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Electrostatics and magnetostatics on the hypersphere

Bakalářská práce

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Ústav teoretické fyziky a astrofyziky

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Abstrakt

Tato bakalářská práce se zabývá Poissonovou rovnicí pro skalární a vektorový potenciál elektrického a magnetického pole. Pomocí jazyka diferenciálních forem převedeme Laplaceův operátor do hypersférických souřadnic. S předpokladem, že řešení Poissonovy rovnice je separabilní najdeme vlastní funkce Laplaceova operátoru - hypersférické kulové funkce. Tyto vlastní funkce tvoří ortogonální bázi na prostoru kvadraticky integrovatelných funkcí na hypersféře. V důsledku toho můžeme vyjadřit elektrostatický potenciál pro jakoukoli funkci hustoty náboje, jejíž střední hodnota an hypersféře je nulová, jako zobecněnou Fourierovu řadu s bází tvořenou hypersférickými harmonickými funkcemi. Ukázalo se, že řešení Poissonova rovnice pro magnetostatický vektorový potenciál je poměrně časově náročný problém a v této práci jsme neposkytli jeho řešení.

Abstract

This bachelor's thesis addresses the Poisson equation for the scalar and vector potentials of the electric and magnetic fields respectively. We express the Laplace operator in the hyperspherical coordinates using the language of differential forms. With the assumption that the solution of the Poisson equation for the electrostatic potential is separable, we find the eigenfunctions of the Laplace operator - hyperspherical harmonics. These eigenfunctions form an orthogonal basis on the space of square-integrable functions. Consequently, we can express the electric potential for any charge density whose mean value over the sphere is zero in the form of a generalized Fourier series with hyperspherical harmonics as the basis. The Poisson equation of the magnetostatic vector potential proved to be a rather time-consuming problem and in this thesis, we do not provide any solution for it.

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Ředitel *ústavu* PřF MU Vám ve smyslu Studijního a zkušebního řádu MU určuje bakalářskou práci s názvem:

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Oficiální zadání:
The student shall formulate electrostatics and magnetostatics on the three-dimensinal hypersphere S^3 equipped with the standard metric field induced by the standard immersion of the hypersphere into four-dimensional Euclidean space. The scalar and vector Poisson equation which arises in this way shall be solved by various methods, e.g. separation of variables. The comparison of solution methods and the limit of infinite hypersphere radius could also be part of the thesis.

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Prohlášení

Prohlašuji, že jsem bakalářskou práci zpracoval samostatně a použil jen prameny uvedené v seznamu literatury.

Michal Valentík

Contents

Introduction

The laws describing the electromagnetic fields are hidden in Maxwell's equations, which have been known to us since the 19$^{\text{th}}$ century. At that time it was a common belief that space is flat and the idea of formulating the laws of electromagnetism for curved space may have seemed pointless. In the same century, mainly thanks to the work of Carl Friedrich Gauss and Bernhard Riemann, differential geometry began to develop, together with the concept of curved spaces. Thanks to Riemann's geometry, Albert Einstein was able to introduce his theory of general relativity in 1905, and suddenly the formulation of physical laws on curved space(time) began to be applied to real-world problems.

This thesis aims to formulate the laws of electrostatics and magnetostatics on the simplest curved manifold that has the same number of space dimensions as our Universe - the three-dimensional hypersphere $\mathbb{S}^3$.

Prerequisites

It is expected that the reader is familiar with the differential forms and the basic operations on them. Throughout the thesis, the basic identities from the vector analysis are used along with the knowledge of electromagnetism at the undergraduate level. If any of these topics are unknown to the reader they should familiarize themselves with them before reading this thesis.

Structure of the thesis

The thesis is divided into 4 chapters. The first and second chapters show the necessary tools that are used throughout the thesis - the first chapter introduces the coordinate maps on the hypersphere $\mathbb{S}^3$ and the second introduces operators on differential forms, mainly the Laplace–de Rham operator expressed in hyperspherical coordinates.

In Chapter 3 with the help of the tools from the first two chapters, the eigenfunctions of the Laplace operator are introduced. The Sturm–Liouville theory allows us to express the solution of the Poisson equation for the electrostatic potential in the form of the generalized Fourier series.

In Chapter 4 the Poisson equation for the magnetostatic vector potential is derived. Unfortunately, the Poisson equation for the vector magnetostatic potential has proved to be a more time-consuming problem than the Poisson equation for the electrostatic potential, and this problem has been left unsolved.

This thesis has four appendices. The first one deals with the solution of the Legendre equation and Chebyshev equation of the second kind using the Frobenius method. The second shows the useful properties of the Gegenbauer polynomial. The third is devoted to the visualization of selected hyperspherical harmonics using stereographic projection. The last appendix briefly describes the functions of the Python script *sas.py*, which is part of this thesis.

1 Coordinate systems of the hypersphere $\mathbb{S}^3$

The standard hypersphere is the set of points in $\mathbb{R}^{n+1}$ which lie a fixed distance from a given point - the center $\mathbf{c}$

$$\mathbb{S}^n = \{\mathbf{p} \in \mathbb{R}^{n+1} | d_2(\mathbf{p} - \mathbf{c}) = r\},$$

where d_2 is the Euclidean metric. In a Cartesian coordinates and for $\mathbf{c} = 0$ (center is at the origin) we have

$$\sqrt{x_1^2 + \cdots + x_{n+1}^2} = r, \quad x_1, \ldots, x_{n+1} \in \langle -r, r \rangle, \quad r \in \mathbb{R}.$$

In this bachelor thesis, we will work with $\mathbb{S}^3$. The radius of our hypersphere shall be one unit $(r = 1)$ unless stated otherwise.

The hypersphere $\mathbb{S}^3$ is a three-dimensional submanifold of $\mathbb{R}^4$ and can be represented in many useful different coordinate systems. In this chapter, we will introduce and derive coordinate systems for $\mathbb{S}^3$, which we will use throughout this thesis.

1.1 Hyperspherical coordinates

Let $\mathbb{R}^n$ be a vector space with Cartesian coordinates, where n is a natural number greater than or equal to 2. Then any general rotation about the origin can be decomposed into rotations in planes given by the Cartesian axes. In this section, we will derive the rotation matrix in a plane, which will help us to derive the hyperspherical coordinates on the hypersphere $\mathbb{S}^3$.

Infinitesimal rotation in the plane can be thought of as a linear transformation

$$\mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \epsilon_1 \\ \epsilon_2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & \epsilon_1 \\ \epsilon_2 & 1 \end{pmatrix} \mathbf{x},$$

where ϵ_1 and ϵ_2 are sufficiently small. The length of the vector before and after the rotation must be conserved. This condition connects ϵ_1 and ϵ_2 by the following relation

$$\|\mathbf{x}\|^2 = \|\mathbf{x}'\|^2,$$
$$x^2 + y^2 = x^2 + 2\epsilon_1 xy + \epsilon_2^2 y^2 + \epsilon_2^2 x^2 + 2\epsilon_2 xy + y^2.$$

Terms with ϵ_1^2 and ϵ_2^2 are small enough to be ignored and thus the relation between the infinitesimal terms is

$$\epsilon_1 = -\epsilon_2.$$

This means that the matrix of infinitesimal rotation has a form

$$\hat{\mathbf{R}} = \begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix} = \hat{\mathbf{1}} + \epsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \hat{\mathbf{1}} + \epsilon\hat{\mathbf{J}}.$$

The rotation by general angle φ can be obtained by taking the limit transition $N \to \infty$ of N compositions of infinitesimal rotations ($\epsilon = \frac{\varphi}{N}$)

$$\hat{\mathbf{R}}(\varphi) = \lim_{N\to\infty} \left(\hat{\mathbf{1}} + \frac{\varphi}{N}\hat{\mathbf{J}}\right)^N = \exp(\varphi\hat{\mathbf{J}}).$$

We can use the Taylor expansion of the exponential matrix

$$\hat{\mathbf{R}}(\varphi) = \sum_{n=0}^{\infty} \frac{\varphi^n}{n!}\hat{\mathbf{J}}^n = \cos(\varphi)\hat{\mathbf{1}} + \sin(\varphi)\hat{\mathbf{J}} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

Multiplying any vector $\mathbf{x}$ by $\hat{\mathbf{R}}(\varphi)$ will rotate $\mathbf{x}$ by an angle φ. This result was obtained simply by exponentiating the term $\varphi\hat{\mathbf{J}}$. We will call the matrix $\hat{\mathbf{J}}$ the rotation generator.

This rotation generation in $\mathbb{R}^2$ can be generalized to $\mathbb{R}^N$. In $\mathbb{R}^N$ there are $\binom{N}{2}$ rotation generators (one for each coordinate plane in which the rotation can take place). These rotations are represented by SO(N) generator matrices. In the case of $\mathbb{R}^4$, there are six rotation generators for six planes

$$\begin{aligned}
\hat{\mathbf{J}}_{12} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \hat{\mathbf{J}}_{13} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \hat{\mathbf{J}}_{14} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
\hat{\mathbf{J}}_{23} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \hat{\mathbf{J}}_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \hat{\mathbf{J}}_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\end{aligned} \tag{1.1}$$

These matrices generate all six possible rotations in planes given by the Cartesian axes (coordinate planes).

Let the vector $\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T$, which represents the north pole of our hypersphere, be denoted by $\mathbf{n}$ and any vector that lies on the $\mathbb{S}^3$ by $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T$.

Let us take such a basis for which vectors $\mathbf{x}$ and $\mathbf{n}$ lie in the plane given by the last two Cartesian axes (this choice can always be made). With this choice of basis, the vector $\mathbf{x}$ will have only two non-zero components $\mathbf{x} = \begin{pmatrix} 0 & 0 & x_3 & x_4 \end{pmatrix}^T$. This means that the dot product of $\mathbf{x}$ and $\mathbf{n}$ is $\langle \mathbf{x}, \mathbf{n} \rangle = \cos\varphi$ and the last two components of the vector $\mathbf{x}$ are $x_3 = -\sin\varphi$ and $x_4 = \cos\varphi$.[1] By applying rotation matrix $\hat{\mathbf{R}}_{34}(\varphi)$ (generated by $\hat{\mathbf{J}}_{34}$) on the $\mathbf{n}$ one can obtain the vector $\mathbf{x}$.

$$\hat{\mathbf{R}}_{34}(\varphi)\mathbf{n} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\sin\varphi \\ \cos\varphi \end{pmatrix}$$

For a general Cartesian basis and a general vector $\mathbf{x}$ (which may or may not lie in the coordinate plane given by the last two Cartesian axes), the north pole $\mathbf{n}$ can be transformed into the vector $\mathbf{x}$ by successive applications of rotations (in the coordinate planes) on the north pole $\mathbf{n}$ which are non-invariant for the $\mathbf{n}$.[2] Let us take non-invariant rotation matrixes $\hat{\mathbf{R}}_{14}(\varphi - \frac{\pi}{2})$, $\hat{\mathbf{R}}_{24}(\vartheta - \frac{\pi}{2})$ and $\hat{\mathbf{R}}_{34}(\zeta - \frac{\pi}{2})$ and apply them on the north pole $\mathbf{n}$.

$$\hat{\mathbf{R}}_{14}\left(\varphi - \frac{\pi}{2}\right) \hat{\mathbf{R}}_{24}\left(\vartheta - \frac{\pi}{2}\right) \hat{\mathbf{R}}_{34}\left(\zeta - \frac{\pi}{2}\right) \mathbf{n} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sin\zeta & \cos\zeta \\ 0 & 0 & -\cos\zeta & \sin\zeta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\vartheta & 0 & \cos\vartheta \\ 0 & 0 & 1 & 0 \\ 0 & -\cos\vartheta & 0 & \sin\vartheta \end{pmatrix} \begin{pmatrix} \sin\varphi & 0 & 0 & \cos\varphi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\cos\varphi & 0 & 0 & \sin\varphi \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\varphi \\ \sin\varphi\cos\vartheta \\ \sin\varphi\sin\vartheta\cos\zeta \\ \sin\varphi\sin\vartheta\sin\zeta \end{pmatrix}.$$

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1. The minus sign in the coefficient x_3 might be confusing for the reader. Later in this section, it will be clear that this choice had to be made to arrive at standardized hyperspherical coordinates.
2. Non-invariant rotations are rotations $\hat{\mathbf{R}}_{14}$, $\hat{\mathbf{R}}_{24}$ and $\hat{\mathbf{R}}_{34}$ generated by $\hat{\mathbf{J}}_{14}$, $\hat{\mathbf{J}}_{24}$ and $\hat{\mathbf{J}}_{34}$ and invariant rotations are rotations $\hat{\mathbf{R}}_{12}$, $\hat{\mathbf{R}}_{13}$ and $\hat{\mathbf{R}}_{23}$ generated by $\hat{\mathbf{J}}_{12}$, $\hat{\mathbf{J}}_{13}$ and $\hat{\mathbf{J}}_{23}$. The invariant rotations for the vector $\mathbf{n}$ are such rotations which, when the rotation is applied to vector $\mathbf{n}$, leave vector $\mathbf{n}$ unchanged.

The vector resulting from this transformation gives us the hyperspherical coordinates,[3] which we can write down as a coordinate map $\alpha : \mathbb{S}^3 \hookrightarrow \mathbb{R}^4$

$$\alpha : \begin{array}{l} x_1 = \cos\varphi \\ x_2 = \sin\varphi\cos\vartheta \\ x_3 = \sin\varphi\sin\vartheta\cos\zeta \\ x_4 = \sin\varphi\sin\vartheta\sin\zeta \end{array} \tag{1.2}$$

We are also interested in the inversion map $\alpha^{-1} : \mathbb{R}^4 \to \mathbb{S}^3$ which can be easily obtained

$$\alpha^{-1} : \begin{array}{l} \varphi = \arctan\frac{\sqrt{x_2^2+x_3^2+x_4^2}}{x_1} + \frac{\pi}{2}, \\ \vartheta = \begin{cases} \arctan\frac{\sqrt{x_3^2+x_4^2}}{x_2} + \frac{\pi}{2}, & \text{if } x_2 \neq 0, x_3 \neq 0 \text{ and } x_4 \neq 0, \\ \text{undefined}, & \text{for } x_2 = x_3 = x_4 = 0, \end{cases} \\ \zeta = \begin{cases} \arctan\left(\frac{x_4}{x_3}\right), & \text{if } x_3 > 0, \\ \arctan\left(\frac{x_4}{x_3}\right) + \pi, & \text{if } x_3 < 0 \text{ and } x_4 \geq 0, \\ \arctan\left(\frac{x_4}{x_3}\right) - \pi, & \text{if } x_3 < 0 \text{ and } x_4 < 0, \\ +\frac{\pi}{2}, & \text{if } x_3 = 0 \text{ and } x_4 > 0, \\ -\frac{\pi}{2}, & \text{if } x_3 = 0 \text{ and } x_4 < 0, \\ \text{undefined}, & \text{if } x_3 = x_4 = 0. \end{cases} \end{array} \tag{1.3}$$

With this inversion map, we chose the domain of our angles $\varphi \in (0, \pi)$, $\vartheta \in (0, \pi)$ and $\zeta \in (-\pi, \pi)$. Note that the set of points on the "meridian" $(\zeta = \pi)$ is not covered by the map α. However, because this set is of measure zero it is negligible for purposes of integration.

1.2 Stereographic projection

Stereographic projection in $\mathbb{R}^{N+1}$ maps a point from a sphere $\mathbb{S}^N$ onto a (hyper)-plane that is perpendicular to the diameter passing through the point on the sphere, i.e. center of projection (it is common to choose the north pole as a center of projection). The projected point from the sphere is connected to the center of the projection by a straight line and the intersection of this line with the plane is the point on which the projected point maps itself. This description of stereographic projection

3. Thanks to our choice of minus signs in the rotation generators (1.1) and a shift by $-\frac{\pi}{2}$, we obtained the hyperspherical coordinates in the standardized form. Had we not made this choice, the sines would have become cosines and vice versa and some later calculations would have been inconvenient.

can be generalized to N-dimensional space $\mathbb{R}^N$.

