Regressor of a 3 Degrees of Freedom planar manipulator modeled in the operational space

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1 Introduction

The dynamic behavior of a n-Degrees of Freedom (DoF) robot manipulator can be derived from the Euler-Lagrange equations of motion

$$L(\mathbf{q},\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^\top \bar{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} - \bar{U}(\mathbf{q}); \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\tau}$$

where $L(\mathbf{q},\dot{\mathbf{q}})$ is the Lagrangian and $\bar{U}(\mathbf{q})$ is the potential energy. $\bar{\mathbf{M}}(\mathbf{q}) \in \mathbb{R}^{n\times n}$ is the inertia matrix and $\dot{\mathbf{q}},\mathbf{q} \in \mathbb{R}^n$ are the joint velocities and positions, respectively. In compact form, these equations can be written as

$$\bar{\mathbf{M}}(\mathbf{q})\ddot{\mathbf{q}} + \bar{\mathbf{C}}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \bar{\mathbf{g}}(\mathbf{q}) = \boldsymbol{\tau} \tag{1}$$

where $\ddot{\mathbf{q}} \in \mathbb{R}^n$ is the joint acceleration vector; $\bar{\mathbf{C}}(\mathbf{q},\dot{\mathbf{q}}) \in \mathbb{R}^{n\times n}$ is the Coriolis and centrifugal effects matrix; $\bar{\mathbf{g}}(\mathbf{q}) := \frac{\partial \bar{U}(\mathbf{q})}{\partial \mathbf{q}} \in \mathbb{R}^n$ is the gravitational force vector, and $\boldsymbol{\tau} \in \mathbb{R}^n$ is a generalized force vector.

It is well-known that (1) enjoys the following fundamental linearity property: *for all* $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$, *dynamics (1) can be written as*

$$\bar{\mathbf{M}}(\mathbf{q})\mathbf{x} + \bar{\mathbf{C}}(\mathbf{q},\dot{\mathbf{q}})\mathbf{y} = \mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{x},\mathbf{y})\boldsymbol{\theta},$$

where $\mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{x},\mathbf{y}) \in \mathbb{R}^{n\times p}$ *is a regressor matrix of known functions and* $\boldsymbol{\theta} \in \mathbb{R}^p$ *is a constant vector that is a function of the manipulator physical parameters (link masses, moments of inertia, etc.) [1].*

This document uses the linearity in the parameters property and presents such factorization for a 3-DoF planar manipulator. The document starts by presenting the joint space model and then such model is used to obtain the operational space model. Finally, the document reports the corresponding regressor matrix and parameters vector pair for this particular 3-DoF case.

2 Joint Space Dynamics

The nonlinear joint space dynamic model of a 3-DoF planar manipulator (Figure 1) follows (1) with $n = 3$.

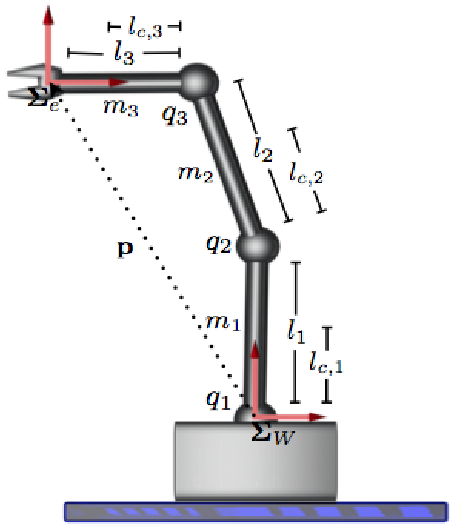

Figure 1: 3-DoF planar manipulator.

The inertia matrix is defined as $\bar{\mathbf{M}}(\mathbf{q}) = [m_{ij}]$*, where*

$$\begin{aligned}
m_{11} &= p_1 + 2p_2\cos(q_2) + 2p_3\cos(q_3) + 2p_4\cos(q_2+q_3);\\
m_{12} &= p_5 + p_2\cos(q_2) + 2p_3\cos(q_3) + p_4\cos(q_2+q_3);\\
m_{13} &= p_6 + p_3\cos(q_3) + p_4\cos(q_2+q_3);\\
m_{21} &= m_{12};\\
m_{22} &= p_5 + 2p_3\cos(q_3);\\
m_{23} &= p_6 + p_3\cos(q_3);\\
m_{31} &= m_{13};\\
m_{32} &= m_{23};\\
m_{33} &= p_6.
\end{aligned}$$

Similarly, the Coriolis and centrifugal matrix is defined as $\bar{\mathbf{C}}(\mathbf{q},\dot{\mathbf{q}}) = [c_{ij}]$*, where*

