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Lectures on fractals + dimension

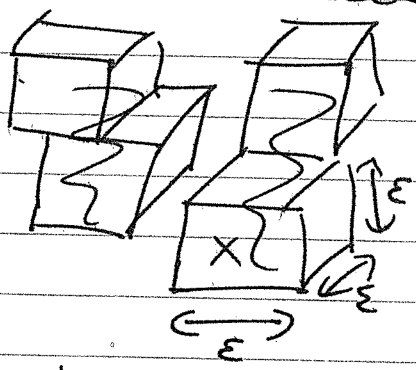
Lecture I

Basic Question. Let $X \subseteq \mathbb{R}^d$ be a closed bounded set: How do we specify its "size"?

We want to introduce a general notion of "dimension". In fact we will consider three different notions:

(a) Box dimension

Definition: For each $\varepsilon > 0$, let $N(\varepsilon)$ be the smallest number of ε-boxes needed to cover X.



We can define the box dimension by

$$\dim_B(X) = \overline{\lim_{\varepsilon \to 0}} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}.$$

Note (a) In many examples the $\overline{\lim}$ is actually a limit

(b) For familiar examples this coincides with top. dimension: $\dim_B(\{0\}) = 0$, $\dim_B([0,1]) = 1$, etc.

Problems

1) Show that for $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ we have $\dim_B(X) = 1/2$ and $\dim_H(X) = 0$

(More generally $\dim_H(Y) = 0$ for any countable set Y)

2) Show that a set $X \subseteq \mathbb{R}$ with $\dim_B(X) < 1$ is necessarily of zero Lebesgue measure.

(Same for Hausdorff Dimension)

Show that if $Leb(X) > 0$ then $\dim_H(X) = d$.

3) Show that $\dim_H(X) \leq \dim_B(X)$

The definition is simple, but often more subtle is the next definition:

(b) Hausdorff Dimension

Definition. For each $\varepsilon > 0$ we can consider covers $\mathcal{U} = \{U_i\}_{i=0}^{\infty}$ for X by open sets U_i with diameter$(U_i) \leq \varepsilon$

- Given $\delta > 0$ we define:

$$H_\varepsilon^\delta(X) = \inf_{\mathcal{U}} \sum_i (\mathrm{diam}(U_i))^\delta \geq 0$$

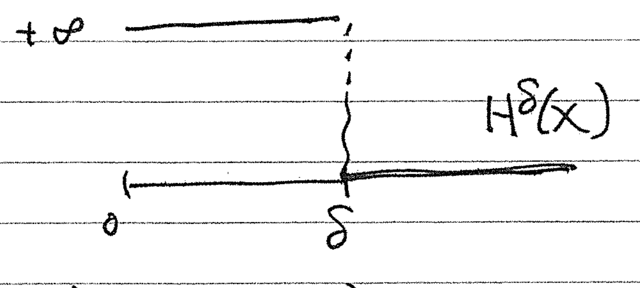
where the infimum is over all such covers.

- We define

$$H^\delta(X) = \lim_{\varepsilon \to 0} H_\varepsilon^\delta(X) \in [0, +\infty]$$

and then the Hausdorff dimension:

$$\dim_H(X) = \inf\{\delta : H^\delta(X) = 0\}$$



Note (i) $\dim_H(X) \leq \dim_B(X)$

(ii) The inequality can be strict e.g. for $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ we have $0 = \dim_H(X) < \dim_B(X) = 1$

Problems

2) Show that for the middle 1/3 Cantor set:

$$\begin{cases} \dim_H(X) = \dim_B(X) = \log 2/\log 3 \\ \dim_F(X) = 0. \end{cases}$$

3) Consider the middle $(1-2\lambda)$-Cantor set:

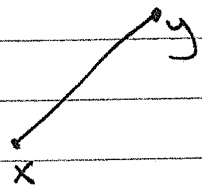
What is its Hausdorff Dimension / Box Dimension?

4) <u>Mass Distribution Principle</u>

Let μ be a probability measure on X (ie, $\mu(X)=1$)

Given $s>0$, the <u>s-energy</u> of μ is:

$$I_s(\mu) = \iint \frac{d\mu(x)\,d\mu(y)}{\|x-y\|^s}$$



<u>Mass Distribution Theorem</u>:

$$\boxed{\text{If } I_s(\mu) < +\infty \text{ then } \dim_H(X) \geq s}$$

Alternatively:
We can also define $\dim_H(X)$ in terms of probability measures μ (ie, $\mu(X)=1$)

- Given a probability measure μ with support $\mathrm{supp}(\mu) \subseteq X$ we define its Fourier transform:

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} \exp(i\langle \xi, x\rangle)\, d\mu(x)$$

where $\begin{cases} \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d, \\ x = (x_1, \dots, x_d) \in X \subseteq \mathbb{R}^d, \end{cases}$ $\langle \xi, x\rangle = \sum_{i=1}^{d} \xi_i x_i$

- One alternative definition of $\dim_H(X)$ is:

$$\dim_H(X) = \sup\Big\{ t>0 : \exists \mu \text{ with } \underbrace{\int \|\xi\|^{t-d} |\hat{\mu}(\xi)|^2 d\xi}_{I_t(\mu)} < +\infty \Big\}$$

(Related to Mass Distribution Principle)

This leads naturally to:

(c) Fourier Dimension.

