Fuzzy Wavelets

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Abstract

A notion of fuzzy wavelets is introduced by using idea of fuzzy transforms. A detailed procedure for analysis and synthesis of functions through fuzzy wavelets is presented.

Keywords: Fuzzy wavelets, wavelets, fuzzy multiresolution analysis, fuzzy orthogonality, fuzzy convolution, fuzzy delta function

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1. Introduction

In classical mathematics, various types of transforms are used for construction of approximation methods and for solution of differential and integral-differential equations. Perfilieva [9] in a seminal paper developed three techniques of direct and inverse fuzzy transform (F-transform) and approximating properties of the inverse F-transform. In this paper, the core idea of the technique of F-transforms is a fuzzy partition of a universe into fuzzy subsets. Perfilieva et al. [9, 10] proved a number of theorems establishing

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best approximation properties of the inverse F-transforms and established that three types of the inverse F-transform are the best approximation in average, from below and from above. The F-transform establishes a correspondence between a set of continuous functions on an interval of real numbers and the set of n-dimensional (real) vectors. The advantage of the inverse formula of the F-transform is a simple approximate representation of the original function. Thus, in complex computations, the inversion formula can be used instead of precise representation of the original function. In addition to the aforementioned characteristics, the inverse F-transform has nice filtering properties. In this paper, this characteristic has been exploited in developing fuzzy wavelets. The proposed fuzzy wavelet theory is based on one of the three fuzzy transforms given in [9], which uses as a basis the ordinary algebra of real numbers.

We have extended fuzzy multiresolution analysis schemes to the fuzzy wavelets. The notion of fuzzy wavelets already exists but in a different sense versus the work presented in this paper. Recently several researchers have attempted to develop "fuzzy wavelets" based models and systems and a few are discussed here. Haung and Zeng [3] developed a fuzzy wavelet algorithm based on fuzzy transforms and wavelets but were used separately. Their algorithm lacks the theory. Moreover, their algorithm utilizes differences between the original function and reconstructed function to calculate the high frequency components of the function. Yong-Guang and Zhi-Xin [5] presented a fuzzy wavelet algorithm for image fusion based on fuzzy rules applied to the calculated wavelet coefficients. Fuzzy wavelet networks [2] introduced a fuzzy model into the wavelet neural networks to improve the

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accuracy of function approximation. In fuzzy wavelet networks, each rule corresponds to a sub-wavelet neural network. Fuzzy wavelet networks [7] have also been used for contrast enhancement. In fuzzy wavelet denoising technique [8], firstly wavelet coefficients are calculated. Then a threshold is defined on these coefficients using fuzzy rules for a trade-off between the quality of noise and quality of the details. Fuzzy wavenets, as discussed in [11], "combine wavelet theory to fuzzy logic and neural networks." Fuzzy wavenets are used for online applications and give better approximation due to adaptive choice of resolution level and continuously increasing number of data points. The number of fuzzy rules required for fuzzy wavelet network is determined using an orthogonal lease square method [13]. In fuzzy wavelet system modeling, proposed in [4], each rule defines fixed scale parameter but changing translation parameter. Fuzzy wavelet used in [12] is a combination of wavelet analysis and fuzzy c-means clustering. Wavelet analysis is carried out for the signal analysis and fuzzy clustering is used for diagnosis. Fuzzy rules [6] are applied to the wavelet coefficients obtained from packet transform to obtain an index for power quality.

In above mentioned research works, the term "fuzzy wavelets" has been used for defining a system which uses wavelets and fuzzy rules, separately. In this paper, we have developed theory of fuzzy wavelets inspired by multiresolution property of the fuzzy basic functions. We show that fuzzy wavelet theory follows conditions like evolved in the theory of wavelets. Fuzzy wavelets are scaled and translated versions of fuzzy basic functions, hence, forming a series of fuzzy wavelets. Fuzzy wavelets are useful mathematical tool for joint time-frequency analysis in signal processing. In Section 2, some prelim-

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inary information is provided. Section 3 deals with first stage fuzzy wavelet analysis. Some definitions like fuzzy delta function, normalized fuzzy basic functions, fuzzy orthogonality, fuzzy convolution are also given here. First stage fuzzy wavelet synthesis is given in Section 4 to approximately recover the original function. In Section 5, the results of the first stage fuzzy wavelets to multiple stage wavelets, for one dimensional signals, are extended. In Section 6, we generalize the results given in Section 5 for two dimensional signals. In Section 7, we summarize the research presented in this paper and future research targets are given. We have adopted the approach of [1] to derive expressions/conditions for fuzzy wavelets, in Sections 3 to 5.

2. Preliminaries

Take an interval $[a, b]$ as a universe. The fuzzy partition of the universe is given by fuzzy subsets of the fuzzy universe $[a, b]$ which have properties given in the following definition.

Definition 1 ([9])**.** Let $x_1 < \cdots < x_n$ be fixed nodes such that $x_1 = a$ and $x_n = b$ and $n \geq 1$. Let $\mathcal{A}_1, \cdots \mathcal{A}_n$ be the fuzzy sets with their membership functions $\mathcal{A}_1(x), \cdots \mathcal{A}_n(x)$ defined on $[a, b]$, form a fuzzy partition of $[a, b]$ if they fulfill the following conditions for $k = 1, \cdots, n$.

1. $\mathcal{A}_k : [a, b] \to [0, 1], \mathcal{A}_k(x_k) = 1$,
2. $\mathcal{A}_k(x) = 0$ if $x \notin (x_{k-1}, x_{k+1})$,
3. $\mathcal{A}_k$ is continuous,
4. $\mathcal{A}_k$, for $k = 2, \cdots, n$, strictly increases on $[x_{k-1}, x_k]$ and strictly decreases on $[x_k, x_{k+1}]$ for $k = 1, \cdots, n-1$,

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5. for all $x \in [a, b]$

$$\sum_{k=1}^{n} \mathcal{A}_k(x) = 1$$

The membership functions $\mathcal{A}_1, \cdots, \mathcal{A}_n$ are called fuzzy basic functions.

