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Isoscalar Factors of the Permutation Group*

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Abstract. We propose a phase convention for the Clebsch-Gordan coefficients of S_n and present tables of isoscalar factors for S_3, S_4, S_5, and S_6. The isoscalar factors have been calculated from a recursion relation due to Hamermesh and by using symmetry properties of the Clebsch-Gordan coefficients. Applications to the study of multiquark systems are discussed.

1 Introduction

The direct product of two irreducible representations $[f']$ and $[f'']$ of the permutation group S_n is also called *inner* product [1], since it refers to the product of irreducible representations of the same S_n.

An important type of physical applications of the inner product is related to the construction of n-particle wave functions when several degrees of freedom are involved. Then, the role of the inner product is to provide an n-particle wave function of a desired permutation symmetry $[f]$ as a linear combination of products of functions, each factor in the product representing a degree of freedom and having a specific permutation symmetry compatible with the symmetry of the total wave function. For example, for a multiquark system, the total wave function is a linear combination of products $R\chi\phi C$, where R is the orbital part, χ the spin part, ϕ the flavour part and C the colour part. It is convenient to construct a wave function in this way whenever one has to calculate matrix elements of a Hamiltonian factorizable into parts corresponding to different degrees of freedom.

Inner products generate Clebsch-Gordan series and Clebsch-Gordan (CG) coefficients of S_n. These coefficients can be factorized into isoscalar factors. Thus, the determination of CG coefficients amounts to the calculation of isoscalar factors. The purpose of this paper is to provide tables with isoscalar factors, based on a consistent phase convention. The basic ingredients of the calculations are the

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symmetry properties of the CG coefficients and a recursion formula for isoscalar factors. The symmetry properties are summarized in Sect. 2. The isoscalar factors are introduced in Sect. 3. The recursion relation due to Hamermesh [1] is discussed in Sect. 4. Our phase convention, together with results for S_3, S_4, S_5, and S_6, are presented in Sect. 5. The results obtained here can be used in any few-fermion system (electrons, nucleons, quarks) containing up to six particles. An application to a six-quark system is considered in the last section.

2 Clebsch-Gordan Coefficients

An inner product is usually reducible into a CG series defined by

$$[f'] \times [f''] = \sum_{m_{[f]}} m_{[f]} [f], \tag{1}$$

where, in the right-hand side, $m_{[f]}$ represents the multiplicity of the irreducible representation $[f]$. Tables of CG series for S_n with $n \leq 8$ can, for example, be found in the review paper of Itzykson and Nauenberg [2], together with a general method of obtaining a series. A simple, practical procedure, is also presented in ref. [3].

From now on, an irreducible representation will shortly be called an *irrep*. Let us denote a basis vector of an irrep $[f]$ of S_n, by $|[f]Y\rangle$, where Y represents a Young tableau or, alternatively, a Yamanouchi symbol $r_n, r_{n-1}, \ldots, r_1$. Here, r_n represents the row of the particle n in the Young tableau Y. A basis vector belonging to the invariant subspace of any $[f]$ in the right-hand side of Eq. (1) is defined as a sum of products of basis vectors of $[f']$ and $[f'']$ by

$$|[f]Y\rangle = \sum_{Y', Y''} S([f']Y'[f'']Y''|[f]Y)|[f']Y'\rangle|[f'']Y''\rangle, \tag{2}$$

where $S([f']Y'[f'']Y''|[f]Y)$ are CG coefficients.

The transformation (2) relates two orthonormal bases, so that the CG coefficients form real orthogonal matrices

$$\sum_{Y'Y''} S([f']Y'[f'']Y''|[f]Y)S([f']Y'[f'']Y''|[f_1]Y_1) = \delta_{ff_1}\delta_{YY_1}, \tag{3}$$

$$\sum_{fY} S([f']Y'[f'']Y''|[f]Y)S\big([f']Y_1'[f'']Y_1''|[f]Y\big) = \delta_{Y'Y_1'}\delta_{Y''Y_1''}. \tag{4}$$

The CG coefficients of S_n have some symmetry properties similar to those of the rotation group coefficients. For $[f'] \neq [f''] \neq [f]$, these are

$$S([f']Y'[f'']Y''|[f]Y) = S([f'']Y''[f']Y'|[f]Y), \tag{5}$$

$$\begin{aligned}\frac{S([f']Y'[f'']Y''|[f]Y)}{\big(d_{[f]}\big)^{1/2}} &= \frac{S([f]Y[f'']Y''|[f']Y')}{\big(d_{[f']}\big)^{1/2}} \\ &= \frac{S([f']Y'[f]Y|[f'']Y'')}{\big(d_{[f'']}\big)^{1/2}},\end{aligned} \tag{6}$$

where $d_{[f']}$, $d_{[f'']}$ and $d_{[f]}$ are the dimensions of the irreps $[f']$, $[f'']$ and $[f]$, respectively. In the case $[f'] = [f'']$, the property (5) must be modified by incorporating a phase factor $\delta_{[f]}$

$$S([f']Y'[f']Y''|[f]Y) = \delta_{[f]} S([f']Y''[f']Y'|[f]Y), \tag{7}$$

where $\delta_{[f]} = 1$ if $[f]Y$ is contained in a symmetrized product and $\delta_{[f]} = -1$ if $|[f]Y\rangle$ is contained in an antisymmetrized product (ref. [1], Sect. 7.14). For example, for S_3 one has

$$S([21]211[21]121|[21]121) = S([21]121[21]211|[21]121) = -\sqrt{\frac{1}{2}} \tag{8}$$

and

$$S\big([21]211[21]121|[1^3]321\big) = -S\big([21]121[21]211|[1^3]321\big) = \sqrt{\frac{1}{2}}. \tag{9}$$

In specifying the CG coefficients above, a phase convention has been used, and this will be discussed in Sect. 4.

There are also symmetry properties specific to S_n. Denoting by $[\tilde{f}]$ the conjugate partition to $[f]$ and by $\tilde{Y}$ the conjugate Young tableau to Y, one has

$$S\big([\tilde{f}]\tilde{Y}'[\tilde{f}'']\tilde{Y}''|[f]Y\big) = (-)^{n^{f'}_{Y'}}(-)^{n^{f''}_{Y''}} S([f']Y'[f'']Y''|[f]Y), \tag{10}$$

$$S\big([f']Y'[\tilde{f}'']\tilde{Y}''|[\tilde{f}]\tilde{Y}\big) = (-)^{n^{f''}_{Y''}}(-)^{n^{f}_{Y}} S([f']Y'[f'']Y''|[f]Y), \tag{11}$$

$$S\big([\tilde{f}']\tilde{Y}'[f'']Y''|[\tilde{f}]\tilde{Y}\big) = (-)^{n^{f'}_{Y'}}(-)^{n^{f}_{Y}} S([f']Y'[f'']Y''|[f]Y), \tag{12}$$

where the coefficient in the left-hand side contains the normal Young tableaux and, in the right-hand side, n^f_Y, etc. is the number of transpositions necessary to bring Y to a normal Young tableau (Tab. 6.1).

