^r$. Since u , u' , and $r < +\infty$, we view the operators $\widetilde{\Phi}(x)$, V and $\widetilde{\mathbf{M}}_{(\lambda_K, \lambda_M)}$ as matrices. Computing V^*V cost $O_t(u^2p)$. Step 1 costs $O_t(r^2u + ru^2)$. Steps 5 (optional) has the same cost except that the sum is done over all pair of $N + U$ points thus it costs $O_t((N + U)^2(r^2u + ru^2))$. Steps 7 costs $O_t(N(ru + up))$. For step 8, the naive inversion of the operator costs $O_t(r^3)$. Eventually the overall complexity of algorithm 3 is

$$O_t \left(ru(r + u) \begin{cases} (N + U)^2 & \text{if } \lambda_M > 0 \\ N & \text{if } \lambda_M = 0 \end{cases} + r^3 + Nu(r + p) \right),$$

while the space complexity is $O_s(r^2)$.

Algorithm 3: Naive close form for the squared error cost.

Input :

- $\mathbf{s} = (x_i, y_i)_{i=1}^N \in (\mathcal{X} \times \mathbb{R}^p)^N$ a sequence of supervised training points,
- $\mathbf{u} = (x_i)_{i=N+1}^{N+U} \in \mathcal{X}^U$ a sequence of unsupervised training points,
- $\tilde{\Phi}(x_i) \in (\mathbb{R}^u, \mathbb{R}^r)$ a feature map defined for all $x_i \in \mathcal{X}$,
- $(M_{ik})_{i,k=1}^{N+U}$ a sequence of data dependent operators (see corollary 3.2),
- $V \in \mathcal{L}(\mathbb{R}^u, \rightarrow \mathbb{R}^p)$ a combination operator,
- $\lambda_K \in \mathbb{R}_{>0}$ the Tychonov regularization term,
- $\lambda_M \in \mathbb{R}_+$ the manifold regularization term.

Output: A model

$$h : \begin{cases} \mathcal{X} \rightarrow \mathbb{R}^p \\ x \mapsto \tilde{\Phi}(x)^* \theta_z, \end{cases}$$

such that θ_z minimize eq. (3.32), where $c(y, y') = \|y - y'\|_2^2$.

```

1  $\mathbf{P} \leftarrow \frac{1}{N} \sum_{i=1}^N \tilde{\Phi}(x_i) V^* V \tilde{\Phi}(x_i)^* \in \mathcal{L}(\mathbb{R}^r, \mathbb{R}^r);$ 
2 if  $\lambda_M = 0$  then
3    $\tilde{\mathbf{M}}_{(\lambda_K, \lambda_M)} \leftarrow I_r \in \mathcal{L}(\mathbb{R}^r, \mathbb{R}^r);$ 
4 else
5    $\tilde{\mathbf{M}}_{(\lambda_K, \lambda_M)} \leftarrow \left( I_r + \frac{\lambda_M}{\lambda_K} \sum_{i,k=1}^{N+U} \tilde{\Phi}(x_i) M_{ik} \tilde{\Phi}(x_k)^* \right) \in \mathcal{L}(\mathbb{R}^r, \mathbb{R}^r);$ 
6 end
7  $\mathbf{Y} \leftarrow \frac{1}{N} \sum_{i=1}^N \tilde{\Phi}(x_i) V^* y_i \in \mathbb{R}^r;$ 
8  $\theta_z \leftarrow \left( \mathbf{P} + \lambda_K \tilde{\mathbf{M}}_{(\lambda_K, \lambda_M)} \right)^{-1} \mathbf{Y} \in \mathbb{R}^r;$ 
9 return  $h : x \mapsto \tilde{\Phi}(x)^* \theta_z;$ 

```

This complexity is to compare with the kernelized solution proposed by Minh, Bazzani, and Murino [31]. Let

$$\mathbf{K} : \begin{cases} \mathcal{U}^{N+U} \rightarrow \mathcal{U}^{N+U} \\ u \mapsto \bigoplus_{i=1}^{N+U} \sum_{j=1}^{N+U} K(x_i, x_j) u_j \end{cases}$$

and

$$\mathbf{M} : \begin{cases} \mathcal{U}^{N+U} \rightarrow \mathcal{U}^{N+U} \\ u \mapsto \bigoplus_{i=1}^{N+U} \sum_{k=1}^{N+U} M_{ik} u_k. \end{cases}$$

When $\mathcal{U} = \mathbb{R}$, \mathbf{K} is called the Gram matrix of K . When $\mathcal{U} = \mathbb{R}^p$, \mathbf{K} is a matrix-valued Gram matrix where each entry $\mathbf{K}_{ij} \in \mathcal{L}(\mathbb{R}^p)$. When $\mathcal{U} =$

