Basis Matters: Better Communication-Efficient Second Order Methods for Federated Learning

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Abstract

Recent advances in distributed optimization have shown that Newton-type methods with proper communication compression mechanisms can guarantee fast local rates and low communication cost compared to first order methods. We discover that the communication cost of these methods can be further reduced, sometimes dramatically so, with a surprisingly simple trick: *Basis Learn (BL)*. The idea is to transform the usual representation of the local Hessians via a change of basis in the space of matrices and apply compression tools to the new representation. To demonstrate the potential of using custom bases, we design a new Newton-type method (BL1), which reduces communication cost via both *BL* technique and bidirectional compression mechanism. Furthermore, we present two alternative extensions (BL2 and BL3) to partial participation to accommodate federated learning applications. We prove local linear and superlinear rates independent of the condition number. Finally, we support our claims with numerical experiments by comparing several first and second order methods.

Contents

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1 Introduction

We consider federated optimization problems of the form

$$\min_{x \in \mathbb{R}^d} f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x), \tag{1}$$

where each function $f_i : \mathbb{R}^d \to \mathbb{R}$ represents the local loss associated with the data owned by device or client $i \in [n] := \{1, 2, \ldots, n\}$ only. This formulation aims to train a single machine learning model $x \in \mathbb{R}^d$ composed of d parameters by minimizing empirical loss $f(x)$ using all n clients' data. We assume f is μ-strongly convex and problem (1) has the unique optimal solution x^* throughout the paper.

Due to the increasing size of the model and the amount of the training data, in practical deployments, methods of choice for solving the problem (1) have been *distributed first-order gradient methods* so far [Liu and Zhang, 2020, Xu et al., 2020]. Among other things, two key considerations in the design of efficient distributed optimization method are *iteration complexity*, which measures how many iterations the method should take to achieve some prescribed accuracy, and *communication cost* per iteration, which measures the number of bits that clients communicate to each other or some parameter server [Bekkerman et al., 2011]. Expectedly, these two quantities are in a trade-off: reducing the size of communicated messages per iteration, potentially increases the total number of iterations. This trade-off then interacts with the problem structure (training data and model) and network properties (bandwidth and latency) to find the best configuration for final deployment.

However, despite their wide applicability, all first-order methods inevitably suffer from a dependence of suitably defined *condition number*. To overcome the curse of the condition number, Newton-type or second-order optimization methods have been gaining considerable attention recently since (at least local) convergence properties of these algorithms are not affected by the condition number of the problem [Dennis and Moré, 1974, Dembo et al., 1982, Griewank, 1981, Nesterov and Polyak, 2006]. On the other hand, the caveat of this approach is that, although it greatly improves iteration complexity, the cost of naively communicating second-order information, such as Hessian matrices, is extremely high and infeasible in practice [Bekkerman et al., 2011]. In this work, we argue that with proper care of second-order information and for ill-conditioned problems, distributed second-order algorithms can offer essentially better trade-offs than first-order algorithms.

1.1 Distributed second order methods.

The straightforward implementation of classical Newton's method in the distributed environment includes communication of local Hessian matrices $\nabla^2 f_i(x^k)$ with d^2 entries in all iterations $k \geq 0$. Consider this naive implementation of Newton's method as the baseline algorithm for distributed second-order optimization, just like the distributed gradient descent algorithm for first-order methods. Below we discuss the main algorithmic designs to reduce the quadratic dependence d^2 of the dimension in per-iteration communication cost and make second-order methods communication efficient for distributed optimization.

One stream of works avoids sending full Hessian matrices and uses second-order information locally to compute Hessian-vector products $\nabla^2 f_i(x^k) v_i^k \in \mathbb{R}^d$ for some (adaptively defined) vectors $v_i^k \in \mathbb{R}^d$. With this approach, second order information is imparted only through such products, which cost d floats of communication instead of d^2. The computation side of this approach is also

efficient since the Hessian matrices are not computed directly but Hessian-vector products only, which are as cheap to compute as gradients $\nabla f_i(x^k)$ [Pearlmutter, 1993]. Methods following this approach are DiSCO [Zhang and Xiao, 2015] (see also [Zhuang et al., 2015, Lin et al., 2014, Roosta et al., 2019]), GIANT [Wang et al., 2018] (see also [Shamir et al., 2014, Reddi et al., 2016]) and DINGO [Crane and Roosta, 2019] (see also [Ghosh et al., 2020]). Inspired by the local first order methods [Gorbunov et al., 2021, Stich, 2020, Khaled et al., 2020, Konečný et al., 2016], local variant of Newton's method was studied in [Gupta et al., 2021]. Typically these methods either guarantee fast rates under stronger assumptions, such as quadratic problems or/and homogeneous data distribution or guarantee only linear rates attainable by first-order methods.

Alternatively, the high cost of Hessian communication can be reduced by compressing second-order information via lossy compression operators acting on matrices (such as low-rank approximations, randomly or greedily sparsifying entries). Again, this idea was originated from the first-order methods employing communication compression [Wangni et al., 2018, Alistarh et al., 2018, 2017, Wen et al., 2017]. Recently developed second-order methods DAN-LA [Zhang et al., 2020], Quantized Newton [Alimisis et al., 2021], NewtonLearn [Islamov et al., 2021] and FedNL [Safaryan et al., 2021] are based on this idea of properly incorporating compression strategies for second-order information. In contrast to the previous approach, this strategy relies on the computation of full Hessian matrices, which might be computationally intensive for some applications. However, the optimization problem these methods address is quite generic (general finite sum structure (1) over arbitrarily heterogeneous data), and theoretical guarantees (global linear with local superlinear convergence rates) are far beyond the reach of all first-order methods.

Motivated by these recent developments on distributed second-order methods with communication compression, we extend the results of FedNL [Safaryan et al., 2021] allowing even more aggressive compression for some applications.

2   Motivation and Contributions

To motivate our approach properly and illustrate the potential of our technique, we discuss three different implementations of *classical Newton's method* for solving the problem (1).

2.1   Naive implementation

For general finite sums (1), Newton's method requires each device $i \in [n]$ to compute gradient vector $\nabla f_i(x)$ and Hessian matrix $\nabla^2 f_i(x)$ at the current point and send them to the parameter server to do the model update. While the convergence of Newton's method is locally quadratic, $\mathcal{O}(d^2)$ communication costs are high due to quadratic dependence from the dimension d. However, we can devise more efficient implementations, given some prior knowledge of the problem or/and data structure.

2.2   Utilizing the problem structure

Suppose the problem (1) models the training of *Generalized Linear Model (GLM)*, such as ridge regression or logistic regression. Then each local loss function has the form

$$f_i(x) = \frac{1}{m} \sum_{j=1}^{m} f_{ij}(x), \tag{2}$$

where $f_{ij}(x) := \varphi_{ij}(a_{ij}^\top x)$ is the loss associated with j^{th} data-vector $a_{ij} \in \mathbb{R}^d$ stored on i^{th} device and $\varphi_{ij} \colon \mathbb{R} \to \mathbb{R}$ is the corresponding loss function.[1] A better implementation of Newton's method taking advantage of the Hessian structure

$$\nabla^2 f_i(x) \;\; = \;\; \frac{1}{m}\sum_{j=1}^{m}\varphi_{ij}''(a_{ij}^\top x)a_{ij}a_{ij}^\top \tag{3}$$

is described in [Islamov et al., 2021, Section 2.2]. In this implementation, the server is assumed to have *access to all training dataset* $\{a_{ij}\}_{ij}$. Then to communicate Hessian matrix of the form (3), it is enough to send m coefficients $\{\varphi_{ij}''(a_{ij}^\top x) \colon j \in [m]\}$ instead of d^2 entries. Hence, in cases when $m \ll d^2$, we can run Newton's method much efficiently with $\mathcal{O}(m+d)$ communication cost.

However, there are two limitations here. First, this approach fails to benefit when local datasets are too big (i.e., $m > d^2$), which is often the case in practice. Second, all devices reveal their local training data to the server, making the approach inapplicable to federated learning applications, where data privacy is crucial.

2.3 Utilizing the data structure

We now describe a strategy that additionally takes advantage of the data structure and dramatically reduces communication costs regardless of the data size and without revealing any local data.

The imposed structural assumption on the data is that local data points $\{a_{ij} \colon j \in [m]\}$ of ith client belong to some linear subspace $G_i \subseteq \mathbb{R}^d$ of dimension $r \in [d]$.[2] Note that this condition is trivially satisfied for $r = d$ for any data. However, in practice, training data points have much smaller intrinsic dimensionality $r \ll d$. Notice that once we fix some basis $\{v_{it}\}_{t=1}^r$ of G_i, we can represent data points a_{ij} as linear combinations

$$a_{ij} = \sum_{t=1}^{r}\alpha_{ijt}v_{it}, \quad j \in [m]. \tag{4}$$

Instead of directly revealing actual data points a_{ij}, each device sends the basis $\{v_{it}\}_{t=1}^r$ to the server *only once* (before the training) with the cost of sending rd floats. Based on the representations (3) from the problem structure and (4) from the data structure, the Hessian of $f_i(x)$ can be transformed into

$$\nabla^2 f_i(x) \stackrel{(3),(4)}{=} \frac{1}{m}\sum_{j=1}^{m}\varphi_{ij}''(a_{ij}^\top x)\sum_{t,l=1}^{r}\alpha_{ijt}\alpha_{ijl}v_{it}v_{il}^\top = \sum_{t,l=1}^{r}\underbrace{\left[\frac{1}{m}\sum_{j=1}^{m}\varphi_{ij}''(a_{ij}^\top x)\alpha_{ijt}\alpha_{ijl}\right]}_{\gamma_{itl}(x)}\underbrace{v_{it}v_{il}^\top}_{\mathbf{V}_{itl}}, \tag{5}$$

where outer products $\mathbf{V}_{itl} := v_{it}v_{il}^\top$ are linearly independent matrices (see Lemma B.1 in the Appendix) and $\gamma_{itl}(x)$ are coefficients in the brackets. The key observation from (5) is that to communicate $\nabla^2 f_i(x)$ we need to send only r^2 coefficients $\{\gamma_{itl}(x) \colon t, l \in [r]\}$ instead of d^2 entries

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[1]For simplicity we assume that each device has the same number of local data points denoted by m. Generally, it could be different for different clients.

[2]To make notations simpler, r is the same for all clients.

Table 1: Key features of different implementation of classical Newton's method in distributed systems. Here m is the number of local training data, r is the intrinsic dimensionality of local data vectors.

Implementation of Newton's method	Standard/Naive	[Islamov et al., 2021]	**Ours**
Problem	General Finite Sum	General Finite Sum	Generalized Linear Model
Data	Arbitrary	Arbitrary	Intrinsically Low-Dimensional
Gradient communication cost per iteration (floats)	d	$\min(m, d)$	r
Hessian communication cost per iteration (floats)	d^2	$\min(m, d^2)$	r^2
Initial communication cost (floats)	–	md	rd
Reveals local training data ?	No	Yes	No

as the server already knows matrices $\mathbf{V}_{itl}$ through the basis $\{v_{it}\}_{t=1}^r$. The takeaway from this observations is that the standard basis of $\mathbb{R}^{d \times d}$ is not always optimal. Indeed, in this case, any basis of $\mathbb{R}^{d \times d}$ containing r^2 matrices $\{\mathbf{V}_{itl}\}_{t,l=1}^r$ is better choice for encoding Hessians $\nabla^2 f_i(x)$ without any loss in precision. Thus, $\mathcal{O}(d^2)$ communication cost is reduced to $\mathcal{O}(r^2 + d)$. In case of $r = \mathcal{O}(\sqrt{d})$, we get Newton's method with $\mathcal{O}(d)$ communication cost and local quadratic convergence.

Analogous to (5), similar representation can be derived for gradients too, namely $\nabla f_i(x) \in G_i$. Hence, we can send $\nabla f_i(x)$ by its r basis coefficients instead of d coordinates. This way we further reduce communication cost up to $\mathcal{O}(r^2)$ (see Table 1 for the summary). In the extreme case of $r = \mathcal{O}(1)$, we run Newton's method with $\mathcal{O}(1)$ cost per iteration!

Note that (5) is a special case of more general Hessian representation $\nabla^2 f_i(x) = \mathbf{Q}_i \Lambda_i(x) \mathbf{Q}_i^\top$, where $\mathbf{Q}_i$ is a fixed invertible matrix (known to the server) and $\Lambda_i(x)$ is a sparse matrix with much less than d^2 (e.g., r^2 for (5)) non-zero entries. Changing the standard basis of $\mathbb{R}^{d \times d}$ via the transition matrix $\mathbf{Q}_i$, we transform potentially dense Hessian $\nabla^2 f_i(x)$ (in the standard basis) into sparse $\Lambda_i(x)$ in the new basis.

Thus, we save in communication for free just by changing the basis in the beginning of the training. Motivated by this idea, we propose a new approach: *Basis Learn.*

2.4 Contributions

Our goal is to further investigate the benefits and possible pitfalls of using custom bases in second-order optimization for general finite sums (1) with arbitrarily heterogeneous data. As, by choosing a suitable basis, we can transform the Hessian into a sparser matrix in a lossless way, we propose and design three new methods, which apply lossy compression strategies afterwards to get even better performance in terms of communication complexity.

(1) Basis learn with bidirectional compression. Our first contribution is the new method BL1, which successfully integrates bidirectional compression with any predefined basis for Hessians. In BL1, both client-to-server and server-to-client communications are compressed via careful application

of compression operators. We allow both unbiased compressors, such as random sparsification (Rand-K) or random dithering, and contractive compressors, such as greedy sparsification (Top-K) or low-rank approximations (Rank-R). In the special case of choosing the standard basis, our method recovers FedNL [Safaryan et al., 2021]. Thus, basis learn can be viewed as a generalization of FedNL.

(2) Extensions to partial device participation. For massively distributed trainings, such as in federated learning, with too many clients, we propose two extensions, BL2 and BL3, to accommodate partial participation of devices. Thus, we unify bidirectional compression and partial participation under basis learn. Furthermore, within these two extensions we propose two options to guarantee the positive definiteness of accumulated Hessian estimator at the server avoiding matrix projection steps of BL1: first option (implemented in BL2) is based on compression error trick of [Safaryan et al., 2021], while the other option (realized in BL3) is to choose bases with positive semidefinite matrices in the symmetric matrix space.

(3) Fast local rates. For all our methods we prove local linear and superlinear rates independent of the condition number and the size of local dataset.

(4) Experiments. By composing low-rank approximation and unbiased compression operators, we propose more efficient compressors for matrices leading to better performance in the experiments.

3   Matrix Compression

Here we adopt two classes of vector compressor operators to matrices. A (possibly) randomized map $\mathcal{C} : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$ is called a *contraction compressor* if there exists a constant $0 < \delta \leq 1$ such that

$$\mathbb{E}\left[\|\mathbf{A} - \mathcal{C}(\mathbf{A})\|_{\mathrm{F}}^2\right] \leq (1 - \delta)\|\mathbf{A}\|_{\mathrm{F}}^2, \quad \forall \mathbf{A} \in \mathbb{R}^{d \times d}. \tag{6}$$

Further, we say that a randomized map $\mathcal{C} : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$ is an *unbiased compressor* if there exists a constant $\omega \geq 0$ ("variance parameter") such that

$$\mathbb{E}\left[\mathcal{C}(\mathbf{A})\right] = \mathbf{A} \quad \text{and} \quad \mathbb{E}\left[\|\mathcal{C}(\mathbf{A})\|_{\mathrm{F}}^2\right] \leq (\omega + 1)\|\mathbf{A}\|_{\mathrm{F}}^2, \quad \forall \mathbf{A} \in \mathbb{R}^{d \times d}. \tag{7}$$

The contraction compressor and unbiased compressor on $\mathbb{R}^d$ can be defined in the same way where the Frobenius norm $\|\cdot\|_{\mathrm{F}}$ is replaced by the standard Euclidean norm $\|\cdot\|$. For more examples of contraction and unbiased compressors, we refer the reader to [Safaryan et al., 2021, Beznosikov et al., 2020]. On the other hand, the compressor on $\mathbb{R}^{d \times d}$ can be regarded as a compressor on $\mathbb{R}^{d^2}$. Hence, compressors on the vector space $\mathbb{R}^{d^2}$ can be applied to the matrix in $\mathbb{R}^{d \times d}$. One can combine two compressors from different classes to get new ones [Qian et al., 2021]. In particular, we consider composition of Rank-R [Safaryan et al., 2021] and unbiased compressors below.

Suppose $\mathcal{Q}_1^i$ and $\mathcal{Q}_2^i$, $i \in [d]$ are unbiased compressors on $\mathbb{R}^d$ with variance parameter ω_1 and ω_2, respectively. For any $\mathbf{A} \in \mathbb{R}^{d \times d}$, let $\mathbf{A} = \sum_{i=1}^{d} \sigma_i u_i v_i^\top$ be the singular value decomposition of $\mathbf{A}$ with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_d \geq 0$. For $R \leq d$, define

$$\mathcal{C}_1(\mathbf{A}) := \sum_{i=1}^{R} \frac{\sigma_i \mathcal{Q}_1^i(a_i u_i) \mathcal{Q}_2^i(b_i v_i)^\top}{a_i b_i (\omega_1 + 1)(\omega_2 + 1)},$$

where $a_i, b_i > 0$ are som constants chosen independently of $\mathcal{Q}_1^i$ and $\mathcal{Q}_2^i$ for $1 \leq i \leq R$. For example, we can set $a_i = b_i = 1$ for all i, or $a_i = b_i = \sqrt{\sigma_i}$ for all i. Notice that if $\mathbf{A}$ is symmetric, $\mathcal{C}_1(\mathbf{A})$ is not necessarily symmetric. However, we can symmetrize the output matrix by defining

$$\mathcal{C}_2(\mathbf{A}) := \begin{cases} \mathcal{C}_1(\mathbf{A}) & \text{if } \mathbf{A} \text{ is not symmetric} \\ \frac{\mathcal{C}_1(\mathbf{A}) + \mathcal{C}_1(\mathbf{A})^\top}{2} & \text{if } \mathbf{A} \text{ is symmetric} \end{cases}.$$

The following lemma says that this symmetrization process preserves contractiveness.

Lemma 3.1. (i) For any $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d \times d}$, if $\mathbf{A}$ is symmetric, then $\left\| (\mathbf{B} + \mathbf{B}^\top)/2 - \mathbf{A} \right\|_\text{F} \leq \|\mathbf{B} - \mathbf{A}\|_\text{F}$.

(ii) For any contraction compressor $\mathcal{C} : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$ with contraction parameter δ, the mapping $\tilde{\mathcal{C}} : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$ defined by

$$\tilde{\mathcal{C}}(\mathbf{A}) := \begin{cases} \mathcal{C}(\mathbf{A}) & \text{if } \mathbf{A} \text{ is not symmetric} \\ \frac{\mathcal{C}(\mathbf{A}) + \mathcal{C}(\mathbf{A})^\top}{2} & \text{if } \mathbf{A} \text{ is symmetric} \end{cases}$$

is also a contraction compressor with contraction parameter δ.

Proposition 3.2. $\mathcal{C}_1$ and $\mathcal{C}_2$ are both contraction compressors with parameter $\frac{R}{d(\omega_1+1)(\omega_2+1)}$.

4 Basis Learning in $\mathbb{R}^{d \times d}$

Let $\{\mathbf{B}_i^{jl} \mid j, l \in [d]\}$ be a basis in $\mathbb{R}^{d \times d}$ and $N := d^2$ be the number of matrices in the basis for any $i \in [n]$. Then for any matrix $\mathbf{A}$ in $\mathbb{R}^{d \times d}$, it can be uniquely represented as $\mathbf{A} = \sum_{j,l} h_{jl}^i(\mathbf{A}) \mathbf{B}_i^{jl}$, where $h_{jl}^i(\mathbf{A}) \in \mathbb{R}$ is the coefficient corresponding to $\mathbf{B}_i^{jl}$.

Define $h^i(\mathbf{A}) \in \mathbb{R}^{d \times d}$ such that $h^i(\mathbf{A})_{jl} := h_{jl}^i(\mathbf{A})$ for $j, l \in [d]$. For any matrix $\mathbf{A} \in \mathbb{R}^{d \times d}$, let $vec : \mathbb{R}^{d \times d} \to \mathbb{R}^{d^2}$ be defined as

$$vec(\mathbf{A}) = (\mathbf{A}_{11}, ..., \mathbf{A}_{d1}, ..., \mathbf{A}_{1i}, ..., \mathbf{A}_{di}, ..., \mathbf{A}_{1d}, ..., \mathbf{A}_{dd})^\top.$$

Define $\mathcal{M}_i := (\mathbf{B}_i^{11}, ..., \mathbf{B}_i^{d1}, ..., \mathbf{B}_i^{1j}, ..., \mathbf{B}_i^{dj}, ..., \mathbf{B}_i^{1d}, ..., \mathbf{B}_i^{dd})$. Then we have $\mathbf{A} = \mathcal{M}_i vec(h^i(\mathbf{A}))$, which is equivalent to

$$vec(\mathbf{A}) = \mathcal{B}_i \cdot vec(h^i(\mathbf{A})), \tag{8}$$

for any matrix $\mathbf{A} \in \mathbb{R}^{d \times d}$, where

$$\mathcal{B}_i := (vec(\mathbf{B}_i^{11}), ..., vec(\mathbf{B}_i^{d1}), ..., vec(\mathbf{B}_i^{1j}), ..., vec(\mathbf{B}_i^{dj}), ..., vec(\mathbf{B}_i^{1d}), ..., vec(\mathbf{B}_i^{dd})) \in \mathbb{R}^{N \times N}.$$

Since the representation (8) is unique, we know $\mathcal{B}_i$ is invertible, and thus

$$vec(h^i(\mathbf{A})) = \mathcal{B}_i^{-1} vec(\mathbf{A}). \tag{9}$$

Example 4.1. The $(j, l)^{th}$ entry of $\mathbf{B}_i^{jl}$ is 1 and the others are 0 for $j, l \in [d]$. Then $\mathbf{A} = h^i(\mathbf{A})$ for any $\mathbf{A} \in \mathbb{R}^{d \times d}$.

Example 4.2. The $(j, l)^{th}$ and $(l, j)^{th}$ entries of $\mathbf{B}_i^{jl}$ are 1 and the others are 0 for $d \geq j \geq l \geq 1$. The $(j, l)^{th}$ entry of $\mathbf{B}_i^{jl}$ is 1, the $(l, j)^{th}$ entry of $\mathbf{B}_i^{jl}$ is -1, and the others are 0 for $1 \leq j < l \leq d$. Then for any symmetric matrix $\mathbf{A} \in \mathbb{R}^{d \times d}$, $h^i(\mathbf{A})$ is the lower triangular part of $\mathbf{A}$.

Algorithm 1 BL1 (Basis Learn with Bidirectional Compression)

1: **Parameters:** Hessian learning rate $\alpha \geq 0$; model learning rate $\eta \geq 0$; gradient compression probability $p \in (0, 1]$; compression operators $\{\mathcal{C}_1^k, \ldots, \mathcal{C}_n^k\}$ and $\mathcal{Q}^k$

2: **Initialization:** $x^0 = w^0 = z^0 \in \mathbb{R}^d$; $\mathbf{L}_i^0 \in \mathbb{R}^{d \times d}$, $\mathbf{H}_i^0 = \sum_{jl}(\mathbf{L}_i^0)_{jl}\mathbf{B}_i^{jl}$, and $\mathbf{H}^0 := \frac{1}{n}\sum_{i=1}^n \mathbf{H}_i^0$; $\xi^0 = 1$

3: **for** each device $i = 1, \ldots, n$ in parallel **do**

4:     **if** $\xi^k = 1$

5:        $w^{k+1} = z^k$, compute local gradient $\nabla f_i(z^k)$ and send to the server

6:     **if** $\xi^k = 0$

7:        $w^{k+1} = w^k$

8:     Compute local Hessian $\nabla^2 f_i(z^k)$ and send $\mathbf{S}_i^k := \mathcal{C}_i^k(h^i(\nabla^2 f_i(z^k)) - \mathbf{L}_i^k)$ to the server

9:     Update local Hessian shifts $\mathbf{L}_i^{k+1} = \mathbf{L}_i^k + \alpha \mathbf{S}_i^k$, $\mathbf{H}_i^{k+1} = \mathbf{H}_i^k + \alpha \sum_{jl}(\mathbf{S}_i^k)_{jl}\mathbf{B}_i^{jl}$

10: **end for**

11: **on** server

12:     **if** $\xi^k = 1$

13:        $w^{k+1} = z^k$, $g^k = \nabla f(z^k)$

14:     **if** $\xi^k = 0$

15:        $w^{k+1} = w^k$, $g^k = \left[\mathbf{H}^k\right]_\mu (z^k - w^k) + \nabla f(w^k)$

16:     $x^{k+1} = z^k - \left[\mathbf{H}^k\right]_\mu^{-1} g^k$

17:     $\mathbf{H}^{k+1} = \mathbf{H}^k + \frac{\alpha}{n}\sum_{i=1}^n \sum_{jl}(\mathbf{S}_i^k)_{jl}\mathbf{B}_i^{jl}$

18:     Send $v^k := \mathcal{Q}^k(x^{k+1} - z^k)$ to all devices $i \in [n]$

19:     Update the model $z^{k+1} = z^k + \eta v^k$

20:     Send $\xi^{k+1} \sim \text{Bernoulli}(p)$ to all devices $i \in [n]$

21: **for** each device $i = 1, \ldots, n$ in parallel **do**

22:     Update the model $z^{k+1} = z^k + \eta v^k$

23: **end for**

We have Algorithm 1 (BL1) as an extension of FedNL-BC in [Safaryan et al., 2021]. BL1 mainly has two differences from FedNL-BC: (i) We use $\mathbf{L}_i^k$ to learn the coefficient matrix $h^i(\nabla^2 f_i(z^k))$ rather than the Hessian; (ii) When $\xi^k = 0$, we use $\left[\mathbf{H}^k\right]_\mu$ rather than $\mathbf{H}^k$ to construct the gradient estimator g^k, where $[\cdot]_\mu$ represents the projection on the set $\{\mathbf{A} \in \mathbb{R}^{d \times d} | \ \mathbf{A} = \mathbf{A}^\top, \mathbf{A} \succeq \mu\mathbf{I}\}$.

