Python Quantitative Project

Ryan Greenup & James Guerra

August 11, 2020

Contents

> **Python**
>
> ```
> 1  code /home/ryan/Dropbox/Studies/QuantProject/Current/Python-Quant/ &
>    ↪  disown
> ```

§ 1 What the unit involves. Ryan:James

Writing and presenting.

§ 2 Python

Two parts:

1. Figure out the python

2. IMPORTANT: MUST have new math

    (a) Dr. Hazrat may have some new math to use.

§ 3 Matrix Exponentiation Ryan

3.3   Implementation in Sympy

The Matrix Exponential is implemented in areas of:

- Graph Centrality modelling [9]

- Systems of Linear Differential Equations [10, Ch. 8.4]

- Theory of Algebraic Lie Groups [2, Ch. 2]

However the method to implement matrix exponentiation provided by the documentation [6] and referenced in the development repository [7] does not appear to be implemented very well, for example the following provides a very long result:

Python

```python
from __future__ import division
from sympy import *
x, y, z, t = symbols('x y z t')
k, m, n = symbols('k m n', integer=True)
f, g, h = symbols('f g h', cls=Function)
init_printing()
init_printing(use_latex='mathjax', latex_mode='equation')


import pyperclip
def lx(expr):
    pyperclip.copy(latex(expr))
    print(expr)
```

```python
1   A = Matrix([
2       [11, 12, 13],
3       [21, 22, 23],
4       [31, 32, 33]
5   ])
6
7       expr = exp(A)
8       expr.doit()
```

$$
\left[-\frac{1}{-\frac{33}{94}+\frac{5\sqrt{1149}}{94}}\left(\frac{-\frac{6552}{-\sqrt{1149}-22}+552}{\left(-\sqrt{1149}-22\right)\left(-\frac{59\sqrt{1149}}{95}-\frac{1837}{95}\right)} - \frac{26}{-\sqrt{1149}-22} \right) - \frac{1}{\frac{33\sqrt{1149}}{2303}+\frac{5745}{2303}}\left(-1 - \frac{1}{-\frac{33}{94}+\frac{5\sqrt{1149}}{94}}\left(\frac{26}{-\sqrt{1149}-22} - \frac{-\frac{6552}{-\sqrt{1149}-22}+552}{\left(-\sqrt{1149}-22\right)\left(-\frac{59\sqrt{1149}}{95}-1\right)} \right) \right. \right.
$$

$$
-2 - \frac{1}{\left(-\frac{1837}{95}+\frac{59\sqrt{1149}}{95}\right)\left(\frac{33\sqrt{1149}}{2303}+\frac{5745}{2303}\right)}e^{-33+\sqrt{1149}}\left(-1 - \frac{1}{-\frac{33}{94}+\frac{5\sqrt{1149}}{94}}\left(\frac{26}{-\sqrt{1149}-22} - \frac{-\frac{6552}{-\sqrt{1149}-22}+552}{\left(-\sqrt{1149}-22\right)\left(-\frac{59\sqrt{1149}}{95}-\frac{1837}{95}\right)} + 2 \right) \right)
$$

$$
-\frac{1}{-\frac{33}{94}+\frac{5\sqrt{1149}}{94}}\left(\frac{-\frac{6552}{-\sqrt{1149}-22}+552}{\left(-\sqrt{1149}-22\right)\left(-\frac{59\sqrt{1149}}{95}-\frac{1837}{95}\right)} - \frac{26}{-\sqrt{1149}-22} \right) - \frac{1}{\frac{33\sqrt{1149}}{2303}+\frac{5745}{2303}}\left(-1 - \frac{33}{94}+ \right.
$$