Definition The Fourier dimension is defined by

$$\dim_F(X) = \sup\{ t>0 : \exists \mu \text{ with } |\hat{\mu}(\xi)| \le C.\|\xi\|^{-t/2} \}$$

Note (i) From the alternative definition: $\dim_F(X) \le \dim_H(X)$

(ii) This inequality can be strict eg. If X = middle third Cantor set then:

$$0 = \dim_F(X) < \dim_H(X) = \frac{\log 2}{\log 3}.$$

Problems

1) Prove existance of IFS. (by applying the fixed point theorem for contractions)

We would like to consider the case that X is "dynamical defined". We will approach this using iterated function schemes

Definition. An iterated function scheme (IFS) on an open set $U \subseteq \mathbb{R}^d$ consists of a finite family of contractions $T_i : U \to U$ $(i=1,\dots,k)$ (ie, $\|T_i(x) - T_i(y)\| \leq c\|x-y\|$ $(i=1,\dots,k)$ for all $x,y \in U$).

We can then associate a compact set $X = X(\{T_i\})$ using the following:

Proposition Let $\{T_i\}_{i=1}$ be an IFS. There exists a unique closed non-empty set $X = X(\{T_i\})$ such that

$$X = \bigcup_{i=1}^{k} T_i(X)$$

called the limit set

An alternative construction is to consider infinite sequences:

$$\Sigma = \{1,2,\dots,k\}^{\mathbb{N}}$$

We can write:

$$X = \left\{ \lim_{N\to\infty} T_{x_0} T_{x_1} \cdots T_{x_N}(\underline{0}) \;\middle|\; \underline{x} = (x_n)_{n=0}^{\infty} \in \Sigma \right\}$$

where $\underline{0} \in U$.

Problems

1) Verify the formulae of Bedford-McMullen for $\dim_B(X)$ ($\dim_H(X)$)

Examples (Middle Third Cantor set)

Let $\begin{cases} T_1 : \mathbb{R} \to \mathbb{R} \\ T_2 : \mathbb{R} \mapsto \mathbb{R} \end{cases}$ by $\begin{cases} T_1 x = x/3 \\ T_2 x = x/3 + 2/3 \end{cases}$

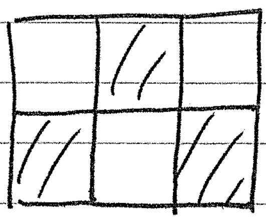
Then $\dim_H(X) = \dim_B(X) = \frac{\log 2}{\log 3}$ $(e \dim_F(C) = 0)$

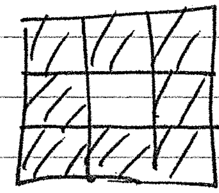
Examples (Bedford McMullen Carpets)

Let $l \geq m \geq 2$.

Let $S \subseteq \{0, 1, \ldots, l-1\} \times \{0, 1, \ldots, m-1\}$

Let $\begin{cases} T_s(x,y) = \left(\frac{x}{l}, \frac{y}{m}\right) + \left(\frac{s_1}{l}, \frac{s_2}{m}\right) \\ \quad \text{where } s = (s_1, s_2) \in S \end{cases}$

eg. 

eg. 

$l = m = 3$

$\# S = 8$.

(Sierpinski Carpet)

$l = 3, m = 2$, $S = \{(0,0), (1,1), (2,0)\}$

Proposition (Bedford, McMullen; 1984)

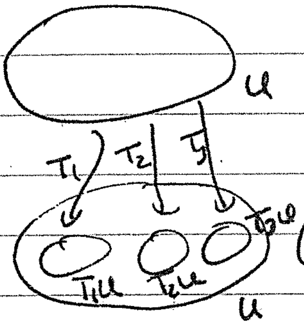
Let $\begin{cases} t_j = \#\{0 \leq i \leq l-1 : (i,j) \in S\}, \ 0 \leq j \leq m-1 \\ a = \# S \ (= t_0 + t_1 + \cdots + t_{m-1}) \end{cases}$

Then $\begin{cases} \dim_H(X) = \log_m \left(\sum_{j=0}^{m-1} t_j^{\log_l m} \right) \\ \dim_B(X) = 1 + \log_l \left(\frac{a}{m}\right) \end{cases}$

(For $l \neq m$ these are typically different)

We can also relate these examples to an expanding map on X denoted $T: X \to X$

Definition We say $\{T_i : U \to U\}$ satisfies the <u>open set condition</u> if there exists an open set $V \subseteq U$ such that



(i) $T_1(V), \ldots, T_n(V) \subseteq V$; and

(ii) $T_1(V), \ldots, T_n(V)$ are disjoint.

When $\{T_i\}_{i=1}^n$ has the open set condition we can define $\begin{cases} T: X \to X \\ Tx = T_i^{-1}x & \text{if } x \in T_i(X) \end{cases}$

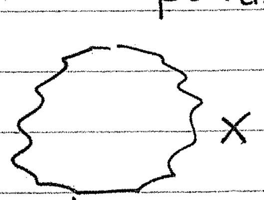
<u>Examples</u>: For middle third Cantor set X define

$$T(x) = 3x \pmod 1$$

<u>Example</u>: Let $\begin{cases} T: \mathbb{C} \to \mathbb{C} \\ T(z) = z^2 + c \end{cases}$ for $|c|$ small

Let $X = \overline{\{T^n x = x : |(T^n)'(x)| > 1\}}$ = closure of repelling periodic points
(Julia Set)

- X can be written as a (generalized) limit set for:



$$\begin{cases} T_1(z) = +\sqrt{z-c} \\ T_2(z) = -\sqrt{z-c} \end{cases}$$ (Extra care needed with domain)

- $T: X \to X$ preserves X.
- Typically there is no explicit formula for $\dim_H(X)$.