\mathbb{R}^p , \mathbf{M} can also be seen as a matrix-valued matrix where each entry is $M_{ik} \in \mathcal{L}(\mathbb{R}^p)$. We also introduce the operators $\mathbf{C}^* \mathbf{C} := I_{N+U} \otimes (\mathbf{C}^* \mathbf{C})$ and

$$\mathbf{P} : \begin{cases} \mathcal{U}^{N+U} \rightarrow \mathcal{U}^{N+U} \\ u \mapsto \left(\bigoplus_{j=1}^N u_j \right) \oplus \left(\bigoplus_{j=N+1}^{N+U} 0 \right) \end{cases}$$

The operator \mathbf{P} is a projection that sets all the terms u_j , $N < j \leq N+U$ of u to zero. Then the equivalent kernelized solution u_z of theorem 3.2 is given by [31]

$$\left(\frac{1}{N} \mathbf{C}^* \mathbf{C} \mathbf{P} \mathbf{K} + \lambda_M \mathbf{M} \mathbf{K} + \lambda_K I_{\bigoplus_{i=1}^{N+U} \mathcal{U}} \right) u_z = \left(\bigoplus_{i=1}^N \mathbf{C}^* y_i \right) \oplus \left(\bigoplus_{i=N+1}^{N+U} 0 \right).$$

which has time complexity $O_t(((N+U)u)^3 + Nup)$ and space complexity $O_s(((N+U)u)^2)$. Hence algorithm 3 is better than its kernelized counterpart when $r = 2Du'$ is small compared to $(N+U)u$. Roughly speaking it is better to use algorithm 3 when the number of features D required is small compared to the number of training point. Notice that computing the data dependent norm is expensive. Indeed when $\lambda_M = 0$, algorithm 3 has a linear complexity with respect to the number of supervised training points N . The complexity becomes quadratic in the number of supervised and unsupervised training points $N+U$ when $\lambda_M > 0$. Moreover suppose that $\lambda_M = 0$, $\mathcal{U} = \mathbb{R}^p$ and $\mathcal{U}' = \mathbb{R}^p$ and the combination operator is $V = I_p$. Then the complexity of algorithm 3 boils down to

$$O_t(p^3(ND^2 + D^3)),$$

which is annoying. Indeed learning p independent models with scalar Random Fourier Features would cost $O_t(p(ND^2 + D^3))$, meaning that introducing has increase the complexity from p to p^3 . However in some cases we can drastically reduce the complexity by viewing the feature-maps as linear operators rather than matrices.

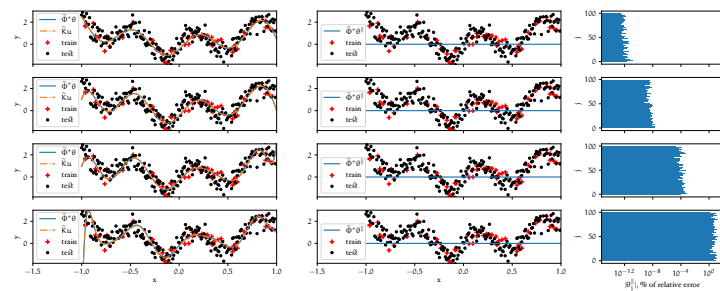


Figure 3: ORFF Representer theorem. We trained a first model named \tilde{K} following

3.4.2 *Efficient linear operators*

When developping algorithm 3 we considered that the feature map $\tilde{\Phi}(x)$ was a matrix from \mathbb{R}^u to \mathbb{R}^r for all $x \in \mathcal{X}$, and therefore that computing $\tilde{\Phi}(x)^* \theta$ has a time complexity of $O(r^2 u)$. While this holds true in the most generic senario, in many cases the feature maps presents some structure or sparsity allowing to reduce the computational cost of evaluating the feature map. We focus on the Operator-valued Random Fourier Feature given by algorithm 1, developped in section 3.2.5 and section 3.2.6 and treat the decomposable kernel, the curl-free kernel and the divergence-free kernel as an example.

3.4.3 *Decomposable kernel*

3.4.4 *Curl-free kernel*

3.4.5 *Divergence-free kernel*

3.5 CONCLUSIONS



4.1 CONSISTENCY OF THE ORFF ESTIMATOR

We are now interested on a non-asymptotic analysis of the ORFF approximation of shift-invariant \mathcal{Y} -Mercer kernels on [LCA](#) group \mathcal{X} endowed with the operation group \star with $\mathcal{X} \subset \mathbb{R}^d$. For a given D , we study how close is the approximation $\tilde{K}(x, z) = \tilde{\Phi}(x)^* \tilde{\Phi}(z)$ to the target kernel $K(x, z)$ for any x, z in \mathcal{X} .