In this thesis, we will work with stereographic projection from $\mathbb{S}^3$ to a hyperplane in $\mathbb{R}^4$. The north pole $\mathbf{n} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T$ will be used as the center of projection and our hyperplane will pass through the origin. The projected point $\mathbf{p} = \begin{pmatrix} x' & y' & z' & w' \end{pmatrix}^T$ on the sphere is connected with the north pole $\mathbf{n}$ by a straight line $\mathbf{l}$

$$\begin{aligned} \mathbf{n} &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T, \\ \mathbf{p} &= \begin{pmatrix} x' & y' & z' & w' \end{pmatrix}^T, \\ \mathbf{l} &= \begin{pmatrix} x & y & z & w \end{pmatrix}^T, \\ \mathbf{l} &= (1-t)\mathbf{n} + t\mathbf{p}, \quad t \in \mathbb{R} \end{aligned} \tag{1.4}$$

The hyperplane that satisfies the conditions set earlier is

$$w = 0. \tag{1.5}$$

Combining the equations (1.4) and (1.5) gives the following set of equations

$$x = x't, \quad y = y't, \quad z = z't, \quad w = 1 - t + w't = 0.$$

The parameter t can be derived from the last equation

$$t = \frac{1}{1-w'},$$

and thus the intersection of the straight line and the hyperplane is

$$x = \frac{x'}{1-w'} = \mu, \quad y = \frac{y'}{1-w'} = \nu, \quad z = \frac{z'}{1-w'} = \xi, \quad w = 0.$$

These equations give us the coordinate map $\beta : \mathbb{R}^4 \to \mathbb{R}^3 \ni \begin{pmatrix} \mu & \nu & \xi \end{pmatrix}$

$$\beta : \begin{aligned} \mu &= \tfrac{x'}{1-w'}, \\ \nu &= \tfrac{y'}{1-w'}, \\ \xi &= \tfrac{z'}{1-w'}. \end{aligned} \tag{1.6}$$

The inversion of the stereographic projection β^{-1} can be obtained similarly. The point on the hyperplane $\mathbf{a}$ is connected to the north pole $\mathbf{n}$ by a straight line $\mathbf{l}$.

$$\begin{aligned} \mathbf{a} &= \begin{bmatrix} \mu & \nu & \xi & 0 \end{bmatrix}, \\ \mathbf{l} &= (1-k)\mathbf{n} + \mathbf{a}k, \quad k \in \mathbb{R}. \end{aligned} \tag{1.7}$$

Intersection point of unit sphere $x'^2 + y'^2 + z'^2 + w'^2 = 1$ and straight line (1.7) is again determined by the parameter k of the line

$$k = \frac{2}{\mu^2 + \nu^2 + \xi^2 + 1}.$$

There is a second valid value for the parameter $k = 0$ but this is the expected intersection with the north pole. Inserting the parameter k into the equation (1.7) gives the following equations which describe the inversion of the stereographic projection $\beta^{-1} : \mathbb{R}^3 \to \mathbb{R}^4$

$$\beta^{-1} : \begin{aligned} x' &= \tfrac{2\mu}{\mu^2+\nu^2+\xi^2+1}, \\ y' &= \tfrac{2\nu}{\mu^2+\nu^2+\xi^2+1}, \\ z' &= \tfrac{2\xi}{\mu^2+\nu^2+\xi^2+1}, \\ w' &= \tfrac{\mu^2+\nu^2+\xi^2-1}{\mu^2+\nu^2+\xi^2+1}. \end{aligned} \tag{1.8}$$

1.2.1 Visualization of functions on $\mathbb{S}^3$

It is not easy for a human mind to visualize four-dimensional space even when we live in one. It is even more difficult to visualize functions on three-dimensional manifolds embedded in four-dimensional space. The stereographic projection helps us with the visualization of functions on $\mathbb{S}^3$ embedded in $\mathbb{R}^4$. Part of this thesis is the Python script *sas.py* that computes stereographic projection of the surfaces of constant value (isosurfaces) and then creates a projection graph in $\mathbb{R}^3$.

Brief description of the stereographic projection function of the script *sas.py*

Input for the script is the function $f(\varphi, \vartheta, \zeta)$ defined on $\mathbb{S}^3$ with hyperspherical angles φ, ϑ and ζ as variables. We start with the fixed point grid with N^3 points in $\mathbb{R}^3$. Each point of the grid is transformed by inverse stereographic projection β^{-1} to hypersphere in $\mathbb{R}^4$ and then assigned hyperspherical angles by the inversion map α^{-1}. Each point is evaluated by input function $f(\varphi, \vartheta, \zeta)$. With function values assigned, it is possible to plot an isosurface graph in $\mathbb{R}^3$ on the N^3 grid.

Examples of visualized functions via stereographical projection

The simplest examples of functions to visualize by stereographic projection are the coordinate functions $f(\varphi, \vartheta, \zeta) = \varphi$ (Figure 1.1), $f(\varphi, \vartheta, \zeta) = \vartheta$ (Figure 1.2) and $f(\varphi, \vartheta, \zeta) = \zeta$ (Figure 1.3). Each of these projections have a symmetry axis. Function $f = \varphi$ has coordinate axis x as a symmetry axis, $f = \theta$ has coordinate axis

y as its coordinate axis, and coordinate axis z is a symmetry axis for the function $f = \zeta$.

One may notice numerical artifacts in Figure 1.3 on a yellow half sphere. This artifact appears due to the function values of the function $f = \zeta$ changing from $-\pi$ to π rapidly on small distances. The plotting package that was used (Pyplot) was not able to plot the isosurfaces accurately due to the relatively low number of grid points N^3. Because of this rapid change of the function values we only showed the isosurfaces of the function values in the interval $\zeta \in (0, \pi)$.

For the more complex example we chose function $f(\varphi, \vartheta, \zeta) = \cos\varphi \cos\vartheta \cos\zeta$. It can be seen in Figure 1.4 that this function has coordinate axis z as its symmetry axis[4] of the stereographic projection. This function (and others) will be used as an example in the following parts of the thesis as well.

4. Note that not all functions have symmetry axis in its stereographic projection. We just happened to choose functions that do.

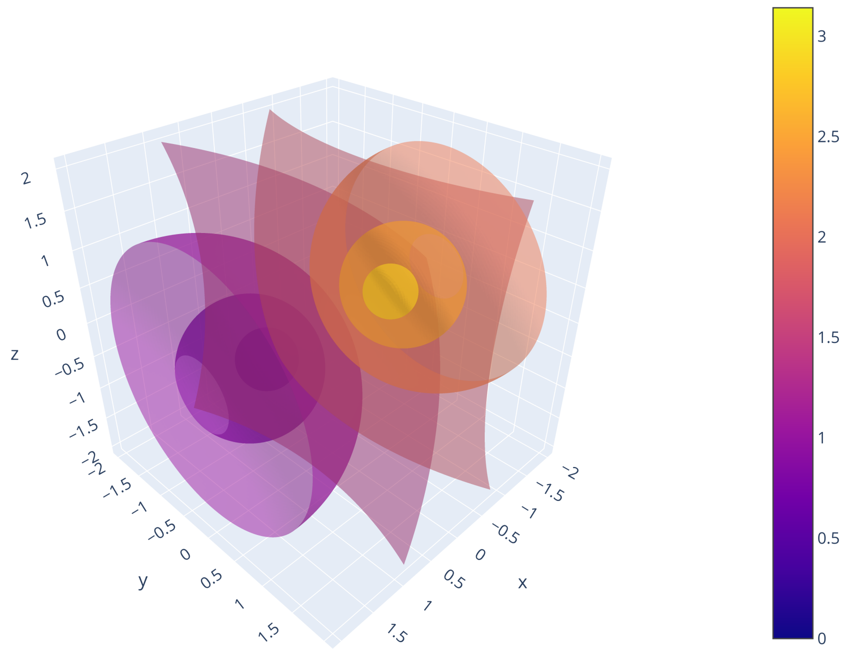

Figure 1.1: Stereographic projection of the surfaces with a constant function value of the function $f(\varphi, \vartheta, \zeta) = \varphi$

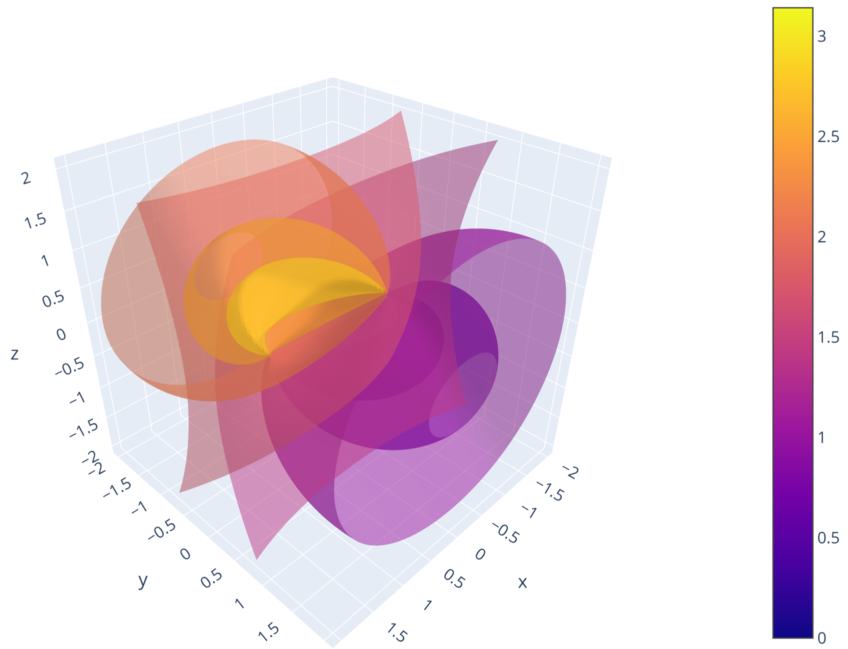

Figure 1.2: Stereographic projection of the surfaces with a constant function value of the function $f(\varphi, \vartheta, \zeta) = \vartheta$

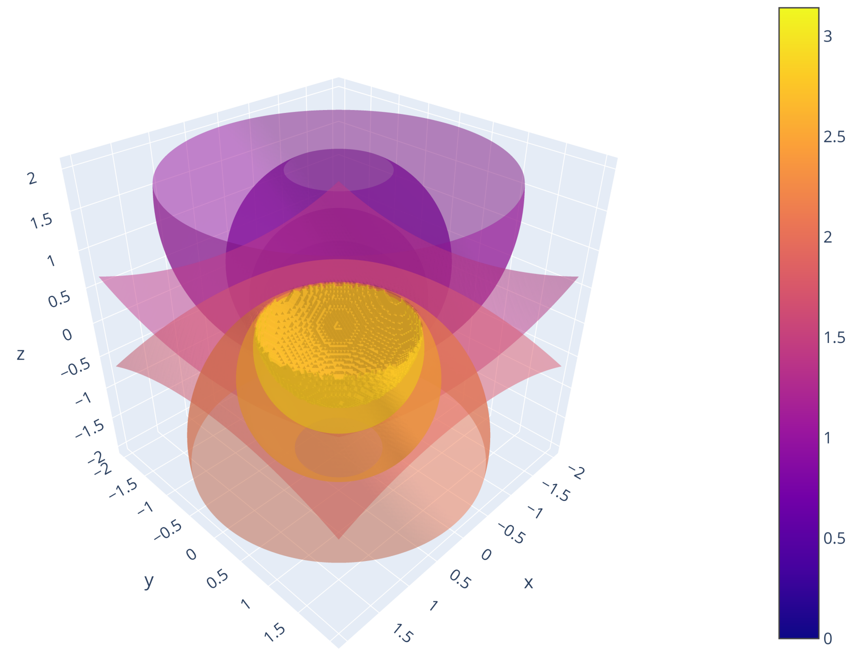

Figure 1.3: Stereographic projection of the surfaces with a constant function value of the function $f(\varphi, \vartheta, \zeta) = \zeta$

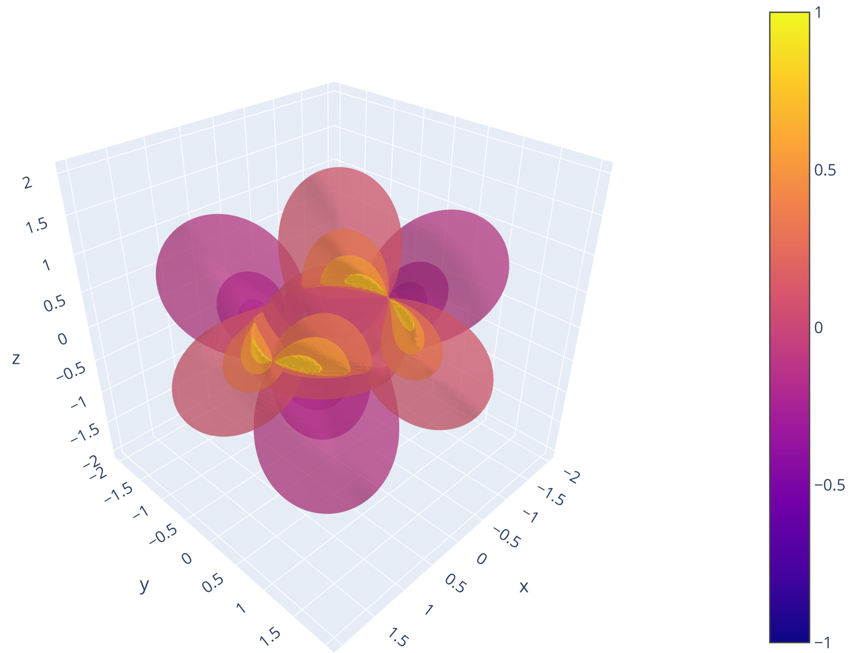

Figure 1.4: Stereographic projection of the surfaces with a constant function value of the function $f = \cos\varphi \cos\vartheta \cos\zeta$

2 Differential forms and the Laplace operator on $\mathbb{S}^3$

The Laplace operator Δ appears in both electrostatics and magnetostatics. Later in this thesis, we will derive the relationship between electrostatic and magnetostatic field potentials and their respective sources. The Laplace operator appears in both relations. This part of the thesis aims to express the Laplace operator for functions (0-forms) and vector fields (1-forms) in hyperspherical coordinates

We can generalize the Laplace operator to the differential forms.[1] If the reader is not familiar with the basics of exterior algebra and differential forms, we recommend that they familiarize themselves with them, for example in GUILLEMIN [2], which provides a good introduction to differential forms. First, let us introduce some important tools that we will need for this generalization.

Inner product of algebraic k-forms. Let V be a n-dimensional vector space with an orthonormal basis $\{\mathbf{e}_i\}_{i=1}^n$ and V^* be its dual vector space with an orthonormal dual basis $\{e^i\}_{i=1}^n$. The inner product of two vectors[2]

$$\mathbf{x} = x^i \mathbf{e}_i,$$
$$\mathbf{y} = y^j \mathbf{e}_j,$$

can be defined (for example in FRANKEL [3]) as

$$(\mathbf{x}|\mathbf{y}) = (x^i\mathbf{e}_i|y^j\mathbf{e}_j) = x^iy^j(\mathbf{e}_i|\mathbf{e}_j) = \delta_{ij}x^iy^j = x_jy^j, \tag{2.1}$$

where

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j \end{cases}$$

is the Kronecker delta in the role of the metric tensor. With the use of this relation for the inner product of two vectors (2.1), we can define the isomorphism $\Psi : V \to V^*$ as

$$\Psi(\mathbf{x}) = (\mathbf{x}|\cdot) = \delta_{ij}x^ie^j = x_je^j = c, \qquad c \in V^* \tag{2.2}$$

where e^j is the dual vector to $\mathbf{e}_j$. The inverse of this map Ψ^{-1} will help us define the inner product on the dual space V^*

$$\Psi^{-1}(c) = \delta^{ij}x_i\mathbf{e}_j = x^i\mathbf{e}_i = (c|\cdot). \tag{2.3}$$

1. The bachelor thesis of Josef Gajdůšek [1] gives a nice overview on the use of differential forms in electrodynamics. Some of the definitions in this section are taken from his thesis.
2. Throughout this thesis we will use the Einstein summation convention.