5) Sketch proof of Moran's Theorem (for 2 contractions) with different contraction rates

ie, If $\begin{cases} T_0 x = \lambda_1 x \\ T_1 x = \lambda_2 x + 1 \end{cases}$ show that

$$\dim_B(X) = d : \quad \lambda_1^d + \lambda_2^d = 1.$$

Example (Bedford-McMullen). Let X = limit set for the Bedford-McMullen carpets then

$$\begin{cases} T : X \to X \\ T(x,y) = (\ell x \pmod 1, mx \pmod 1) \end{cases}$$

preserves X.

Lecture 2

Q: When can we hope to have a relatively simple expression for $\dim_H(X)$?

We need to make additional assumptions:

Definition. We call $T_i : U \to U$ $(i=1,\ldots,k)$ **similarities** if there exists $1 > c_i > 0$ $(i=1,\ldots,k)$ such that

$$\|T_i(x) - T_i(y)\| = c_i \|x-y\|, \quad \forall x,y \in U.$$

This leads to the following:

Theorem (Moran) If $\{T_i\}_{i=1}^k$ is an iterated function scheme with

(i) The $\{T_i\}_{i=1}^k$ are similarities; and

(ii) $\{T_i\}_{i=1}^n$ satisfy the open set condition

Then $d = \dim_H(X)$ is the unique solution to

$$c_1^d + c_2^d + \cdots + c_k^d = 1$$

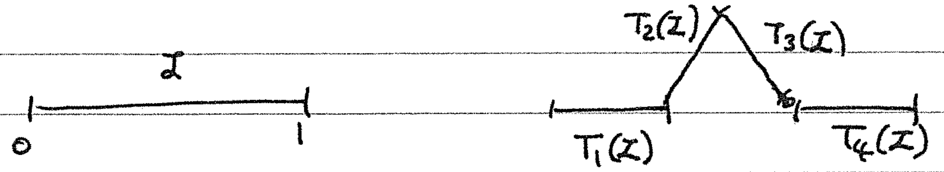
Examples (Middle third Cantor set). Recall $\begin{cases} T_1 x = x/3 \\ T_2 x = x/3 + 2/3 \end{cases}$

Thus $c_1 = c_2 = 1/3$ and $d = \dfrac{\log 2}{\log 3}$.

Example (Sierpinski Carpet). Let $T_{\underline{i}}(x,y) = \left(\frac{x}{3}, \frac{y}{3}\right) + \left(\frac{i_1}{3}, \frac{i_2}{3}\right)$ where $\underline{i} = (i_1, i_2)$ with $i_1, i_2 \in \{0,1,2\}$ and $(i_1,i_2) \neq (1,1)$ (thus 8 contractions and $c_{\underline{i}} = 1/3$).

Thus $d = \dfrac{\log 8}{\log 3}$.

Example (Koch Snowflake).



Then $n = 4$ and $c = 1/3$ and $d = \dfrac{\log 4}{\log 3}$.

Two Questions on generalizations:

Question 1: Can we replace the similarities assumption with something more general?

Question 2: Can we drop the hypothesis of the open set condition?

We will deal with the first question now and postpone the second question to later.

The generalization of similarities is conformality:

Definition We call $T_i : U \to U$ $(i = 1, \dots, n)$ contractions conformal if they are C^1 and $D_x T_i = c_i(x) \Theta_i(x)$ $(x \in U)$ with
$$\begin{cases} 0 < |c_i(x)| < 1 & \text{(contractions)} \\ \Theta_i(x) \in SO(n-1) & \text{(rotations)} \end{cases}$$

Notation: For convenience write $D_x T_i = T_i'(x)$

Examples (E_2): Nonlinear Cantor sets

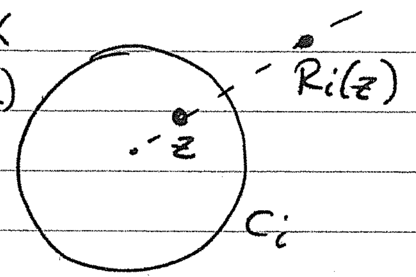
<u>Examples</u> (Hyperbolic Julia sets) As before

<u>Examples</u> (Schottky groups).

