A SUBDIVISION ALGORITHM FOR SMOOTHING DOWN IRREGULAR SHAPED POLYHEDRONS

D. W. H. DOO

Brunel University

Uxbridge, Middlesex, England.

In most of the existing surface fitting and smoothing techniques, a rectangular mesh system is used. If non-4-sided regions are unavoidable in a design, difficulties will arise when joining them to the rectilinear systems.

A new type of techniques that overcomes the n-sided problem is being described here. By an extension of Chaikin's Algorithm to 3-D, using linear combinations of the vertices of a polyhedron, new vertices are defined to form a smoother polyhedron which contains a finer grid of faces. Repetition of this process will eventually produce a 'smooth' surface within a specified tolerance.

Further investigation has shown that a closed form can be obtained, and the final surface will become a 'B-spline like' surface with everywhere smooth in the tangent plane.

1. INTRODUCTION

In most of the exist patching and smoothing techniques for surfaces, a rectangular mesh is used. The parametric values in two directions (u and v) form continuous smoothness across the boundaries, and the Cartesian Product form is taken to form a 3-D surface.

Coon's Patch[1] has been applied in industries since the mid-sixties in the design of engineering surfaces. Subsequently, various forms of computer aided design of space curves and surfaces have been developed. Professor Bezier of Renault[2] has developed a free form curve theory from the Bernstein polynomial and extended it to form a Bezier Net bivariate function for surfaces. Gordon and Riesenfeld's B-spline technique[3] gives a very flexible and well behaved method for curves and surface designs.

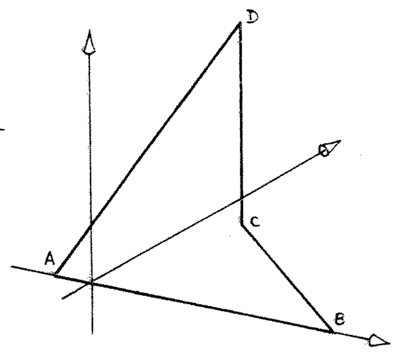
These methods are well developed for rectangular mesh systems, but in the actual design process, it is sometimes necessary to include non-4-sided patches which gives rise to difficulties in the present rectilinear systems. This n-sided problem is a 'Topological' one since they can all be resolved into 3-sided patches associated with 4-sided ones. If there were solutions to 3-sided patches, the corresponding problem can be solved.

Boolean Sum theory[4] has been used to smoothly interpolate triangles against the sides and vertices, but no one has yet successfully incorporated this theory into rectangular mesh systems.

Traditionally in the industry, 3-sided patches arising from the design are approximated by degenerated 4-sided patches with one boundary diminished to zero. However, when joining such a surface to another patch, problem will arise in the doubled corner where the continuity across the patches lies. In practice, this local flatness is usually reduced manually.

In a 4-sided parametric patching technique, it is necessary for the parametrization to be continuous across the patch boundaries. When a large curve segment is adjacent to a small curve segment, the large tangent magnitude may cause a loop in the small curve segment[5]. The choice of regularly spaced boundary data may minimize this effect, but a patch arranged as in Fig.1 may have a folding effect.

Fig.1



In this paper, the author has taken a completely controversal view on surface smoothing techniques. The resulting approach will not have the problems in the existing surface defining techniques as mentioned above.

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2. SMOOTHING A CURVE

In general, a conic section tangential to two intersecting straight lines can be fitted to smooth out the corner formed between them. The final figure is continuous in both zero and first derivatives. One degree of freedom therefore remains, and this decides the type of conic section to be fitted, ie. whether an ellipse, a parabola or a hyperbola. In special cases a circle can be fitted, or a degenerated hyperbola may give the two original straight lines.

There is a common trick long adopted by carpenters to obtain a smooth corner with the minimum of tools. Suppose a corner C is to be smoothed out from A to B, (Fig. 2) the carpenter will first divide AC and CB into equal numbers of portions. By sawing along the straight lines joining the opposite markings, as shown in the figure, a fairly smooth curve can be obtained. It can be proved that if the markings are infinitesimally small, the resulting curve will be parabolic[6]. This is in accordance to the inverse of the theorem that any two tangents to a parabola will be cut proportionally by another tangent[7].