3 Isoscalar Factors

Any CG coefficient of S_n can be factorized into an isoscalar factor, called K matrix [1], and a CG coefficient of S_{n-1}. This factorization property is a particular case of Racah's factorization lemma specific to the chain

$$S_n \supset S_{n-1} \supset S_{n-2} \supset \cdots \supset S_2 \tag{13}$$

and it can be applied to any link in the chain such that a CG coefficient of S_n becomes a product of isoscalar factors, associated to successive links, and a CG of S_2 which is trivially equal to unity. Therefore, the knowledge of CG of S_n amounts to the knowledge of isoscalar factors of S_n, S_{n-1}, $\ldots$, S_3.

To apply the factorization property to the link $S_n \supset S_{n-1}$, it is necessary to specify the row p of the n-th particle, and the row q of the $(n-1)$-st particle in the Yamanouchi symbol Y. If y is the distribution of the $n-2$ remaining particles, one writes

$$Y = (pqy). \tag{14}$$

Then, the isoscalar factor denoted by K is defined by

$$S([f']p'q'y'[f'']p''q''y''|[f]pqy) = K([f']p'[f'']p''|[f]p)S\Big([f'_{p'}]q'y'[f''_{p''}]q''y''|[f_p]qy\Big), \tag{15}$$

where S in the left-hand side is a CG coefficient of S_n and in the right-hand side of S_{n-1}. In the latter, $[f_p]$, etc., is a partition associated to S_{n-1} obtained from $[f]$ from the removal of the n-th particle.

The orthogonality relations (3) and (4) generate orthogonality relations for the isoscalar factors K, which read

$$\sum_{p'p''} K([f']p'[f'']p''|[f]p)\, K\,([f']p'[f'']p''|[f_1]p_1) = \delta_{ff_1}\delta_{pp_1}, \tag{16}$$

$$\sum_{fp} K([f']p'[f'']p''|[f]p)\, K\,\big([f']p'_1[f'']p''_1|[f]p\big) = \delta_{p'p'_1}\delta_{p''p''_1}. \tag{17}$$

The relation (16) holds for f and f_1 $(f \neq f_1)$ which branch from the same f_p, after the removal of the last particle. It corresponds to orthogonality of any two columns in a given table, as one can see in the next section. Also, Eq. (16) has no meaning for $f = f_1$ but $p \neq p_1$. In Eq. (17), the sum over f includes only partitions which also branch from the same f_p after the last particle has been removed. It therefore corresponds to orthogonality of rows in any table of the next section.

4 Recursion Formula

Before introducing the recursion formula, we first recall some properties of irreps of S_n in the standard Young-Yamanouchi representation which we need below. In general, any permutation can be written as a product of transpositions and any transposition can be written as a product of transpositions $(i-1, i)$ of adjacent numbers. Therefore, the problem of finding the irreps of S_n reduces to the knowledge of the matrices of adjacent transpositions. In the Young-Yamanouchi representation, these have the form [3]

$$(i-1,\, i)|[f]pqy\rangle = \alpha^f_{pq}|[f]pqy\rangle + \beta^f_{pq}|[f]qpy\rangle, \tag{18}$$

where, in $|[f]pqy\rangle$, the particle i is in row p and $i-1$ in row q, and in $|[f]qpy\rangle$, i and $i-1$ exchange their places. The coefficients α and β can be expressed in terms of the *axial distance* μ from $i-1$ to i in a Young tableau, which is given by the number of steps obtained from trying to reach i from $i-1$ by a rectangular route: Any time one crosses a line going upwards (downwards) or to the right (left), one counts $+1$ (-1). There are two situations:

1. If $i-1$ and i are in the same row or column, one has

$$\alpha^f_{pq} = \mu, \qquad \beta^f_{pq} = 0. \tag{19}$$

2. If $i-1$ and i are neither in the same row, nor in the same column,

$$\alpha^f_{pq} = \frac{1}{\mu}, \qquad \beta^f_{pq} = \left[1 - \left(\alpha^f_{pq}\right)^2\right]^{1/2}. \tag{20}$$

For example, acting with $(i-1,\, i) = (34)$ on the basis vector $|[31]2111\rangle$, the relation (18) becomes

$$(34)|[31]2111\rangle = -\frac{1}{3}|[31]2111\rangle + \frac{2\sqrt{2}}{3}|[31]1211\rangle. \tag{21}$$

By using relations similar to those for the rotation group but resulting from the coupling (1) and thus containing irreps and CG of S_n, Hamermesh applied twice the factorization property (15) and obtained, in this way, a recursion relation which allows the determination of isoscalar factors of S_n from the isoscalar factors of S_{n-1}. This relation reads

$$\begin{aligned} &K([f']p'[f'']p''|[f]p)\, K\left([f'_{p'}]q'[f''_{p''}]q''|[f_p]q\right)\left(\alpha^{f'}_{p'q'}\alpha^{f''}_{p''q''} - \alpha^{f}_{pq}\right) \\ &\quad + K([f']p'[f'']q''|[f]p)\, K\left([f'_{p'}]q'[f''_{q''}]p''|[f_p]q\right)\alpha^{f'}_{p'q'}\beta^{f''}_{p''q''} \\ &\quad + K([f']q'[f'']p''|[f]p)\, K\left([f'_{q'}]p'[f''_{p''}]q''|[f_p]q\right)\beta^{f'}_{p'q'}\alpha^{f''}_{p''q''} \\ &\quad + K([f']q'[f'']q''|[f]p)\, K\left([f'_{q'}]p'[f''_{q''}]p''|[f_p]q\right)\beta^{f'}_{p'q'}\beta^{f''}_{p''q''} \\ &= K([f']p'[f'']p''|[f]q)\, K\left([f'_{p'}]q'[f''_{p''}]q''|[f_q]p\right)\beta^{f}_{pq}, \end{aligned} \tag{22}$$

where α^f_{pq} and β^f_{pq}, etc., are given by Eqs. (19) or (20). In each term, the first K matrix refers to S_n and the second to S_{n-1}. Combining this relation with the orthogonality relations (16) and (17), one can determine the isoscalar factors of S_n once those for S_{n-1} are known. The advantage of this algorithm over that of Chen [4] is that it can be easily used by anyone with a basic knowledge of Young diagrams while Chen's approach is more specialized. Of course, a consistent phase convention is very important. This will be discussed in the next section.

5 Phase Convention and Results

To our knowledge, the problem of phase convention for CG of S_n is very little discussed in the literature. Hamermesh gives some examples but does not specify a phase convention. Like for the rotation group, a phase convention is necessary and it must be the same for all S_n. Whenever transformations between different coupling schemes are derived, a consistent phase convention is of crucial importance [5].

Here, we propose a phase convention similar to that of Chen [4]. It is represented by the requirement

$$S\big([f']Y'_n[f'']Y''_n|[f]Y_n\big) > 0, \tag{23}$$

where Y'_n, etc., are normal Young tableaux. If, for a given $[f'], [f'']$, and $[f]$, the coefficient associated to the normal Young tableaux is zero, then one has to order the Young tableaux according to Tab. 6.1 and take the first non-vanishing coefficient in that order to be positive. In the coefficient above, there are three distinct Young tableaux which can be varied. Our rule is that at fixed Y_n, we first keep Y'_n fixed and vary Y''_n until a non-vanishing coefficient is reached. If that is not the case yet, then we decrease Y'_n according to Tab. 6.1 and at each step, we search for the first non-vanishing coefficient. Such a phase convention seems natural for the derivation of CG coefficients based on Eq. (22).