$$\begin{aligned}
c_{11} &= -p_2\sin(q_2)\dot{q}_2 - p_3\sin(q_3)\dot{q}_3 - p_4\sin(q_2+q_3)(\dot{q}_2+\dot{q}_3);\\
c_{12} &= -p_2\sin(q_2)(\dot{q}_1+\dot{q}_2) - p_3\sin(q_3)\dot{q}_3 - p_4\sin(q_2+q_3)(\dot{q}_1+\dot{q}_2+\dot{q}_3);\\
c_{13} &= -p_3\sin(q_3)(\dot{q}_1+\dot{q}_2+\dot{q}_3) - p_4\sin(q_2+q_3)(\dot{q}_1+\dot{q}_2+\dot{q}_3);\\
c_{21} &= p_2\sin(q_2)\dot{q}_1 - p_3\sin(q_3)\dot{q}_3 + p_4\sin(q_2+q_3)(\dot{q}_1);\\
c_{22} &= -p_3\sin(q_3)\dot{q}_3;\\
c_{23} &= -p_3\sin(q_3)(\dot{q}_1+\dot{q}_2+\dot{q}_3);\\
c_{31} &= p_3\sin(q_3)(\dot{q}_1+\dot{q}_2) + p_4\sin(q_2+q_3)\dot{q}_1;\\
c_{32} &= p_3\sin(q_3)(\dot{q}_1+\dot{q}_2);\\
c_{33} &= 0.
\end{aligned}$$

The gravity vector is defined as $\bar{\mathbf{g}}(\mathbf{q}) = [g_i]$*, where*

$$\begin{aligned}
g_1 &= p_7\cos(q_1) + p_8\cos(q_1+q_2) + p_9\cos(q_1+q_2+q_3);\\
g_2 &= p_8\cos(q_1+q_2) + p_9\cos(q_1+q_2+q_3);\\
g_3 &= p_9\cos(q_1+q_2+q_3).
\end{aligned}$$

The parameters (p_j*,* $j = 1\ \ldots\ 9$) *[2], that are used in the previous definitions are given by*

$$\begin{aligned}
p_1 &= m_1l_{c,1}^2 + In_1 + m_2l_1^2 + m_2l_{c,2}^2 + In_2 + m_3l_1^2 + m_3l_2^2 + m_3l_{c,3}^2 + In_3;\\
p_2 &= m_2l_1l_{c,2} + m_3l_1l_2\\
p_3 &= m_3l_2l_{c,3};\\
p_4 &= m_3l_1l_{c,3};\\
p_5 &= m_2l_{c,2}^2 + In_2 + m_3l_2^2 + m_3l_{c,3}^2 + In_3;\\
p_6 &= m_3l_{c,3}^2 + In_3;\\
p_7 &= m_2gl_{c,2} + m_3gl_2;\\
p_8 &= m_2gl_{c,2} + m_3gl_2;\\
p_9 &= m_3gl_{c,3}.
\end{aligned}$$

The physical meaning of each physical parameter is depicted in Table 1.

Parameters ($i \in [1,3]$)	**Physical Meaning**
m_i	link mass
In_i	link inertia
l_i	link length
$l_{c,i}$	Center of mass distance

Table 1: System parameters

The analytical Jacobian which, in this case, is the same as the geometric Jacobian [3], is defined as $\mathbf{J_A}(\mathbf{q}) = [j_{ij}] \in \mathbb{R}^{3\times 3}$*, where*

$$\begin{aligned}
j_{11} &= -l_1\sin(q_1) - l_2\sin(q_1+q_2) - l_3\sin(q_1+q_2+q_3);\\
j_{12} &= -l_2\sin(q_1+q_2) - l_3\sin(q_1+q_2+q_3);\\
j_{13} &= -l_3\sin(q_1+q_2+q_3);\\
j_{21} &= l_1\cos(q_1) + l_2\cos(q_1+q_2) + l_3\cos(q_1+q_2+q_3);\\
j_{22} &= l_2\cos(q_1+q_2) + l_3\cos(q_1+q_2+q_3);\\
j_{23} &= l_3\cos(q_1+q_2+q_3);\\
j_{31} &= 1;\\
j_{32} &= 1;\\
j_{33} &= 1.
\end{aligned}$$

3 Operational Space Dynamics

The compact form of the nonlinear operational space dynamics can be derived from (1) and it is given by

$$\mathbf{M}(\mathbf{r})\ddot{\mathbf{r}} + \mathbf{C}(\mathbf{r}, \dot{\mathbf{r}})\dot{\mathbf{r}} + \mathbf{g}(\mathbf{r}) = \mathbf{u} \tag{2}$$

where $\mathbf{r} \in \mathbb{R}^3$ *is the operational space coordinate vector, describing the position (*$\mathbf{p} \in \mathbb{R}^2$*) and orientation (*$\boldsymbol{\varphi} \in \mathbb{R}$*) of the manipulator end-effector,* $\mathbf{M}(\mathbf{r}) \in \mathbb{R}^{3\times3}$*,* $\mathbf{C}(\mathbf{r}, \dot{\mathbf{r}}) \in \mathbb{R}^{3\times3}$*,* $\mathbf{g}(\mathbf{r}) \in \mathbb{R}^3$*, and* $\mathbf{u} \in \mathbb{R}^3$*. These operators and signals are defined as*

$$\begin{aligned}
\dot{\mathbf{r}} &= \mathbf{J}_A(\mathbf{q})\dot{\mathbf{q}} \\
\ddot{\mathbf{r}} &= \mathbf{J}_A(\mathbf{q})\ddot{\mathbf{q}} + \dot{\mathbf{J}}_A(\mathbf{q})\dot{\mathbf{q}} \\
\mathbf{M}(\mathbf{r}) &= \mathbf{J}_A^{-\top}\bar{\mathbf{M}}(\mathbf{q})\mathbf{J}_A^{-1} \\
\mathbf{C}(\mathbf{r}, \dot{\mathbf{r}}) &= \mathbf{J}_A^{-\top}(\bar{\mathbf{M}}(\mathbf{q})\dot{\mathbf{J}}_A^{-1} + \bar{\mathbf{C}}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{J}_A^{-1}) \\
\mathbf{g}(\mathbf{r}) &= \mathbf{J}_A^{-\top}\mathbf{g}(\mathbf{q}) \\
\mathbf{u} &= \mathbf{J}_A^{-\top}\boldsymbol{\tau}
\end{aligned}$$