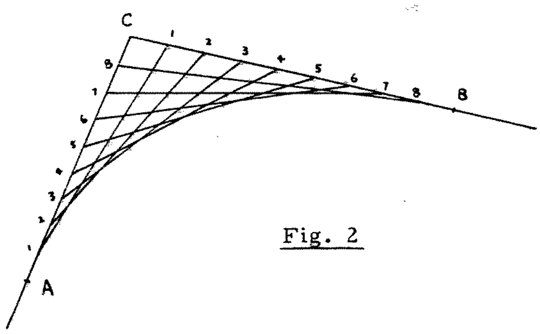


Fig. 2

In 1974, G. M. Chaikin derived a high speed curve generating algorithm for data point pairs[8]. The algorithm is simple but effective. Consider a curve described by 4 points $P_1 P_2 P_3$ and P_4 (Fig. 3) the generated Chaikin's curve is tangential to the end points P_1 and P_4, and is also tangential to the line $P_2 P_3$ at its mid-point P_5. This figure is then separated into two triangles $P_1 P_2 P_5$ and $P_5 P_3 P_4$, each of which can then be subdivided into halves to form a new quadrilateral; e.g. $P_1 P_6 P_7 P_5$ and $P_5 P_8 P_9 P_4$, where P_6, P_7, P_8 and P_9 are the mid-points of the lines $P_1 P_2$, $P_2 P_5$, $P_5 P_3$ and $P_3 P_4$ respectively. Each of these quadrilaterals can be subdivided repeatedly in similiar process as to form more quadrilaterals until the desired smoothness is achieved.

This process is similar to the carpenters' trick, as it can be seen that each new line segment generated by Chaikin's method is in effect a portion of the straight line dividing the edges of the corner in proportional ratios, and the final Chaikin's curve will be made up by parabolic segments, this final curve is known as the quadratic B-spline[9].

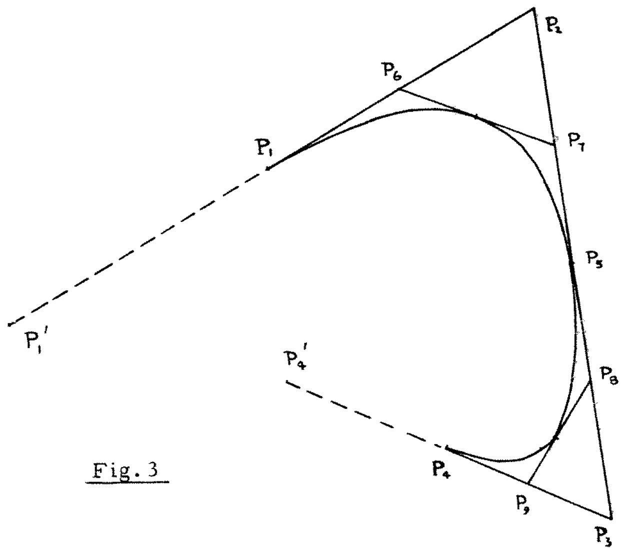


Fig. 3

In Fig. 3, if we extend P_1 away from P_2 to a point P_1' with $P_1'P_2$ = twice $P_1 P_2$, and extend P_4 to P_4' in the same way, we have an opened polygon $P_1' P_2 P_3 P_4'$ and we can restate Chaikin's method in a different way: Given a polygon $P_1' P_2 P_3 P_4'$, the corners and the end points can be cut away at the $\frac{1}{4}$ length of the segments from the vertices, and a new polygon will be formed. The process can then be repeated on the resulting polygon until, in the limit, the polygon approaches a parabolic B-spline.

A similar process extended to 3-D can be used to smooth down a polyhedron in the same sense.

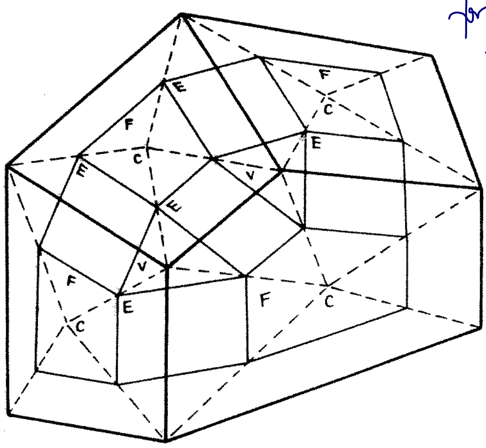
3. SMOOTHING A POLYHEDRON

Consider any polyhedron made up of vertices and faces. On each face of the polyhedron, there exists a centroid. A new face can be formed by joining the mid-points of all the lines linking the centroid to the vertices on each face. By connecting all these new faces accordingly, a new polyhedron will be obtained (Fig. 4). This subdivision process is similar to chaikin's method of joining the centroids of the segments of the polygon to the corners, taking their halves as new vertices, and joining all the adjacent new vertices in order to form a new polygon.