Definition 2 ([9]). Let $\mathbb{S}_{\Delta x, n} = \{\mathcal{A}_1, \cdots, \mathcal{A}_n\}_{\Delta x}$ be a set of fuzzy basic functions which form a fuzzy partition on $[a, b]$, g be any function from $C[a, b]$ and Δx be the support of each basic function. Fuzzy transform of g with respect to $\mathbb{S}_{\Delta x, n}$ is given by:

$$G_k = \frac{\int_a^b g(x)\mathcal{A}_k(x)dx}{\int_a^b \mathcal{A}_k(x)dx} \tag{1}$$

for $k = 1, \cdots, n$.

Corresponding inverse fuzzy transform to approximate the original function is given by following definition.

Definition 3 ([9]). Let $\mathbb{S}_{\Delta x, n} = \{\mathcal{A}_1, \cdots, \mathcal{A}_n\}$ be a set of fuzzy basic functions which form a fuzzy partition on $[a, b]$ and g be any function from $C[a, b]$ and corresponding fuzzy transform of g with respect to $\mathbb{S}_{\Delta x, n}$ be $\mathbf{G}_n[g] = [G_1, \cdots, G_n]$. Then

$$g_{F,n}(x) = \sum_{k=1}^{n} G_k \mathcal{A}_k(x) \tag{2}$$

is called the inverse fuzzy transform.

We can reconstruct the function from its fuzzy coefficients G_k but this is an approximation and we lose some information.

Consider that n defines the maximum possible number of fuzzy basic functions to approximate $g(x)$, and n is divisible by 2^p, where $p \in \mathbb{N}$. If we double

the support of the basic functions, the number of basic functions is halved. Therefore, we can generally write $\mathbb{S}_{2^{j-1}\Delta x, n/2^{j-1}} = \{\mathcal{A}_1, \cdots, \mathcal{A}_{n/2^{j-1}}\}_{2^{j-1}\Delta x}$. See Fig. 1 for details. For different values of j, we get different approximations of the functions and such a process is called multiresolution analysis of a function. For $j = 1$, $\mathbb{S}_{2^{j-1}\Delta x, n/2^{j-1}}$ becomes a set of fuzzy basic functions as given in Definition 3. We call this set as a set of fuzzy basic functions of the first stage in fuzzy multiresolution analysis. Therefore, $\left\{\mathbb{S}_{2^{j-1}\Delta x, n/2^{j-1}}\right\}_{j=1}^{p}$ is a set of basic functions for a complete fuzzy multiresolution analysis of p stages. Wavelets are functions that satisfy certain mathematical require-
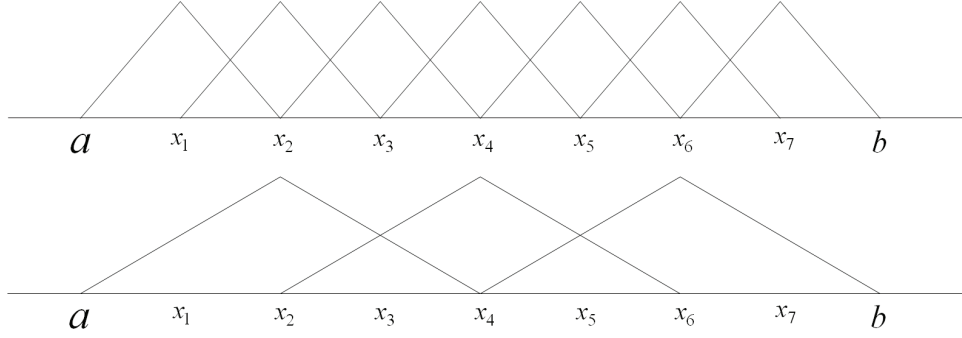

Figure 1: Fuzzy basic functions for fuzzy multiresolution analysis

ments, as stated in the definition given below.

Definition 4 ([1]). Suppose $M \in \mathbb{N}$ and $N = 2M$. let $u, v \in l^2(\mathbb{C}^N)$, where $l^2(\mathbb{C}^N)$ is an N-dimensional vector space over complex numbers $\mathbb{C}$ with Euclidean norm. Then $B = \{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ is a first stage wavelet basis for $l^2(\mathbb{C}^N)$ if and only if the followings hold for all $n = 0, 1, \cdots, M - 1$.

$$|\hat{u}(n)|^2 + |\hat{u}(n+M)|^2 = 2 \tag{3}$$

$$|\hat{v}(n)|^2 + |\hat{v}(n+M)|^2 = 2 \tag{4}$$

$$\hat{u}(n)\overline{\hat{v}(n)} + \hat{u}(n+M)\overline{\hat{v}(n+M)} \;\; = \;\; 0 \tag{5}$$

where $\hat{u}(n)$ and $\hat{v}(n)$ the are Fourier transforms of u and v, respectively. Wavelet conditions are given by Equations 3–5. For details see [1].

3. Fuzzy Wavelets: Analysis

Definition 5. If $\mathcal{A}_k(x)$ is a fuzzy basic function satisfying all the conditions in Definition 1, then *normalized fuzzy basic function* $\mathcal{B}_k(x)$ is given by

$$\mathcal{B}_k(x) = \frac{\mathcal{A}_k(x)}{\int_{-\infty}^{\infty} \mathcal{A}_k(x)dx} \tag{6}$$

Definition 6. Let $\mathcal{A}_k$ be a fuzzy basic function. A fuzzy basic function $\delta_k(x)$ which satisfies all the conditions given in Definition 1. Let there exists $p \in \mathbb{N}$ and $p > 1$ such that the following holds.

$$\delta_k(x) = \mathcal{A}_k^p(x) \tag{7}$$

Where $\mathcal{A}_k^p(x) = \mathcal{A}_k(x) \cdots \mathcal{A}_k(x)$ (p times), and $\delta_k(x)$ called *fuzzy delta function*.