If $A \in \mathcal{L}_+(\mathcal{Y})$ we denote $\|A\|_{\mathcal{Y}, \mathcal{Y}}$ its operator norm, which amounts the square root of the largest eigenvalue of A . For x and z in some compact $\mathcal{C} \subset \mathbb{R}^d$, we consider: $F(x \star z^{-1}) = \tilde{K}(x, z) - K(x, z)$ and study how the uniform norm

$$\|\tilde{K} - K\|_{\mathcal{C} \times \mathcal{C}} = \|F\|_{\infty} = \sup_{(x, z) \in \mathcal{C} \times \mathcal{C}} \|\tilde{K}(x, z) - K(x, z)\|_{\mathcal{Y}, \mathcal{Y}} \quad (4.1)$$

behaves according to D . Figure 4 empirically shows convergence of three different [OVK](#) approximations for x, z sampled from the compact $[-1, 1]^4$ and using an increasing number of sample points D . The log-log plot shows that all three kernels have the same convergence rate, up to a multiplicative factor.

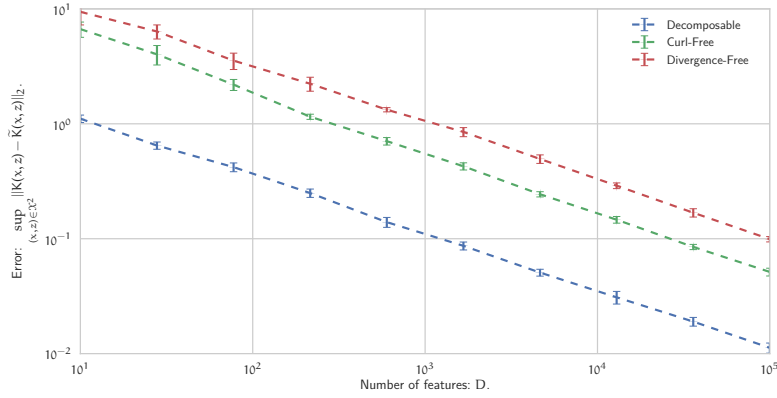


Figure 4: Error reconstructing the target operator-valued kernel k with ORFF approximation \tilde{K} for the decomposable, curl-free and divergence-free kernel.

In order to bound the error with high probability, we turn to concentration inequalities devoted to random matrices [6]. Prior to the presentation of general results, we briefly recall the uniform convergence of [RFF](#) approximation for a scalar shift invariant kernel on the additive [LCA](#) group \mathbb{R}^d and introduce a direct corollary about decomposable shift-invariant [OVK](#) on the [LCA](#) group $(\mathbb{R}^d, +)$.

4.1.1 Scalar random Fourier Features and decomposable kernel

Rahimi and Recht [33] proved the uniform convergence of Random Fourier Feature (RFF) approximation for a scalar shift invariant kernel on the LCA group \mathbb{R}^d endowed with the group operation $\star = +$. In the case of the shift-invariant decomposable OVK, an upper bound on the error can be directly obtained as a consequence of the result in the scalar case obtained by Rahimi and Recht [33] and other authors [39, 41] since in this case

Theorem 4.1 (Uniform error bound for RFF, Rahimi and Recht [33]). *Let \mathcal{C} be a compact subset of \mathbb{R}^d of diameter $|\mathcal{C}|$. Let k be a shift invariant kernel, differentiable with a bounded second derivative and μ its normalized Inverse Fourier transform such that it defines a probability measure. Let D the dimension of the Fourier feature vectors. Then for the mapping $\tilde{\Phi}$ described in Section 1, we have*

$$\Pr_{\mu} \left\{ (\omega_j)_{j=1}^D \mid \|\tilde{k} - k\|_{\mathcal{C} \times \mathcal{C}} \geq \epsilon \right\} \leq 2^8 \left(\frac{d\sigma|\mathcal{C}|}{\epsilon} \right)^2 \exp \left(-\frac{\epsilon^2 D}{4(d+2)} \right)$$

From theorem 4.1, we can deduce the following corollary about the uniform convergence of the ORFF approximation of the decomposable kernel. We recall that for a given pair x, z in \mathcal{C} , $\tilde{K}(x, z) = \tilde{\Phi}(x)^* \tilde{\Phi}(z) = A\tilde{k}(x, z)$ and $K_0(x - z) = A\mathbb{E}_{\mu}[\tilde{k}(x, z)]$.

