Mathematic Modeling the Outline
– 数学建模知识点大纲 –

$e^{\imath\theta} = \cos(\theta) + \imath\sin(\theta)$

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Contents

1 Optimization Model

1.1 Lineaer Programming

1.1.1 Standard Form

The object is

$$Minimize\, Z = C^T X$$

Subject to

$$Ax = b$$

Bounds

$$x \geqslant 0$$

Where

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}, A = \begin{bmatrix} a_{11} & \dots \\ \dots & a_{nn} \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$$

1.1.2 Standarlization Methods

- 最大最小转换：$Z' = -Z$
- 不等式约束变为等式约束：不等式中添加松弛变量/剩余变量x_{n+i}。
- $b_i \geq 0, (i = 1, 2, \dots, n)$，乘$-1$反转公式以标准化。
- X_i 无限制时，引入非负变量m, n，令$x_i = m - n$ 并代入目标方程，转化为非负限制。

1.1.3 Solving

图解法。单纯形法（需求标准式或矩阵式问题）。

1.1.4 Application

各类资源规划，投资，生产/存储控制，下料。

1.2 Integer Linear Programming

分支界定法 Branch&Cut

1.3 Mixed Integer Programming

1.4 Dynamic Programming

1.4.1 Basic Concept

1. Step $k = 1 \dots n$
2. Status S_k
3. Decision $u_k(x_k)$
4. Policy
$$P_{kj} = u_k(x_k), \dots, u_j(x_j)$$
5. Status Transition Equation
$$x_{k+1} = T_k(x_k, u_k(x_k)), k = 1, \dots, n$$
6. Objective function
$$V_{k,n}(x_k, p_{k,n}(x_k)) = \Phi_k(x_k, u_k, V_{k+1,n})$$
7. Step Profit
$$V_{k,n}(x_k, p_{k,n}(x_k) = opt_{k \leq j \leq n} V_j(x_j, u_j)$$
8. Optimal Value Function
$$f_k(x_k) = opt V_{k,n}(x_k, p_{k,n}(x_k))$$

1.4.2 Optimization Theory

○ Bellman Optimization Theory

1.4.3 逆序法，正序法

1.4.4 Applications

When put Dynamic Programming into practice, pay attention to 1. definition of steps; 2. policies allowed within each step; 3. status transition equation.

- Shortest path problem
- Knapsack Problem
- Resource Assignment
- ...

1.5 Non-Linear Programming

1.5.1 Optimal problem without constraint

General form

$$Min\, f(x), x = (x_1, x_2, \dots, x_n)^T \in R^n$$

Local solution optimized → 一阶必要

$$\nabla f(x^*) = 0$$

Local solution optimized → 二阶充分

$$\nabla f(x^*) = 0\, and\, \nabla^2 f(x^*)\, is\, Positive\, definite$$

1.5.2 Optimal problem with constraints

General form

$$Min\, f(x)\,, x \in R$$

$$s.t.\ c_i(x) = 0, i \in E = \{1, 2, \ldots, l\}$$

$$c_i(x) = 0, i \in I = \{l+1, l+2, \ldots, l+m\}$$

Local solution optimized $\rightarrow$ 必要

$$\exists vector\, \lambda^*$$

$$\nabla_x L(x^*, \lambda^*) = \nabla f(x^*) + \sum_{i=1}^{l+m} \lambda_i^* \nabla c_i(x^*) = 0$$

$$c_i(x^*) = 0, i \in E$$

$$c_i(x^*) \leq 0, i \in I$$

$$\lambda_i^* \geq 0, i \in I$$

$$\lambda_i^* c_i(x^*) = 0, i \in I$$

Where the L is the Lagronge function

$$L(x, \lambda) = f(x) + \sum_{i=1}^{l+m} \lambda_i c_i(x)$$

1.5.3 Application

订购/存储模型，投资/组合问题
lingo例子
e.g.

$$minf(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

```
sets:
  var/1..2/: x;
endsets
[OBJ] min = x(1)^2 + x(2);
[C1] x(1)^2 + x(2)^2 <= 9;
[C2] x(1) + x(2) <= 1;
@for(var: @free(x));
```

1.6 Software/Tools

Matlab/Octave, Lingo/Lindo, IBM ILOG CPLEX.