4 Regressor Matrix and Physical Parameters Vector

Using the linearity in the parameters property, the operational space model (2) can be described as

$$\mathbf{M}(\mathbf{r})\ddot{\mathbf{r}} + \mathbf{C}(\mathbf{r}, \dot{\mathbf{r}})\dot{\mathbf{r}} + \mathbf{g}(\mathbf{r}) \quad = \quad \mathbf{Y}(\ddot{\mathbf{r}}, \dot{\mathbf{r}}, \mathbf{r})\boldsymbol{\Theta}$$

in this case, there exist 25 different parameters and hence $\mathbf{Y}(\ddot{\mathbf{r}}, \dot{\mathbf{r}}, \mathbf{r}) \in \mathbb{R}^{3\times25}$ *and* $\boldsymbol{\Theta} \in \mathbb{R}^{25}$*. Each of these parameters is shown in Table 2.*

$\Theta_1 = p_7/l_1$	$\Theta_{10} = p_5/l_1^2$	$\Theta_{19} = p_9$
$\Theta_2 = p_8/l_2$	$\Theta_{11} = (l_3p_1)/l_1^2$	$\Theta_{20} = (l_3^2p_5)/l_1^2$
$\Theta_3 = p_2/(l_1l_2)$	$\Theta_{12} = (l_3p_5)/l_2^2$	$\Theta_{21} = (l_3^2p_2)/(l_1l_2)$
$\Theta_4 = p_3/l_2$	$\Theta_{13} = (l_3p_6)/l_2^2$	$\Theta_{22} = (l_3p_3)/l_2$
$\Theta_5 = p_4/l_1$	$\Theta_{14} = (l_3p_5)/l_1^2$	$\Theta_{23} = (l_3p_4)/l_1$
$\Theta_6 = (l_3p_2)/(l_1l_2)$	$\Theta_{15} = (l_3^2p_6)/l_2^2$	$\Theta_{24} = (l_3p_8)/l_2$
$\Theta_7 = p_6/l_2^2$	$\Theta_{16} = (l_3^2p_1)/l_1^2$	$\Theta_{25} = l_3p_7)/l_1$
$\Theta_8 = p_5/l_2^2$	$\Theta_{17} = p_6$	
$\Theta_9 = p_1/l_1^2$	$\Theta_{18} = (l_3^2p_5)/l_2^2$	

Table 2: Elements of the parameters vector $\boldsymbol{\Theta}$

The components of the regressor matrix ($\mathbf{Y}(\dot{\mathbf{r}}, \mathbf{r}_o, \dot{\mathbf{q}}, \mathbf{q})$) are