Actually, to our understanding, the difference with Chen comes from the fact that the requirement (23) is here imposed only on a restricted number of CG coefficients. The phases of the others are automatically determined through the symmetry properties (5)–(7) or (10)–(12). The recursion relation (22) gives relative phases. Our results for the isoscalar factors are presented in Tabs. 6.2–6.5, where the isoscalar factors have been grouped together in square tables, according to the discussion following Eqs. (16) and (17). We mark with an asterisk all coefficients where the requirement (23) has been imposed. Blanc space means zero value for K. Table 6.2 gives the isoscalar factors of S_3. Here, our phases are the same as those of Hamermesh and Chen. Tab. 6.3 covers all possible inner products of S_4. From a comparison with Chen's tables, we find that Chen does not implement the properties (10)–(12). For example, according to Eq. (11) and Tab. 6.1, we require

$$S([31]2111[22]2211|[211]3211) = -S([31]2111[22]2121|[31]1121), \tag{24}$$

where the $(-)$ sign in the right-hand side is responsible for the negative phase of $K([31]2[22]2|[211]3)$ while Chen chooses this isoscalar factor to be positive straight from Eq. (23). Tab. 6.4 contains all inner products of S_5 where the requirement (23) has to be imposed and many others. The few remaining coefficients can be determined by using the symmetry properties (10)–(12) relating conjugate partitions. This table covers most of Chen's results. There are cases where our phases are different from those of Chen, due to our direct or indirect use (via Tab. 6.3) of properties (10)–(12). Tab. 6.5 exhibits results for S_6. Most of them are entirely new. There are many inner products of S_6 and we made a selection among them. These are (Tab. 4.6, ref. [3])

$$\begin{aligned}
[51]\times[51] &= [6]+[51]+[42]+[411],\\
[51]\times[42] &= [51]+[42]+[411]+[33]+[321],\\
[51]\times[411] &= [51]+[42]+[411]+[321]+[3111],\\
[51]\times[33] &= [42]+[321],\\
[51]\times[321] &= [42]+[411]+[33]+2[321]+[222]+[3111]+[2211],\\
[42]\times[42] &= [6]+[51]+2[42]+[411]+2[321]+[222]+[3111],\\
[42]\times[411] &= [51]+[42]+2[411]+[33]+2[321]+[3111]+[2211],\\
[42]\times[33] &= [51]+[411]+[33]+[321]+[2211],\\
[42]\times[222] &= [42]+[321]+[222]+[3111]+[21111].
\end{aligned} \tag{25}$$

At the beginning of this work we were especially interested in the products relevant for the NN problem [5, 6] as, for example $[51]\times[222]$, $[42]\times[33]$, or $[42]\times[222]$. Hence, few inner products are common to those considered in ref. [7], where no phase convention is specified.

Most of the calculated isoscalar factors correspond to the case where the multiplicity $m_{[f]}$ of Eq. (1) is equal to one. In all these cases, the values of K are straightforwardly obtained from the recursion relation (22). It should be mentioned that, in cases where $m_{[f]} \neq 1$, Eq. (22) has $m_{[f]}$ solutions. There is no general criterion for the selection of the solutions. Here, we considered $m_{[f]} = 2$ at most (see Eqs. (25)). In such cases, the two solutions are denoted by α and β. Our procedure was to find the solution α from the recursion relation (22) and to determine the solution β by the orthogonality relation (16), i.e. the orthogonality of the unknown column on the known columns of the corresponding square table. We failed in solving the case of the couplings $[42] \times [411] \to [411]$ and $[42] \times [321] \to [321]$. In both cases, the solution β satisfies the recursion relation (22) and the orthogonality relation (16) at the same time. The solution α does not satisfy these relations simultaneously in neither case. For starting we had a table with four rows and five columns, so we expected to eliminate one column only, but we could not find which one. That is why we did not include the corresponding table.

6 Applications

As mentioned in the introduction, the CG coefficients of S_n are very handy in constructing totally antisymmetric states for a system of n identical fermions, especially when the system possesses more than two degrees of freedom. Particularly interesting cases are the few quark systems. For $4 \leq n \leq 6$, applications have been considered elsewhere [8–13]. Here we briefly recall the technique of obtaining the so-called two-body fractional parentage coefficients (cfp) from isoscalar factors and illustrate the procedure by an example of a wave function for $n = 6$. Note that the two-body cfp can be used for the kinetic term as well (ref. [11]).

In the quark case, we work in the FS coupling scheme, where the flavour F and the spin S are coupled together to form an intermediate representation $[f]$ and the orbital O and colour C degrees of freedom are also coupled together to form a representation $[\tilde{f}]$, conjugate to $[f]$. Then, irrespective of n, a totally antisymmetric wave function can be written as

$$\psi_{[1^n]} = \frac{1}{{d_f}^{1/2}} \sum_Y (-)^{n_Y^f} |fY\rangle |\tilde{f}\tilde{Y}\rangle \tag{26}$$

where n_Y^f are given in Tab. 6.1 and d_f is the dimension of the irrep $[f]$ of S_n. Note that the product ${d_f}^{-1/2}(-)^{n_Y^f}$ represents the CG coefficients appearing in the inner product $[f] \times [\tilde{f}] \to [1^n]$. The next step is to make explicit the content of the intermediate couplings $[f] = [f'] \times [f'']$ for FS and OC through

$$|[f]Y\rangle = \sum_{Y',Y''} S([f']Y'[f'']Y''|[f]Y)|[f']Y'\rangle|[f'']Y''\rangle. \tag{27}$$

In calculating matrix elements of two-body operators it is convenient to rewrite $|[f]Y\rangle$ and $|[\tilde{f}]\tilde{Y}\rangle$ in Eq. (26) in the diagonalized Rutherford-Young-Yamanouchi representation [3, 7], because in this representation the last pair of particles has a definite permutation symmetry. We next use the factorization property (15) twice for each CG coefficient of S_n and introduce the $\bar{K}$ matrix (see, e.g., ref. [3])

Table 6.1. Ordering of basis vectors for irreps of S_n ($n = 3$, 4, 5, and 6)

$[f]$	Young tableaux $[f]\,Y$ and their phases $(-)^{n_Y^f}$					
[21]	1) 12 3 (+)	2) 13 2 (−)				
[31]	1) 123 4 (+)	2) 124 3 (−)	3) 134 2 (+)			
[22]	1) 12 34 (+)	2) 13 24 (−)				
[211]	1) 12 3 4 (+)	2) 13 2 4 (−)	3) 14 2 3 (+)			
[41]	1) 1234 5 (+)	2) 1235 4 (−)	3) 1245 3 (+)	4) 1345 2 (−)		
[32]	1) 123 45 (+)	2) 124 35 (−)	3) 134 25 (+)	4) 125 34 (+)	5) 135 24 (−)	
[311]	1) 123 4 5 (+)	2) 124 3 5 (−)	3) 134 2 5 (+)	4) 125 3 4 (+)	5) 135 2 4 (−)	6) 145 2 3 (+)
[221]	1) 12 34 5 (+)	2) 13 24 5 (−)	3) 12 35 4 (−)	4) 13 25 4 (+)	5) 14 25 3 (−)	
$[21^3]$	1) 12 3 4 5 (+)	2) 13 2 4 5 (−)	3) 14 2 3 5 (+)	4) 15 2 3 4 (−)		
[51]	1) 12345 6 (+)	2) 12346 5 (−)	3) 12356 4 (+)	4) 12456 3 (−)	5) 13456 2 (+)	
[42]	1) 1234 56 (+)	2) 1235 46 (−)	3) 1245 36 (+)	4) 1345 26 (−)	5) 1236 45 (+)	6) 1246 35 (−)
	7) 1346 25 (+)	8) 1256 34 (+)	9) 1356 24 (−)			
[411]	1) 1234 5 6 (+)	2) 1235 4 6 (−)	3) 1245 3 6 (+)	4) 1345 2 6 (−)	5) 1236 4 5 (+)	6) 1246 3 5 (−)