This is obvious that

$$\int_{-\infty}^{\infty} \delta_k(x)dx \ll \int_{-\infty}^{\infty} \mathcal{A}_k(x)dx \tag{8}$$

and:

$$\int_{-\infty}^{\infty} \mathcal{A}_k(x)dx = 1 \tag{9}$$

Definition 7. Let $\mathcal{A}_k(x)$ (for $k = 0, \cdots, n$) be the fuzzy basic functions. $\{\mathcal{A}_k(x)\}$ are *fuzzy orthogonal*, if

$$\int_{-\infty}^{\infty} \mathcal{A}_j(x)\mathcal{A}_k(x)dx = \begin{cases} \delta_k(x) & j = k \\ \epsilon(x) & |j - k| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where $\epsilon(x)$ is a function such that

$$\int_{-\infty}^{\infty} \epsilon(x)dx = \alpha \ll \int_{-\infty}^{\infty} \delta_k(x)dx \qquad (11)$$

where α is an arbitrarily small non-negative real number close to zero.

In the remaining paper, we shall use the word fuzzy basic functions for normalized fuzzy membership functions.

Definition 8. Consider a fuzzy basic function $\mathcal{A}(x)$ but centered at first node, i.e., $k = 0$. We define a *shift operator* (R_k) as follows.

$$\mathcal{A}_k(x) = R_k \mathcal{A}(x) \qquad (12)$$

Definition 9. *Fuzzy inner product* is defined by:

$$\langle \mathcal{A}, R_k \mathcal{A} \rangle = \oplus_{m=-\infty}^{\infty} \mathcal{A} \otimes \mathcal{A}_k \qquad (13)$$

where

$$(\mathcal{A} \otimes \mathcal{A}_k)(m) = \mathcal{A}(m)\mathcal{A}_k(m) \qquad (14)$$

is an ordinary product. Moreover, summations of any two terms in Equation (13) is calculated as follows.

$$
\begin{aligned}
(\mathcal{A} \otimes \mathcal{A}_k)(m) \oplus (\mathcal{A} \otimes \mathcal{A}_k)(n) &= (\mathcal{A} \otimes \mathcal{A}_k)(m) + (\mathcal{A} \otimes \mathcal{A}_k)(n) \\
&\quad - (\mathcal{A} \otimes \mathcal{A}_k)(m)(\mathcal{A} \otimes \mathcal{A}_k)(n) \qquad (15)
\end{aligned}
$$

Definition 10. Let $\mathcal{A}_k(x) = R_k \mathcal{A}(x)$ (for $k = 0, \cdots, n$) be the fuzzy basic functions satisfying Equations (9–10). Then $\{\mathcal{A}_k(x)\}$ are *fuzzy orthonormal*. This implies:

$$\langle \mathcal{A}(x), R_k\mathcal{A}(x)\rangle = \begin{cases} \delta_k(x) & k = 0 \\ \epsilon(x) & |k| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

where $\langle \cdot, \cdot \rangle$ is an inner product. As area under $\epsilon(x)$ is much smaller than that of $\delta_k(x)$, we approximate that

$$\langle \mathcal{A}(x), R_k\mathcal{A}(x)\rangle = \begin{cases} \delta_k(x) & k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

Definition 11. For any fuzzy basic function $\mathcal{A}$, we define *reflection of a membership function*, $\tilde{\mathcal{A}}$, by:

$$\tilde{\mathcal{A}}(x) = \mathcal{A}(-x) \tag{18}$$

for $x \in [a, b]$. In discrete form

$$\tilde{\mathcal{A}}(n) = \mathcal{A}(-n) = \mathcal{A}(N - n) \tag{19}$$

for all $n \leq N$ and $N \in \mathbb{N}$.

Definition 12. Consider two basic functions $\mathcal{A}(x)$ and $\mathcal{B}(x)$. *Fuzzy convolution* of $\mathcal{A}(x)$ and $\mathcal{B}(x)$ is defined as:

$$(\mathcal{A}(x) * \mathcal{B}(x))(k) = \oplus_{x=-\infty}^{\infty} \mathcal{A}(x) \otimes \mathcal{B}(k - x) \tag{20}$$

for all k, where k represents the index of nodes as given in Definition 1.

Definition 13. Let $\mathcal{A}(x)$ be a fuzzy basic function. Then the discrete Fourier transform of $\mathcal{A}$ is denoted by $\hat{\mathcal{A}}$. We denote the inverse discrete Fourier transform of $\mathcal{A}$ by $\check{\mathcal{A}}$.

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We follow the conventional definitions of Fourier transform and inverse Fourier transforms [1] with their standard properties.

$$(\hat{\mathcal{A}})\check{} = \mathcal{A} \tag{21}$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two basic functions. Then

$$(\mathcal{A}(x) * \mathcal{B}(x))\hat{} = \hat{\mathcal{A}}(x)\hat{\mathcal{B}}(x) \tag{22}$$

and we define:

$$\hat{\delta}_k(x) = 1 \tag{23}$$

Remark 1. Let $\mathcal{A}$ be a fuzzy basic function and its discrete Fourier transform be $\hat{\mathcal{A}}$. Then *discrete Fourier transform of a reflected fuzzy basic function* is defined as:

$$(\tilde{\mathcal{A}})\hat{} = \bar{\hat{\mathcal{A}}} \tag{24}$$

Where $\bar{\hat{\mathcal{A}}}$ represents conjugate of $\hat{\mathcal{A}}$.

