# MOCK EXAM FOR

# MATHEMATICS FOR COMPUTER SCIENTISTS

Try to do this mock exam in three hours.

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***Answer all FOUR questions***

Try to do this mock exam without help from other students or electronic devices. This mock exam is designed to take 3 hours. You are allowed on the real exam to use the lecture notes. Relying heavily on lecture notes costs time, so make sure you practice the exercises, which will help you to apply the notes without having to look them up. If this mock exam takes more than 3 hours, then having to look up the information is the most likely cause for delay.

**Question 1** This question is about Logic and Set Theory.

[overall 29 marks]

a) Use a truth table to prove that $P \vee \neg R) \Leftrightarrow (Q \Rightarrow \bot)$ is equivalent to $(P \Rightarrow \neg Q) \wedge (Q \Rightarrow R) \wedge (R \Rightarrow (\neg Q \Rightarrow P))$. [6 marks]

b) Let $A = \{a, b, 1, 2\}$, $B = \{a, b, c\}$ and $C = \{1, 2, 3\}$.

(i) What is the powerset of $A \cap B$?

(ii) Draw an Euler diagram of $A$, $B$ and $C$.

(iii) Draw a Venn diagram of $A$, $B$ and $C$.

(iv) Let $D = \mathcal{P}(A \setminus B) \times C$. What is $|D|$?

(v) Using $A$, $B$, $C$, and the basic set operations $(\cup, \cap, \setminus, \mathcal{P}, \times)$ construct $\{(c, 3)\}$.

[$5 \cdot 3 = 15$ marks]

c) Consider the statement $\exists^1_{n \in \mathbb{N}}\big(5 \mid n \wedge \forall_{k \in \mathbb{N}}(k \mid n \Rightarrow (k = 1 \vee k = n))\big)$.

(i) What does this statement mean? Is it true?

(ii) Rewrite the formula only using *universal* quantification. (Use De Morgan and the definition of $\exists^1$. Write steps for partial credit in case of a mistake.)

[$3 + 5 = 8$ marks]

**Question 2** This question is about Functions and Relations.

[overall 20 marks]

a Bijections:

(i) Can you construct a bijective function $f : \mathbb{Z} \to \mathbb{Z}$, other than the function $f(a) = a$?

(ii) We can take a relation $\mathbf{R}$, which is defined as $(b, a) \in \mathbf{R}$ whenever $(a, b) \in f$ [i.e. whenever $f(a) = b$]. Is the relation $\mathbf{R}$ a function, partial function, multi-valued function, or not? Why (not)?

(iii) Recall the bijective function from integers to natural numbers. Let's call this function $g$. What are the domain, image and codomain of $g \circ g$?

(iv) Let $g^{-1}$ be a bijective function from natural numbers to integers, with the property that $g(g^{-1}(n)) = n$. Using your function $f$, can you compute $(g \circ f \circ g^{-1})(5)$.

(v) Does there exist a bijection $h$ between natural numbers, such that $h(h(n)) = n$? [$3 + 3 + 2 + 2 + 2 = 12$ marks] b Take the relation $\mathbf{S}$ as, $(s,t) \in \mathbf{S}$ is $(s,t)$ is a matching pair of socks.

(i) Is this relationship heterogeneous or homogeneous?

(ii) Alice is a sloppy person, and Bob is very neat. Call Alice's sock drawer $A$, and Bob's sock drawer $B$. We say that a relation $\mathbf{T} \subseteq X \times Y$ is *serial* if for every element in $x \in X$, there exists an element $y \in Y$, such that $(x,y) \in \mathbf{T}$. Whose sock drawer is more likely to be serial? Why?

(iii) Charlie's sock drawer contains one pair of black, white and blue socks, a pair of Christmas socks, a pair of striped socks, and single red sock. Is the relation $\mathbf{S}$ defined over Charlie's sock drawer a function, a partial function, a multi-valued function, or none? [$2+3+3=8$ marks]

**Question 3** This question is about Relations and Graphs.
[overall 21 marks]

a Let $\mathbf{R} \subseteq \mathbb{N} \times \mathbb{N}$, and $(n,m) \in \mathbf{R}$ iff $2 \mid n \wedge 3 \mid m$.

