Relations

Cartesian products

A *pair* consisting of a first and b second is written (a, b).

A pair is ordered, so $(a, b) \neq (b, a)$ (unless $a = b$).

Two pairs (a, b) and (x, y) are considered equal if and only if $a = x$ and $b = y$.

A *n-tuple* is an extension of a pair, allowing for more than two elements.

E.g. (a, b, c, d) is a 4-tuple.

Tuples can recursively be defined using pairing, so that, e.g. (a, b, c, d) is just shorthand for $(a, (b, (c, d)))$.

The Cartesian product $(X \times Y)$ between two sets X and Y is the set of all pairs of X's and Y's.

Formally: $X \times Y = \{(x, y) | x \in X, y \in Y\}$.

Cartesian products can be defined over tuples, e.g. denoted $X \times Y \times Z$ for a 3-tuple.

The Cartesian product $X \times X$ can also be written X^2 (and $X \times X \times X = X^3$, etc.)

Writing $\forall_{x \in X, y \in Y}(\dots)$ is the same as $\forall_{(x,y) \in X \times Y}(\dots)$.

And similarly for existential quantification: $\exists_{x \in X, y \in Y}(\dots)$ is $\exists_{(x,y) \in X \times Y}(\dots)$

Relations

A relation R between X's and Y's is a set $\{(x,y)|x \in X \wedge y \in Y \wedge P(x,y)\}$.

The relation R between x and y holds, iff $(x,y) \in R$.

A relation R between X's and Y's is a subset of a Cartesian product; $R \subseteq X \times Y$.

E.g. let $X = \{\text{mary}, \text{john}, \text{fred}\}$ and $Y = \{\text{fluffy}, \text{snowball}, \text{felix}\}$, an ownership relation R is $\{(\text{mary}, \text{snowball}), (\text{john}, \text{fluffy}), (\text{fred}, \text{snowball})\}$. In this case, Mary and Fred both own snowball (perhaps they share a household), and felix is a stray cat.

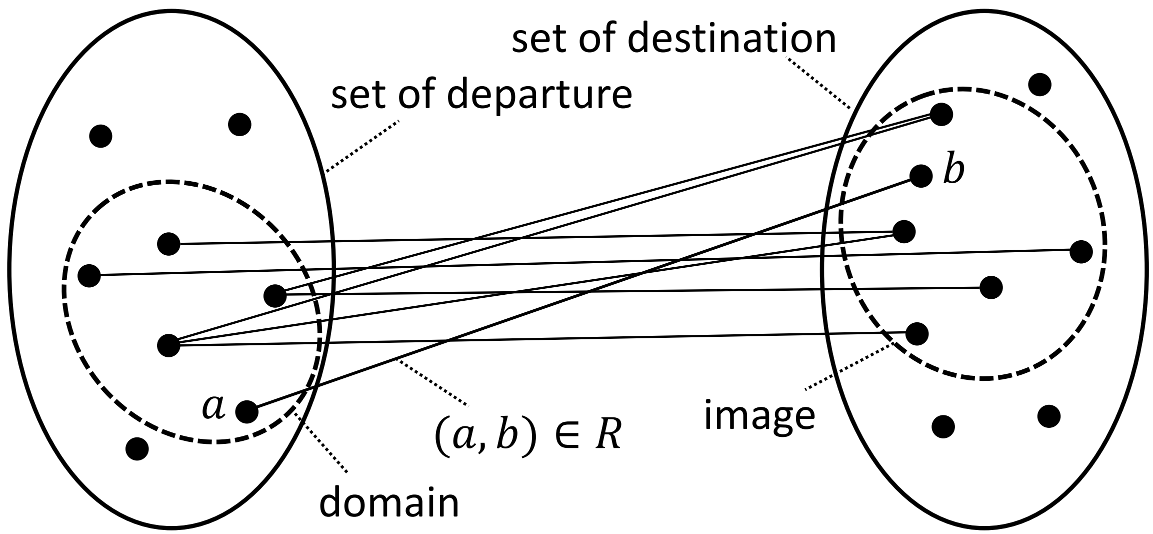

Figure 1: Graphical representation of a relation.

For $R \subseteq X \times Y$, X is called the set of departure and Y is called the set of destination or codomain.

The set $D = \{x \in X | \exists_{y \in Y}((x, y) \in R)\}$ of x related to some element in Y, is called the domain.

The set $I = \{y \in Y | \exists_{x \in X}((x, y) \in R)\}$ of y related to some element in X, is called the image.

The domain is a subset of the set of departure, and the image is a subset of the set of destination.

If the set of departure is equal to the set of destination, then the relation is called *homogeneous*.

Important properties of a homogeneous relation $R \subseteq X \times X$:

- Reflexivity: $\forall_{x \in X}((x, x) \in R)$,
- Symmetry: $\forall_{x \in X, y \in X}((x, y) \in R \Rightarrow (y, x) \in R)$,
- Transitivity: $\forall_{x \in X, y \in X, z \in X}(((x, y) \in R \wedge (y, z) \in R) \Rightarrow (x, z) \in R)$,
- Irreflexivity: $\forall_{x \in X}((x, x) \notin R)$,
- Antisymmetry: $\forall_{x \in X, y \in X}(((x, y) \in R \wedge (y, x) \in R) \Rightarrow x = y)$,
- Connexity: $\forall_{x \in X, y \in X}((x, y) \in R \vee (y, x) \in R)$.