Table 6.1 (continued)

	7) 1346 2 5 (+)	8) 1256 3 4 (+)	9) 1356 2 4 (−)	10) 1456 2 3 (+)		
[33]	1) 123 456 (+)	2) 124 356 (−)	3) 134 256 (+)	4) 125 346 (+)	5) 135 246 (−)	
[321]	1) 123 45 6 (+)	2) 124 35 6 (−)	3) 134 25 6 (+)	4) 125 34 6 (+)	5) 135 24 6 (−)	6) 123 46 5 (−)
	7) 124 36 5 (+)	8) 134 26 5 (−)	9) 125 36 4 (−)	10) 135 26 4 (+)	11) 145 26 3 (−)	12) 126 34 5 (−)
	13) 136 24 5 (+)	14) 126 35 4 (+)	15) 136 25 4 (−)	16) 146 25 3 (+)		
[222]	1) 12 34 56 (+)	2) 13 24 56 (−)	3) 12 35 46 (−)	4) 13 25 46 (+)	5) 14 25 36 (−)	
$[31^3]$	1) 123 4 5 6 (+)	2) 124 3 5 6 (−)	3) 134 2 5 6 (+)	4) 125 3 4 6 (+)	5) 135 2 4 6 (−)	6) 145 2 3 6 (+)
	7) 126 3 4 5 (−)	8) 136 2 4 5 (+)	9) 146 2 3 5 (−)	10) 156 2 3 4 (+)		
$[2^2 1^2]$	1) 12 34 5 6 (+)	2) 13 24 5 6 (−)	3) 12 35 4 6 (−)	4) 13 25 4 6 (+)	5) 14 25 3 6 (−)	6) 12 36 4 5 (+)
	7) 13 26 4 5 (−)	8) 14 26 3 5 (+)	9) 15 26 3 4 (−)			
$[21^4]$	1) 12 3 4 5 6 (+)	2) 13 2 4 5 6 (−)	3) 14 2 3 5 6 (+)	4) 15 2 3 4 6 (−)	5) 16 2 3 4 5 (+)	

Table 6.2. Isoscalar factors for S_3. The rows give $[f']p'$ and $[f'']p''$ associated to Eq. (1). The column select those $[f]p$ from the right-hand side of Eq. (1) which branch from the same f_p. See text after Eqs. (16)–(17)

	[3]1	[21]2
[21]2 [21]2	$\sqrt{1/2}$	$\sqrt{1/2}$
[21]1 [21]1	$\sqrt{1/2}$	$-\sqrt{1/2}$

	[21]1	$[1^3]3$
[21]2 [21]1	$-\sqrt{1/2}$	$\sqrt{1/2}$
[21]1 [21]2	$-\sqrt{1/2}$	$-\sqrt{1/2}$

Table 6.3. Same as Table 6.2 but for S_4

	[4]1	[31]2
[31]2 [31]2	$\sqrt{1/3}^*$	$\sqrt{2/3}^*$
[31]1 [31]1	$\sqrt{2/3}$	$-\sqrt{1/3}$

	[31]1	[22]2	[211]3
[31]2 [31]1	$-\sqrt{1/6}$	$\sqrt{1/3}^*$	$\sqrt{1/2}^*$
[31]1 [31]2	$-\sqrt{1/6}$	$\sqrt{1/3}$	$-\sqrt{1/2}$
[31]1 [31]1	$\sqrt{2/3}$	$\sqrt{1/3}$	

	[211]1
[31]1 [31]1	1

	[31]2
[31]1 [22]2	1

	[211]1
[31]1 [22]2	1

	[31]2
[31]1 [211]3	1

	[31]1	[211]3
[31]2 [22]2	$\sqrt{1/2}$	$-\sqrt{1/2}$
[31]1 [22]2	$\sqrt{1/2}$	$\sqrt{1/2}$

	[211]1	$[1^4]4$
[31]2 [211]1	$\sqrt{2/3}$	$\sqrt{1/3}$
[31]1 [211]3	$\sqrt{1/3}$	$-\sqrt{2/3}$

	[31]1	[22]2	[211]3
[31]2 [211]3	$-\sqrt{1/2}$	$-\sqrt{1/3}$	$-\sqrt{1/6}$
[31]1 [211]3		$\sqrt{1/3}$	$-\sqrt{4/6}$
[31]1 [211]1	$\sqrt{1/2}$	$-\sqrt{1/3}$	$-\sqrt{1/6}$

	[4]1
[22]2 [22]2	1^*

	[22]2
[22]2 [22]2	1^*

	$[1^4]4$
[22]2 [22]2	1

	[31]2
[22]2 [211]3	-1

Table 6.3 (continued)

	[31]1	[211]3
[22]2 [211]3	$\sqrt{1/2}$	$-\sqrt{1/2}$
[22]2 [211]1	$\sqrt{1/2}$	$\sqrt{1/2}$

	[211]1
[22]2 [211]3	-1

	[4]1	[31]2
[211]3 [211]3	$\sqrt{2/3}^*$	$-\sqrt{1/3}$
[211]1 [211]1	$\sqrt{1/3}$	$\sqrt{2/3}$

	[211]1
[211]3 [211]3	1

	[31]1	[22]2	[211]3
[211]3 [211]3	$-\sqrt{4/6}$	$-\sqrt{1/3}$	
[211]3 [211]1	$\sqrt{1/6}$	$-\sqrt{1/3}$	$\sqrt{1/2}$
[211]1 [211]3	$\sqrt{1/6}$	$-\sqrt{1/3}$	$-\sqrt{1/2}$

resulting from the two sequent factorizations. Then one has

$$\begin{aligned} |[f]Y\rangle = \sum \bar{K}([f']p'q'[f'']p''q''|[f]pq)S(f'_{p'q'}y'f''_{p''q''}y''|f_{pq}y) \\ \times |[f']p'q'y'\rangle|[f'']p''q''y''\rangle, \end{aligned} \tag{28}$$

where the summation runs over $p'q'y'$ and $p''q''y''$ (see the notation of Eq. (14)). The CG coefficients of S_{n-2} appearing in the right-hand side of Eq. (28) are summed up through orthonormality relations in the calculation of the two-body operator matrix elements. Each matrix element will then be a sum of products of two-body matrix elements and two-body cfp coefficients. In the quark context, for a given $[f]$, each of these coefficients is therefore a product of three $\bar{K}$ matrices, one associated to the coupling (26) and two to Eq. (27). Each separate $\bar{K}$ matrix can be called a cfp also.