Assumption 4.3. *(i)* $\mathcal{Q}^k$ *($\mathcal{Q}_i^k$) is an unbiased compressor with parameter* ω_{M} *and* $0 < \eta \leq 1/(\omega_{\mathrm{M}}+1)$. *(ii) For all* $j \in [d]$, $(z^k)_j$ *in Algorithm 1* $((z_i^k)_j$ *in Algorithm 2 or Algorithm 3) is a convex combination of* $\{(x^t)_j\}_{t=0}^k$ *for* $k \geq 0$.

Assumption 4.4. *(i)* $\mathcal{Q}^k$ *($\mathcal{Q}_i^k$) is a contraction compressor with parameter* δ_{M} *and* $\eta = 1$. *(ii)* $\mathcal{Q}^k$ *($\mathcal{Q}_i^k$) is deterministic, i.e.,* $\mathbb{E}[\mathcal{Q}^k(x)] = \mathcal{Q}^k(x)$ *for any* $x \in \mathbb{R}^d$.

Assumption 4.5. *(i)* $\mathcal{C}_i^k$ *is an unbiased compressor with parameter* ω *and* $0 < \alpha \leq 1/(\omega+1)$. *(ii) For all* $i \in [n]$ *and* $j, l \in [d]$, $(\mathbf{L}_i^k)_{jl}$ *is a convex combination of* $\{h^i(\nabla^2 f_i(z^t))_{jl}\}_{t=0}^k$ *in Algorithm 1* $(\{h^i(\nabla^2 f_i(z_i^t))_{jl}\}_{t=0}^k$ *in Algorithm 2 or* $\{\tilde{h}^i(\nabla^2 f_i(z_i^t))_{jl}\}_{t=0}^k$ *in Algorithm 3) for* $k \geq 0$.

Assumption 4.6. *(i)* $\mathcal{C}_i^k$ *is a contraction compressor with parameter* δ *and* $\alpha = 1$. *(ii)* $\mathcal{C}_i^k$ *is deterministic, i.e.,* $\mathbb{E}[\mathcal{C}_i^k(\mathbf{A})] = \mathcal{C}_i^k(\mathbf{A})$ *for any* $\mathbf{A} \in \mathbb{R}^{d \times d}$.

Assumption 4.7. *We have* $\|\nabla^2 f_i(x) - \nabla^2 f_i(y)\| \leq H\|x - y\|$, $\|\nabla^2 f_i(x) - \nabla^2 f_i(y)\|_F \leq H_1\|x - y\|$, $\|h^i(\nabla^2 f_i(x)) - h^i(\nabla^2 f_i(y))\|_F \leq M_1\|x - y\|$, $\max_{jl}\{|h^i(\nabla^2 f_i(x))_{jl} - h^i(\nabla^2 f_i(y))_{jl}|\} \leq M_2\|x - y\|$, $\max_{jl}\{\|\mathbf{B}_i^{jl}\|_F\} \leq R$ *for any* $x, y \in \mathbb{R}^d$ *and* $i \in [n]$. *For Algorithm 2, we assume each* f_i *is* μ*-strongly convex.*

We estimate M_1 and M_2 in Assumption 4.7 in the following lemma.

Lemma 4.8. *Assume* $\|\nabla^2 f_i(x) - \nabla^2 f_i(y)\|_F \leq H_1\|x - y\|$ *and* $\max_{jl}\{|(\nabla^2 f_i(x))_{jl} - (\nabla^2 f_i(y))_{jl}|\} \leq \nu\|x - y\|$ *for any* $x, y \in \mathbb{R}^d$, *and* $i \in [n]$. *Then we have* $M_1 \leq \max_i\{\|\mathcal{B}_i^{-1}\|\}H_1$ *and* $M_2 \leq \nu \max_i\{\|\mathcal{B}_i^{-1}\|_\infty\}$.

To present our theory in a unified manner, we define

$$N_B := \begin{cases} 1 & \text{if the bases } \{\mathbf{B}_i^{jl}\}_{j,l \in [d]} \text{ are all orthogonal} \\ d^2 & \text{otherwise} \end{cases} \tag{10}$$

$$(A_M, B_M) := \begin{cases} (\eta, \eta) & \text{if Asm. 4.3(i) holds} \\ \left(\frac{\delta_M}{4}, \frac{6}{\delta_M} - \frac{7}{2}\right) & \text{if Asm. 4.4(i) holds} \end{cases} \tag{11}$$

$$(A, B) := \begin{cases} (\alpha, \alpha) & \text{if Asm. 4.5(i) holds} \\ \left(\frac{\delta}{4}, \frac{6}{\delta} - \frac{7}{2}\right) & \text{if Asm. 4.6(i) holds} \end{cases} \tag{12}$$

and $\mathcal{H}^k := \frac{1}{n}\sum_{i=1}^n \|\mathbf{L}_i^k - \mathbf{L}_i^*\|_F^2$, $\Phi_1^k := \|z^k - x^*\|^2 + \frac{A_M(1-p)}{2p}\|w^k - x^*\|^2$, where $\mathbf{L}_i^* := h^i(\nabla^2 f_i(x^*))$, for $k \geq 0$.

Theorem 4.9 (Linear convergence of BL1)**.** *Let Assumption 4.7 hold. Let Assumption 4.3 (i) or Assumption 4.4 (i) hold. Assume* $\|z^k - x^*\|^2 \leq \frac{A_M\mu^2}{4H^2 B_M}$ *and* $\mathcal{H}^k \leq \frac{A_M\mu^2}{16N_B R^2 B_M}$ *for* $k \geq 0$. *Then we have*

$$\mathbb{E}[\Phi_1^k] \leq \left(1 - \frac{\min\{A_M, p\}}{2}\right)^k \Phi_1^0,$$

for $k \geq 0$.

Define $\Phi_2^k := \mathcal{H}^k + \frac{4BM_1^2}{A_M}\|x^k - x^*\|^2$ for $k \geq 0$.

Theorem 4.10 (Superlinear convergence of BL1)**.** *Let* $\eta = 1$, $\xi^k \equiv 1$ *and* $\mathcal{Q}^k(x) \equiv x$ *for any* $x \in \mathbb{R}^d$ *and* $k \geq 0$. *Let Assumption 4.7 hold. Let Assumption 4.5 (i) or Assumption 4.6 (i) hold. Assume* $\|z^k - x^*\|^2 \leq \frac{A_M\mu^2}{4H^2 B_M}$ *and* $\mathcal{H}^k \leq \frac{A_M\mu^2}{16N_B R^2 B_M}$ *for* $k \geq 0$. *Then we have*

$$\mathbb{E}[\Phi_2^k] \leq \theta_1^k \Phi_2^0,$$

and

$$\mathbb{E}\left[\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2}\right] \leq \theta_1^k \left(\frac{A_M H^2}{8BM_1^2\mu^2} + \frac{2N_B R^2}{\mu^2}\right)\Phi_2^0,$$

for $k \geq 0$, *where* $\theta_1 := \left(1 - \frac{\min\{4A, A_M\}}{4}\right)$.

Next, we explore under what conditions we can guarantee the boundness of $\|z^k - x^*\|$ and $\mathcal{H}^k$.

Theorem 4.11. *Let Assumption 4.7 hold. Then we have the following results.*

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(i) *Let Assumption 4.3 and Assumption 4.5 (ii) hold. If* $\|x^0 - x^*\|^2 \leq \tilde{c}_1 := \min\left\{\frac{\mu^2}{4d^2H^2}, \frac{\mu^2}{16d^4 N_\mathrm{B} R^2 M_2^2}\right\}$, *then* $\|z^k - x^*\|^2 \leq d\tilde{c}$ *and* $\mathcal{H}^k \leq \frac{\mu^2}{16dN_\mathrm{B} R^2}$ *for* $k \geq 0$.

(ii) *Let Assumption 4.4 and Assumption 4.6 hold. If* $\|z^0 - x^*\|^2 \leq \tilde{c}_2 := \min\left\{\frac{A_\mathrm{M}\mu^2}{4H^2 B_\mathrm{M}}, \frac{AA_\mathrm{M}\mu^2}{16N_\mathrm{B} R^2 B_\mathrm{M} BM_1^2}\right\}$ *and* $\mathcal{H}^0 \leq \frac{A_\mathrm{M}\mu^2}{16N_\mathrm{B} R^2 B_\mathrm{M}}$, *then* $\|z^k - x^*\|^2 \leq \tilde{c}_2$ *and* $\mathcal{H}^k \leq \frac{A_\mathrm{M}\mu^2}{16N_\mathrm{B} R^2 B_\mathrm{M}}$ *for* $k \geq 0$.

We can also unify the bidirectional compression and partial participation to have BL2 (Algorithm 2), where $[\cdot]_s$ represents an operator on $\mathbb{R}^{d\times d}$ such that $[\mathbf{A}]_s = (\mathbf{A} + \mathbf{A}^\top)/2$ for any $\mathbf{A} \in \mathbb{R}^{d\times d}$. Since each node has a local model w_i^k, we introduce z_i^k to apply the bidirectional compression, and $\mathbf{L}_i^k$ is expected to learn $h^i(\nabla^2 f_i(z_i^k))$ iteratively. Like in FedNL-PP [Safaryan et al., 2021], the key relation

$$g_i^k = ([\mathbf{H}_i^k]_s + l_i^k \mathbf{I})w_i^k - \nabla f_i(w_i^k) \tag{13}$$

need to be maintained in the design of BL2. The reason to keep relation (13) is that the update of x^k follows the structure of Stochastic Newton [Kovalev et al., 2019], where g_i^k is supposed to be $\nabla^2 f_i(w_i^k) - \nabla f_i(w_i^k)$, and naturally $\nabla^2 f_i(w_i^k)$ is replaced by the Hessian estimator $[\mathbf{H}_i^k]_s + l_i^k\mathbf{I}$. Here we use $l_i^k = \|[\mathbf{H}_i^k]_s - \nabla^2 f_i(z_i^k)\|_\mathrm{F}$ to guarantee the positive definiteness of $[\mathbf{H}_i^k]_s + l_i^k\mathbf{I}$ like in [Safaryan et al., 2021]. From (13), it is easy to see that on server, for the $\xi_i^k = 0$ case, $g_i^{k+1} - g_i^k = ([\mathbf{H}_i^{k+1}]_s - [\mathbf{H}_i^k]_s + l_i^{k+1}\mathbf{I} - l_i^k\mathbf{I})w_i^{k+1}$ since $w_i^{k+1} = w_i^k$. We give the convergence results of BL2 in the following two theorems.

Let $\Phi_3^k := \mathcal{W}^k + \frac{2p}{A_\mathrm{M}}\left(1 - \frac{\tau A_\mathrm{M}}{n}\right)\mathcal{Z}^k$, where $\mathcal{Z}^k := \frac{1}{n}\sum_{i=1}^n \|z_i^k - x^*\|^2$, for $k \geq 0$.

Theorem 4.12 (Linear convergence of BL2). *Let Assumption 4.7 hold. Let Assumption 4.3 (i) or Assumption 4.4 (i) hold. Assume* $\|z_i^k - x^*\|^2 \leq \frac{A_\mathrm{M}\mu^2}{(6H^2 + 24H_1^2)B_\mathrm{M}}$ *and* $\mathcal{H}^k \leq \frac{A_\mathrm{M}\mu^2}{96N_\mathrm{B} R^2 B_\mathrm{M}}$ *for all* $i \in [n]$ *and* $k \geq 0$. *Then we have*

$$\mathbb{E}[\Phi_3^k] \leq \left(1 - \frac{\tau\min\{p, A_\mathrm{M}\}}{2n}\right)^k \Phi_3^0,$$

for $k \geq 0$.

Define $\Phi_4^k := \mathcal{H}^k + \frac{4BM_1^2}{A_\mathrm{M}}\|x^k - x^*\|^2$ *for* $k \geq 0$.

Theorem 4.13 (Superlinear convergence of BL2). *Let* $\eta = 1$, $\xi^k \equiv 1$, $S^k \equiv [n]$, *and* $\mathcal{Q}_i^k(x) \equiv x$ *for any* $x \in \mathbb{R}^d$ *and* $k \geq 0$. *Let Assumption 4.7 hold. Let Assumption 4.5 (i) or Assumption 4.6 (i) hold. Assume* $\|z_i^k - x^*\|^2 \leq \frac{A_\mathrm{M}\mu^2}{(6H^2 + 24H_1^2)B_\mathrm{M}}$ *and* $\mathcal{H}^k \leq \frac{A_\mathrm{M}\mu^2}{96N_\mathrm{B} R^2 B_\mathrm{M}}$ *for all* $i \in [n]$ *and* $k \geq 0$. *Then we have*

$$\mathbb{E}[\Phi_4^k] \leq \theta_2^k \Phi_4^0,$$

and

$$\mathbb{E}\left[\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2}\right] \leq \theta_2^k \left(\frac{A_\mathrm{M}(3H^2 + 12H_1^2)}{16BM_1^2\mu^2} + \frac{12N_\mathrm{B} R^2}{\mu^2}\right)\Phi_4^0,$$

for $k \geq 0$, *where* $\theta_2 := 1 - \frac{\min\{2A, A_\mathrm{M}\}}{2}$.

Now, we explore under what conditions we can guarantee the boundedness of $\|z_i^k - x^*\|$ and $\mathcal{H}^k$.

Theorem 4.14. *Let Assumption 4.7 hold. Then we have the following results.*

Algorithm 2 BL2 (Basis Learn with Bidirectional Compression and Partial Participation)

1: **Parameters:** $\alpha > 0$; $\eta > 0$; matrix compression operators $\{\mathcal{C}_1^k, \ldots, \mathcal{C}_n^k\}$; $p \in (0,1]$; $0 < \tau \leq n$

2: **Initialization:** $w_i^0 = z_i^0 = x^0 \in \mathbb{R}^d$; $\mathbf{L}_i^0 \in \mathbb{R}^{d \times d}$; $\mathbf{H}_i^0 = \sum_{jl} (\mathbf{L}_i^0)_{jl} \mathbf{B}_i^{jl}$; $l_i^0 = \|[\mathbf{H}_i^0]_s - \nabla^2 f_i(w_i^0)\|_{\mathrm{F}}$; $g_i^0 = ([\mathbf{H}_i^0]_s + l_i^0 \mathbf{I})w_i^0 - \nabla f_i(w_i^0)$; Moreover: $\mathbf{H}^0 = \frac{1}{n}\sum_{i=1}^n \mathbf{H}_i^0$; $l^0 = \frac{1}{n}\sum_{i=1}^n l_i^0$; $g^0 = \frac{1}{n}\sum_{i=1}^n g_i^0$

3: **on** server

4: $\quad x^{k+1} = \left([\mathbf{H}^k]_s + l^k\mathbf{I}\right)^{-1} g^k$, choose a subset $S^k \subseteq [n]$ such that $\mathbb{P}[i \in S^k] = \tau/n$ for all $i \in [n]$

5: $\quad v_i^k = \mathcal{Q}_i^k(x^{k+1} - z_i^k)$, $z_i^{k+1} = z_i^k + \eta v_i^k$ for $i \in S^k$

6: $\quad z_i^{k+1} = z_i^k, \quad w_i^{k+1} = w_i^k$ for $i \notin S^k$

7: $\quad$ Send v_i^k to the selected devices $i \in S^k$

8: **for** each device $i = 1, \ldots, n$ in parallel **do**

9: $\quad$ **for participating devices** $i \in S^k$ **do**

10: $\quad\quad z_i^{k+1} = z_i^k + \eta v_i^k$, $\mathbf{S}_i^k := \mathcal{C}_i^k(h^i(\nabla^2 f_i(z_i^{k+1})) - \mathbf{L}_i^k)$

11: $\quad\quad \mathbf{L}_i^{k+1} = \mathbf{L}_i^k + \alpha \mathbf{S}_i^k$, $\mathbf{H}_i^{k+1} = \mathbf{H}_i^k + \alpha\sum_{jl}(\mathbf{S}_i^k)_{jl}\mathbf{B}_i^{jl}$

12: $\quad\quad l_i^{k+1} = \|[\mathbf{H}_i^{k+1}]_s - \nabla^2 f_i(z_i^{k+1})\|_{\mathrm{F}}$

13: $\quad\quad$ Sample $\xi_i^{k+1} \sim \mathrm{Bernoulli}(p)$

14: $\quad\quad$ **if** $\xi_i^k = 1$

15: $\quad\quad\quad w_i^{k+1} = z_i^{k+1}$, $g_i^{k+1} = ([\mathbf{H}_i^{k+1}]_s + l_i^{k+1}\mathbf{I})w_i^{k+1} - \nabla f_i(w_i^{k+1})$, send $g_i^{k+1} - g_i^k$ to server

16: $\quad\quad$ **if** $\xi_i^k = 0$

17: $\quad\quad\quad w_i^{k+1} = w_i^k$, $g_i^{k+1} = ([\mathbf{H}_i^{k+1}]_s + l_i^{k+1}\mathbf{I})w_i^{k+1} - \nabla f_i(w_i^{k+1})$

18: $\quad\quad$ Send $\mathbf{S}_i^k$, $l_i^{k+1} - l_i^k$, and ξ_i^k to server

19: $\quad$ **for non-participating devices** $i \notin S^k$ **do**

20: $\quad\quad z_i^{k+1} = z_i^k$, $w_i^{k+1} = w_i^k$, $\mathbf{L}_i^{k+1} = \mathbf{L}_i^k$, $\mathbf{H}_i^{k+1} = \mathbf{H}_i^k$, $l_i^{k+1} = l_i^k$, $g_i^{k+1} = g_i^k$

21: **end for**

22: **on** server

23: $\quad$ **if** $\xi_i^k = 1$

24: $\quad\quad w_i^{k+1} = z_i^{k+1}$, receive $g_i^{k+1} - g_i^k$

25: $\quad$ **if** $\xi_i^k = 0$

26: $\quad\quad w_i^{k+1} = w_i^k$, $g_i^{k+1} - g_i^k = \alpha\left[\sum_{jl}(\mathbf{S}_i^k)_{jl}\mathbf{B}_i^{jl}\right]_s w_i^{k+1} + (l_i^{k+1} - l_i^k)w_i^{k+1}$

27: $\quad g^{k+1} = g^k + \frac{1}{n}\sum_{i \in S^k}\left(g_i^{k+1} - g_i^k\right)$

28: $\quad \mathbf{H}^{k+1} = \mathbf{H}^k + \frac{\alpha}{n}\sum_{i \in S^k}\sum_{jl}(\mathbf{S}_i^k)_{jl}\mathbf{B}_i^{jl}$

29: $\quad l^{k+1} = l^k + \frac{1}{n}\sum_{i \in S^k}\left(l_i^{k+1} - l_i^k\right)$

(i) Let Assumption 4.3 and Assumption 4.5 (ii) hold. If

$$\|x^0 - x^*\|^2 \leq \tilde{c}_3 := \min\left\{\frac{\mu^2}{d^2(6H^2 + 24H_1^2)}, \frac{\mu^2}{96d^4 N_{\mathrm{B}} R^2 M_2^2}\right\},$$

then $\|z_i^k - x^*\|^2 \leq d\tilde{c}_3$ and $\mathcal{H}^k \leq \frac{\mu^2}{96d N_{\mathrm{B}} R^2}$ for $i \in [n]$ and $k \geq 0$.

(ii) Let Assumption 4.4 and Assumption 4.6 hold. If $\|z_i^0 - x^*\|^2 \leq \tilde{c}_4$, where

$$\tilde{c}_4 := \min\left\{\frac{A_{\mathrm{M}}\mu^2}{B_{\mathrm{M}}(6H^2 + 24H_1^2)}, \frac{AA_{\mathrm{M}}\mu^2}{96 N_{\mathrm{B}} R^2 B_{\mathrm{M}} B M_1^2}\right\},$$

and $\|\mathbf{L}_i^0 - \mathbf{L}_i^*\|_F^2 \leq \frac{A_M \mu^2}{96 N_B R^2 B_M}$ for all $i \in [n]$, then $\|z_i^k - x^*\|^2 \leq \tilde{c}_4$ and $\|\mathbf{L}_i^k - \mathbf{L}_i^*\|_F^2 \leq \frac{A_M \mu^2}{96 N_B R^2 B_M}$ for all $i \in [n]$ and $k \geq 0$.

5 Basis Learning in $\mathcal{S}^d$

Let $\{\mathbf{B}_i^{jl} \mid j, l \in [d], j \geq l\}$ be a basis in the symmetric subspace $\mathcal{S}^d$ of $\mathbb{R}^{d \times d}$ that consists of all the symmetric matrices for $i \in [n]$. In this case, the number of symmetric matrices in the basis is $\tilde{N} := \frac{d(d+1)}{2}$. Then for any symmetric matrix $\mathbf{A}$ in $\mathbb{R}^{d \times d}$, it can be uniquely represented as

$$\mathbf{A} = \sum_{j \geq l} \tilde{h}_{jl}^i(\mathbf{A}) \mathbf{B}^{jl},$$

where $\tilde{h}_{jl}^i(\mathbf{A}) \in \mathbb{R}$ is the coefficient corresponding to $\mathbf{B}_i^{jl}$. Let $\mathbf{B}_i^{lj} := \mathbf{B}_i^{jl}$ for $j > l$ and define $\tilde{h}^i(\mathbf{A})$ as a symmetric matrix in $\mathbb{R}^{d \times d}$ such that $\tilde{h}^i(\mathbf{A})_{jl} := \frac{1}{2}\tilde{h}_{jl}^i$ for $j > l$ and $\tilde{h}^i(\mathbf{A})_{jl} := \tilde{h}_{jl}^i$ for $j = l$. Let $svec : \mathcal{S}^d \to \mathbb{R}^{\tilde{N}}$ be defined as

$$svec(\mathbf{A}) := (\mathbf{A}_{11}, 2\mathbf{A}_{21}..., 2\mathbf{A}_{d1}, ..., \mathbf{A}_{jj}, ..., 2\mathbf{A}_{dj}, ..., \mathbf{A}_{dd})^\top,$$

and $\tilde{\mathcal{M}}_i := (\mathbf{B}_i^{11}, ..., \mathbf{B}_i^{d1}, ..., \mathbf{B}_i^{jj}, ..., \mathbf{B}_i^{dj}, ..., \mathbf{B}_i^{dd})$. Then we have $\mathbf{A} = \tilde{\mathcal{M}}_i svec(\tilde{h}^i(\mathbf{A}))$, which is equivalent to

$$svec(\mathbf{A}) = \tilde{\mathcal{B}}_i \cdot svec(\tilde{h}^i(\mathbf{A})), \tag{14}$$

for any symmetric matrix $\mathbf{A}$, where

$$\tilde{\mathcal{B}}_i := (svec(\mathbf{B}_i^{11}), ..., svec(\mathbf{B}_i^{d1}), ..., svec(\mathbf{B}_i^{jj}), ..., svec(\mathbf{B}_i^{dj}), ..., svec(\mathbf{B}_i^{dd})) \in \mathbb{R}^{\tilde{N} \times \tilde{N}}.$$

Since the representation (14) is unique, we know $\tilde{\mathcal{B}}_i$ is invertible, and thus

$$svec(\tilde{h}^i(\mathbf{A})) = (\tilde{\mathcal{B}}_i)^{-1} svec(\mathbf{A}). \tag{15}$$

Example 5.1. *We choose $\mathbf{B}_i^{jl} \in \mathcal{S}^d$ such that for $j \neq l$, $(\mathbf{B}_i^{jl})_{jl} = (\mathbf{B}_i^{jl})_{lj} = (\mathbf{B}_i^{jl})_{jj} = (\mathbf{B}_i^{jl})_{ll} = 1$ and the other entries are 0; for $j = l$, $(\mathbf{B}_i^{jj})_{jj} = 1$ and the other entries are 0. It is easy to verify it is a basis in $\mathcal{S}^d$, and we also have $\mathbf{B}_i^{jl} \succeq 0$.*