Simplifying this result doesn't seem to help either:

```python
1   simplify(expr)
```

$$
\left[\frac{1}{12\left(-1065889+33298\sqrt{1149}\right)e^{\sqrt{1149}}}\left(-8625947e^{33+2\sqrt{1149}} - 2131778e^{\sqrt{1149}} - 2032943e^{33} + 74651\sqrt{1149}e^{33} + 66596\sqrt{1149}e^{\sqrt{1149}} + 258329\sqrt{1149}e^{3} \right. \right.
$$

$$
\frac{1}{6\left(-1065889+33298\sqrt{1149}\right)e^{\sqrt{1149}}}\left(-66949e^{33+2\sqrt{1149}} - 66596\sqrt{1149}e^{\sqrt{1149}} - 2064829e^{33} + 61128\sqrt{1149}e^{33} + 2131778e^{\sqrt{1149}} + 5468\sqrt{1149}e^{33+2} \right.
$$

$$
\frac{1}{12\left(-1065889+33298\sqrt{1149}\right)e^{\sqrt{1149}}}\left(-236457\sqrt{1149}e^{33+2\sqrt{1149}} - 6226373e^{33} - 2131778e^{\sqrt{1149}} + 66596\sqrt{1149}e^{\sqrt{1149}} + 169861\sqrt{1149}e^{33} + 8358151e^{3} \right.
$$

Methods suggested online only provide numerical solutions or partial sums:

- python - Sympy Symbolic Matrix Exponential - Stack Overflow

- python - Exponentiate symbolic matrix expression using SymPy - Stack Overflow

- Calculate state transition matrix in python - Stack Overflow

Instead this will need to be implemented from first principles.

¶ 3 Theory

3.3   Matrix Exponentiation

A Matrix Exponential is defined by using the ordinary exponential power series [2, Ch. 2],[10, Ch. 8.4] (should we prove the power series generally?):

3

$$e^{\mathbf{X}} = \sum_{k=0}^{\infty} \left[\frac{1}{k!} \cdot \mathbf{X^k} \right] \tag{1}$$

This definition can be expanded upon however by using properties of logarithms:

$$b = e^{\log_e (b)}, \quad \forall b \in \mathbb{C} \tag{2}$$

$$\implies b^{\mathbf{X}} = \left(e^{\log_e (b)} \right)^{\mathbf{X}} \tag{3}$$

$$\implies b^{\mathbf{X}} = e^{\log_e b \mathbf{X}} \tag{4}$$

The identity in (2) is justified by the definition of the complex log. However some discussion is required for (3) because it is not clear that the exponential will generally distribute throught he parenthesis like so $(a \cdot b)^k = a^k \cdot b^k$, for example consider $\left([-1]^2 \cdot 3 \right)^{\frac{1}{2}} \neq [-1]^{\frac{2}{2}} \cdot 3^{\frac{1}{2}}$.

A sufficient condition for this identity is $k \in \mathbb{Z}^*$, consider this example which will be important later:

$$(\log_e (b)\mathbf{X})^k, \quad \forall k \in \mathbb{Z}^* \tag{5}$$

Because multiplication is commutative $\forall z \in \mathbb{C}$, this could be re-expressed in the form:

$$(\log_e (b)\mathbf{X})^k = \underbrace{\log_e (b) \cdot \log_e (b) \cdot \log_e (b) \dots}_{k \text{ times}} \times \underbrace{\mathbf{XXX} \dots}_{k \text{ times}}$$

$$= \log_e^k (b)\mathbf{X}^k \tag{6}$$

Now consider the the following by applying (6):

$$e^X = \sum_{k=0}^{\infty} \left[\frac{1}{k!} \mathbf{X}^k \right]$$

$$\implies e^{bX} = \sum_{k=0}^{\infty} \left[\frac{1}{k!} (b\mathbf{X})^k \right] \quad \forall b \in \mathbb{C}$$

$$= \sum_{k=0}^{\infty} \left[\frac{1}{k!} b^k \mathbf{X}^k \right]$$

$$= \left(e^b \right)^{\mathbf{X}}$$

$$\implies e^{b\mathbf{X}} = e^{\mathbf{X}b} = \left(e^b \right)^{\mathbf{X}} = \left(e^{\mathbf{X}} \right)^b \qquad \square \tag{7}$$