Let $C_1, \ldots, C_r$ be circles in $\mathbb{C}$ with disjoint interiors. If $C_i = \{z \in \mathbb{C} : |z - c_i| = r_i\}$ then we define

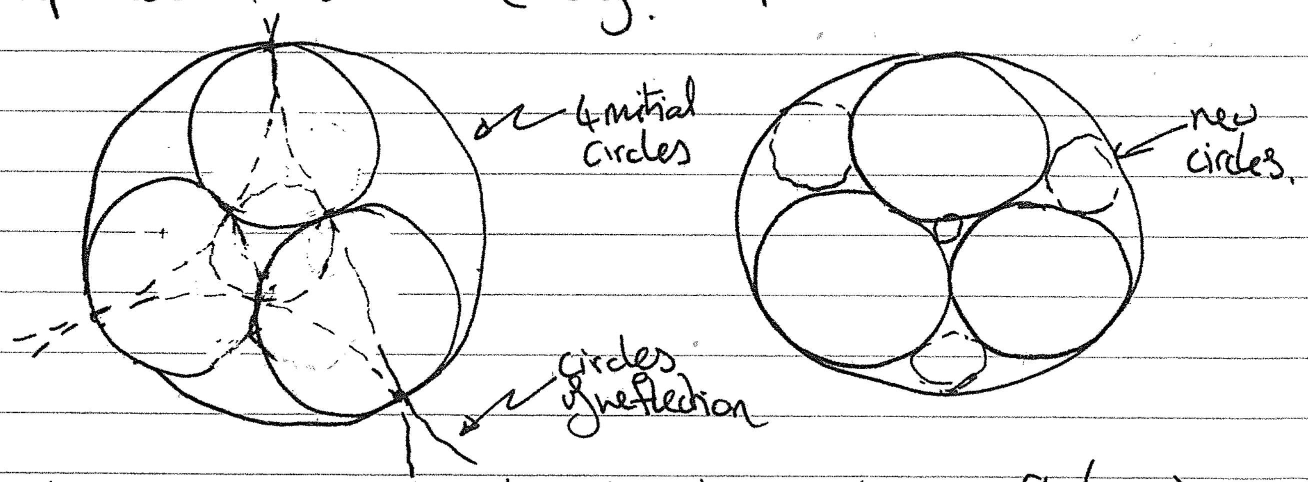
$$\begin{cases} R_i : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\} & (i = 1, \ldots, r) \\ R_i(z) = \dfrac{z - c_i}{|z - c_i|^2} + c_i & \text{(hyperbolic reflection)} \end{cases}$$

Let $T : X \to X$ by $Tx = R_i(x)$ if x inside C_i



If the circles are disjoint then X will be a Cantor set.

<u>Remark</u>. If the circles touch then X may have a complicated structure eg. $r = 4$: A



The dimension of the limit set (Apolonian circle packing) can only be numerically computed as $d = 1{\cdot}31\ldots$

- If $T_i'(x) \in [\alpha_i, \beta_i]$ then show $a \leq \dim_\mu(x) \leq b$ where $\sum_i \alpha_i^a = 1$ and $\sum_i \beta_i^b = 1$.

- If $f : X \to \mathbb{R}$ is Hölder continuous, show that the limit exists in the definition of $P(f)$

<u>Remark.</u> We can also define pressure by a variational principle, generalizing that for entropy:

$$P(f) = \sup\{h_\cdot(\mu) + \int f\,d\mu : \mu = T\text{-invariant probability}\}$$

- Show that if $f : X \to \mathbb{R}$ is Hölder continuous then we can replace $\overline{\lim}$ by $\lim$ in the definition of pressure.

Let $T : X \xrightarrow{C^2} X$ be an expanding map on the limit set corresponding to $T|_{T_i X} = T_i^{-1}$. The main tool is the following.

Thermodynamic Formalism.

Let $f : X \to \mathbb{R}$ be a continuous function.

Definition. We define the pressure $P(f) \in \mathbb{R}$ by:

$$P(f) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log \Big(\underbrace{\sum_{\substack{T^n x = x \\ x\in X}}}_{\text{Sum over periodic points.}} \exp\big(f(x) + f(Tx) + \cdots + f(T^{n-1}x)\big) \Big)$$

Application to Dimension: We want to consider a family of functions:

$$f_t = -t \log |T'(x)| , \quad t \in \mathbb{R}.$$

We then consider the function:

$$[0,d] \ni t \longmapsto P(f_t) \in \mathbb{R}.$$

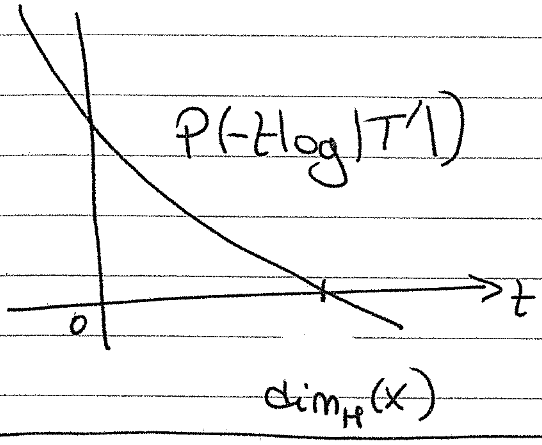
$$\qquad\qquad \| $$

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \Big(\sum_{\substack{T^n x = x \\ x\in X}} \frac{1}{|(T^n)'(x)|^t} \Big)$$

Theorem (Bowen 1978, Ruelle 1982). Let $T : X \to X$ be a C^2 conformal expanding map. There is a unique solution $0 \le t \le d$ to

$$P(-t \log |T'|) = 0$$

which occurs at $t = \dim_H (X)$.



Note: (i) $t \mapsto P(f_t)$ is monotone decreasing.
(ii) $t \mapsto P(f_t)$ is analytic on $[0,d]$

Corollary. If T_λ $(-\varepsilon < \lambda < \varepsilon)$ is an analytic family of expanding maps then $\lambda \mapsto \dim_H(\Lambda_\lambda)$ is analytic.