The new polyhedron now has a larger number of faces and vertices, but the size of the faces are smaller and the polyhedron is smoother, as the segments in sectional views have a smaller change of direction than the original polyhedron.

Fig.4 Smoothing of a polyhedron. The thicker lines represent the original figure, the thinner lines represent the new one, with their types specified , C is centroid.

for each V
for each edge



In this process, 3 types of new faces can be formed ;

a) Type F: A 5-sided face will give a new and smaller 5-sided face within itself and bears a similar shape, this type of new face is termed Type F (formed by face).

b) Type V: A vertex common to 3 faces, i.e. a corner where 3 faces joined together having three common boundaries, will produce a 3-sided face, this is termed Type V (formed by vertex).

c) Type E: On each common boundary of two adjacent faces, a 4-sided face will be formed, this is termed Type E (formed by edge).

The new polyhedron will consist of these 3 types of new faces. A n-sided face will provide a basis for a smaller n-sided F type new face, it will remain n-sided as the subdivision carries on and will gradually converge to the centroid and diminish to an acceptable size. A common edge will always produce a 4-sided new face, and a m-spoked vertex will produce a m-sided V type face which will, in turn, become the basis of a smaller m-sided F type face in the next subdivision process.

These 3 types of faces linked together sharing common edges form a new polyhedron for further subdivisions. The process can be used for any number of sides on a face of the polyhedron, and the new faces automatically joined up by the next subdivision.

4. NEW FACES AND VERTICES GENERATION

Euler's formula for polyhedron states that:

$$F + V - E = 2 \tag{4.1}$$

Where F = number of faces,
E = number of edges,
V = number of vertices.

For the original polyhedron,

$$F_0 + V_0 - E_0 = 2$$

After the first subdivision, the total number of new faces will be the total for F, V and E type faces, i.e.,

$$F_1 = F_0 + V_0 + E_0 \tag{4.2}$$

Each old boundary will give a 4-sided polygon with 4 new vertices, however, any two adjacent edges on a face will generate only one new vertwx on that face, therefore the number of new vertices formed will be,

$$V_1 = 4\,E_0 / 2 = 2\,E_0 \tag{4.3}$$

And from (4.1), the number of new edges will be,

$$E_1 = F_1 + V_1 - 2$$

$$= F_0 + V_0 + 3\,E_0 - 2 \tag{4.4}$$

For the next cycle, after the 2nd subdivision,

$$F_2 = F_1 + V_1 + E_1$$

$$= 2\,F_0 + 2\,V_0 + 6\,E_0 - 2 \tag{4.5}$$

$$V_2 = 2\,E_1$$

$$= 2\,F_0 + 2\,V_0 + 6\,E_0 - 4 \tag{4.6}$$

$$E_2 = F_2 + V_2 - 2$$

$$= 4\,F_0 + 4\,V_0 + 12\,E_0 - 8 \tag{4.7}$$

Judging from equation (4.6) and (4.7), ther exists a relationship $V = \frac{1}{2} E$; irrespective of the number of faces and the number of sides on each face, which have not been taken into account in the above derivations. The only solution that is possible for $V = \frac{1}{2} E$ in this case is for each vertex to become common to 4 edges, that is, after the subdivisions, all the new vertices will become regular 4-spoked vertices. This is helpful since we can be sure that there will not be any unexpected irregular sided faces being generated after the first subdivision.