We choose a basis $\{\mathbf{B}_i^{jl}\}$ in $\mathcal{S}^d$ such that $\mathbf{B}_i^{jl} \succeq 0$ for BL3 (Algorithm 3). The way to guarantee the positive definiteness of the Hessian estimator is similar to the approach of Islamov et al. [2021]. From the definition of γ_i^k, we know $(\mathbf{L}_i^k)_{jl} + 2\gamma_i^k \geq c > 0$. Noticing that $\nabla^2 f_i(z_i^k)$ can be expressed in the form

$$\sum_{jl} \left(\frac{\tilde{h}^i(\nabla^2 f_i(z_i^k))_{jl} + 2\gamma_i^k}{(\mathbf{L}_i^k)_{jl} + 2\gamma_i^k} \cdot ((\mathbf{L}_i^k)_{jl} + 2\gamma_i^k) - 2\gamma_i^k \right) \mathbf{B}_i^{jl},$$

for β_i^k in Option 2, we have the inequality

$$\sum_{jl} \left(\beta^k ((\mathbf{L}_i^k)_{jl} + 2\gamma_i^k) - 2\gamma_i^k \right) \mathbf{B}_i^{jl} - \nabla^2 f_i(z_i^k) = \sum_{jl} \left(\beta^k - \frac{\tilde{h}^i(\nabla^2 f_i(z_i^k))_{jl} + 2\gamma_i^k}{(\mathbf{L}_i^k)_{jl} + 2\gamma_i^k} \right) \cdot ((\mathbf{L}_i^k)_{jl} + 2\gamma_i^k) \mathbf{B}_i^{jl} \succeq \mathbf{0}.$$

Algorithm 3 BL3

Parameters: learning rate $\alpha > 0$, positive constant $c > 0$, minibatch size $\tau \in \{1, 2, \ldots, n\}$

Initialization: $\mathbf{B}_i^{jl} \succeq 0$; $w_i^0 = z_i^0 = x^0$ for $i \in [n]$; $\mathbf{L}_i^0 \in \mathbb{R}^{d \times d}$; $\gamma_i^0 = \max_{jl}\{c, |(\mathbf{L}_i^0)_{jl}|\}$; $\mathbf{A}_i^0 = \sum_{jl}((\mathbf{L}_i^0)_{jl} + 2\gamma_i^0)\mathbf{B}_i^{jl}$; $\mathbf{C}_i^0 = \sum_{jl} 2\gamma_i^0 \mathbf{B}_i^{jl}$; $\mathbf{A}^0 = \frac{1}{n}\sum_{i=1}^n \mathbf{A}_i^0$; $\mathbf{C}^0 = \frac{1}{n}\sum_{i=1}^n \mathbf{C}_i^0$; $\beta_i^0 = \max_{jl} \frac{\tilde{h}^i(\nabla^2 f_i(w_i^0))_{jl} + 2\gamma_i^0}{(\mathbf{L}_i^0)_{jl} + 2\gamma_i^0}$; $\beta^0 = \max_i\{\beta_i^0\}$; $g_{i,1}^0 = \mathbf{A}_i^0 w_i^0$; $g_{i,2}^0 = \mathbf{C}_i^0 w_i^0 + \nabla f_i(w_i^0)$; $g_1^0 = \frac{1}{n}\sum_{i=1}^n g_{i,1}^0$; $g_2^0 = \frac{1}{n}\sum_{i=1}^n g_{i,2}^0$; $\mathbf{H}^0 = \beta^0 \mathbf{A}^0 - \mathbf{C}^0$; $g^0 = \beta^0 g_1^0 - g_2^0$

on server
    $x^{k+1} = (\mathbf{H}^k)^{-1} g^k$ Main step: Update the global model
    Choose a subset $S^k \subseteq \{1, \ldots, n\}$ such that $\mathbb{P}[i \in S^k] = \tau/n$ for all $i \in [n]$
    $v_i^k = \mathcal{Q}_i^k(x^{k+1} - z_i^k)$, $z_i^{k+1} = z_i^k + \eta v_i^k$ for $i \in S^k$
    $z_i^{k+1} = z_i^k$, $w_i^{k+1} = w_i^k$ for $i \notin S^k$
    Send v_i^k to the selected devices $i \in S^k$ Communicate to selected clients

for each node $i = 1, \ldots, n$ **do**
  for participating devices $i \in S^k$ **do**
  $z_i^{k+1} = z_i^k + \eta v_i^k$, $\mathbf{L}_i^{k+1} = \mathbf{L}_i^k + \alpha \mathcal{C}_i^k\left(\tilde{h}^i(\nabla^2 f_i(z_i^{k+1})) - \mathbf{L}_i^k\right)$, $\gamma_i^{k+1} = \max_{jl}\{c, |(\mathbf{L}_i^{k+1})_{jl}|\}$
  Option 1: $\beta_i^{k+1} = \max_{jl} \frac{\tilde{h}^i(\nabla^2 f_i(z_i^k))_{jl} + 2\gamma_i^{k+1}}{(\mathbf{L}_i^{k+1})_{jl} + 2\gamma_i^{k+1}}$ *Option 2:* $\beta_i^{k+1} = \max_{jl} \frac{\tilde{h}^i(\nabla^2 f_i(z_i^{k+1}))_{jl} + 2\gamma_i^{k+1}}{(\mathbf{L}_i^{k+1})_{jl} + 2\gamma_i^{k+1}}$
  $\mathbf{A}_i^{k+1} = \mathbf{A}_i^k + \sum_{jl}\left((\mathbf{L}_i^{k+1})_{jl} - (\mathbf{L}_i^k)_{jl} + 2\gamma_i^{k+1} - 2\gamma_i^k\right)\mathbf{B}_i^{jl}$
  $\mathbf{C}_i^{k+1} = \mathbf{C}_i^k + \sum_{jl}\left(2\gamma_i^{k+1} - 2\gamma_i^k\right)\mathbf{B}_i^{jl}$,   Sample $\xi_i^{k+1} \sim \text{Bernoulli}(p)$
  if $\xi_i^k = 1$
    $w_i^{k+1} = z_i^{k+1}$, $g_{i,1}^{k+1} = \mathbf{A}_i^{k+1} w_i^{k+1}$, $g_{i,2}^{k+1} = \mathbf{C}_i^{k+1} w_i^{k+1} + \nabla f_i(w_i^{k+1})$
    Send $g_{i,1}^{k+1} - g_{i,1}^k$, $g_{i,2}^{k+1} - g_{i,2}^k$ to server
  if $\xi_i^k = 0$
    $w_i^{k+1} = w_i^k$, $g_{i,1}^{k+1} = \mathbf{A}_i^{k+1} w_i^{k+1}$, $g_{i,2}^{k+1} = \mathbf{C}_i^{k+1} w_i^{k+1} + \nabla f_i(w_i^{k+1})$
  Send $\mathbf{L}_i^{k+1} - \mathbf{L}_i^k$, β_i^{k+1}, ξ_i^k, $\gamma_i^{k+1} - \gamma_i^k$ to server
  for non-participating devices $i \notin S^k$ **do**
  $z_i^{k+1} = z_i^k$, $w_i^{k+1} = w_i^k$, $\mathbf{L}_i^{k+1} = \mathbf{L}_i^k$, $\gamma_i^{k+1} = \gamma_i^k$, $\beta_i^{k+1} = \beta_i^k$, $\mathbf{A}_i^{k+1} = \mathbf{A}_i^k$, $\mathbf{C}_i^{k+1} = \mathbf{C}_i^k$, $g_{i,1}^{k+1} = g_{i,1}^k$, $g_{i,2}^{k+1} = g_{i,2}^k$
end for
on server
  if $\xi_i^k = 1$
    $w_i^{k+1} = z_i^{k+1}$, Receive $g_{i,1}^{k+1} - g_{i,1}^k$, $g_{i,2}^{k+1} - g_{i,2}^k$,
  if $\xi_i^k = 0$
    $w_i^{k+1} = w_i^k$, $g_{i,1}^{k+1} - g_{i,1}^k = \sum_{jl}(\mathbf{L}_i^{k+1} - \mathbf{L}_i^k)_{jl}\mathbf{B}_i^{jl} w_i^{k+1} + 2(\gamma_i^{k+1} - \gamma_i^k)w_i^{k+1}$
    $g_{i,2}^{k+1} - g_{i,2}^k = \sum_{jl} 2(\gamma_i^{k+1} - 2\gamma_i^k)\mathbf{B}_i^{jl} w_i^{k+1}$
  $g_1^{k+1} = g_1^k + \frac{1}{n}\sum_{i \in S^k}\left(g_{i,1}^{k+1} - g_{i,1}^k\right)$, $g_2^{k+1} = g_2^k + \frac{1}{n}\sum_{i \in S^k}\left(g_{i,2}^{k+1} - g_{i,2}^k\right)$, $\beta^{k+1} = \max_i\{\beta_i^{k+1}\}$
  $g^{k+1} = \beta^{k+1} g_1^{k+1} - g_2^{k+1}$, $\mathbf{A}^{k+1} = \mathbf{A}^k + \frac{1}{n}\sum_{i \in S^k}\sum_{jl}\left((\mathbf{L}_i^{k+1})_{jl} - (\mathbf{L}_i^k)_{jl} + 2\gamma_i^{k+1} - 2\gamma_i^k\right)\mathbf{B}_i^{jl}$
  $\mathbf{C}^{k+1} = \mathbf{C}^k + \frac{1}{n}\sum_{i \in S^k}\sum_{jl}\left(2\gamma_i^{k+1} - 2\gamma_i^k\right)\mathbf{B}_i^{jl}$, $\mathbf{H}^{k+1} = \beta^{k+1}\mathbf{A}^{k+1} - \mathbf{C}^{k+1}$

Thus, if we can maintain the Hessian estimator in the form $\mathbf{H}_i^k := \sum_{jl} \left(\beta^k((\mathbf{L}_i^k)_{jl} + 2\gamma_i^k) - 2\gamma_i^k\right)\mathbf{B}_i^{jl}$, then $\mathbf{H}_i^k \succeq \nabla^2 f_i(z_i^k)$ (we can get $\mathbf{H}_i^k \succeq \nabla^2 f_i(z_i^{k-1})$ for Option 1 similarly). To achieve this goal, we use two auxiliary matrices $\mathbf{A}_i^k$, $\mathbf{C}_i^k$, and maintain $\mathbf{A}_i^k = \sum_{jl}((\mathbf{L}_i^k)_{jl} + 2\gamma_i^k)\mathbf{B}_i^{jl}$, $\mathbf{C}_i^k = \sum_{jl} 2\gamma_i^k \mathbf{B}_i^{jl}$, and $\mathbf{H}_i^k = \beta^k \mathbf{A}_i^k - \mathbf{C}_i^k$. BL3 follows the same structure of BL2, thus we also need to keep the relation $g_i^k = \mathbf{H}_i^k w_i^k - \nabla f_i(w_i^k)$, which is actually $g_i^k = \beta^k \mathbf{A}_i^k w_i^k - \mathbf{C}_i^k w_i^k - \nabla f_i(w_i^k)$. Since for non-participating devices, β^k usually changes at each step, we split g_i^k to two parts by using two auxiliary vectors $g_{i,1}^k$, $g_{i,2}^k$, and keeping $g_{i,1}^k = \mathbf{A}_i^k w_i^k$, $g_{i,2}^k = \mathbf{C}_i^k w_i^k - \nabla f_i(w_i^k)$, and $g_i^k = \beta^k g_{i,1}^k - g_{i,2}^k$. The rest of BL3 is the same as BL2. We need the following assumption for BL3.

Assumption 5.2. *Assume* $\|\nabla^2 f_i(x) - \nabla^2 f_i(y)\| \le H\|x - y\|$ *for any* $x, y \in \mathbb{R}^d$ *and* $i \in [n]$. *Assume* $\max_{jl}\{|(\mathbf{L}_i^k)_{jl}|\} \le M_3$ *for all* $i \in [n]$ *and* $k \ge 0$. *Assume* $\|\tilde{h}^i(\nabla^2 f_i(x)) - \tilde{h}^i(\nabla^2 f_i(y))\|_F \le M_4\|x - y\|$, $\max_{jl}\{|\tilde{h}^i(\nabla^2 f_i(x))_{jl} - \tilde{h}^i(\nabla^2 f_i(y))_{jl}|\} \le M_5\|x - y\|$ *for any* $x, y \in \mathbb{R}^d$ *and* $i \in [n]$, *and* $\max_{jl}\{\|\mathbf{B}_i^{jl}\|_F\} \le R$ *for* $i \in [n]$. *Assume each* f_i *is* μ-*strongly convex.*

We estimate M_3, M_4, and M_5 in Assumption 5.2 in the following lemma.

Lemma 5.3. *We have the following bounds on* M_3, M_4, *and* M_5:

(i) *Assume* $\|\nabla^2 f_i(x) - \nabla^2 f_i(y)\|_F \le H_1\|x - y\|$ *and* $\max_{jl}\{|(\nabla^2 f_i(x))_{jl} - (\nabla^2 f_i(y))_{jl}|\} \le \nu\|x - y\|$ *for any* $x, y \in \mathbb{R}^d$, *and* $i \in [n]$. *Then we have* $M_4 \le \sqrt{2}\max_i\{\|(\tilde{\mathcal{B}}_i)^{-1}\|\}H_1$ *and* $M_5 \le 2\nu\max_i\{\|(\tilde{\mathcal{B}}_i)^{-1}\|_\infty\}$.

(ii) *Assume* $\max_{jl}\{|(\nabla^2 f_i(x))_{jl}|\} \le \gamma$ *for any* $x \in \mathbb{R}^d$ *and* $i \in [n]$. *If Assumption 4.5 (ii) holds, then* $M_3 \le 2\gamma\max_i\{\|(\tilde{\mathcal{B}}_i)^{-1}\|_\infty\}$.

(iii) *Assume* $\|\nabla^2 f_i(x)\|_F \le \tilde{\gamma}$ *for any* $x \in \mathbb{R}^d$ *and* $i \in [n]$. *If Assumption 4.6 holds and* $\|\mathbf{L}_i^0\|_F \le \frac{\sqrt{2B}}{\sqrt{A}}\|(\tilde{\mathcal{B}}_i)^{-1}\|\tilde{\gamma}$ *for all* $i \in [n]$, *then we have* $\|\mathbf{L}_i^k\|_F \le \frac{\sqrt{2B}}{\sqrt{A}}\|(\tilde{\mathcal{B}}_i)^{-1}\|\tilde{\gamma}$ *for* $k \ge 0$ *and* $i \in [n]$, *and* $M_3 \le \frac{\sqrt{2B}\tilde{\gamma}}{\sqrt{A}}\max_i\{\|(\tilde{\mathcal{B}}_i)^{-1}\|\}$.

Let $\Phi_5^k := \mathcal{W}^k + \frac{2p}{A_M}\left(1 - \frac{\tau A_M}{n}\right)\mathcal{Z}^k$, where $\mathcal{Z}^k := \frac{1}{n}\sum_{i=1}^n \|z_i^k - x^*\|^2$, for $k \ge 0$.

Theorem 5.4 (Linear convergence of BL3). *Let Assumption 5.2 hold. Let Assumption 4.3 (i) or Assumption 4.4 (i) hold. Assume* $\|z_i^k - x^*\|^2 \le \frac{A_M\mu^2}{4(H^2 + 4c_1)B_M}$ *and* $\mathcal{H}^k \le \frac{A_M\mu^2}{16c_2 B_M}$ *for all* $i \in [n]$ *and* $k \ge 0$, *where* $c_1 := \frac{4N^2 R^2 M_5^2(M_3 + 2\max\{c, M_3\})^2}{c^2}$ *and* $c_2 := 2NR^2\left(1 + \frac{2N(M_3 + 2\max\{c, M_3\})^2}{c^2}\right)$. *Then we have*

$$\mathbb{E}[\Phi_5^k] \le \left(1 - \frac{\tau\min\{p, A_M\}}{2n}\right)^k \Phi_5^0,$$

for $k \ge 0$.

Define $\Phi_6^k := \mathcal{H}^k + \frac{4BM_4^2}{A_M}\|x^k - x^*\|^2$ for $k \ge 0$.

Theorem 5.5 (Superlinear convergence of BL3). *Let* $\eta = 1$, $\xi^k \equiv 1$, $S^k \equiv [n]$, *and* $\mathcal{Q}_i^k(x) \equiv x$ *for any* $x \in \mathbb{R}^d$ *and* $k \ge 0$. *Let Assumption 5.2 hold. Let Assumption 4.5 (i) or Assumption 4.6 (i) hold. Assume* $\|z_i^k - x^*\|^2 \le \frac{A_M\mu^2}{4(H^2 + 4c_1)B_M}$ *and* $\mathcal{H}^k \le \frac{A_M\mu^2}{16c_2 B_M}$ *for all* $i \in [n]$ *and* $k \ge 0$. *Then we have*

$$\mathbb{E}[\Phi_6^k] \le \theta_3^k \Phi_6^0,$$

for $k \geq 0$, where $\theta_3 := \left(1 - \frac{\min\{2A, A_\mathrm{M}\}}{2}\right)$. Moreover, for Option 1, we have

$$\mathbb{E}\left[\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2}\right] \leq \theta_3^k \left(\frac{A_\mathrm{M}(H^2\theta_3 + 4c_1)}{8BM_4^2\mu^2\theta_3} + \frac{2c_2}{\mu^2}\right)\Phi_6^0,$$

and for Option 2, we have

$$\mathbb{E}\left[\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2}\right] \leq \theta_3^k \left(\frac{A_\mathrm{M}(H^2 + 4c_1)}{8BM_4^2\mu^2} + \frac{2c_2}{\mu^2}\right)\Phi_6^0,$$

for $k \geq 0$.

Next, we explore under what conditions we can guarantee the boundedness of $\|z_i^k - x^*\|^2$ and $\mathcal{H}^k$.

Theorem 5.6. *Let Assumption 5.2 hold. Then we have the following results.*

(i) *Let Assumption 4.3 and Assumption 4.5 (ii) hold. If*

$$\|x^0 - x^*\|^2 \leq \min\left\{\frac{\mu^2}{4d^2(H^2 + 4c_1)}, \frac{\mu^2}{16d^4c_2M_5^2}\right\},$$

then $\|z_i^k - x^*\|^2 \leq \min\left\{\frac{\mu^2}{4d(H^2+4c_1)}, \frac{\mu^2}{16d^3c_2M_5^2}\right\}$ and $\mathcal{H}^k \leq \frac{\mu^2}{16dc_2}$ for $i \in [n]$ and $k \geq 0$.

(ii) *Let Assumption 4.4 and Assumption 4.6 hold. If $\|z_i^0 - x^*\|^2 \leq \min\left\{\frac{A_\mathrm{M}\mu^2}{4B_\mathrm{M}(H^2+4c_1)}, \frac{AA_\mathrm{M}\mu^2}{16c_2B_\mathrm{M}BM_4^2}\right\}$ and $\|\mathbf{L}_i^0 - \mathbf{L}_i^*\|_\mathrm{F}^2 \leq \frac{A_\mathrm{M}\mu^2}{16c_2B_\mathrm{M}}$ for all $i \in [n]$, then $\|z_i^k - x^*\|^2 \leq \min\left\{\frac{A_\mathrm{M}\mu^2}{4B_\mathrm{M}(H^2+4c_1)}, \frac{AA_\mathrm{M}\mu^2}{16c_2B_\mathrm{M}BM_4^2}\right\}$ and $\|\mathbf{L}_i^k - \mathbf{L}_i^*\|_\mathrm{F}^2 \leq \frac{A_\mathrm{M}\mu^2}{16c_2B_\mathrm{M}}$ for all $i \in [n]$ and $k \geq 0$.*

6   Experiments

We conduct numerical experiments to compare the performance of BL methods with various efficient methods in federated learning. We consider regularized logistic regression problem

$$\min_{x \in \mathbb{R}^d}\left\{f(x) := \frac{1}{n}\sum_{i=1}^{n} f_i(x) + \frac{\lambda}{2}\|x\|^2\right\}, \tag{16}$$

where

$$f_i(x) := \frac{1}{m}\sum_{j=1}^{m}\log\left(1 + \exp\left(-b_{ij}a_{ij}^\top x\right)\right).$$

Here $\{a_{ij}, b_{ij}\}_{j \in [m]}$ are data points stored on i-th device. In our experiments, we used the following datasets from LibSVM [Chang and Lin, 2011]: a1a, a9a, phishing, w2a, w8a, covtype, madelon. We use two values of the regularization parameter: $\lambda \in \{10^{-3}, 10^{-4}\}$. In the figures we plot the relation of the optimality gap $f(x^k) - f(x^*)$ and the number of communicated bits per node. The optimal value $f(x^*)$ is chosen as the function value at the 20-th iterate of standard Newton's method.

6.1 Basis computation for BL

One of the most popular types of data preprocessing in classical machine learning is dimension reduction. One of such techniques is based on SVD of the feature matrix. We want to point out that SVD could also be used to find a basis for each client. In our experiments, we use linalg.orth function from SciPy module [Jones et al., 2001–]. In other words, such data preprocessing could be used not only for the stability of certain machine learning models but also for improving the optimization process.

6.2 Comparison with second-order methods

We compare the performance of BL1 with DINGO [Crane and Roosta, 2019], FedNL [Safaryan et al., 2021], NL1 [Islamov et al., 2021], N0 [Safaryan et al., 2021] in terms of communication complexity. For FedNL, NL1, and BL1 we use $\nabla^2 f_i(x^0)$ as the initialization of $\mathbf{H}_i^0$. Besides, the stepsize $\alpha = 1$, Rank-1 compression for matrices, and option 1 (projection) were used for FedNL. For NL1 compression mechanism is Rand-1 with stepsize $\alpha = 1/(\omega+1)$. Backtracking linesearch for DINGO selects the largest stepsize from $\{1, 2^{-1}, 2^{-2}, \ldots, 2^{-10}\}$. We set the authors' choice for other parameters of the method: $\theta = 10^{-4}, \phi = 10^{-6}, \rho = 10^{-4}$. Compression operator $\mathcal{C}_i^k$ in BL1 is Top-K, where $K = r$ (r is the dimension of the local data). We set $p = 1$ and use identity compression for $\mathcal{Q}^k$ with stepsize $\eta = 1$ for models (backside compression is not used). According the results in Figure 1 (1st row), BL1 is the most efficient method in all cases.
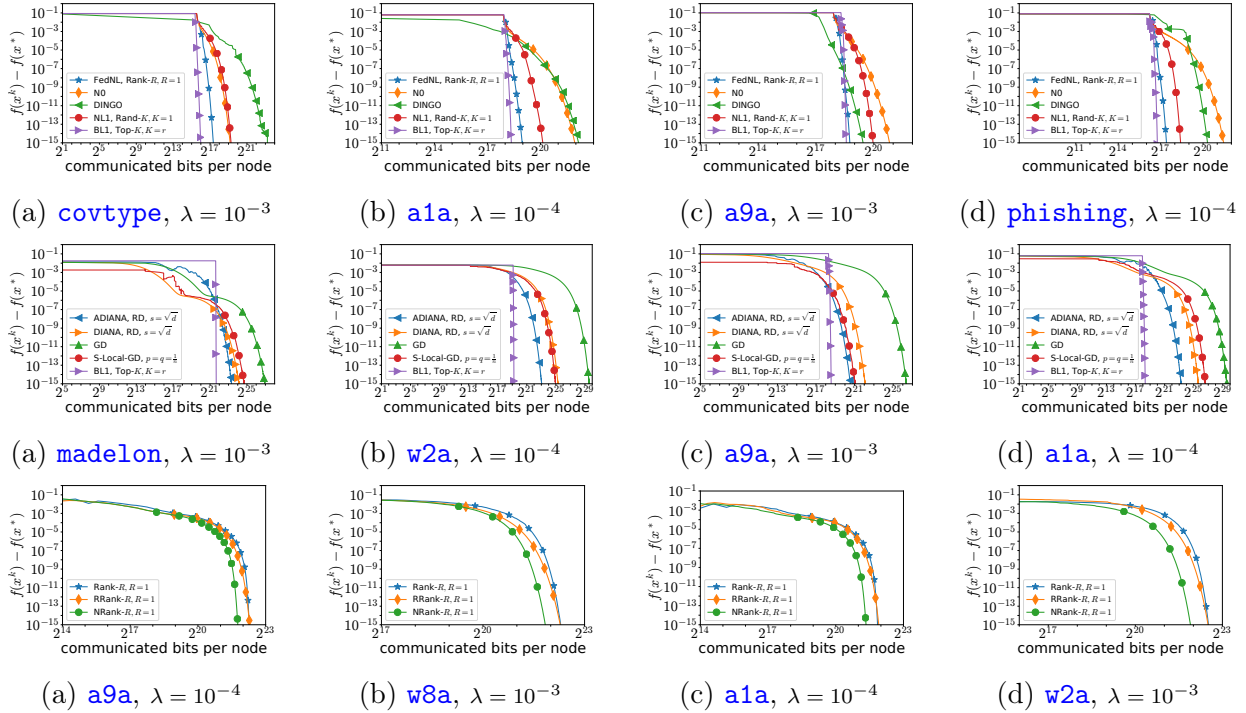

(a) covtype, $\lambda = 10^{-3}$ (b) a1a, $\lambda = 10^{-4}$ (c) a9a, $\lambda = 10^{-3}$ (d) phishing, $\lambda = 10^{-4}$

(a) madelon, $\lambda = 10^{-3}$ (b) w2a, $\lambda = 10^{-4}$ (c) a9a, $\lambda = 10^{-3}$ (d) a1a, $\lambda = 10^{-4}$

(a) a9a, $\lambda = 10^{-4}$ (b) w8a, $\lambda = 10^{-3}$ (c) a1a, $\lambda = 10^{-4}$ (d) w2a, $\lambda = 10^{-3}$

Figure 1: Comparison of BL1 with N0, FedNl, NL1, DINGO (**first row**), DIANA, ADIANA, GD, S-Local-GD (**second row**) and the performance of BL2 with compressors Rank-R, RRank-R, and NRank-R (**third row**) in terms of communication complexity.

