A SIMPLE PROOF OF TAYLOR'S THEOREM

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We provide a simple inductive proof of Taylor's Theorem. The key step in our proof relies upon the Fundamental Theorem of Calculus (FTC) and a judicious choice of auxiliary functions.

Theorem 1. *(Taylor's Theorem) Suppose f is a single-variable real-valued function that can be differentiated n+1 times in an interval, I, containing x_0 with the $n+1^{st}$ derivative integrable on I. If $x_0 + h \in I$, then we have:*
$f(x_0 + h) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)h^k}{k!} + \int_{x_0}^{x_0+h} f^{(n+1)}(t)\frac{(x_0+h-t)^n}{n!}dt.$

proof. We begin with the $n = 1$ case. By the FTC we have,
$f(x_0+h) = f(x_0)+\int_{x_0}^{x_0+h} f^{'}(t)dt = f(x_0)+\int_{x_0}^{x_0+h} f^{'}(x_0)dt+\int_{x_0}^{x_0+h}(f^{'}(t)-f^{'}(x_0))dt.$
The next two observations offer the central insight.
First, note if $g(t) = (f^{'}(x_0) - f^{'}(t))(x_0 + h - t)$ then $g^{'}(t) = f^{'}(t) - f^{'}(x_0) - f^{(2)}(t)(x_0 + h - t)$ and $g(x_0 + h) = g(x_0)$. Hence by the FTC this implies, $\int_{x_0}^{x_0+h}((f^{'}(t) - f^{'}(x_0)) - f^{(2)}(t)(x_0 + h - t))dt = 0$ or $\int_{x_0}^{x_0+h}(f^{'}(t) - f^{'}(x_0))dt = \int_{x_0}^{x_0+h} f^{(2)}(t)(x_0 + h - t)dt$. Next note that, $\int_{x_0}^{x_0+h} f^{'}(x_0)dt = f^{'}(x_0)h$. Thus, $f(x_0 + h) = f(x_0) + f^{'}(x_0)h + \int_{x_0}^{x_0+h} f^{(2)}(t)(x_0 + h - t)dt.$

The following two lemmas generalize the key ideas that arise in the n=1 case.

Lemma 1. $\int_{x_0}^{x_0+h}((f^{(k)}(t) - f^{(k)}(x_0))\frac{(x_0+h-t)^{k-1}}{(k-1)!} - f^{(k+1)}(t)\frac{(x_0+h-t)^k}{k!})dt = 0$

Proof. Consider $g(t) = (f^{(k)}(x_0) - f^{(k)}(t))\frac{(x_0+h-t)^k}{k!}$. Note that $g^{'}(t)$ equals the integrand and $g(x_0 + h) = g(x_0)$. Hence claim follows by the FTC. □

Lemma 2. $\int_{x_0}^{x_0+h} f^{(k)}(x_0)\frac{(x_0+h-t)^{k-1}}{(k-1)!}dt = \frac{f^{(k)}(x_0)h^k}{k!}$

Remainder of the proof of Theorem 1. From these lemmas we can now easily provide the proof of the inductive step. We have,
$f(x_0 + h) = f(x_0) + \sum_{k=1}^{n-1} \frac{f^{(k)}(x_0)h^k}{k!} + \int_{x_0}^{x_0+h} f^{(n)}(t)\frac{(x_0+h-t)^{n-1}}{(n-1)!}dt = f(x_0)+$
$\sum_{k=1}^{n-1} \frac{f^{(k)}(x_0)h^k}{k!}+\int_{x_0}^{x_0+h} f^{(n)}(x_0)\frac{(x_0+h-t)^{n-1}}{(n-1)!}dt+\int_{x_0}^{x_0+h}(f^{(n)}(t)-f^{(n)}(x_0))\frac{(x_0+h-t)^{n-1}}{(n-1)!}dt$
$=f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)h^k}{k!} + \int_{x_0}^{x_0+h} f^{(n+1)}(t)\frac{(x_0+h-t)^n}{n!}dt.$ □

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