Equality $(=)$ is reflexive, symmetric and transitive.

At most $(\leq)$ is reflexive, antisymmetric, transitive and connex.

Less than $(<)$ is irreflexive, antisymmetric, transtive and semi-connex (connex when $x \neq y$).

Subset $(\subseteq)$ is reflexive, antisymmetric and transitive.

Often, we write xRy to mean $(x, y) \in R$, e.g. $x = y$, $x < y$ or $x \subseteq y$.

If, for every element x in X, there is at most one $(x, y) \in R$, then R is called a *partial function*.

Formally: $\forall_{x \in X, y \in Y, z \in Y}(((x, y) \in R \wedge (x, z) \in R) \Rightarrow y = z)$.

If, for every element x in X, there is at least one $(x, y) \in R$, then R is called a *multi-valued function*.

Formally: $\forall_{x \in X}(\exists_{y \in Y}((x, y) \in R))$.

If, for every element x in X, there is exactly one $(x, y) \in R$, then R is called a *function*.

Formally: $\forall_{x \in X}(\exists^1_{y \in Y}((x, y) \in R))$.

Functions

The letters f, g, h are often used to denote functional relations.

Rather than writing $f \subseteq A \times B$, we write $f : A \to B$.

Let f be a functional relation. Typically, we write $f(a) = b$ instead of $(a, b) \in f$.

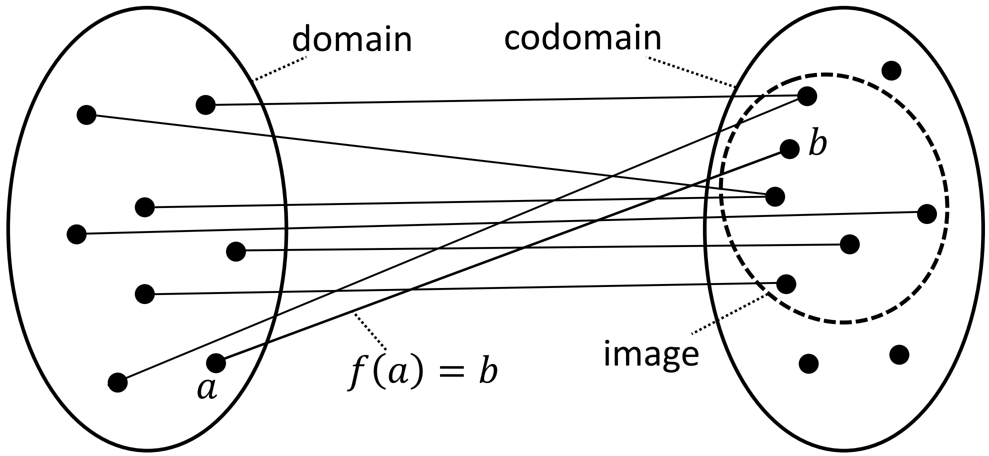

Figure 2: Graphical representation of function.

There are three special kinds of functions:

- Injective functions: $\forall_{a \in A, b \in A}(f(a) = f(b) \Rightarrow a = b)$,
- Surjective functions: $\forall_{x \in X}(\exists_{a \in A}(x = f(a)))$,
- Bijective functions: $\forall_{x \in X}(\exists^1_{a \in A}(x = f(a)))$.

A function that is injective and surjective is bijective.

The *inverse* of a function f is a relation f^{-1} that has (a, b) iff $(b, a) \in f$.

If f is bijective, then f^{-1} is a function.

If f is injective/surjective, then f^{-1} is a partial/multivalued function.

Function composition: if $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$.

$(g \circ f)(x) = y$ is defined to be $g(f(a))$.

An *operation* is a function of type $X^n \to X$.

Examples of *binary operations* $(X \times X \to X)$ are $+$, $\cdot$, $\times$, $\wedge$.

Binary operations are usually written, e.g., $x \oplus y = z$ instead of $\oplus(x, y) = z$ or $(x, y, z) \in \oplus$

A set X is called *closed* under operation $f : Y \to Z$, when $\forall_{x \in X}(f(x) \in X)$.

E.g., the result of adding two natural numbers is always a natural number: $\forall_{(n,m) \in \mathbb{N}^2}(n + m \in \mathbb{N})$.

So we can say that the natural numbers are closed under addition.

But, e.g., the natural numbers are *not* closed under subtraction, since $3 - 5 \notin \mathbb{N}$.

The closure of a set X under an operation $\oplus$ is the smallest superset of X which is closed under $\oplus$.

For example, the closure of the natural numbers $(\mathbb{N})$ under subtraction is integers $(\mathbb{Z})$.

The transitive closure $\mathbf{S}$ of a relation $\mathbf{R}$ is: $(a, b) \in \mathbf{R} \Rightarrow (a, b) \in \mathbf{S}$ and $(a, b) \in \mathbf{S} \wedge (b, c) \in \mathbf{S} \Rightarrow (a, c) \in \mathbf{S}$.