

DATA ARE NOT REAL!

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Large-scale learning on structured input-output data with operator-valued kernels

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## ABSTRACT

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Short summary of the contents... a great guide by Kent Beck how to write good abstracts can be found here:

<https://plg.uwaterloo.ca/~migod/research/beck00PSLA.html>



## PUBLICATIONS

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Some ideas and figures have appeared previously in the following publications:

Put your publications from the thesis here. The packages `multibib` or `bibtopic` etc. can be used to handle multiple different bibliographies in your document.





*We have seen that computer programming is an art,  
because it applies accumulated knowledge to the world,  
because it requires skill and ingenuity, and especially  
because it produces objects of beauty.*

## ACKNOWLEDGEMENTS

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Put your acknowledgements here.

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<sup>1</sup> Members of GuIT (Gruppo Italiano Utilizzatori di T<sub>E</sub>X e L<sup>A</sup>T<sub>E</sub>X)



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## LISTINGS

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## ACRONYMS

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OVK Operator-Valued Kernel

ORFF Operator-valued Random Fourier Feature

RKHS Reproducing Kernel Hilbert Space

vv-RKHS vector-valued Reproducing Kernel Hilbert Space

LCA Locally Compact Abelian





## Part I

### INTRODUCTION

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## MOTIVATIONS

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## BACKGROUND

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## 2.1 NOTATIONS

The euclidean inner product in  $\mathbb{R}^d$  is denoted  $\langle \cdot, \cdot \rangle$  and the euclidean norm is denoted  $\|\cdot\|$ . The unit pure imaginary number  $\sqrt{-1}$  is denoted  $i$ .  $\mathcal{B}(\mathbb{R}^d)$  is the Borel  $\sigma$ -algebra on  $\mathbb{R}^d$ . If  $\mathcal{X}$  and  $\mathcal{Y}$  are two vector spaces, we denote by  $\mathcal{F}(\mathcal{X}; \mathcal{Y})$  the vector space of functions  $f : \mathcal{X} \rightarrow \mathcal{Y}$  and  $\mathcal{C}(\mathcal{X}; \mathcal{Y}) \subset \mathcal{F}(\mathcal{X}; \mathcal{Y})$  the subspace of continuous functions. If  $\mathcal{H}$  is an Hilbert space we denote its scalar product by  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  and its norm by  $\|\cdot\|_{\mathcal{H}}$ . We set  $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}; \mathcal{H})$  to be the space of linear operators from  $\mathcal{H}$  to itself. If  $W \in \mathcal{L}(\mathcal{H})$ ,  $\text{Ker } W$  denotes the nullspace,  $\text{Im } W$  the image and  $W^* \in \mathcal{L}(\mathcal{H})$  the adjoint operator (transpose when  $W$  is a real matrix). All these notations are summarized in table 1.

## 2.2 ABOUT STATISTICAL LEARNING

## 2.3 ON LARGE-SCALE LEARNING

## 2.4 ELEMENTS OF ABSTRACT HARMONIC ANALYSIS

### 2.4.1 Locally compact Abelian groups

**Definition 1.** *Locally Compact Abelian group.* A group  $(\mathcal{X}, \star)$  is said to be Locally Compact Abelian if it is a topological commutative group  $\mathcal{X}$  for which every point has a compact neighborhood and is Hausdorff.

Locally Compact Abelian (LCA) groups are central to the general definition of Fourier Transform which is related to the concept of Pontryagin duality [8]. Let  $(\mathcal{X}, \star)$  be a LCA group with  $e$  its neutral element and the notation,  $x^{-1}$ , for the inverse of  $x \in \mathcal{X}$ . A *character* is a complex continuous homomorphism  $\omega : \mathcal{X} \rightarrow \mathbb{U}$  from  $\mathcal{X}$  to the set of complex numbers of unit module  $\mathbb{U}$ . The set of all characters of  $\mathcal{X}$  forms the Pontryagin *dual group*  $\hat{\mathcal{X}}$ . The dual group of an LCA group is an LCA group and the dual group operation is defined by