6.3 Comparison with first-order methods

Next we compare the performance of BL1 with vanilla gradient descent (GD), DIANA [Mishchenko et al., 2019], ADIANA [Li et al., 2020], and shifted local gradient descent (S-Local-GD) [Gorbunov et al., 2021] in terms of communication complexity. Theoretical stepsizes were chosen for first-order methods. For DIANA and ADIANA we use random dithering compression [Alistarh et al., 2017, Horváth et al., 2019] with $s = \sqrt{d}$ levels. Probabilities p and q are equal to $1/n$ for S-Local-GD. Parameters of BL1 are the same as in the previous section. We clearly see in Figure 1 (2nd row) that BL1 is more communication efficient than all gradient type methods by *several orders in magnitude.*

6.4 Composition of compressors

In our next experiment we analyse the composition of Rank-R and unbiased compression operators; see Section 3 for more details. We consider BL2 with 3 compression mechanisms: Rank-R, RRank-R (composition of Rank-R and random dithering with $s = \sqrt{d}$ levels), and NRank-R (composition of Rank-R and natural compression). For all three compressors $R = 1$, and initializaion is $\mathbf{H}^0 = \nabla^2 f(x^0)$. Besides, the parameters of BL2 are the following: $\tau = n$, $p = \frac{1}{10}$. Finally, we use Top-K with $K = \lfloor \frac{d}{10} \rfloor$ for $\mathcal{Q}_i^k$. In this experiment we use standard basis in the space of matrices which means that BL2 turns to be FedNL. According to numerical results presented in Figure 1 (3rd row), composition is indeed useful.

7 Extensions

In this paper, we consider the basis in $\mathbb{R}^{d \times d}$ and $\mathcal{S}^d$. It is actually possible to extend Basis Learn to the case where $\{\mathbf{B}_i^{jl}\}$ is not necessarily a basis in some space. More precisely, if there exist a set $\{\mathbf{B}_i^j\}_{j \in S^i}$ and a map $h^i : \mathbb{R}^d \to \mathbb{R}^{|S^i|}$ such that for any $x \in \mathbb{R}^d$, $\nabla^2 f_i(x)$ can be represented by $\sum_j h^i(x)_j \mathbf{B}_i^j$ and h^i is L-Lipschitz continuous, i.e., $\|h^i(x) - h^i(y)\| \leq L\|x - y\|$ for any $x, y \in \mathbb{R}^d$, then we can get the corresponding algorithm and convergence results in the same way.

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Appendix

A   Extra experiments

In this section we demonstrate additional numerical experiments comparing BL with relevant benchmarks and with state-of-the-art methods. We consider regularized logistic regression problem

$$\min_{x \in \mathbb{R}^d} \left\{ \frac{1}{n} \sum_{i=1}^n f_i(x) + \frac{\lambda}{2} \|x\|^2 \right\}, \quad \text{where} \quad f_i(x) = \frac{1}{m} \sum_{j=1}^m \log \left(1 + \exp(-b_{ij} a_{ij}^\top x) \right),$$

and $\{a_{ij}, b_{ij}\}_{j \in [m]}$ are data samples belonging to the i-th node.

A.1   Parameters setting and data sets

The data sets were taken from LibSVM library [Chang and Lin, 2011]: a1a, a9a, phishing, covtype, madelon, w2a, w8a. Each data set was partitioned across several nodes to cover a variety of scenarios. See Table 2 for more detailed description.

Table 2: Data sets used in the experiments with the number of worker nodes n used in each case.

data set	# workers n	# data points $(= nm)$	# features d	average dimension r
a1a	16	1600	123	64
a9a	80	32560	123	82
phishing	100	110	68	35
covtype	200	581000	54	24
madelon	10	2000	500	200
w2a	50	3450	300	59
w8a	142	49700	300	133

Theoretical parameters were used for gradient type methods: vanilla gradient descent (GD), DIANA [Mishchenko et al., 2019], ADIANA [Li et al., 2020], and local gradient descent (Local-GD). The parameter constants for DINGO [Crane and Roosta, 2019] were chosen following authors' choice: $\theta = 10^{-4}, \phi = 10^{-6}, \rho = 10^{-4}$. Backtracking line search was used for DINGO to find the largest stepsize from $\{1, 2^{-1}, \ldots, 2^{-10}\}$. The initialization of $\mathbf{H}_i^0$ for NL1 [Islamov et al., 2021] and vanilla FedNL [Safaryan et al., 2021] is $\nabla^2 f_i(x^0)$. Besides, for NL1 we use Rand-K compressor with $K = 1$ and the stepsize $\alpha = \frac{1}{\omega + 1}$, where $\omega = \frac{m}{K} - 1$. For FedNL we use option 1 to make the Hessian approximation to be positive definite (projection onto the cone of positive definite matrices), stepsize $\alpha = 1$, and compression operator Rank-R with $R = 1$. For BL3, we use option 2.

We carry out experiments for two values of regularization parameter $\lambda \in \{10^{-3}, 10^{-4}, 10^{-5}\}$. In the figures we plot the optimality gap $f(x^k) - f(x^*)$ versus the number of communicated bits per node. The optimal value $f(x^*)$ is chosen as the function value at the 20-th iterate of standard Newton's method.

A.2 Compression operators

Unbiased compression operator: random dithering. In all experiments with ADIANA and DIANA the compression operator applied on gradient differences is random dithering [Alistarh et al., 2017, Horváth et al., 2019]. This compressor has the parameter s (number of levels) and can be defined via the formula

$$\mathcal{C}(x) := \text{sign}(x) \cdot \|x\|_q \cdot \frac{\xi_s}{s}, \tag{17}$$

where $\|x\|_q := \left(\sum_i |x_i|^q\right)^{1/q}$ and $\xi_s \in \mathbb{R}^d$ is a random vector whose i-th entire defind as follows

$$(\xi_s)_i = \begin{cases} l+1 & \text{with probability } \frac{|x_i|}{\|x\|_q} s - l, \\ l & \text{otherwise.} \end{cases} \tag{18}$$

Here $s \in \mathbb{N}_+$ denotes the levels of rounding, and l satisfies $\frac{|x_i|}{\|x\|_q} \in \left[\frac{l}{s}, \frac{l+1}{s}\right]$. This compressor has variance parameter satsfying $\omega \leq 2 + \frac{d^{1/2}+d^{1/q}}{s}$ [Horváth et al., 2019]. However, for Euclidean norm ($q=2$) one can improve the bound to $\omega \leq \min\left\{\frac{d}{s^2}, \frac{\sqrt{d}}{s}\right\}$ [Alistarh et al., 2017].

Examples of contractive compression operators for matrices. One of the examples of contractive compression operators is low-rank approximation or Rank-R compressor. This compression operator is based on singular value decomposition of the matrix and belongs to the class of contractive compressors with $\delta = \frac{R}{d}$ [Safaryan et al., 2021]. Let $\mathbf{X} \in \mathbb{R}^{d \times d}$ and singular value decomposition of $\mathbf{X}$ is

$$\mathbf{X} = \sum_{i=1}^{d} \sigma_i u_i v_i^\top, \tag{19}$$

where the singular values σ_i are sorted in non-increasing order: $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_d$. Then, the Rank-$R$ compressor, for $R \leq d$, is defined by

$$\mathcal{C}(\mathbf{X}) := \sum_{i=1}^{R} \sigma_i u_i v_i^\top. \tag{20}$$

Note that if the input of Rank-R compressor is a symmetric matrix, then its output is automatically symmetric matrix.

Another popular choice of contractive compressors in practice is Top-K. This compressor applied on matrices sorts the entires of input in non-increasing order by magnitude, and then selects K maximal elements. Top-K compressor belongs to the class of contractive compressors with $\delta = \frac{d^2}{K}$. For arbitrary matrix $\mathbf{X} \in \mathbb{R}^{d \times d}$ let sort its entires in non-increasing order by magnitude, i.e., $X_{i_k j_k}$ is the k-th maximal element of $\mathbf{X}$ by magnitude. Let $\{\mathbf{B}_{ij}\}_{i,j=1}^{d}$ be a standard basis in the space of matrices. Then, the Top-K compression operator can be defined via

$$\mathcal{C}(\mathbf{X}) := \sum_{k=1}^{K} X_{i_k j_k} \cdot \mathbf{B}_{i_k j_k}. \tag{21}$$

One way how to make the output of this compressor to be a symmetric matrix is to apply Top-K on upper triangular part of the input.

A.3 Example of unbiased compression operators for matrices

The simplest example of unbiased compressor which could be applied on matrices is random sparsification operator or Rand-K. This compressor belongs to the class of unbiased compressors with $\omega = \frac{d^2}{K} - 1$. For the input matrix $\mathbf{X} \in \mathbb{R}^{d \times d}$ we choose a set $\mathcal{S}_K$ of indexes (i,j) of cardinality K uniformly at random. Then Rand-K compressor can be defined via

$$\mathcal{C}\left(\mathbf{X}\right)_{ij} := \begin{cases} \frac{d^2}{K} X_{ij} & \text{if } (i,j) \in \mathcal{S}_K, \\ 0 & \text{if } (i,j) \notin \mathcal{S}_K. \end{cases} \tag{22}$$

The way how to make the output of Rand-K to be a symmetric matrix is exaclty the same as for Top-K.

A.4 The performance of Newton's method in different basis

First, we investigate how the performance of Newton's method is influenced by the choice of the basis. We compare the efficiency of Newton's method on two bases: the one that was described in Section 2.3 and the standard one. The results are presented in Figure 2. We clearly see that Newton's method in the specific basis is approximately 4 times more communication-efficient than in standard one.
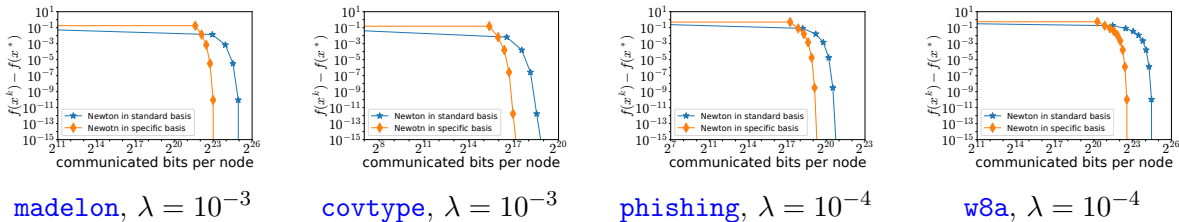

madelon, $\lambda = 10^{-3}$ covtype, $\lambda = 10^{-3}$ phishing, $\lambda = 10^{-4}$ w8a, $\lambda = 10^{-4}$

Figure 2: The performance of Newton's method in different basis in terms of communication complexity.

A.5 Composition of Top-K and unbiased compressor

Next, we study other type of composition of compression operators. We investigate how composition of Top-K and unbiased compression operator [Qian et al., 2021] influences the performance of BL2. We compare the performance of BL2 with Top-K $(K = r)$, RTop-K $(K = r)$ (composition of Top-K and random dithering with $s = \sqrt{K}$), and NTop-K $(K = r)$ (composition of Top-K and natural compression). The initialization of $\mathbf{H}^0$ is $\nabla^2 f(x^0)$. Besides, we use the basis that was decribed in Section 2.3. We set the following parameters for BL2: $p = \frac{r}{2d}$, $\tau = n$, and Top-K with $K = \lfloor \frac{r}{2} \rfloor$ for models in the experiments on w2a, a1a data sets. In the experiments on a9a, phishing data sets, these parameters are $p = \frac{r}{4d}$, $\tau = n$, and Top-K, $(K = \lfloor \frac{r}{4} \rfloor)$ compressor for models. The results are presented in Figure 3. According to numerical results, we can conclude that composition of Top-K and natural compression is the most efficient compressor in all cases. However, RTop-K have almost the same performance as Top-K on data sets a1a, a9a.
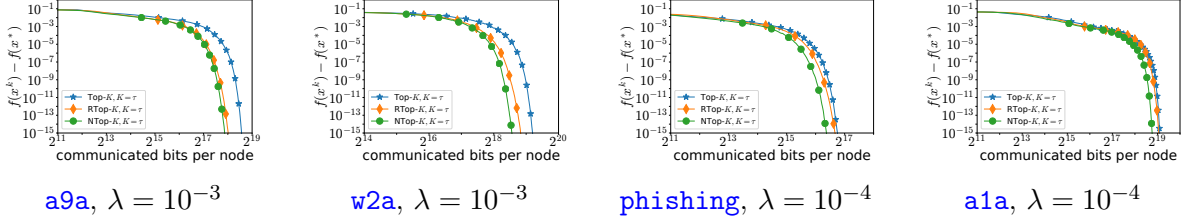
a9a, $\lambda = 10^{-3}$ w2a, $\lambda = 10^{-3}$ phishing, $\lambda = 10^{-4}$ a1a, $\lambda = 10^{-4}$

Figure 3: The performance of BL2 with different types of compression operators: Top-K, RTop-K (composition of Top-K and random dithering with $s = \sqrt{K}$), and NTop-K (composition of Top-K and natrual compression).

A.6 The effect of partial participation

a1a, $\lambda = 10^{-4}$ a1a, $\lambda = 10^{-5}$ a9a, $\lambda = 10^{-4}$ phishing, $\lambda = 10^{-5}$
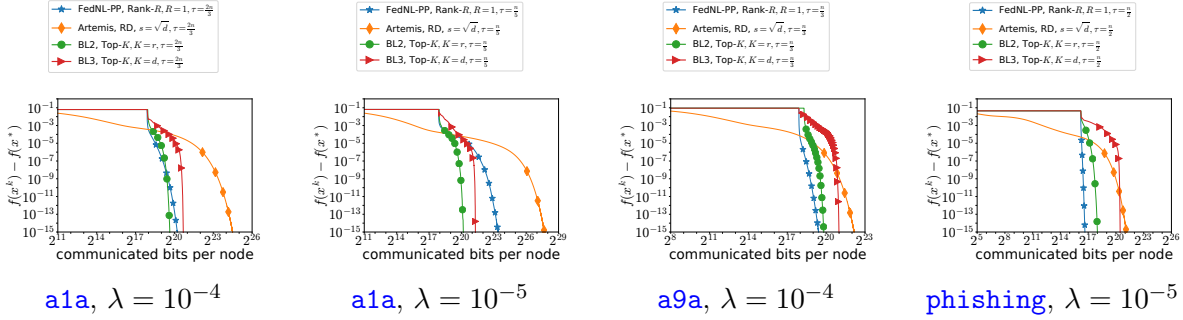
Figure 4: The comparison of FedNL-PP, BL2, BL3, and Artemis with partial device participation in terms of communication complexity.

In this section we study the effect of partial participation. For FedNL-PP [Safaryan et al., 2021] we use stepsize $\alpha = 1$ and Rank-R ($R = 1$) compression operator. The specific basis described in Section 2.3 were used for BL2. Besides, the parameters of this method are the following: compression operator $\mathcal{C}_i^k$ is Top-K with $K = r$, $p = 1$. The basis for BL3 were chosen from the Example 4.15. We use Top-K compressor with $K = d$ for $\mathcal{C}_i^k$, and set $p = 1$ for this method. Both for BL2 and BL3 stepsizes are $\alpha = \eta = 1$, model comressor $\mathcal{Q}_i^k$ is identity. Random dithering with $s = \sqrt{d}$ levels was used for Artemis [Philippenko and Dieuleveut, 2021]. In different cases we set the number of active devices τ equal to various fractions of n. The results of the experiment are presented in Figure 4. According to the plots, BL2 and FedNL-PP are the best methods, they outperform each depending on data set. BL3 also outperform FedNL-PP on a1a ($\lambda = 10^{-5}$) data set. In almost all cases FedNL-PP and BL2 outperform Artemis be *many orders in magnitude*. We can conclude that specific for the problem basis could be beneficial.

A.7 Bidirectional compression

In our next test we compare FedNL-BC [Safaryan et al., 2021], BL1, BL2, BL3, and DORE [Liu et al., 2020]. The parameters of FedNL-BC are the following: matrix compression operator is Top-K,

$K = \lfloor \frac{d}{2} \rfloor$; model compression operator is Top-K, $K = \lfloor \frac{d}{2} \rfloor$; stepsizes are $\alpha = \eta = 1$; probability $p = 1$. We use option 1 (projection) to make Hessian approximation to be positive definite. Next, we use the basis described in Section 2.3 for BL1 and BL2. We use Top-K, $K = \lfloor \frac{r}{2} \rfloor$, for matrices and models compression, probability $p = \frac{r}{2d}$, and stepsizes $\alpha = \eta = 1$. The basis for BL3 is described in Example 4.15 in the main paper. Besides, this method has the following parameters: Top-K, $K = \lfloor \frac{d}{2} \rfloor$ for models and Hessians compression; stepsize $\alpha = \eta = 1$; probability $p = \frac{1}{2}$. Finally, all devices are active for BL2 and BL3, i.e. $\tau = n$. The results of this test can be found in Figure 5.
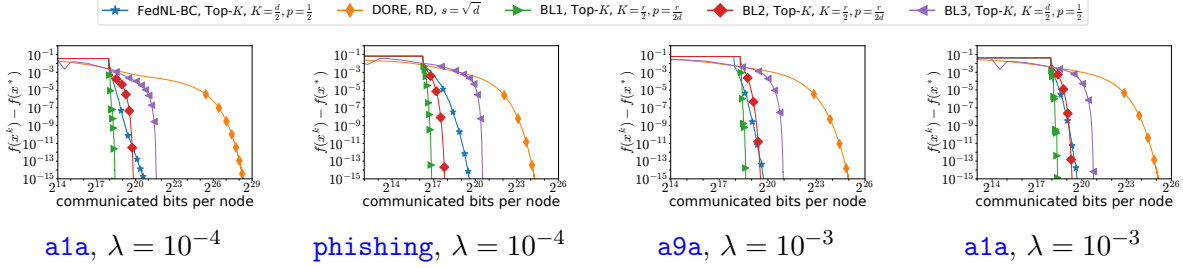

Figure 5: The comparison of FedNL-BC, BL1, BL2, BL3 and DORE with bidirectional compression in terms of communication complexity.

We see that all second-order methods outperform DORE in terms of communication complexity by *many orders in magnitude*. Moreover, we can conclude that specific to the problem basis is helpful since BL1 and BL2 outperform FedNL-BC.

A.8   Comparison of BL2 and BL3

Finally, we compare BL2 and BL3 with bidirectional compression and partial participation simultaneously. We set the number of active devices to $\frac{n}{2}$. For BL2 we use standard basis in the space of matrices, for BL3 the basis is one that was given in the example 4.15. For both methods the compression operator is Top-K, $K = \lfloor pd \rfloor$, both for models and matrices. The gradient compressor is lazy Bernoulli compressor with parameter p. We set $p \in \{1, 1/3, 1/5\}$. In the Figure 6 we plot the optimality gap $f(x^k) - f(x^*)$ versus the average number of communicated bits per node.
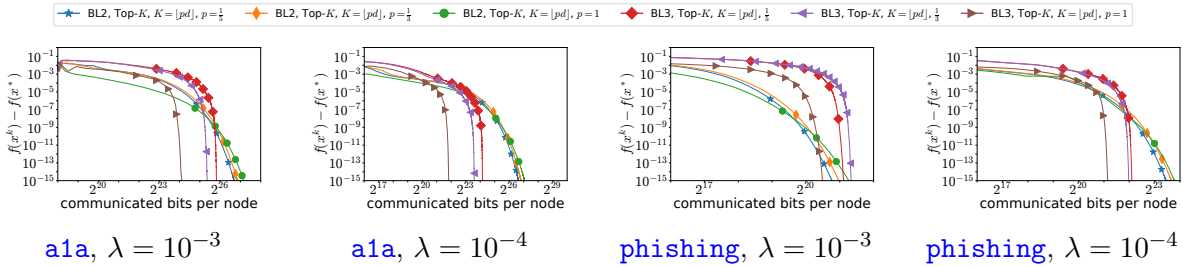

Figure 6: The comparison of BL2 and BL3 with bidirectional compression and partial participation in terms of communication complexity.

The first observation from the numerical results is that BL2 is less communication-efficient method than BL3. However, if we use specific basis for BL2, then it improves the performance of the

method; in Figure 5 BL2 is better than BL3. Besides, we clearly see that bicompression improves the performance of BL2 in partial participation setting. However, this is not the case for BL3.

B Proofs of Lemma 3.1 and Proposition 3.2

B.1 Proof of Lemma 3.1

(i) We have

$$
\begin{aligned}
&\left\| \frac{(\mathbf{B} + \mathbf{B}^\top)}{2} - \mathbf{A} \right\|_{\mathrm{F}}^2 - \|\mathbf{B} - \mathbf{A}\|_{\mathrm{F}}^2 \\
&= \frac{1}{4}\|\mathbf{B} + \mathbf{B}^\top\|_{\mathrm{F}}^2 + \|\mathbf{A}\|_{\mathrm{F}}^2 - \langle \mathbf{B} + \mathbf{B}^\top, \mathbf{A} \rangle - \|\mathbf{B}\|_{\mathrm{F}}^2 - \|\mathbf{A}\|_{\mathrm{F}}^2 + 2\langle \mathbf{B}, \mathbf{A} \rangle \\
&= \frac{1}{4}\|\mathbf{B}\|_{\mathrm{F}}^2 + \frac{1}{4}\|\mathbf{B}^\top\|_{\mathrm{F}}^2 + \frac{1}{2}\langle \mathbf{B}, \mathbf{B}^\top \rangle - \|\mathbf{B}\|_{\mathrm{F}}^2 + \langle \mathbf{B} - \mathbf{B}^\top, \mathbf{A} \rangle \\
&= \frac{1}{2}\langle \mathbf{B}, \mathbf{B}^\top \rangle - \frac{1}{2}\|\mathbf{B}\|_{\mathrm{F}}^2 + \langle \mathbf{B} - \mathbf{B}^\top, \mathbf{A} \rangle \\
&\leq \langle \mathbf{B} - \mathbf{B}^\top, \mathbf{A} \rangle \\
&= 0,
\end{aligned}
$$

where the first inequality comes from the Cauchy-Schwartz inequality, and the last equality comes from the fact that $\mathbf{A}$ is symmetric.

(ii) From (i), for any $\mathbf{A} \in \mathbb{R}^{d \times d}$ we have

$$
\mathbb{E}\|\tilde{\mathcal{C}}(\mathbf{A}) - \mathbf{A}\|_{\mathrm{F}}^2 \leq \mathbb{E}\|\mathcal{C}(\mathbf{A}) - \mathbf{A}\|_{\mathrm{F}}^2 \leq (1 - \delta)\|\mathbf{A}\|_{\mathrm{F}}^2.
$$

B.2 Proof of Proposition 3.2

From the definition of $\mathcal{C}_1$, we have

$$
\begin{aligned}
\mathbb{E}[\|\mathcal{C}_1(\mathbf{A}) - \mathbf{A}\|_{\mathrm{F}}^2] &= \mathbb{E}\|\mathcal{C}_1(\mathbf{A})\|_{\mathrm{F}}^2 + \|\mathbf{A}\|_{\mathrm{F}}^2 - 2\mathbb{E}[\langle \mathcal{C}_1(\mathbf{A}), \mathbf{A} \rangle] \\
&= \|\mathbf{A}\|_{\mathrm{F}}^2 + \sum_{i=1}^{R} \mathbb{E}\left[\frac{\sigma_i^2 \mathcal{Q}_2^i(b_i v_i)^\top \mathcal{Q}_2^i(b_i v_i) \mathcal{Q}_1^i(a_i u_i)^\top \mathcal{Q}_1^i(a_i u_i)}{a_i^2 b_i^2 (\omega_1 + 1)^2 (\omega_2 + 1)^2} \right] \\
&\quad + \sum_{i,j\in[R], i\neq j} \mathbb{E}\left[\frac{\sigma_i \sigma_j \mathcal{Q}_2^i(b_i v_i)^\top \mathcal{Q}_2^j(b_j v_j) \mathcal{Q}_1^j(a_j u_j)^\top \mathcal{Q}_1^i(a_i u_i)}{a_i a_j b_i b_j (\omega_1 + 1)^2 (\omega_2 + 1)^2} \right] \\
&\quad - 2\left\langle \sum_{i=1}^{R} \frac{\sigma_i u_i v_i^\top}{(\omega_1 + 1)(\omega_2 + 1)}, \mathbf{A} \right\rangle \\
&= \|\mathbf{A}\|_{\mathrm{F}}^2 + \sum_{i=1}^{R} \frac{\sigma_i^2 \mathbb{E}\|\mathcal{Q}_2^i(b_i v_i)\|^2 \cdot \mathbb{E}\|\mathcal{Q}_1^i(a_i u_i)\|^2}{a_i^2 b_i^2 (\omega_1 + 1)^2 (\omega_2 + 1)^2} - 2\left\langle \sum_{i=1}^{R} \frac{\sigma_i u_i v_i^\top}{(\omega_1 + 1)(\omega_2 + 1)}, \mathbf{A} \right\rangle,
\end{aligned}
$$

where in the last two equalities, we use the independence of each $\mathcal{Q}_1^i$, $\mathcal{Q}_2^i$, and the fact that $u_j^\top u_i = 0$ and $v_i^\top v_j = 0$ for $i \neq j$. From the definition of unbiased compressors, we further have

$$
\mathbb{E}[\|\mathcal{C}_1(\mathbf{A}) - \mathbf{A}\|_{\mathrm{F}}^2] \leq \|\mathbf{A}\|_{\mathrm{F}}^2 + \sum_{i=1}^R \frac{\sigma_i^2 \|u_i\|^2 \|v_i\|^2}{(\omega_1 + 1)(\omega_2 + 1)} - 2 \left\langle \sum_{i=1}^R \frac{\sigma_i u_i v_i^\top}{(\omega_1 + 1)(\omega_2 + 1)}, \mathbf{A} \right\rangle
$$

$$
= \left(1 - \frac{1}{(\omega_1 + 1)(\omega_2 + 1)}\right) \|\mathbf{A}\|_{\mathrm{F}}^2
$$

$$
+ \frac{1}{(\omega_1 + 1)(\omega_2 + 1)} \left(\|\mathbf{A}\|_{\mathrm{F}}^2 + \sum_{i=1}^R \sigma_i^2 \|u_i\|^2 \|v_i\|^2 - 2 \left\langle \sum_{i=1}^R \sigma_i u_i v_i^\top, \mathbf{A} \right\rangle \right)
$$

$$
= \left(1 - \frac{1}{(\omega_1 + 1)(\omega_2 + 1)}\right) \|\mathbf{A}\|_{\mathrm{F}}^2 + \frac{1}{(\omega_1 + 1)(\omega_2 + 1)} \left\| \sum_{i=1}^R \sigma_i u_i v_i^\top - \mathbf{A} \right\|_{\mathrm{F}}^2
$$

$$
\leq \left(1 - \frac{1}{(\omega_1 + 1)(\omega_2 + 1)}\right) \|\mathbf{A}\|_{\mathrm{F}}^2 + \frac{(1 - R/d)}{(\omega_1 + 1)(\omega_2 + 1)} \|\mathbf{A}\|_{\mathrm{F}}^2
$$

$$
= \left(1 - \frac{R}{d(\omega_1 + 1)(\omega_2 + 1)}\right) \|\mathbf{A}\|_{\mathrm{F}}^2,
$$

where in the last inequality we use the fact that Rank-R is a contraction compressor with parameter R/d [Safaryan et al., 2021].