It is useful to illustrate this procedure with the six-quark state $[42]_O[51]_{FS}$ appearing as the dominant state in the description of the NN system at short separation distances within the Goldstone boson exchange (GBE) model [11]. In this case, $[f] = [51]_{FS}$ and we rewrite Eq. (26) in the form

$$\begin{aligned} |[42]_O[51]_{FS}\rangle = \sqrt{1/5}\; |[51]_{FS}1\tilde{2}\rangle\, |[21111]_{OC}1\bar{5}\rangle \\ - \sqrt{1/5}\; |[51]_{FS}1\bar{2}\rangle\, |[21111]_{OC}1\tilde{5}\rangle \\ + \sqrt{3/5}\; |[51]_{FS}1\bar{1}\rangle\, |[21111]_{OC}4\tilde{5}\rangle, \end{aligned} \tag{29}$$

where $p\bar{q}$ ($p\tilde{q}$) means symmetry (antisymmetry) for the 6th and 5th particles located in the rows p and q, respectively. The coefficients in the right-hand side are the $\bar{K}$ matrices which can be easily built by using Tabs. 6.4 and 6.5 and the symmetry properties (10)–(12).

Table 6.4. Same as Table 6.2 but for S_5

	[5]1	[41]2
[41]2 [41]2	$\sqrt{1/4}^*$	$\sqrt{3/4}^*$
[41]1 [41]1	$\sqrt{3/4}$	$-\sqrt{1/4}$

	[41]1	[32]2	[311]3
[41]2 [41]1	$-\sqrt{1/12}$	$\sqrt{5/12}^*$	$\sqrt{1/2}^*$
[41]1 [41]2	$-\sqrt{1/12}$	$\sqrt{5/12}$	$-\sqrt{1/2}$
[41]1 [41]1	$\sqrt{5/6}$	$\sqrt{1/6}$	

	[311]1
[41]1 [41]1	1

	[32]1
[41]1 [41]1	1

	[41]2
[41]1 [32]2	1

	[41]1	[32]2	[311]3
[41]2 [32]2	$\sqrt{5/15}$	$\sqrt{2/12}^*$	$\sqrt{10/20}^*$
[41]1 [32]2	$\sqrt{2/15}$	$\sqrt{5/12}$	$-\sqrt{9/20}$
[41]1 [32]1	$\sqrt{8/15}$	$-\sqrt{5/12}$	$-\sqrt{1/20}$

	[32]1	[221]3
[41]2 [32]1	$-\sqrt{3/8}$	$\sqrt{5/8}^*$
[41]1 [32]2	$-\sqrt{5/8}$	$-\sqrt{3/8}$

	[311]1	[221]2
[41]1 [32]2	$-\sqrt{1/4}$	$\sqrt{3/4}$
[41]1 [32]1	$-\sqrt{3/4}$	$-\sqrt{1/4}$

	[41]2
[41]1 [311]3	-1

	[41]1	[32]2	[311]3
[41]2 [311]3	$\sqrt{1/3}$	$\sqrt{10/24}$	$\sqrt{2/8}^*$
[41]1 [311]3		$-\sqrt{9/24}$	$\sqrt{5/8}$
[41]1 [311]1	$-\sqrt{2/3}$	$\sqrt{5/24}$	$\sqrt{1/8}$

	[32]1	[221]3
[41]1 [311]3	$-\sqrt{1/16}$	$\sqrt{15/16}$
[41]1 [311]1	$-\sqrt{15/16}$	$-\sqrt{1/16}$

	[311]1	[221]2	[2111]4
[41]2 [311]1	$-\sqrt{2/8}$	$\sqrt{10/24}$	$\sqrt{1/3}^*$
[41]1 [311]3	$-\sqrt{1/8}$	$\sqrt{5/24}$	$-\sqrt{2/3}$
[41]1 [311]1	$-\sqrt{5/8}$	$-\sqrt{9/24}$	

	[2111]1
[41]1 [311]1	1

	[32]2	[311]3
[41]1 [221]3	$-\sqrt{1/4}$	$\sqrt{3/4}$
[41]1 [221]2	$-\sqrt{3/4}$	$-\sqrt{1/4}$

	[32]1	[221]3
[41]2 [221]3	$\sqrt{5/8}$	$-\sqrt{3/8}$
[41]1 [221]2	$-\sqrt{3/8}$	$-\sqrt{5/8}$

Table 6.4 (continued)

	[311]1	[221]2	[2111]4
[41]2 [221]2	$\sqrt{10/20}$	$\sqrt{2/12}$	$-\sqrt{5/15}$
[41]1 [221]3	$-\sqrt{1/20}$	$-\sqrt{5/12}$	$-\sqrt{8/15}$
[41]1 [221]2	$-\sqrt{9/20}$	$\sqrt{5/12}$	$-\sqrt{2/15}$

	[2111]1
[41]1 [221]2	1

	[221]3
[41]1 [2111]4	-1

	[311]3
[41]1 [2111]4	1

	[311]1	[221]2	[2111]4
[41]2 [2111]4	$\sqrt{1/2}$	$-\sqrt{5/12}$	$-\sqrt{1/12}$
[41]1 [2111]4		$-\sqrt{2/12}$	$\sqrt{10/12}$
[41]1 [2111]1	$\sqrt{1/2}$	$\sqrt{5/12}$	$\sqrt{1/12}$

	[2111]1	$[1^5]5$
[41]2 [2111]1	$\sqrt{3/4}$	$\sqrt{1/4}$
[41]1 [2111]4	$\sqrt{1/4}$	$-\sqrt{3/4}$

	[5]1	[41]2
[32]2 [32]2	$\sqrt{3/5}$	$\sqrt{2/5}$
[32]1 [32]1	$\sqrt{2/5}$	$-\sqrt{3/5}$

	[41]1	[32]2	[311]3
[32]2 [32]2	$\sqrt{1/3}$	$\sqrt{4/6}^*$	
[32]2 [32]1	$-\sqrt{1/3}$	$\sqrt{1/6}$	$\sqrt{1/2}^*$
[32]1 [32]2	$-\sqrt{1/3}$	$\sqrt{1/6}$	$-\sqrt{1/2}$

	[32]1	[221]3
[32]2 [32]2	$\sqrt{1/4}$	$\sqrt{3/4}^*$
[32]1 [32]1	$\sqrt{3/4}$	$-\sqrt{1/4}$

	[311]1	[221]2	[2111]4
[32]2 [32]2	$\sqrt{4/10}$		$\sqrt{3/5}$
[32]2 [32]1	$-\sqrt{3/10}$	$-\sqrt{1/2}$	$\sqrt{1/5}$
[32]1 [32]2	$\sqrt{3/10}$	$-\sqrt{1/2}$	$-\sqrt{1/5}$

	[2111]1
[32]1 [32]1	1

	[221]2	[2111]4	[311]1
[32]2 [221]3	$\sqrt{1/6}$	$\sqrt{1/3}$	$\sqrt{1/2}$
[32]2 [221]2	$\sqrt{4/6}$	$-\sqrt{1/3}$	
[32]1 [221]2	$\sqrt{1/6}$	$\sqrt{1/3}$	$-\sqrt{1/2}$