The inner product on dual space V^* of two algebraic forms

$$a = a_i e^i,$$
$$b = b_j e^j,$$

is defined as

$$(a|b) = (a_i e^i | b_j e^j) = a_i b_j (e^i | e^j) = \delta^{ij} a_i b_j = a^j b_j. \tag{2.4}$$

With a definition (2.4) of the inner product of two algebraic forms, we can now formulate a definition for the inner product of two algebraic k-forms. Let a and b be the two algebraic k-forms $\bigwedge^k(V^*)$ and a^i with b^j be the algebraic 1-forms V^*.

$$a = a^{i_1} \wedge \cdots \wedge a^{i_k}$$
$$b = b^{j_1} \wedge \cdots \wedge b^{j_k}.$$

Then the inner product of k-forms is defined by the following relation

$$(a|b) = \det \begin{pmatrix} (a^{i_1}|b^{j_1}) & \dots & (a^{i_k}|b^{j_k}) \\ \vdots & \ddots & \vdots \\ (a^{i_k}|b^{j_1}) & \dots & (a^{i_k}|b^{j_k}) \end{pmatrix} \tag{2.5}$$

Volume form. Volume form $\omega \in \bigwedge^n(V^*)$ on n-dimensional manifold M is defined as a n-form that satisfy following condition

$$(\omega|\omega) = 1.$$

Two n-forms are compatible with the condition, differing only by a sign

$$\omega \ = \ \pm\, e^1 \ \wedge \ \dots \ \wedge \ e^n.$$

We will make a choice and define ω as

$$\omega = e^1 \ \wedge \ \dots \ \wedge \ e^n. \tag{2.6}$$

Hodge (star) operator. The Hodge operator $\star$ is a linear map, mapping $\bigwedge^k(V^*)$ to $\bigwedge^{n-k}(V^*)$. The defining relation for the Hodge operator $\star$ is

$$v \wedge \star u = (v|u)\omega, \quad \text{for all } v, u \in \textstyle\bigwedge^k(V^*). \tag{2.7}$$

Inverse Hodge operator. Let us take a k-form $v = e^1 \wedge \cdots \wedge e^k \in \bigwedge^k(V^*)$. From the defining relation of the Hodge operator $\star$ (2.7) it must be true that

$$v \wedge \star v = (v|v)\omega,$$
$$\star v = e^{k+1} \wedge \cdots \wedge e^n = u.$$

Let us apply the Hodge operator once more to the last equation

$$\star\star v = \star u$$

Again from the defining relation of the Hodge operator, it must be true that

$$u \wedge \star(\star v) = u \wedge \star u = (u|u)\omega,$$
$$e^{k+1} \wedge \cdots \wedge e^n \wedge \star(e^{k+1} \wedge \cdots \wedge e^n) = e^1 \wedge \cdots \wedge e^n,$$
$$e^{k+1} \wedge \cdots \wedge e^n \wedge e^1 \wedge \cdots \wedge e^k = e^1 \wedge \cdots \wedge e^n,$$
$$\star\star v = (-1)^{k(n-k)} v. \tag{2.8}$$

Because every k-form of the form $e^{i_1} \wedge \ldots \wedge e^{i_k}$ can be transformed into the form $e^1 \wedge \cdots \wedge e^k$ by appropriate relabeling and permutation of the indices the relation (2.8) is true for every k-form. Now if we introduce the inverse Hodge operator $\star^{-1}$ and apply it on the equation (2.8) we can write $\star^{-1}$ as

$$\star^{-1} = (-1)^{k(n-k)} \star \tag{2.9}$$

Inner product of differential forms. Now, let us have the vector space $\Omega^k(M)$ of differential k-forms on n-dimensional manifold M. We will define the inner product of two differential k-forms $f, g \in \Omega^k(M)$ as

$$\langle f, g \rangle = \int_M \overline{f} \wedge \star g. \tag{2.10}$$

Adjoint operator to exterior derivative. The adjoint operator $\mathrm{d}^\dagger$ to the exterior derivative d with respect to the global inner product can be expressed by the Hodge operator $\star$, the inverse of the Hodge operator $\star^{-1}$, and exterior derivative d

$$\begin{aligned}\langle \mathrm{d}\alpha, \beta \rangle &= \int_M \mathrm{d}\overline{\alpha} \wedge \star\beta = \int_M \mathrm{d}(\overline{\alpha} \wedge \star\beta) + (-1)^p \int_M \overline{\alpha} \wedge \mathrm{d}(\star\beta) \\ &= \int_{\partial M} \overline{\alpha} \wedge \star\beta + \int_M \overline{\alpha} \wedge \star\left((-1)^p \star^{-1} \mathrm{d}\star\beta\right) \\ &= \int_M \overline{\alpha} \wedge \star \mathrm{d}^\dagger \beta = \langle \alpha, \mathrm{d}^\dagger \beta \rangle\end{aligned}$$

where $\beta \in \Omega^p(M)$ and $\alpha \in \Omega^{p-1}(M)$. From the relation above we can see that the adjoint operator $\mathrm{d}^\dagger$ to the exterior derivative can be expressed as

$$\mathrm{d}^\dagger = (-1)^p \star^{-1} \mathrm{d}\star. \tag{2.11}$$

2.1 Laplace–de Rham operator

The generalization of the Laplace operator to differential forms on the Riemannian manifold is called the Laplace–de Rham operator and is commonly defined (for example in NAKAHARA [4]) as

$$\overline{\Delta} = \mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\mathrm{d}^\dagger. \tag{2.12}$$

In mathematics and physics, many definitions are not standardized, and two seemingly identical definitions can be different. Unfortunately, this is also the case with the "standard" physical Laplace operator which in the cartesian coordinates in $\mathbb{R}^n$ has a form of

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

and the Laplace–de Rham operator defined by relation (2.12). For our use case, we will only need the Laplace-de Rham operator for 0-forms and 1-forms on the 3-dimensional manifold (hypersphere). We will verify if the Laplace–de Rham operator coincides with the Laplace operator on $\mathbb{R}^3$ for both 0-forms and 1-forms.

Let us start with a 0-form (function) $f \in \Omega^0(\mathbb{R}^3)$

$$f = f(x_1, x_2, x_3),$$

where x_1, x_2 and x_3 are the cartesian coordinates, which induce the orthonormal basis $\mathbf{e}_1 = \boldsymbol{\partial}_{x_1}$, $\mathbf{e}_2 = \boldsymbol{\partial}_{x_2}$ and $\mathbf{e}_3 = \boldsymbol{\partial}_{x_3}$ and also the dual orthonormal basis $e^1 = \mathrm{d}x_1$, $e^2 = \mathrm{d}x_2$ and $e^3 = \mathrm{d}x_3$. Let us apply the Laplace–de Rham operator $\overline{\Delta}$ on the function f

$$\begin{aligned} \overline{\Delta} f &= \left(-\star^{-1}\mathrm{d}\star\mathrm{d} + \mathrm{d}\star^{-1}\mathrm{d}\star\right) f = -\star^{-1}\mathrm{d}\star\mathrm{d}f, \\ \overline{\Delta} f &= -\frac{\partial^2 f}{\partial x_1^2} - \frac{\partial^2 f}{\partial x_2^2} - \frac{\partial^2 f}{\partial x_3^2}. \end{aligned} \tag{2.13}$$

If we compare this result with the Laplace operator applied on the function f

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}$$

we can see that they differ by a minus sign.

Now, let us take an 1-form $\alpha \in \Omega^1(\mathbb{R}^3)$ as

$$\alpha = A(x_1, x_2, x_3)\mathrm{d}x_1 + B(x_1, x_2, x_3)\mathrm{d}x_2 + C(x_1, x_2, x_3)\mathrm{d}x_3,$$

where x_1, x_2 and x_3 are the cartesian coordinates, which induce the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and also the dual orthonormal basis $\{e^1, e^2, e^3\}$. We can uniquely associate the vector field $\mathbf{F}$ on $\mathbb{R}^3$ with this 1-form α (and conversely, we can uniquely associate 1-form α to the vector field $\mathbf{F}$) with the use of the map Ψ^{-1} defined by the relation (2.3) (or Ψ defined by the relation (2.2) respectively).

$$\Psi^{-1}(\alpha) = \Psi^{-1}(Ae^1 + Be^2 + Ce^3) = A\mathbf{e}_1 + B\mathbf{e}_2 + C\mathbf{e}_3 = \mathbf{F},$$
$$\Psi(\mathbf{F}) = \Psi(A\mathbf{e}_1 + B\mathbf{e}_2 + C\mathbf{e}_3) = Ae^1 + Be^2 + Ce^3 = \alpha.$$

The result of applying the Laplace operator Δ to a vector field $\mathbf{F}$ is well known

$$\Delta\mathbf{F} = \left(\frac{\partial^2 A}{\partial x_1^2} + \frac{\partial^2 A}{\partial x_2^2} + \frac{\partial^2 A}{\partial x_3^2}\right)\mathbf{e}_1 + \left(\frac{\partial^2 B}{\partial x_1^2} + \frac{\partial^2 B}{\partial x_2^2} + \frac{\partial^2 B}{\partial x_3^2}\right)\mathbf{e}_2 + \\ + \left(\frac{\partial^2 C}{\partial x_1^2} + \frac{\partial^2 C}{\partial x_2^2} + \frac{\partial^2 C}{\partial x_3^2}\right)\mathbf{e}_3.$$

Let us apply the Laplace–de Rham operator on the 1-form α

$$\overline{\Delta}\alpha = \left(\star^{-1}\mathrm{d}\star\mathrm{d} - \mathrm{d}\star^{-1}\mathrm{d}\star\right)\alpha = \left(\star\mathrm{d}\star\mathrm{d} - \mathrm{d}\star\mathrm{d}\star\right)\alpha,$$
$$\overline{\Delta}\alpha = -\left(\frac{\partial^2 A}{\partial x_1^2} + \frac{\partial^2 A}{\partial x_2^2} + \frac{\partial^2 A}{\partial x_3^2}\right)\mathrm{d}x_1 - \left(\frac{\partial^2 B}{\partial x_1^2} + \frac{\partial^2 B}{\partial x_2^2} + \frac{\partial^2 B}{\partial x_3^2}\right)\mathrm{d}x_2 - \\ - \left(\frac{\partial^2 C}{\partial x_1^2} + \frac{\partial^2 C}{\partial x_2^2} + \frac{\partial^2 C}{\partial x_3^2}\right)\mathrm{d}x_3.$$

The unique vector field associated with the 1-form $\overline{\Delta}\alpha$ should be the same as $\Delta\mathbf{F}$ if the operators Δ and $\overline{\Delta}$ are equivalent. We can see that they differ by the minus sign.

We showed that the Laplace operator and the Laplace–de Rham operator differ by the minus sign if applied to 0-from or 1-form on $\mathbb{R}^3$. Fortunately, we have noticed these differences and for our case (0-forms and 1-forms on a 3-dimensional manifold) we can write the relationship between the Laplace and the Laplace–de Rham operators as

$$\Delta = -\overline{\Delta} = -\mathrm{d}^\dagger\mathrm{d} - \mathrm{d}\mathrm{d}^\dagger. \tag{2.14}$$

The mathematicians chose this definition due to their preference for the Laplace operator to be non-negative definite.

Note that both the Laplace and the Laplace–de Rham operators are self-adjoint operators

$$\begin{aligned} \overline{\Delta}^{\dagger} &= (\mathrm{d}^{\dagger}\mathrm{d} + \mathrm{d}\mathrm{d}^{\dagger})^{\dagger} = \mathrm{d}^{\dagger}\mathrm{d} + \mathrm{d}\mathrm{d}^{\dagger} = \overline{\Delta}, \\ \Delta^{\dagger} &= (-\mathrm{d}^{\dagger}\mathrm{d} - \mathrm{d}\mathrm{d}^{\dagger})^{\dagger} = -\mathrm{d}^{\dagger}\mathrm{d} - \mathrm{d}\mathrm{d}^{\dagger} = \Delta, \end{aligned} \tag{2.15}$$

which is an important property that we will utilize later in this thesis.

2.1.1 Laplace–de Rham for 0-forms (functions) on $\mathbb{S}^3$

In our case of a hypersphere $\mathbb{S}^3$, the dimension of the manifold is $n = 3$. If we apply the Laplace–de Rham operator on a function (0-form) f we end up with

$$\overline{\Delta} f = (-\star^{-1}\mathrm{d}\star\mathrm{d} + \mathrm{d}\star^{-1}\mathrm{d}\star) f = -\star\mathrm{d}\star\mathrm{d} f. \tag{2.16}$$

We can notice that the operator $\mathrm{d}\mathrm{d}^{\dagger}$ applied on a function f is zero because the Hodge operator applied on the function $\star f$ is a 3-form and in the exterior derivative of 3-form $\mathrm{d}\star f$ is zero on 3-dimensional manifold $\mathbb{S}^3$.

With the use of the hyperspherical coordinate map α defined by the relation (1.2) we can now express the Laplace–de Rham operator in hyperspherical coordinates. First, let us write down the tangent vectors of the coordinate map α in the standard orthonormal basis of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_4$

$$\begin{aligned} \boldsymbol{\partial}_{\varphi} &= -\sin\varphi\,\mathbf{e}_1 + \cos\varphi\cos\vartheta\,\mathbf{e}_2 + \cos\varphi\sin\vartheta\cos\zeta\,\mathbf{e}_3 + \cos\varphi\sin\vartheta\sin\zeta\,\mathbf{e}_4, \\ \boldsymbol{\partial}_{\vartheta} &= -\sin\varphi\sin\vartheta\,\mathbf{e}_2 + \sin\varphi\cos\vartheta\cos\zeta\,\mathbf{e}_3 + \sin\varphi\cos\vartheta\sin\zeta\,\mathbf{e}_4, \\ \boldsymbol{\partial}_{\zeta} &= -\sin\varphi\sin\vartheta\sin\zeta\,\mathbf{e}_3 + \sin\varphi\sin\vartheta\cos\zeta\,\mathbf{e}_4. \end{aligned}$$

These tangent vectors are orthogonal with respect to the standard inner product $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$ on $\mathbb{R}^4$. This inner product induces the norm $\|\boldsymbol{\partial}_i\| = \langle \boldsymbol{\partial}_i, \boldsymbol{\partial}_i \rangle$ which allow us to write down the normalized tangent vectors $\mathbf{e}_{\varphi}$, $\mathbf{e}_{\vartheta}$ and $\mathbf{e}_{\zeta}$ (which form the orthonormal basis on $T_x\mathbb{S}^3$) and their dual vectors e^{φ}, e^{ϑ} and e^{ζ} (which form the orthonormal dual basis)

$$\begin{array}{lll} \|\boldsymbol{\partial}_{\varphi}\| = 1 & \mathbf{e}_{\varphi} = \boldsymbol{\partial}_{\varphi} & e^{\varphi} = \mathrm{d}\varphi, \\ \|\boldsymbol{\partial}_{\vartheta}\| = \sin\varphi & \mathbf{e}_{\vartheta} = \frac{1}{\sin\varphi}\boldsymbol{\partial}_{\vartheta} & e^{\vartheta} = \sin\varphi\,\mathrm{d}\vartheta, \\ \|\boldsymbol{\partial}_{\zeta}\| = \sin\varphi\sin\vartheta & \mathbf{e}_{\zeta} = \frac{1}{\sin\varphi\sin\vartheta}\boldsymbol{\partial}_{\zeta} & e^{\zeta} = \sin\varphi\sin\vartheta\,\mathrm{d}\zeta, \end{array} \tag{2.17}$$

where $\mathrm{d}\varphi$, $\mathrm{d}\vartheta$ and $\mathrm{d}\zeta$ are differential forms induced by $\boldsymbol{\partial}_{\varphi}$, $\boldsymbol{\partial}_{\vartheta}$ and $\boldsymbol{\partial}_{\zeta}$. The last column in the array of relations (2.17) will help us express the Laplace–de Rham operator (and in our case the Laplace operator because we apply the Laplace–de Rham operator on a 0-form $f(\varphi, \vartheta, \zeta)$) in hyperspherical coordinates defined by coordinate map (1.2).

$$\begin{aligned}
\overline{\Delta} f &= -\star \mathrm{d} \star \left(\frac{\partial f}{\partial \varphi} \mathrm{d}\varphi + \frac{\partial f}{\partial \vartheta} \mathrm{d}\vartheta + \frac{\partial f}{\partial \zeta} \mathrm{d}\zeta \right) \\
&= -\star \mathrm{d} \star \left(\frac{\partial f}{\partial \varphi} e^{\varphi} + \frac{1}{\sin\varphi} \frac{\partial f}{\partial \vartheta} e^{\vartheta} + \frac{1}{\sin\varphi \sin\vartheta} \frac{\partial f}{\partial \zeta} e^{\zeta} \right) \\
&= -\star \mathrm{d} \left(\frac{\partial f}{\partial \varphi} e^{\vartheta} \wedge e^{\zeta} + \frac{1}{\sin\varphi} \frac{\partial f}{\partial \vartheta} e^{\zeta} \wedge e^{\varphi} + \frac{1}{\sin\varphi \sin\vartheta} \frac{\partial f}{\partial \zeta} e^{\varphi} \wedge e^{\vartheta} \right) \\
&= -\star \mathrm{d} \left(\sin^2\varphi \sin\vartheta \frac{\partial f}{\partial \varphi} \mathrm{d}\vartheta \wedge \mathrm{d}\zeta + \sin\vartheta \frac{\partial f}{\partial \vartheta} \mathrm{d}\zeta \wedge \mathrm{d}\varphi + \frac{1}{\sin\vartheta} \frac{\partial f}{\partial \zeta} \mathrm{d}\varphi \wedge \mathrm{d}\vartheta \right) \\
&= -\star \left[\left(\sin\vartheta \frac{\partial \sin^2\varphi \frac{\partial f}{\partial \varphi}}{\partial \varphi} + \frac{\partial \sin\vartheta \frac{\partial f}{\partial \vartheta}}{\partial \vartheta} + \frac{1}{\sin\vartheta} \frac{\partial^2 f}{\partial \zeta^2} \right) \underbrace{\mathrm{d}\varphi \wedge \mathrm{d}\vartheta \wedge \mathrm{d}\zeta}_{\frac{1}{\sin^2\varphi \sin\vartheta} e^{\varphi} \wedge e^{\vartheta} \wedge e^{\zeta}} \right], \\
&= -\frac{1}{\sin^2\varphi} \frac{\partial \sin^2\varphi \frac{\partial f}{\partial \varphi}}{\partial \varphi} - \frac{1}{\sin^2\varphi \sin\vartheta} \frac{\partial \sin\vartheta \frac{\partial f}{\partial \vartheta}}{\partial \vartheta} - \frac{1}{\sin^2\varphi \sin^2\vartheta} \frac{\partial^2 f}{\partial \zeta^2}
\end{aligned} \tag{2.18}$$

From the previous equation (2.18) and the relation (2.14) between the Laplace operator and the Laplace–de Rham operator it follows that the Laplace operator Δ on the hypersphere $\mathbb{S}^3$ expressed in hyperspherical coordinates will take the following form

$$\Delta = \frac{1}{\sin^2\varphi} \frac{\partial \sin^2\varphi \frac{\partial}{\partial \varphi}}{\partial \varphi} + \frac{1}{\sin^2\varphi \sin\vartheta} \frac{\partial \sin\vartheta \frac{\partial}{\partial \vartheta}}{\partial \vartheta} + \frac{1}{\sin^2\varphi \sin^2\vartheta} \frac{\partial^2}{\partial \zeta^2}. \tag{2.19}$$