So the matrix exponential for an arbitrary base could be given by:

$$b = e^{\log_e(b)}, \quad \forall b \in \mathbb{C}$$

$$\implies b^{\mathbf{X}} = \left(e^{\log_e(b)} \right)^{\mathbf{X}}$$

as per (7)

$$b^{\mathbf{X}} = e^{\log_e(b)\mathbf{X}}$$

$$b^{\mathbf{X}} = \sum_{k=0}^{\infty} \left[\frac{(\log_e(b)\mathbf{X})^k}{k!} \right]$$

$$= \sum_{k=0}^{\infty} \left[\frac{\log_e^k(b)}{k!}\mathbf{X}^k \right] \tag{8}$$

This is also consistent with the *McLaurin Series* expansion of $b^{\mathbf{X}}$ ($\forall b \in \mathbb{C}$):

$$f(x) = \sum_{k=0}^{\infty} \left[\frac{f^{(n)}(0)}{k!} x^k \right]$$

$$\implies b^x = \sum_{k=0}^{\infty} \left[\frac{\frac{d^n}{dx^n}(b^x)\,|_{x=0}}{k!} x^k \right]$$

$$\implies b^{\mathbf{X}} = \sum_{k=0}^{\infty} \left[\frac{\frac{d^n}{d\mathbf{X}^n}(b^{\mathbf{X}})\,|_{\mathbf{X}=\mathbf{O}}}{k!} \mathbf{X}^k \right]$$

By ordinary calculus identities we have $f(x) = b^x \implies f^{(n)}(x) = b^x \log_e^n(b)$ which distribute through a matrix and hence:

$$b^x = \sum_{k=0}^{\infty} \left[\frac{b^0 \log_e^k(b)}{k!} x^k \right]$$

$$\implies b^{\mathbf{X}} = \sum_{k=0}^{\infty} \left[\frac{b^0 \log_e^k(b)}{k!} \mathbf{X}^k \right]$$

By the previous identity:

$$\implies b^{\mathbf{X}} = \sum_{k=0}^{\infty} \left[\frac{(\log_e(b)\mathbf{X})^k}{k!} \right]$$

$$= e^{\log_e(b)\mathbf{X}}$$

3.3 Matrix-Matrix Exponentiation

Matrix-Matrix exponentiation has applications in quantum mechanics [1, p. 84].
   As for Matrices with the requirements:

1. Square

2. Normal:

- Commutes with it's congugate transpose

3. Non Singular

4. Non Zero Determinant

$$||A - I|| < 1 \implies e^{\log_e (\mathbf{A})} = \mathbf{A} \text{ (By Lie Groups Springer Textbook)}$$
$$\implies \mathbf{A^B} = \left(e^{\log_e (\mathbf{A})}\right)^{\mathbf{B}}$$
$$\text{Similar justification as (7)}$$
$$\implies \mathbf{A^B} = e^{\log_e (\mathbf{A})\mathbf{B}}$$

However the following identities are by **Definition** anyway: [1]

$$\mathbf{A^B} = e^{\log_e (\mathbf{A})\mathbf{B}} \tag{9}$$
$$\mathbf{^B A} = e^{\mathbf{B}\log_e (\mathbf{A})} \tag{10}$$

¶ 3 An alternative Implementation in Sympy

```python
def matexp(mat, base = E):
    """
    Return the Matrix Exponential of a square matrix
    """
    import copy
    import sympy
# Should realy test for sympy vs numpy array
# Test for Square Matrix
    if mat.shape[0] != mat.shape[1]:
        print("ERROR: Only defined for Square matrices")
        return
    m = zeros(mat.shape[0])
    for i in range(m.shape[0]):
        for j in range(m.shape[1]):
            m[i,j] = Sum((mat[i,j]*ln(base))**k/factorial(k), (k, 0,
              oo)).doit()
    return m
```

```python
matexp(A, pi)
```

$$\begin{bmatrix} \pi^{11} & \pi^{12} & \pi^{13} \\ \pi^{21} & \pi^{22} & \pi^{23} \\ \pi^{31} & \pi^{32} & \pi^{33} \end{bmatrix}$$

But it would be nice to expand this to matrix bases for there uses in quantum mechanics.