Corollary 4.1 (Uniform error bound for decomposable ORFF). *Let \mathcal{C} be a compact subset of \mathbb{R}^d of diameter $|\mathcal{C}|$. Let K be a decomposable kernel built from a positive operator A , and k a shift invariant kernel with $\sigma^2 = \int_{\mathbb{R}^d} \|\omega\|^2 d\mu(\omega) < \infty$. Then*

$$\begin{aligned} \Pr_{\mu} \left\{ (\omega_j)_{j=1}^D \mid \|\tilde{K} - K\|_{\mathcal{C} \times \mathcal{C}} \geq \epsilon \right\} \\ \leq 2^8 \left(\frac{d\sigma\|A\|_{y,y}|\mathcal{C}|}{\epsilon} \right)^2 \exp \left(-\frac{\epsilon^2 D}{4\|A\|_2^2(d+2)} \right) \end{aligned}$$

Proof The proof directly extends theorem 4.1 given by [33]. Let \tilde{k} the Random Fourier approximation for the scalar-valued kernel k . Since

$$\sup_{(x,z) \in \mathcal{C} \times \mathcal{C}} \|\tilde{K}(x, z) - K(x, z)\|_{y,y} = \sup_{(x,z) \in \mathcal{C} \times \mathcal{C}} \|A\|_{y,y} |\tilde{k}(x, z) - k(x, z)|,$$

taking $\epsilon' = \|A\|_{y,y} \epsilon$ gives the following result for all positive ϵ' :

$$\begin{aligned} \Pr_{\mu} \left\{ (\omega_j)_{j=1}^D \mid \sup_{x,z \in \mathcal{C}} \|A(\tilde{k}(x, z) - k(x, z))\|_{y,y} \geq \epsilon' \right\} \\ \leq 2^8 \left(\frac{d\sigma\|A\|_{y,y}|\mathcal{C}|}{\epsilon'} \right)^2 \exp \left(-\frac{(\epsilon')^2 D}{4\|A\|_{y,y}^2(d+2)} \right) \end{aligned}$$

which concludes the proof. \square

Please note that a similar corollary could have been obtained for the recent result of Sutherland and Schneider [41] who refined the bound proposed by Rahimi and Recht by using a Bernstein concentration inequality instead of the Hoeffding inequality. More recently Sriperumbudur and Szabo [39] showed an optimal bound for Random Fourier Feature,

4.1.2 Uniform convergence of ORFF approximation on LCA groups

In this analysis, we assume that \mathcal{Y} is finite dimensional, in remark 4.1, we discuss how the proof could be extended to infinite dimensional output Hilbert spaces. We propose a bound for Operator-valued Random Fourier Feature approximation in the general case. It relies on two main ideas:

1. matrix-Bernstein concentration inequality for random matrices need to be used instead of concentration inequality for scalar random variables,
2. a general theorem valid for random matrices with bounded norms such as decomposable kernel ORFF approximation as well as unbounded norms such as the ORFF approximation we proposed for curl and divergence-free kernels that behave as subexponential random variables.

Before introducing the new theorem, we give the definition of the Orlicz norm which gives a proxy-bound on the norm of subexponential random variables.

Definition 4.1 (Orlicz norm [42]). Let $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-decreasing convex function with $\psi(0) = 0$. For a random variable X on a measured space $(\Omega, \mathcal{T}(\Omega), \mu)$, the quantity

$$\|X\|_\psi = \inf \{ C > 0 \mid \mathbf{E}_\mu[\psi(|X|/C)] \leq 1 \}.$$

is called the Orlicz norm of X .

Here, the function ψ is chosen as $\psi(u) = \psi_\alpha(u)$ where $\psi_\alpha(u) := e^{u^\alpha} - 1$. When $\alpha = 1$, a random variable with finite Orlicz norm is called a *subexponential variable* because its tails decrease at an exponential rate. Let X be a Hermitian random operator. We call *variance* of X the quantity $\mathbf{Var}_\mu[X] = \mathbf{E}_\mu[X - \mathbf{E}_\mu[X]]^2$. With this convention if X is a $p \times p$ Hermitian matrix, $\mathbf{Var}_\mu[X]_{\ell m} = \sum_{r=1}^p \mathbf{Cov}_\mu[X_{\ell r}, X_{r m}]$.

Theorem 4.2 Assume K is a shift-invariant \mathcal{Y} -Mercer kernel on \mathcal{C} a compact subset of \mathbb{R}^d of diameter $|\mathcal{C}|$, where \mathbb{R}^d is seen as an Abelian group endowed with the group operation \star . Suppose that the decomposition $\mu(\cdot)A(\cdot)$ is equal to the Inverse Fourier transform of the kernel's signature¹⁵. Let \tilde{K} be the ORFF approximation¹⁶ of K , K_0 be the kernel signature of K . Define the constants $b, \sigma_p^2, m \in \mathbb{R}_+$ as