2 Dynamics Model

2.1 Differential Function Model

2.1.1 Exponential Model

$$x(t+\Delta t) - x(t) = kx(t)\Delta t$$

when $\Delta x \to 0$

$$\begin{cases} \frac{dx}{dt} = kx \\ x(0) = x_0 \end{cases}$$

the solution

$$x(t) = x_0 e^{kt}$$

2.1.2 SI Model: susceptible infective : Logistic

$$s(t) + i(t) = N$$

$$k(s) = \frac{ks}{N} = k(1 - \frac{i}{n})$$

$$\begin{cases} \frac{di}{dt} = k(1 - \frac{i}{n})s \\ x(0) = x_0 \end{cases}$$

the solution

$$x(t) = \frac{n}{1 + (\frac{n}{x_0})e^{-kt} - 1}$$

2.1.3 SIS Model

$$\begin{cases} \frac{dx}{dt} = k(1 - \frac{x}{n})x - lx \\ x(0) = x_0 \end{cases}$$

2.1.4 SIR Model

$$\begin{cases} \frac{dx}{dt} = \frac{ksx}{n} - lx \\ \frac{ds}{dt} = -\frac{ksx}{n} \\ \frac{dr}{dt} = lx \\ x(0) = x_0,\ r(0) = r_0,\ s(0) = s_0 \end{cases}$$

2.2 Stability

定理: 设x_0 是微分方程 $dx(t)/dt = f(x)$ 的平衡点且 $f'(x_0) \neq 0$， 若$f'(x_0) \leq 0$ 则x_0 稳定；若 $f'(x_0) \geq 0$，则 x_0 不稳定。

2.3 Application

Best fishing strategy.

2.4 Software

Matlab:
dsolve, solver = { ode45, ode... }

3 Possibility

3.1 Probability Base

Include [1]

3.2 Computer Emulating - Monte Carlo

Statistical simulation method: Combine the emulation of random events with the probability feature of different kinds of random events.
e.g. Solve $I = \int_a^b g(x)dx$ with Monte Carlo
sol. Cast uniformly randomized points to the rectangle range, then count how many point dropped under the curve $g(x)$.
1. Build the probability model; 2. extract samples from known probability distributions; 3. Setup statistic variables in need.

3.2.1 Monte Carlo Precision

CLT: Central-Limit Theorem
1. Randomly Casting Points

$$E(X) \approx \bar{p} = \frac{k}{n}$$

2. Mean Value

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \ldots + x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

3.3 Random Number Generator

1. $x_i = \lambda x_{i-1}(modM)$.
2. $x_i = (\lambda x_i + c)(modM)$.
Tests:

1. Parameter test
2. Uniformity test
3. Independency test

In matlab:
rand*, normrand

3.3.1 Probability - Reference

References

[1] 概率论与数理统计，浙江大学出版社

4 Graph Theory

4.1 Basic concept of graph theory

- 矩阵
  邻接矩阵 node - node
  关联矩阵 node - edge
  赋权矩阵 weight

- 各种类型的图
  完全图，立方体（偶图），完全偶图，赋权图，有向图,无向图
  子图
  $$V' \subset V, E' \subset E$$
  生成子图
  $$V' = V, E' \subset E$$
  补图

- 树
  树，生成树，无向生成树

4.2 一些定理

1. Given simple graph G contains no isolated node, and contains m edges, then the number of all generated subgraph is 2^m
   $$\binom{m}{0} + \binom{m}{1} + \ldots + \binom{m}{m} = 2^m$$
2. For $\forall$ undirected graph
   $$\sum_{v \in V} d(v) = 2m$$
3. For $\forall$ directed graph
   $$\sum_{v \in V} d^+(v) + \sum_{v \in V} d^-(v) = m$$

4.3 Shortest Path Problem

4.3.1 Dijkstra Algorithm

Base of this algorithm

$$d(u_0, \bar{S}) = min_{u \in S, v \in \bar{S}} \{d(u_0, u) + w(u, v)\}$$

where u, v denotes the source and destination node, $d(u, v)$ denotes the distance between u and v, $S \in V$ and $u_o \in S$ and $\bar{S} = V$
S.
This algorithm can not only find the shortest path from u_o to v_0, but also all the shortest path from u_o to any other nodes in graph G.

4.3.2 Floyd Algorithm

Get the minimum distance between to given nodes.

4.4 Application

1. 运输问题
2. 转运问题
3. 最优指派问题: 匈牙利算法
4. 中国邮递员问题: Fleury Algorithm

4.5 Euler Graph and Hamilton Cycle

4.5.1 Traveling Salesman Problem

Figure out the Hamilton cycle which possess the minimum weight.
Assume that:

- w_{ij} denotes the distance between city i to city j
- x_{ij} denotes the decision if going from city i

The solution is:

$$MinimizeC = \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} x_{ij} \tag{1}$$

s.t. :

$$\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, \ldots, n$$
$$\sum_{j=1}^{n} x_{ij} = 1, j = 1, 2, \ldots, n$$
$$u_i - u_j + n x_{ij} \leqslant n - 1; i \neq j; i, j = 2, 3, \ldots, n$$
$$x_{ij} = 0 OR 1; i \neq j; i, j = 1, 2, \ldots, n$$
$$u_j \geqslant 0; j = 1, 2, \ldots, n$$