The built in method for a**mat is not implemented.

there is exp(mat) but this returns garbage (see github issue), (see other solution on stack exchange that is numeric and example)

show our method with proofs of

cauchy power taylor then exp

then show our code

```Python
A = Matrix([ [11,12,13], [21,22,23], [31,32,33] ])

B = Matrix([
    [1,2,3],
    [4,5,6],
    [7,8,9]
])


A**B
```

§ 4 Recursive Relations   Ryan

A linear recurrence relation is of the form:

$$\sum_{n=0}^{\infty} [c_i \cdot a_n] = 0, \quad \exists c \in \mathbb{R}, \ \forall i < k \in \mathbb{Z}^+ \tag{11}$$

In order to find a solution for a_n, the following one-to-one correspondence can be used to relate the vector space of the sequence to the power series ring:(cite stackExchange[1]):

$$g : (a_n)_{n \in \mathbb{Z}^+} \to \mathbb{C}[[x]] : g(a_n) = \sum_{n=0}^{\infty} \left[\frac{x^n}{n!} a_n \right] \tag{12}$$

This technique is referred to as generating functions. [4]

¶ 4 Generating Functions

4.4   Generating Functions

A Generating Function, as shown in (12) is a way of encoding an infinite series of numbers (a_n) by treating them as the coefficients of a power series $(\sum_{n=0}^{\infty} [a_n x^n])$ where the variable remains in an

indeterminate form, they were first introduced by Abraham De Moivre in 1730 in order to solve the linear recurrence relations [3], as shown in (11) (such as the *Fibonacci Sequence*).

4.4 Example

Given the Linear Recurrence Relation:

$$a_0 = 1$$
$$a_0 = 1$$
$$a_{n+2} = a_{n+1+2a_n}, \quad n \geq 0$$

To solve this we can use what's known as a Generating Function, see the disucssion below
We will make consider the function $f(x)$ as shown below in:

$$f(x) = \sum_{n=0}^{\infty} [a_n x^n] \tag{13}$$

It can be shown (see (**??**)) that:

$$\sum_{n=0}^{\infty} [a_{n+1} x^n] = \frac{f(x) - a_0}{x} \tag{14}$$

$$\sum_{n=0}^{\infty} [a_{n+2} x^n] = \frac{f(x) - a_0 - a_1 x}{x^2} \tag{15}$$

So to use the generating Function consider:

$$2a_n + a_{n+1} = a_{n+2}$$
$$2a_n x^n + a_{n+1} x^n = a_{n+2} x^n$$
$$\sum_{n=0}^{\infty} [2a_n x^n] + \sum_{n=0}^{\infty} [a_{n+1} x^n] = \sum_{n=0}^{\infty} [a_{n+2} x^n] \tag{16}$$

Observe that in (**??**) tuse te
By applying the previous identity shown in (13), (14) and (15):

$$2f(x) + \frac{f(x) - a_0}{x} = \frac{f(x) - a_0}{-a_1 x} x^2$$
$$\implies f(x) = \frac{1}{1 - x - x^2} \tag{17}$$

WARNING

I accidently dropped the 2 here, it doesn't matter but it does show that how this could be dealt with algebraically

The function $f(x)$ in (17) can be solved by way of a power series, (see for example 11_{Series}), but first it is necessary to use partial fractions to split it up:

By partial fractions it is known:

$$
\begin{aligned}
f(x) &= \frac{1}{1 - x - x^2} \\
&= \frac{-1}{x^2 + x - 1} \\
&= \frac{-1}{(x - 2)(x - 1)} \\
&= \frac{A_1}{x - 2} + \frac{A_2}{x - 1}, \quad A_i \in \mathbb{R}, i \in \mathbb{Z}^+ \\
\implies -1 &= A_1(x - 1) + A_2(x - 2)
\end{aligned}
$$