Lemma 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two fuzzy basic functions. For any $k \in \mathbb{Z}$,*

$$(\mathcal{A} * \tilde{\mathcal{B}})(k) = \langle \mathcal{A}, R_k \mathcal{B} \rangle \tag{25}$$

$$(\mathcal{A} * \mathcal{B})(k) = \left\langle \mathcal{A}, R_k \tilde{\mathcal{B}} \right\rangle \tag{26}$$

Proof:

Let $\mathcal{A}$ and $\mathcal{B}$ be two discrete fuzzy basic functions. For any k:

$$\begin{aligned}
\langle \mathcal{A}, R_k \mathcal{B} \rangle &= \oplus_{n=0}^{N-1} \mathcal{A}(n) \otimes R_k \mathcal{B}(n) \\
&= \oplus_{n=0}^{N-1} \mathcal{A}(n) \otimes \mathcal{B}(n-k) \\
&= \oplus_{n=0}^{N-1} \mathcal{A}(n) \otimes \tilde{\mathcal{B}}(k-n) \\
&= (\mathcal{A} * \tilde{\mathcal{B}})(k)
\end{aligned}$$

Similarly,

$$
\begin{aligned}
\left\langle \mathcal{A}, R_k \tilde{\mathcal{B}} \right\rangle &= \oplus_{n=0}^{N-1} \mathcal{A}(n) \otimes R_k \tilde{\mathcal{B}}(n) \\
&= \oplus_{n=0}^{N-1} \mathcal{A}(n) \otimes \tilde{\mathcal{B}}(n-k) \\
&= \oplus_{n=0}^{N-1} \mathcal{A}(n) \otimes \mathcal{B}(k-n) \\
&= (\mathcal{A} * \mathcal{B})(k)
\end{aligned}
$$

Lemma 2. *Let $\mathcal{A}(x)$ be a normal fuzzy basic function. If $\{R_k \mathcal{A}(x)\}_{k=0}^{N-1}$ is an orthonormal basis then $|(\mathcal{A}(x))\Upsilon(k)| = 1$ for all $k \in \mathbb{Z}$.*

Proof: Using Lemma 1,

$$
\langle \mathcal{A}(x), R_k \mathcal{A}(x) \rangle = (\mathcal{A}(x) * \tilde{\mathcal{A}}(x))(k) \tag{27}
$$

Using fuzzy orthonormality and Equation 27:

$$
(\mathcal{A}(x) * \tilde{\mathcal{A}}(x))(k) = \delta_k(x) \tag{28}
$$

This equation holds for all $k \in \mathbb{Z}$. Taking discrete Fourier transform on both the sides:

$$
\begin{aligned}
1 &= \hat{\delta}_k(x) = (\mathcal{A}(x) * \tilde{\mathcal{A}}(x))\Upsilon(k) = (\mathcal{A}(x))\Upsilon(k)(\tilde{\mathcal{A}}(x))\Upsilon(k) \\
&= (\mathcal{A}(x))\Upsilon(k)\overline{(\mathcal{A}(x))\Upsilon(k)} = |(\mathcal{A}(x))\Upsilon(k)|^2 \tag{29}
\end{aligned}
$$

Taking square root on both the sides ($|(\mathcal{A}(x))\Upsilon(k)|$ is magnitude and it is always non-negative):

$$
|(\mathcal{A}(x))\Upsilon(k)| = 1
$$

Remark 2. From filtering theory, it is well known that fuzzy basic functions have the property that they can extract low frequency harmonics in a function. To extract high frequency harmonics of a function, we shift the spectrum by π and extract the low frequency harmonics of the new signal.

For this purpose, we define a spectral shift function f as follows.

Definition 14. Let us define a spectral shift function, f, given below:

$$f(x) = e^{-j\pi x} \tag{30}$$

where $j^2 = -1$. For discrete signals:

$$f(n) = (-1)^n \tag{31}$$

Lemma 3. *Let $x(n)$ be a discrete function. We define a function $x^*(n) = f(n)x(n) = (-1)^n x(n)$. For $M \in \mathbb{Z}$ and $N = 2M$,*

$$(\widehat{x^*})(n) = \hat{x}(n + M) \tag{32}$$

Proof:

$$
\begin{aligned}
(\widehat{x^*})(n) &= \sum_{k=0}^{N-1} x^*(k) e^{-j2\pi kn/N} = \sum_{k=0}^{N-1} (-1)^k x(k) e^{-j2\pi kn/N} \\
&= \sum_{k=0}^{N-1} x(k) e^{-j\pi k} e^{-j2\pi kn/N} = \sum_{k=0}^{N-1} x(k) e^{-j2\pi k(n+M)/N} = \hat{x}(n + M)
\end{aligned}
$$

Corollary 1. *Let $x(n)$ be a function and $f(n)$ is given by Equation 31. Then*

$$
\begin{aligned}
x(n) + x^*(n) &= x(n) + x(n)f(n) = x(n)\left[1 + (-1)^n\right] \\
&= \begin{cases} 2x(n) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}
\end{aligned} \tag{33}
$$

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Note: In the remaining part of the paper, we shall use $\mathcal{A}$, $\tilde{\mathcal{A}}$ and δ_k in place of $\mathcal{A}(x)$, $\tilde{\mathcal{A}}(x)$ and $\delta_k(x)$ respectively.

Lemma 4. *Let $\mathcal{A}$ be a fuzzy basic function. For $k \in \mathbb{Z}$, $\{R_k\mathcal{A}\}_{k=0}^{N-1} \bigcup \{R_k\mathcal{A}f\}_{k=0}^{N-1}$ is a set of fuzzy orthonormal basis.*

Proof: For $j, k \in \mathbb{Z}$

$$\langle R_j\mathcal{A}, R_k\mathcal{A}f \rangle = \langle \mathcal{A}_j, \mathcal{A}_k f \rangle = \int \mathcal{A}_j(x)\mathcal{A}_k(x)f(x)dx$$

Integrating we get:

$$\langle R_j\mathcal{A}, R_k\mathcal{A}f \rangle = 0 \tag{34}$$

Lemma 5. *Let $M \in \mathbb{Z}$, $N = 2M$, and $\mathcal{A}$ be a fuzzy basic function. If $\{R_{2k}\mathcal{A}\}_{k=0}^{M-1}$ is a set of fuzzy orthonormal basis then the following holds for $n = 0, 1, \cdots, M - 1$.*

$$\left|\hat{\mathcal{A}}(n)\right|^2 + \left|\hat{\mathcal{A}}(n + M)\right|^2 = 2 \tag{35}$$