This follows from the implicit function theorem.

Application (Quadratic maps).

Let $\left\{ \begin{array}{l} T_c : \mathbb{C} \to \mathbb{C} \\ T_c(z) = z^2 + c \end{array} \right\}$ for $c \simeq 0$.

Let $X_c = X(T_c)$ be the Julia set for T_c.

Ruelle (1982): For c close to 0: $\dim_H(X_c) = 1 + \frac{|c|^2}{4\log 2} + o(|c|^2)$

Remark: If $\{T_i\}_{i=1}^d$ are conformal contractions satisfying the open set condition then: $\dim_H(X) = \dim_B(X)$.

(Uses Mass distribution Principle).

Computing Hausdorff Dimension (via zeta functions).

Question: If we don't have a simple explicit formula for $\dim_H(X)$ then can we estimate it?

Assume that $T : X \to X$ satisfies:

(i) Hyperbolicity ($\exists \lambda > 1$, $|T'(x)| \geq \lambda$, $\forall x \in X$);

(ii) $T : X \to X$ is conformal; and

(iii) Local maximality: $\exists$ open nhd $U \supseteq X$ with
$$X = \bigcap_{n=0}^{\infty} T^{-n}(U).$$

(iv) Markov structure: so that inverse branches are an IFS.

Examples: Hyperbolic rational map.

Examples: Schottky group limit set.

(I) First approach (after C. McMullen).

Fix $x \in X$. For each $n \geq 1$ we choose $s_n > 0$:
$$\sum_{T^n y = x} \left| (T^n)'(y) \right|^{-s_n} = 1.$$

Then $s_n = \dim_H(X_n) + O(\theta^n)$.

(for some $0 < \theta < 1$)

(II) Second approach (Jenkinson + P.)

Assume additionally:

(v) Analyticity - assume that $T : U \to \mathbb{R}^d$ is (real) analytic.

The algorithm (a) Let us define a sequence $(a_m)_{m=1}^{\infty}$:

$$a_m = a_m(s) = \sum_{T^m x = x} \frac{|(T^m)'(x)|^{-s}}{\det(I - (T^m)'(x))}; \quad \text{then}$$

(b) a sequence $(b_n(s))_{n=1}^{\infty}$ by

Dynamical ζ-function $\zeta(z,s)$

$$1 + \sum_{n=1}^{\infty} b_n(s) z^n = \exp\left(- \sum_{m=1}^{\infty} \frac{a_m(s) z^m}{m} \right)$$

(Take $z = 1$)

Finally, (c) we define approximations $(s_N)_{N=1}^{\infty}$ to $\dim_H(X)$ by s_N = solution to:

$$1 + \sum_{n=1}^{N} b_n(s) = 0.$$

$$\boxed{\begin{array}{l} \underline{\text{Theorem}}: \ \dim_H(X) = s_N + O(\theta^{N^2}) \\ (\text{for some } 0 < \theta < 1). \end{array}}$$

Example (E_2): Let $X = \left\{ x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \;\middle|\; a_n \in \{1,2\},\ n \geq 1 \right\}$

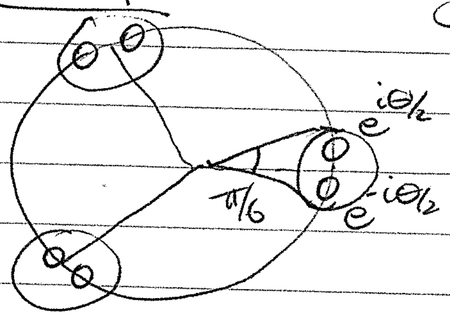
then we can estimate (with $N = 25$) (Friday seminar)

$\dim_H(E_2) = 0.5312\ldots$ (to 100 decimal places)

Examples (Julia set). Let $\begin{cases} T : \mathbb{C} \to \mathbb{C} \\ T(z) = z^2 + c \text{ with } c = -0.0 \end{cases}$

Then $\dim_H(X) = 1.001213662481746462\ldots$

Examples (Schottky groups)



Consider circles $C_0, C_1, C_2 \subseteq \mathbb{C}$ symmetrically placed on circle, meeting it orthogonally, with circular arc $\theta = \pi/6$

Let X = limit set by reflection ($\subseteq$ circle).

$\dim(X) = 0.1839$ ——— (38 decimal places)

Lecture 3

Measures, dimension and multifractal analysis

Let μ be a probability measure on X. (ie, $\mu(X)=1$)

We can define the Hausdorff dimension of μ (in terms of the dimension of the sets on which it sits)

Definition. We define the Hausdorff Dimension of μ by:

$$\dim_H(\mu) = \inf\{\dim_H(Y) : \mu(Y)=1\}$$

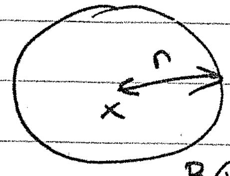
(Clearly, $\dim_H(\mu) \leq \dim_H(X)$)

We can also define a local notion of dimension:

Definition. We define the pointwise dimension of μ at $x\in X$ by:

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

when it exists!



$B(x,r)$

For IFSs there is a natural way to construct measures on X.

Let $\Sigma = \{1, 2, \ldots, k\}^{\mathbb{N}_0}$ be the space of infinite sequences, $\underline{x} = (x_n)_{n=0}^{\infty} \in \Sigma$.