Let $x = 2$:

$$
\begin{aligned}
-1 &= A_1(2 - 1) + 0 \\
&= A_1 = -1
\end{aligned}
$$

Let $x = 1$:

$$
\begin{aligned}
-1 &= 0 + A_2(1 - 2) \\
\implies A_2 &= 1
\end{aligned}
$$

Hence:

$$
f(x) = \frac{1}{x - 1} - \frac{1}{x - 2}
$$

By definition of the power series:

$$
\sum_{n=0}^{\infty} [rx^n] = \frac{1}{1 - rx^n} \tag{18}
$$

we can conclude that:

$$
\begin{aligned}
\frac{1}{x - 1} &= -\frac{1}{1 - (1)x} \\
&= -\sum_{n=0}^{\infty} [x^n] \\
\frac{-1}{x - 2} &= \frac{1}{2 - x} \\
&= \frac{1}{2} \frac{1}{1 - \frac{1}{2}x} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}x \right)^n \right]
\end{aligned}
$$

and so:

$$f(x) = \frac{1}{2} \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}x \right)^n \right] - \sum_{n=0}^{\infty} [x^n]$$

$$f(x) = \sum_{n=0}^{\infty} \left[\frac{1}{2} \left(\frac{1}{2}x \right)^n - x^n \right]$$

$$f(x) = \sum_{n=0}^{\infty} \left[\frac{1}{2 \cdot 2^n} x^n - x^n \right]$$

$$f(x) = \sum_{n=0}^{\infty} \left[x^n \left(\frac{1}{2 \cdot 2^n} - 1 \right) \right]$$

$$\implies a_n = \frac{1}{2 \cdot 2^n} - 1 \tag{19}$$

¶ 4 Exponential Generating Function

4.4 Motivation

Consider the *Fibonacci Sequence*:

$$a_n = a_{n-1} + a_{n-2}$$
$$\iff a_{n+2} = a_{n+1} + a_n \tag{20}$$

Solving this outright is quite difficult, a power series generating function can be used to solve it as shown in section 4.4 , which provides a corresponding equation to the effect of:

$$x^2 f(x) - x f(x) - f(x) = 0$$

This however still requires a little intuition, however, just from observation, this appears similar in structure to the following *ordinary differential equation*, which would be fairly easy to deal with:

$$f''(x) - f'(x) - f(x) = 0$$

This would imply that $f(x) \propto e^{mx}, \quad \exists m \in \mathbb{Z}$ because $\frac{\mathrm{d}(e^x)}{\mathrm{d}x} = e^x$, but that's fine because we have a power series for that already:

$$f(x) = e^{rx} = \sum_{n=0}^{\infty} \left[r \frac{x^n}{n!} \right]$$

So this would give an easy means by which to solve the linear recurrence relation.

4.4 Example

Consider using the following generating function, (the derivative of the generating function as in (22) and (23) is provided in section 4.4)

$$f(x) = \sum_{n=0}^{\infty} \left[a_n \cdot \frac{x^n}{n!} \right] = e^x \tag{21}$$

$$f'(x) = \sum_{n=0}^{\infty} \left[a_{n+1} \cdot \frac{x^n}{n!} \right] = e^x \tag{22}$$

$$f''(x) = \sum_{n=0}^{\infty} \left[a_{n+2} \cdot \frac{x^n}{n!} \right] = e^x \tag{23}$$

So the recursive relation from 20 could be expressed :

$$a_{n+2} = a_{n+1} + a_n$$
$$\frac{x^n}{n!} a_{n+2} = \frac{x^n}{n!} (a_{n+1} + a_n)$$
$$\sum_{n=0}^{\infty} \left[\frac{x^n}{n!} a_{n+2} \right] = \sum_{n=0}^{\infty} \left[\frac{x^n}{n!} a_{n+1} \right] + \sum_{n=0}^{\infty} \left[\frac{x^n}{n!} a_n \right]$$
$$f''(x) = f'(x) + f(x)$$

Using the theory of higher order linear differential equations with constant coefficients it can be shown:

$$f(x) = c_1 \cdot \exp \left[\left(\frac{1 - \sqrt{5}}{2} \right) x \right] + c_2 \cdot \exp \left[\left(\frac{1 + \sqrt{5}}{2} \right) \right]$$