2.1.2 Laplace–de Rham operator for 1-forms (vector fields) on $\mathbb{S}^3$

Let us take an 1-form

$$\alpha = A(\varphi, \vartheta, \zeta) \mathrm{d}\varphi + B(\varphi, \vartheta, \zeta) \sin\varphi \mathrm{d}\vartheta + C(\varphi, \vartheta, \zeta) \sin\varphi \sin\vartheta \mathrm{d}\zeta \tag{2.20}$$

and apply the Laplace–de Rham operator $\overline{\Delta}$ on it

$$\overline{\Delta}\alpha = \left(\star^{-1} \mathrm{d} \star \mathrm{d} - \mathrm{d} \star^{-1} \mathrm{d} \star \right) \alpha = \left(\star \mathrm{d} \star \mathrm{d} - \mathrm{d} \star \mathrm{d} \star \right) \alpha. \tag{2.21}$$

Again with the use of the last column in the array of relations (2.17) we will express the Laplace–de Rham operator in the hyperspherical coordinates for 1-forms

$$
\begin{aligned}
\overline{\Delta}\alpha = \Bigg(& \left\{ \frac{\partial}{\partial\vartheta} \left[\left(\frac{\partial\left(B\sin\varphi\right)}{\partial\varphi} - \frac{\partial A}{\partial\vartheta} \right) \sin\vartheta \right] + \right. \\
& \left. + \frac{\partial}{\partial\zeta} \left[\left(\frac{\partial\left(C\sin\varphi\sin\vartheta\right)}{\partial\varphi} - \frac{\partial A}{\partial\zeta} \right) \frac{1}{\sin\vartheta} \right] \right\} \frac{1}{\sin^2\varphi\sin\vartheta} - \\
& - \frac{\partial}{\partial\varphi} \left[\left(\frac{\partial\left(A\sin^2\varphi\sin\vartheta\right)}{\partial\varphi} + \frac{\partial\left(B\sin\varphi\sin\vartheta\right)}{\partial\vartheta} + \frac{\partial\left(C\sin\varphi\right)}{\partial\zeta} \right) \frac{1}{\sin^2\varphi\sin\vartheta} \right] \Bigg) e^{\varphi} + \\
& + \Bigg(\left\{ \frac{\partial}{\partial\zeta} \left[\left(\frac{\partial\left(C\sin\varphi\sin\vartheta\right)}{\partial\vartheta} - \frac{\partial\left(B\sin\varphi\right)}{\partial\zeta} \right) \frac{1}{\sin^2\varphi\sin\vartheta} \right] + \right. \\
& \left. + \frac{\partial}{\partial\varphi} \left[\left(\frac{\partial A}{\partial\vartheta} - \frac{\partial\left(B\sin\varphi\right)}{\partial\varphi} \right) \sin\vartheta \right] \right\} \frac{1}{\sin\varphi\sin\vartheta} - \\
& - \frac{\partial}{\partial\vartheta} \left[\left(\frac{\partial\left(A\sin^2\varphi\sin\vartheta\right)}{\partial\varphi} + \frac{\partial\left(B\sin\varphi\sin\vartheta\right)}{\partial\vartheta} + \frac{\partial\left(C\sin\varphi\right)}{\partial\zeta} \right) \frac{1}{\sin^2\varphi\sin\vartheta} \right] \Bigg) \frac{e^{\vartheta}}{\sin\varphi} + \\
& + \Bigg(\left\{ \frac{\partial}{\partial\varphi} \left[\left(\frac{\partial A}{\partial\zeta} - \frac{\partial\left(C\sin\varphi\sin\vartheta\right)}{\partial\varphi} \right) \frac{1}{\sin\vartheta} \right] + \right. \\
& \left. + \frac{\partial}{\partial\vartheta} \left[\left(\frac{\partial\left(B\sin\varphi\right)}{\partial\zeta} - \frac{\partial\left(C\sin\varphi\sin\vartheta\right)}{\partial\vartheta} \right) \frac{1}{\sin^2\varphi\sin\vartheta} \right] \right\} \frac{1}{\sin\varphi} - \\
& - \frac{\partial}{\partial\zeta} \left[\left(\frac{\partial\left(A\sin^2\varphi\sin\vartheta\right)}{\partial\varphi} + \frac{\partial\left(B\sin\varphi\sin\vartheta\right)}{\partial\vartheta} + \frac{\partial\left(C\sin\varphi\right)}{\partial\zeta} \right) \frac{1}{\sin^2\varphi\sin\vartheta} \right] \Bigg) \frac{e^{\zeta}}{\sin\varphi\sin\vartheta}.
\end{aligned}
\tag{2.22}
$$

3 Electrostatics

Electrostatics in a vacuum in $\mathbb{R}^3$ is described by two Maxwell's equations (Gauss's law and Faraday's law of induction respectively)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{3.1}$$

$$\nabla \times \mathbf{E} = \mathbf{0}, \tag{3.2}$$

where $\mathbf{E}$ is the intensity of the electric field, ρ is the charge density and ε_0 is the vacuum permittivity. For the electric field $\mathbf{E}$, which obeys equations (3.1) and (3.2), there exists a scalar potential ϕ such that

$$\mathbf{E} = -\nabla\phi. \tag{3.3}$$

The relation between electric potential ϕ and intensity of electric field $\mathbf{E}$ allows us to write the Gauss's law as

$$\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -\frac{\rho}{\varepsilon_0}, \tag{3.4}$$

which is called the Poisson equation. Now let us generalize these laws to the hypersphere $\mathbb{S}^3$ using the language of differential forms.

Generalization[1] of Gauss's law (3.1) and Faraday's law (3.2) to the hypersphere $\mathbb{S}^3$ expressed by the differential forms is

$$\mathrm{d}\star E = \frac{\sigma}{\varepsilon_0}, \tag{3.5}$$

$$\mathrm{d}E = 0, \tag{3.6}$$

where $\sigma = \rho(\varphi, \vartheta, \zeta) \cdot e^{\varphi} \wedge e^{\vartheta} \wedge e^{\zeta}$ is a 3-form $\Omega^3(\mathbb{S}^3)$ of charge density and E is the 1-form $\Omega^1(\mathbb{S}^3)$ of electric intensity. Just like for $\mathbb{R}^3$, we have a relation between the 0-form of electrostatic potential ϕ and the 1-form of electric intensity E

$$E = -\mathrm{d}\phi, \tag{3.7}$$

1. This generalization can be applied to any Riemannian manifold. If instead of $\mathbb{S}^3$ we applied this formulation of Maxwell's equations to $\mathbb{R}^3$, we would get the formulation equivalent to Gauss's law (3.1) and Faraday's law (3.2) expressed by the differential operators of vector analysis.

which is a direct consequence of Faraday's law (3.6). Let us plug the relation (3.7) into the Gauss's law (3.5)

$$\begin{aligned} \mathrm{d}\star\mathrm{d}\phi &= -\frac{\sigma}{\varepsilon_0}, \\ \star\mathrm{d}\star\mathrm{d}\phi &= -\star\frac{\sigma}{\varepsilon_0}, \\ \star\mathrm{d}\star\mathrm{d}\phi &= -\frac{\rho}{\varepsilon_0}. \end{aligned}$$

Because the operator $-\mathrm{d}\mathrm{d}^\dagger = -\mathrm{d}\star\mathrm{d}\star$ applied to any 0-form is zero, we can, without loss of generality, add this term to the left side of the previous equation

$$\begin{aligned} (\star\mathrm{d}\star\mathrm{d} - \mathrm{d}\star\mathrm{d}\star)\phi &= -\frac{\rho}{\varepsilon_0}, \\ -\overline{\Delta}\phi = \Delta\phi &= -\frac{\rho}{\varepsilon_0}. \end{aligned} \tag{3.8}$$

From the generalized Maxwell's equations (3.5) and (3.6) we obtained the Poisson equation for the electrostatic potentials on the hypersphere $\mathbb{S}^3$.

3.1 Separation of variables

It would be convenient if we could find a set of eigenvalues and eigenfunctions for the Laplace operator and express function ϕ as a linear combination of such eigenfunctions. Let us assume a solution in the form of $\phi = \Phi(\varphi)\Theta(\vartheta)Z(\zeta)$

$$\Delta(\Phi\Theta Z) = \lambda\Phi\Theta Z. \tag{3.9}$$

We can use our expression for the Laplace operator in hyperspherical coordinates defined in equation (2.19) on the left-hand side of the equation (3.9) and after some slight rearrangement we end up with

$$\frac{1}{\Phi}\frac{\partial}{\partial\varphi}\left(\sin^2\varphi\frac{\partial\Phi}{\partial\varphi}\right) + \frac{1}{\Theta\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Theta}{\partial\vartheta}\right) + \frac{1}{Z\sin^2\vartheta}\frac{\partial^2 Z}{\partial\zeta^2} = \lambda\sin^2\varphi.$$

We can move terms that are functions of φ only to the left-hand side and the rest of the terms to the right-hand side. Because one side of the equation is a function of ϑ and ζ and the other side is a function of φ then both sides must be equal to some constant μ

$$\lambda\sin^2\varphi - \frac{1}{\Phi}\frac{\partial}{\partial\varphi}\left(\sin^2\varphi\frac{\partial\Phi}{\partial\varphi}\right) = \frac{1}{\Theta\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Theta}{\partial\vartheta}\right) + \frac{1}{Z\sin^2\vartheta}\frac{\partial^2 Z}{\partial\zeta^2} = \mu. \tag{3.10}$$

This separation can be done again. Let us do it for the last two parts of the equation (3.10). And again, because both sides of the equation are functions of different variables, both of them must be equal to some constant $-\nu$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial \zeta^2} = -\frac{\sin\vartheta}{\Theta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Theta}{\partial\vartheta}\right) + \mu\sin^2\vartheta = -\nu.$$

We now have a system of three differential equations for each variable φ, ϑ and ζ

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\varphi^2} + \nu Z = 0, \tag{3.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\sin\vartheta\frac{\mathrm{d}\Theta}{\mathrm{d}\vartheta}\right) - \left(\mu\sin\vartheta + \frac{\nu}{\sin\vartheta}\right)\Theta = 0, \tag{3.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\sin^2\varphi\frac{\mathrm{d}\Phi}{\mathrm{d}\varphi}\right) - \left(\lambda\sin^2\varphi - \mu\right)\Phi = 0. \tag{3.13}$$

3.1.1 Azimuthal equation

We will call equation (3.11) azimuthal equation because the hyperspherical coordinate ζ is the generalization of the azimuthal angle known from the spherical coordinates. Solving the azimuthal equation is straightforward and its solution is

$$Z = Ae^{i\sqrt{\nu}\zeta}. \tag{3.14}$$

Since ζ represents the azimuthal angle, we require Z to be periodic with period of 2π i.g. $Z(\zeta + 2\pi) = Z(\zeta)$. This condition will be satisfied only if $\sqrt{\nu} = m \in \mathbb{Z}$. We will replace ν by m^2 in the equation (3.12) to account for this condition.

3.1.2 General Legendre equation

Let us take $x = \cos\vartheta$ and substitute that into the equation (3.12). From the substitution, it follows that

$$\sin\vartheta = \sqrt{1-x^2},$$

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta} = \frac{\mathrm{d}z}{\mathrm{d}\vartheta}\frac{\mathrm{d}}{\mathrm{d}x} = -\sin\vartheta\frac{\mathrm{d}}{\mathrm{d}x},$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2} = \frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\frac{\mathrm{d}x}{\mathrm{d}\vartheta}\frac{\mathrm{d}}{\mathrm{d}x}\right) = \left(\frac{\mathrm{d}x}{\mathrm{d}\vartheta}\right)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 x}{\mathrm{d}\vartheta^2}\frac{\mathrm{d}}{\mathrm{d}x} = \sin^2\vartheta\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \cos\vartheta\frac{\mathrm{d}}{\mathrm{d}x},$$

$$(1-x^2)\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}\Theta}{\mathrm{d}x} - \left(\mu + \frac{m^2}{1-x^2}\right)\Theta = 0. \tag{3.15}$$

Equation (3.15) is commonly referred to as general Legendre equation. To find the solution to this equation, we will first set the constant $m = 0$ and find a solution to the equation

$$(1 - x^2)\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}\Theta}{\mathrm{d}x} - \mu\Theta = 0, \tag{3.16}$$

which is usually called the Legendre equation. In the appendix A.1 we found the solution of the Legendre equation to be the Legendre polynomials $P_l(z)$ defined by the recurrent relation

$$\begin{gathered} P_l(x) = \sum_{m=0}^{l} a_m x^m, \quad a_{m+2} = \frac{(m+1)m - (l+1)l}{(m+2)(m+1)} a_m, \\ \mu = (l+1)l, \quad l \in \mathbb{N}. \end{gathered} \tag{3.17}$$

Any other solution can be written as a linear combination of $P_l(x)$ for various l. To have a uniquely defined set of Legendre polynomials we need to set the values of the first two coefficients. Standard values[2] of the first two coefficients are $a_0 = 1$ and $a_1 = 1$. The polynomials defined by equation (3.17) and the condition for the first two coefficients are the standard Legendre polynomials $P_l(x)$. Note that depending on the parity of l, the Legendre polynomials are either even or odd.

Now that we know that $\mu = (l+1)l$, we can solve the general Legendre equation (3.15) with the help of the associated Gegenbauer equation

$$(1 - x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - (2\alpha + 1)x\frac{\mathrm{d}y}{\mathrm{d}x} + \left[n(n + 2\alpha) - \frac{(j + 2\alpha - 1)j}{1 - x^2}\right] y = 0. \tag{3.18}$$

This equation and its solution are discussed in the appendix B.2. We can notice that the general Legendre equation is the special case of the associated Gegenbauer equation for $\alpha = \frac{1}{2}$ and $j = m$. The solution to the equation (3.18) are the associated Gegenbauer polynomials $C_n^{\alpha,j}(x)$ which are defined by the following relation

$$C_n^{\alpha,j}(x) = (1 - x^2)^{j/2}\frac{\mathrm{d}^j}{\mathrm{d}x^j}C_n^{\alpha}(x), \tag{3.19}$$

where $C_n^{\alpha}(x)$ are the Gegenbauer polynomials. Therefore, the associated Gegenbauer polynomials $C_l^{1/2,m}(x)$ are the solutions to the general Legendre equation (3.15).[3]

2. This convention is used, for example, in ABRAMOWITZ [5]
3. These solutions are also valid for negative integers m, because the associated Gegenbauer equation is invariant under the substitution $m = -m$ for $\alpha = \frac{1}{2}$.

3.1.3 General Chebyshev equation of the second kind

To solve the equation (3.13) we will proceed similarly to the previous section. We will take the substitution $x = \cos\varphi$ and substitute it into the equation (3.13). From the substitution, as in the previous section, it follows that

$$\sin\varphi = \sqrt{1-x^2},$$
$$\frac{\mathrm{d}}{\mathrm{d}\varphi} = \frac{\mathrm{d}x}{\mathrm{d}\varphi}\frac{\mathrm{d}}{\mathrm{d}x} = -\sin\varphi\frac{\mathrm{d}}{\mathrm{d}x},$$
$$\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2} = \frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{\mathrm{d}x}{\mathrm{d}\varphi}\frac{\mathrm{d}}{\mathrm{d}x}\right) = \left(\frac{\mathrm{d}x}{\mathrm{d}\varphi}\right)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2x}{\mathrm{d}\varphi^2}\frac{\mathrm{d}}{\mathrm{d}x} = \sin^2\varphi\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \cos\varphi\frac{\mathrm{d}}{\mathrm{d}x},$$
$$(1-x^2)\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} - 3s\frac{\mathrm{d}\Phi}{\mathrm{d}x} - \left(\lambda + \frac{(l+1)l}{1-x^2}\right)\Phi = 0. \tag{3.20}$$

We will call this equation the general Chebyshev equation of the second kind, because when $l = 0$ the equation (3.20) becomes the Chebyshev equation of the second kind

$$(1-s^2)\frac{\mathrm{d}^2\Phi}{\mathrm{d}s^2} - 3s\frac{\mathrm{d}\Phi}{\mathrm{d}s} - \lambda\Phi = 0. \tag{3.21}$$

As with the Legendre equation, we found the solution to the Chebyshev equation of the second kind (3.21) in the appendix A.2 to be Chebyshev polynomials of the second kind

$$\begin{gathered} U_n(s) = \sum_{k=0}^{n} b_k s^k, \quad b_{k+2} = \frac{(k+2)k - n(n+2)}{(k+2)(k+1)} b_k, \\ \lambda = -(n+2)n, \quad n \in \mathbb{N} \cup \{0\}. \end{gathered} \tag{3.22}$$

Standard Chebyshev polynomials of the second kind are defined by the value of the first two coefficients $a_0 = 1$ and $a_1 = 1$.[4] Now that we know that the eigenvalues of the Laplace operator are $\lambda = -n(n+2)$, we can solve the general equation (3.20).