	[2111]1	$[1^5]5$
[32]2 [221]2	$\sqrt{2/5}$	$\sqrt{3/5}$
[32]1 [221]3	$-\sqrt{3/5}$	$\sqrt{2/5}$

	[221]3	[32]1
[32]2 [221]2	$\sqrt{1/4}$	$-\sqrt{3/4}$
[32]1 [221]3	$-\sqrt{3/4}$	$-\sqrt{1/4}$

Table 6.4 (continued)

	[41]1	[32]2	[311]3
[32]2 [221]3	$-\sqrt{1/5}$	$\sqrt{1/2}$	$\sqrt{3/10}$
[32]2 [221]2	$\sqrt{3/5}$		$\sqrt{4/10}$
[32]1 [221]2	$-\sqrt{1/5}$	$-\sqrt{1/2}$	$\sqrt{3/10}$

	[41]2
[32]1 [221]3	1

	[2111]4	[221]2	[311]1
[32]2 [2111]4	$-\sqrt{2/15}$	$-\sqrt{5/12}$	$\sqrt{9/20}$
[32]2 [2111]1	$\sqrt{1/3}$	$\sqrt{2/12}$	$\sqrt{10/20}$
[32]1 [2111]4	$-\sqrt{8/15}$	$\sqrt{5/12}$	$\sqrt{1/20}$

	[221]3	[32]1
[32]2 [2111]4	$\sqrt{5/8}$	$\sqrt{3/8}$
[32]1 [2111]1	$\sqrt{3/8}$	$-\sqrt{5/8}$

	[2111]1
[32]2 [2111]4	1

	[32]2	[311]3
[32]2 [2111]4	$-\sqrt{3/4}$	$-\sqrt{1/4}$
[32]1 [2111]4	$-\sqrt{1/4}$	$\sqrt{3/4}$

	[32]1	[221]3
[32]2 [311]3	$\sqrt{5/8}$	$\sqrt{3/8}$
[32]2 [311]1	$-\sqrt{3/8}$	$\sqrt{5/8}$

	[41]1	[32]2	$[311]3_{\alpha}$	$[311]3_{\beta}$
[32]2 [311]3	$-\sqrt{9/30}$		$\sqrt{12/20}^{*}$	$\sqrt{1/10}^{*}$
[32]2 [311]1	$-\sqrt{5/30}$	$-\sqrt{4/12}$		$-\sqrt{5/10}$
[32]1 [311]3	$-\sqrt{1/30}$	$-\sqrt{5/12}$	$-\sqrt{3/20}$	$\sqrt{4/10}$
[32]1 [311]1	$-\sqrt{15/30}$	$\sqrt{3/12}$	$-\sqrt{5/20}$	

	[2111]1
[32]2 [311]1	1

	[41]2
[32]2 [311]3	1

	$[311]1_{\alpha}$	$[311]1_{\beta}$	[221]2	[2111]4
[32]2 [311]3		$\sqrt{5/10}$	$-\sqrt{4/12}$	$\sqrt{5/30}$
[32]2 [311]1	$\sqrt{12/20}$	$-\sqrt{1/10}$		$\sqrt{9/30}$
[32]1 [311]3	$-\sqrt{5/20}$		$\sqrt{3/12}$	$\sqrt{15/30}$
[32]1 [311]1	$-\sqrt{3/20}$	$-\sqrt{4/10}$	$-\sqrt{5/12}$	$\sqrt{1/30}$

Table 6.4 (continued)

	[5]1	[41]2
[311]3 [311]3	$\sqrt{1/2}^*$	$\sqrt{1/2}^*$
[311]1 [311]1	$\sqrt{1/2}$	$-\sqrt{1/2}$

	[2111]1	$[1^5]5$
[311]3 [311]1	$\sqrt{1/2}$	$\sqrt{1/2}$
[311]1 [311]3	$-\sqrt{1/2}$	$\sqrt{1/2}$

	[41]1	$[32]2_\alpha$	$[32]2_\beta$	[311]3
[311]3 [311]3	$\sqrt{5/12}^*$	$\sqrt{1/2}$	$\sqrt{1/12}$	
[311]3 [311]1	$-\sqrt{1/12}$		$\sqrt{5/12}$	$\sqrt{1/2}^*$
[311]1 [311]3	$-\sqrt{1/12}$		$\sqrt{5/12}$	$-\sqrt{1/2}$
[311]1 [311]1	$-\sqrt{5/12}$	$\sqrt{1/2}$	$-\sqrt{1/12}$	

	$[32]1_\alpha$	$[32]1_\beta$	$[221]3_\alpha$	$[221]3_\beta$
[311]3 [311]3	$-\sqrt{3/16}$	$\sqrt{1/2}$	$\sqrt{5/16}$	
[311]3 [311]1	$-\sqrt{5/16}$		$-\sqrt{3/16}$	$\sqrt{1/2}$
[311]1 [311]3	$-\sqrt{5/16}$		$-\sqrt{3/16}$	$-\sqrt{1/2}$
[311]1 [311]1	$-\sqrt{3/16}$	$-\sqrt{1/2}$	$\sqrt{5/16}$	

	[311]1	$[221]2_\alpha$	$[221]2_\beta$	[2111]4
[311]3 [311]3	$\sqrt{1/2}$		$\sqrt{5/12}$	$\sqrt{1/12}$
[311]3 [311]1		$\sqrt{1/2}$	$\sqrt{1/12}$	$-\sqrt{5/12}$
[311]1 [311]3		$\sqrt{1/2}$	$-\sqrt{1/12}$	$\sqrt{5/12}$
[311]1 [311]1	$\sqrt{1/2}$		$-\sqrt{5/12}$	$-\sqrt{1/12}$

Table 6.5. Same as Table 6.2 but for S_6

	[6]1	[51]2
[51]2 [51]2	$\sqrt{1/5}^*$	$\sqrt{4/5}^*$
[51]1 [51]1	$\sqrt{4/5}$	$-\sqrt{1/5}$

	[51]1	[42]2	[411]3
[51]2 [51]1	$-\sqrt{1/20}$	$\sqrt{9/20}^*$	$\sqrt{1/2}^*$
[51]1 [51]2	$-\sqrt{1/20}$	$\sqrt{9/20}$	$-\sqrt{1/2}$
[51]1 [51]1	$\sqrt{18/20}$	$\sqrt{2/20}$	

Table 6.5 (continued)

	[42]1
[51]1 [51]1	1

	[411]1
[51]1 [51]1	1

	[51]2
[51]1 [42]2	1

	[51]1	[42]2	[411]3
[51]2 [42]2	$\sqrt{9/36}$	$\sqrt{1/4}^*$	$\sqrt{9/18}^*$
[51]1 [42]2	$\sqrt{2/36}$	$\sqrt{2/4}$	$-\sqrt{8/18}$
[51]1 [42]1	$\sqrt{25/36}$	$-\sqrt{1/4}$	$-\sqrt{1/18}$