By equating this to the power series:

$$f(x) = \sum_{n=0}^{\infty} \left[\left(c_1 \left(\frac{1 - \sqrt{5}}{2} \right)^n + c_2 \cdot \left(\frac{1 + \sqrt{5}}{2} \right)^n \right) \cdot \frac{x^n}{n} \right]$$

Now given that:

$$f(x) = \sum_{n=0}^{\infty} \left[a_n \frac{x^n}{n!} \right]$$

We can conclude that:

$$a_n = c_1 \cdot \left(\frac{1 - \sqrt{5}}{2} \right)^n + c_2 \cdot \left(\frac{1 + \sqrt{5}}{2} \right)$$

By applying the initial conditions:

$$a_0 = c_1 + c_2 \implies c_1 = -c_2$$

$$a_1 = c_1 \left(\frac{1 + \sqrt{5}}{2} \right) - c_1 \frac{1 - \sqrt{5}}{2} \implies c_1 = \frac{1}{\sqrt{5}}$$

And so finally we have the solution to the *Fibonacci Sequence* 20:

$$
\begin{aligned}
a_n &= \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right] \\
&= \frac{\varphi^n - \psi^n}{\sqrt{5}} \\
&= \frac{\varphi^n - \psi^n}{\varphi - \psi}
\end{aligned}
\tag{24}
$$

where:

- $\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.61 \ldots$

- $\psi = 1 - \varphi = \frac{1 - \sqrt{5}}{2} \approx 0.61 \ldots$

4.4 Derivative of the Exponential Generating Function

Differentiating the exponential generating function has the effect of shifting the sequence to the backward: [4]

$$f(x) = \sum_{n=0}^{\infty} \left[a_n \frac{x^n}{n!} \right] \tag{25}$$

$$
\begin{aligned}
f'(x) \, ht) &= \frac{\mathrm{d}}{\mathrm{d}x} \left(\sum_{n=0}^{\infty} \left[a_n \frac{x^n}{n!} \right] \right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x} \left(a_0 \frac{x^0}{0!} + a_1 \frac{x^1}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \ldots \frac{x^k}{k!} \right) \\
&= \sum_{n=0}^{\infty} \left[\frac{\mathrm{d}}{\mathrm{d}x} \left(a_n \frac{x^n}{n!} \right) \right] \\
&= \sum_{n=0}^{\infty} \left[\frac{a_n}{(n-1)!} x^{n-1} \right] \\
&= \sum_{n=0}^{\infty} \left[\frac{a_n}{(n-1)!} x^{n-1} \right] \\
\implies &= \sum_{n=0}^{\infty} \left[\frac{x^n}{n!} a_{n+1} \right]
\end{aligned}
\tag{26}
$$

If $f(x) = \sum_{n=0}^{\infty} \left[a_n \frac{x^n}{n!} \right]$ can it be shown by induction that $\frac{\mathrm{d}^k}{\mathrm{d}x^k} (f(x)) = f^k(x) \sum_{n=0}^{\infty} \left[x^n \frac{a_{n+k}}{n!} \right]$

4.4   Homogeneous Proof

An equation of the form:

$$\sum_{n=0}^{\infty} \left[c_i \cdot f^{(n)}(x) \right] = 0 \tag{27}$$

is said to be a homogenous linear ODE:

Linear because the equation is linear with respect to $f(x)$

Ordinary because there are no partial derivatives (e.g. $\frac{\partial}{\partial x}(f(x))$)

Differential because the derivates of the function are concerned

Homogenous because the ***RHS*** is 0

- A non-homogeous equation would have a non-zero RHS

There will be k solutions to a kth order linear ODE, each may be summed to produce a superposition which will also be a solution to the equation, [10, Ch. 4] this will be considered as the desired complete solution (and this will be shown to be the only solution for the recurrence relation (11)). These k solutions will be in one of two forms:

1. $f(x) = c_i \cdot e^{m_i x}$

2. $f(x) = c_i \cdot x^j \cdot e^{m_i x}$

where:

- $\sum_{i=0}^{k} \left[c_i m^{k-i} \right] = 0$

    - This is referred to the characteristic equation of the recurrence relation or ODE [5]

- $\exists i, j \in \mathbb{Z}^+ \cap [0, k]$

    - These is often referred to as repeated roots [5, 11] with a multiplicity corresponding to the number of repetitions of that root [8, §3.2]

Unique Roots of Characteristic Equation

Example   An example of a recurrence relation with all unique roots is the fibonacci sequence, as described in section 4.4 .