Again, as in the previous section with the general Legendre equation, we can see that the general Chebyshev equation of the second kind (3.20) is also the special case of the associated Gegenbauer equation (3.18) for $\alpha = 1$ and $j = l$. As discussed in the appendix B.2, solutions to the associated Gegenbauer equation are the associated Gegenbauer polynomials $C_n^{\alpha,j}(x)$ and therefore solutions to the general Chebyshev equation of the second kind (3.20) are the associated Gegenbauer polynomials $C_n^{1,l}(x)$.

4. This standardization is from ABRAMOWITZ [5].

3.1.4 Hyperspherical harmonics

Now that we have solved the equations (3.11), (3.12) and (3.13) we have the set of eigenfunctions of the Laplace operator on the $\mathbb{S}^3$. Let us denote this set of eigenfunctions by H_{nlm} and call it the hyperspherical harmonics

$$H_{nlm}(\varphi, \vartheta, \zeta) = N_{nlm} C_n^{1,l}(\cos\varphi) C_l^{1/2,m}(\cos\vartheta) e^{im\zeta}, \tag{3.23}$$
$$N_{nlm} \in \mathbb{C}, \quad n \geq l \geq |m|, \tag{3.24}$$
$$\Delta H_{nlm} = -(n+2)nH_{nlm}. \tag{3.25}$$

We would like to find such a normalization constant N_{nlm} for which the hyperspherical harmonics H_{nlm} are normalized with respect to the inner product (integral over the hypersphere $\mathbb{S}^3$)

$$\langle H_{nlm}, H_{nlm} \rangle = \int_0^\pi \sin^2\varphi \mathrm{d}\varphi \int_0^\pi \sin\vartheta \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\zeta \overline{H_{nlm}} H_{nlm} = 1.$$

This integral can be broken down into three parts

$$\int_0^{2\pi} \mathrm{d}\zeta, \tag{3.26}$$
$$\int_0^\pi \left(C_l^{1/2,m}(\cos\vartheta)\right)^2 \sin\vartheta \mathrm{d}\vartheta, \tag{3.27}$$
$$\int_0^\pi \left(C_n^{1,l}(\cos\varphi)\right)^2 \sin^2\varphi \mathrm{d}\varphi. \tag{3.28}$$

The result of the first integral (3.26) is straightforward

$$\int_0^{2\pi} \mathrm{d}\zeta = 2\pi.$$

After the substitution $x = \cos\vartheta$ and $x = \cos\varphi$, respectively, the integrals (3.27) and (3.28) transform into

$$\int_{-1}^1 \left(C_l^{1/2,m}(x)\right)^2 \mathrm{d}x, \tag{3.29}$$
$$\int_{-1}^1 (1-x^2)^{1/2} \left(C_n^{1,l}(x)\right)^2 \mathrm{d}x. \tag{3.30}$$

Both of these integrals are special cases of the norm of the associated Gegenbauer polynomial

$$\int_{-1}^1 (1-x^2)^{\alpha-1/2} \left[C_n^{\alpha,j}(x)\right]^2 \mathrm{d}x = \frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha+j)}{(n-j)!(n+\alpha)\left[\Gamma(\alpha)\right]^2}, \tag{3.31}$$

which we derived in the appendix B.3. Let us apply the above result to the integrals (3.29) and (3.30)

$$\int_{-1}^{1} \left(C_l^{1/2,m}(x)\right)^2 \mathrm{d}x = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}, \tag{3.32}$$

$$\int_{-1}^{1} (1-x^2)^{1/2} \left(C_n^{1,l}(x)\right)^2 \mathrm{d}x = \frac{\pi}{2(n+1)}\frac{(n+l+1)!}{(n-l)!}. \tag{3.33}$$

In order for the hyperspherical harmonics H_{nlm} to be normalized, the normalization constant N_{nlm} must take the following form

$$N_{nlm} = \sqrt{\frac{(2l+1)(n+1)(l-m)!(n-l)!}{2\pi^2(l+m)!(n+l+1)!}} \cdot \sqrt{e^{i\xi}}, \tag{3.34}$$

where $\xi \in [0, 2\pi)$. For the rest of the thesis we have chosen $\xi = 0$.

In the appendix C we provide the reader with the table of hyperspherical harmonics H_{nlm} for the lowest values of the coefficient numbers n, l, and m. Using the stereographic projection, we were able to visualize some of the selected hyperspherical harmonics.

3.2 Decomposition into eigenfunctions

We would like to show that the set of hyperspherical functions H_{nlm} forms an orthogonal basis in the space of square-integrable functions on $\mathbb{S}^3$. This would allow us to use H_{nlm} as a basis for the generalized Fourier series. Therefore, we would be able to express any square-integrable function on $\mathbb{S}^3$ as a linear combination of the eigenfunctions of the Laplace operator.

Using results of Sturm–Liouville theory, we will show that the set of hyperspherical functions H_{nlm} in fact forms an orthogonal basis! We will not prove the results of the Sturm–Liouville theory but only refer to the first chapter of GERLACH [6] on linear mathematics in infinite dimensions.

3.2.1 Application of Sturm–Liouville theory

Recall the system of linear ordinary differential equations (3.11), (3.12) and (3.13) with their respective solutions $e^{im\zeta}$, $C_l^{1/2,m}(\vartheta)$ and $C_n^{1,l}(\varphi)$ on closed intervals $\zeta \in [0, 2\pi]$, $\vartheta \in [0, \pi]$ and $\varphi \in [0, \pi]$. All these equations can be rewritten into the form of Sturm–Liouville differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + q(x)y = -\lambda w(x)y. \tag{3.35}$$

If all the following conditions are met, we call this problem the regular Sturm–Liouville problem.

- Problem is restricted to a closed domain of definition $x \in [a, b]$.
- The coefficient functions $p(x)$, $\frac{\mathrm{d}p(x)}{\mathrm{d}x}$, $q(x)$ and $w(x)$ are continuous on the domain of definition $x \in [a, b]$.
- The coefficient functions $p(x) > 0$ and $w(x) > 0$ on the domain of definition $x \in [a, b]$.
- The endpoint conditions are of the mixed Dirichlet-Neumann type

$$\begin{aligned} \alpha y(a) + \alpha' y'(a) = 0, \qquad & \alpha^2 + \alpha'^2 \neq 0, \qquad & \alpha, \alpha' \in \mathbb{R}, \\ \beta y(b) + \beta' y'(b) = 0, \qquad & \beta^2 + \beta'^2 \neq 0, \qquad & \beta, \beta' \in \mathbb{R}. \end{aligned}$$

The Sturm–Liouville theory states that the solutions to the regular Sturm–Liouville problem form an orthogonal basis with respect to the inner product

$$\int_a^b w(x) y_n(x) y_m(x) \mathrm{d}x. \tag{3.36}$$

This basis on the space of the square-integrable functions consists of eigenfunctions of the Sturm–Liouville differential operator

$$L = -\frac{1}{w(x)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right] + q(x)\right). \tag{3.37}$$

Note that λ in the equation (3.35) is the eigenvalue of the differential operator L. Now we will show that the conditions for the regular Sturm–Liouville problem hold for our differential equations (3.11), (3.12), and (3.13), and therefore that their solutions (which we already know!) form an orthogonal basis.

The equation (3.11) is already in the form of Sturm–Liouville differential equation

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\zeta^2} = -m^2 Z$$

with coefficient functions $p(\zeta) = 1$, $q(\zeta) = 0$ and $w(\zeta) = 1$. Solutions to this equation (3.2.1) are already known

$$Z = e^{im\zeta}, \qquad m \in \mathbb{N}.$$

For all constants $\alpha = -im\alpha'$ and $\beta = -im\beta'$ the mixed Dirichlet-Neumann boundary conditions are always satisfied. From the results of the regular Sturm–Liouville problem it follows that the solutions of this equation are eigenfunctions of the Sturm–Liouville differential operator

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}$$

with the eigenvalues $\lambda_m = m^2, m \in \mathbb{Z}$. Thus these eigenfunctions form a basis in the space of square-integrable functions on $\zeta \in [0, 2\pi]$.

We already know that after the substitution $x = \cos\varphi$ and $x = \cos\vartheta$ the differential equations (3.12) and (3.13) transform into the special cases of the associated Gegenbauer equation (3.18). We can rewrite the associated Gegenbauer equation in the Sturm–Liouville form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)^{\alpha+\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}x}\right] - (j+2\alpha-1)j(1-x^2)^{\alpha-\frac{3}{2}}y = -n(n+2\alpha)(1-x^2)^{\alpha-\frac{1}{2}}y \tag{3.38}$$

with coefficient functions $p(x) = (1-x^2)^{\alpha+\frac{1}{2}}$, $q(x) = -(j+2\alpha-1)j(1-x^2)^{\alpha-\frac{3}{2}}$ and $w(x) = (1-x^2)^{\alpha-\frac{1}{2}}$. Solutions to this equation are the associated Gegenbauer polynomials

$$y = C_n^{\alpha,j}(x), \qquad n, j, 2\alpha \in \mathbb{N}.^5 \tag{3.39}$$

Let us discuss the boundary conditions for the case $j = 0$. When $j = 0$ then the associated Gegenbauer polynomials $C_n^{\alpha,0}(x)$ are equal to the Gegenbuaer polynomials $C_n^{\alpha}(x)$ and because the Gegenbauer polynomials are bounded on the interval $[-1, 1]$ we are guaranteed that there exist such constants α, α', β, and β' for which the boundary conditions are satisfied. For the case of $j \neq 0$ the boundary conditions are always satisfied because

$$C_n^{\alpha,j}(-1) = C_n^{\alpha,j}(-1) = 0.$$

Although the coefficient function $q(s)$ is not continuous at the boundary points $x = \pm 1$, the results of the regular Sturm–Liouville problem still hold, and therefore $C_n^{\alpha,j}(x)$ must be the eigenfunctions of the Sturm–Liouville differential operator

$$L = -\frac{1}{(1-x^2)^{\alpha-\frac{1}{2}}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)^{\alpha+\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}\right] - (j+2\alpha-1)j(1-x^2)^{\alpha-\frac{3}{2}}\right)$$

5. In special case $\alpha = \frac{1}{2}$ the coefficient j can take the negative integer values.

with eigenvalues $\lambda_n = n(n+2\alpha), n \in \mathbb{N} \cup \{0\}$. Therefore the associated Gegenbauer polynomials $C_n^{1,l}(\cos\varphi)$ and $C_n^{1/2,l}(\cos\vartheta)$ form a basis on the space of square-integrable functions on $\varphi \in [0,\pi]$ and $\vartheta \in [0,\pi]$ respectively.

Hyperspherical harmonics $H_{nlm}(\varphi,\vartheta,\zeta)$ are the product of $e^{im\zeta}$, $C_n^{1/2,l}(\cos\vartheta)$ and $C_n^{1,l}(\cos\varphi)$ normalized by a normalization constant N_{nlm}. Each of these three sets of functions forms an orthogonal basis on the space of square-integrable functions on their respective domains of definition, and therefore hyperspherical harmonics with N_{nlm} must also form an orthonormal basis on the space of square-integrable functions defined on the hypersphere!

3.2.2 Generalized Fourier series

Because eigenfunctions H_{nlm} of the Laplace operator are complete and orthonormal with respect to the inner product

$$\langle f, g\rangle = \int_0^\pi \sin^2\varphi \mathrm{d}\varphi \int_0^\pi \sin\vartheta \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\zeta \overline{f(\varphi,\vartheta,\zeta)} g(\varphi,\vartheta,\zeta). \tag{3.40}$$

we are able to express any square-integrable function $f(\varphi,\vartheta,\zeta)$ defined on $\mathbb{S}^3$ as generalized Fourier series

$$f(\varphi,\vartheta,\zeta) = \sum_{nlm} c_{nlm} H_{nlm}(\varphi,\vartheta,\zeta), \tag{3.41}$$

where c_{nlm} are the Fourier coefficients given by the relation

$$c_{nlm} = \langle H_{nlm}, f\rangle. \tag{3.42}$$

If we know the charge density function $\rho(\varphi,\vartheta,\zeta)$ we can solve the Poisson equation (2.19) for the electric potential $\phi(\varphi,\vartheta,\zeta)$. The equality

$$\langle H_{nlm}, \Delta\phi\rangle = \langle H_{nlm}, -\frac{\rho}{\varepsilon_0}\rangle$$

follows from the Poisson equation (2.19). Let us utilize the fact that the Laplace operator is self-adjoint (we have shown that in the equation (2.15))[6] and apply this property on the inner product of $\Delta\phi$ and H_{nlm}

$$\langle H_{nlm}, \Delta\phi\rangle = \langle \Delta H_{nlm}, \phi\rangle = \langle H_{nlm}, -\frac{\rho}{\varepsilon_0}\rangle.$$

6. We showed that the Laplace operator is a self-adjoint operator with respect to the inner product (2.10). The inner product (3.40) for functions that we use here is equivalent to the inner product (2.10) for 0-forms.

We also showed that H_{nlm} is the eigenfunction of the Laplace operator with the eigenvalue $\lambda = -(n+2)n$. Applying this to the previous equation we get

$$\langle \Delta H_{nlm}, \phi \rangle = -(n+2)n\langle H_{nlm}, \phi \rangle = \langle H_{nlm}, -\frac{\rho}{\varepsilon_0} \rangle,$$
$$\langle H_{nlm}, \phi \rangle = \frac{1}{(n+2)n}\langle H_{nlm}, \frac{\rho}{\varepsilon_0} \rangle. \tag{3.43}$$

Let us calculate Fourier series with respect to H_{nlm} of the electric potential ϕ from the Poisson equation (2.19)

$$\phi(\varphi, \vartheta, \zeta) = \sum_{nlm} \langle H_{nlm}, \phi \rangle H_{nlm}.$$

We can plug in the inner product from equation (3.43) and finally we arrive at the solution to the Poisson equation

$$\phi(\varphi, \vartheta, \zeta) = \sum_{nlm} \frac{1}{\varepsilon_0 (n+2)n} \langle H_{nlm}, \rho \rangle H_{nlm}. \tag{3.44}$$

One may notice that if the mean value of charge density ρ is not zero, then the Poisson equation (2.19) does not have a solution. This is because in the sum (3.44) we divide by n with the possible value of zero (we would like to avoid dividing by 0). If $n = 0$ then $H_{nlm} = 1$ and the inner product $\langle 1, \rho \rangle$ is equal to the mean value of ρ on $\mathbb{S}^3$ multiplied by the volume of the hypersphere ($2\pi^2$). If the mean value of ρ is zero, we can omit this term and avoid dividing by zero.

3.2.3 Numerical solutions

Now that we have found a solution to the Poisson equation we can compute the electrostatic potential ϕ for the charge density distribution using the Python script *sas.py* that is part of this thesis[7]. The script for a charge density function ρ computes a given number[8] of Fourier components $\langle H_{nlm}, \rho \rangle$ of the sum (3.44). Then we divide each coefficient by the appropriate eigenvalue $-n(n+2)$ and sum all the terms. Finally, we visualize the resulting electrostatic potential ϕ using the stereographic projection described earlier in 1.2.1.

For the first example, we will take

$$\rho_1 = 30A \sin\vartheta \cos\zeta - A\left(\varphi - \frac{\pi}{2}\right)^3, \qquad [A] = \mathrm{Cm}^{-3},$$

7. Brief description of all the functions of the script *sas.py* is in the appendix D.
8. We set this number by choosing the highest possible value for n. In all solutions, we used 7 as the highest possible number giving us 203 terms in the sum (3.44).

as a charge distribution function. The stereographic projection of this charge density function is in Figure 3.1. The stereographic projection of the electrostatic potential ϕ_1 multiplied by the electric permittivity ε_0 for this charge density ρ_1 can be seen in Figure 3.2.

For the second example, we chose the charge density function

$$\rho_2 = A\left(\varphi - \frac{\pi}{2} + \sin 2\vartheta\right), \qquad [A] = \text{ Cm}^{-3}.$$

Stereographic projection of this charge density function ρ_2 can be seen in Figure 3.3 and stereographical projection of the electrostatic potential ψ multiplied by ε_0 given by the charge density ρ_2 is shown in Figure 3.4.

The charge density function

$$\rho_3 = A\left(\cos\varphi + 2\cos\vartheta\right)\sin 2\varphi \sin\zeta, \qquad [A] = \text{ Cm}^{-3}$$

was chosen as the final example. The stereographic projection of the charge density ρ_3 is shown in Figure 3.5 and the stereographic projection of electrostatic potential ϕ_3 multiplied by ε_0 can be seen in Figure 3.6.

Note that for the purposes of the numerical solutions, we chose the radius of the hypersphere to be $r = 1$ m.