Proof: From Equation 28:

$$(\mathcal{A} * \tilde{\mathcal{A}})(2k) = \delta_k \tag{36}$$

Using Corollary 1

$$\left((\mathcal{A} * \tilde{\mathcal{A}}) + (\mathcal{A} * \tilde{\mathcal{A}})f\right)(n) = \begin{cases} 2(\mathcal{A} * \tilde{\mathcal{A}})(n) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Replacing n by $2k$ (for all k) and using Equation 36:

$$\left((\mathcal{A} * \tilde{\mathcal{A}}) + (\mathcal{A} * \tilde{\mathcal{A}})f\right)(2k) = 2(\mathcal{A} * \tilde{\mathcal{A}})(2k) = 2\delta_k$$

$$(\mathcal{A} * \tilde{\mathcal{A}}) + (\mathcal{A} * \tilde{\mathcal{A}})f = 2\delta_k$$

Taking the discrete Fourier transform of the above equation,

$$(\widehat{\mathcal{A} * \tilde{\mathcal{A}}})(n) + (\widehat{(\mathcal{A} * \tilde{\mathcal{A}})f})(n) = 2 \tag{37}$$

$$(\widehat{\mathcal{A} * \tilde{\mathcal{A}}})(n) = \hat{\mathcal{A}}(n)(\widehat{\tilde{\mathcal{A}}})(n) = \hat{\mathcal{A}}(n)\overline{\hat{\mathcal{A}}(n)} = \left|\hat{\mathcal{A}}(n)\right|^2 \tag{38}$$

$$(\widehat{(\mathcal{A} * \tilde{\mathcal{A}})f})(n) = (\widehat{\mathcal{A} * \tilde{\mathcal{A}}})(n + M) = \left|\hat{\mathcal{A}}(n + M)\right|^2 \tag{39}$$

Using Equations 38 and 39 in Equation 37, we get:

$$\left|\hat{\mathcal{A}}(n)\right|^2 + \left|\hat{\mathcal{A}}(n + M)\right|^2 = 2$$

This proves Equation 35.

Corollary 2. *Equation 35 is periodic with $N = 2M$. Therefore,*

$$\left|\hat{\mathcal{A}}(n + 2M)\right|^2 + \left|\hat{\mathcal{A}}(n + 3M)\right|^2 = 2$$

$$|(\widehat{\mathcal{A}f})(n + M)|^2 + |(\widehat{\mathcal{A}f})(n + 2M)|^2 = 2$$

$$|(\widehat{\mathcal{A}f})(n + M)|^2 + |(\widehat{\mathcal{A}f})(n)|^2 = 2$$

$$|(\widehat{\mathcal{A}f})(n)|^2 + |(\widehat{\mathcal{A}f})(n + M)|^2 = 2 \tag{40}$$

Theorem 1. *Suppose $M \in \mathbb{N}$ and $N = 2M$. Let $\mathcal{A}$ be a fuzzy basic function. If*

$$S = \{R_{2k}\mathcal{A}\}_{k=0}^{M-1}\bigcup\{R_{2k}\mathcal{A}f\}_{k=0}^{M-1} \tag{41}$$

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is a set of orthonormal fuzzy basis then

$$\left|\hat{\mathcal{A}}(n)\right|^2 + \left|\hat{\mathcal{A}}(n+M)\right|^2 = 2$$

$$|(\mathcal{A}f)\hat{\ }(n)|^2 + |(\mathcal{A}f)\hat{\ }(n+M)|^2 = 2$$

$$\hat{\mathcal{A}}(n)\overline{(\mathcal{A}f)\hat{\ }(n)} + \hat{\mathcal{A}}(n+M)\overline{(\mathcal{A}f)\hat{\ }(n+M)} = 0$$

Proof: First two conditions of the theorem have been proved in Lemma 5 and Corollary 2. Third condition is proved as follows.

From Lemma 4, for even values of k:

$$\langle \mathcal{A}, R_k \mathcal{A}f \rangle = 0$$

From Equations 27 and 34 (for even values of k):

$$(\mathcal{A} * (\mathcal{A}f)\hat{\ })(k) = \langle \mathcal{A}, R_k \mathcal{A}f \rangle$$
$$= 0 \tag{42}$$

For odd values of k, we use Equation 33 with $x = \mathcal{A} * (\mathcal{A}f)\hat{\ }$:

$$(\mathcal{A} * (\mathcal{A}f)\hat{\ })(k) + (\mathcal{A} * (\mathcal{A}f)\hat{\ })(k)f(k) = 0 \tag{43}$$

From the above two equations, we can write:

$$(\mathcal{A} * (\mathcal{A}f)\hat{\ }) + (\mathcal{A} * (\mathcal{A}f)\hat{\ })f = 0 \tag{44}$$

for all k.

Taking the discrete Fourier transform of the above equation:

$$(\mathcal{A} * (\mathcal{A}f)\hat{\ })\hat{\ }(n) + ((\mathcal{A} * (\mathcal{A}f)\hat{\ })f)\hat{\ }(n) = 0 \tag{45}$$

Taking the first term of the above equation:

$$(\mathcal{A} * \widehat{(\mathcal{A}f)})(n) = \hat{\mathcal{A}}(n)\overline{(\mathcal{A}f)}(n) \qquad (46)$$

Now, taking the second term,

$$((\mathcal{A} * \widehat{(\mathcal{A}f)})f)(n) = (\mathcal{A} * \widehat{(\mathcal{A}f)})(n+M) = \hat{\mathcal{A}}(n+M)\overline{(\mathcal{A}f)}(n+M) \qquad (47)$$

Putting Equations 46 and 47 in Equation 45:

$$\hat{\mathcal{A}}(n)\overline{(\mathcal{A}f)(n)} + \hat{\mathcal{A}}(n+M)\overline{(\mathcal{A}f)(n+M)} = 0 \qquad (48)$$

This proves the third condition for all $n = 0, 1, \cdots, M - 1$.

Definition 15. Let $M \in \mathbb{N}$ and $N = 2M$, and $\mathcal{A}$ be a fuzzy basic function. A set of fuzzy orthonormal basis of the form $\{R_{2k}\mathcal{A}\}_{k=0}^{M-1} \bigcup \{R_{2k}\mathcal{A}f\}_{k=0}^{M-1}$ is called first stage fuzzy wavelets.

$\mathcal{A}$ is called the generator of the first stage fuzzy wavelet basis.