For $\mathcal{C}_2$, the result follows from Lemma 3.1 (ii).

B.3 Linear Independence of Outer Products

Lemma B.1. *Let vectors $\{v_1, v_2, \ldots, v_r\} \subset \mathbb{R}^d$ are linearly independent. Then outer products $\{v_i v_j^\top : i, j = 1, 2, \ldots, r\}$ are linearly independent matrices in $\mathbb{R}^{d \times d}$.*

Proof. Let $\{e_1, e_2, \ldots, e_d\}$ be the standard basis in $\mathbb{R}^d$. Then, for all $i \in [r]$

$$
v_i = \sum_{t=1}^r v_{it} e_t.
$$

Denote $\mathbf{E}_{tl} = e_t e_l^\top$. Suppose linear combination of matrices $\{v_i v_j^\top : i, j = 1, 2, \ldots, r\}$ with some coefficients c_{ij} is zero matrix. After simple transformations, we get

$$
\mathbf{0} = \sum_{i,j=1}^r c_{ij} v_i v_j^\top = \sum_{i,j=1}^r c_{ij} \sum_{t,l=1}^d v_{it} v_{jl} \mathbf{E}_{tl} = \sum_{t,l=1}^d \left[\sum_{i,j=1}^r c_{ij} v_{it} v_{jl} \right] \mathbf{E}_{tl},
$$

which implies that

$$
\sum_{i,j=1}^r c_{ij} v_{it} v_{jl} = 0, \quad \text{for all } t, l \in [d].
$$

Then notice that

$$
0 = \sum_{i,j=1}^r c_{ij} v_{it} v_{jl} = \sum_{i=1}^r \left[\sum_{j=1}^r c_{ij} v_{jl} \right] v_{it} = \sum_{i=1}^r c_{il}' v_{it}
$$

28

holds for all $t \in [d]$, which implies that $\sum_{i=1}^{r} c'_{il} v_i = 0$ (where that last 0 is a vector of size d). Since v_i's are linearly independent, we get $c'_{il} = 0$ for all $i \in [d]$ and $l \in [r]$. By definition $c'_{il} = \sum_{j=1}^{r} c_{ij} v_{jl}$, hence $\sum_{j=1}^{r} c_{ij} v_j = 0$. Again using linear independence of v_i's, we get $c_{ij} = 0$ for all $i, j \in [d]$. Therefore outer products $v_i v_j^\top$ are also independent. $\qquad\square$

C   Proofs for BL1

We denote $\mathbb{E}_k[\cdot]$ as the conditional expectation on z^k, w^k, and $\mathbf{H}_i^k$.

C.1   Proof of Lemma 4.8

If $\|\nabla^2 f_i(x) - \nabla^2 f_i(y)\|_{\mathrm{F}} \leq H_1 \|x - y\|$ for any $x, y \in \mathbb{R}^d$, and $i \in [n]$, then from (9) we have

$$
\begin{aligned}
\|h^i(\nabla^2 f_i(x)) - h^i(\nabla^2 f_i(y))\|_{\mathrm{F}} &= \|vec(h^i(\nabla^2 f_i(x))) - vec(h^i(\nabla^2 f_i(y)))\| \\
&\leq \|\mathcal{B}_i^{-1}\| \cdot \|vec(\nabla^2 f_i(x)) - vec(\nabla^2 f_i(y))\| \\
&= \|\mathcal{B}_i^{-1}\| \cdot \|\nabla^2 f_i(x) - \nabla^2 f_i(y)\|_{\mathrm{F}} \\
&\leq \|\mathcal{B}_i^{-1}\| H_1 \|x - y\|,
\end{aligned}
$$

which implies that M_1 in Assumption 4.7 satisfies $M_1 \leq \max_i \{\|\mathcal{B}_i^{-1}\|\} H_1$.

If $|(\nabla^2 f_i(x))_{jl} - (\nabla^2 f_i(y))_{jl}| \leq \nu \|x - y\|$ for any $x, y \in \mathbb{R}^d$, $i \in [n]$, and $j, l \in [d]$, then from (9), every entry of $h^i(\nabla^2 f_i(x)) - h^i(\nabla^2 f_i(y))$ will be bounded by $\nu \|\mathcal{B}_i^{-1}\|_\infty \|x-y\|$. Hence M_2 in Assumption 4.7 satisfies $M_2 \leq \nu \max_i \{\|\mathcal{B}_i^{-1}\|_\infty\}$.

C.2   Lemmas

The proofs of Lemma C.1 and Lemma C.2 are the same as that of Lemma B.1 in [Safaryan et al., 2021]. Thus we omit them.

Lemma C.1. *Let $\mathcal{Q}$ be a compressor and $\eta > 0$. For any $x, y, z \in \mathbb{R}^d$, we have following results.*

(i) *If $\mathcal{Q}$ is an unbiased compressor with parameter ω_{M} and $\eta \leq 1/(\omega_{\mathrm{M}}+1)$, then*

$$
\mathbb{E}\|z + \eta \mathcal{Q}(x - z) - y\|^2 \leq (1 - \eta)\|z - y\|^2 + \eta \|x - y\|^2,
$$

where $\mathbb{E}[\cdot]$ is the expectation with respect to $\mathcal{Q}$.

(ii) *If $\mathcal{Q}$ is a contraction compressor with parameter δ_{M} and $\eta = 1$, then*

$$
\mathbb{E}\|z + \eta \mathcal{Q}(x - z) - y\|^2 \leq \left(1 - \frac{\delta_{\mathrm{M}}}{4}\right)\|z - y\|^2 + \left(\frac{6}{\delta_{\mathrm{M}}} - \frac{7}{2}\right)\|x - y\|^2,
$$

Lemma C.2. *Let $\mathcal{C}$ be a compressor and $\alpha > 0$. For any matrix $\mathbf{L} \in \mathbb{R}^{d \times d}$ and $y, z \in \mathbb{R}^d$, we have the following results.*

(i) *If $\mathcal{C}$ is an unbiased compressor with parameter ω and $\alpha \leq 1/\omega+1$, then*

$$
\mathbb{E}\|\mathbf{L} + \alpha \mathcal{C}(h^i(\nabla^2 f_i(y)) - \mathbf{L}) - h^i(\nabla^2 f_i(z))\|_{\mathrm{F}}^2 \leq (1 - \alpha)\|\mathbf{L} - h^i(\nabla^2 f_i(z))\|_{\mathrm{F}}^2 + \alpha M_1^2 \|y - z\|^2,
$$

where $\mathbb{E}[\cdot]$ is the expectation with respect to $\mathcal{C}$.

(ii) *If $\mathcal{C}$ is a contraction compressor with parameter δ and $\alpha = 1$, then*

$$
\mathbb{E}\|\mathbf{L} + \alpha \mathcal{C}(h^i(\nabla^2 f_i(y)) - \mathbf{L}) - h^i(\nabla^2 f_i(z))\|_{\mathrm{F}}^2 \leq \left(1 - \frac{\delta}{4}\right)\|\mathbf{L} - h^i(\nabla^2 f_i(z))\|_{\mathrm{F}}^2 + \left(\frac{6}{\delta} - \frac{7}{2}\right) M_1^2 \|y - z\|^2.
$$

Lemma C.3. *We consider four cases:*

(i) *If Assumption 4.3 (ii) holds, $\|x^0 - x^*\|^2 \leq \min\{\frac{\mu^2}{4d^2H^2}, \frac{M}{d}\}$, $\|z^k - x^*\|^2 \leq \min\{\frac{\mu^2}{4dH^2}, M\}$, and $\mathcal{H}^k \leq \frac{\mu^2}{4dN_{\mathrm{B}}R^2}$ for $k \leq K$ and any $M > 0$, then $\|z^{K+1} - x^*\|^2 \leq \min\{\frac{\mu^2}{4dH^2}, M\}$.*

(ii) *If Assumption 4.4 holds, $\mathcal{H}^K \leq \frac{A_{\mathrm{M}}\mu^2}{4N_{\mathrm{B}}R^2B_{\mathrm{M}}}$, $\|z^k - x^*\|^2 \leq \min\{\frac{A_{\mathrm{M}}\mu^2}{4H^2B_{\mathrm{M}}}, M\}$ for $k \leq K$ and any $M > 0$, then $\|z^{K+1} - x^*\|^2 \leq \min\{\frac{A_{\mathrm{M}}\mu^2}{4H^2B_{\mathrm{M}}}, M\}$.*

(iii) *If Assumption 4.5(ii) holds, and $\|z^k - x^*\|^2 \leq \frac{M}{d^2M_2^2}$ for $k \leq K$ and any $M > 0$, then $\mathcal{H}^K \leq M$.*

(iv) *If Assumption 4.6 holds, $\mathcal{H}^K \leq M$, and $\|z^K - x^*\|^2 \leq \frac{AM}{BM_1^2}$ for any any $M > 0$, then $\mathcal{H}^{K+1} \leq M$.*

Proof. (i) If $\xi^k = 1$, from (23), (24), and (26), we have

$$\|x^{k+1} - x^*\|^2 \leq \frac{1}{\mu^2}\left(\frac{H^2}{2}\|z^k - x^*\|^2 + 2N_{\mathrm{B}}R^2\mathcal{H}^k\right)\|z^k - x^*\|^2$$

$$\leq \frac{1}{d}\|z^k - x^*\|^2$$

$$\leq \min\left\{\frac{\mu^2}{4d^2H^2}, \frac{M}{d}\right\},$$

for $0 \leq k \leq K$.

If $\xi^k = 0$, since we also have $\|w^k - x^*\|^2 \leq \min\{\frac{\mu^2}{4dH^2}, M\}$, then from (23), (25), and (26), we can get the above inequality in the same way.

Since $\|x^0 - x^*\|^2 \leq \min\{\frac{\mu^2}{4d^2H^2}, \frac{M}{d}\}$, we know $\|x^k - x^*\|^2 \leq \min\{\frac{\mu^2}{4d^2H^2}, \frac{M}{d}\}$ for all $0 \leq k \leq K + 1$. Then from Assumption 4.3 (ii), we can get

$$\|z^{K+1} - x^*\|^2 \leq d\max_j|z_j^{K+1} - x_j^*|^2$$

$$\leq d\max_{0 \leq t \leq K+1}\|x^t - x^*\|^2$$

$$\leq \min\left\{\frac{\mu^2}{4dH^2}, M\right\}.$$

(ii) First, from the update rule of w^k, we know $\|w^k - x^*\|^2 \leq \min\{\frac{A_{\mathrm{M}}\mu^2}{4H^2B_{\mathrm{M}}}, M\}$ for $k \leq K$. If $\xi^K = 1$, from (23), (24), and (26), we have

$$\|x^{K+1} - x^*\|^2 \leq \frac{1}{\mu^2}\left(\frac{H^2}{2}\|z^K - x^*\|^2 + 2N_{\mathrm{B}}R^2\mathcal{H}^K\right)\|z^K - x^*\|^2$$

$$\leq \left(\frac{A_{\mathrm{M}}}{8B_{\mathrm{M}}} + \frac{A_{\mathrm{M}}}{2B_{\mathrm{M}}}\right)\|z^K - x^*\|^2$$

$$\leq \frac{A_{\mathrm{M}}}{B_{\mathrm{M}}}\min\left\{\frac{A_{\mathrm{M}}\mu^2}{4H^2B_{\mathrm{M}}}, M\right\}.$$

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If $\xi^K = 0$, from $\|w^K - x^*\|^2 \leq \min\{\frac{A_{\mathrm{M}}\mu^2}{4H^2 B_{\mathrm{M}}}, M\}$ and (25), we can obtain the above inequality similarly. Then from Lemma C.1 (ii), we arrive at

$$
\begin{aligned}
\|z^{K+1} - z^*\|^2 &\leq (1 - A_{\mathrm{M}})\|z^K - x^*\|^2 + B_{\mathrm{M}}\|x^{K+1} - x^*\|^2 \\
&\leq (1 - A_{\mathrm{M}}) \min\left\{\frac{A_{\mathrm{M}}\mu^2}{4H^2 B_{\mathrm{M}}}, M\right\} + A_{\mathrm{M}} \min\left\{\frac{A_{\mathrm{M}}\mu^2}{4H^2 B_{\mathrm{M}}}, M\right\} \\
&= \min\left\{\frac{A_{\mathrm{M}}\mu^2}{4H^2 B_{\mathrm{M}}}, M\right\}.
\end{aligned}
$$

(iii) From Assumption 4.5(ii), we have

$$
\begin{aligned}
\mathcal{H}^K &= \frac{1}{n}\sum_{i=1}^{n}\|\mathbf{L}_i^K - \mathbf{L}_i^*\|_{\mathrm{F}}^2 \\
&\leq \frac{1}{n}\sum_{i=1}^{n} d^2 \max_{jl}\{|(\mathbf{L}_i^K)_{jl} - (\mathbf{L}_i^*)_{jl}|^2\} \\
&\leq d^2 M_2^2 \max_{0\leq t\leq K}\|z^t - x^*\|^2 \\
&\leq M.
\end{aligned}
$$

(iv) From Assumption 4.6 and Lemma C.2 (ii), we have

$$
\begin{aligned}
\|\mathbf{L}_i^{K+1} - \mathbf{L}_i^*\|_{\mathrm{F}}^2 &\leq (1 - A)\|\mathbf{L}_i^K - \mathbf{L}_i^*\|_{\mathrm{F}}^2 + B M_1^2 \|z^K - x^*\|^2 \\
&\leq (1 - A)\|\mathbf{L}_i^K - \mathbf{L}_i^*\|_{\mathrm{F}}^2 + AM,
\end{aligned}
$$

which implies that

$$
\mathcal{H}^{K+1} = \frac{1}{n}\sum_{i=1}^{n}\|\mathbf{L}_i^{K+1} - \mathbf{L}_i^*\|_{\mathrm{F}}^2 \leq (1 - A)M + AM \leq M.
$$

$\square$

C.3   Proof of Theorem 4.9

First we have

$$
\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|z^k - x^* - [\mathbf{H}^k]_\mu^{-1} g^k\|^2 \\
&= \left\|[\mathbf{H}^k]_\mu^{-1}\left([\mathbf{H}^k]_\mu(z^k - x^*) - (g^k - \nabla f(x^*))\right)\right\|^2 \\
&\leq \frac{1}{\mu^2}\left\|[\mathbf{H}^k]_\mu(z^k - x^*) - (g^k - \nabla f(x^*))\right\|^2, \quad (23)
\end{aligned}
$$

where we use $\nabla f(x^*) = 0$ in the second equality, and $\|[\mathbf{H}^k]_\mu^{-1}\| \leq \frac{1}{\mu}$ in the last inequality.

If $\xi^k = 1$, then

$$\left\| [\mathbf{H}^k]_\mu (z^k - x^*) - (g^k - \nabla f(x^*)) \right\|^2$$

$$= \left\| \nabla f(z^k) - \nabla f(x^*) - \nabla^2 f(x^*)(z^k - x^*) + (\nabla^2 f(x^*) - [\mathbf{H}^k]_\mu)(z^k - x^*) \right\|^2$$

$$\leq 2 \left\| \nabla f(z^k) - \nabla f(x^*) - \nabla^2 f(x^*)(z^k - x^*) \right\|^2 + 2 \left\| (\nabla^2 f(x^*) - [\mathbf{H}^k]_\mu)(z^k - x^*) \right\|^2$$

$$\leq \frac{H^2}{2} \|z^k - x^*\|^4 + 2\|[\mathbf{H}^k]_\mu - \nabla^2 f(x^*)\|^2 \cdot \|z^k - x^*\|^2$$

$$\leq \frac{H^2}{2} \|z^k - x^*\|^4 + 2\|\mathbf{H}^k - \nabla^2 f(x^*)\|_F^2 \|z^k - x^*\|^2$$

$$= \frac{H^2}{2} \|z^k - x^*\|^4 + 2 \left\| \frac{1}{n}\mathbf{H}_i^k - \frac{1}{n}\nabla^2 f_i(x^*) \right\|_F^2 \|z^k - x^*\|^2$$

$$\leq \frac{H^2}{2} \|z^k - x^*\|^4 + \frac{2}{n} \sum_{i=1}^n \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_F^2 \|z^k - x^*\|^2, \tag{24}$$

where in the second inequality, we use the Lipschitz continuity of the Hessian of f, and in the last inequality, we use the convexity of $\| \cdot \|_F^2$.

If $\xi^k = 0$, then

$$\left\| [\mathbf{H}^k]_\mu (z^k - x^*) - (g^k - \nabla f(x^*)) \right\|^2$$

$$= \left\| [\mathbf{H}^k]_\mu (z^k - w^k) + \nabla f(w^k) - \nabla f(x^*) - [\mathbf{H}^k]_\mu (z^k - x^*) \right\|^2$$

$$= \left\| [\mathbf{H}^k]_\mu (x^* - w^k) + \nabla f(w^k) - \nabla f(x^*) \right\|^2$$

$$= \left\| \nabla f(w^k) - \nabla f(x^*) - \nabla^2 f(x^*)(w^k - x^*) + (\nabla^2 f(x^*) - [\mathbf{H}^k]_\mu)(w^k - x^*) \right\|^2$$

$$\leq \frac{H^2}{2} \|w^k - x^*\|^4 + 2\|\mathbf{H}^k - \nabla^2 f(x^*)\|_F^2 \|w^k - x^*\|^2$$

$$\leq \frac{H^2}{2} \|w^k - x^*\|^4 + \frac{2}{n} \sum_{i=1}^n \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_F^2 \|w^k - x^*\|^2. \tag{25}$$

From the above three inequalities, we can obtain

$$\mathbb{E}_k \|x^{k+1} - x^*\|^2 \leq \frac{H^2 p}{2\mu^2} \|z^k - x^*\|^4 + \frac{2p}{n\mu^2} \sum_{i=1}^n \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_F^2 \|z^k - x^*\|^2$$

$$+ \frac{H^2(1-p)}{2\mu^2} \|w^k - x^*\|^4 + \frac{2(1-p)}{n\mu^2} \sum_{i=1}^n \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_F^2 \|w^k - x^*\|^2.$$

From the definition of (A_M, B_M) and Lemma C.1, by choosing $z = z^k$, $x = x^{k+1}$, and $y = x^*$ in Lemma C.1, we can obtain

$$\mathbb{E}_k \|z^{k+1} - x^*\|^2 = \mathbb{E}_k \|z^k + \eta \mathcal{Q}^k (x^{k+1} - z^k) - x^*\|^2$$

$$\leq (1 - A_M)\|z^k - x^*\|^2 + B_M \mathbb{E}_k \|x^{k+1} - x^*\|^2.$$

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Combining the above two inequalities, we arrive at

$$\mathbb{E}_k \|z^{k+1} - x^*\|^2 \leq (1 - A_\mathrm{M})\|z^k - x^*\|^2 + \frac{B_\mathrm{M}p}{\mu^2}\left(\frac{H^2}{2}\|z^k - x^*\|^2 + \frac{2}{n}\sum_{i=1}^{n}\|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_\mathrm{F}^2\right)\|z^k - x^*\|^2$$

$$+ \frac{B_\mathrm{M}(1-p)}{\mu^2}\left(\frac{H^2}{2}\|w^k - x^*\|^2 + \frac{2}{n}\sum_{i=1}^{n}\|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_\mathrm{F}^2\right)\|w^k - x^*\|^2.$$

From the update rule of $\mathbf{H}_i^k$, we know $\mathbf{H}_i^k = \sum_{jl}(\mathbf{L}_i^k)_{jl}\mathbf{B}_i^{jl}$. Denote $\mathbf{L}_i^* = h^i(\nabla^2 f_i(x^*))$. Then we have

$$\|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_\mathrm{F}^2 = \left\|\sum_{jl}(\mathbf{L}_i^k - \mathbf{L}_i^*)_{jl}\mathbf{B}_i^{jl}\right\|_\mathrm{F}^2 \leq N_\mathrm{B}\sum_{jl}\|(\mathbf{L}_i^k - \mathbf{L}_i^*)_{jl}\mathbf{B}_i^{jl}\|_\mathrm{F}^2$$

$$\leq N_\mathrm{B}R^2\|\mathbf{L}_i^k - \mathbf{L}_i^*\|_\mathrm{F}^2. \tag{26}$$

Define $\mathcal{H}^k := \frac{1}{n}\sum_{i=1}^{n}\|\mathbf{L}_i^k - \mathbf{L}_i^*\|_\mathrm{F}^2$. Then we have

$$\mathbb{E}_k \|z^{k+1} - x^*\|^2 \leq (1 - A_\mathrm{M})\|z^k - x^*\|^2 + \frac{B_\mathrm{M}p}{\mu^2}\left(\frac{H^2}{2}\|z^k - x^*\|^2 + 2N_\mathrm{B}R^2\mathcal{H}^k\right)\|z^k - x^*\|^2$$

$$+ \frac{B_\mathrm{M}(1-p)}{\mu^2}\left(\frac{H^2}{2}\|w^k - x^*\|^2 + 2N_\mathrm{B}R^2\mathcal{H}^k\right)\|w^k - x^*\|^2.$$

Assume $\|z^k - x^*\|^2 \leq \frac{A_\mathrm{M}\mu^2}{4H^2 B_\mathrm{M}}$ and $\mathcal{H}^k \leq \frac{A_\mathrm{M}\mu^2}{16N_\mathrm{B}R^2 B_\mathrm{M}}$ for $k \geq 0$. Then from the update rule of w^k, we know $\|w^k - x^*\|^2 \leq \frac{A_\mathrm{M}\mu^2}{4H^2 B_\mathrm{M}}$ for $k \geq 0$. Thus we have

$$\mathbb{E}_k \|z^{k+1} - x^*\|^2 \leq \left(1 - A_\mathrm{M} + \frac{A_\mathrm{M}p}{4}\right)\|z^k - x^*\|^2 + \frac{A_\mathrm{M}(1-p)}{4}\|w^k - x^*\|^2. \tag{27}$$

From the update rule of w^k, we have

$$\mathbb{E}_k \|w^{k+1} - x^*\|^2 = p\|z^k - x^*\|^2 + (1-p)\|w^k - x^*\|^2. \tag{28}$$

Define $\Phi_1^k := \|z^k - x^*\|^2 + \frac{A_\mathrm{M}(1-p)}{2p}\|w^k - x^*\|^2$. Then we can get

$$\begin{aligned}
\mathbb{E}_k[\Phi_1^{k+1}] &= \mathbb{E}_k\|z^{k+1} - x^*\|^2 + \frac{A_\mathrm{M}(1-p)}{2p}\mathbb{E}_k\|w^{k+1} - x^*\|^2 \\
&\overset{(27)}{\leq} \left(1 - A_\mathrm{M} + \frac{A_\mathrm{M}p}{4}\right)\|z^k - x^*\|^2 + \frac{A_\mathrm{M}(1-p)}{4}\|w^k - x^*\|^2 + \frac{A_\mathrm{M}(1-p)}{2p}\mathbb{E}_k\|w^{k+1} - x^*\|^2 \\
&\overset{(28)}{\leq} \left(1 - \frac{A_\mathrm{M}}{2}\right)\|z^k - x^*\|^2 + \left(1 - \frac{p}{2}\right)\frac{A_\mathrm{M}(1-p)}{2p}\|w^k - x^*\|^2 \\
&\leq \left(1 - \frac{\min\{A_\mathrm{M}, p\}}{2}\right)\Phi_1^k.
\end{aligned}$$

By applying the tower property, we have

$$\mathbb{E}[\Phi_1^{k+1}] \leq \left(1 - \frac{\min\{A_\mathrm{M}, p\}}{2}\right)\mathbb{E}[\Phi_1^k].$$

Unrolling the recursion, we can get the result.