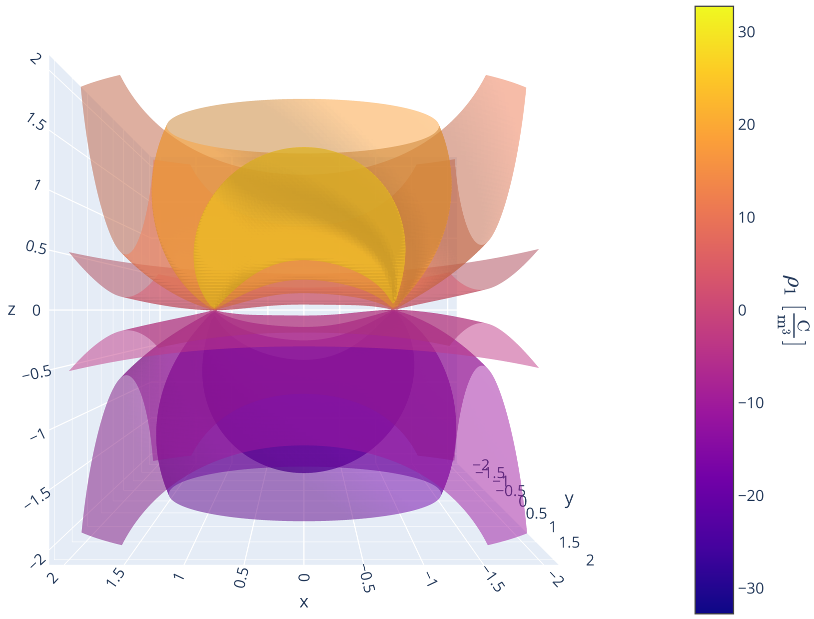

Figure 3.1: Example 1: stereographic projection of surfaces of constant charge density $\rho_1 = 30A \sin\vartheta \cos\zeta - A\left(\varphi - \frac{\pi}{2}\right)^3$.

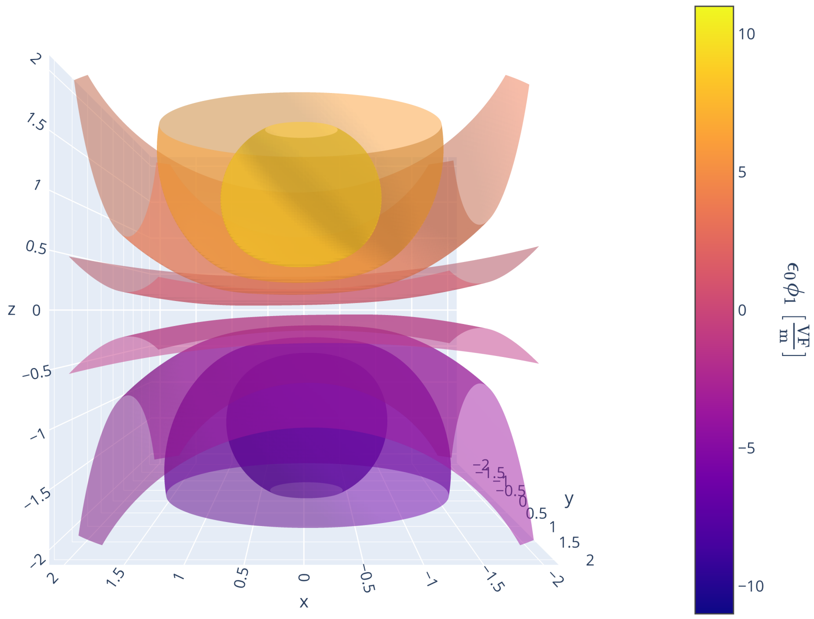

Figure 3.2: Example 1: stereographic projection of surfaces of constant value of the electric potential ϕ_1 multiplied by ε_0 given by the charge density $\rho_1 = 30\ A\ \sin\vartheta \cos\zeta - A\left(\varphi - \frac{\pi}{2}\right)^3$ calculated numerically.

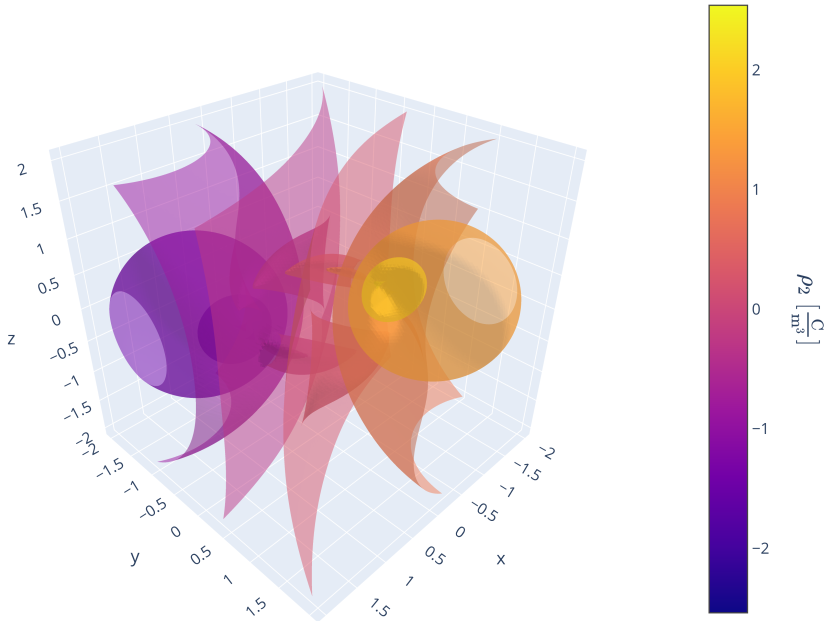

Figure 3.3: Example 2: stereographic projection of surfaces of constant charge density $\rho_2 = A\left(\varphi - \frac{\pi}{2} + \sin 2\vartheta\right)$.

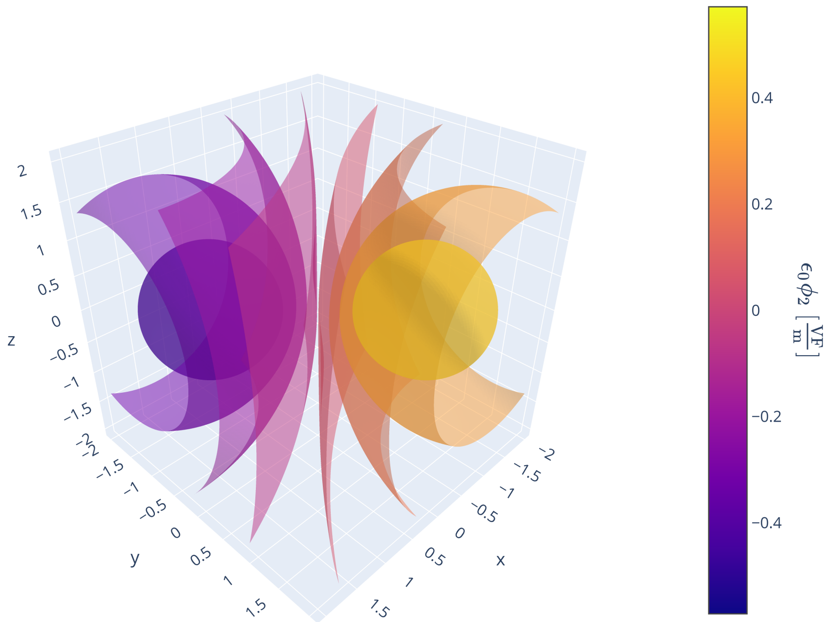

Figure 3.4: Example 2: stereographic projection of surfaces of constant value of the electric potential ϕ_2 multiplied by ε_0 given by the charge density $\rho_2 = A\ \left(\varphi - \frac{\pi}{2} + \sin 2\vartheta\right)$ calculated numerically.

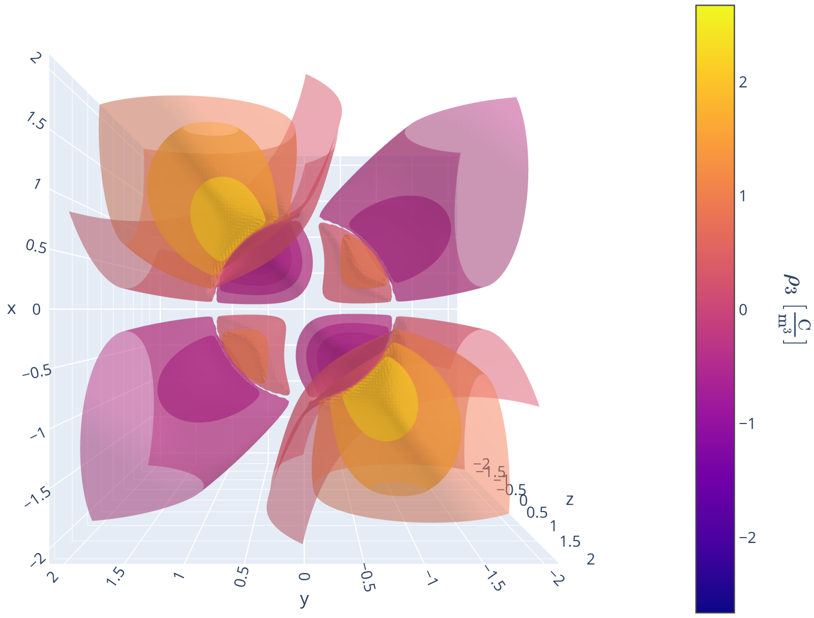

Figure 3.5: Example 3: stereographic projection of surfaces of constant charge density $\rho_3 = A\,(\cos\varphi + 2\cos\vartheta)\sin 2\varphi \sin\zeta$.

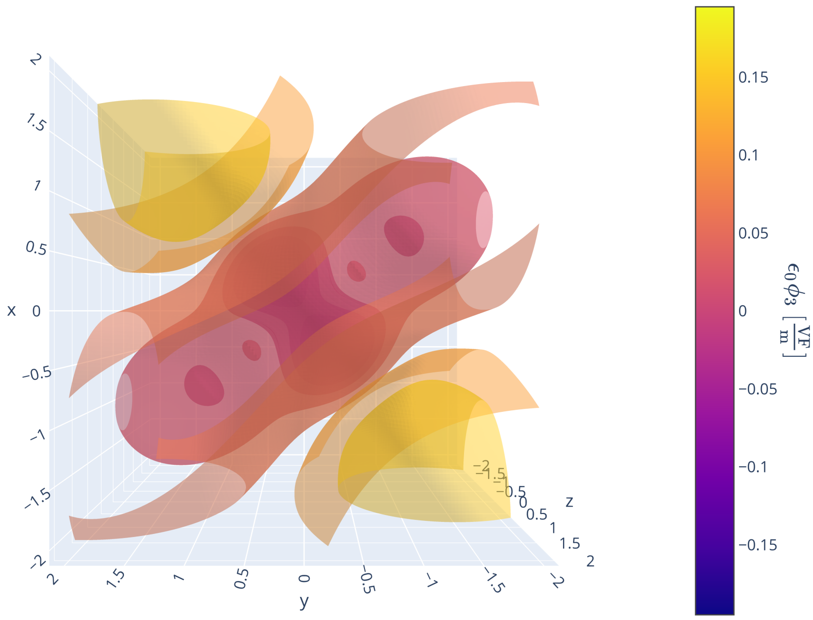

Figure 3.6: Example 3: stereographic projection of surfaces of constant value of the electric potential ϕ_3 multiplied by ε_0 given by the charge density $\rho_3 = A\,(\cos\varphi + 2\cos\vartheta)\sin 2\varphi \sin\zeta$ calculated numerically.

4 Magnetostatics

In the previous chapter, we described the electrostatics in the vacuum by two Maxwell's equations. The other two of Maxwell's equations describe the magnetostatics in the vacuum in $\mathbb{R}^3$ - Gauss's law for magnetism and Ampere's law

$$\nabla \cdot \mathbf{B} = 0, \tag{4.1}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \tag{4.2}$$

where $\mathbf{B}$ is the magnetic induction, $\mathbf{j}$ is the current desity and μ_0 is the vacuum permeability. Thanks to the fact that the divergence of the magnetic induction $\mathbf{B}$ is zero, we can express the magnetic induction $\mathbf{B}$ as a curl of the vector potential $\mathbf{A}$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{4.3}$$

This relation does not define the vector potential uniquely. Let us have two vector potentials $\mathbf{A}$ and $\mathbf{A}'$ that have the same curl and give the same magnetic induction $\mathbf{B}$

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \mathbf{A}'.$$

Therefore

$$\nabla \times \mathbf{A} - \nabla \times \mathbf{A}' = \nabla \times \left(\mathbf{A} - \mathbf{A}'\right).$$

From vector analysis, it follows that if the curl of the vector field (in our case the vector field is $\mathbf{A} - \mathbf{A}'$) is zero, then it must be the gradient of some scalar field ψ. That means that if the vector field $\mathbf{A}$ is satisfactory for some problem then

$$\mathbf{A}' = \mathbf{A} + \nabla\psi \tag{4.4}$$

will also be satisfactory and giving the same magnetic induction $\mathbf{B}$ as the vector potential $\mathbf{A}$.

Let us plug in the relation (4.3) into Ampere's law (4.2)

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \Delta\mathbf{A} = \mu_0 \mathbf{j}. \tag{4.5}$$

It is convenient to place some condition on the vector potential $\mathbf{A}$. In the case of magnetostatics, the natural choice is

$$\nabla \cdot \mathbf{A} = 0. \tag{4.6}$$

Two vector potentials $\mathbf{A}$ and $\mathbf{A}'$ which have the same curl (and therefore give the same magnetic induction $\mathbf{B}$) do not necessarily have to have the same divergence. Applying the divergence operator on the equation (4.4) we end up with

$$\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} + \Delta\psi.$$

With the suitable choice of the scalar field ψ we can make the vector potential $\nabla \cdot \mathbf{A}'$ equal anything we wish (within reason, of course), and therefore the choice of $\nabla \cdot \mathbf{A}' = 0$ is also possible. With this condition applied on the vector potential the term $\nabla (\nabla \cdot \mathbf{A})$ vanishes in the equation (4.5) and we end up with the Poisson equation for the vector potential

$$\Delta \mathbf{A} = -\mu_0 \mathbf{j}.$$

As in the previous chapter we will use the language of differential forms to generalize the Gauss's law for magnetism (4.1) and Amperes's law (4.2) to the hypersphere $\mathbb{S}^3$

$$\mathrm{d}B = 0, \qquad B \in \Omega^2(\mathbb{S}^3), \tag{4.7}$$
$$\mathrm{d}\star B = \mu_0 j, \qquad j \in \Omega^2(\mathbb{S}^3), \tag{4.8}$$

where B is a 2-form of magnetic induction and j is 2-form of current density. The relation between the 2-form of magnetic induction B and the 1-form magnetostatic potential $A \in \Omega^1(\mathbb{S}^3)$ follows from Gauss's law for magnetism

$$B = \mathrm{d}(A + \mathrm{d}f)\,. \tag{4.9}$$

The term $\mathrm{d}f$ (an exterior derivative of any function f) does not change the 2-form of the magnetic induction B and can thus be used to calibrate the 1-from potential. Let us choose the relation

$$\mathrm{d}^\dagger A = 0 \tag{4.10}$$

as the calibration condition. With the right choice of f, this relation can always be satisfied. Now let us plug the relation (4.9) into the Ampere's law (4.8)

$$\begin{aligned} \mathrm{d}\star\mathrm{d}A &= \mu_0 j, \\ \star\mathrm{d}\star\mathrm{d}A &= \star\mu_0 j. \end{aligned}$$

With our calibration relation (4.10) we can add the operator $-\mathrm{d}\mathrm{d}^\dagger = -\mathrm{d}\star\mathrm{d}\star$ on the left-hand side of the previous equation

$$\begin{aligned} (\star\mathrm{d}\star\mathrm{d} - \mathrm{d}\star\mathrm{d}\star)A &= \star\mu_0 j, \\ (-\star\mathrm{d}\star\mathrm{d} + \mathrm{d}\star\mathrm{d}\star)A &= -\star\mu_0 j, \\ \Delta A &= -\star\mu_0 j. \end{aligned} \tag{4.11}$$

And again we obtained the Poisson equation for the 1-form potential A with calibration relation for 1-form potential $\mathrm{d}^\dagger A = 0$.