	[42]1	[33]2	[321]3
[51]2 [42]1	$-\sqrt{4/25}$	$\sqrt{9/45}^*$	$\sqrt{144/225}^*$
[51]1 [42]2	$-\sqrt{5/25}$	$\sqrt{20/45}$	$-\sqrt{80/225}$
[51]1 [42]1	$\sqrt{16/25}$	$\sqrt{16/45}$	$\sqrt{1/225}$

	[411]1	[321]2
[51]1 [42]2	$-\sqrt{1/9}$	$\sqrt{8/9}$
[51]1 [42]1	$\sqrt{8/9}$	$\sqrt{1/9}$

	[321]1
[51]1 [42]1	1

	[51]2
[51]1 [411]3	−1

	[51]1	[42]2	[411]3
[51]2 [411]3	$\sqrt{1/4}$	$\sqrt{9/20}$	$\sqrt{3/10}^*$
[51]1 [411]3		$-\sqrt{8/20}$	$\sqrt{6/10}$
[51]1 [411]1	$\sqrt{3/4}$	$-\sqrt{3/20}$	$-\sqrt{1/10}$

	[42]1	[321]3
[51]1 [411]3	$-\sqrt{1/25}$	$\sqrt{24/25}^*$
[51]1 [411]1	$\sqrt{24/25}$	$\sqrt{1/25}$

	[411]1	[321]2	[3111]4
[51]2 [411]1	$-\sqrt{2/15}$	$\sqrt{8/15}$	$\sqrt{1/3}^*$
[51]1 [411]3	$-\sqrt{1/15}$	$\sqrt{4/15}$	$-\sqrt{2/3}$
[51]1 [411]1	$\sqrt{12/15}$	$\sqrt{3/15}$	

	[321]1
[51]1 [411]1	1

	[3111]1
[51]1 [411]3	1

	[42]1	[321]3
[51]2 [33]2	$\sqrt{9/25}$	$\sqrt{16/25}^*$
[51]1 [33]2	$\sqrt{16/25}$	$-\sqrt{9/25}$

Table 6.5 (continued)

	[42]2
[51]1 [33]2	1

	[321]2
[51]1 [33]2	1

	[321]1
[51]1 [33]2	-1

	[42]2	[411]3
[51]1 [321]3	$-\sqrt{1/4}$	$\sqrt{3/4}$
[51]1 [321]2	$\sqrt{3/4}$	$\sqrt{1/4}$

	[42]1	[33]2	[321]3_α	[321]3_β
[51]2 [321]3	$\sqrt{144/400}$	$\sqrt{16/80}$	$\sqrt{3/25}^{*}$	$\sqrt{64/200}^{*}$
[51]1 [321]3	$\sqrt{1/400}$	$-\sqrt{9/80}$	$-\sqrt{12/25}$	$\sqrt{81/200}$
[51]1 [321]2	$\sqrt{30/400}$	$\sqrt{30/80}$	$-\sqrt{10/25}$	$-\sqrt{30/200}$
[51]1 [321]1	$\sqrt{225/400}$	$-\sqrt{25/80}$		$-\sqrt{25/200}$

	[321]1_α	[321]1_β	[222]3	[2211]4
[51]2 [321]1	$\sqrt{3/25}$	$\sqrt{64/200}$	$-\sqrt{16/80}$	$\sqrt{144/400}$
[51]1 [321]3		$\sqrt{25/200}$	$-\sqrt{25/80}$	$-\sqrt{225/400}$
[51]1 [321]2	$\sqrt{10/25}$	$\sqrt{30/200}$	$\sqrt{30/80}$	$-\sqrt{30/400}$
[51]1 [321]1	$-\sqrt{12/25}$	$\sqrt{81/200}$	$\sqrt{9/80}$	$\sqrt{1/400}$

	[411]1	[321]2_α	[321]2_β	[3111]4
[51]2 [321]2	$\sqrt{16/48}$	$-\sqrt{1/3}$		$-\sqrt{16/48}$
[51]1 [321]3	$\sqrt{1/48}$	$-\sqrt{1/3}$	$\sqrt{1/8}$	$\sqrt{25/48}$
[51]1 [321]2	$\sqrt{6/48}$		$-\sqrt{6/8}$	$\sqrt{6/48}$
[51]1 [321]1	$\sqrt{25/48}$	$\sqrt{1/3}$	$\sqrt{1/8}$	$\sqrt{1/48}$

	[3111]1	[2211]2
[51]1 [321]2	$\sqrt{1/4}$	$\sqrt{3/4}$
[51]1 [321]1	$\sqrt{3/4}$	$-\sqrt{1/4}$

	[2211]4	[321]1
[51]2 [222]3	$-\sqrt{9/25}$	$-\sqrt{16/25}$
[51]1 [222]3	$-\sqrt{16/25}$	$\sqrt{9/25}$

	[321]3
[51]1 [222]3	-1

	[321]2
[51]1 [222]3	1

	[2211]2
[51]1 [222]3	-1

Table 6.5 (continued)

	$[6]1$	$[51]2$
$[42]2\ [42]2$	$\sqrt{4/9}^*$	$\sqrt{5/9}$
$[42]1\ [42]1$	$\sqrt{5/9}$	$-\sqrt{4/9}$

	$[51]1$	$[42]2_\alpha$	$[42]2_\beta$	$[411]3$
$[42]2\ [42]2$	$\sqrt{10/36}$	$\sqrt{2/4}^*$	$\sqrt{2/9}^*$	
$[42]2\ [42]1$	$-\sqrt{5/36}$	$\sqrt{1/4}$	$-\sqrt{1/9}$	$\sqrt{1/2}^*$
$[42]1\ [42]2$	$-\sqrt{5/36}$	$\sqrt{1/4}$	$-\sqrt{1/9}$	$-\sqrt{1/2}$
$[42]1\ [42]1$	$\sqrt{16/36}$		$-\sqrt{5/9}$	

	$[42]1_\alpha$	$[42]1_\beta$	$[321]3_\alpha$	$[321]3_\beta$
$[42]2\ [42]2$	$\sqrt{1/5}$	$-\sqrt{4/45}$		$\sqrt{64/90}^*$
$[42]2\ [42]1$		$-\sqrt{20/45}$	$\sqrt{1/2}^*$	$-\sqrt{5/90}$
$[42]1\ [42]2$		$-\sqrt{20/45}$	$-\sqrt{1/2}$	$-\sqrt{5/90}$
$[42]1\ [42]1$	$\sqrt{4/5}$	$\sqrt{1/45}$		$-\sqrt{16/90}$

	$[321]1_\alpha$	$[321]1_\beta$	$[222]3$
$[42]2\ [42]1$	$-\sqrt{1/2}$	$\sqrt{1/18}$	$\sqrt{4/9}^*$
$[42]1\ [42]2$	$\sqrt{1/2}$	$\sqrt{1/18}$	$\sqrt{4/9}$
$[42]1\ [42]1$		$\sqrt{16/18}$	$-\sqrt{1/9}$

	$[321]2_\alpha$	$[321]2_\beta$	$[411]1$	$[3111]4$
$[42]2\ [42]2$	$\sqrt{32/162}$		$\sqrt{25/81}$	$\sqrt{40/81}^*$
$[42]2\ [42]1$	$\sqrt{25/162}$	$\sqrt{1/2}$	$\sqrt{8/81}$	$-\sqrt{20/81}$
$[42]1\ [42]2$	$-\sqrt{25/162}$	$\sqrt{1/2}$	$-\sqrt{8/81}$	$\sqrt{20/81}$
$[42]1\ [42]1$	$-\sqrt{80/162}$		$\sqrt{40/81}$	$-\sqrt{1/81}$