The calculation for the number of F, V and E E after a number of subdivisions can be formulated in the following matrix form,

$$\begin{pmatrix} F_1 \\ E_1 \\ V_1 \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_0 \\ E_0 \\ V_0 \\ D \end{pmatrix} \qquad (4.8)$$

where D= -2 for closed polyhedrons.

$$\begin{pmatrix} F_2 \\ E_2 \\ V_2 \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_1 \\ E_1 \\ V_1 \\ D \end{pmatrix} \qquad (4.9)$$

i.e.

$$= \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^2 \begin{pmatrix} F_0 \\ E_0 \\ V_0 \\ \end{pmatrix}$$

in general,

$$\begin{pmatrix} F_n \\ E_n \\ V_n \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^n \begin{pmatrix} F_0 \\ E_0 \\ V_0 \\ D \end{pmatrix} \qquad (4.10)$$

5. WEIGHT FUNCTIONS

For a n-sided polygon, having vertices P_1 P_2 P_3 ......P_n, the centroid P_c is,

$$P_c = \frac{\sum_{i=1}^{n} P_i}{n} \qquad (5.1)$$

The mid-point from the centroid to a vertex P_r is,

$$P_r' = \frac{P_r + \dfrac{\sum_{i=1}^{n} P_i}{n}}{2}$$

i.e.,

$$P_r' = \frac{n+1}{2n} P_r + \sum_{\substack{i=1 \\ i \neq r}}^{n} \frac{P_i}{2n} \qquad (5.2)$$

where P_r' is the new vertex formed on the original polygon on the vertex P_r and the n P_r's will define the new F type face being generated. These P 's joined to those generated by adjacent fa faces will make up the V type and E type new faces. Eqn. (5.2) can be written in the form of a weight distribution function,

$$P_r' = \alpha_r P_r + \sum_{\substack{i=1 \\ i \neq r}}^{n} \alpha_i P_i \qquad (5.3)$$

where, in this case, $\alpha_r = \dfrac{n+1}{2n}$

and $\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_{r-1} = \alpha_{r+1} = \ldots\ldots = \alpha_n = \dfrac{1}{2n}$

6. VARIATIONS ON THE DIVIDING ROUTINE

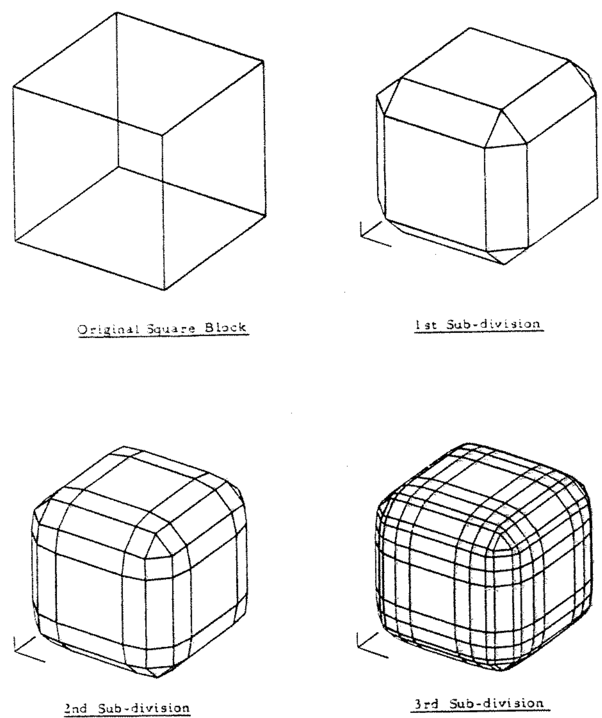


Fig. 5 Dividing down from a unit cube with new vertices located at ½ the distance from the old vertices to the centroid on each face. (Hidden lines removed on the Sub-divisions)

The weight distribution may be easily varied to get different effects as long as the sum of all the linear combination weights is equal to 1 and that they are symmetrically distributed. For example, the new vertex P_r' can be taken at the $\frac{1}{3}$ length of the centroid to the old vertex line instead of $\frac{1}{2}$. The effect of the new vertices being nearer to the old will give a slower conversion at the centre of a face, but a sharper change along its edges and vertices, as illustrated in Fig. 5 and Fig. 6. If a constant $\frac{2}{3}$ is being chosen, the effect is just the reverse, as in Fig. 7.

Although the new polygon of each face will eventually converge to the centroid by these subdividing methods, in the case of a highly twisted original face, the new F type polygon will still be highly twisted since it takes on a similar shape. Hence, after several subdivisions, the generated F type face will have the same degree of 'out-of-plane' at the vertices. That is, through a magnifier, a picture roughly the same as the original will be observed.