$$(\omega_1 \star \omega_2)(x) = \omega_1(x)\omega_2(x) \in \mathbb{U}.$$

The Pontryagin duality theorem states that  $\hat{\hat{\mathcal{X}}} \cong \mathcal{X}$ . I.e. there is a canonical isomorphism between any LCA group and its double dual. To emphasize this duality the following notation is usually adopted:  $\omega(x) = (x, \omega) = (\omega, x)$ , where  $x \in \mathcal{X}$ ,  $\omega \in \hat{\mathcal{X}}$ . Another important property involves the complex conjugate of the pairing which is defined as  $\overline{(x, \omega)} = (x^{-1}, \omega)$ .

Table 1: Mathematical symbols used throughout the paper and their signification.

Symbol	Meaning
$i$	Unit pure imaginary number $\sqrt{-1}$ .
$e$	Euler constant.
$\langle \cdot, \cdot \rangle$	Euclidean inner product.
$\ \cdot\ $	Euclidean norm.
$\mathcal{X}$	Input space $()$ .
$\hat{\mathcal{X}}$	The Pontryagin dual of $\mathcal{X}$ .
$\mathcal{Y}$	Output space (Hilbert space).
$\mathcal{H}$	Feature space (Hilbert space).
$\langle \cdot, \cdot \rangle_{\mathcal{Y}}$	The canonical inner product of the Hilbert space $\mathcal{Y}$ .
$\ \cdot\ _{\mathcal{Y}}$	The canonical norm induced by the inner product of the Hilbert space $\mathcal{Y}$ .
$\mathcal{F}(\mathcal{X}; \mathcal{Y})$	Vector space of function from $\mathcal{X}$ to $\mathcal{Y}$ .
$\mathcal{C}(\mathcal{X}; \mathcal{Y})$	The vector subspace of $\mathcal{F}$ of continuous function from $\mathcal{X}$ to $\mathcal{Y}$ .
$\mathcal{L}(\mathcal{H}; \mathcal{Y})$	The set of bounded linear operator from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{Y}$ .
$\mathcal{L}(\mathcal{Y})$	The set of bounded linear operator from a Hilbert space $\mathcal{H}$ to itself.
$\mathcal{B}(\mathcal{X})$	Borel $\sigma$ -algebra on $\mathcal{X}$ .
$\mu(\mathcal{X})$	A scalar positive measure of $\mathcal{X}$ .
$p_{\mu}(x)$	The Radon-Nikodym derivative of $\mu$ w.r.t. the Lebesgue measure.
$dx, d\omega$	The canonical Haar measure of the LCA group $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ . (resp. $(\hat{\mathcal{X}}, \mathcal{B}(\hat{\mathcal{X}}))$ ).
$L^p(\mathcal{X}, dx)$	The Banach space of $ \cdot ^p$ -integrable function from $(\mathcal{X}, \mathcal{B}(\mathcal{X}, dx))$ to $\mathbb{C}$ .
$L^p(\mathcal{X}, dx; \mathcal{Y})$	The Banach space of $\ \cdot\ _{\mathcal{Y}}^p$ (Bochner)-integrable function from $(\mathcal{X}, \mathcal{B}(\mathcal{X}), dx)$ to $\mathcal{Y}$ .

We notice that for any pairing depending of  $\omega$ , there exists a function  $h_{\omega} : \mathcal{X} \rightarrow \mathbb{R}$  such that:  $(x, \omega) = \exp(-ih_{\omega}(x))$  since any pairing maps into  $\mathbb{U}$ . Moreover,

$$\begin{aligned} (x \star z^{-1}, \omega) &= \omega(x)\omega(z^{-1}) = \exp(-ih_{\omega}(x)) \exp(-ih_{\omega}(z^{-1})) \\ &= \exp(-ih_{\omega}(x)) \exp(+ih_{\omega}(z)). \end{aligned}$$

Table 2 provide an explicit list of pairings for various groups based on  $\mathbb{R}^d$  or its subsets. We especially mention the duality pairing asso-

Table 2: Classification of Fourier transforms in terms of their domain and transform domain.