Let $\pi : \Sigma \to X$

$$\pi(\underline{x}) = \lim_{N \to \infty} T_{x_0} T_{x_1} \cdots T_{x_N}(\underline{o}) \quad (\text{for any } \underline{o} \in U)$$

Given a probability measure μ on Σ we project it down to a probability measure $\nu := \mu \pi^{-1}$ on Σ

(ie, $\nu(B) = \mu(\pi^{-1}B)$ for $B \subseteq X$ Borel).

<u>Example</u>. For any probability vector $\underline{P} = (P_1, \ldots, P_k)$ we can associate the Bernoulli measure $\mu_{\underline{P}}$ on Σ, and then ν on X.

A more general class of measures on Σ are:

<u>Definition</u>: We say that a probability measure μ on Σ is a <u>Gibbs measure</u>, if:

(i) It is invariant under $\sigma : \Sigma \to \Sigma$ given by $\sigma(x_n) = (x_{n+1})$, ie, $\mu(A) = \mu(\sigma^{-1}A)$, $\forall A \subseteq \Sigma$ Borel.

(ii) The forward derivative:

$$\frac{d\mu\sigma}{d\mu}(\underline{x}) = \lim_{n \to \infty} \frac{\mu[x_1, x_2, \ldots, x_n]}{\mu[x_0, \ldots, x_n]}$$

is Hölder continuous.

(eg. The Bernoulli measure is Gibbs, with
$\frac{d\mu\sigma}{d\mu}(\underline{x}) = \frac{1}{p_{x_0}}$)

For Gibbs measures μ and their projections $\nu = \mu\pi^{-1}$ we have the following:

Theorem Let $\{T_i\}_{i=1}^k$ be an iterated function scheme of conformal C^2 contractions satisfying the open set condition.

Let ν be the projection of a Gibbs measure.

Then:

(i) The limit $d_\nu(x)$ exists for a.e. (ν) $x \in X$ and is equal to $\dim_H(\nu)$;

(ii) We have $\dim_H(\nu) = \frac{h_\nu(T)}{\int \log|T'| d\nu}$.

But we only know $d_\nu(x) = \dim_H(\nu)$ for almost every $x \in X$.

Question. Given $\alpha \neq \dim_H(\nu)$ how large is the set of x with $d_\nu(x) = \alpha$?

Let us define:

$$f(\alpha) := \dim_H(\{x \in X : d_\nu(x) = \alpha\})$$

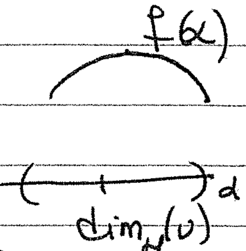
The following striking result was proved by Pesin + Weiss.

Theorem (Multifractal Analysis)

In a neighbourhood $(\dim_H(\nu)-\varepsilon, \dim_H(\nu)+\varepsilon)$ $(\varepsilon>0)$:

(a) $\alpha \longmapsto f(\alpha)$ is convex.

(b) $\alpha \longmapsto f(\alpha)$ is analytic



Sketch of Proof

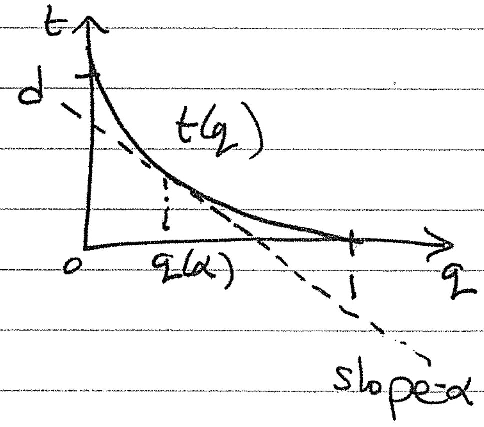
Given $q \in \mathbb{R}$ we can try to find $t = t(q)$:

$$P\left(-t.\log|T'| - q\log\left(\frac{d\mu\sigma}{d\mu}\right)\right) = 0$$

(When $q=0$, $t=\dim_H(\Lambda)$; when $q=1$, $t=0$)

(a) Given α we can try to find $q=q(\alpha)$:

$$\alpha = -\frac{\partial t}{\partial q}(q)$$



(b) Given q, $t(q)$ and $\alpha(q)$ we compute:

$$f(\alpha) = t(q) + q.\alpha(q)$$

- Works for α close to $\dim_H(\nu)$
- Analyticity comes from properties of pressure.

Overlaps and transversality.

Question: What happens if we don't assume the open set condition?

Consider the following IFS: $T_1, T_2, T_3 : \mathbb{R} \to \mathbb{R}$ by

$$\left.\begin{array}{l} T_1 x = \lambda x \\ T_2 x = \lambda x + 1 \\ T_3 x = \lambda x + 3 \end{array}\right\} \text{ where } 0 < \lambda < 1.$$

Let X_λ be the associated limit set.

For $0 < \lambda < 1/4$ the open set condition holds and $\dim_H(X_\lambda) = \frac{\log 3}{\log(1/\lambda)}$.