Definition 16. Let $M \in \mathbb{N}$ and $N = 2M$. Define $\mathcal{D} : l^2(\mathbb{R}_N) \to l^2(\mathbb{R}_M)$, such that

$$(\mathcal{D}x)(n) = x(2n) \qquad (49)$$

for $n = 0, 1, \cdots, M - 1$. $\mathcal{D}$ is called *decimation operator*.

Figure 2 shows a complete analysis process for the first stage fuzzy wavelets.

4. Fuzzy Wavelets: Synthesis

Definition 17. Let $M \in \mathbb{N}$ and $N = 2M$. Define $\mathcal{U} : l^2(\mathbb{R}_M) \to l^2(\mathbb{R}_N)$, such that

$$(\mathcal{U}x)(n) = \begin{cases} x(\frac{n}{2}) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \qquad (50)$$

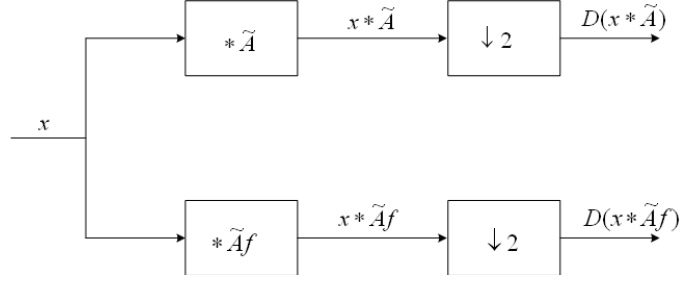for $n = 0, 1, \cdots, M - 1$. $\mathcal{U}$ is called *upsampling operator*.

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Figure 2: Analysis stage of first stage fuzzy wavelets

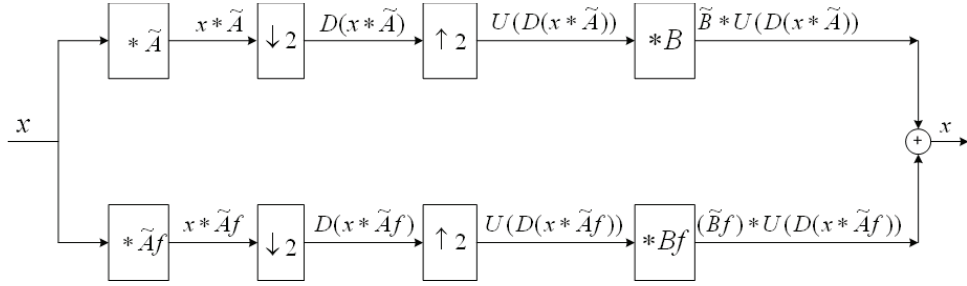Figure 3 shows a complete analysis and synthesis processes for the first stage fuzzy wavelets.

Figure 3: Analysis and synthesis stage of Fuzzy wavelets

As:

$$\mathcal{U}(\mathcal{D}(x))(n) = \frac{1}{2}(x + xf)(n) \tag{51}$$

Replacing x with $x * \tilde{\mathcal{A}}f$.

$$\mathcal{U}(\mathcal{D}(x * \tilde{\mathcal{A}}f))(n) = \frac{1}{2}\left((x * \tilde{\mathcal{A}}f) + (x * \tilde{\mathcal{A}}f)f\right)(n) \tag{52}$$

Similarly:

$$\mathcal{U}(\mathcal{D}(x * \tilde{\mathcal{A}}))(n) = \frac{1}{2}\left((x * \tilde{\mathcal{A}}) + (x * \tilde{\mathcal{A}})f\right)(n) \tag{53}$$

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Taking the discrete Fourier transform on both the sides of the above equations, i.e., Equations 52 and 53.

$$(\mathcal{U}(\mathcal{D}(x * \tilde{\mathcal{A}}f)))\hat{}(n) = \frac{1}{2}\left(\hat{x}(n)\overline{(\mathcal{A}f)\hat{}(n)} + \hat{x}(n+M)\overline{(\mathcal{A}f)\hat{}(n+M)}\right) \quad (54)$$

Similarly:

$$(\mathcal{U}(\mathcal{D}(x * \tilde{\mathcal{A}})))\hat{}(n) = \frac{1}{2}\left(\hat{x}(n)\overline{\hat{\mathcal{A}}(n)} + \hat{x}(n+M)\overline{\hat{\mathcal{A}}(n+M)}\right) \quad (55)$$

Convolving left hand side of Equation 52 with $\tilde{\mathcal{B}}f$ and Equation 53 with $\tilde{\mathcal{B}}$, and taking the discrete Fourier transform of the summation of the resulting expressions:

$$\begin{aligned}
\left(\tilde{\mathcal{B}}f \quad * \quad \mathcal{U}(\mathcal{D}(x * \tilde{\mathcal{A}}f)) + \tilde{\mathcal{B}} * \mathcal{U}(\mathcal{D}(x * \tilde{\mathcal{A}}))\right)^{\wedge}(n) \\
= \quad & \overline{(\mathcal{B}f)\hat{}(n)}.\frac{1}{2}\left(\hat{x}(n)\overline{(\mathcal{A}f)\hat{}(n)} + \hat{x}(n+M)\overline{(\mathcal{A}f)\hat{}(n+M)}\right) \\
+ \quad & \overline{\hat{\mathcal{B}}(n)}.\frac{1}{2}\left(\hat{x}(n)\overline{\hat{\mathcal{A}}(n)} + \hat{x}(n+M)\overline{\hat{\mathcal{A}}(n+M)}\right) \quad (56) \\
= \quad & \frac{1}{2}\left(\overline{(\mathcal{B}f)\hat{}(n)(\mathcal{A}f)\hat{}(n)} + \overline{\hat{\mathcal{B}}(n)\hat{\mathcal{A}}(n)}\right)\hat{x}(n) \\
+ \quad & \frac{1}{2}\left(\overline{(\mathcal{B}f)\hat{}(n)(\mathcal{A}f)\hat{}(n+M)} + \overline{\hat{\mathcal{B}}(n)\hat{\mathcal{A}}(n+M)}\right)\hat{x}(n+M) \quad (57)
\end{aligned}$$