C.4 Proof of Theorem 4.10

Since $\xi^k \equiv 1$, $\eta = 1$, and $\mathcal{Q}^k(x) \equiv x$ for any $x \in \mathbb{R}^d$, it is easy to see that $z^k \equiv x^k$ for $k \geq 0$. In this case, we can view $\mathcal{Q}^k$ as an unbiased compressor with $\omega_M = 0$ or a contraction compressor with $\delta_M = 1$. Then from (27), we have

$$\mathbb{E}_k\|x^{k+1} - x^*\|^2 \leq \left(1 - \frac{A_M}{2}\right)\|x^k - x^*\|^2.$$

From Lemma C.2, we can obtain

$$\mathbb{E}_k[\mathcal{H}^{k+1}] \leq (1 - A)\mathcal{H}^k + BM_1^2\|x^k - x^*\|^2.$$

Thus,

$$\begin{aligned}
\mathbb{E}_k[\Phi_2^{k+1}] &= \mathbb{E}_k[\mathcal{H}^{k+1}] + \frac{4BM_1^2}{A_M}\mathbb{E}_k\|x^{k+1} - x^*\|^2 \\
&\leq (1 - A)\mathcal{H}^k + BM_1^2\|x^k - x^*\|^2 + \frac{4BM_1^2}{A_M}\left(1 - \frac{A_M}{2}\right)\|x^k - x^*\|^2 \\
&\leq \left(1 - \frac{\min\{4A, A_M\}}{4}\right)\Phi_2^k.
\end{aligned}$$

By applying the tower property, we have $\mathbb{E}[\Phi_2^{k+1}] \leq \theta_1\mathbb{E}[\Phi_2^k]$. Unrolling the recursion, we have $\mathbb{E}[\Phi_2^k] \leq \theta_1^k\Phi_2^0$.

Then we further have $\mathbb{E}[\mathcal{H}^k] \leq \theta_1^k\Phi_2^0$ and $\mathbb{E}\|x^k - x^*\|^2 \leq \frac{A_M}{4BM_1^2}\theta_1^k\Phi_2^0$. From $z^k \equiv x^k$, (23), and (24), we can get

$$\|x^{k+1} - x^*\|^2 \leq \frac{1}{\mu^2}\left(\frac{H^2}{2}\|x^k - x^*\|^2 + 2N_B R^2\mathcal{H}^k\right)\|x^k - x^*\|^2.$$

Assume $x^k \neq x^*$ for all $k \geq 0$. Then we have

$$\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2} \leq \frac{1}{\mu^2}\left(\frac{H^2}{2}\|x^k - x^*\|^2 + 2N_B R^2\mathcal{H}^k\right),$$

and by taking expectation, we arrive at

$$\begin{aligned}
\mathbb{E}\left[\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2}\right] &\leq \frac{H^2}{2\mu^2}\mathbb{E}\|x^k - x^*\|^2 + \frac{2N_B R^2}{\mu^2}\mathbb{E}[\mathcal{H}^k] \\
&\leq \theta_1^k\left(\frac{A_M H^2}{8BM_1^2\mu^2} + \frac{2N_B R^2}{\mu^2}\right)\Phi_2^0.
\end{aligned}$$

C.5 Proof of Theorem 4.11

(i) Noticed that under Assumption 4.3, we have $A_M = B_M = \eta$. We prove this by mathematical induction. First, since $z^0 = x^0$, we know $\|z^0 - x^*\|^2 \leq \min\{\frac{\mu^2}{4dH^2}, \frac{\mu^2}{16d^3 N_B R^2 M_2^2}\}$. Then from Lemma C.3 (iii), we have $\mathcal{H}^0 \leq \frac{\mu^2}{16dN_B R^2}$. Next, assume

$$\|z^k - x^*\|^2 \leq \min\left\{\frac{\mu^2}{4dH^2}, \frac{\mu^2}{16d^3 N_B R^2 M_2^2}\right\} \quad \text{and} \quad \mathcal{H}^k \leq \frac{\mu^2}{16dN_B R^2},$$

for $k \leq K$. By choosing $M = \frac{\mu^2}{16d^3 N_{\mathrm{B}} R^2 M_2^2}$ in Lemma C.3 (i), we have $\|z^{K+1} - x^*\|^2 \leq \min\left\{\frac{\mu^2}{4dH^2}, \frac{\mu^2}{16d^3 N_{\mathrm{B}} R^2 M_2^2}\right\}$. By further using Lemma C.3 (iii), we can get $\mathcal{H}^{K+1} \leq \frac{\mu^2}{16d N_{\mathrm{B}} R^2}$.

(ii) We prove the result by induction. Assume $\|z^k - x^*\|^2 \leq \min\left\{\frac{A_{\mathrm{M}}\mu^2}{4H^2 B_{\mathrm{M}}}, \frac{A A_{\mathrm{M}}\mu^2}{16 N_{\mathrm{B}} R^2 B_{\mathrm{M}} B M_1^2}\right\}$ and $\mathcal{H}^k \leq \frac{A_{\mathrm{M}}\mu^2}{16 N_{\mathrm{B}} R^2 B_{\mathrm{M}}}$ for $k \leq K$. Then by Lemma C.3 (iv), we have $\mathcal{H}^{K+1} \leq \frac{A_{\mathrm{M}}\mu^2}{16 N_{\mathrm{B}} R^2 B_{\mathrm{M}}}$. Moreover, by Lemma C.3 (ii), we have $\|z^{K+1} - x^*\|^2 \leq \min\left\{\frac{A_{\mathrm{M}}\mu^2}{4H^2 B_{\mathrm{M}}}, \frac{A A_{\mathrm{M}}\mu^2}{16 N_{\mathrm{B}} R^2 B_{\mathrm{M}} B M_1^2}\right\}$.

D   Proofs for BL2

We denote $\mathbb{E}_k[\cdot]$ as the conditonal expectation on z_i^k, w_i^k, l_i^k, $\mathbf{L}_i^k$, and $\mathbf{H}_i^k$.

D.1   A lemma

Lemma D.1. *We consider four cases:*

(i) *If Assumption 4.3 (ii) holds, $\|x^0 - x^*\|^2 \le \min\{\frac{\mu^2}{d^2(6H^2 + 24H_1^2)}, \frac{M}{d}\}$, $\|z_i^k - x^*\|^2 \le \min\{\frac{\mu^2}{d(6H^2 + 24H_1^2)}, M\}$, and $\mathcal{H}^k \le \frac{\mu^2}{24dN_{\mathrm{B}}R^2}$ for $k \le K$, $i \in [n]$, and any $M > 0$, then $\|z_i^{K+1} - x^*\|^2 \le \min\{\frac{\mu^2}{d(6H^2 + 24H_1^2)}, M\}$ for $i \in [n]$.*

(ii) *If Assumption 4.4 holds, $\mathcal{H}^K \le \frac{A_{\mathrm{M}}\mu^2}{24N_{\mathrm{B}}R^2 B_{\mathrm{M}}}$, $\|z_i^k - x^*\|^2 \le \min\{\frac{A_{\mathrm{M}}\mu^2}{B_{\mathrm{M}}(6H^2 + 24H_1^2)}, M\}$ for $k \le K$, $i \in [n]$, and any $M > 0$, then $\|z_i^{K+1} - x^*\|^2 \le \min\{\frac{A_{\mathrm{M}}\mu^2}{B_{\mathrm{M}}(6H^2 + 24H_1^2)}, M\}$ for $i \in [n]$.*

(iii) *If Assumption 4.5(ii) holds, and $\|z_i^k - x^*\|^2 \le \frac{M}{d^2 M_2^2}$ for $k \le K$, $i \in [n]$, and any $M > 0$, then $\mathcal{H}^K \le M$.*

(iv) *If Assumption 4.6 holds, $\|\mathbf{L}_i^K - \mathbf{L}_i^*\|_{\mathrm{F}}^2 \le M$, and $\|z_i^K - x^*\|^2 \le \frac{AM}{BM_1^2}$ for $i \in [n]$ and any $M > 0$, then $\|\mathbf{L}_i^{K+1} - \mathbf{L}_i^*\|_{\mathrm{F}}^2 \le M$ for $i \in [n]$.*

Proof. (i) First, from the update rule of w_i^k, we know $\mathcal{Z}^k \le \min\{\frac{\mu^2}{d(6H^2 + 24H_1^2)}, M\}$ and $\mathcal{W}^k \le \min\{\frac{\mu^2}{d(6H^2 + 24H_1^2)}, M\}$ for $k \le K$. Then from (30), we have

$$\|x^{k+1} - x^*\|^2 \le \frac{3H^2}{4\mu^2}(\mathcal{W}^k)^2 + \frac{12N_{\mathrm{B}}R^2}{\mu^2}\mathcal{H}^k\mathcal{W}^k + \frac{3H_1^2}{\mu^2}\mathcal{Z}^k\mathcal{W}^k$$
$$\le \frac{1}{d}\mathcal{W}^k$$
$$\le \min\left\{\frac{\mu^2}{d^2(6H^2 + 24H_1^2)}, \frac{M}{d}\right\},$$

for $0 \le k \le K$.
Since $\|x^0 - x^*\|^2 \le \min\{\frac{\mu^2}{d^2(6H^2 + 24H_1^2)}, \frac{M}{d}\}$, we know $\|x^k - x^*\|^2 \le \min\{\frac{\mu^2}{d^2(6H^2 + 24H_1^2)}, \frac{M}{d}\}$ for all $0 \le k \le K + 1$. Then for $i \in S^k$, from Assumption 4.3 (ii), we can get

$$\|z_i^{K+1} - x^*\|^2 \le d\max_j |(z_i^{K+1})_j - x_j^*|^2$$
$$\le d\max_{0 \le t \le K+1} \|x^t - x^*\|^2$$
$$\le \min\left\{\frac{\mu^2}{d(6H^2 + 24H_1^2)}, M\right\}.$$

For $i \notin S^k$, we have

$$\|z_i^{K+1} - x^*\|^2 = \|z_i^K - x^*\|^2 \le \min\left\{\frac{\mu^2}{d(6H^2 + 24H_1^2)}, M\right\}.$$

(ii) First, from the update rule of w^k, we know $\mathcal{Z}^k \leq \min\{\frac{A_\mathrm{M}\mu^2}{B_\mathrm{M}(6H^2+24H_1^2)}, M\}$ and $\mathcal{W}^k \leq \min\{\frac{A_\mathrm{M}\mu^2}{B_\mathrm{M}(6H^2+24H_1^2)}, M\}$ for $k \leq K$. Then from (30), we have

$$
\begin{aligned}
\|x^{K+1} - x^*\|^2 &\leq \frac{3H^2}{4\mu^2}(\mathcal{W}^K)^2 + \frac{12N_\mathrm{B}R^2}{\mu^2}\mathcal{H}^K\mathcal{W}^K + \frac{3H_1^2}{\mu^2}\mathcal{Z}^K\mathcal{W}^K \\
&\leq \frac{A_\mathrm{M}}{B_\mathrm{M}}\mathcal{W}^K \\
&\leq \frac{A_\mathrm{M}}{B_\mathrm{M}} \min\left\{\frac{A_\mathrm{M}\mu^2}{B_\mathrm{M}(6H^2+24H_1^2)}, M\right\}.
\end{aligned}
$$

Then for $i \in S^k$, from Lemma C.1 (ii), we arrive at

$$
\begin{aligned}
\|z_i^{K+1} - z^*\|^2 &\leq (1 - A_\mathrm{M})\|z_i^K - x^*\|^2 + B_\mathrm{M}\|x^{K+1} - x^*\|^2 \\
&\leq (1 - A_\mathrm{M}) \min\left\{\frac{A_\mathrm{M}\mu^2}{B_\mathrm{M}(6H^2+24H_1^2)}, M\right\} + A_\mathrm{M} \min\left\{\frac{A_\mathrm{M}\mu^2}{B_\mathrm{M}(6H^2+24H_1^2)}, M\right\} \\
&= \min\left\{\frac{A_\mathrm{M}\mu^2}{B_\mathrm{M}(6H^2+24H_1^2)}, M\right\}.
\end{aligned}
$$

For $i \notin S^k$, we have

$$
\|z_i^{K+1} - x^*\|^2 = \|z_i^K - x^*\|^2 \leq \min\left\{\frac{A_\mathrm{M}\mu^2}{B_\mathrm{M}(6H^2+24H_1^2)}, M\right\}.
$$

(iii) From Assumption 4.5(ii), we have

$$
\begin{aligned}
\mathcal{H}^K &= \frac{1}{n}\sum_{i=1}^{n}\|\mathbf{L}_i^K - \mathbf{L}_i^*\|_\mathrm{F}^2 \\
&\leq \frac{1}{n}\sum_{i=1}^{n} d^2 \max_{jl}\{|(\mathbf{L}_i^K)_{jl} - (\mathbf{L}_i^*)_{jl}|^2\} \\
&\leq d^2 M_2^2 \max_{i\in[n], 0\leq t\leq K}\|z_i^t - x^*\|^2 \\
&\leq M.
\end{aligned}
$$

(iv) For $i \in S^k$, from Assumption 4.6 and Lemma C.2 (ii), we have

$$
\begin{aligned}
\|\mathbf{L}_i^{K+1} - \mathbf{L}_i^*\|_\mathrm{F}^2 &\leq (1 - A)\|\mathbf{L}_i^K - \mathbf{L}_i^*\|_\mathrm{F}^2 + BM_1^2\|z_i^K - x^*\|^2 \\
&\leq (1 - A)M + AM \\
&= M.
\end{aligned}
$$

For $i \notin S^k$, we also have

$$
\|\mathbf{L}_i^{K+1} - \mathbf{L}_i^*\|_\mathrm{F}^2 = \|\mathbf{L}_i^K - \mathbf{L}_i^*\|_\mathrm{F}^2 \leq M.
$$

$\square$

D.2 Proof of Theorem 4.12

First, from $[\mathbf{H}_i^k]_s + l_i^k \mathbf{I} \succeq \nabla^2 f_i(z_i^k) \succeq \mu\mathbf{I}$, we know $[\mathbf{H}^k]_s + l^k\mathbf{I} = \frac{1}{n}\sum_{i=1}^n([\mathbf{H}_i^k]_s + l_i^k\mathbf{I}) \succeq \mu\mathbf{I}$. Then we have

$$
\begin{aligned}
\|x^{k+1} - x^*\| &= \left\| \left([\mathbf{H}^k]_s + l^k\mathbf{I}\right)^{-1} \left(g^k - \left([\mathbf{H}^k]_s + l^k\mathbf{I}\right)x^* + \nabla f(x^*) \right) \right\| \\
&\leq \frac{1}{\mu} \left\| \frac{1}{n}\sum_{i=1}^n g_i^k - \frac{1}{n}\sum_{i=1}^n([\mathbf{H}_i^k]_s + l_i^k\mathbf{I})x^* + \frac{1}{n}\sum_{i=1}^n \nabla f_i(x^*) \right\| \\
&\leq \frac{1}{n\mu}\sum_{i=1}^n \|g_i^k - ([\mathbf{H}_i^k]_s + l_i^k)x^* + \nabla f_i(x^*)\| \\
&= \frac{1}{n\mu}\sum_{i=1}^n \|([\mathbf{H}_i^k]_s + l_i^k\mathbf{I})w_i^k - \nabla f_i(w_i^k) - ([\mathbf{H}_i^k]_s + l_i^k\mathbf{I})x^* + \nabla f_i(x^*)\| \\
&\leq \frac{1}{n\mu}\sum_{i=1}^n \|\nabla f_i(w_i^k) - \nabla f_i(x^*) - \nabla^2 f_i(x^*)(w_i^k - x^*)\| \\
&\quad + \frac{1}{n\mu}\sum_{i=1}^n \|([\mathbf{H}_i^k]_s + l_i^k\mathbf{I} - \nabla^2 f_i(x^*))(w_i^k - x^*)\|,
\end{aligned}
$$

where we use $\nabla f(x^*) = 0$ in the first equality and $g_i^k = ([\mathbf{H}_i^k]_s + l_i^k\mathbf{I})w_i^k - \nabla f_i(w_i^k)$ in the second equality. Since $\|\nabla^2 f_i(x) - \nabla^2 f_i(y)\| \leq H\|x - y\|$ for any $x, y \in \mathbb{R}^d$, we further have

$$
\begin{aligned}
\|x^{k+1} - x^*\| &\leq \frac{H}{2n\mu}\sum_{i=1}^n \|w_i^k - x^*\|^2 + \frac{1}{n\mu}\sum_{i=1}^n \|([\mathbf{H}_i^k]_s + l_i^k\mathbf{I} - \nabla^2 f_i(x^*))\| \cdot (w_i^k - x^*)\| \\
&\leq \frac{H}{2\mu}\mathcal{W}^k + \frac{1}{n\mu}\sum_{i=1}^n \left(\|[\mathbf{H}_i^k]_s - \nabla^2 f_i(x^*)\|_{\mathrm{F}} + l_i^k \right)\|w_i^k - x^*\|,
\end{aligned}
$$

where $\mathcal{W}^k := \frac{1}{n}\sum_{i=1}^n \|w_i^k - x^*\|^2$.

Since $\|\nabla^2 f_i(x) - \nabla^2 f_i(y)\|_{\mathrm{F}} \leq H_1\|x - y\|$ for any $x, y \in \mathbb{R}^d$, we can get

$$
\begin{aligned}
l_i^k &= \|[\mathbf{H}_i^k]_s - \nabla^2 f_i(z_i^k)\|_{\mathrm{F}} \\
&\leq \|[\mathbf{H}_i^k]_s - \nabla^2 f_i(x^*)\|_{\mathrm{F}} + \|\nabla^2 f_i(z_i^k) - \nabla^2 f_i(x^*)\|_{\mathrm{F}} \\
&\leq \|[\mathbf{H}_i^k]_s - \nabla^2 f_i(x^*)\|_{\mathrm{F}} + H_1\|z_i^k - x^*\|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|x^{k+1} - x^*\| &\leq \frac{H}{2\mu}\mathcal{W}^k + \frac{1}{n\mu}\sum_{i=1}^n \left(\|[\mathbf{H}_i^k]_s - \nabla^2 f_i(x^*)\| + \|[\mathbf{H}_i^k]_s - \nabla^2 f_i(x^*)\|_{\mathrm{F}} + H_1\|z_i^k - x^*\| \right)\|w_i^k - x^*\| \\
&\leq \frac{H}{2\mu}\mathcal{W}^k + \frac{2}{n\mu}\sum_{i=1}^n \|[\mathbf{H}_i^k]_s - \nabla^2 f_i(x^*)\|_{\mathrm{F}}\|w_i^k - x^*\| + \frac{H_1}{n\mu}\sum_{i=1}^n \|z_i^k - x^*\|\|w_i^k - x^*\| \\
&\overset{Lemma\ 3.1}{\leq} \frac{H}{2\mu}\mathcal{W}^k + \frac{2}{n\mu}\sum_{i=1}^n \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}\|w_i^k - x^*\| + \frac{H_1}{n\mu}\sum_{i=1}^n \|z_i^k - x^*\|\|w_i^k - x^*\| \\
&\leq \frac{H}{2\mu}\mathcal{W}^k + \frac{2}{n\mu}\left(\sum_{i=1}^n \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 \right)^{\frac{1}{2}}\left(n\mathcal{W}^k \right)^{\frac{1}{2}} + \frac{H_1}{n\mu}\left(n\mathcal{Z}^k \right)^{\frac{1}{2}}\left(n\mathcal{W}^k \right)^{\frac{1}{2}},
\end{aligned}
$$

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where we use the Cauchy-Schwarz inequality in the last inequality and $\mathcal{Z}^k := \frac{1}{n} \sum_{i=1}^{n} \|z_i^k - x^*\|^2$. Since $\mathbf{H}_i^k = \sum_{jl} (\mathbf{L}_i^k)_{jl} \mathbf{B}_i^{jl}$, same as (26), we have

$$\|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 \leq N_{\mathrm{B}} R^2 \|\mathbf{L}_i^k - \mathbf{L}_i^*\|_{\mathrm{F}}^2, \tag{29}$$

where $\mathbf{L}_i^* = h^i(\nabla^2 f_i(x^*))$ and N_{B} is defined in 10. Then from the convexity of $\|\cdot\|^2$, we further bound $\|x^{k+1} - x^*\|^2$ as

$$\|x^{k+1} - x^*\|^2 \leq \frac{3H^2}{4\mu^2} (\mathcal{W}^k)^2 + \frac{12\mathcal{W}^k}{n\mu^2} \sum_{i=1}^{n} \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|_{\mathrm{F}}^2 + \frac{3H_1^2}{\mu^2} \mathcal{Z}^k \mathcal{W}^k$$

$$\overset{(29)}{\leq} \frac{3H^2}{4\mu^2} (\mathcal{W}^k)^2 + \frac{12 N_{\mathrm{B}} R^2 \mathcal{W}^k}{n\mu^2} \sum_{i=1}^{n} \|\mathbf{L}_i^k - \mathbf{L}_i^*\|_{\mathrm{F}}^2 + \frac{3H_1^2}{\mu^2} \mathcal{Z}^k \mathcal{W}^k$$

$$= \frac{3H^2}{4\mu^2} (\mathcal{W}^k)^2 + \frac{12 N_{\mathrm{B}} R^2}{\mu^2} \mathcal{H}^k \mathcal{W}^k + \frac{3H_1^2}{\mu^2} \mathcal{Z}^k \mathcal{W}^k, \tag{30}$$

where $\mathcal{H}^k = \frac{1}{n} \sum_{i=1}^{n} \|\mathbf{L}_i^k - \mathbf{L}_i^*\|_{\mathrm{F}}^2$.

For $i \in S^k$, we have $z_i^{k+1} = z_i^k + \eta \mathcal{Q}_i^k (x^{k+1} - z_i^k)$. Then from the definition of $(A_{\mathrm{M}}, B_{\mathrm{M}})$ and Lemma C.1, by choosing $z = z_i^k$, $x = x^{k+1}$, and $y = x^*$ in Lemma C.1, we can obtain

$$\mathbb{E}_k[\|z_i^{k+1} - x^*\|^2 \mid i \in S^k] = \mathbb{E}_k[\|z_i^k + \eta \mathcal{Q}^k (x^{k+1} - z_i^k) - x^*\|^2 \mid i \in S^k]$$

$$\leq (1 - A_{\mathrm{M}}) \|z_i^k - x^*\|^2 + B_{\mathrm{M}} \mathbb{E}_k \|x^{k+1} - x^*\|^2$$

$$= (1 - A_{\mathrm{M}}) \|z_i^k - x^*\|^2 + B_{\mathrm{M}} \|x^{k+1} - x^*\|^2.$$

Noticing that $\mathbb{P}[i \in S^k] = \tau/n$ and $z_i^{k+1} = z_i^k$ for $i \notin S^k$, we further have

$$\mathbb{E}_k \|z_i^{k+1} - x^*\|^2 = \frac{\tau}{n} \mathbb{E}_k[\|z_i^{k+1} - x^*\|^2 \mid i \in S^k] + \left(1 - \frac{\tau}{n}\right) \mathbb{E}_k[\|z_i^{k+1} - x^*\|^2 \mid i \notin S^k]$$

$$\leq \frac{\tau}{n} (1 - A_{\mathrm{M}}) \|z_i^k - x^*\|^2 + \frac{\tau B_{\mathrm{M}}}{n} \|x^{k+1} - x^*\|^2 + \left(1 - \frac{\tau}{n}\right) \|z_i^k - x^*\|^2$$

$$= \left(1 - \frac{\tau A_{\mathrm{M}}}{n}\right) \|z_i^k - x^*\|^2 + \frac{\tau B_{\mathrm{M}}}{n} \|x^{k+1} - x^*\|^2,$$

which implies that

$$\mathbb{E}_k[\mathcal{Z}^{k+1}] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_k \|z_i^{k+1} - x^*\|^2$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} \left(1 - \frac{\tau A_{\mathrm{M}}}{n}\right) \|z_i^k - x^*\|^2 + \frac{\tau B_{\mathrm{M}}}{n} \|x^{k+1} - x^*\|^2$$

$$= \left(1 - \frac{\tau A_{\mathrm{M}}}{n}\right) \mathcal{Z}^k + \frac{\tau B_{\mathrm{M}}}{n} \|x^{k+1} - x^*\|^2. \tag{31}$$

For $i \in S^k$, from the update rule of w_i^{k+1}, we have

$$\mathbb{E}_k[\|w_i^{k+1} - x^*\|^2 \mid i \in S^k] = p \mathbb{E}_k[\|z_i^{k+1} - x^*\|^2] + (1 - p)\|w_i^k - x^*\|^2.$$

For $i \notin S^k$, we have $w_i^{k+1} = w_i^k$. Thus,

$$
\begin{aligned}
\mathbb{E}_k \|w_i^{k+1} - x^*\|^2 &= \frac{\tau}{n} \mathbb{E}_k[\|w_i^{k+1} - x^*\|^2 \mid i \in S^k] + \left(1 - \frac{\tau}{n}\right) \mathbb{E}_k[\|w_i^{k+1} - x^*\|^2 \mid i \notin S^k] \\
&= \left(1 - \frac{\tau p}{n}\right) \|w_i^k - x^*\|^2 + \frac{\tau p}{n} \mathbb{E}_k \|z_i^{k+1} - x^*\|^2,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
\mathbb{E}_k[\mathcal{W}^{k+1}] &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}_k \|w_i^{k+1} - x^*\|^2 \\
&= \frac{1}{n} \sum_{i=1}^n \left(1 - \frac{\tau p}{n}\right) \|w_i^k - x^*\|^2 + \frac{1}{n} \sum_{i=1}^n \frac{\tau p}{n} \mathbb{E}_k \|z_i^{k+1} - x^*\|^2 \\
&= \left(1 - \frac{\tau p}{n}\right) \mathcal{W}^k + \frac{\tau p}{n} \mathbb{E}_k[\mathcal{Z}^{k+1}] \\
&\overset{(31)}{\leq} \left(1 - \frac{\tau p}{n}\right) \mathcal{W}^k + \frac{\tau p}{n} \left(1 - \frac{\tau A_{\mathrm{M}}}{n}\right) \mathcal{Z}^k + \frac{\tau^2 B_{\mathrm{M}} p}{n^2} \|x^{k+1} - x^*\|^2. \qquad (32)
\end{aligned}
$$