Sometimes it is necessary for the out-of-plane to diminish faster than the size of the face in the subdivision algorithm. It is therefore, probably inadequate to scale just by a factor of 2 on the distances between the centroid and the vertices.

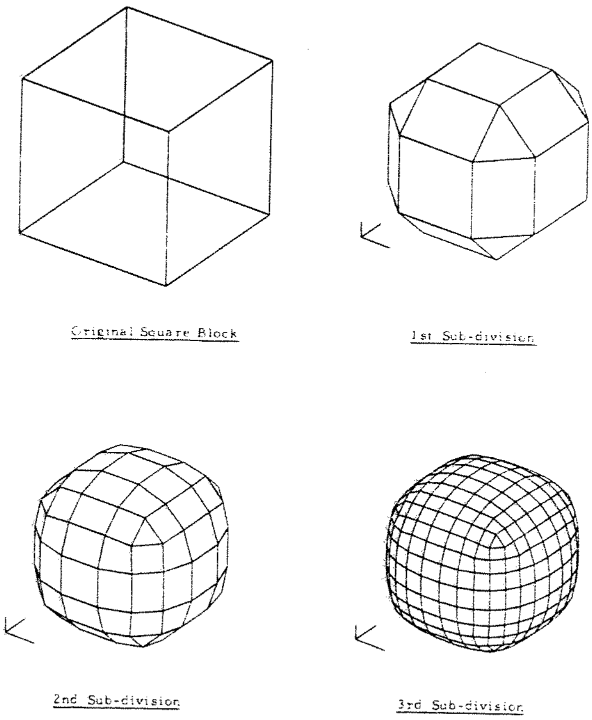


Fig. 6 Dividing down from a unit cube with new vertices located at half the distance between the centroid and old vertices on each face. (Hidden lines removed on the Sub-divisions)

A way of getting over this is to form the new vertices on the average plane of the face concerned,

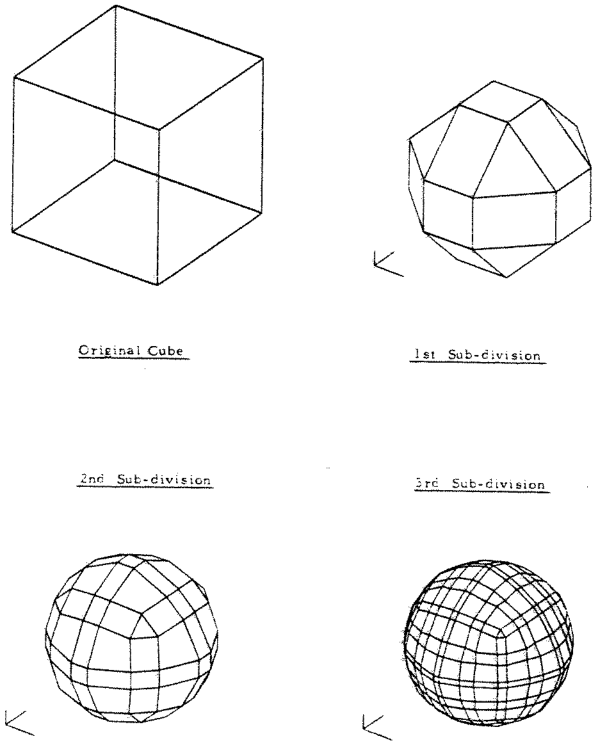


Fig. 7 Dividing down from a unit cube with new vertices located at $\frac{2}{3}$ the distance from the old vertices to the centroid on each face. (Hidden lines removed on the Sub-divisions)

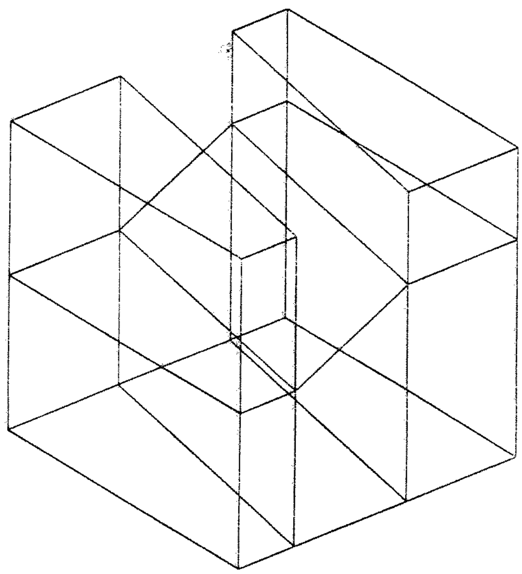


Fig. 8 The original figure of a U block with 22 faces and 24 vertices.