$\mathcal{X}$	$\hat{\mathcal{X}}$	Operation	Pairing
$\mathbb{R}^d$	$\mathbb{R}^d$	$+$	$(x, \omega) = \exp(i\langle x, \omega \rangle)$
$\mathbb{R}_{*,+}^d$	$\mathbb{R}^d$	$\cdot$	$(x, \omega) = \exp(i\langle \log(x), \omega \rangle)$
$(-c; +\infty)^d$	$\mathbb{R}^d$	$\odot$	$(x, \omega) = \exp(i\langle \log(x+c), \omega \rangle)$

ciated to the skewed multiplicative [LCA](#) group  $\mathcal{X} = ((-c; +\infty)^d, \odot)$ . Hence  $h_\omega(x) = \sum_{k=1}^d \omega_k \log(x_k + c)$ . This group together with the operation  $\odot$  has been proposed by [\[10\]](#) to handle histograms features especially useful in image recognition applications.

#### 2.4.2 The Fourier transform

For a function with values in a separable Hilbert space  $f \in L^1(\mathcal{X}, dx; \mathcal{Y})$ , where  $dx$  is the Haar measure on  $\mathcal{X}$ , we denote  $\mathcal{F}[f]$  its Fourier transform which is defined by

$$\forall \omega \in \hat{\mathcal{X}}, \quad \mathcal{F}[f](\omega) = \int_{\hat{\mathcal{X}}} \overline{(x, \omega)} f(x) dx.$$

For a measure defined on  $\mathcal{X}$ , there exists a unique suitably normalized measure  $d\omega$  on  $\hat{\mathcal{X}}$  such that  $\forall f \in L^1(\mathcal{X}, dx; \mathcal{Y})$  and if  $\mathcal{F}[f] \in L^1(\hat{\mathcal{X}}, d\omega, \mathcal{Y})$  we have

$$\forall x \in \mathcal{X}, \quad f(x) = \int_{\hat{\mathcal{X}}} \mathcal{F}[f](\omega)(x, \omega) d\omega. \quad (1)$$

Moreover if  $d\omega$  is normalized,  $\mathcal{F}$  extends to a unitary operator from  $L^2(\mathcal{X}, dx, \mathcal{Y})$  onto  $L^2(\hat{\mathcal{X}}, d\omega, \mathcal{Y})$ . Then the inverse Fourier transform of a function  $g \in L^1(\hat{\mathcal{X}}, d\omega, \mathcal{Y})$  (where  $d\omega$  is a Haar measure on  $\hat{\mathcal{X}}$  suitably normalized w. r. t. the Haar measure  $dx$ ) is noted  $\mathcal{F}^{-1}[g]$  defined by

$$\forall x \in \mathcal{X}, \quad \mathcal{F}^{-1}[g](x) = \int_{\hat{\mathcal{X}}} (x, \omega) g(\omega) d\omega,$$

For the familiar case of a scalar-valued function  $f$  on the [LCA](#) group  $(\mathbb{R}^d, +)$ , we have:

$$\forall \omega \in \hat{\mathcal{X}}, \quad \mathcal{F}[f](\omega) = \int_{\mathbb{R}^d} e^{-i\langle \omega, x-z \rangle} f(x) dx, \quad (2)$$

the Haar measure being here the Lebesgue measure.

#### 2.5 ON OPERATOR-VALUED KERNELS

We now introduce the theory of vector-valued Reproducing Kernel Hilbert Space ([vv-RKHS](#)) that provides a flexible framework to study and learn vector-valued functions.



### 2.5.1 Definitions and properties

An operator-valued kernel is defined here as a  $\mathcal{Y}$ -reproducing kernel Carmeli et al. [5].