Note that, there are some TSP example program available in the LINGO's user manual[1].

```
MODEL:
! traveling seller;
SETS:
city/Pe T Pa M N L/: u;
link(city, city): w,x;
endsets
data:
!to Pe  T Pa  M  N  L;
w =  0 13 51 77 68 50
13  0 60 70 67 59
```

```
51 60  0 57 36  2
77 70 57  0 20 55
68 67 36 20  0 34
50 59  2 55 34  0;
enddata

n = @size(city);
min = @sum(link: w * x);
@for(city(k):
@sum(city(i)|i #ne# k: x(i,k)) = 1;
@sum(city(j)|j #ne# k: x(k,j)) = 1;
);

@for(link(i,j)|i #gt# 1 #and# j #gt# 1 #and# i #ne# j:
u(i) - u(j) + n*x(i,j) <= n-1;
);

@for(link: @bin(x));
end
```

4.6 Trees and spinning trees

4.6.1 无向生成树

避圈法

4.6.2 最优连线问题

Kruskal

4.6.3 最大流问题

4.7 Python3

python3-networkx

4.8 Matlab

graph* function set:
graphminspantree
graphshortpath

5 Statistics and Curve Fitting, Regression, Interpolation

5.1 Clustering

拟合，分段
K-means：原则上需要预先知道类别数量。

5.2 Regression, Curve Fitting

Find a smooth curve witch matches data best, i.e. Minimize MSE. No requirement that curve must cover all data.

5.2.1 主要内容

- 相关关系.数学表达式
- 回归方程，回归预测
- 估计的标准误差

5.2.2 Single Variable Linear Regression

$$Y = a_0 + a_1 X + \varepsilon$$

where

$$\varepsilon_i \sim N(0, \sigma^2)$$

Object:

$$Q = \sum (y - y_c)^2$$

Get the parameters: Least square Method

5.2.3 Multi Variable Linear Regression

TODO: svm :: svr ?

$$Y = a_0 + a_1 X_1 + a_2 X_2 + \ldots + a_n X_n + \varepsilon$$

5.2.4 Multi Variable Non-Linear Regression

$$Y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

5.2.5 keys

- 选择主成分
- 相关系数
- 置信区间

5.3 Interpolation

Find function f, which matches all data, and f is similar to the true function. e.g. Polinomial interpolation

$$P_n(x) = a_0 + a_1 x + \ldots + a_n x^n$$

5.3.1 1D Interpolation

拉格朗日插值
牛顿插值
分段插值
三次样条插值
e.g. 一次插值，线性插值：点斜式

$$L_1(x) = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

e.g. 拉格朗日插值多项式
hint. 三次样条插值(三次多项，节点二阶倒数连续)比分段线性插值更加光滑。
matlab: interp1

5.3.2 2D Interpolation

2 cases: Grid data points, Scattered data points.
Method: Nearest
Method: Piecewise interpolation
matlab: grid data: interp2
matlab: scatterd data: griddata

5.3.3 Priciples

Min RSS (Residual Sum of Squares).

6 Markov Chain

6.1 placeholder

7 Fuzzy Mathematics

7.1 Fuzzy Subset

While the feature function of certain subset can be represented as the mapping

$$X_A : U \to \{0, 1\}$$

where $X_A(x) = 1$ when $x \in A$.
秃头悖论。
The fuzzy subset $A \in U$ can be represented as the mapping

$$A(x) : U \to [0, 1]$$

7.1.1 λ Cut Set

$$A_\lambda = \{x | A(x) \geqslant \lambda\}$$

7.1.2 Fuzzy Relationship

While the classical 2 variable relationship can be represented as the mapping

$$R : X \times Y \to \{0, 1\}$$

which is in fact the feature function of subset R of $X \times Y$.
The fuzzy relationship can be represented as the mapping

$$R : X \times Y \to [0, 1]$$

The fuzzy matrix means

$$R_{m\times n}(i, j) = R(x_i, y_j)$$

7.1.3 Fuzzy Relation Synthesis

$$R_1 \circ R_2 = (c_{ij})_{m\times n}$$

where

$$c_{ij} = \vee\{(a_{ik} \wedge b_{kj} | 1 \le k \le s\}$$

7.1.4 Fuzzy Clustering

- Data Standarlizing
  1. 平移：标准差变换，极差变换
- Fuzzy Likeness Matrix
  1. Cosine of included angle
  2. Correlation coefficient
- Distances $r_{ij} = 1 - cd(x_i, x_j)$
  1. Hamming Distance
  2. Euclidean Distance
  3. Chebyshev Distance
- Fuzzy equivalant matrix

8 排队论

A Paper composing

1. Background of problem
2. Assumptions
3. Setup the Mathematical Model
4. Get the solution of model
5. Model Analyzation
6. Model Validation

A.1 Reference Structure

摘要 问题重述与分析 问题假设 符号说明 模型建立与求解 结果分析 模型检验 模型推广 模型评价参考文献和附录

B Common References

References

[1] LINGO 官方文档 http://www.lindo.com/downloads/PDF/LINGO.pdf

[2] 数学建模基础（第二版），薛毅，科学出版社

[3] 本文的Git Repo，以及我写的一些数模相关程序

[4] Octave PDF Document http://www.gnu.org/software/octave/octave.pdf