	$[3111]1$
$[42]1\ [42]1$	-1

Table 6.5 (continued)

	[42]1	[33]2	$[321]3_\alpha$	$[321]3_\beta$
[42]2 [411]3	$\sqrt{27/75}$	$\sqrt{8/45}^*$	$\sqrt{64/225}^*$	$\sqrt{24/135}^*$
[42]2 [411]1	$\sqrt{8/75}$	$\sqrt{12/45}$	$-\sqrt{6/225}$	$-\sqrt{81/135}$
[42]1 [411]3		$-\sqrt{10/45}$	$\sqrt{125/225}$	$-\sqrt{30/135}$
[42]1 [411]1	$\sqrt{40/75}$	$-\sqrt{15/45}$	$-\sqrt{30/225}$	

	$[321]1_\alpha$	$[321]1_\beta$	[2211]4
[42]2 [411]1	$-\sqrt{10/15}$	$\sqrt{15/225}$	$\sqrt{20/75}^*$
[42]1 [411]3	$\sqrt{3/15}$	$-\sqrt{18/225}$	$\sqrt{54/75}$
[42]1 [411]1	$\sqrt{2/15}$	$\sqrt{192/225}$	$\sqrt{1/75}$

	[3111]1	[2211]2
[42]2 [411]1	$\sqrt{1/3}$	$\sqrt{2/3}$
[42]1 [411]1	$\sqrt{2/3}$	$-\sqrt{1/3}$

	[51]2
[42]2 [411]3	1^*

	[51]1	[42]2	$[411]3_\alpha$	$[411]3_\beta$
[42]2 [411]3	$-\sqrt{8/36}$		$\sqrt{100/135}^*$	$\sqrt{6/162}$
[42]2 [411]1	$\sqrt{3/36}$	$\sqrt{25/60}$		$\sqrt{81/162}$
[42]1 [411]3	$-\sqrt{1/36}$	$-\sqrt{27/60}$	$-\sqrt{8/135}$	$\sqrt{75/162}$
[42]1 [411]1	$\sqrt{24/36}$	$-\sqrt{8/60}$	$\sqrt{27/135}$	

	[51]1	[411]3
[42]2 [33]2	$\sqrt{5/9}$	$\sqrt{4/9}^*$
[42]1 [33]2	$\sqrt{4/9}$	$-\sqrt{5/9}$

	[411]1	[321]2
[42]2 [33]2	$\sqrt{4/9}$	$\sqrt{5/9}$
[42]1 [33]2	$-\sqrt{5/9}$	$\sqrt{4/9}$

	[51]2
[42]1 [33]2	1

	[33]2	[321]3
[42]2 [33]2	$\sqrt{4/9}^*$	$\sqrt{5/9}^*$
[42]1 [33]2	$\sqrt{5/9}$	$-\sqrt{4/9}$

	[321]1	[2211]4
[42]2 [33]2	$\sqrt{1/5}$	$\sqrt{4/5}$
[42]1 [33]2	$-\sqrt{4/5}$	$\sqrt{1/5}$

	[2211]2
[42]1 [33]2	1

Table 6.5 (continued)

	[42]2
[42]1 [222]3	1

	[42]1	[321]3
[42]2 [222]3	$\sqrt{4/5}$	$\sqrt{1/5}$
[42]1 [222]3	$-\sqrt{1/5}$	$\sqrt{4/5}$

	[321]1	[222]3
[42]2 [222]3	$\sqrt{5/9}$	$\sqrt{4/9}$
[42]1 [222]3	$-\sqrt{4/9}$	$\sqrt{5/9}$

	[321]2	[3111]4
[42]2 [222]3	$\sqrt{5/9}$	$\sqrt{4/9}$
[42]1 [222]3	$-\sqrt{4/9}$	$\sqrt{5/9}$

	[3111]1	[21^4]5
[42]2 [222]3	$\sqrt{4/9}$	$-\sqrt{5/9}$
[42]1 [222]3	$-\sqrt{5/9}$	$-\sqrt{4/9}$

	[21^4]1
[42]1 [222]3	1

Next we write explicitly the decoupling $[21111]_{OC} \rightarrow [42]_O \times [222]_C$ by using the $\bar{K}$ matrices of Tab. 5 of ref. [5]. This gives[1]

$$
\begin{aligned}
|[42]_O[51]_{FS}\rangle = \sqrt{1/5}\,(&\sqrt{2/3}|[42]_O\overline{11}\rangle\,|[222]_C\overline{33}\rangle \\
&+ \sqrt{1/3}\,|[42]_O\widetilde{12}\rangle\,|[222]_C\widetilde{23}\rangle)|[51]_{FS}\widetilde{12}\rangle \\
&- \sqrt{1/5}\,|[42]_O\overline{12}\rangle\,|[222]_C\widetilde{23}\rangle\,|[51]_{FS}\overline{12}\rangle \\
&+ \sqrt{3/5}(-\sqrt{4/27}|[42]_O\overline{11}\rangle\,|[222]_C\widetilde{23}\rangle\,|[51]_{FS}\overline{11}\rangle \\
&+ \sqrt{6/27}\,|[42]_O\overline{12}\rangle\,|[222]_C\widetilde{23}\rangle\,|[51]_{FS}\overline{11}\rangle \\
&+ \sqrt{5/27}\,|[42]_O\overline{22}\rangle\,|[222]_C\widetilde{23}\rangle\,|[51]_{FS}\overline{11}\rangle \\
&- \sqrt{12/27}\,|[42]_O\widetilde{12}\,\rangle|[222]_C\overline{33}\rangle\,|[51]_{FS}\overline{11}\rangle). \qquad (30)
\end{aligned}
$$

In a similar way, one can also introduce the decoupling $[51]_{FS} \rightarrow [33]_F \times [42]_S$, for isospin $I = 0$ and spin $S = 1$ for example, by using the $\bar{K}$ matrices of Tab. 1 of ref. [5]. Note that at any stage the last pair of particles must be in a totally antisymmetric state. For example, in the first term of Eq. (30) the pair is symmetric in the orbital and colour space but antisymmetric in the FS space.

Finally, it is necessary to test the consistency of the phase convention introduced here. A possibility is to look at the off-diagonal matrix elements calculated in the truncated space chosen for the diagonalization of a Hamiltonian model. For example in ref. [11], where the GBE model [14–16] has been used, in

[1] In Eq. (30) we used $\bar{K}([42]\widetilde{12}[222]\widetilde{23}|[21111]\widetilde{15}) = \sqrt{1/3}$ instead of $-\sqrt{1/3}$ inadvertently printed in ref. [5]

the $SU_F(3)$ exact limit, the π, K, and η exchanges should contribute equally. Then, in this limit, the off-diagonal matrix elements, neglecting the contribution of η', should be identically zero. Another test is the construction of unitary transformations between different coupling schemes, as mentioned in Sect. 5 (see, e.g., ref. [5]). These two tests have been satisfied in our studies.

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