$$
\begin{aligned}
y_{1,1} &= (g(\cos(2q_1+q_2)+\cos(q_2)))/(2\sin(q_2));\\
y_{1,2} &= -(g(\cos(2q_1+q_2)+\cos(q_2)))/(2\sin(q_2));\\
y_{1,3} &= (\dot{V}_1\cos(q_2)^2+V_2\dot{q}_1+(V_2\dot{q}_2)/2)/(\cos(q_2)^2-1)-((\dot{V}_2\cos(2q_1+3q_2))/4\\
&\quad -(\dot{V}_2\cos(2q_1-q_2))/4+(\dot{V}_1\sin(2q_1-q_2))/4-(\dot{V}_1\sin(2q_1+3q_2))/4\\
&\quad +(3V_1\dot{q}_2\cos(2q_1+q_2))/4+(3V_2\dot{q}_2\sin(2q_1+q_2))/4\\
&\quad +(V_1\dot{q}_1\cos(2q_1-q_2))/4-(V_1\dot{q}_1\cos(2q_1+3q_2))/4\\
&\quad +(V_1\dot{q}_2\cos(2q_1-q_2))/4+(V_2\dot{q}_1\sin(2q_1-q_2))/4\\
&\quad -(V_2\dot{q}_1\sin(2q_1+3q_2))/4+(V_2\dot{q}_2\sin(2q_1-q_2))/4\\
&\quad +V_1\dot{q}_2\cos(q_2))/(\sin(q_2)(\cos(q_2)^2-1));\\
y_{1,4} &= (\cos(q_1)(V_3\dot{q}_1\sin(q_3)-\dot{V}_3\cos(q_3)+V_3\dot{q}_2\sin(q_3)+V_3\dot{q}_3\sin(q_3)))/\sin(q_2);\\
y_{1,5} &= -(\cos(q_1+q_2)(V_3\dot{q}_1\sin(q_2+q_3)-\dot{V}_3\cos(q_2+q_3)+V_3\dot{q}_2\sin(q_2+q_3)\\
&\quad +V_3\dot{q}_3\sin(q_2+q_3)))/\sin(q_2);\\
y_{1,6} &= (\dot{V}_3\cos(q_1)\cos(q_2)\sin(q_3)+\dot{V}_3\cos(q_1+q_2)\sin(q_2+q_3)\cos(q_2)\\
&\quad +V_3\dot{q}_2\cos(q_1+q_2)\cos(q_2+q_3)\cos(q_2)+V_3\dot{q}_3\cos(q_1+q_2)\cos(q_2+q_3)\cos(q_2)\\
&\quad +V_3\dot{q}_3\cos(q_1)\cos(q_2)\cos(q_3))/(\cos(q_2)^2-1)\\
&\quad -((V_3\dot{q}_2\sin(2q_2+q_3)\cos(q_1))/4-(V_3\dot{q}_2\sin(2q_2-q_3)\cos(q_1))/4\\
&\quad +V_3\dot{q}_1\cos(q_1+q_2)\sin(q_2+q_3)+(3V_3\dot{q}_2\cos(q_1+q_2)\sin(q_2+q_3))/2\\
&\quad -V_3\dot{q}_1\cos(q_1)\sin(q_3)+(V_3\dot{q}_2\cos(q_1)\sin(q_3))/2\\
&\quad -(V_3\dot{q}_2\sin(q_2-q_3)\cos(q_1+q_2))/4+(V_3\dot{q}_2\sin(3q_2+q_3)\cos(q_1+q_2))/4\\
&\quad +V_3\dot{q}_1\cos(q_1)\cos(q_2)^2\sin(q_3)-V_3\dot{q}_1\cos(q_1+q_2)\sin(q_2+q_3)\cos(q_2)^2\\
&\quad -V_3\dot{q}_2\cos(q_1+q_2)\sin(q_2+q_3)\cos(q_2)^2)/(\sin(q_2)(\cos(q_2)^2-1));\\
y_{1,7} &= (\cos(q_1)(V_1\dot{q}_2\cos(q_1)\cos(q_2)+V_2\dot{q}_2\cos(q_2)\sin(q_1)))/\sin(q_2)^3\\
&\quad -(\cos(q_1)(\dot{V}_1\cos(q_1)+\dot{V}_2\sin(q_1)-V_1\dot{q}_1\sin(q_1)+V_2\dot{q}_1\cos(q_1)))/\sin(q_2)^2;\\
y_{1,8} &= (\cos(q_1)(\dot{V}_1\cos(q_1)+\dot{V}_2\sin(q_1)-V_1\dot{q}_1\sin(q_1)+V_2\dot{q}_1\cos(q_1)))/\sin(q_2)^2\\
&\quad -(\cos(q_1)(V_1\dot{q}_2\cos(q_1)\cos(q_2)+V_2\dot{q}_2\cos(q_2)\sin(q_1)))/\sin(q_2)^3;\\
y_{1,9} &= (2\dot{V}_1+2\dot{V}_1\cos(2q_1+2q_2)+2V_2\dot{q}_1+2V_2\dot{q}_2\\
&\quad +2\dot{V}_2\sin(2q_1+2q_2))/(4\sin(q_2)^2)+(V_1\dot{q}_1\cos(2q_1+q_2)\\
&\quad +2V_1\dot{q}_2\cos(2q_1+q_2)+V_2\dot{q}_1\sin(2q_1+q_2)+2V_2\dot{q}_2\sin(2q_1+q_2)\\
&\quad -V_1\dot{q}_1\cos(2q_1+3q_2)-V_2\dot{q}_1\sin(2q_1+3q_2)\\
&\quad +2V_1\dot{q}_2\cos(q_2))/(\sin(3q_2)-3\sin(q_2));
\end{aligned}
$$