Let $\Phi_3^k := \mathcal{W}^k + \frac{2p}{A_{\mathrm{M}}} \left(1 - \frac{\tau A_{\mathrm{M}}}{n}\right) \mathcal{Z}^k$ for $k \geq 0$. Then from the above inequality we have

$$
\begin{aligned}
\mathbb{E}_k[\Phi_3^{k+1}] &\leq \left(1 - \frac{\tau p}{n}\right) \mathcal{W}^k + \frac{\tau p}{n} \left(1 - \frac{\tau A_{\mathrm{M}}}{n}\right) \mathcal{Z}^k + \frac{\tau^2 B_{\mathrm{M}} p}{n^2} \|x^{k+1} - x^*\|^2 + \frac{2p}{A_{\mathrm{M}}} \left(1 - \frac{\tau A_{\mathrm{M}}}{n}\right) \mathbb{E}_k[\mathcal{Z}^{k+1}] \\
&\overset{(31)}{\leq} \left(1 - \frac{\tau p}{n}\right) \mathcal{W}^k + \frac{2p}{A_{\mathrm{M}}} \left(1 - \frac{\tau A_{\mathrm{M}}}{n}\right)\left(1 - \frac{\tau A_{\mathrm{M}}}{2n}\right) \mathcal{Z}^k + \frac{2\tau p B_{\mathrm{M}}}{n A_{\mathrm{M}}} \left(1 - \frac{\tau A_{\mathrm{M}}}{2n}\right) \|x^{k+1} - x^*\|^2 \\
&\overset{30}{\leq} \left(1 - \frac{\tau p}{n} + \frac{2\tau p B_{\mathrm{M}}}{n A_{\mathrm{M}}} \left(\frac{3H^2}{4\mu^2} \mathcal{W}^k + \frac{12 N_{\mathrm{B}} R^2}{\mu^2} \mathcal{H}^k + \frac{3H_1^2}{\mu^2} \mathcal{Z}^k \right)\right) \mathcal{W}^k \\
&\quad + \frac{2p}{A_{\mathrm{M}}} \left(1 - \frac{\tau A_{\mathrm{M}}}{n}\right)\left(1 - \frac{\tau A_{\mathrm{M}}}{2n}\right) \mathcal{Z}^k.
\end{aligned}
$$

If $\|z_i^k - x^*\|^2 \leq \frac{A_{\mathrm{M}} \mu^2}{(6H^2 + 24 H_1^2) B_{\mathrm{M}}}$ and $\mathcal{H}^k \leq \frac{A_{\mathrm{M}} \mu^2}{96 N_{\mathrm{B}} R^2 B_{\mathrm{M}}}$ for all $k \geq 0$, then we have

$$
\frac{3H^2}{4\mu^2} \mathcal{W}^k + \frac{12 N_{\mathrm{B}} R^2}{\mu^2} \mathcal{H}^k + \frac{3H_1^2}{\mu^2} \mathcal{Z}^k \leq \frac{A_{\mathrm{M}}}{4 B_{\mathrm{M}}},
$$

which implies that

$$
\begin{aligned}
\mathbb{E}_k[\Phi_3^{k+1}] &\leq \left(1 - \frac{\tau p}{2n}\right) \mathcal{W}^k + \frac{2p}{A_{\mathrm{M}}} \left(1 - \frac{\tau A_{\mathrm{M}}}{n}\right)\left(1 - \frac{\tau A_{\mathrm{M}}}{2n}\right) \mathcal{Z}^k \\
&\leq \left(1 - \frac{\tau \min\{p, A_{\mathrm{M}}\}}{2n}\right) \Phi_3^k.
\end{aligned}
$$

By applying the tower property, we have

$$
\mathbb{E}[\Phi_3^{k+1}] \leq \left(1 - \frac{\tau \min\{p, A_{\mathrm{M}}\}}{2n}\right) \mathbb{E}[\Phi_3^k].
$$

Unrolling the recursion, we can obtain the result.

D.3   Proof of Theorem 4.13

Since $\xi^k \equiv 1$, $\eta = 1$, $S^k \equiv [n]$, and $\mathcal{Q}_i^k(x) \equiv x$ for any $x \in \mathbb{R}^d$, it is easy to see that $w_i^k = z_i^k \equiv x^k$ for all $i \in [n]$ and $k \geq 0$. In this case, we can view $\mathcal{Q}_i^k$ as an unbiased compressor with $\omega_M = 0$ or a contraction compressor with $\delta_M = 1$. Then from (31), we have

$$\mathbb{E}_k\|x^{k+1} - x^*\|^2 \leq (1 - A_M)\|x^k - x^*\|^2 + B_M\|x^{k+1} - x^*\|^2$$

$$\overset{(30)}{\leq} (1 - A_M)\|x^k - x^*\|^2 + \frac{1}{4}A_M\|x^k - x^*\|^2$$

$$= \left(1 - \frac{3A_M}{4}\right)\|x^k - x^*\|^2.$$

From Lemma C.2, we can obtain

$$\mathbb{E}_k[\mathcal{H}^{k+1}] \leq (1 - A)\mathcal{H}^k + BM_1^2\|x^k - x^*\|^2.$$

Thus,

$$\mathbb{E}_k[\Phi_4^{k+1}] = \mathbb{E}_k[\mathcal{H}^{k+1}] + \frac{4BM_1^2}{A_M}\mathbb{E}_k\|x^{k+1} - x^*\|^2$$

$$\leq (1 - A)\mathcal{H}^k + BM_1^2\|x^k - x^*\|^2 + \frac{4BM_1^2}{A_M}\left(1 - \frac{3A_M}{4}\right)\|x^k - x^*\|^2$$

$$\leq \left(1 - \frac{\min\{2A, A_M\}}{2}\right)\Phi_4^k.$$

By applying the tower property, we have $\mathbb{E}[\Phi_4^{k+1}] \leq \theta_2\mathbb{E}[\Phi_4^k]$. Unrolling the recursion, we have $\mathbb{E}[\Phi_4^k] \leq \theta_2^k\Phi_4^0$.

Then we further have $\mathbb{E}[\mathcal{H}^k] \leq \theta_2^k\Phi_4^0$ and $\mathbb{E}\|x^k - x^*\|^2 \leq \frac{A_M}{4BM_1^2}\theta_2^k\Phi_4^0$. From $w_i^k = z_i^k \equiv x^k$ and (30), we can get

$$\|x^{k+1} - x^*\|^2 \leq \left(\frac{3H^2 + 12H_1^2}{4\mu^2}\|x^k - x^*\|^2 + \frac{12N_B R^2}{\mu^2}\mathcal{H}^k\right)\|x^k - x^*\|^2.$$

Assume $x^k \neq x^*$ for all $k \geq 0$. Then we have

$$\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2} \leq \frac{3H^2 + 12H_1^2}{4\mu^2}\|x^k - x^*\|^2 + \frac{12N_B R^2}{\mu^2}\mathcal{H}^k,$$

and by taking expectation, we arrive at

$$\mathbb{E}\left[\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2}\right] \leq \frac{3H^2 + 12H_1^2}{4\mu^2}\mathbb{E}\|x^k - x^*\|^2 + \frac{12N_B R^2}{\mu^2}\mathbb{E}[\mathcal{H}^k]$$

$$\leq \theta_2^k\left(\frac{A_M(3H^2 + 12H_1^2)}{16BM_1^2\mu^2} + \frac{12N_B R^2}{\mu^2}\right)\Phi_4^0.$$

D.4 Proof of Theorem 4.14

(i) Noticed that under Assumption 4.3, we have $A_{\mathrm{M}} = B_{\mathrm{M}} = \eta$. We prove this by mathematical induction. First, since $z_i^0 = x^0$, we know $\|z_i^0 - x^*\|^2 \leq \min\{\frac{\mu^2}{d(6H^2+24H_1^2)}, \frac{\mu^2}{96d^3 N_{\mathrm{B}} R^2 M_2^2}\}$ for $i \in [n]$. Then from Lemma D.1 (iii), we have $\mathcal{H}^0 \leq \frac{\mu^2}{96d N_{\mathrm{B}} R^2}$. Next, assume

$$\|z_i^k - x^*\|^2 \leq \min\left\{ \frac{\mu^2}{d(6H^2+24H_1^2)}, \frac{\mu^2}{96d^3 N_{\mathrm{B}} R^2 M_2^2} \right\} \text{ for } i \in [n] \quad \text{and} \quad \mathcal{H}^k \leq \frac{\mu^2}{96d N_{\mathrm{B}} R^2},$$

for $k \leq K$. By choosing $M = \frac{\mu^2}{96d^3 N_{\mathrm{B}} R^2 M_2^2}$ in Lemma D.1 (i), we have

$$\|z_i^{K+1} - x^*\|^2 \leq \min\left\{ \frac{\mu^2}{d(6H^2+24H_1^2)}, \frac{\mu^2}{96d^3 N_{\mathrm{B}} R^2 M_2^2} \right\},$$

for $i \in [n]$. By further using Lemma D.1 (iii), we can get $\mathcal{H}^{K+1} \leq \frac{\mu^2}{96d N_{\mathrm{B}} R^2}$.

(ii) We prove the result by induction. Assume $\|z_i^k - x^*\|^2 \leq \min\left\{ \frac{A_{\mathrm{M}}\mu^2}{B_{\mathrm{M}}(6H^2+24H_1^2)}, \frac{AA_{\mathrm{M}}\mu^2}{96 N_{\mathrm{B}} R^2 B_{\mathrm{M}} B M_1^2} \right\}$ and $\|\mathbf{L}_i^k - \mathbf{L}_i^*\|_{\mathrm{F}}^2 \leq \frac{A_{\mathrm{M}}\mu^2}{96 N_{\mathrm{B}} R^2 B_{\mathrm{M}}}$ for all $i \in [n]$ and $k \leq K$. Then by Lemma D.1 (iv), we have $\|\mathbf{L}_i^{K+1} - \mathbf{L}_i^*\|_{\mathrm{F}}^2 \leq \frac{A_{\mathrm{M}}\mu^2}{96 N_{\mathrm{B}} R^2 B_{\mathrm{M}}}$. Moreover, by Lemma D.1 (ii), we have $\|z_i^{K+1} - x^*\|^2 \leq \min\left\{ \frac{A_{\mathrm{M}}\mu^2}{B_{\mathrm{M}}(6H^2+24H_1^2)}, \frac{AA_{\mathrm{M}}\mu^2}{96 N_{\mathrm{B}} R^2 B_{\mathrm{M}} B M_1^2} \right\}$ for $i \in [n]$.

E   Proofs for BL3

We denote $\mathbb{E}_k[\cdot]$ as the conditonal expectation on z_i^k, w_i^k, $\mathbf{L}_i^k$, γ_i^k, β_i^k, $\mathbf{A}_i^k$ and $\mathbf{C}_i^k$.

E.1   Proof of Lemma 5.3

(i) If $\|\nabla^2 f_i(x) - \nabla^2 f_i(y)\|_{\mathrm{F}} \leq H_1 \|x - y\|$ for any $x, y \in \mathbb{R}^d$, and $i \in [n]$, then from (15) we have

$$
\begin{aligned}
\|\tilde{h}^i(\nabla^2 f_i(x)) - \tilde{h}^i(\nabla^2 f_i(y))\|_{\mathrm{F}} &\leq \|svec(\tilde{h}^i(\nabla^2 f_i(x))) - svec(\tilde{h}^i(\nabla^2 f_i(y)))\| \\
&\leq \|(\tilde{\mathcal{B}}_i)^{-1}\| \cdot \|svec(\nabla^2 f_i(x)) - svec(\nabla^2 f_i(y))\| \\
&\leq \sqrt{2}\|(\tilde{\mathcal{B}}_i)^{-1}\| \cdot \|\nabla^2 f_i(x) - \nabla^2 f_i(y)\|_{\mathrm{F}} \\
&\leq \sqrt{2}\|(\tilde{\mathcal{B}}_i)^{-1}\| H_1 \|x - y\|,
\end{aligned}
$$

which implies that M_4 in Assumption 5.2 satisfies $M_4 \leq \sqrt{2}\max_i\{\|(\tilde{\mathcal{B}}_i)^{-1}\|\}H_1$.

If $|(\nabla^2 f_i(x))_{jl} - (\nabla^2 f_i(y))_{jl}| \leq \nu \|x - y\|$ for any $x, y \in \mathbb{R}^d$, $i \in [n]$, and $j, l \in [d]$, then from (15), every entry of $\tilde{h}^i(\nabla^2 f_i(x)) - \tilde{h}^i(\nabla^2 f_i(y))$ will be bounded by $2\nu\|(\tilde{\mathcal{B}}_i)^{-1}\|_\infty \|x - y\|$. Hence M_5 in Assumption 5.2 satisfies $M_5 \leq 2\nu \max_i\{\|(\tilde{\mathcal{B}}_i)^{-1}\|_\infty\}$.

(ii) If $|(\nabla^2 f_i(x))_{jl}| \leq \gamma$ for any $x \in \mathbb{R}^d$, $i \in [n]$, and $j, l \in [d]$, then from (15), every entry of $\tilde{h}^i(\nabla^2 f_i(x))$ will be bounded by $2\gamma\|(\tilde{\mathcal{B}}_i)^{-1}\|_\infty$, i.e., $\max_{jl}\{|\tilde{h}^i(\nabla^2 f_i(x))_{jl}|\} \leq 2\gamma\|(\tilde{\mathcal{B}}_i)^{-1}\|_\infty$. In particular, under Assumption 4.5 (ii), $(\mathbf{L}_i^k)_{jl}$ is a convex combination of $\{\tilde{h}^i(\nabla^2 f_i(z_i^t))_{jl}\}_{t \leq k}$, and thus M_3 in Assumption 5.2 satisfies $M_3 \leq 2\gamma \max_i\{\|(\tilde{\mathcal{B}}_i)^{-1}\|_\infty\}$.

(iii) First, from $\|\nabla^2 f_i(x)\|_{\mathrm{F}} \leq \tilde{\gamma}$ and (15), we have

$$
\begin{aligned}
\|\tilde{h}^i(\nabla^2 f_i(x))\|_{\mathrm{F}} &\leq \|svec(\tilde{h}^i(\nabla^2 f_i(x)))\| \\
&\leq \|(\tilde{\mathcal{B}}_i)^{-1}\| \cdot \|svec(\nabla^2 f_i(x))\| \\
&\leq \sqrt{2}\|(\tilde{\mathcal{B}}_i)^{-1}\| \cdot \|\nabla^2 f_i(x)\|_{\mathrm{F}} \\
&\leq \sqrt{2}\|(\tilde{\mathcal{B}}_i)^{-1}\|\tilde{\gamma},
\end{aligned}
$$

for any $x \in \mathbb{R}^d$ and $i \in [n]$. Assume $\|\mathbf{L}_i^K\|_{\mathrm{F}}^2 \leq \frac{2B}{A}\|(\tilde{\mathcal{B}}_i)^{-1}\|^2\tilde{\gamma}^2$. Then under Assumption 4.6, same as Lemma E.1 (ii), we have

$$
\begin{aligned}
\|\mathbf{L}_i^{K+1}\|_{\mathrm{F}}^2 &= \|\mathbf{L}_i^K + \mathcal{C}_i^K\left(\tilde{h}^i(\nabla^2 f_i(z_i^{K+1})) - \mathbf{L}_i^K\right)\|_{\mathrm{F}}^2 \\
&\leq \left(1 - \frac{\delta}{4}\right)\|\mathbf{L}_i^K\|_{\mathrm{F}}^2 + \left(\frac{6}{\delta} - \frac{7}{2}\right)\|\tilde{h}^i(\nabla^2 f_i(z_i^{K+1}))\|_{\mathrm{F}}^2 \\
&= (1 - A)\|\mathbf{L}_i^K\|_{\mathrm{F}}^2 + B\|\tilde{h}^i(\nabla^2 f_i(z_i^{K+1}))\|_{\mathrm{F}}^2 \\
&\leq (1 - A)\cdot\frac{2B}{A}\|(\tilde{\mathcal{B}}_i)^{-1}\|^2\tilde{\gamma}^2 + B\cdot 2\|(\tilde{\mathcal{B}}_i)^{-1}\|^2\tilde{\gamma}^2 \\
&= \frac{2B}{A}\|(\tilde{\mathcal{B}}_i)^{-1}\|^2\tilde{\gamma}^2.
\end{aligned}
$$

Since $\|\mathbf{L}_i^0\|_{\mathrm{F}}^2 \leq \frac{2B}{A}\|(\tilde{\mathcal{B}}_i)^{-1}\|^2\tilde{\gamma}^2$, by mathematical induction, we can get $\|\mathbf{L}_i^k\|_{\mathrm{F}}^2 \leq \frac{2B}{A}\|(\tilde{\mathcal{B}}_i)^{-1}\|^2\tilde{\gamma}^2$ for $k \geq 0$. At last, from $\max_{jl}\{|(\mathbf{L}_i^k)_{jl}|\} \leq \|\mathbf{L}_i^k\|_{\mathrm{F}} \leq \frac{\sqrt{2B}}{\sqrt{A}}\|(\tilde{\mathcal{B}}_i)^{-1}\|\tilde{\gamma}$, we can obtain $M_3 \leq \frac{\sqrt{2B}\tilde{\gamma}}{\sqrt{A}}\max_i\{\|(\tilde{\mathcal{B}}_i)^{-1}\|\}$.

E.2 Lemmas

The proof of Lemma E.1 is the same as that of Lemma B.1 in [Safaryan et al., 2021]. Hence we omit it.

Lemma E.1. *Let $\mathcal{C}$ be a compressor and $\alpha > 0$. For any matrix $\mathbf{L} \in \mathbb{R}^{d \times d}$ and $y, z \in \mathbb{R}^d$, we have the following results.*

(i) If $\mathcal{C}$ is an unbiased compressor with parameter ω and $\alpha \leq \frac{1}{\omega+1}$, then

$$\mathbb{E}\|\mathbf{L} + \alpha\mathcal{C}(\tilde{h}^i(\nabla^2 f_i(y)) - \mathbf{L}) - \tilde{h}^i(\nabla^2 f_i(z))\|_{\mathrm{F}}^2 \leq (1-\alpha)\|\mathbf{L} - \tilde{h}^i(\nabla^2 f_i(z))\|_{\mathrm{F}}^2 + \alpha M_4^2\|y-z\|^2,$$

where $\mathbb{E}[\cdot]$ is the expectation with respect to $\mathcal{C}$.

(ii) If $\mathcal{C}$ is a contraction compressor with parameter δ and $\alpha = 1$, then

$$\mathbb{E}\|\mathbf{L} + \alpha\mathcal{C}(\tilde{h}^i(\nabla^2 f_i(y)) - \mathbf{L}) - \tilde{h}^i(\nabla^2 f_i(z))\|_{\mathrm{F}}^2 \leq \left(1 - \frac{\delta}{4}\right)\|\mathbf{L} - \tilde{h}^i(\nabla^2 f_i(z))\|_{\mathrm{F}}^2 + \left(\frac{6}{\delta} - \frac{7}{2}\right)M_4^2\|y-z\|^2.$$

The constants c_1 and c_2 in the following lemma are defined in (33).

Lemma E.2. *We consider four cases:*

(i) If Assumption 4.3 (ii) holds, $\|x^0 - x^*\|^2 \leq \min\{\frac{\mu^2}{4d^2(H^2+4c_1)}, \frac{M}{d}\}$, $\|z_i^k - x^*\|^2 \leq \min\{\frac{\mu^2}{4d(H^2+4c_1)}, M\}$, and $\mathcal{H}^k \leq \frac{\mu^2}{4dc_2}$ for $k \leq K$, $i \in [n]$, and any $M > 0$, then $\|z_i^{K+1} - x^*\|^2 \leq \min\{\frac{\mu^2}{4d(H^2+4c_1)}, M\}$ for $i \in [n]$.

(ii) If Assumption 4.4 holds, $\mathcal{H}^K \leq \frac{A_{\mathrm{M}}\mu^2}{4c_2 B_{\mathrm{M}}}$, $\|z_i^k - x^*\|^2 \leq \min\{\frac{A_{\mathrm{M}}\mu^2}{4B_{\mathrm{M}}(H^2+4c_1)}, M\}$ for $k \leq K$, $i \in [n]$, and any $M > 0$, then $\|z_i^{K+1} - x^*\|^2 \leq \min\{\frac{A_{\mathrm{M}}\mu^2}{4B_{\mathrm{M}}(H^2+4c_1)}, M\}$ for $i \in [n]$.

(iii) If Assumption 4.5(ii) holds, and $\|z_i^k - x^*\|^2 \leq \frac{M}{d^2 M_5^2}$ for $k \leq K$, $i \in [n]$, and any $M > 0$, then $\mathcal{H}^K \leq M$.

(iv) If Assumption 4.6 holds, $\|\mathbf{L}_i^K - \mathbf{L}_i^*\|_{\mathrm{F}}^2 \leq M$, and $\|z_i^K - x^*\|^2 \leq \frac{AM}{BM_4^2}$ for $i \in [n]$ and any $M > 0$, then $\|\mathbf{L}_i^{K+1} - \mathbf{L}_i^*\|_{\mathrm{F}}^2 \leq M$ for $i \in [n]$.