the out-of-plane is then eliminated. This subdivision method is termed Type 2 while the previous direct $\frac{1}{2}$ distance method is termed Type 1. The Type 2 method gives a smooth centre on the minimal limit face, but the fact that the new vertices formed are being 'pulled' down to the average plane may induce an undesirable twist on the V type and E type faces. Another method is introduced by forming the new vertices that gradually converge towards the average plane, but at the

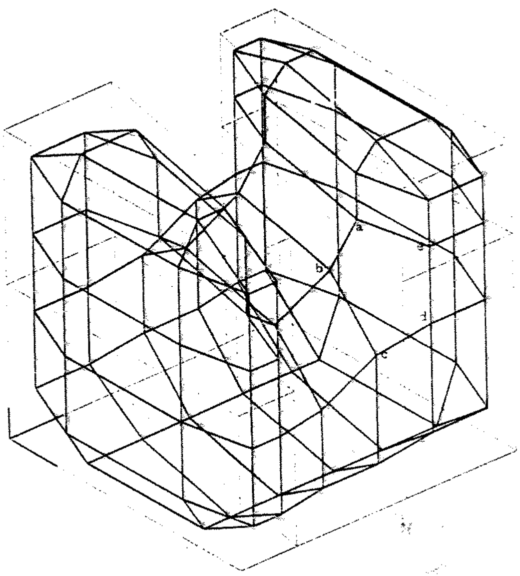


Fig. 9 The 1st Sub-division of the U block. the polygon (5 sided) abcde is one of the highly twisted V type face.

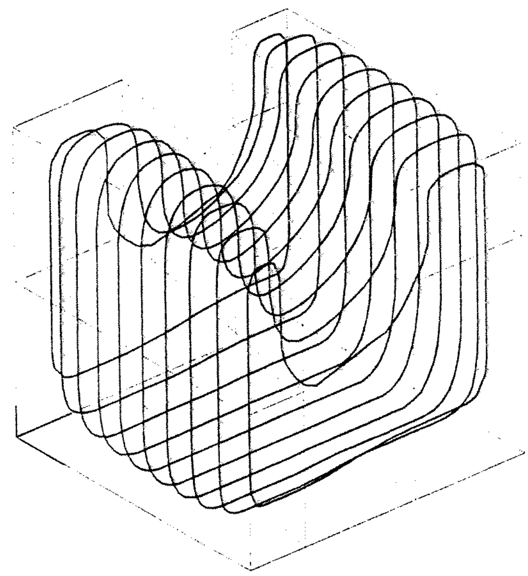


Fig. 11 The 10 sectional views on the 3rd Sub-division by the Type 2 method. the figure is almost the same as Fig. 10 except that it is smoother on the segments through the highly twisted portion.

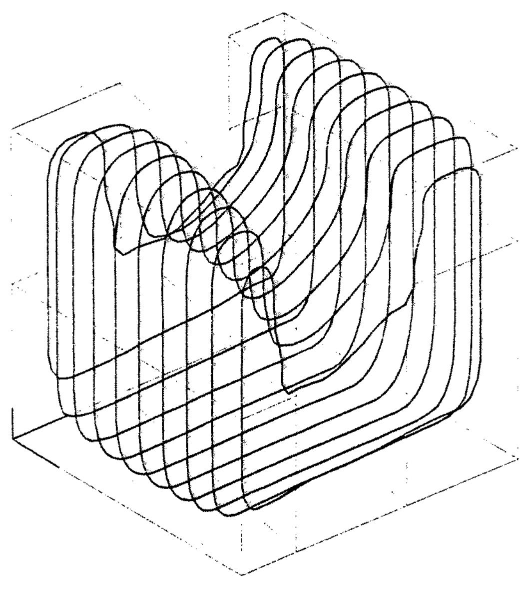


Fig. 10 The 10 sectional views on the 3rd Sub-division by the Type 1 method, the sections through the highly twisted faces are still quite rough.

