On Rowland's Sequence

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Joint work with Fernando Chamizo and Dulcinea Raboso

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Rowland's Sequence 2008

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k	1	2	3	4	5	6	7	8	9	10	11	...
a_k	7	8	9	10	15	18	19	20	21	22	33	...
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The sequence $\{a_k - a_{k-1}\}_{k>1}$ has a surprising property:

$\{a_k - a_{k-1}\} =$
1, 1, 1, 5, 3, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 23, 3, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 47, 3, 1, 5, 3, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 101, 3, 1, 1, 7,
1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 13, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 233, 3, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
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1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 101, 3, 1, 1, 7,
1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 13, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 233, 3, 1, 1, 1,
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Every term distinct than 1 is a prime number.

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Theorem

$a_k - a_{k-1}$ is 1 or prime for $k > 1$.

Auxiliary sequences

We have an elementary proof for this theorem.

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$$\begin{cases} c_1^* = 5 \\ c_n^* = c_{n-1}^* + \mathsf{lpf}(c_{n-1}^*) - 1,\ n > 1 \end{cases} ; \qquad r_n^* = \frac{c_n^* + 1}{2},\ n \geq 1.$$

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$$a_k - a_{k-1} = \begin{cases} \mathrm{lpf}(c_{n-1}^*), & \text{if } k = r_n^* \text{ for some } n > 1. \\ 1, & \text{otherwise} \end{cases}$$

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Using this proof, we obtain another amazing result:

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- Also, $a_1 = 1$ and $a_1 = 3$ lead to the sequences $a_k = k$ and $a_k = k + 2$, respectively.
- Unfortunately, under these conditions, $a_k - a_{k-1}$ is not necessarily prime. For instance, if $a_1 = 533$, then $a_{18} - a_{17} = 9$.

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$$\begin{cases} r_1 = 1 \\ r_{n+1} = \min\{p + p\lfloor r_n/p \rfloor : p|c_n\},\ n \geq 1 \end{cases}$$

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Equivalently, $r_{n+1} = \min\{k > r_n : (k, c_n) \neq 1\}$.

Proposition

For any odd $a_1 > 3$, the sequence $\{a_k\}$ satifies $a_k = c_n + k + 1$, for $r_n \leq k < r_{n+1}$. Moreover,

$$a_k - a_{k-1} = \begin{cases} \gcd(c_{n-1}, r_n), & \text{if } k = r_n \text{ for some } n > 1. \\ 1, & \text{otherwise.} \end{cases}$$

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Conjecture (A)

For any Generalized Rowland's Sequence, there exists a positive integer N such that $a_k - a_{k-1}$ is 1 or prime for every $k > N$.

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n	1	2	3	4	5	6	7	8	9	10	$\dots$
r_n	1						$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
c_n	115						$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

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r_n	1	5	7	10	12	131	...	...	...	...	...
c_n	115	119	125	129	131	261	...	...	...	...	...

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n	1	2	3	4	5	6	7	8	9	10	...
r_n	1	5	7	10	12	131	132	263	264	272	...
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In this example, c_5 is the first prime term in the sequence $\{c_n\}$.

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If $r_n = c_{n-1}$, then $c_n = 2r_n - 1$.

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r_n	1	3	5	6	41	42	83	84	167	168	...
c_n	33	35	39	41	81	83	165	167	333	335	...

n	1	2	3	4	5	6	7	8	9	10	11
r_n	1	7	11	17	18	20	21	29	30	35	587
c_n	511	517	527	543	545	549	551	579	581	587	1173

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Conjecture (A) holds if $c_n = 2r_n - 1$ for some $n \in \mathbb{Z}^+$, or if c_m is prime for some $m \in \mathbb{Z}^+$. Computations suggest that this happens for any initial condition a_1. Moreover, it seems that the minimal choices of m and n in these claims are always consecutive.

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Conjecture (B)

For an odd $a_1 > 3$, we define (writing $\inf \emptyset = \infty$)

$$\begin{aligned} n_0 &= \inf\{n \in \mathbb{Z}^+ : c_n = 2r_n - 1\}, \\ m_0 &= \inf\{n \in \mathbb{Z}^+ : c_n \text{ is prime}\}. \end{aligned}$$

Then

(i) $n_0 < \infty$, (ii) $m_0 < \infty$, (iii) $n_0 = m_0 + 1 < \infty$

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As we saw before, (ii) implies (i). Trivially, (iii) implies (ii) and (i).

Relation between the conjectures

If Conjecture (B) is true, then so it is Conjecture (A):

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Proposition

For any odd $a_1 > 3$, $r_n \leq (c_n + 1)/2$, for every $n > 1$. Moreover, the equality occurs if and only if $\gcd(c_{n-1}, r_n)$ is prime p and $p\lfloor r_{n-1}/p \rfloor = (c_{n-1} - p)/2$.

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Under (iii), there exists a prime p such that

$$\begin{aligned} \frac{p+1}{2} &= \inf\{k : a_k = 3k\}, \\ p &= \inf\{k : a_k = 3k,\ a_k - a_{k-1} > 1\}. \end{aligned}$$

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Given a_1, with $m_0 < \infty$; and taking $a_1' = a_1 + c_{m_0}!$, it can be proved that $m_0' > m_0$.

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1, 1, 11, 3, 1, 1, 1, 1, 1, 13, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
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Removing the ones, we obtain a sequence of primes:
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In general, a Rowland's Chain is any subsequence of concatenated primes inside a sequence $\{a_k - a_{k-1}\}_{k \geq n_0}$, for any odd $a_1 > 3$.

We associate to the chain $C_k = \{p_1, p_2, \ldots, p_k\}$ the shifted partial sums

$$S(n) = \sum_{j<n} (p_j - 1), \quad S(1) = 0.$$

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The following is a characterization of Rowland's chains:

Proposition

C_k is a Rowland's Chain if and only if it verifies these conditions:

- $S(m) \equiv S(n) \pmod{p_n}$, when $p_n = p_m$.
- $S(m) \not\equiv S(n) \pmod{p_n}$, when $p_n < p_m$.
- For any prime q, the set $\{S(j) \pmod{q} : p_j > q\}$ does not contain all residue classes modulo q.

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- For any prime q, the set $\{S(j) \pmod{q} : p_j > q\}$ does not contain all residue classes modulo q.

There are lots of restrictions. For instance, $\{11, 5, p\}$ is not a Rowland's chain for any $p > 3$.

Corollary

If $p_1, p_2, \dots, p_k$ are distinct primes, then
$C_{2k} = \{p_1, p_2, \dots, p_k, p_1, p_2, \dots, p_k\}$ is not a Rowland's chain.

Corollary

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In spite of the last Corollary, large non-consecutive repetitions can be found. For instance,

$$\begin{aligned} C_{27} &= \{3, 5, 3, 23, 3, 5, 3, 653, 3, 5, 3, 23, 3, 5, 3, 3603833, \\ &\quad 3, 5, 3, 23, 3, 5, 3, 653, 3, 5, 3\} \end{aligned}$$

has length 27 but just 5 primes. This Chain corresponds to $a_1 = 155030303168220 5$.

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