**Definition 2.** Given  $\mathcal{X}$ , a Polish space and  $\mathcal{Y}$ , a Hilbert Space, a map  $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y})$  is called a  $\mathcal{Y}$ -reproducing kernel if

$$\sum_{i,j=1}^N \langle K(x_i, x_j) y_j, y_i \rangle_{\mathcal{Y}} \geq 0,$$

for all  $x_1, \dots, x_N$  in  $\mathcal{X}$ , all  $y_1, \dots, y_N$  in  $\mathcal{Y}$  and  $N \geq 1$ . Given  $x \in \mathcal{X}$ ,  $K_x : \mathcal{Y} \rightarrow \mathcal{F}(\mathcal{X}; \mathcal{Y})$  denotes the linear operator whose action on a vector  $y$  is the function  $K_x y \in \mathcal{F}(\mathcal{X}; \mathcal{Y})$  defined by  $(K_x y)(z) = K(z, x)y$ , for all  $z \in \mathcal{X}$ .

Additionally, given a  $\mathcal{Y}$ -reproducing kernel  $K$ , there is a unique Hilbert space  $\mathcal{H}_K \subset \mathcal{F}(\mathcal{X}; \mathcal{Y})$  satisfying  $K_x \in \mathcal{L}(\mathcal{Y}; \mathcal{H}_K)$ , for all  $x \in \mathcal{X}$  and  $\forall x \in \mathcal{X}, \forall f \in \mathcal{H}_K, f(x) = K_x^* f$ , where  $K_x^* : \mathcal{H}_K \rightarrow \mathcal{Y}$  is the adjoint of  $K_x$ . The space  $\mathcal{H}_K$  is called the *vector-valued Reproducing Kernel Hilbert Space* associated with  $K$ . The corresponding product and norm are denoted by  $\langle \cdot, \cdot \rangle_K$  and  $\|\cdot\|_K$ , respectively. As a consequence [5] we have:

$$K(x, z) = K_x^* K_z \quad \forall x, z \in \mathcal{X}$$

$$\mathcal{H}_K = \overline{\text{span}} \{K_x y \mid \forall x \in \mathcal{X}, \forall y \in \mathcal{Y}\}$$

Another way to describe functions of  $\mathcal{H}_K$  consists in using a suitable feature map.

**Proposition 3** (Feature Operator Carmeli et al. [5]). Let  $\mathcal{H}$  be a Hilbert space and  $\Phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y}; \mathcal{H})$ , with  $\Phi_x := \Phi(x)$ . Then the operator  $W : \mathcal{H} \rightarrow \mathcal{F}(\mathcal{X}; \mathcal{Y})$  defined for all  $g \in \mathcal{H}$ , and for all  $x \in \mathcal{X}$  by  $(Wg)(x) = \Phi_x^* g$  is a partial isometry from  $\mathcal{H}$  onto the *vv-RKHS*  $\mathcal{H}_K$  with reproducing kernel

$$K(x, z) = \Phi_x^* \Phi_z, \quad \forall x, z \in \mathcal{X}.$$

$W^*W$  is the orthogonal projection onto

$$\text{Ker } W^\perp = \overline{\text{span}} \{\Phi_x y \mid \forall x \in \mathcal{X}, \forall y \in \mathcal{Y}\}.$$

Then  $\|f\|_K = \inf \{\|g\|_{\mathcal{H}} \mid \forall g \in \mathcal{H}, Wg = f\}$ .

We call  $\Phi$  a *feature map*,  $W$  a *feature operator* and  $\mathcal{H}$  a *feature space*.

### 2.5.2 Examples of operator-valued kernels

Operator-valued kernels have been first introduced in Machine Learning to solve multi-task regression problems. Multi-task regression is encountered in many fields such as structured classification when classes belong to a hierarchy for instance. Instead of solving independently  $p$  single output regression task, one would like to take advantage of the relationships between output variables when learning and making a decision.

Some authors also refer to as separable kernels.