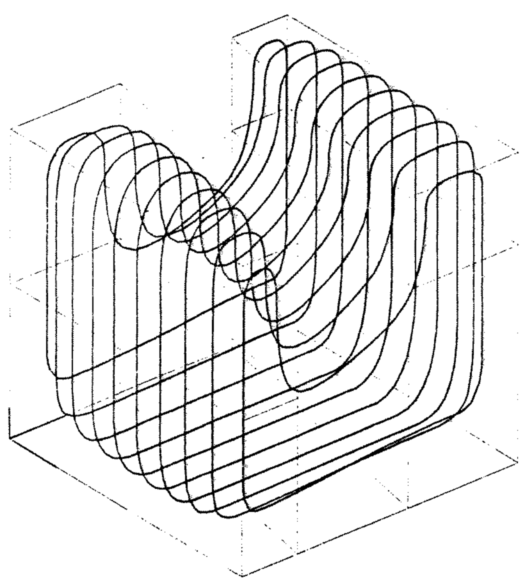


Fig. 12 The 10 sectional views on the 3rd Sub-division by the Type 3 method. The section lines are much smoother than those in Fig. 10 and in Fig. 11 .

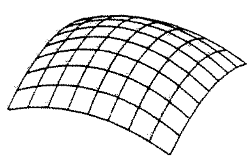
same time avoiding too much twist being formed on V type and E type faces. This method of subdivision is termed Type 3.

The Type 3 method is proved to be the most successful, as illustrated in Fig. 10, 11, and 12, which are the parallel plane sectioning views on the 3rd subdivision to the closed polyhedron in Fig. 8. The sectioning planes through the un-twisted faces exibiting almost the same lines in these figures, but on the highly twisted faces, e.g., abcde in Fig.9, the Type 3 figure gives a much smoother shape than the other two.

7. SURFACE FITTING

In a subdivision process, if four 4-sided faces are formed next to one another shearing one common vertex, a bi-quadratic surface patch in two parametric variables can be fitted over the four centroids of the faces. An ajoining set of 4-sided faces will provide another bi-quadratic patch ajoining to this one with both coordinates and parametric slopes continuities across their boundary. Further more, this system of surface patches are proved to be invariant for further subdivisions by the type 3 method.

This result is hardly surprising since the Type 3 method generates each new 4-sided face on the lofted surface of the old polygon, and the process gives a surface in effect is a Cartesian Product of two Chaikin's curve, which is a bi-quadratic B-spline surface.



The 4 patches fitted over the 9 centroids drawn with 3 intermediate lines between the boundaries.

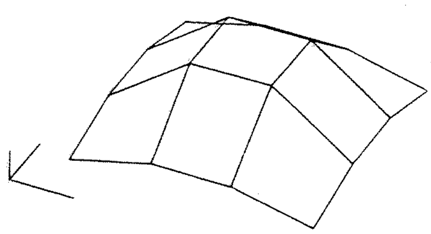
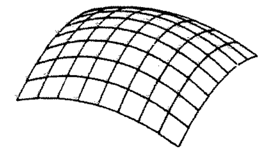


Fig. 13 The 9 faces resulting from the 1st Subdivision of the four 4-sided set and the corresponding 4 biquadratic surfaces fitted on their centroids.

As illustrated in Fig. 13 and 14, four bi-quadratic surface patches fitted to 9 faces which resulted from the 1st subdivision on the original four 4-sided faces, being drawn with the right density, is exactly the same as that obtained by the 64 biquadratic patches fitted to the 81 faces resulting from the third subdivision of the original figure.



The 64 patches fitted over the 81 faces with the boundaries being drawn only.

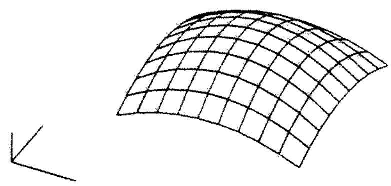
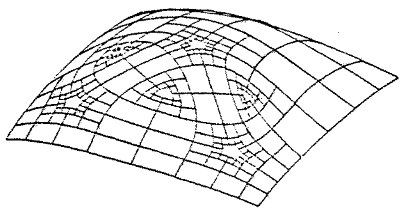


Fig. 14 The 81 faces resulting from the 3rd Subdivision and the corresponding biquadratic patches fitted onto them.