1. $b = D \|\mathbf{Var}_\mu[\tilde{K}]\|_{\mathcal{C} \times \mathcal{C}} < \infty$,
2. $\sigma_p^2 = \mathbf{E}_\mu \left[H_\omega^2 \|A(\omega)\|_{\mathcal{Y}, \mathcal{Y}}^2 \right] < \infty$,
3. $m = 4 \left(\| \|A(\omega)\|_{\mathcal{Y}, \mathcal{Y}} \|_{\psi_1} + \|K\|_{\mathcal{C} \times \mathcal{C}} \right) < \infty$,

¹⁵ in the sense of proposition 3.3

¹⁶ \tilde{K} depends on D .

where $\omega \sim \mu$. Then for all ϵ in \mathbb{R}_+ ,

$$\Pr_{\mu} \left\{ (\omega_j)_{j=1}^D \mid \|\tilde{K} - K\|_{\mathcal{C} \times \mathcal{C}} \geq \epsilon \right\} \\ \leq C_{d,p} \left(\frac{\sigma_p |\mathcal{C}|}{\epsilon} \right)^{\frac{2}{1+2/d}} \begin{cases} \exp \left(-\frac{\epsilon^2 D}{8(d+2)(b + \frac{\epsilon \bar{u}}{6})} \right) & \text{if } \epsilon \bar{u} \leq 2(e-1)b \\ \exp \left(-\frac{\epsilon D}{(d+2)(e-1)\bar{u}} \right) & \text{otherwise,} \end{cases}$$

where $\bar{u} = 2m \log \left(2^{\frac{3}{2}} \left(\frac{m}{b} \right)^2 \right)$ and $C_{d,p} = p \left(\left(\frac{d}{2} \right)^{\frac{-d}{d+2}} + \left(\frac{d}{2} \right)^{\frac{2}{d+2}} \right) 2^{\frac{6d+2}{d+2}}$ and H_{ω} is the Lipschitz constant¹⁷ of $h_{\omega}(x)$.

Sketch of proof In the following, let $F(\delta) = F(x \star z^{-1}) = \tilde{K}(x, z) - K(x, z)$. Let $\mathcal{C}_{\Delta} = \{x \star z^{-1} \mid x, z \in \mathcal{C}\}$. Since \mathcal{C} is supposed compact, so is \mathcal{C}_{Δ} . Its diameter is at most $2|\mathcal{C}|$ where $|\mathcal{C}|$ is the diameter of \mathcal{C} . It is then possible to find an ϵ -net covering \mathcal{C}_{Δ} with at most $T = (4|\mathcal{C}|/r)^d$ balls of radius r . We call δ_i for $i \in \{1, \dots, T\}$ the center of the i -th ball, called anchors of the ϵ -net. Denote L_F the Lipschitz constant of F . We introduce the following lemma proved in [33].

Lemma 4.1

To apply the lemma, we must check assumptions (H1) and (H2).

Sketch of proof (H1) We bound the Lipschitz constant by noticing that F is differentiable, so

$$L_F = \left\| \frac{\partial F}{\partial \delta}(\delta^*) \right\|_{y,y} \quad \text{where } \delta^* = \arg \max_{\delta \in \mathcal{C}_{\Delta}} \left\| \frac{\partial F}{\partial \delta}(\delta) \right\|_{y,y}.$$

Using Jensen's inequality and applying Markov's inequality yields

$$\Pr_{\mu} \left\{ \omega \mid L_F \geq \frac{\epsilon}{2r} \right\} = \Pr_{\mu} \left\{ \omega \mid L_F^2 \geq \left(\frac{\epsilon}{2r} \right)^2 \right\} \\ \leq \mathbb{E}_{\mu} \left[\frac{\partial}{\partial \delta} h_{\omega}(\delta) \|A(\omega)\|_{y,y}^2 \right] \left(\frac{2r}{\epsilon} \right)^2.$$

We set $\sigma_p^2 = \mathbb{E}_{\mu} \left[H_{\omega}^2 \|A(\omega)\|_2^2 \right]$ where H_{ω} is the Lipschitz-constant of h_{ω} and suppose its existence. \square

Sketch of proof (H2) To obtain a bound on the anchors we apply theorem 4 of Koltchinskii et al. [21]. We suppose the existence of the two constants

$$b = \sup_{\delta \in \mathcal{C}_{\Delta}} D \|\text{Var}_{\mu} [\tilde{K}_e(\delta)]\|_{y,y}$$

and

$$\bar{u} = \log \left(2 \left(\frac{m}{b} \right)^2 + 1 \right),$$

where $m = 4 \left(\|A(\omega)\|_{y,y} \|\psi_1\| + \sup_{\delta \in \mathcal{C}_{\Delta}} \|K_e(\delta)\|_{y,y} \right)$ and $\omega \sim \mu$. Then $\forall i \in \{1, \dots, T\}$,

$$\Pr_{\mu} \left\{ (\omega_j)_{j=1}^D \mid \|F(\delta_i)\|_{y,y} \geq \epsilon \right\} \leq 2p \begin{cases} \exp \left(-\frac{D\epsilon^2}{4b+2\epsilon\bar{u}/3} \right) & \text{if } \epsilon \bar{u} \leq 2(e-1)b \\ \exp \left(-\frac{D\epsilon}{(e-1)\bar{u}} \right) & \text{otherwise.} \end{cases}$$

¹⁷ Remember that $\exp(ih_{\omega}(x)) = (x, \omega)$, so H_{ω} depends on the group operation.