$$
\begin{aligned}
y_{1,10} &= -(2\dot{V}_1 + 2\dot{V}_1\cos(2q_1+2q_2) + 2V_2\dot{q}_1 + 2V_2\dot{q}_2 \\
&\quad +2\dot{V}_2\sin(2q_1+2q_2))/(4\sin(q_2)^2) - (V_1\dot{q}_1\cos(2q_1+q_2) \\
&\quad +2V_1\dot{q}_2\cos(2q_1+q_2) + V_2\dot{q}_1\sin(2q_1+q_2) \\
&\quad +2V_2\dot{q}_2\sin(2q_1+q_2) - V_1\dot{q}_1\cos(2q_1+3q_2) \\
&\quad -V_2\dot{q}_1\sin(2q_1+3q_2) + 2V_1\dot{q}_2\cos(q_2))/(\sin(3q_2) - 3\sin(q_2)); \\
y_{1,11} &= (\cos(q_1+q_2)\sin(q_2)(\dot{V}_3\sin(q_3) + V_3\dot{q}_3\cos(q_3)) \\
&\quad -V_3\dot{q}_2\cos(q_1+q_2)\cos(q_2)\sin(q_3))/\sin(q_2)^3; \\
y_{1,12} &= (\cos(q_1)(\dot{V}_3\sin(q_2+q_3) + V_3\dot{q}_2\cos(q_2+q_3) + V_3\dot{q}_3\cos(q_2+q_3)))/\sin(q_2)^2 \\
&\quad -(V_3\dot{q}_2\sin(q_2+q_3)\cos(q_1)\cos(q_2))/\sin(q_2)^3; \\
y_{1,13} &= (V_3\dot{q}_2\sin(q_2+q_3)\cos(q_1)\cos(q_2))/\sin(q_2)^3 - (\cos(q_1)(\dot{V}_3\sin(q_2+q_3) \\
&\quad +V_3\dot{q}_2\cos(q_2+q_3) + V_3\dot{q}_3\cos(q_2+q_3)))/\sin(q_2)^2; \\
y_{1,14} &= -(\cos(q_1+q_2)\sin(q_2)(\dot{V}_3\sin(q_3) + V_3\dot{q}_3\cos(q_3)) \\
&\quad -V_3\dot{q}_2\cos(q_1+q_2)\cos(q_2)\sin(q_3))/\sin(q_2)^3; \\
y_{1,15} &= y_{1,16} = y_{1,17} = y_{1,18} = y_{1,19} = 0; \\
y_{1,20} &= y_{1,21} = y_{1,22} = y_{1,23} = y_{1,24} = 0; \\
y_{1,25} &= 0; \\
y_{2,1} &= g/2 + (g\sin(2q_1+q_2))/(2\sin(q_2)); \\
y_{2,2} &= g/2 - (g\sin(2q_1+q_2))/(2\sin(q_2)); \\
y_{2,3} &= -(-\dot{V}_2\cos(q_2)^2 + V_1\dot{q}_1 + (V_1\dot{q}_2)/2)/(\cos(q_2)^2 - 1) - ((\dot{V}_1\cos(2q_1+3q_2))/4 \\
&\quad -(\dot{V}_1\cos(2q_1-q_2))/4 - (\dot{V}_2\sin(2q_1-q_2))/4 + (\dot{V}_2\sin(2q_1+3q_2))/4 \\
&\quad -(3V_2\dot{q}_2\cos(2q_1+q_2))/4 + (3V_1\dot{q}_2\sin(2q_1+q_2))/4 \\
&\quad -(V_2\dot{q}_1\cos(2q_1-q_2))/4 + (V_2\dot{q}_1\cos(2q_1+3q_2))/4 \\
&\quad -(V_2\dot{q}_2\cos(2q_1-q_2))/4 + (V_1\dot{q}_1\sin(2q_1-q_2))/4 \\
&\quad -(V_1\dot{q}_1\sin(2q_1+3q_2))/4 + (V_1\dot{q}_2\sin(2q_1-q_2))/4 \\
&\quad +V_2\dot{q}_2\cos(q_2))/(\sin(q_2)(\cos(q_2)^2 - 1)); \\
y_{2,4} &= (\sin(q_1)(V_3\dot{q}_1\sin(q_3) - \dot{V}_3\cos(q_3) + V_3\dot{q}_2\sin(q_3) + V_3\dot{q}_3\sin(q_3)))/\sin(q_2); \\
y_{2,5} &= -(\sin(q_1+q_2)(V_3\dot{q}_1\sin(q_2+q_3) - \dot{V}_3\cos(q_2+q_3) + V_3\dot{q}_2\sin(q_2+q_3) \\
&\quad +V_3\dot{q}_3\sin(q_2+q_3)))/\sin(q_2); \\
y_{2,6} &= -(V_3\dot{q}_1\sin(q_1)\sin(q_3) - V_3\dot{q}_1\sin(q_1+q_2)\sin(q_2+q_3) \\
&\quad +V_3\dot{q}_2\sin(q_1)\sin(q_3))/\sin(q_2) - (V_3\dot{q}_2\sin(q_1+q_2)\sin(q_2+q_3) \\
&\quad -\cos(q_2)(\dot{V}_3\sin(q_1+q_2)\sin(q_2+q_3)\sin(q_2) + \dot{V}_3\sin(q_1)\sin(q_2)\sin(q_3) \\
&\quad +V_3\dot{q}_2\cos(q_2+q_3)\sin(q_1+q_2)\sin(q_2) \\
&\quad +V_3\dot{q}_3\cos(q_2+q_3)\sin(q_1+q_2)\sin(q_2) + V_3\dot{q}_3\cos(q_3)\sin(q_1)\sin(q_2)) \\
&\quad +V_3\dot{q}_2\sin(q_1)\sin(q_3))/(\sin(q_2)(\cos(q_2)^2 - 1));
\end{aligned}
$$