[as expected]

However for $\lambda > 1/4$ the Open Set Condition doesn't hold:

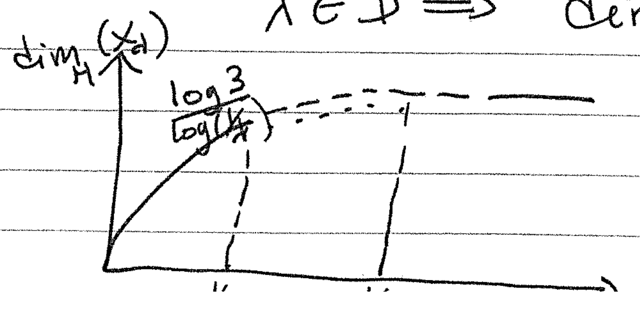
Theorem (P. + Simon, '94)

(a) For a.e. (Leb.) $\lambda \in [1/4, 1/3]$ we have

$$\dim_H(X_\lambda) = \dim_B(X_\lambda) = \frac{\log 3}{\log(1/\lambda)}$$

(b) For a dense subset $D \subseteq [1/4, 1/3]$:

$$\lambda \in D \Rightarrow \dim_B(X_\lambda) < \frac{\log 3}{\log(1/\lambda)}$$



Sketch of Proof.

Let $\mu = (1/3, 1/3, 1/3)^{\mathbb{N}_0}$ be the Bernoulli measure on $\Sigma = \{1,2,3\}^{\mathbb{N}_0}$.

Let $\pi_\lambda : \Sigma \longrightarrow X_\lambda$ be the natural map

Given $\varepsilon > 0$ let $s = \dfrac{\log 3}{\log 1/\lambda} + \varepsilon$.

By the potential definition of $\dim_H(X_\lambda)$ it suffices to show:

For a.e.(leb) $\lambda \in [1/4, 1/3]$; $\displaystyle\int_{X_\lambda}\int_{X_\lambda} \frac{d\nu_\lambda(x)\, d\nu_\lambda(y)}{|x-y|^s} < +\infty$

But by Fubini's theorem this could follow by:

$$\int_{1/4}^{1/3} \left(\int_{X_\lambda}\int_{X_\lambda} \frac{d\nu_\lambda(x)\, d\nu_\lambda(x)}{|x-y|^s} \right) d\lambda < +\infty$$

$$= \int_{1/4}^{1/3} \left(\int_\Sigma\int_\Sigma \frac{d\mu(\underline{x})\, d\mu(\underline{y})}{|\pi_\lambda(\underline{x}) - \pi_\lambda(\underline{y})|^s} \right) d\lambda$$

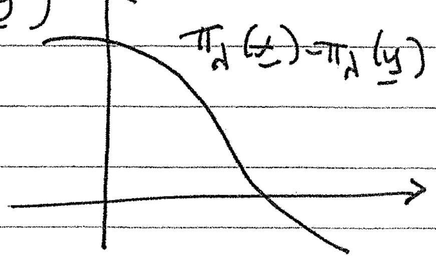
$$= \int_\Sigma\int_\Sigma \left(\int_{1/4}^{1/3} \frac{d\lambda}{|\pi_\lambda(\underline{x}) - \pi_\lambda(\underline{y})|^s} \right) d\mu(\underline{x})\, d\mu(\underline{y})$$

(by another application of Fubini)

But bounds on the inner integral come from behaviour of $\lambda \longmapsto \pi_\lambda(\underline{x}) - \pi_\lambda(\underline{y})$

("transversality")

i.e. crosses horizontal axis with some slope.



A similar analysis applies to a classical problem of Erdös

Let $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ be given by

$$\begin{cases} T_1 x = \lambda x \\ T_2 x = \lambda x + 1 \end{cases} \qquad (0 < \lambda < 1)$$

<u>Definition</u> The <u>self-similar measure</u> for these contractions is the unique probability measure ν_λ on $\mathbb{R}$ such that

$$\nu_\lambda = \frac{1}{2}\left(\nu_\lambda T_1^{-1} + \nu_\lambda T_2^{-1}\right)$$

(i.e., $\nu_\lambda(B) = \frac{1}{2}\left(\nu(T_1^{-1}B) + \nu(T_2^{-1}B)\right)$ for $B \subseteq \mathbb{R}$ Borel)

- For $\lambda < 1/2$ the measure ν_λ is supported on a (zero Lebesgue measure) Cantor set.

<u>Question</u> (Erdös, 1939): Is the measure ν_λ absolutely continuous for a.e. (leb) $\lambda \in (1/2, 1)$?

- Erdös showed there are some $\lambda \in (1/2, 1)$ with ν_λ singular (λ = reciprocal of a Pisot number)

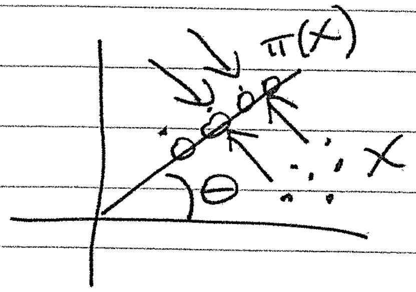
<u>Theorem</u> (Solomyak '95). The Erdös Conjecture holds.

The proof uses "transversality".