**Definition 4** (Decomposable kernel). *Let  $A$  be a positive semi-definite operator of  $\mathcal{L}(\mathcal{Y})$ .  $K$  is said to be a  $\mathcal{Y}$ -Mercer decomposable kernel if for all  $(x, z) \in \mathcal{X}^2$ ,*

$$K(x, z) = k(x, z)A,$$

where  $k$  is a scalar Mercer kernel.

When  $\mathcal{Y} = \mathbb{R}^p$ , the matrix  $A$  is interpreted as encoding the relationships between the outputs coordinates. If a graph coding for the proximity between tasks is known, then it is shown in Álvarez, Rosasco, and Lawrence [1], Baldassarre et al. [2], and Evgeniou, Micchelli, and Pontil [7] that  $A$  can be chosen equal to the pseudo inverse  $L^\dagger$  of the graph Laplacian such that the norm in  $\mathcal{H}_K$  is a graph-regularizing penalty for the outputs (tasks). When no prior knowledge is available,  $A$  can be set to the empirical covariance of the output training data or learned with one of the algorithms proposed in the literature [6, 11, 14]. Another interesting property of the decomposable kernel is its universality (a kernel which may approximate an arbitrary continuous target function uniformly on any compact subset of the input space). A reproducing kernel  $K$  is said *universal* if the associated [vv-RKHS](#)  $\mathcal{H}_K$  is dense in the space  $\mathcal{C}(\mathcal{X}, \mathcal{Y})$ . The conditions for a kernel to be universal have been discussed in Caponnetto et al. [4] and Carmeli et al. [5]. In particular they show that a decomposable kernel is universal provided that the scalar kernel  $k$  is universal and the operator  $A$  is injective.

Curl-free and divergence-free kernels provide an interesting application of operator-valued kernels [3, 12, 13] to *vector field* learning, for which input and output spaces have the same dimensions ( $d = p$ ). Applications cover shape deformation analysis [13] and magnetic fields approximations [15]. These kernels discussed in [9] allow encoding input-dependent similarities between vector-fields.

**Definition 5** (Curl-free and Div-free kernel). *Assume  $\mathcal{X} = (\mathbb{R}^d, +)$  and  $\mathcal{Y} = \mathbb{R}^p$  with  $d = p$ . The divergence-free kernel is defined as*

$$K^{\text{div}}(x, z) = K_0^{\text{div}}(\delta) = (\nabla \nabla^T - \Delta I)k_0(\delta)$$

and the curl-free kernel as

$$K^{\text{curl}}(x, z) = K_0^{\text{curl}}(\delta) = -\nabla \nabla^T k_0(\delta),$$

where  $\nabla \nabla^T$  is the Hessian operator and  $\Delta$  is the Laplacian operator.

Although taken separately these kernels are not universal, a convex combination of the curl-free and divergence-free kernels allows to learn any vector field that satisfies the Helmholtz decomposition theorem [3, 12].

## Part II

### CONTRIBUTIONS

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## OPERATOR-VALUED RANDOM FOURIER FEATURES

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### 3.1 MOTIVATIONS

Random Fourier Features have been proved useful to implement efficiently kernel methods in the scalar case, allowing to learn a linear model based on an approximated feature map. In this work, we are interested to construct approximated operator-valued feature maps to learn vector-valued functions. With an explicit (approximated) feature map, one converts the problem of learning a function  $f$  in the vector-valued Reproducing Kernel Hilbert Space  $\mathcal{H}_K$  into the learning of a linear model  $\tilde{f}$  defined by:

$$\tilde{f}(x) = \tilde{\Phi}(x) * \theta,$$

where  $\Phi : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{Y})$  and  $\theta \in \mathcal{H}$ . The methodology we propose works for operator-valued kernels defined on any Locally Compact Abelian (LCA) group, noted  $(\mathcal{X}, *)$ , for some operation noted  $*$ . This allows us to use the general context of Pontryagin duality for Fourier transforms of functions on LCA groups. Building upon a generalization of Bochner's theorem for operator-valued measures, an operator-valued kernel is seen as the *Fourier transform* of an operator-valued positive measure. From that result, we extend the principle of Random Fourier Feature for scalar-valued kernels and derive a general methodology to build Operator Random Fourier Feature when operator-valued kernels are shift-invariant according to the chosen group operation.