Combining (H1) and (H2). Now applying the lemma and taking the union bound over the centers of the ϵ -net yields

$$\Pr_{\mu} \left\{ (\omega_j)_{j=1}^D \left| \sup_{\delta \in \mathcal{C}_{\Delta}} \|F(\delta)\|_{\mathcal{Y}, \mathcal{Y}} \leq \epsilon \right. \right\} \geq 1 - \kappa_1 r^{-d} - \kappa_2 r^2,$$

with $\kappa_2 = 4\sigma_p^2 \epsilon^{-2}$ and

$$\kappa_1 = 2p(4|\mathcal{C}|)^d \begin{cases} \exp\left(-\frac{\epsilon^2 D}{16(b + \frac{\epsilon}{6}\bar{u})}\right) & \text{if } \epsilon\bar{u} \leq 2(e-1)b \\ \exp\left(-\frac{\epsilon D}{2(e-1)\bar{u}}\right) & \text{otherwise.} \end{cases}$$

We choose r such that $d\kappa_1 r^{-d-1} - 2\kappa_2 r = 0$, i. e. $r = \left(\frac{d\kappa_1}{2\kappa_2}\right)^{\frac{1}{d+2}}$. The bound becomes

$$\begin{aligned} & \Pr_{\mu} \left\{ (\omega_j)_{j=1}^D \left| \sup_{\delta \in \mathcal{C}_{\Delta}} \|F(\delta)\| \geq \epsilon \right. \right\} \\ & \leq p C'_d 2^{\frac{6d+2}{d+2}} \left(\frac{\sigma_p |\mathcal{C}|}{\epsilon}\right)^{\frac{2}{1+2/d}} \begin{cases} \exp\left(-\frac{\epsilon^2}{8(d+2)(b + \frac{\epsilon}{6}\bar{u})}\right) & \text{if } \epsilon\bar{u} \leq 2(e-1)b \\ \exp\left(-\frac{\epsilon}{(d+2)(e-1)\bar{u}}\right) & \text{otherwise.} \end{cases} \end{aligned}$$

where $C'_d = \left(\left(\frac{d}{2}\right)^{\frac{-d}{d+2}} + \left(\frac{d}{2}\right)^{\frac{2}{d+2}}\right)$. Conclude by taking $C_{d,p} = p C'_d 2^{\frac{6d+2}{d+2}}$. \square

We give a comprehensive full proof of the theorem in the appendix. It follows the usual scheme derived in Rahimi and Recht [33] and Sutherland and Schneider [41] and involves Bernstein concentration inequality for unbounded symmetric matrices (theorem 3.3 in the supplements).

Remark 4.1 (dealing with infinite dimensional operators). We studied the concentration of ORFFs under the assumption that \mathcal{Y} is finite dimensional. This assumption is required in the proof when bounding the Lipschitz constant. The proof require the quantity \tilde{K} to be smooth enough to be able to interchange integral and derivative such that $E \frac{\partial}{\partial \delta} \tilde{K} = \frac{\partial}{\partial \delta} E \tilde{K}$. In the proof we show, component-wise, that that is K is twice differentiable, then the interchange is possible. To do the same for infinite dimensional operator-valued kernels, one could use Hille's theorem¹⁸ and could deduce the necessary condition on K to allow the interchange. One could also deduced a bound on the Lipschitz constant without computing the derivative of $\tilde{K} - K$. Additionally one should study whether Koltchinskii et al. [21, theorem 4] could apply to infinite dimensional operators and under which assumption.

¹⁸ See <http://fa.its.tudelft.nl/~neerven/publications/papers/ISEM.pdf>.

4.1.3 Variance of the ORFF approximation

We now provide a bound on the norm of the variance of \tilde{K} , required to apply theorem 4.2.