Proof   Consider the linear recurrence relation (11):

$$\sum_{n=0}^{\infty} [c_i \cdot a_n] = 0, \quad \exists c \in \mathbb{R}, \ \forall i < k \in \mathbb{Z}^+ \tag{11^2}$$

By implementing the exponential generating function as shown in (21), this provides:

$$\sum_{i=0}^{k} [c_i \cdot a_n] = 0$$

By Multiplying through and summing:

$$\implies \sum_{i=0}^{k} \left[\sum_{n=0}^{\infty} \left[c_i a_n \frac{x^n}{n!} \right] \right] = 0$$

$$\sum_{i=0}^{k} \left[c_i \sum_{n=0}^{\infty} \left[a_n \frac{x^n}{n!} \right] \right] = 0 \tag{28}$$

Recall from (21) xxx the generating function $f(x)$:

$$\sum_{i=0}^{k} \left[c_i f^{(k)}(x) \right] = 0 \tag{29}$$

Now assume that the solution exists and all roots of the characteristic polynomial are unique (i.e. the solution is of the form $f(x) \propto e^{m_i x} : \quad m_i \neq m_j \forall i \neq j$), this implies that [10, Ch. 4] :

$$f(x) = \sum_{i=0}^{k} [k_i e^{m_i x}], \quad \exists m, k \in \mathbb{C}$$

This can be re-expressed in terms of the exponential power series, in order to relate the solution of the function $f(x)$ back to a solution of the sequence a_n, (see section ¶ 5 for a derivation of the exponential power series):

$$\sum_{i=0}^{k} [k_i e^{m_i x}] = \sum_{i=0}^{k} \left[k_i \sum_{n=0}^{\infty} \frac{(m_i x)^n}{n!} \right]$$

$$= \sum_{i=0}^{k} \sum_{n=0}^{\infty} k_i m_i^n \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{k} k_i m_i^n \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left[\frac{x^n}{n!} \sum_{i=0}^{k} [k_i m_i^n] \right], \quad \exists k_i \in \mathbb{C}, \ \forall i \in \mathbb{Z}^+ \cap [1, k] \tag{30}$$

14

Recall the definition of the generating function from 29, by relating this to (30):

$$f(x) = \sum_{n=0}^{\infty} \left[\frac{x^n}{n!} a_n \right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{x^n}{n!} \sum_{i=0}^{k} [k_i m_i^n] \right]$$

$$\implies a_n = \sum_{n=0}^{k} [k_i m_i^n]$$

$\square$

This can be verified by the fibonacci sequence as shown in section xxx, the solution to the characteristic equation is $m_1 = \varphi, m_2 = (1 - \varphi)$ and the corresponding solution to the linear ODE and recursive relation are:

$$f(x) = c_1 e^{\varphi x} + c_2 e^{(1-\varphi)x}, \quad \exists c_1, c_2 \in \mathbb{R} \subset \mathbb{C}$$

$$\iff a_n = k_1 n^{\varphi} + k_2 n^{1-\varphi}, \quad \exists k_1, k_2 \in \mathbb{R} \subset \mathbb{C}$$

Repeated Roots of Characteristic Equation

Example

Proof

General Proof

Consider a Recursive relation with constant coefficients:

$$\sum_{n=0}^{\infty} [c_i \cdot a_n] = 0, \quad \exists c \in \mathbb{R}, \ \forall i < k \in \mathbb{Z}^+$$

This can be expressed in terms of the exponential generating function:

$$\sum_{n=0}^{\infty} [c_i \cdot a_n] = 0 \implies \sum_{n=0}^{\infty} \left[\sum_{n=0}^{\infty} [c_i \cdot a_n] \right] = 0$$

¶ 4 Links to references

1. https://math.stackexchange.com/a/1775226

2. https://math.stackexchange.com/a/593553

3. https://www.maa.org/sites/default/files/pdf/upload_library/22/Ford/IvanNiven.pdf

