How to use Generating Functions to Solve Recursive Linear Relation

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Problem Questions

Given the Linear Recurrence Relation:

$$\begin{aligned} a_0 &= 1 \\ a_0 &= 1 \\ a_{n+2} = a_{n+1+2a_n}, \quad n &\geq 0 \end{aligned}$$

To solve this we can use what's known as a Generating Function, see the disucssion below

We will make consider the function $f(x)$ such that:

$$f(x) = \sum_{n=0}^{\infty} [a_n x^n]$$

It can be shown (see below) that:

$$\begin{aligned} \sum_{n=0}^{\infty} [a_{n+1} x^n] &= \frac{f(x) - a_0}{x} \\ \sum_{n=0}^{\infty} [a_{n+2} x^n] &= \frac{f(x) - a_0 - a_1 x}{x^2} \end{aligned}$$

So to use the generating Function consider:

$$2a_n + a_{n+1} = a_{n+2}$$
$$2a_n x^n + a_{n+1} x^n = a_{n+2} x^n$$
$$\sum_{n=0}^{\infty} [2a_n x^n] + \sum_{n=0}^{\infty} [a_{n+1} x^n] = \sum_{n=0}^{\infty} [a_{n+2} x^n]$$

By applying the previous identity:

$$\begin{aligned} 2f(x) + \frac{f(x) - a_0}{x} &= \frac{f(x) - a_0}{-a_1 x} x^2 \\ \implies f(x) &= \frac{1}{1 - x - x^2} \end{aligned}$$

WARNING
I accidently dropped the 2 here, it doesn't matter but it does show that how this could be dealt with algebraically

Now this can be solved by way of a power series, (see for example 11_Series), but first it is necessary to use partial fractions to split it up.

By partial fractions it is known:

$$\begin{aligned} f(x) &= \frac{1}{1 - x - x^2} \\ &= \frac{-1}{x^2 + x - 1} \\ &= \frac{-1}{(x-2)(x-1)} \\ &= \frac{A_1}{x-2} + \frac{A_2}{x-1}, \quad A_i \in \mathbb{R}, i \in \mathbb{Z}^+ \\ \implies -1 &= A_1 (x-1) + A_2 (x-2) \end{aligned}$$

Let $x = 2$:

$$\begin{aligned} -1 &= A_1 (2-1) + 0 \\ &= A_1 = -1 \end{aligned}$$

Let $x = 1$:

$$\begin{aligned} -1 &= 0 + A_2 (1-2) \\ \implies A_2 &= 1 \end{aligned}$$

Hence:

$$f(x) = \frac{1}{x-1} - \frac{1}{x-2}$$

Now because it is known that:

$$\sum_{n=0}^{\infty}[rx^n] = \frac{1}{1-rx^n}$$

we can conclude that:

$$\begin{aligned}
\frac{1}{x-1} &= -\frac{1}{1-(1)\,x} \\
&= -\sum_{n=0}^{\infty}[x^n] \\
\frac{-1}{x-2} &= \frac{1}{2-x} \\
&= \frac{1}{2}\frac{1}{1-\frac{1}{2}x} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left[\left(\frac{1}{2}x\right)^n\right]
\end{aligned}$$

and so:

$$\begin{aligned}
f(x) &= \frac{1}{2}\sum_{n=0}^{\infty}\left[\left(\frac{1}{2}x\right)^n\right] - \sum_{n=0}^{\infty}[x^n] \\
f(x) &= \sum_{n=0}^{\infty}\left[\frac{1}{2}\left(\frac{1}{2}x\right)^n - x^n\right] \\
f(x) &= \sum_{n=0}^{\infty}\left[\frac{1}{2\cdot 2^n}x^n - x^n\right] \\
f(x) &= \sum_{n=0}^{\infty}\left[x^n\left(\frac{1}{2\cdot 2^n} - 1\right)\right] \\
\implies a_n &= \frac{1}{2\cdot 2^n} - 1
\end{aligned}$$

Generating Functions

A Generating Function is a way of encoding an infinite series of numbers (a_n) by treating them as the coeficcients of a power series $(\sum_{n=0}^{\infty}[a_n x^n])$.

The variable remains in an indeterminate form and they were first introduced by Abraham De Moivre in 1730 in order to solve the general linear recurrence problem [1]

[1] Donald E. Knuth, The Art of Computer Programming, Volume 1 Fundamental Algorithms

Using the Power series for the Exponential Function

Motivation

Consider the *Fibonacci Sequence*:

$$\begin{aligned} & a_n = a_{n-1} + a_{n-2} \\ \iff & a_{n+2} = a_{n+1} + a_n \end{aligned}$$

Solving this outright is quite difficult, previously we used a power series generating function to solve it, something to the effect of:

$$x^2 f(x) - x f(x) - f(x) = 0$$

This however is still a little tricky, however, just from observation, the following would be fairly easy to deal with:

$$f''(x) - f'(x) - f(x) = 0$$

This would however imply that $f(x) = e^x$ because $\frac{\mathrm{d}(e^x)}{\mathrm{d}x} = e^x$, but that's fine because we have a power series for that already:

Fix

$$f(x) = e x = \sum_{n=0}^{\infty} \left[r \frac{x^n}{n!} \right]$$

So this would give an easy means by which to solve the linear recurrence relation.

