How to use Generating Functions to Solve Recursive Linear Relation

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Problem Questions

Given the Linear Recurrence Relation:

$$\begin{aligned} a_0 &= 1 \\ a_0 &= 1 \\ a_{n+2} = a_{n+1+2a_n}, \quad n &\geq 0 \end{aligned}$$

To solve this we can use what's known as a Generating Function, see the disucssion below

We will make consider the function $f(x)$ such that:

$$f(x) = \sum_{n=0}^{\infty} [a_n x^n]$$

It can be shown (see below) that:

$$\begin{aligned} \sum_{n=0}^{\infty} [a_{n+1} x^n] &= \frac{f(x) - a_0}{x} \\ \sum_{n=0}^{\infty} [a_{n+2} x^n] &= \frac{f(x) - a_0 - a_1 x}{x^2} \end{aligned}$$

So to use the generating Function consider:

$$
\begin{aligned}
2a_n + a_{n+1} &= a_{n+2} \\
2a_n x^n + a_{n+1}x^n &= a_{n+2}x^n \\
\sum_{n=0}^{\infty}\left[2a_n x^n\right] + \sum_{n=0}^{\infty}\left[a_{n+1}x^n\right] &= \sum_{n=0}^{\infty}\left[a_{n+2}x^n\right]
\end{aligned}
$$

By applying the previous identity:

$$
\begin{aligned}
2f(x) + \frac{f(x) - a_0}{x} &= \frac{f(x) - a_0}{-a_1 x} x^2 \\
\implies f(x) &= \frac{1}{1 - x - x^2}
\end{aligned}
$$

WARNING
I accidently dropped the 2 here, it doesn't matter but it does show that how this could be dealt with algebraically

Now this can be solved by way of a power series, (see for example 11_Series), but first it is necessary to use partial fractions to split it up.

By partial fractions it is known:

$$
\begin{aligned}
f(x) &= \frac{1}{1 - x - x^2} \\
&= \frac{-1}{x^2 + x - 1} \\
&= \frac{-1}{(x-2)(x-1)} \\
&= \frac{A_1}{x-2} + \frac{A_2}{x-1}, \quad A_i \in \mathbb{R}, i \in \mathbb{Z}^+ \\
\implies -1 &= A_1(x-1) + A_2(x-2) \\
\text{Let } x &= 2\text{:} \\
-1 &= A_1(2-1) + 0 \\
&= A_1 = -1 \\
\text{Let } x &= 1\text{:} \\
-1 &= 0 + A_2(1-2) \\
\implies A_2 &= 1 \\
\text{Hence:} & \\
f(x) &= \frac{1}{x-1} - \frac{1}{x-2}
\end{aligned}
$$

Now because it is known that:

$$\sum_{n=0}^{\infty} [rx^n] = \frac{1}{1 - rx^n}$$

we can conclude that:

$$\begin{aligned}
\frac{1}{x-1} &= -\frac{1}{1-(1)\,x} \\
&= -\sum_{n=0}^{\infty} [x^n] \\
\frac{-1}{x-2} &= \frac{1}{2-x} \\
&= \frac{1}{2}\frac{1}{1-\frac{1}{2}x} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left[\left(\frac{1}{2}x\right)^n\right]
\end{aligned}$$

and so:

$$\begin{aligned}
f(x) &= \frac{1}{2}\sum_{n=0}^{\infty}\left[\left(\frac{1}{2}x\right)^n\right] - \sum_{n=0}^{\infty}[x^n] \\
f(x) &= \sum_{n=0}^{\infty}\left[\frac{1}{2}\left(\frac{1}{2}x\right)^n - x^n\right] \\
f(x) &= \sum_{n=0}^{\infty}\left[\frac{1}{2\cdot 2^n}x^n - x^n\right] \\
f(x) &= \sum_{n=0}^{\infty}\left[x^n\left(\frac{1}{2\cdot 2^n} - 1\right)\right] \\
\implies a_n &= \frac{1}{2\cdot 2^n} - 1
\end{aligned}$$

Generating Functions

A Generating Function is a way of encoding an infinite series of numbers (a_n) by treating them as the coeficcients of a power series ($\sum_{n=0}^{\infty} [a_n x^n]$).

The variable remains in an indeterminate form and they were first introduced by Abraham De Moivre in 1730 in order to solve the general linear recurrence problem [1]

[1] Donald E. Knuth, The Art of Computer Programming, Volume 1 Fundamental Algorithms

Using the Power series for the Exponential Function

Now this is all well and good but if we could relate this to $f(x) = e^x$ we'd really be cooking with fire because we could connect linear recurrence relations to non-homogenous linear differential equations.

(Third Edition) Addison-Wesley. ISBN 0-201-89683-4. Section 1.2.9: "Generating Functions".