We want to equate the right hand side of the above equation to $x(n)$ to get an approximation of $x(n)$. One possibility (the simplest) gives the following two conditions.

$$(\mathcal{B}f)\hat{}(n)(\mathcal{A}f)\hat{}(n) + \hat{\mathcal{B}}(n)\hat{\mathcal{A}}(n) = 2 \quad (58)$$

$$(\mathcal{B}f)\hat{}(n)(\mathcal{A}f)\hat{}(n+M) + \hat{\mathcal{B}}(n)\hat{\mathcal{A}}(n+M) = 0 \quad (59)$$

Rewriting Equations 58:

$$\hat{\mathcal{A}}(n)\hat{\mathcal{B}}(n) + \hat{\mathcal{A}}(n+M)\hat{\mathcal{B}}(n+M) = 2 \quad (60)$$

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Comparing Equation 60 with Equation 35:

$$\hat{\mathcal{B}}(n) = \overline{\hat{\mathcal{A}}(n)} \qquad (61)$$

and this implies:

$$\mathcal{B}(n) = \tilde{\mathcal{A}}(n) \qquad (62)$$

Rewriting Equations 59:

$$\hat{\mathcal{A}}(n)(\mathcal{B}f)\hat{}(n) + \hat{\mathcal{A}}(n+M)(\mathcal{B}f)\hat{}(n+M) = 0 \qquad (63)$$

Comparing Equation 63 with third condition given in Theorem 1 given by Equation 48:

$$(\mathcal{B}f)\hat{}(n) = \overline{(\mathcal{A}f)\hat{}(n)} \qquad (64)$$

and this implies:

$$(\mathcal{B}f)(n) = (\tilde{\mathcal{A}}f)(n) \qquad (65)$$

5. Multistage Fuzzy Wavelets

In this section, we generalize the results obtained for the first stage fuzzy wavelets to multiple stage wavelets. Let N be divisible by 2^p and $p \in \mathbb{N}$.

Analysis

We know that $\{R_{2k}\mathcal{A}\}_{k=0}^{M-1}$ extracts low frequency components of the function whereas $\{R_{2k}\mathcal{A}f\}_{k=0}^{M-1}$ extracts high frequency components of the the function. Like wavelets, we apply the fuzzy wavelet analysis further to the low frequency components computed using $\{R_{2k}\mathcal{A}\}_{k=0}^{M-1}$ and that is $\mathcal{D}(x * \tilde{\mathcal{A}})$.

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This means, in the second stage, we are to compute $\mathcal{D}(\mathcal{D}(x * \tilde{\mathcal{A}}) * \tilde{\mathcal{A}})$ and $\mathcal{D}(\mathcal{D}(x * \tilde{\mathcal{A}}) * \tilde{\mathcal{A}}f)$. Such a process can be carried out p-times. Note that after the analysis in the second stage, we are left with three vectors, namely, $\mathcal{D}(\mathcal{D}(x * \tilde{\mathcal{A}}) * \tilde{\mathcal{A}})$, $\mathcal{D}(\mathcal{D}(x * \tilde{\mathcal{A}}) * \tilde{\mathcal{A}}f)$ and $\mathcal{D}(x * \tilde{\mathcal{A}}f)$.

Consider a set of vectors $\{x_1, y_1, x_2, y_2, \cdots, x_p, y_p\}$. These vectors are computed by the following equations.

$$x_1 = \mathcal{D}(x * \tilde{\mathcal{A}}) \in l^2(\mathbb{R}^{N/2}) \tag{66}$$

$$y_1 = \mathcal{D}(x * \tilde{\mathcal{A}}f) \in l^2(\mathbb{R}^{N/2}) \tag{67}$$

$$x_2 = \mathcal{D}(x_1 * \tilde{\mathcal{A}}) \in l^2(\mathbb{R}^{N/2^2}) \tag{68}$$

$$y_2 = \mathcal{D}(x_1 * \tilde{\mathcal{A}}f) \in l^2(\mathbb{R}^{N/2^2}) \tag{69}$$

Continuing this way,

$$x_l = \mathcal{D}(x_{l-1} * \tilde{\mathcal{A}}) \in l^2(\mathbb{R}^{N/2^l}) \tag{70}$$

$$y_l = \mathcal{D}(x_{l-1} * \tilde{\mathcal{A}}f) \in l^2(\mathbb{R}^{N/2^l}) \tag{71}$$

The set of vectors $\{y_1, y_2, \cdots, y_p, x_p\}$, at output of the pth analysis stage, is called p^{th} *stage fuzzy wavelet filter bank*. A two stage fuzzy wavelet filter bank is shown in Fig. 4.

Definition 18. Let $M \in \mathbb{N}$ and $N = 2M$, and $\mathcal{A}$ be a fuzzy basic function, and N be divisible by 2^p. A set of fuzzy orthonormal basis of the form $\{R_{2^j k}\mathcal{A}_{2^{j-1}\Delta}\}_{k=0}^{M-1} \bigcup \{R_{2^j k}\mathcal{A}_{2^{j-1}\Delta}f\}_{k=0}^{M-1}$ is called j^{th} stage fuzzy wavelets, where Δ is the support of $\mathcal{A}$, and $j = 1, \cdots, p$.