Proposition 4.1 (Bounding the variance of \tilde{K}). *Let K be a \mathcal{Y} -Mercer kernel on \mathcal{C} , a compact subset of \mathbb{R}^d . K is supposed to be translation invariant for the group operation \star on \mathbb{R}^d . Let \tilde{K} be the ORFF approximation of K (as defined in ??) and $\mathcal{C}_\Delta = \{ \chi \star z^{-1} \mid (\chi, z) \in \mathcal{C} \times \mathcal{C} \}$. Then for all δ in \mathcal{C}_Δ ,*

$$\begin{aligned} & \|\text{Var}_\mu [\tilde{K}_e(\delta)]\|_{\mathcal{Y}, \mathcal{Y}} \\ & \leq \frac{\| (K_e(2\delta) + K_e(0))\mathbb{E}_\mu[A(\omega)] - 2K_e(\delta)^2 \|_{\mathcal{Y}, \mathcal{Y}} + 2\|\text{Var}_\mu[A(\omega)]\|_{\mathcal{Y}, \mathcal{Y}}}{2D}. \end{aligned}$$

Proof *It relies on the i. i. d. property of the random vectors ω_j and trigonometric identities (see the proof in ?? of the supplementary material).* \square

An empirical illustration of these bounds is shown in ??. We generated a random point in $[-1, 1]^4$ and computed the empirical variance of the estimator (blue line). We also plotted (red line) the theoretical bound in proposition 4.1.

4.1.4 Application on decomposable, curl-free and div-free OVKs

First, the two following example discuss the form of H_ω for the additive group and the skewed-multiplicative group.

Example 4.1 (Additive group). *On the additive group, $h_\omega(\delta) = \langle \omega, \delta \rangle$. Hence $H_\omega = \|\omega\|_2$.*

Example 4.2 (Skewed-multiplicative group). *On the skewed multiplicative group, $h_\omega(\delta) = \langle \omega, \log(\delta + c) \rangle$. Therefore*

$$\sup_{\delta \in \mathcal{C}} \|\nabla h_\omega(\delta)\| = \sup_{\delta \in \mathcal{C}} \|\omega / (\delta + c)\|_2.$$

Eventually \mathcal{C} is compact and finite dimensional thus \mathcal{C} is closed and bounded. Thus $H_\omega = \|\omega\|_2 / (\min_{\delta \in \mathcal{C}} \|\delta\|_2 + c)$.

Now we compute upper bounds on the norm of the variance and Orlicz norm of the three ORFFs we took as examples.

DECOMPOSABLE KERNEL: notice that in the case of the Gaussian decomposable kernel, i.e. $A(\omega) = A$, $e = 0$, $K_0(\delta) = Ak_0(\delta)$, $k_0(\delta) \geq 0$ and $k_0(\delta) = 1$, then we have

$$D \|\text{Var}_\mu [\tilde{K}_0(\delta)]\|_{\mathcal{Y}, \mathcal{Y}} \leq (1 + k_0(2\delta)) \|A\|_{\mathcal{Y}, \mathcal{Y}} / 2 + k_0(\delta)^2.$$

CURL-FREE AND DIV-FREE KERNELS: recall that in this case $p = d$. For the (Gaussian) curl-free kernel, $A(\omega) = \omega\omega^*$ where $\omega \in \mathbb{R}^d \sim \mathcal{N}(0, \sigma^{-2}I_d)$ thus $\mathbb{E}_\mu[A(\omega)] = I_d/\sigma^2$ and $\text{Var}_\mu[A(\omega)] = (d+1)I_d/\sigma^4$. Hence,

$$D \|\text{Var}_\mu [\tilde{K}_0(\delta)]\|_{\mathcal{Y}, \mathcal{Y}} \leq \frac{1}{2} \left\| \frac{1}{\sigma^2} K_0(2\delta) - 2K_0(\delta)^2 \right\|_{\mathcal{Y}, \mathcal{Y}} + \frac{(d+1)}{\sigma^4}.$$

This bound is illustrated by fig. 4 B, for a given datapoint. Eventually for the Gaussian divergence-free kernel, $A(\omega) = I\|\omega\|_2^2 - \omega\omega^*$, thus $E_\mu[A(\omega)] = I_d(d-1)/\sigma^2$ and $\text{Var}_\mu[A(\omega)] = d(4d-3)I_d/\sigma^4$. Hence,

$$D\|\text{Var}_\mu[\tilde{K}_0(\delta)]\|_{y,y} \leq \frac{1}{2} \left\| \frac{(d-1)}{\sigma^2} K_0(2\delta) - 2K_0(\delta)^2 \right\|_{y,y} + \frac{d(4d-3)}{\sigma^4}.$$

Eventually, we ensure that the random variable $\|A(\omega)\|$ has a finite Orlicz norm with $\psi = \psi_1$ in these three cases.