$$
\begin{aligned}
y_{2,7} &= (-V_2\dot{q}_2\cos(q_2)\cos(q_1)^2 + V_1\dot{q}_2\cos(q_2)\sin(q_1)\cos(q_1) \\
&\quad +V_2\dot{q}_2\cos(q_2))/(\sin(q_2) - \cos(q_2)^2\sin(q_2)) \\
&\quad +(\dot{V}_2 - V_1\dot{q}_1 - \dot{V}_2\cos(q_1)^2 + V_1\dot{q}_1\cos(q_1)^2 + \dot{V}_1\cos(q_1)\sin(q_1) \\
&\quad +V_2\dot{q}_1\cos(q_1)\sin(q_1))/(\cos(q_2)^2 - 1); \\
y_{2,8} &= -(-V_2\dot{q}_2\cos(q_2)\cos(q_1)^2 + V_1\dot{q}_2\cos(q_2)\sin(q_1)\cos(q_1) \\
&\quad +V_2\dot{q}_2\cos(q_2))/(\sin(q_2) - \cos(q_2)^2\sin(q_2)) - (\dot{V}_2 - V_1\dot{q}_1 - \dot{V}_2\cos(q_1)^2 \\
&\quad +V_1\dot{q}_1\cos(q_1)^2 + \dot{V}_1\cos(q_1)\sin(q_1) + V_2\dot{q}_1\cos(q_1)\sin(q_1))/(\cos(q_2)^2 - 1); \\
y_{2,9} &= (V_1\dot{q}_1\sin(2q_1 + q_2) - 2V_2\dot{q}_2\cos(2q_1 + q_2) - V_2\dot{q}_1\cos(2q_1 + q_2) \\
&\quad +2V_1\dot{q}_2\sin(2q_1 + q_2) + V_2\dot{q}_1\cos(2q_1 + 3q_2) - V_1\dot{q}_1\sin(2q_1 + 3q_2) \\
&\quad +2V_2\dot{q}_2\cos(q_2))/(\sin(3q_2) - 3\sin(q_2)) - (2\dot{V}_2\cos(2q_1 + 2q_2) \\
&\quad -2\dot{V}_2 + 2V_1\dot{q}_1 + 2V_1\dot{q}_2 - 2\dot{V}_1\sin(2q_1 + 2q_2))/(4\sin(q_2)^2); \\
y_{2,10} &= (2\dot{V}_2\cos(2q_1 + 2q_2) - 2\dot{V}_2 + 2V_1\dot{q}_1 + 2V_1\dot{q}_2 \\
&\quad -2\dot{V}_1\sin(2q_1 + 2q_2))/(4\sin(q_2)^2) - (V_1\dot{q}_1\sin(2q_1 + q_2) \\
&\quad -2V_2\dot{q}_2\cos(2q_1 + q_2) - V_2\dot{q}_1\cos(2q_1 + q_2) \\
&\quad +2V_1\dot{q}_2\sin(2q_1 + q_2) + V_2\dot{q}_1\cos(2q_1 + 3q_2) \\
&\quad -V_1\dot{q}_1\sin(2q_1 + 3q_2) + 2V_2\dot{q}_2\cos(q_2))/(\sin(3q_2) - 3\sin(q_2)); \\
y_{2,11} &= (\sin(q_1 + q_2)\sin(q_2)(\dot{V}_3\sin(q_3) - V_3\dot{q}_3(2\sin(q_3/2)^2 - 1)) \\
&\quad +V_3\dot{q}_2\sin(q_1 + q_2)\sin(q_3)(2\sin(q_2/2)^2 - 1))/\sin(q_2)^3; \\
y_{2,12} &= -(\sin(q_1)((\dot{V}_3\cos(2q_2 + q_3))/2 - (\dot{V}_3\cos(q_3))/2 + V_3\dot{q}_2\sin(q_3) \\
&\quad +(V_3\dot{q}_3\sin(q_3))/2 - (V_3\dot{q}_3\sin(2q_2 + q_3))/2))/\sin(q_2)^3; \\
y_{2,13} &= (\sin(q_1)((\dot{V}_3\cos(2q_2 + q_3))/2 - (\dot{V}_3\cos(q_3))/2 + V_3\dot{q}_2\sin(q_3) \\
&\quad +(V_3\dot{q}_3\sin(q_3))/2 - (V_3\dot{q}_3\sin(2q_2 + q_3))/2))/\sin(q_2)^3; \\
y_{2,14} &= -(\sin(q_1 + q_2)\sin(q_2)(\dot{V}_3\sin(q_3) - V_3\dot{q}_3(2\sin(q_3/2)^2 - 1)) \\
&\quad +V_3\dot{q}_2\sin(q_1 + q_2)\sin(q_3)(2\sin(q_2/2)^2 - 1))/\sin(q_2)^3; \\
y_{2,15} &= y_{2,16} = y_{2,17} = y_{2,18} = y_{2,19} = 0; \\
y_{2,20} &= y_{2,21} = y_{2,22} = y_{2,23} = y_{2,24} = 0; \\
y_{2,25} &= 0; \\
y_{3,1} &= y_{3,2} = y_{3,3} = 0; \\
y_{3,4} &= (\cos(q_2)(V_1\dot{q}_2\cos(q_1)\cos(q_3)\sin(q_2) \\
&\quad +V_2\dot{q}_2\cos(q_3)\sin(q_1)\sin(q_2)))/(\sin(q_2) - \cos(q_2)^2\sin(q_2)) - (\dot{V}_1\cos(q_1)\cos(q_3) \\
&\quad +\dot{V}_2\cos(q_3)\sin(q_1) + V_2\dot{q}_1\cos(q_1)\cos(q_3) + V_1\dot{q}_1\cos(q_1)\sin(q_3) \\
&\quad -V_1\dot{q}_1\cos(q_3)\sin(q_1) + V_1\dot{q}_2\cos(q_1)\sin(q_3) + V_2\dot{q}_1\sin(q_1)\sin(q_3) \\
&\quad +V_2\dot{q}_2\sin(q_1)\sin(q_3))/\sin(q_2);
\end{aligned}
$$