Definition 19. Let $M \in \mathbb{N}$ and $N = 2M$, and $\mathcal{A}$ be a fuzzy basic function, and N be divisible by 2^p. A set of fuzzy orthonormal basis of the
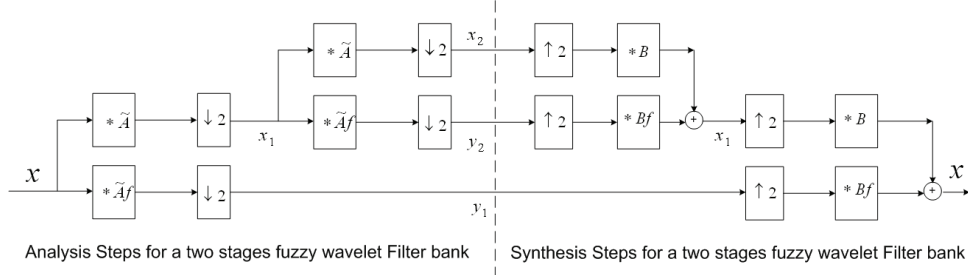
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Figure 4: A two stage fuzzy wavelet filter bank for one dimensional discrete signals. It includes analysis and synthesis stages.

form $\{R_{2^p k}\mathcal{A}_{2^{p-1}\Delta}\}_{k=0}^{M-1} \bigcup \{R_{2^j k}\mathcal{A}_{2^{j-1}\Delta}f\}_{j=1,k=0}^{j=p,k=M-1}$ is called p^{th} stage fuzzy wavelet filter bank, where Δ is the support of $\mathcal{A}$.

Remark 3. Sum of the number of components of all the output vectors of the pth stage wavelet filter bank is

$$\frac{N}{2} + \frac{N}{4} + \cdots + \frac{N}{2^{p-1}} + \frac{N}{2^p} + \frac{N}{2^p} = N \tag{72}$$

Synthesis

Consider that set of vectors $\{y_1, y_2, \cdots, y_p, x_p\}$, at output of pth stage wavelet filter bank is available and we have to construct an approximation of x. We construct x_{p-1} using x_p and y_p.

$$x_{p-1} = \mathcal{U}(x_p) * \mathcal{A} + \mathcal{U}(y_p) * \mathcal{A}f \tag{73}$$

Similarly, for the jth stage:

$$x_j = \mathcal{U}(x_{j+1}) * \mathcal{A} + \mathcal{U}(y_{j+1}) * \mathcal{A}f \tag{74}$$

Continuing in this way, we get the approximation of the original signal as:

$$x = \mathcal{U}(x_1) * \mathcal{A} + \mathcal{U}(y_1) * \mathcal{A}f \tag{75}$$

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6. Two Dimensional Multistage Fuzzy Wavelets

A two dimensional discrete function $\mathbf{x}$, from $C([a,b] \times [c,d])$, can be considered as a matrix of size $M_1 \times M_2$, for a rectangular universe $[a,b] \times [c,d]$. M_1 and M_2 are the number of nodes for $[a,b]$ and $[c,d]$, respectively, and are divisible by 2^p. Let $\mathcal{A}$ be a fuzzy basic function. The elements of the sets of functions $\{R_{2k}\mathcal{A}\}_{k=0}^{M_1-1}$ and $\{R_{2k}\mathcal{A}\}_{k=0}^{M_2-1}$ form a fuzzy partition of the universes $[a,b]$ and $[c,d]$, along rows and columns of the matrix, respectively.

A two stage fuzzy wavelet analysis scheme is shown in Fig. 5 for two dimensional discrete signals. We apply the fuzzy wavelet analysis scheme to the rows of the discrete two dimensional matrix $\mathbf{x}$ and then down sample. This process yields two matrices x_1 and x_2, both of size $\frac{M_1}{2} \times M_2$. An execution of fuzzy wavelet analysis scheme to the columns of x_1 and x_2 and then down sample again, generating four matrices x_{11}, x_{12}, x_{21} and x_{22}, each of size $\frac{M_1}{2} \times \frac{M_2}{2}$, thus completing the first stage analysis.

7. Conclusions

We have developed fuzzy wavelets theory based on a direct and inverse fuzzy transform based on the ordinary algebra of reals. Fuzzy wavelets are useful for harmonic analysis of the functions and for construction of various approximating models depending on the choice of fuzzy basic functions. We have developed conditions for fuzzy wavelets. While developing conditions for fuzzy wavelets, we have developed terms like fuzzy multiresolution analysis, fuzzy delta function, normalized fuzzy basic functions, fuzzy orthogonality, fuzzy convolution, and orthonormal fuzzy basic functions, with their required properties. These terms are synonymous with the terms existing in
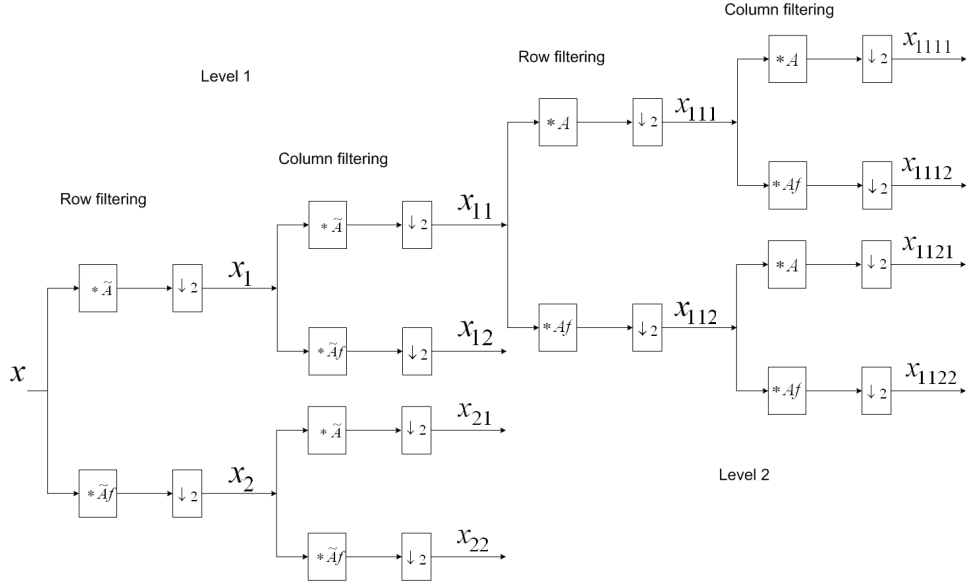
Figure 5: A two stage fuzzy wavelet analysis for two dimensional discrete signals.

the signal/image processing theory. These terms will be useful in developing the theory of fuzzy signal processing. We also intend to use fuzzy wavelets for joint time frequency analysis of discrete and continuous functions.

Acknowledgements

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