Misc

1. https://brilliant.org/wiki/generating-functions-solving-recurrence-relations/

2. https://www.math.cmu.edu/~af1p/Teaching/Combinatorics/Slides/Generating-Functions.pdf

3. https://www.math.cmu.edu/~af1p/Teaching/Combinatorics/Slides/Generating-Functions.pdf

§ 5 Prove the Power Series

¶ 5 General Power Series

¶ 5 Exponential Power Series

¶ 5 Taylor Series

When n is set to 0, the *Taylor Series* reduces to the *Mean Value Theorem*, when a is set to 0 the series is referred to as the *Maclaurin Series*.

References

[1] I. Barradas and J. E. Cohen. "Iterated Exponentiation, Matrix-Matrix Exponentiation, and Entropy". In: *Journal of Mathematical Analysis and Applications* 183.1 (Apr. 1, 1994), pp. 76–88. ISSN: 0022-247X. DOI: 10.1006/jmaa.1994.1132. URL: http://www.sciencedirect.com/science/article/pii/S0022247X84711322 (visited on 08/02/2020) (cit. on p. 5).

[2] Brian C. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Second edition. Graduate Texts in Mathematics 222. OCLC: ocn910324548. Cham ; New York: Springer, 2015. 449 pp. ISBN: 978-3-319-13466-6 (cit. on pp. 1, 3).

[3] Donald Ervin Knuth. *The Art of Computer Programming*. 3rd ed. Reading, Mass: Addison-Wesley, 1997. 3 pp. ISBN: 978-0-201-89683-1 978-0-201-89684-8 978-0-201-89685-5 (cit. on p. 7).

[4] Eric Lehman, Tom Leighton, and Albert Meyer. *Readings | Mathematics for Computer Science | Electrical Engineering and Computer Science | MIT OpenCourseWare*. Sept. 8, 2010. URL: https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2010/readings/ (visited on 08/10/2020) (cit. on pp. 7, 12).

[5] Oscar Levin. *Solving Recurrence Relations*. Jan. 29, 2018. URL: http://discrete.openmathbooks.org/dmoi2/sec_recurrence.html (visited on 08/11/2020) (cit. on p. 13).

[6] *Matrices SymPy 1.6.1 Documentation*. July 2, 2020. URL: https://docs.sympy.org/latest/tutorial/matrices.html (visited on 08/02/2020) (cit. on p. 2).

[7] *Matrix Exponential ŭ Issue #6218 ŭ Sympy/Sympy*. Feb. 9, 2019. URL: https://github.com/sympy/sympy/issues/6218 (visited on 08/02/2020) (cit. on p. 2).

[8] Olympia Nicodemi, Melissa A. Sutherland, and Gary W. Towsley. *An Introduction to Abstract Algebra with Notes to the Future Teacher*. OCLC: 253915717. Upper Saddle River, NJ: Pearson Prentice Hall, 2007. 436 pp. ISBN: 978-0-13-101963-8 (cit. on p. 13).

[9] Laurence A. F. Park and Simeon Simoff. "Power Walk: Revisiting the Random Surfer". In: *Proceedings of the 18th Australasian Document Computing Symposium*. ADCS '13. Brisbane, Queensland, Australia: Association for Computing Machinery, Dec. 5, 2013, pp. 50–57. ISBN: 978-1-4503-2524-0. DOI: 10.1145/2537734.2537749. URL: http://doi.org/10.1145/2537734.2537749 (visited on 07/31/2020) (cit. on p. 1).

[10] Dennis G Zill and Michael R Cullen. *Differential Equations*. 7th ed. Brooks/Cole, 2009 (cit. on pp. 1, 3, 13, 14).

[11] Dennis G. Zill and Michael R. Cullen. "8.4 Matrix Exponential". In: *Differential Equations with Boundary-Value Problems*. 7th ed. Belmont, CA: Brooks/Cole, Cengage Learning, 2009. ISBN: 978-0-495-10836-8 (cit. on p. 13).