COMPUTING THE ORLICZ NORM: for a random variable with strictly monotonic moment generating function (MGF), one can characterize its inverse ψ_1 Orlicz norm by taking the functional inverse of the MGF evaluated at 2 (see ?? of the supplementary material). In other words $\|X\|_{\psi_1}^{-1} = \text{MGF}(x)_X^{-1}(2)$. For the Gaussian curl-free and divergence-free kernel,

$$\|A^{\text{div}}(\omega)\|_{y,y} = \|A^{\text{curl}}(\omega)\|_{y,y} = \|\omega\|_2^2,$$

where $\omega \sim \mathcal{N}(0, I_d/\sigma^2)$, hence $\|A(\omega)\|_2 \sim \Gamma(p/2, 2/\sigma^2)$. The MGF of this gamma distribution is $\text{MGF}(x)(t) = (1 - 2t/\sigma^2)^{-(p/2)}$. Eventually

$$\|\|A^{\text{div}}(\omega)\|_{y,y}\|_{\psi_1}^{-1} = \|\|A^{\text{curl}}(\omega)\|_{y,y}\|_{\psi_1}^{-1} = \frac{\sigma^2}{2} \left(1 - 4^{-\frac{1}{p}}\right).$$

5.1 INTRODUCTION

5.2 TIME SERIES MODELLING

5.3 FUNCTIONAL DATA ANALYSIS

5.3.0.1 *One class SVM revisited*5.3.0.2 *Many quantile regression*

5.4 NEURAL NETWORKS, DEEP LEARNING

5.5 CONCLUSIONS



Part III

FINAL WORDS AND PERSPECTIVES

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CONCLUSIONS

Part IV

APPENDIX



OPERATOR-VALUED FUNCTIONS AND INTEGRATION

PROOFS OF THEOREMS

B.I PROOF OF

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DECLARATION

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15, Rue Plumet, 75015 - Paris, France, Septembre 2016

Romain Brault

LARGE-SCALE LEARNING ON STRUCTURED INPUT-OUTPUT DATA WITH OPERATOR-VALUED KERNELS

ABSTRACT: Many problems in Machine Learning can be cast into vector-valued functions approximation. Operator-Valued Kernels *Operator-Valued Kernels* and vector-valued Reproducing Kernel Hilbert Spaces provide a theoretical and practical framework to address that issue, extending nicely the well-known framework of scalar-valued kernels. However large scale applications are usually not affordable with these tools that require an important computational power along with a large memory capacity. In this thesis, we propose and study scalable methods to perform regression with *Operator-Valued Kernels*. To achieve this goal, we extend Random Fourier Features, an approximation technique originally introduced for scalar-valued kernels, to *Operator-Valued Kernels*. The idea is to take advantage of an approximated operator-valued feature map in order to come up with a linear model in a finite-dimensional space.

This thesis is structured as follows. First we develop a general framework devoted to the approximation of shift-invariant Mercer kernels on Locally Compact Abelian groups and study their properties along with the complexity of the algorithms based on them. Second we show theoretical guarantees by bounding the error due to the approximation, with high probability. Third, we study various applications of Operator Random Fourier Features to different tasks of Machine learning such as multi-class classification, multi-task learning, time serie modeling, functional regression and anomaly detection. We also compare the proposed framework with other state of the art methods. Fourth, we conclude by drawing short-term and mid-term perspectives of this work.

RÉSUMÉ: Dans cette thèse nous nous intéressons au passage à l'échelle des méthodes de régression à *noyaux à valeurs opérateurs*. Dans le contexte des noyaux scalaires, un noyau peut être vu comme une mesure de similarité entre deux points. Une solution d'un problème d'apprentissage prend alors généralement la forme d'une combinaison linéaire de ces similarités où les pondérations sont les inconnus à déterminer afin de trouver un modèle rendant compte au "mieux" des données. Dans le cadre des noyaux à noyaux à valeurs opérateurs l'évaluation d'un noyau entre deux points n'est plus une similarité scalaire, mais une fonction agissant sur des vecteurs. Une solution s'exprime alors comme une combinaison linéaire d'opérateurs vis-à-vis de poids vectoriels.

Bien que les noyaux à valeurs opérateurs généralisent strictement leurs confrères scalaires, ceux-ci on le désavantage d'être des méthodes couteuses en mémoire et en temps de calcul. Dans le contexte actuel, où les données sont nombreuses, il est intéressant de pouvoir traiter celles-ci dans leur totalité afin d'obtenir de meilleurs résultats. Pour ce faire nous remplaçons les noyaux à valeur opérateurs par une construction stochastique équivalente appelé fonction de redescription.

Dans un premier temps nous généralisons le principe de construction de ces fonctions de redescription fondé sur une généralisation du théorème de Bochner pour les noyaux de Mercer invariants par translation et étudions leurs propriétés ainsi que la complexité des algorithmes en découlant. Nous nous intéressons ensuite à donner des garanties théoriques en montrant que l'erreur commise par l'approximation stochastique est contrôlable avec forte probabilité. Nous concluons par plusieurs exemples applications à l'apprentissage de données fonctionnelles, multitâche, semi-supervisé, et l'apprentissage de séries temporelles tout en comparant notre approche à d'autres méthodes de l'état de l'art.

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