(i) Which of the following properties hold for $\mathbf{R}$: Reflexivity, irreflexivity, symmetry, antisymmetry, transitivity and connex. Motivate your answers.

(ii) Given $n$ and $m$, such that $(n,m) \in \mathbf{R}$, but $(n,n) \notin \mathbf{R}$. Is the statement $3 \mid n+m$ necessarily true, necessarily false, or does it depend? Explain why. [$2 \cdot 3 = 6$ marks]

b Consider the labelled graph in Figure 1.

(i) Write this graph down formally by specifying the set of vertices $V$ and the set of edges $E$.

(ii) Can you identify the shortest path from $b$ to $e$?

(iii) Is this graph strongly connected? If yes, show a cycle through all vertices. If no, provide the largest strongly connected component, and a cycle through all vertices in the SCC.

(iv) Provide a planar embedding of this graph.
[$3+3+2+3=11$ marks]

c Let $\mathbf{S}$ be a relation that contains a pair of nodes $(x,y)$, whenever the graph in Figure 1 has a path from $x$ and $y$. What are the set of departure, set of destination, domain and image of $S$?
[4 marks]

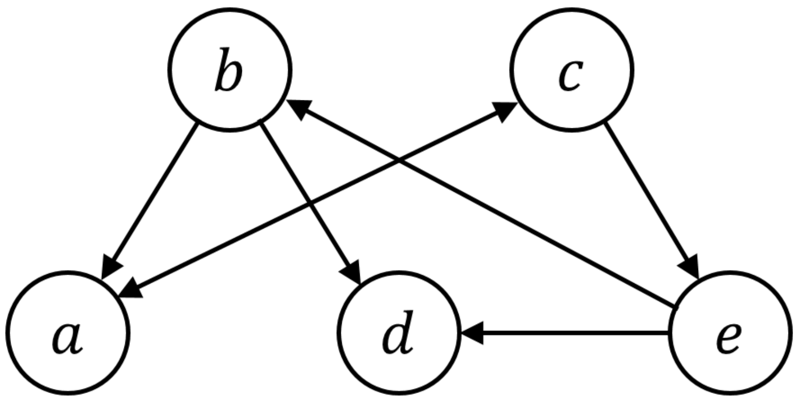


Figure 1: A graph with directed edges.

**Question 4** This question is about Proofs

[overall 30 marks]

a We want to *formally* prove that for every odd number, there exists a strictly smaller natural number. A natural number $n$ is called odd when: $\exists_{k \in \mathbb{N}}(2 \cdot k + 1 = n)$.
Note, as this is a formal proof question, you can only use the rules of logic and the AA axioms.

(i) Write down the formal lemma that we should prove (a formula with only logic and arithmetic symbols, and no free variables).

(ii) Prove the lemma formally. [$4 + 7 = 11$ marks]

b Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

A recursive definition of rectorial is: $\begin{cases} f(0) = 0 \\ f(s(n)) = n + n \cdot f(n). \end{cases}$

A recursive definition of factorial is: $\begin{cases} 0! = 1 \\ s(n)! = n! \cdot s(n) \end{cases}$

(i) Use induction to show that for $n \in \mathbb{N}$, $n \leq n!$. Write down your base case and induction step clearly.

(ii) Use induction to show that for $n \in \mathbb{N}$, $f(n) \leq n!$. Write down your base case and induction step clearly. [$2 \cdot 7 = 14$ marks]

c You are given a list of numbers $(n_0, n_1, \ldots, n_k)$ on a blackboard. You can select any pair of numbers $n_i$ and $n_j$, and replace it with the number computed by $n_i \cdot n_j + n_i + n_j$. Prove that the order in which you replace the numbers does not matter. [5 marks]