### 3.2 CONSTRUCTION

We present a construction of Operator-valued Random Fourier Feature (ORFF) such that  $f : x \mapsto \tilde{\Phi}(x) * \theta$  is a continuous function that maps an arbitrary LCA group  $\mathcal{X}$  as input space to an arbitrary output Hilbert space  $\mathcal{Y}$ . First we define a functional *Fourier feature map*, and then propose a Monte-Carlo sampling from this feature map to construct an approximation of a shift-invariant  $\mathcal{Y}$ -Mercer kernel. Then, we prove the convergence of the kernel approximation  $\tilde{K}(x, z) = \tilde{\Phi}(x) * \tilde{\Phi}(z)$  with high probability on *compact* subsets of the LCA  $\mathcal{X}$ , when  $\mathcal{Y}$  is *finite dimensional*. Eventually we conclude with some numerical experiments.

The following proposition of Carmeli et al. [5] and Zhang, Xu, and Zhang [16] extends Bochner's theorem to any shift-invariant  $\mathcal{Y}$ -Mercer kernel.

**Proposition 6** (Operator-valued Bochner's theorem [16]). *If a continuous function  $K$  from  $\mathcal{X} \times \mathcal{X}$  to  $\mathcal{Y}$  is a shift-invariant  $\mathcal{Y}$ -Mercer kernel on  $\mathcal{X}$ ,*

then there exists a unique positive operator-valued measure  $M : \mathcal{B}(\mathcal{X}) \rightarrow \mathcal{L}_+(\mathcal{Y})$  such that:

$$\forall x, z \in \mathcal{X}, K(x, z) = \int_{\hat{\mathcal{X}}} \overline{(x \star z^{-1}, \omega)} dM(\omega), \quad (3)$$

where  $M$  belongs to the set of all the  $\mathcal{L}_+(\mathcal{Y})$ -valued measures of bounded variation on the  $\sigma$ -algebra of Borel subsets of  $\hat{\mathcal{X}}$ . Conversely, from any positive operator-valued measure  $M$ , a shift-invariant kernel  $K$  can be defined by eq. (3).

Although this theorem is central to the spectral decomposition of shift-invariant  $\mathcal{Y}$ -Mercer [OVK](#), the following results proved by Carmeli et al. [\[5\]](#) provides insights about this decomposition that are more relevant in practise. It first shows how to build shift-invariant  $\mathcal{Y}$ -Mercer kernel but more importantly, also states that any operator-valued spectral decomposition of such [OVKs](#) when  $\mathcal{Y}$  is finite dimensional or  $\mathcal{X}$  is compact can be written using a pair  $(A, \mu)$  where  $A$  is an operator-valued function on  $\hat{\mathcal{X}}$  and  $\mu$  is a real-valued positive measure on  $\hat{\mathcal{X}}$ . Note that obviously such a pair is not unique and the choice of this paper may have an impact on theoretical properties as well as practical computations.

**Proposition 7** (Carmeli et al. [\[5\]](#)). *Let  $\mu$  be a positive measure on  $\mathcal{B}(\hat{\mathcal{X}})$  and  $A : \hat{\mathcal{X}} \rightarrow \mathcal{L}(\mathcal{Y})$  such that  $\langle A(\cdot)y, y' \rangle \in L^1(\mathcal{X}, d\mu)$  for all  $y, y' \in \mathcal{Y}$  and  $A(\omega) \succcurlyeq 0$  for  $\mu$ -almost all  $\omega$ . Then, for all  $\delta \in \mathcal{X}$  and for all  $y, y' \in \mathcal{Y}$ ,*

$$\langle y, K_e(\delta)y \rangle_{\mathcal{Y}} = \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y, A(\omega)y' \rangle_{\mathcal{Y}} d\mu(\omega) \quad (4)$$