$$
\begin{aligned}
y_{3,5} &= (\dot{V}_1 \cos(q_1+q_2)\cos(q_2+q_3) + \dot{V}_2 \cos(q_2+q_3)\sin(q_1+q_2) \\
&\quad + V_2\dot{q}_1 \cos(q_1+q_2)\cos(q_2+q_3) + V_2\dot{q}_2 \cos(q_1+q_2)\cos(q_2+q_3) \\
&\quad + V_1\dot{q}_1 \cos(q_1+q_2)\sin(q_2+q_3) - V_1\dot{q}_1 \cos(q_2+q_3)\sin(q_1+q_2) \\
&\quad - V_1\dot{q}_2 \cos(q_2+q_3)\sin(q_1+q_2) + V_2\dot{q}_1 \sin(q_1+q_2)\sin(q_2+q_3))/\sin(q_2) \\
&\quad - (\cos(q_2)(V_1\dot{q}_2 \cos(q_1+q_2)\cos(q_2+q_3)\sin(q_2) \\
&\quad + V_2\dot{q}_2 \cos(q_2+q_3)\sin(q_1+q_2)\sin(q_2)))/(\sin(q_2) - \cos(q_2)^2\sin(q_2)); \\
y_{3,6} &= (\dot{V}_1 \cos(q_1)\cos(q_2)\sin(q_3) + \dot{V}_2 \cos(q_2)\sin(q_1)\sin(q_3) \\
&\quad + \dot{V}_1 \cos(q_1+q_2)\sin(q_2+q_3)\cos(q_2) + \dot{V}_2 \sin(q_1+q_2)\sin(q_2+q_3)\cos(q_2) \\
&\quad + V_2\dot{q}_1 \cos(q_1+q_2)\sin(q_2+q_3)\cos(q_2) + V_2\dot{q}_2 \cos(q_1+q_2)\sin(q_2+q_3)\cos(q_2) \\
&\quad - V_1\dot{q}_1 \sin(q_1+q_2)\sin(q_2+q_3)\cos(q_2) - V_1\dot{q}_2 \sin(q_1+q_2)\sin(q_2+q_3)\cos(q_2) \\
&\quad + V_2\dot{q}_1 \cos(q_1)\cos(q_2)\sin(q_3) - V_1\dot{q}_1 \cos(q_2)\sin(q_1)\sin(q_3))/(\cos(q_2)^2 - 1) \\
&\quad - (V_1\dot{q}_1 \cos(q_1)\sin(q_3) - V_2\dot{q}_1 \sin(q_1+q_2)\sin(q_2+q_3) \\
&\quad - V_1\dot{q}_1 \cos(q_1+q_2)\sin(q_2+q_3) + V_1\dot{q}_2 \cos(q_1)\sin(q_3) \\
&\quad + V_2\dot{q}_1 \sin(q_1)\sin(q_3) + V_2\dot{q}_2 \sin(q_1)\sin(q_3) \\
&\quad - V_1\dot{q}_1 \cos(q_1)\cos(q_2)^2\sin(q_3) - V_2\dot{q}_1 \cos(q_2)^2\sin(q_1)\sin(q_3) \\
&\quad + V_1\dot{q}_1 \cos(q_1+q_2)\sin(q_2+q_3)\cos(q_2)^2 + V_1\dot{q}_2 \cos(q_1+q_2)\sin(q_2+q_3)\cos(q_2)^2 \\
&\quad + V_2\dot{q}_1 \sin(q_1+q_2)\sin(q_2+q_3)\cos(q_2)^2 \\
&\quad + V_2\dot{q}_2 \sin(q_1+q_2)\sin(q_2+q_3)\cos(q_2)^2)/(\sin(q_2)(\cos(q_2)^2 - 1)); \\
y_{3,7} &= y_{3,8} = y_{3,9} = y_{3,10} = 0; \\
y_{3,11} &= -(\sin(q_3)((\dot{V}_1 \sin(q_1))/2 - (\dot{V}_2 \cos(q_1))/2 + (\dot{V}_2 \cos(q_1+2q_2))/2 \\
&\quad - (\dot{V}_1 \sin(q_1+2q_2))/2 + (V_2\dot{q}_1 \sin(q_1))/2 + V_2\dot{q}_2 \sin(q_1) \\
&\quad - (V_1\dot{q}_1 \cos(q_1+2q_2))/2 - (V_2\dot{q}_1 \sin(q_1+2q_2))/2 \\
&\quad + (V_1\dot{q}_1 \cos(q_1))/2 + V_1\dot{q}_2 \cos(q_1)))/\sin(q_2)^3; \\
y_{3,12} &= (\sin(q_2+q_3)(\dot{V}_1 \cos(q_1) + \dot{V}_2 \sin(q_1) - V_1\dot{q}_1 \sin(q_1) \\
&\quad + V_2\dot{q}_1 \cos(q_1)))/\sin(q_2)^2 - (\sin(q_2+q_3)(V_1\dot{q}_2 \cos(q_1)\cos(q_2) \\
&\quad + V_2\dot{q}_2 \cos(q_2)\sin(q_1)))/\sin(q_2)^3; \\
y_{3,13} &= (\sin(q_2+q_3)(V_1\dot{q}_2 \cos(q_1)\cos(q_2) + V_2\dot{q}_2 \cos(q_2)\sin(q_1)))/\sin(q_2)^3 \\
&\quad - (\sin(q_2+q_3)(\dot{V}_1 \cos(q_1) + \dot{V}_2 \sin(q_1) - V_1\dot{q}_1 \sin(q_1) \\
&\quad + V_2\dot{q}_1 \cos(q_1)))/\sin(q_2)^2; \\
y_{3,14} &= (\sin(q_3)((\dot{V}_1 \sin(q_1))/2 - (\dot{V}_2 \cos(q_1))/2 + (\dot{V}_2 \cos(q_1+2q_2))/2 \\
&\quad - (\dot{V}_1 \sin(q_1+2q_2))/2 + (V_2\dot{q}_1 \sin(q_1))/2 + V_2\dot{q}_2 \sin(q_1) \\
&\quad - (V_1\dot{q}_1 \cos(q_1+2q_2))/2 - (V_2\dot{q}_1 \sin(q_1+2q_2))/2 \\
&\quad + (V_1\dot{q}_1 \cos(q_1))/2 + V_1\dot{q}_2 \cos(q_1)))/\sin(q_2)^3;
\end{aligned}
$$