This effectively reduces the subdivisions that is required over any four 4-sided set of faces, and will reduce the memory necessary to store the information. As the surface patches fitted stay invariant in further process, it is therefore only necessary to perform the subdivisions on those that cannot form a four 4-sided face sets, namely, those joined by irregular sided faces which resulted from non-4-sided regions and non-4-spoked vertices in the original polyhedron.

As the subdivision algorithm keep generating mainly 4-sided faces, (Chapter 4) and the size of the grid becomes smaller in each step, these non-patchable areas will reduce to isolated singularities or call it 'holes', with a non-4-sided face at the centre and surrounded by a ring of 4-sided faces (Fig.15). Each further subdivision will generate more 4-sided faces around the holes, and more patches hence can be fitted. The area of each non-patchable hole is reduced by a factor of 4 in each step and therefore shrink towards the centroid of the non-4-sided face. By an analogy of Fourier technique applying to the eigenvector problem of the subdivision method[10], it is possible to prove that the limiting final surface is actually slope continuous and holds a somewhat constant curvature over these singularity regions.



The patches being fitted after the 4th Subdivision.

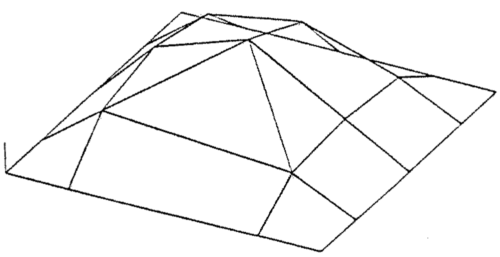


Fig. 15 The original irregular sided open polyhedron and the patches being fitted in the subdivisions.

8. CONCLUSIONS

The subdivision algorithms have no problem on arbitrary specific polyhedrons and no difficulty in matching irregularly sided faces together. The limiting surface converges towards the centroid of each face and becomes a figure enclosed by the tangent planes within the convex hull.

The algorithm will generate a grid approximately four times finer than the previous step in every loop, although the size of the faces are being quartered, the corresponding storage required to hold the information is four times as much. But because the process is localized, i.e., the generation of new face is governed by the old one and its neighbours only, and that the algorithm works equally well with opened polyhedrons, the figure can therefore be broken down into smaller parts for calculations, in this way a much finer grid may be obtained on a limited storage space. The algorithm is particular suitable for greyscale display purpose, although in the actual manufacturing process, one would obtain the tool pathes from the surface patches rather than the subdivision itself.

A mathematical representation of the complete surface is probably not obtainable, because the final surface is consisting of infinite number of patches, and the non-patchable area merely becomes infinitely small, (Fig. 16) nevertheless, this surface is proved to have the right quadratic behavior even at these non-patchable regions in the limit[10].

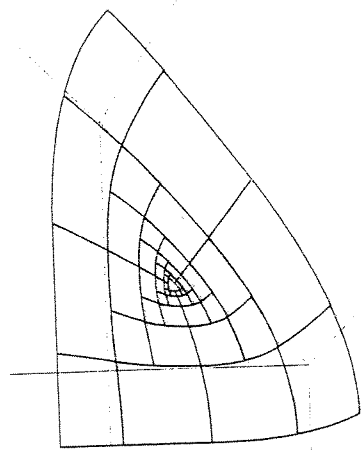


Fig. 16. The patches fitted on to the original figure (fainter lines) after the 5th subdivision, the non-patchable region is contracting towards the centroid of the non-4-sided face.

As far as practical application is concern, only a finite subdivision is required for the area of the holes to shrink down to a limit which exceeds the accuracy limit of the manufacturing machine. The tool path through the surface patches on the object to be produced can be easily found by including some existing surface contouring routines, and in most of the cases, the cutting path may not pass through the holes, therefore, only a few steps of subdivisions is required for obtaining a complete tool path for the NC machine.

9. FURTHER WORK

This subdivision algorithm is guaranteed to converge within the convex hull, a property lacking in some other methods of fitting a surface by solving non-linear equations.

The boundaries of the complete surface fitted onto an opened polyhedron are governed by the centroids of the boundary faces, this creats a difficulty to the designers. Although doubled edges may tie a boundary down to a quadratic curve, this still lacks the freedom required as far as arbitrarily specified boundaries are concerned.

As multiple vertices produce cusps, and the possibility of extending Chaikin's algorithm to higher order curves by including several adjacent vertices exists; further investigations may review the possibility of fitting a higher order surface to a general and arbitrarily defined polyhedron.

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