is the kernel signature of a shift-invariant  $\mathcal{Y}$ -Mercer kernel  $K$  such that  $K(x, z) = K_e(x \star z^{-1})$ . In other terms, each function  $K_e(\cdot)$  is the Fourier transform of  $A(\cdot)p_{\mu}(\cdot)$  where the integral converges in the weak operator topology and  $p_{\mu}(\omega) = \frac{d\mu}{d\omega}$  is the Radon-Nikodym derivative (density) of the measure  $\mu$ . If  $\mathcal{Y}$  is finite dimensional or  $\mathcal{X}$  is compact, any shift-invariant kernel is of the above form for some pair  $(A(\omega), \mu(\omega))$ .

This theorem is more interesting than eq. (3) in the sense that it shows that we are certain of the existence of a *scalar* measure  $\mu$  and a positive operator  $A(\omega)$ , provided that  $\mathcal{X}$  is compact or  $\mathcal{Y}$  is finite dimensional. When  $p = 1$  one can always assume  $A$  is reduced to the scalar 1,  $\mu$  is still a bounded positive measure and we retrieve the Bochner theorem applied to the scalar case [\(??\)](#).

While [??](#) gives some insights on how to build an approximation of a  $\mathcal{Y}$ -Mercer kernel, we need a theorem that provides an explicit construction of the pair  $A(\omega), \mu(\omega)$  from the kernel signature. Proposition 14 in Carmeli et al. [\[5\]](#) gives the solution, and also provide a sufficient condition for theorem [7](#) to apply.

**Proposition 8** (Carmeli et al. [5]). *Let  $K$  be a shift-invariant  $\mathcal{Y}$ -Mercer kernel. Suppose that  $\forall z \in \mathcal{X}$  and  $\forall y, y' \in \mathcal{Y}$ ,  $\langle K_e(\cdot)y, y' \rangle \in L^1(\mathcal{X}, dx)$  where  $dx$  denotes the Haar measure on  $(\mathcal{X}, \star)$ . Define  $C$  such that for all  $\omega \in \hat{\mathcal{X}}$  and for all  $y, y'$  in  $\mathcal{Y}$ ,*

$$\begin{aligned} \langle y, C(\omega)y' \rangle &= \int_{\mathcal{X}} (\delta, \omega) \langle y, K_e(\delta)y' \rangle d\delta \\ &= \mathcal{F}^{-1} [\langle y, K_e(\cdot)y' \rangle] (\omega) \end{aligned} \quad (5)$$

Then

- i)  $C(\omega)$  is a bounded non-negative operator for all  $\omega \in \hat{\mathcal{X}}$ ,
- ii)  $\langle y, C(\cdot)y' \rangle \in L^1(\hat{\mathcal{X}}, d\omega)$  for all  $y, y' \in \mathcal{Y}$ ,
- iii) for all  $\delta \in \mathcal{X}$  and for all  $y, y'$  in  $\mathcal{Y}$ ,

$$\langle y, K_e(\delta)y' \rangle = \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y, C(\omega)y' \rangle d\omega.$$

Gathering the two propositions, we present now the following property that allows to build a spectral decomposition of a shift-invariant  $\mathcal{Y}$ -Mercer kernel on a LCA group  $(\mathcal{X}, \star)$ .

**Proposition 9** (Spectral decomposition of shift-invariant  $\mathcal{Y}$ -Mercer kernel). *Let  $K_e$  be the signature of a shift-invariant  $\mathcal{Y}$ -Mercer kernel on  $(\mathcal{X}, \star)$ .*