Lecture 4

1. Projections of sets

Let $\begin{cases} X \subseteq \mathbb{R}^2 \text{ be a subset} \\ \theta \in [0,\pi] \\ L_\theta = \underbrace{\{(r\cos\theta, r\sin\theta) : r \in \mathbb{R}\}}_{\text{line at angle } \theta \text{ to } x\text{-axis}} \end{cases}$



Let $\begin{cases} \Pi_\theta : \mathbb{R}^2 \to L_\theta \\ \Pi_\theta(x,y) = x\cos\theta + y\sin\theta \end{cases}$

be the orthogonal projection onto L_θ

It is easy to see from the definitions:

$$\dim_H(\Pi_\theta X) \leq \min\{\dim_H(X), 1\}$$

Moreover we have equality in "most" directions

<u>Theorem</u> (Marstrand, 1954).
For ae (Lebesgue) $\theta \in [0,\pi]$ we have

$$\dim_H(\Pi_\theta X) = \min\{\dim_H X, 1\}$$

(The proof uses Fubini's theorem (again))

<u>Question</u>. Are there special examples of X where we get equality in every direction.

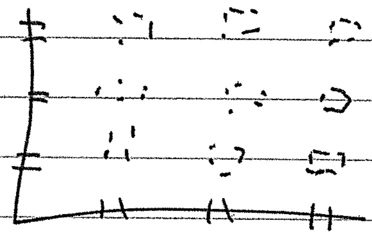
One result in this direction is for products of Cantor sets:

Let X_a = limit set for $\begin{cases} T_0 x = ax \\ T_1 x = ax + (1-a) \end{cases}$

Let X_b = limit set for $\begin{cases} T_0 x = bx \\ T_1 x = bx + (1-b) \end{cases}$

where $0 < a, b < 1$ (ie, middle $(1-2a)$-Cantor set, and middle $(1-2b)$-Cantor set, respectively)

Let $X = X_a \times X_b$ (ie, product of Cantor sets)



Then
$$\dim_H(X) = \dim_H(X_a) + \dim_H(X_b)$$
$$= \frac{\log 2}{\log 1/a} + \frac{\log 2}{\log 1/b}$$

<u>Theorem</u> (Peres–Shmerkin).

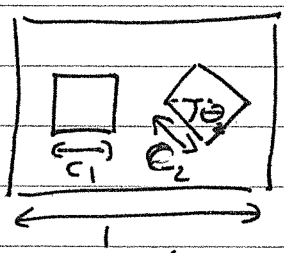
Assume that $\frac{\log a}{\log b}$ is irrational, then providing $\theta \neq 0, \pi/2$ then
$$\dim_H(\pi_\theta X) = \min\{\dim_H(X), 1\}$$

Alternatively one can find (other) construction of a set $X \subseteq \mathbb{R}^2$ with the same properties

Theorem (Peres-Shmerkin, Hochman-Shmerkin)

Let $\{T_i\}_{i=1}^{2}$ be similarities in $\mathbb{R}^2$ of the form

$$\begin{cases} T_1\begin{pmatrix} x \\ y \end{pmatrix} = A_1\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \\ T_2\begin{pmatrix} x \\ y \end{pmatrix} = A_2\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \end{cases}$$



where: $A_1 = c_1\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix}$ & $A_2 = c_2\begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix}$,

with $0 < c_1, c_2 < 1$

If $\frac{\theta_1 - \theta_2}{2\pi}$ is irrational then

$$\dim_H(\pi_\theta X) = \min\{\dim_H(X), 1\}$$

The generalization to higher dimensions $\mathbb{R}^d$ uses ideas of Furstenberg on the scenery flow, ie, scaling up small pieces of the set X and studying their behavior.

2. Fourier dimension revisited

Recall the definition of the Fourier Dimension of a set $X \subseteq \mathbb{R}$:

$$\dim_F(X) = \sup\left\{ t \geq 0 \;\middle|\; \begin{array}{l} \exists \text{ prob. measure } \mu \text{ on } X: \\ \text{with } |\hat{\mu}(u)| = O\left(\frac{1}{1+|u|^{t/2}}\right) \end{array} \right\}$$

(cf. T. Wolff, "Lectures in Harmonic Analysis")

where $\hat{\mu}(u) = \int_{-\infty}^{+\infty} e^{iut}\, d\mu(t)$

Question: Can we find any sets $X \subseteq \mathbb{R}$ with $\dim_F(X) > 0$?

The simplest example is the nonlinear Cantor set

$$X = E_2 := \{ \underbrace{[a_1, a_2, a_3, \ldots]}_{\text{continued fraction expansion}} \mid a_1, a_2, a_3, \ldots \in \{1,2\} \}$$

Theorem (Kaufman e Queffélec-Ramaré)

$$\boxed{\dim_F(E_2) > 0}$$

Idea of Proof We consider a linear operator:

$$\begin{cases} \mathcal{L} : C(E_2) \longrightarrow C(E_2) \\ \mathcal{L}h(x) = \frac{1}{(1+x)^\delta} h\left(\frac{1}{1+x}\right) + \frac{1}{(2+x)^\delta} h\left(\frac{1}{2+x}\right) \end{cases}$$

where $h \in C(E_2)$ and $\delta = \dim_H(E_2)$

There exists μ on E_2 with $\mathcal{L}^* \mu = \mu$ then

$$|\hat{\mu}(u)| \leq \int |(\mathcal{L}^n e^{iu\cdot})(t)| \, d\mu(t),$$

where $n = n(u)$.

The bounds on $|\hat{\mu}(u)|$ come from properties of $\mathcal{L}$

Moreover, it is crucial that $\delta = 0.53\ldots > 1/2$.