$$
\begin{aligned}
y_{3,15} &= -(\dot{V}_3 - \dot{V}_3(\cos(2q_2+2q_3)/2+1/2))/\sin(q_2)^2 - ((V_3\dot{q}_2\cos(q_2+2q_3))/2 \\
&\quad +(V_3\dot{q}_3\cos(q_2+2q_3))/4 - (V_3\dot{q}_3\cos(3q_2+2q_3))/4 \\
&\quad -(V_3\dot{q}_2\cos(q_2))/2)/\sin(q_2)^3; \\
y_{3,16} &= (\dot{V}_3 - \dot{V}_3(\cos(2q_3)/2+1/2) + (V_3\dot{q}_3\sin(2q_3))/2)/\sin(q_2)^2 \\
&\quad -(V_3\dot{q}_2\cos(q_2) - V_3\dot{q}_2\cos(q_2)(\cos(2q_3)/2+1/2))/\sin(q_2)^3; \\
y_{3,17} &= \dot{V}_3;
\end{aligned}
$$

$$
\begin{aligned}
y_{3,18} &= (\dot{V}_3 - \dot{V}_3(\cos(2q_2+2q_3)/2+1/2))/\sin(q_2)^2 + ((V_3\dot{q}_2\cos(q_2+2q_3))/2 \\
&\quad +(V_3\dot{q}_3\cos(q_2+2q_3))/4 - (V_3\dot{q}_3\cos(3q_2+2q_3))/4 \\
&\quad -(V_3\dot{q}_2\cos(q_2))/2)/\sin(q_2)^3; \\
y_{3,19} &= g\cos(q_1+q_2+q_3); \\
y_{3,20} &= (V_3\dot{q}_2\cos(q_2) - V_3\dot{q}_2\cos(q_2)(\cos(2q_3)/2+1/2))/\sin(q_2)^3 - (\dot{V}_3 \\
&\quad -\dot{V}_3(\cos(2q_3)/2+1/2) + (V_3\dot{q}_3\sin(2q_3))/2)/\sin(q_2)^2; \\
y_{3,21} &= -(-4\dot{V}_3\sin(q_2)\cos(q_2)^2 + 4V_3\dot{q}_2\cos(q_2) + \dot{V}_3\sin(3q_2+2q_3) \\
&\quad +\dot{V}_3\sin(q_2-2q_3) - V_3\dot{q}_2\cos(q_2-2q_3) - 3V_3\dot{q}_2\cos(q_2+2q_3) \\
&\quad -V_3\dot{q}_3\cos(q_2-2q_3) + V_3\dot{q}_3\cos(3q_2+2q_3))/(4\sin(q_2)(\cos(q_2)^2-1)); \\
y_{3,22} &= -(V_3\dot{q}_2\cos(q_2-2q_3) - V_3\dot{q}_2\cos(q_2+2q_3) + V_3\dot{q}_3\cos(q_2-2q_3) \\
&\quad -2V_3\dot{q}_3\cos(q_2+2q_3) + V_3\dot{q}_3\cos(3q_2+2q_3))/(\sin(3q_2) - 3\sin(q_2)) \\
&\quad -(2\dot{V}_3 - 2\dot{V}_3\cos(2q_2+2q_3) - 2\dot{V}_3\cos(2q_2) \\
&\quad +2\dot{V}_3\cos(2q_3))/(4\sin(q_2)^2); \\
y_{3,23} &= (V_3\dot{q}_2\cos(q_2-2q_3) - V_3\dot{q}_2\cos(q_2+2q_3) + V_3\dot{q}_3\cos(q_2-2q_3) \\
&\quad -2V_3\dot{q}_3\cos(q_2+2q_3) + V_3\dot{q}_3\cos(3q_2+2q_3))/(\sin(3q_2) - 3\sin(q_2)) \\
&\quad -(2\dot{V}_3 + 2\dot{V}_3\cos(2q_2+2q_3) - 2\dot{V}_3\cos(2q_2) \\
&\quad -2\dot{V}_3\cos(2q_3))/(4\sin(q_2)^2); \\
y_{3,24} &= (g(\sin(q_1-q_3) - \sin(q_1+2q_2+q_3)))/(2\sin(q_2)); \\
y_{3,25} &= (g\cos(q_1)\sin(q_3))/\sin(q_2);
\end{aligned}
$$

5 Conclusions

In this report was presented the regressor and the parameters vector of a three degrees of freedom planar manipulator. The joint space as well as the operational space nonlinear dynamic models were also shown as part of the regressor derivation process.

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