Solving the Sequence

Now this is all well and good but if we could relate this to $f(x) = e^x$ we'd really be cooking with fire because we could connect linear recurrence relations to non-homogenous linear differential equations.

Consider using the following generating function:

(Third Edition) Addison-Wesley. ISBN 0-201-89683-4. Section 1.2.9: "Generating Functions".

$$f(x) = \sum_{n=0}^{\infty} \left[a_n \cdot \frac{x^n}{n!} \right] \qquad = e^x$$

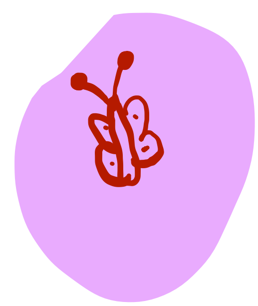
$\mathcal{TODO}$ :: The real trick is showing this derivative property

$$f'(x) = \sum_{n=0}^{\infty} \left[a_{n+1} \cdot \frac{x^n}{n!} \right] \qquad = e^x$$

$$f''(x) = \sum_{n=0}^{\infty} \left[a_{n+2} \cdot \frac{x^n}{n!} \right] \qquad = e^x$$

So the recursive relation from above could be expressed:

$$\begin{aligned} a_{n+2} &= a_{n+1} + a_n \\ \frac{x^n}{n!} a_{n+2} &= \frac{x^n}{n!} \left(a_{n+1} + a_n \right) \\ \sum_{n=0}^{\infty} \left[\frac{x^n}{n!} a_{n+2} \right] &= \sum_{n=0}^{\infty} \left[\frac{x^n}{n!} a_{n+1} \right] + \sum_{n=0}^{\infty} \left[\frac{x^n}{n!} a_n \right] \\ f''(x) &= f'(x) + f(x) \end{aligned}$$



Using the theory of higher order linear differential equations with constant coefficients it can be shown:

$$f(x) = c_1 \cdot \exp\left[\left(\frac{1-\sqrt{5}}{2} \right) x \right] + c_2 \cdot \exp\left[\left(\frac{1+\sqrt{5}}{2} \right) \right]$$

By equating this to the power series:

$$f(x) = \sum_{n=0}^{\infty} \left[\left(c_1 \left(\frac{1-\sqrt{5}}{2} \right)^n + c_2 \cdot \left(\frac{1+\sqrt{5}}{2} \right)^n \right) \cdot \frac{x^n}{n} \right]$$

Now given that:

$$f(x) = \sum_{n=0}^{\infty} \left[a_n \frac{x^n}{n!} \right]$$

We can conclude that:

$$a_n = c_1 \cdot \left(\frac{1-\sqrt{5}}{2} \right)^n + c_2 \cdot \left(\frac{1+\sqrt{5}}{2} \right)$$

By applying the initial conditions:

$$a_0 = c_1 + c_2 \implies c_1 = -c_2$$
$$a_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) - c_1\frac{1-\sqrt{5}}{2} \implies c_1 = \frac{1}{\sqrt{5}}$$

And so finally we have:

$$\begin{aligned} a_n &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \\ &= \frac{\varphi^n - \psi^n}{\sqrt{5}} \\ &= \frac{\varphi^n - \psi^n}{\varphi - \psi} \end{aligned}$$

1.abcde....
0.abcde ....
.6180....

$\varphi = \frac{1-\sqrt{5}}{2}$

$\psi = 1 - \varphi$

where:

- $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61\ldots$
- $\psi = 1 - \varphi = \frac{1-\sqrt{5}}{2} \approx 0.61\ldots$

0,1,1,2,3,5,8,13..

Open Questions:

- Show that the derivitive of the power series is a_{n+2}
- Redo the initial problem for the Fibonacci Sequence
- Extend this to a non-homogenous equation
- Extend this to all linear recursion problems with first order ODES
- Show that this is an isomorphism Lindear ODEs with constant coefficients to recursive relations with constant coefficients.

References

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Misc

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5. https://www.math.cmu.edu/~af1p/Teaching/Combinatorics/Slides/Generating-Functions.pdf
6. https://www.math.cmu.edu/~af1p/Teaching/Combinatorics/Slides/Generating-Functions.pdf