We can express 1-form potential A and current density 2-form j in an orthonormal basis

$$\begin{aligned} A &= A_1(\varphi,\vartheta,\zeta)e^{\varphi} + A_2(\varphi,\vartheta,\zeta)e^{\vartheta} + A_3(\varphi,\vartheta,\zeta)e^{\zeta}, \\ j &= j_1(\varphi,\vartheta,\zeta)e^{\vartheta}\wedge e^{\zeta} + j_2(\varphi,\vartheta,\zeta)e^{\zeta}\wedge e^{\varphi} + j_3(\varphi,\vartheta,\zeta)e^{\varphi}\wedge e^{\vartheta}, \end{aligned} \tag{4.12}$$

and with together with the calibration condition (4.10) we can plug it into the Poisson equation (4.11)

$$\begin{aligned} &\left\{\frac{\partial}{\partial\vartheta}\left[\left(\frac{\partial\left(A_2\sin\varphi\right)}{\partial\varphi} - \frac{\partial A_1}{\partial\vartheta}\right)\sin\vartheta\right] + \right. \\ &\left. +\frac{\partial}{\partial\zeta}\left[\left(\frac{\partial\left(A_3\sin\varphi\sin\vartheta\right)}{\partial\varphi} - \frac{\partial A_1}{\partial\zeta}\right)\frac{1}{\sin\vartheta}\right]\right\}\frac{e^{\varphi}}{\sin^2\varphi\sin\vartheta} \\ &\left\{\frac{\partial}{\partial\zeta}\left[\left(\frac{\partial\left(A_3\sin\varphi\sin\vartheta\right)}{\partial\vartheta} - \frac{\partial\left(A_2\sin\varphi\right)}{\partial\zeta}\right)\frac{1}{\sin^2\varphi\sin\vartheta}\right] + \right. \\ &\left. +\frac{\partial}{\partial\varphi}\left[\left(\frac{\partial A_1}{\partial\vartheta} - \frac{\partial\left(A_2\sin\varphi\right)}{\partial\varphi}\right)\sin\vartheta\right]\right\}\frac{e^{\vartheta}}{\sin\varphi\sin\vartheta} \\ &\left\{\frac{\partial}{\partial\varphi}\left[\left(\frac{\partial A_1}{\partial\zeta} - \frac{\partial\left(A_3\sin\varphi\sin\vartheta\right)}{\partial\varphi}\right)\frac{1}{\sin\vartheta}\right] + \right. \\ &\left. +\frac{\partial}{\partial\vartheta}\left[\left(\frac{\partial\left(A_2\sin\varphi\right)}{\partial\zeta} - \frac{\partial\left(A_3\sin\varphi\sin\vartheta\right)}{\partial\vartheta}\right)\frac{1}{\sin^2\varphi\sin\vartheta}\right]\right\}\frac{e^{\zeta}}{\sin\varphi} = \mu_0 j_1 e^{\varphi} + j_2 e^{\vartheta} + j_3 e^{\zeta}. \end{aligned}$$

The above equation forms a considerably more complex problem than the Poisson equation for the scalar potential ϕ we solved in the previous chapter. Due to the complex nature of this problem, which we did not anticipate at the time of the thesis assignment, we did not solve the problem of magnetostatics on the hypersphere $\mathbb{S}^3$ in this thesis.

5 Conclusion

The main objective of this thesis was to find the solution for the Poisson equation of both scalar potential ϕ of electric intensity and vector potential $\mathbf{A}$ of magnetic induction in the static case.

The first chapter is dedicated to coordinate systems on the hypersphere $\mathbb{S}^3$. We derived the hyperspherical map α and stereographic projection β and their respective inverse maps.

For the solution of the Poisson equations, we needed to express the Laplace operator Δ in hyperspherical coordinates (which we chose as the natural coordinate system on the hypersphere $\mathbb{S}^3$). In the second chapter, we introduced mathematical tools for differential forms including Laplace–de Rham operator

$$\overline{\Delta} = \mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\mathrm{d}^\dagger$$

which is generally considered as the generalization of the Laplace operator Δ to differential forms. Unfortunately due to the lack of standardizations (or rather the large number of non-compatible standardizations) of definitions we had to verify that the definitions of the Laplace–de Rham operator $\overline{\Delta}$ and the Laplace operator Δ are compatible with each other. We found out that this is not the case and for our purposes (0-forms and 1-forms on $\mathbb{S}^3$) the relation between the Laplace operator and the Laplace–de Rham operator is

$$\Delta = -\overline{\Delta}.$$

With this relation in mind, we were able to express the Laplace–de Rham operator on the hypersphere $\mathbb{S}^3$ in hyperspherical coordinates for both 0-forms (functions) and 1-forms (vector fields). The reason mathematicians have a minus sign in their definition of the Laplace–de Rham operator is to make the Laplace–de Rham operator non-negative definite.

In solving the Poisson equation for the electric potential

$$\Delta\phi = -\frac{\rho}{\epsilon_0}$$

we assumed that the solution is separable and tried to find the eigenfunctions of the Laplace operator. In the third chapter, we found the sought eigenfunctions (and eigenvalues $\lambda = -(n+2)n$, $n \in \mathbb{N} \cup \{0\}$) and due to their resemblance to the spherical harmonics Y_l^m we call them the hyperspherical harmonics

$$H_{nlm}(\varphi,\vartheta,\zeta) = N_{nlm} C_n^{1,l}(\cos\varphi) C_l^{1/2,m}(\cos\vartheta) e^{im\zeta},$$
$$N_{nlm} = \sqrt{\frac{(2l+1)(n+1)(l-m)!(n-l)!}{2\pi^2(l+m)!(n+l+1)!}}.$$

Thanks to the fact that the Laplace operator is self-adjoint we were able to express the electrostatic potential ϕ as a generalized Fourier series (with the hyperspherical harmonics H_{nlm} as an orthonormal basis) of the charge density ρ divided by the corresponding eigenvalue and vacuum permittivity ε_0

$$\phi(\varphi,\vartheta,\zeta) = \sum_{nlm} \frac{1}{\varepsilon_0(n+2)n} \langle H_{nlm}, \rho \rangle H_{nlm}.$$

Unfortunately, this method constrains us only to the charge density functions ρ whose mean value over the hypersphere $\mathbb{S}^3$ is zero.

This result, allowed us to numerically calculate the electrostatic potential ϕ for some charge densities ρ by taking a finite number of terms from the solution expressed by the generalized Fourier series. With the use of the stereographic projection β (1.6) we were able to visualize the electrostatic potential ϕ in Figures 3.2, 3.4 and 3.6 for some charge densities ρ.

In the last chapter, we derived the Poisson equation for the 1-form magnetostatic potential $\mathbf{A}$. The complex form of the series of differential equations resulting from the Poisson equation proved to be a significant challenge and we do not provide the solution for this problem.

Summary

The thesis leaves the problem of magnetostatics open. This problem could serve as a basis for further work building on our findings - mainly the hyperspherical harmonics - which we believe may appear in some form in the solution of the Poisson equation for the vector potential $\mathbf{A}$.

The main result of this thesis is the defining relation of the hyperspherical harmonics H_{nlm}. In the course of writing this thesis, we discovered that this problem has been addressed before. In chapter 10 of TALMAN [7] the spherical harmonics in four dimensions are defined with the use of Gegenbauer polynomials (as in this thesis). A different approach is offered in the article FOLLAND [8] in which the Laplacian for n-dimensional hypersphere $\mathbb{S}^n$ is calculated.

A Frobenius method

If we do not worry about the technical details of the Frobenius method, we can say that it is a technique for solving linear second-order differential equations by assuming the solution in the form of a power series

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n, \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \sum_{n=1}^{\infty} n a_n x^{n-1}, \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.
\end{aligned} \tag{A.1}
$$

In our efforts to find the eigenfunctions of the Laplace operator on the hypersphere $\mathbb{S}^3$ we encountered two differential equations that can be solved by the Frobenius method - Legendre equation and Chebyshev equation of the second kind

$$
\text{Legendre equation:} \qquad (1-z^2)\frac{\mathrm{d}^2\Theta}{\mathrm{d}z^2} - 2z\frac{\mathrm{d}\Theta}{\mathrm{d}z} - \mu\Theta = 0, \tag{A.2}
$$

$$
\text{Chebyshev equation of the 2}^{\text{nd}}\text{ kind:} \qquad (1-s^2)\frac{\mathrm{d}^2\Phi}{\mathrm{d}s^2} - 3s\frac{\mathrm{d}\Phi}{\mathrm{d}s} - \lambda\Phi = 0. \tag{A.3}
$$

A.1 Legendre equation

To find a solution of the Legendre equation $\Theta(z)$ on our domain of definition $[-1, 1]$ (because these are the possible values of $z = \cos\vartheta \in [-1, 1]$) we can plug our assumed solution of the power series (A.1) into the equation (A.2)

$$
\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} z^n - \sum_{m=2}^{\infty} m(m-1) a_m z^m - \sum_{m=1}^{\infty} 2m a_m z^m - \sum_{m=0}^{\infty} \mu a_m z^m = 0,
$$

$$
\begin{aligned}
&2a_2 - \mu a_0 + (6a_3 - 2a_1 - \mu a_1)z + \\
&\qquad + \sum_{m=2}^{\infty} \{(m+2)(m+1) a_{m+2} - [(m+1)m + \mu]\, a_m\} z^m = 0.
\end{aligned}
$$

This equation gives us the recurrent relation for the coefficients of the power series

$$
a_{m+2} = \frac{(m+1)m + \mu}{(m+2)(m+1)} a_m, \quad m \in \mathbb{N} \cup \{0\}
$$

This recurrent relation defines the Legendre polynomials $P_l(z)$ (up to the normalization constant) which are the solution to the Legendre equation (A.2). We demand convergence on our domain of definition $z \in [-1,1]$. The series converges on the open interval $(-1,1)$. For the power series to also converge on points $z = \pm 1$, it must have a finite number of terms - it must be a polynomial. This can be achieved if the numerator $(m+1)m - \mu$ in the equation (A.2) is zero for some m and therefore $\mu = -(l+1)l$, where $l \in \mathbb{N} \cup \{0\}$.

In conclusion, the solution of the Legendre equation (A.2) are the Legendre polynomials

$$\Theta(z) = P_l(z) = \sum_{m=0}^{l} a_m z^m, \quad a_{m+2} = \frac{(m+1)m - (l+1)l}{(m+2)(m+1)} a_m. \tag{A.4}$$

A.2 Chebyshev equation of the second kind

To find the solution of the equation (A.3) $\Phi(s)$ on the domain $s \in [-1,1]$ we will apply the identical process as for the Legendre equation. Let us plug the assumed solution into the Chebyshev equation of the second kind (A.3)

$$\sum_{k=0}^{\infty}(k+2)(k+1)b_{k+2}s^n - \sum_{k=2}^{\infty}k(k-1)b_k s^k - \sum_{k=1}^{\infty}3kb_k s^k - \sum_{k=0}^{\infty}\lambda b_k s^n = 0,$$

$$2b_2 - \lambda b_0 + (6b_3 - 3b_1 - \lambda b_1)s + \\ + \sum_{k=2}^{\infty}\{(k+2)(k+1)b_{k+2} - [(k+2)k + \lambda]\, b_k\}s^k = 0.$$

Again we arrive at the recurrent relation for coefficients

$$b_{k+2} = \frac{(k+2)k + \lambda}{(k+2)(k+1)} b_k, \quad k \in \mathbb{N} \cup \{0\}.$$

This time the recurrent relation defines the Chebyshev polynomials of the second kind $U_n(s)$. For the series to converge on the whole domain of definition $s \in [-1,1]$ the numerator in the recurrent relation must be equal to zero for some k. From this condition we arrive at $\lambda = -(n+2)n, n \in \mathbb{N} \cup \{0\}$.

We found the solution of the Chebyshev equation of the second kind (A.3) to be the Chebyshev polynomials of the second kind $U_n(s)$

$$\Phi(s) = U_n(s) = \sum_{k=0}^{n} b_k s^k, \quad b_{k+2} = \frac{(k+2)k - (n+2)n}{(k+2)(k+1)} b_k \tag{A.5}$$

B Hypergeometric function and Gegenbauer polynomials

All definitions in this section are taken from [5].

B.1 Differentiating as parameter shifting

We would like to show the connection between the hypergeometric function

$$F(a,b,c,z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}, \quad (\beta)_m = \beta(\beta+1)\ldots(\beta+m-1)^1, \tag{B.1}$$

and differentiation of Gegenabuer polynomials $C_n^\alpha(x)$ (which are generalizations of both Legendre and Chebyshev polynomials). The hypergeometric function is a solution to the hypergeometric differential equation

$$z(1-z)\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + [c-(a+b+1)z]\,\frac{\mathrm{d}y}{\mathrm{d}z} - aby = 0. \tag{B.2}$$

The hypergeometric function can be used to express Gegenbauer polynomials

$$C_n^\alpha(x) = \frac{(n+2\alpha-1)!}{n!(2\alpha-1)!} F(-n, n+2\alpha, \alpha+\frac{1}{2}, \frac{1-x}{2}), \tag{B.3}$$

where α is integer multiple of $\frac{1}{2}$ and $n \in \mathbb{N}$. This can be shown by plugging in the corresponding parameters ($a = -n$, $b = n+2\alpha$, $c = \alpha + \frac{1}{2}$) and variable ($z = \frac{1-x}{2}$) to the hypergeometric differential equation (B.2). This will transform the hypergeometric differential equation into the Gegenbauer differential equation

$$(1-x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - (2\alpha+1)x\frac{\mathrm{d}y}{\mathrm{d}x} + (n+2\alpha)ny = 0. \tag{B.4}$$

Its solutions are the Gegenbauer polynomials $C_n^\alpha(x)$. If the coefficient $\alpha = \frac{1}{2}$, then the Gegenbauer differential equation (B.4) becomes the associated Legendre equation (3.16) and for $\alpha = 1$, the Gegenbauer differential equation becomes the Chebyshev equation of the second kind (3.21).

---

1. This product is commonly referred to as rising factorial or Pochhammer function.

Let us differentiate the Gegenbauer differential equation (B.4) j-times term by term

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j}\left[(1-x^2)y^{(2)}\right] = (1-x^2)y^{(j+2)} - 2jxy^{(j+1)} - 2j(j-1)y^{(j)}, \tag{B.5}$$

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j}\left[-(2\alpha+1)xy^{(j)}\right] = -(2\alpha+1)xy^{(j+1)} - j(2\alpha+1)y^{(j)}, \tag{B.6}$$

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j}\left[(n+2\alpha)ny\right] = (n+2\alpha)ny^{(j)}. \tag{B.7}$$

After differentiation and slight readjustment, we arrive at the equation

$$(1-x^2)\frac{\mathrm{d}^2y^{(j)}}{\mathrm{d}x^2} - (2j+2\alpha+1)x\frac{\mathrm{d}y^{(j)}}{\mathrm{d}x} + (n-j)(n+j+2\alpha)y^{(j)} = 0 \tag{B.8}$$

which on closer inspection is the same differential equation as the Gegenbauer differential equation (B.4) for $C_{n-j}^{\alpha+j}(x)$. This means that there is a relation

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j}\left[C_n^{\alpha}(x)\right] = K_j^{\alpha}C_{n-j}^{\alpha+j}(x), \quad K_j^{\alpha} \in \mathbb{R} \tag{B.9}$$

between j times differentiated Gegenbauer polynomial $C_n^{\alpha}(x)$ and $C_{n-j}^{\alpha+j}(x)$ because they are both solutions to the same differential equation.

To find the proportionality constant K_j^{α} we will make use of the hypergeometric function (B.1) and its relation with Gegenbeuaer polynomials (B.3)

$$\begin{aligned}
\frac{\mathrm{d}^j}{\mathrm{d}x^j}\left[C_n^{\alpha}(x)\right]\bigg|_{x=1} &= \frac{\mathrm{d}^j}{\mathrm{d}x^j}\left[\frac{(n+2\alpha-1)!}{n!(2\alpha-1)!}F(-n, n+2\alpha, \alpha+\frac{1}{2}, \frac{1-x}{2})\right]\bigg|_{x=1} \\
&= \frac{(n+2\alpha-1)!}{n!(2\alpha-1)!}\sum_{m=0}^{\infty}\frac{(-n)_m(n+2\alpha)_m}{m!(\alpha+\frac{1}{2})_m}\frac{\mathrm{d}^j}{\mathrm{d}x^j}\left[\left(\frac{1-x}{2}\right)^m\right]\bigg|_{x=1} \\
&= \frac{(n+2\alpha-1)!}{n!(2\alpha-1)!}\frac{(-n)_j(n+2\alpha)_j}{j!(\alpha+\frac{1}{2})_j}j!\left(-\frac{1}{2}\right)^j
\end{aligned}$$

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j}\left[C_n^{\alpha}(x)\right]\bigg|_{x=1} = \frac{(n+2\alpha+j-1)!}{(n-j)!(2\alpha-1)!(\alpha+\frac{1}{2})_j 2^j}, \tag{B.10}$$

$$C_{n-j}^{\alpha+j}(1) = \frac{(n+j+2\alpha-1)!}{(n-j)!(2\alpha+2j-1)}. \tag{B.11}$$

We made our work easier by quantifying the expressions at $x = 1$, where all of the terms in the hypergeometric function vanish except for one. By comparing equations (B.10) and (B.11) we can see that the proportionality constant is in the form

$$K_j^\alpha = \frac{(2\alpha + 2j - 1)!}{2^j (2\alpha - 1)! (\alpha + \frac{1}{2})_j} = \frac{(2\alpha)_{2j}}{2^j (\alpha + \frac{1}{2})_j} = 2^j (\alpha)_j \tag{B.12}$$

and the relation (B.9) will take the following form

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j} \left[C_n^\alpha(x) \right] = K_j^\alpha C_{n-j}^{\alpha+j}(x) = 2^j (\alpha)_j C_{n-j}^{\alpha+j}(x). \tag{B.13}$$

B.2 Associated Gegenbauer equation

Let us call the equation

$$(1 - x^2) \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - (2\alpha + 1) x \frac{\mathrm{d}y}{\mathrm{d}x} + \left[n(n + 2\alpha) - \frac{(j + 2\alpha - 1)j}{1 - x^2} \right] y = 0 \tag{B.14}$$

the associated Gegenbauer equation. We will now prove that the associated Gegenbauer polynomials

$$C_n^{\alpha,j}(x) = (1 - x^2)^{j/2} \frac{\mathrm{d}^j}{\mathrm{d}x^j} C_n^\alpha(x) \tag{B.15}$$

are solutions to this equation.