Proof. (i) First, from the update rule of w_i^k, we know $\mathcal{Z}^k \leq \min\{\frac{\mu^2}{4d(H^2+4c_1)}, M\}$ and $\mathcal{W}^k \leq \min\{\frac{\mu^2}{4d(H^2+4c_1)}, M\}$ for $k \leq K$. Then for Option 1, from (34), we have

$$\|x^{k+1} - x^*\|^2 \leq \frac{H^2}{2\mu^2}(\mathcal{W}^k)^2 + \frac{2c_1}{\mu^2}\mathcal{Z}^{k-1}\mathcal{W}^k + \frac{2c_2}{\mu^2}\mathcal{H}^k\mathcal{W}^k$$
$$\leq \frac{1}{d}\mathcal{W}^k$$
$$\leq \min\left\{\frac{\mu^2}{4d^2(H^2+4c_1)}, \frac{M}{d}\right\},$$

for $0 \leq k \leq K$. For Option 2, we can get the same bound for $\|x^{k+1} - x^*\|^2$ as above from (35). Since $\|x^0 - x^*\|^2 \leq \min\{\frac{\mu^2}{4d^2(H^2+4c_1)}, \frac{M}{d}\}$, we know $\|x^k - x^*\|^2 \leq \min\{\frac{\mu^2}{4d^2(H^2+4c_1)}, \frac{M}{d}\}$ for all $0 \leq k \leq K+1$. Then for $i \in S^k$, from Assumption 4.3 (ii), we can get

$$\|z_i^{K+1} - x^*\|^2 \leq d \max_j |(z_i^{K+1})_j - x_j^*|^2$$

$$\leq d \max_{0 \leq t \leq K+1} \|x^t - x^*\|^2$$

$$\leq \min\left\{\frac{\mu^2}{4d(H^2+4c_1)}, M\right\}.$$

For $i \notin S^k$, we have

$$\|z_i^{K+1} - x^*\|^2 = \|z_i^K - x^*\|^2 \leq \min\left\{\frac{\mu^2}{4d(H^2+4c_1)}, M\right\}.$$

(ii) First, from the update rule of w^k, we know $\mathcal{Z}^k \leq \min\{\frac{A_{\mathrm{M}}\mu^2}{4B_{\mathrm{M}}(H^2+4c_1)}, M\}$ and $\mathcal{W}^k \leq \min\{\frac{A_{\mathrm{M}}\mu^2}{4B_{\mathrm{M}}(H^2+4c_1)}, M\}$ for $k \leq K$. Then for Option 1, from (34), we have

$$\|x^{K+1} - x^*\|^2 \leq \frac{H^2}{2\mu^2}(\mathcal{W}^K)^2 + \frac{2c_1}{\mu^2}\mathcal{Z}^{K-1}\mathcal{W}^K + \frac{2c_2}{\mu^2}\mathcal{H}^K\mathcal{W}^K$$

$$\leq \frac{A_{\mathrm{M}}}{B_{\mathrm{M}}}\mathcal{W}^K$$

$$\leq \frac{A_{\mathrm{M}}}{B_{\mathrm{M}}} \min\left\{\frac{A_{\mathrm{M}}\mu^2}{4B_{\mathrm{M}}(H^2+4c_1)}, M\right\}.$$

For Option 2, we can get the same bound for $\|x^{K+1} - x^*\|^2$ as above from (35). Then for $i \in S^k$, from Lemma C.1 (ii), we arrive at

$$\|z_i^{K+1} - z^*\|^2 \leq (1 - A_{\mathrm{M}})\|z_i^K - x^*\|^2 + B_{\mathrm{M}}\|x^{K+1} - x^*\|^2$$

$$\leq (1 - A_{\mathrm{M}}) \min\left\{\frac{A_{\mathrm{M}}\mu^2}{4B_{\mathrm{M}}(H^2+4c_1)}, M\right\} + A_{\mathrm{M}} \min\left\{\frac{A_{\mathrm{M}}\mu^2}{4B_{\mathrm{M}}(H^2+4c_1)}, M\right\}$$

$$= \min\left\{\frac{A_{\mathrm{M}}\mu^2}{4B_{\mathrm{M}}(H^2+4c_1)}, M\right\}.$$

For $i \notin S^k$, we have

$$\|z_i^{K+1} - x^*\|^2 = \|z_i^K - x^*\|^2 \leq \min\left\{\frac{A_{\mathrm{M}}\mu^2}{4B_{\mathrm{M}}(H^2+4c_1)}, M\right\}.$$

(iii) From Assumption 4.5(ii), we have

$$\mathcal{H}^K = \frac{1}{n}\sum_{i=1}^n \|\mathbf{L}_i^K - \mathbf{L}_i^*\|_{\mathrm{F}}^2$$

$$\leq \frac{1}{n}\sum_{i=1}^n d^2 \max_{jl}\{|(\mathbf{L}_i^K)_{jl} - (\mathbf{L}_i^*)_{jl}|^2\}$$

$$\leq d^2 M_5^2 \max_{i\in[n], 0\leq t\leq K} \|z_i^t - x^*\|^2$$

$$\leq M.$$

(iv) For $i \in S^k$, from Assumption 4.6 and Lemma E.1 (ii), we have

$$\|\mathbf{L}_i^{K+1} - \mathbf{L}_i^*\|_F^2 \le (1 - A)\|\mathbf{L}_i^K - \mathbf{L}_i^*\|_F^2 + BM_4^2\|z_i^K - x^*\|^2$$
$$\le (1 - A)M + AM$$
$$= M.$$

For $i \notin S^k$, we also have

$$\|\mathbf{L}_i^{K+1} - \mathbf{L}_i^*\|_F^2 = \|\mathbf{L}_i^K - \mathbf{L}_i^*\|_F^2 \le M.$$

$\square$

E.3 Proof of Theorem 5.4

Define $\mathbf{H}_i^k := \beta^k \mathbf{A}_i^k - \mathbf{C}_i^k$ for $i \in [n]$ and $k \ge 0$. First, it is easy to verify that $\mathbf{A}^k = \frac{1}{n}\sum_{i=1}^n \mathbf{A}_i^k$, $\mathbf{C}^k = \frac{1}{n}\sum_{i=1}^n \mathbf{C}_i^k$, $\mathbf{H}^k = \frac{1}{n}\sum_{i=1}^n \mathbf{H}_i^k$, $g_1^k = \frac{1}{n}\sum_{i=1}^n g_{i,1}^k$, and $g_2^k = \frac{1}{n}\sum_{i=1}^n g_{i,2}^k$ for $k \ge 0$. Then we have

$$g^k = \beta^k g_1^k - g_2^k$$
$$= \frac{1}{n}\sum_{i=1}^n \left(\beta^k g_{i,1}^k - g_{i,2}^k\right)$$
$$= \frac{1}{n}\sum_{i=1}^n \left(\beta^k \mathbf{A}_i^k w_i^k - \mathbf{C}_i^k w_i^k - \nabla f_i(w_i^k)\right)$$
$$= \frac{1}{n}\sum_{i=1}^n \left(\mathbf{H}_i^k w_i^k - \nabla f_i(w_i^k)\right).$$

Thus, from

$$x^{k+1} = \left(\mathbf{H}^k\right)^{-1} g^k = \left(\mathbf{H}^k\right)^{-1}\left[\frac{1}{n}\sum_{i=1}^n \left(\mathbf{H}_i^k w_i^k - \nabla f_i(w_i^k)\right)\right],$$

and

$$x^* = \left(\mathbf{H}^k\right)^{-1}\left[\mathbf{H}^k x^* - \nabla f(x^*)\right] = \left(\mathbf{H}^k\right)^{-1}\left[\frac{1}{n}\sum_{i=1}^n \left(\mathbf{H}_i^k x^* - \nabla f_i(x^*)\right)\right],$$

we can obtain

$$x^{k+1} - x^* = \left(\mathbf{H}^k\right)^{-1}\left[\frac{1}{n}\sum_{i=1}^n \left(\mathbf{H}_i^k(w_i^k - x^*) - (\nabla f_i(w_i^k) - \nabla f_i(x^*))\right)\right].$$

Then from the triangle inequality and the fact that $\mathbf{H}^k \succeq \mu \mathbf{I}$, we have

$$\|x^{k+1} - x^*\| \le \frac{1}{\mu n} \sum_{i=1}^{n} \left\| \nabla f_i(w_i^k) - \nabla f_i(x^*) - \mathbf{H}_i^k(w_i^k - x^*) \right\|$$

$$\le \frac{1}{\mu n} \sum_{i=1}^{n} \left\| \nabla f_i(w_i^k) - \nabla f_i(x^*) - \nabla^2 f_i(x^*)(w_i^k - x^*) \right\| + \frac{1}{\mu n} \sum_{i=1}^{n} \left\| (\mathbf{H}_i^k - \nabla^2 f_i(x^*)(w_i^k - x^*)) \right\|$$

$$\le \frac{H}{2\mu n} \sum_{i=1}^{n} \|w_i^k - x^*\|^2 + \frac{1}{\mu n} \sum_{i=1}^{n} \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\| \cdot \|w_i^k - x^*\|$$

$$= \frac{H}{2\mu} \mathcal{W}^k + \frac{1}{\mu n} \sum_{i=1}^{n} \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\| \cdot \|w_i^k - x^*\|.$$

We further use Young's inequality to bound $\|x^{k+1} - x^*\|^2$ as

$$\|x^{k+1} - x^*\|^2 \le \frac{H^2}{2\mu^2} (\mathcal{W}^k)^2 + \frac{2}{\mu^2 n^2} \left(\sum_{i=1}^{n} \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\| \cdot \|w_i^k - x^*\| \right)^2$$

$$\le \frac{H^2}{2\mu^2} (\mathcal{W}^k)^2 + \frac{2}{\mu^2} \left(\frac{1}{n} \sum_{i=1}^{n} \|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|^2 \right) \mathcal{W}^k,$$

where we use Cauchy-Schwarz inequality in the last inequality. Next we estimate $\|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|^2$. Denote $\mathbf{L}_i^* = \tilde{h}^i(\nabla^2 f_i(x^*))$ and assume $\max_{jl}\{\|\mathbf{B}_i^{jl}\|_F\} \le R$ for $i \in [n]$. Then

$$\|\mathbf{H}_i^k - \nabla^2 f_i(x^*)\|^2 = \|\beta^k \mathbf{A}_i^k - \mathbf{C}_i^k - \nabla^2 f_i(x^*)\|^2$$

$$= \left\| \sum_{jl} \left[\beta^k((\mathbf{L}_i^k)_{jl} + 2\gamma_i^k) - 2\gamma_i^k - (\mathbf{L}_i^*)_{jl} \right] \mathbf{B}^{jl} \right\|^2$$

$$\le N R^2 \sum_{jl} \left| \beta^k((\mathbf{L}_i^k)_{jl} + 2\gamma_i^k) - ((\mathbf{L}_i^*)_{jl} + 2\gamma_i^k) \right|^2.$$

Assuming $\max_{jl}\{|(\mathbf{L}_i^k)_{jl}|\} \le M_3$ for $i \in [n]$, we have

$$\left| \beta^k((\mathbf{L}_i^k)_{jl} + 2\gamma_i^k) - ((\mathbf{L}_i^*)_{jl} + 2\gamma_i^k) \right|^2 = \left| (\beta^k - 1)((\mathbf{L}_i^k)_{jl} + 2\gamma_i^k) + (\mathbf{L}_i^k - \mathbf{L}_i^*)_{jl} \right|^2$$

$$\le 2 \left| (\beta^k - 1)((\mathbf{L}_i^k)_{jl} + 2\gamma_i^k) \right|^2 + 2 \left| (\mathbf{L}_i^k - \mathbf{L}_i^*)_{jl} \right|^2$$

$$\le 2(M_3 + 2\max\{c, M_3\})^2 \left| \beta^k - 1 \right|^2 + 2 \left| (\mathbf{L}_i^k - \mathbf{L}_i^*)_{jl} \right|^2.$$

For Option 1 in Algorithm 3, we have $\beta_i^k = \max_{jl} \frac{\tilde{h}^i(\nabla^2 f_i(z_i^{k-1}))_{jl} + 2\gamma_i^k}{(\mathbf{L}_i^k)_{jl} + 2\gamma_i^k}$, where we define $z_i^{-1} = z_i^0$.

For any $j, l \in [d]$, we have

$$
\left| \frac{\tilde{h}^i(\nabla^2 f_i(z_i^{k-1}))_{jl} + 2\gamma_i^k}{(\mathbf{L}_i^k)_{jl} + 2\gamma_i^k} - 1 \right|^2 = \left| \frac{(\tilde{h}^i(\nabla^2 f_i(z_i^{k-1})) - \mathbf{L}_i^k)_{jl}}{(\mathbf{L}_i^k + 2\gamma_i^k \mathbf{I})_{jl}} \right|^2
$$

$$
\leq \frac{1}{c^2} \left| (\tilde{h}^i(\nabla^2 f_i(z_i^{k-1})) - \mathbf{L}_i^k)_{jl} \right|^2
$$

$$
\leq \frac{2}{c^2} \left| (\tilde{h}^i(\nabla^2 f_i(z_i^{k-1})) - \mathbf{L}_i^*)_{jl} \right|^2 + \frac{2}{c^2} \left| (\mathbf{L}_i^k - \mathbf{L}_i^*)_{jl} \right|^2
$$

$$
\leq \frac{2M_5^2}{c^2} \| z_i^{k-1} - x^* \|^2 + \frac{2}{c^2} \left| (\mathbf{L}_i^k - \mathbf{L}_i^*)_{jl} \right|^2,
$$

where we use $(\mathbf{L}_i^k + 2\gamma_i^k \mathbf{I})_{jl} \geq (\mathbf{L}_i^k)_{jl} + |(\mathbf{L}_i^k)_{jl}| + c \geq c$ in the first inequality, in the second inequality, we use the Young's inequality, and the last inequality comes from $\max_{jl}\{|(\tilde{h}^i(\nabla^2 f_i(x)) - \mathbf{L}_i^*)_{jl}|\} \leq M_5 \|x - x^*\|$. Then from the definition of β^k, we arrive at

$$
\left| \beta^k - 1 \right|^2 \leq \max_{jl} \left\{ \frac{2M_5^2}{c^2} \| z_i^{k-1} - x^* \|^2 + \frac{2}{c^2} \left| (\mathbf{L}_i^k - \mathbf{L}_i^*)_{jl} \right|^2 \right\}
$$

$$
\leq \frac{2M_5^2}{c^2} \| z_i^{k-1} - x^* \|^2 + \frac{2}{c^2} \| \mathbf{L}_i^k - \mathbf{L}_i^* \|^2.
$$

For Option 2 in Algorithm 3, we can have the following bound in the same way.

$$
\left| \beta^k - 1 \right|^2 \leq \frac{2M_5^2}{c^2} \| z_i^k - x^* \|^2 + \frac{2}{c^2} \| \mathbf{L}_i^k - \mathbf{L}_i^* \|^2.
$$

For Option 1, from the above inequalities, we can get

$$
\| \mathbf{H}_i^k - \nabla^2 f_i(x^*) \|^2 \leq \frac{4N^2 R^2 M_5^2 (M_3 + 2\max\{c, M_3\})^2}{c^2} \| z_i^{k-1} - x^* \|^2
$$

$$
+ 2NR^2 \left(1 + \frac{2N(M_3 + 2\max\{c, M_3\})^2}{c^2} \right) \| \mathbf{L}_i^k - \mathbf{L}_i^* \|_{\mathrm{F}}^2,
$$

which implies that

$$
\frac{1}{n} \sum_{i=1}^n \| \mathbf{H}_i^k - \nabla^2 f_i(x^*) \|^2 \leq \frac{4N^2 R^2 M_5^2 (M_3 + 2\max\{c, M_3\})^2}{c^2} \mathcal{Z}^{k-1}
$$

$$
+ 2NR^2 \left(1 + \frac{2N(M_3 + 2\max\{c, M_3\})^2}{c^2} \right) \mathcal{H}^k,
$$

where $\mathcal{H}^k := \frac{1}{n} \sum_{i=1}^n \| \mathbf{L}_i^k - \mathbf{L}_i^* \|_{\mathrm{F}}^2$. Similarly, for Option 2, we can get

$$
\frac{1}{n} \sum_{i=1}^n \| \mathbf{H}_i^k - \nabla^2 f_i(x^*) \|^2 \leq \frac{4N^2 R^2 M_5^2 (M_3 + 2\max\{c, M_3\})^2}{c^2} \mathcal{Z}^k
$$

$$
+ 2NR^2 \left(1 + \frac{2N(M_3 + 2\max\{c, M_3\})^2}{c^2} \right) \mathcal{H}^k.
$$

Let

$$
c_1 := \frac{4N^2 R^2 M_5^2 (M_3 + 2\max\{c, M_3\})^2}{c^2}, \quad c_2 := 2NR^2 \left(1 + \frac{2N(M_3 + 2\max\{c, M_3\})^2}{c^2} \right) \quad (33)
$$

Then for Option 1, we have

$$\|x^{k+1} - x^*\|^2 \leq \frac{H^2}{2\mu^2}(\mathcal{W}^k)^2 + \frac{2}{\mu^2}\mathcal{W}^k\left(c_1\mathcal{Z}^{k-1} + c_2\mathcal{H}^k\right)$$

$$= \frac{H^2}{2\mu^2}(\mathcal{W}^k)^2 + \frac{2c_1}{\mu^2}\mathcal{Z}^{k-1}\mathcal{W}^k + \frac{2c_2}{\mu^2}\mathcal{H}^k\mathcal{W}^k, \tag{34}$$

and for Option 2, we have

$$\|x^{k+1} - x^*\|^2 \leq \frac{H^2}{2\mu^2}(\mathcal{W}^k)^2 + \frac{2c_1}{\mu^2}\mathcal{Z}^k\mathcal{W}^k + \frac{2c_2}{\mu^2}\mathcal{H}^k\mathcal{W}^k. \tag{35}$$

From the update rule of w_i^k and z_i^k, the results in (31) and (32) also hold for Algorithm 3. Then for Option 1, we have

$$\mathbb{E}_k[\Phi_5^{k+1}] \overset{(32)}{\leq} \left(1 - \frac{\tau p}{n}\right)\mathcal{W}^k + \frac{\tau p}{n}\left(1 - \frac{\tau A_M}{n}\right)\mathcal{Z}^k + \frac{\tau^2 B_M p}{n^2}\|x^{k+1} - x^*\|^2 + \frac{2p}{A_M}\left(1 - \frac{\tau A_M}{n}\right)\mathbb{E}_k[\mathcal{Z}^{k+1}]$$

$$\overset{(31)}{\leq} \left(1 - \frac{\tau p}{n}\right)\mathcal{W}^k + \frac{2p}{A_M}\left(1 - \frac{\tau A_M}{n}\right)\left(1 - \frac{\tau A_M}{2n}\right)\mathcal{Z}^k + \frac{2\tau p B_M}{n A_M}\left(1 - \frac{\tau A_M}{2n}\right)\|x^{k+1} - x^*\|^2$$

$$\overset{34}{\leq} \left(1 - \frac{\tau p}{n} + \frac{2\tau p B_M}{n A_M}\left(\frac{H^2}{2\mu^2}\mathcal{W}^k + \frac{2c_2}{\mu^2}\mathcal{H}^k + \frac{2c_1}{\mu^2}\mathcal{Z}^{k-1}\right)\right)\mathcal{W}^k$$

$$+ \frac{2p}{A_M}\left(1 - \frac{\tau A_M}{n}\right)\left(1 - \frac{\tau A_M}{2n}\right)\mathcal{Z}^k.$$

If $\|z_i^k - x^*\|^2 \leq \frac{A_M\mu^2}{4(H^2+4c_1)B_M}$ and $\mathcal{H}^k \leq \frac{A_M\mu^2}{16c_2 B_M}$ for all $k \geq 0$, then we have

$$\frac{H^2}{2\mu^2}\mathcal{W}^k + \frac{2c_2}{\mu^2}\mathcal{H}^k + \frac{2c_1}{\mu^2}\mathcal{Z}^{k-1} \leq \frac{A_M}{4B_M},$$

which implies that

$$\mathbb{E}_k[\Phi_5^{k+1}] \leq \left(1 - \frac{\tau p}{2n}\right)\mathcal{W}^k + \frac{2p}{A_M}\left(1 - \frac{\tau A_M}{n}\right)\left(1 - \frac{\tau A_M}{2n}\right)\mathcal{Z}^k$$

$$\leq \left(1 - \frac{\tau \min\{p, A_M\}}{2n}\right)\Phi_5^k.$$

By applying the tower property, we have

$$\mathbb{E}[\Phi_5^{k+1}] \leq \left(1 - \frac{\tau \min\{p, A_M\}}{2n}\right)\mathbb{E}[\Phi_5^k].$$

Unrolling the recursion, we can obtain the result. For Option 2, we can have the same result.

E.4   Proof of Theorem 5.5

Since $\xi^k \equiv 1$, $\eta = 1$, $S^k \equiv [n]$, and $\mathcal{Q}_i^k(x) \equiv x$ for any $x \in \mathbb{R}^d$, it is easy to see that $w_i^k = z_i^k \equiv x^k$ for all $i \in [n]$ and $k \geq 0$. In this case, we can view $\mathcal{Q}_i^k$ as an unbiased compressor with $\omega_M = 0$ or a

contraction compressor with $\delta_{\mathrm{M}} = 1$. Since (31) also holds for Algorithm 3, for Option 1, we have

$$
\begin{aligned}
\mathbb{E}_k \|x^{k+1} - x^*\|^2 &\leq (1 - A_{\mathrm{M}}) \|x^k - x^*\|^2 + B_{\mathrm{M}} \|x^{k+1} - x^*\|^2 \\
&\overset{(34)}{\leq} (1 - A_{\mathrm{M}}) \|x^k - x^*\|^2 + \frac{1}{4} A_{\mathrm{M}} \|x^k - x^*\|^2 \\
&= \left(1 - \frac{3A_{\mathrm{M}}}{4}\right) \|x^k - x^*\|^2.
\end{aligned}
$$

For Option 2, we can get the same bound for $\mathbb{E}_k \|x^{k+1} - x^*\|^2$ as above from (35).

From Lemma E.1, we can obtain

$$
\mathbb{E}_k[\mathcal{H}^{k+1}] \leq (1 - A)\mathcal{H}^k + BM_4^2 \|x^k - x^*\|^2.
$$

Thus,

$$
\begin{aligned}
\mathbb{E}_k[\Phi_6^{k+1}] &= \mathbb{E}_k[\mathcal{H}^{k+1}] + \frac{4BM_4^2}{A_{\mathrm{M}}} \mathbb{E}_k \|x^{k+1} - x^*\|^2 \\
&\leq (1 - A)\mathcal{H}^k + BM_4^2 \|x^k - x^*\|^2 + \frac{4BM_4^2}{A_{\mathrm{M}}} \left(1 - \frac{3A_{\mathrm{M}}}{4}\right) \|x^k - x^*\|^2 \\
&\leq \left(1 - \frac{\min\{2A, A_{\mathrm{M}}\}}{2}\right) \Phi_6^k.
\end{aligned}
$$

By applying the tower property, we have $\mathbb{E}[\Phi_6^{k+1}] \leq \theta_2 \mathbb{E}[\Phi_6^k]$. Unrolling the recursion, we have $\mathbb{E}[\Phi_6^k] \leq \theta_3^k \Phi_6^0$.

Then we further have $\mathbb{E}[\mathcal{H}^k] \leq \theta_3^k \Phi_6^0$ and $\mathbb{E}\|x^k - x^*\|^2 \leq \frac{A_{\mathrm{M}}}{4BM_4^2} \theta_3^k \Phi_6^0$. For Option 1, from $w_i^k = z_i^k \equiv x^k$ and (34), we can get

$$
\|x^{k+1} - x^*\|^2 \leq \left(\frac{H^2}{2\mu^2} \|x^k - x^*\|^2 + \frac{2c_1}{\mu^2} \|x^{k-1} - x^*\|^2 + \frac{2c_2}{\mu^2} \mathcal{H}^k\right) \|x^k - x^*\|^2.
$$

Assume $x^k \neq x^*$ for all $k \geq 0$. Then we have

$$
\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2} \leq \frac{H^2}{2\mu^2} \|x^k - x^*\|^2 + \frac{2c_1}{\mu^2} \|x^{k-1} - x^*\|^2 + \frac{2c_2}{\mu^2} \mathcal{H}^k,
$$

and by taking expectation, we arrive at

$$
\begin{aligned}
\mathbb{E}\left[\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2}\right] &\leq \frac{H^2}{2\mu^2} \mathbb{E}\|x^k - x^*\|^2 + \frac{2c_1}{\mu^2} \mathbb{E}\|x^{k-1} - x^*\|^2 + \frac{2c_2}{\mu^2} \mathbb{E}[\mathcal{H}^k] \\
&\leq \theta_3^k \left(\frac{A_{\mathrm{M}}(H^2 \theta_3 + 4c_1)}{8BM_4^2 \mu^2 \theta_3} + \frac{2c_2}{\mu^2}\right) \Phi_6^0.
\end{aligned}
$$

For Option 2, from $w_i^k = z_i^k \equiv x^k$ and (35), we can get

$$
\|x^{k+1} - x^*\|^2 \leq \left(\frac{H^2 + 4c_1}{2\mu^2} \|x^k - x^*\|^2 + \frac{2c_2}{\mu^2} \mathcal{H}^k\right) \|x^k - x^*\|^2.
$$

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Assume $x^k \neq x^*$ for all $k \geq 0$. Then we have

$$\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2} \leq \frac{H^2 + 4c_1}{2\mu^2} \|x^k - x^*\|^2 + \frac{2c_2}{\mu^2} \mathcal{H}^k,$$

and by taking expectation, we arrive at

$$\mathbb{E}\left[\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2}\right] \leq \frac{H^2 + 4c_1}{2\mu^2} \mathbb{E}\|x^k - x^*\|^2 + \frac{2c_2}{\mu^2} \mathbb{E}[\mathcal{H}^k]$$

$$\leq \theta_3^k \left(\frac{A_{\mathrm{M}}(H^2 + 4c_1)}{8BM_4^2\mu^2} + \frac{2c_2}{\mu^2}\right) \Phi_6^0.$$

E.5 Proof of Theorem 5.6

(i) Notice that under Assumption 4.3, we have $A_{\mathrm{M}} = B_{\mathrm{M}} = \eta$. We prove this by mathematical induction. First, since $z_i^0 = x^0$, we know $\|z_i^0 - x^*\|^2 \leq \min\{\frac{\mu^2}{4d(H^2+4c_1)}, \frac{\mu^2}{16d^3c_2M_5^2}\}$ for $i \in [n]$. Then from Lemma E.2 (iii), we have $\mathcal{H}^0 \leq \frac{\mu^2}{16dc_2}$. Next, assume

$$\|z_i^k - x^*\|^2 \leq \min\left\{\frac{\mu^2}{4d(H^2 + 4c_1)}, \frac{\mu^2}{16d^3c_2M_5^2}\right\} \text{ for } i \in [n] \quad \text{and} \quad \mathcal{H}^k \leq \frac{\mu^2}{16dc_2},$$

for $k \leq K$. By choosing $M = \frac{\mu^2}{16d^3c_2M_5^2}$ in Lemma E.2 (i), we have

$$\|z_i^{K+1} - x^*\|^2 \leq \min\left\{\frac{\mu^2}{4d(H^2 + 4c_1)}, \frac{\mu^2}{16d^3c_2M_5^2}\right\},$$

for $i \in [n]$. By further using Lemma E.2 (iii), we can get $\mathcal{H}^{K+1} \leq \frac{\mu^2}{16dc_2}$.

(ii) We prove the result by induction. Assume $\|z_i^k - x^*\|^2 \leq \min\left\{\frac{A_{\mathrm{M}}\mu^2}{4B_{\mathrm{M}}(H^2+4c_1)}, \frac{AA_{\mathrm{M}}\mu^2}{16c_2B_{\mathrm{M}}BM_4^2}\right\}$ and $\|\mathbf{L}_i^k - \mathbf{L}_i^*\|_{\mathrm{F}}^2 \leq \frac{A_{\mathrm{M}}\mu^2}{16c_2B_{\mathrm{M}}}$ for all $i \in [n]$ and $k \leq K$. Then by Lemma E.2 (iv), we have $\|\mathbf{L}_i^{K+1} - \mathbf{L}_i^*\|_{\mathrm{F}}^2 \leq \frac{A_{\mathrm{M}}\mu^2}{16c_2B_{\mathrm{M}}}$. Moreover, by Lemma E.2 (ii), we have $\|z_i^{K+1} - x^*\|^2 \leq \min\left\{\frac{A_{\mathrm{M}}\mu^2}{4B_{\mathrm{M}}(H^2+4c_1)}, \frac{AA_{\mathrm{M}}\mu^2}{16c_2B_{\mathrm{M}}BM_4^2}\right\}$ for $i \in [n]$.