*Suppose that for all  $y, y' \in \mathcal{Y}$ ,  $\langle K_e(\cdot)y, y' \rangle \in L^1(\mathcal{X}, dx)$ . If  $\mathcal{Y}$  is of finite dimension or  $\mathcal{X}$  is compact, then there exists  $\mu$ , a positive measure on  $\mathcal{B}(\hat{\mathcal{X}})$ , and  $A : \hat{\mathcal{X}} \rightarrow \mathcal{L}_+(\mathcal{Y})$ , a operator-valued functions such that  $\langle A(\cdot)y, y' \rangle \in L^1(\mathcal{X}, d\mu)$  for all  $y, y' \in \mathcal{Y}$  and*

$$\forall (y, y') \in \mathcal{Y}^2, \quad \langle y, K_e(\delta)y' \rangle = \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y, A(\omega)y' \rangle p_{\mu}(\omega) d\omega.$$

where  $\langle y, A(\omega)y' \rangle p_{\mu}(\omega) = \mathcal{F}^{-1} [\langle y, K_e(x \star z^{-1})y' \rangle]$ .

*Proof.* From eq. (4) and eq. (5), if  $\mathcal{X}$  is compact or  $\mathcal{Y}$  is finite dimensional, we can write the following equality concerning the operator-valued kernel signature  $K_e$ . For all  $\delta \in \mathcal{X}$  and for all  $y, y'$  in  $\mathcal{Y}$

$$\int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y, C(\omega)y' \rangle d\omega = \int_{\hat{\mathcal{X}}} \overline{(\delta, \omega)} \langle y, A(\omega)y' \rangle d\mu(\omega).$$

Since both sides of the equation define continuous functions, the following equality holds  $\mu$ -almost everywhere. For all  $\omega \in \hat{\mathcal{X}}$  and for all  $y, y' \in \mathcal{Y}$ ,

$$\langle y, C(\omega)y' \rangle = \langle y, A(\omega)y' \rangle p_{\mu}(\omega) = \mathcal{F}^{-1} [\langle y, K_e(\cdot)y' \rangle] (\omega), \quad (6)$$

where  $p_{\mu}(\omega) = \frac{d\mu}{d\omega}$  is the Radon-Nikodym derivative of the measure  $\mu$ , e.g. its density.  $\square$



In the case where  $\mathcal{Y} = \mathbb{R}^p$ , we rewrite eq. (6) coefficient-wisely by choosing the orthonormal basis of  $\mathcal{Y}$ ,  $(e_1, \dots, e_p)$ , such that for all  $i, j \in \{1, \dots, p\}$ ,

$$\langle e_i, C(\omega) e_j \rangle = C(\omega)_{ij} = A(\omega)_{ij} p_\mu(\omega) = \mathcal{F}^{-1} [K_e(\delta)_{ij}]. \quad (7)$$

It follows that

$$\forall i, j \in \{1, \dots, p\}, \quad K_e(x \star z^{-1})_{ij} = \mathcal{F} [A(\cdot)_{ij}] \quad (8)$$

**Remark 10.** Note that although the inverse Fourier transform of  $K_e$  yields a unique operator-valued function  $C(\cdot)$ , the decomposition of  $C(\omega)$  into  $A(\omega)p_\mu(\omega)$  is not unique. The choice of the decomposition may be justified by the computational cost or by the nature of the constants involved in the uniform convergence of the estimator.

### 3.2.1 Construction of ORFF

Without loss of generality we assume that  $\int_{\mathcal{X}} d\mu(\omega) = 1$  and thus,  $\mu$  is a probability distribution and  $p_\mu$ , a probability density. Note that this is always possible through an appropriate rescaling of the kernel.

## 3.3 UNIFORM BOUND ON THE APPROXIMATION

### 3.4 LEARNING WITH OPERATOR-VALUED RANDOM-FOURIER FEATURES

### 3.5 CONSISTENCY AND GENERALIZATION BOUNDS

### 3.6 CONCLUSIONS





#### 4.1 BACKGROUND

#### 4.2 THE NYSTRÖM METHOD

#### 4.3 SUB-SAMPLING THE DATA

#### 4.4 CONCLUSIONS

## Part III

### FINAL WORDS

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## CONCLUSIONS

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## Part IV

## APPENDIX





## OPERATOR-VALUED FUNCTIONS AND INTEGRATION

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## DECLARATION

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Romain Brault



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