First, we will differentiate the Gegenbauer differential equation (B.4) m-times. Thankfully we have already done that in the previous chapter and the result is written down as equation (B.8). Let us make a substitution $y^{(j)} = u$ and plug it into the equation (B.8)

$$(1 - z^2) u'' - (2l + 2\alpha + 1) x u' + (n - l)(n + l + 2\alpha) u = 0. \tag{B.16}$$

Instead of u, we can substitute $u = v(1 - x^2)^{-l/2}$. Before we plug in our substitution, we will calculate u' and u''

$$u' = (1 - x^2)^{-l/2} \left(v' + \frac{jx}{1 - x^2} v \right), \tag{B.17}$$

$$u'' = (1 - x^2)^{-l/2} \left[v'' + \frac{2jx}{1 - x^2} v' + \frac{j}{1 - x^2} v + \frac{j(j + 2)x^2}{(1 - x^2)^2} v \right]. \tag{B.18}$$

Finally, after we plug everything in and multiply the equation by $(1 - x^2)^{l/2}$, we arrive at the associated Gegenbauer equation

$$(1 - x^2) v'' - (2\alpha + 1) x v' + \left[(n + 2)n - \frac{(j + 2\alpha - 1)j}{1 - x^2} \right] v = 0.$$

It can be seen that $v = (1-x^2)^{j/2} u = C_n^{\alpha,j}(x)$ are solutions of the associated Gegenbauer equation (B.14).

B.3 Normalization of associated Gegenbauer polynomials

The norm[2] of the Gegenbauer polynomials with respect to the weight function $w(x) = (1-x^2)^{\alpha-1/2}$ is

$$\int_{-1}^{1} (1-x^2)^{\alpha-1/2} \left[C_n^{\alpha}(x)\right]^2 \mathrm{d}x = \frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(n+\alpha)\left[\Gamma(\alpha)\right]^2}. \tag{B.19}$$

To find the norm of the associated Gegenbauer polynomials $C_n^{\alpha,j}(x)$ we will make use of the formula (B.13) for differentiation of the Gegenbauer polynomials. Let us write down the expression for the norm of the associated Gegenbauer polynomials and apply the formula (B.13) on it

$$\begin{aligned} &\int_{-1}^{1} (1-x^2)^{\alpha+j-1/2} \left[\frac{\mathrm{d}^j}{\mathrm{d}x^j} C_n^{\alpha}(x)\right]^2 \mathrm{d}x = \\ &= 2^{2j}\left[(\alpha)_j\right]^2 \int_{-1}^{1} (1-x^2)^{\alpha+j-1/2} \left[C_{n-j}^{\alpha+j}(x)\right]^2 \mathrm{d}x. \end{aligned} \tag{B.20}$$

We can see that the integral on the left is equal to the norm of the Gegenbauer polynomial $C_{n-j}^{\alpha+j}(x)$. Using this result, we can write down the formula for the norm of the associated Gegenbauer polynomials.

$$\int_{-1}^{1} (1-x^2)^{\alpha-1/2} \left[C_n^{\alpha,j}(x)\right]^2 \mathrm{d}x = 2^{2j}\left[(\alpha)_j\right]^2 \frac{\pi 2^{1-2(\alpha+j)}\Gamma(n+2\alpha+j)}{(n-j)!(n+\alpha)\left[\Gamma(\alpha+j)\right]^2}. \tag{B.21}$$

The rising factorial can be expressed by a Gamma function

$$(\alpha)_j = \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)} \tag{B.22}$$

and after plugging this expression in, we finally arrive at the norm of the associated Gegenbauer polynomials

$$\int_{-1}^{1} (1-x^2)^{\alpha-1/2} \left[C_n^{\alpha,j}(x)\right]^2 \mathrm{d}x = \frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha+j)}{(n-j)!(n+\alpha)\left[\Gamma(\alpha)\right]^2}. \tag{B.23}$$

2. This relation for the norm is from ABRAMOWITZ [5]

C Hyperspherical harmonics H_{nlm}

We have defined the hyperspherical harmonics as

$$H_{nlm}(\varphi,\vartheta,\zeta)=\sqrt{\frac{(2l+1)(n+1)(l-m)!(n-l)!}{2\pi^2(l+m)!(n+l+1)!}}C_n^{1,l}(\cos\varphi)C_l^{1/2,m}(\cos\vartheta)e^{im\zeta}, \tag{C.1}$$

where $C_n^{\alpha,l}$ are the associated Gegenbauer polynomials defined by the relation (B.15). The first few hyperspherical harmonics H_{nlm} for the lowest coefficients can be seen in the Table C.1 on the next page.

We used stereographic projection (see the section 1.2) on some selected hyperspherical harmonics H_{nlm} to visualize them. The selected hyperspherical harmonics that are visualized in the following figures are of two types - hyperspherical harmonics of low coefficient numbers n, l, m and those hyperspherical harmonics whose stereographic projection is subjectively (for the author) interesting. Note that we projected only the real part of the function for those hyperspherical harmonics H_{nlm} which are complex-valued ($m \neq 0$).

The case of $n, l, m = 0, 0, 0$ is trivial (as seen in the Table C.1) and there is no point in generating a graph of isosurfaces for constant function. By taking a look at the Table C.1 one can convince oneself that the real part of the hyperspherical harmonics which differ only by a sign of the coefficient number m are the same functions (differing at most by a minus sign). That is why we did not use the hyperspherical harmonics with a negative value of m for the stereographic projection. Selected stereographic projections of low coefficient number hyperspherical harmonics H_{nlm} are depicted in the Figures C.1, C.2, C.3, C.4 and C.5.

The Figures C.6, C.7 and C.8 show the visually interesting (at least for the author) stereographic projections of the hyperspherical harmonics

$$\begin{aligned}
H_{10,5,5} = &\quad \frac{2\sqrt{462}\left(24\cos^4\varphi-12\cos^2\varphi+1\right)\sin^5\varphi\sin^5\vartheta\cos\varphi e^{5i\zeta}}{\sqrt{13}\pi},\\
H_{12,6,0} = &\quad -2\sqrt{26}\cdot\frac{315\sin^4\vartheta-525\sin^2\vartheta-231\cos^6\vartheta+215}{\sqrt{3553}\pi}\cdot\\
&\qquad\cdot\frac{\left(704\cos^6\varphi-440\cos^4\varphi+60\cos^2\varphi-1\right)\sin^6\varphi}{\sqrt{3553}\pi},\\
H_{15,15,15} = &\quad \frac{2\sqrt{2}\sin^{15}\varphi\sin^{15}\vartheta e^{15i\zeta}}{\pi}.
\end{aligned}$$

There are many more hyperspherical harmonics H_{nlm} whose stereographic projections are fascinating, unfortunately we have not discovered them all and we do not have enough space to show all of those we have found.

Table C.1: The first hyperspherical harmonics H_{nlm} for the lowest coefficients n, l, m

n	l	m	$H_{nlm}(\varphi, \vartheta, \zeta)$	n	l	m	$H_{nlm}(\varphi, \vartheta, \zeta)$
0	0	0	$\frac{1}{\sqrt{2}\pi}$	3	1	-1	$-\frac{\left(12\cos^2\varphi - 2\right)\sin\varphi\sin\vartheta e^{-i\zeta}}{\sqrt{5}\pi}$
1	0	0	$\frac{\sqrt{2}\cos\varphi}{\pi}$	3	1	0	$\frac{\left(24\cos^2\varphi - 4\right)\sin\varphi\cos\vartheta}{\sqrt{10}\pi}$
1	1	-1	$-\frac{\sin\varphi\sin\vartheta e^{-i\zeta}}{\pi}$	3	1	1	$\frac{\left(12\cos^2\varphi - 2\right)\sin\varphi\sin\vartheta e^{i\zeta}}{\sqrt{5}\pi}$
1	1	0	$\frac{\sqrt{2}\sin\varphi\cos\vartheta}{\pi}$	3	2	-2	$\frac{2\sqrt{3}\sin^2\varphi\sin^2\vartheta\cos\varphi e^{-2i\zeta}}{\pi}$
1	1	1	$\frac{\sin\varphi\sin\vartheta e^{i\zeta}}{\pi}$	3	2	-1	$-\frac{4\sqrt{3}\sin^2\varphi\cos\varphi\cos\vartheta\sin\vartheta e^{-i\zeta}}{\pi}$
2	0	0	$\frac{4\cos^2\varphi - 1}{\sqrt{2}\pi}$	3	2	0	$\frac{\sqrt{2}\left(4 - 6\sin^2\vartheta\right)\sin^2\varphi\cos\varphi}{\pi}$
2	1	-1	$-\frac{\sqrt{6}\cos\varphi\sin\varphi\sin\vartheta e^{-i\zeta}}{\pi}$	3	2	1	$\frac{4\sqrt{3}\sin^2\varphi\cos\varphi\cos\vartheta\sin\vartheta e^{i\zeta}}{\pi}$
2	1	0	$\frac{2\sqrt{3}\cos\varphi\sin\varphi\cos\vartheta}{\pi}$	3	2	2	$\frac{2\sqrt{3}\sin^2\varphi\sin^2\vartheta\cos\varphi e^{2i\zeta}}{\pi}$
2	1	1	$\frac{\sqrt{6}\cos\varphi\sin\varphi\sin\vartheta e^{i\zeta}}{\pi}$	3	3	-3	$-\frac{\sqrt{2}\sin^2\varphi\sin^2\vartheta\sin\varphi\sin\vartheta e^{-3i\zeta}}{\pi}$
2	2	-2	$\frac{\sqrt{6}\sin^2\varphi\sin^2\vartheta e^{-2i\zeta}}{2\pi}$	3	3	-2	$\frac{2\sqrt{3}\sin^2\varphi\sin^2\vartheta\sin\varphi\cos\vartheta e^{-2i\zeta}}{\pi}$
2	2	-1	$-\frac{\sqrt{6}\sin^2\varphi\cos\vartheta\sin\vartheta e^{-i\zeta}}{\pi}$	3	3	-1	$\frac{\sqrt{6}\left(5\sin^2\vartheta - 4\right)\sin^2\varphi\sin\varphi\sin\vartheta e^{-i\zeta}}{\sqrt{5}\pi}$
2	2	0	$\frac{\left(2 - 3\sin^2\vartheta\right)\sin^2\varphi}{\pi}$	3	3	0	$\frac{\left(3\cos\vartheta + 5\cos 3\vartheta\right)\sin^2\varphi\sin\varphi}{\sqrt{10}\pi}$
2	2	1	$\frac{\sqrt{6}\sin^2\varphi\cos\vartheta\sin\vartheta e^{i\zeta}}{\pi}$	3	3	1	$\frac{\sqrt{6}\left(4 - 5\sin^2\vartheta\right)\sin^2\varphi\sin\varphi\sin\vartheta e^{i\zeta}}{\sqrt{5}\pi}$
2	2	2	$\frac{\sqrt{6}\sin^2\varphi\sin^2\vartheta e^{2i\zeta}}{2\pi}$	3	3	2	$\frac{2\sqrt{3}\sin^2\varphi\sin^2\vartheta\sin\varphi\cos\vartheta e^{2i\zeta}}{\pi}$
3	0	0	$\frac{8\cos^3\varphi - 4\cos\varphi}{\sqrt{2}\pi}$	3	3	3	$\frac{\sqrt{2}\sin^2\varphi\sin^2\vartheta\sin\varphi\sin\vartheta e^{3i\zeta}}{\pi}$

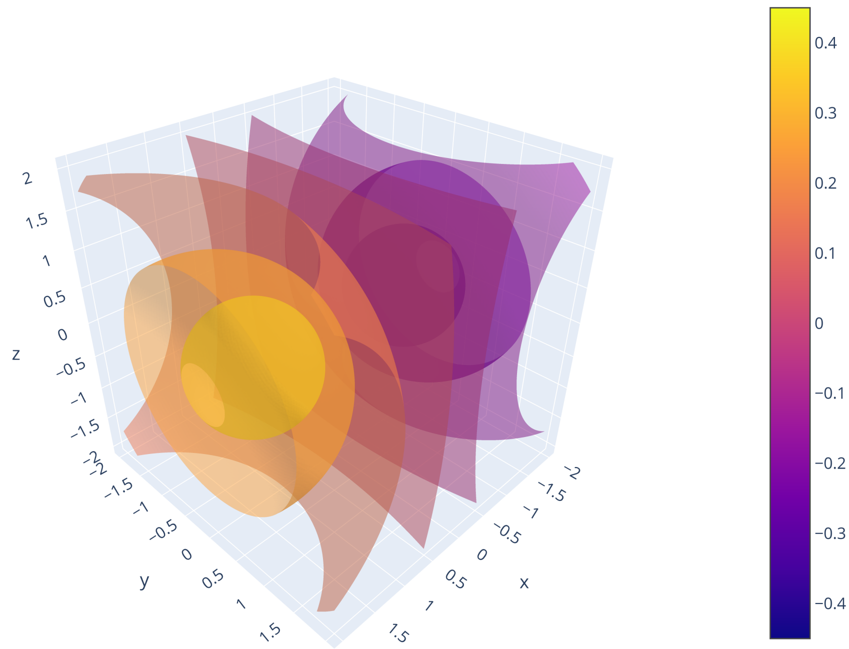

Figure C.1: Stereographic projection of surfaces of constant value of the hyperspherical harmonic $H_{1,0,0}$

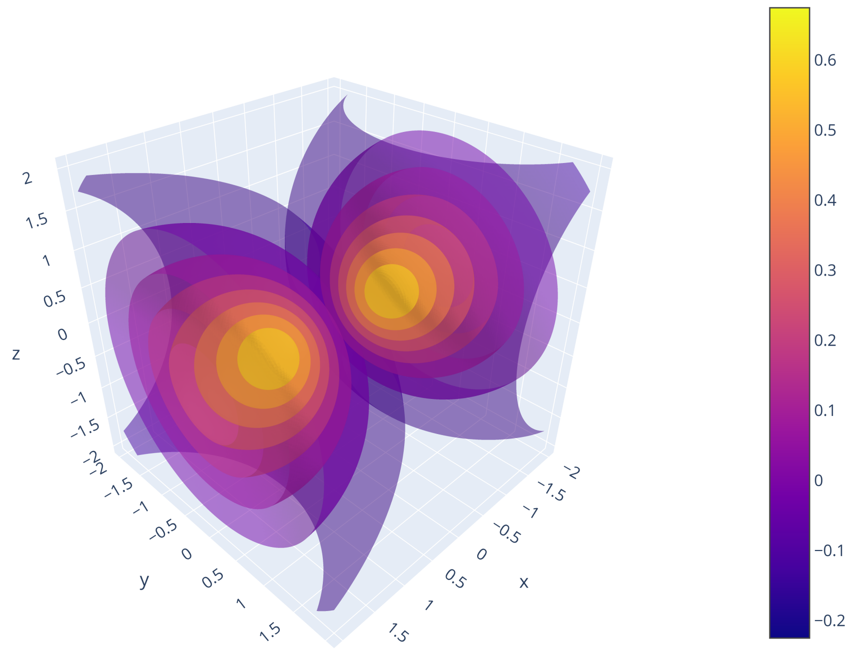

Figure C.2: Stereographic projection of surfaces of constant value of the hyperspherical harmonic $H_{2,0,0}$

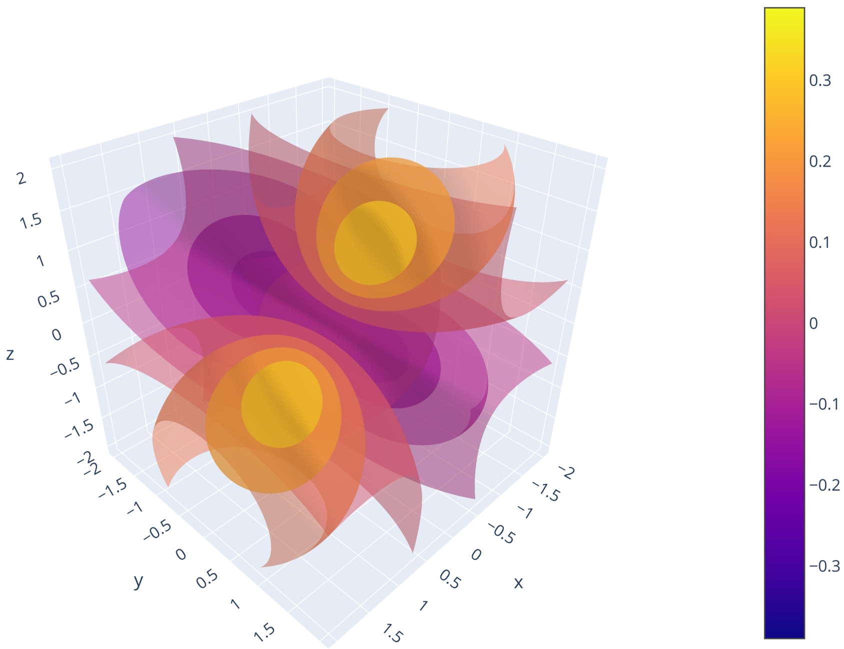

Figure C.3: Stereographic projection of surfaces of constant value of the hyperspherical harmonic $H_{2,1,1}$

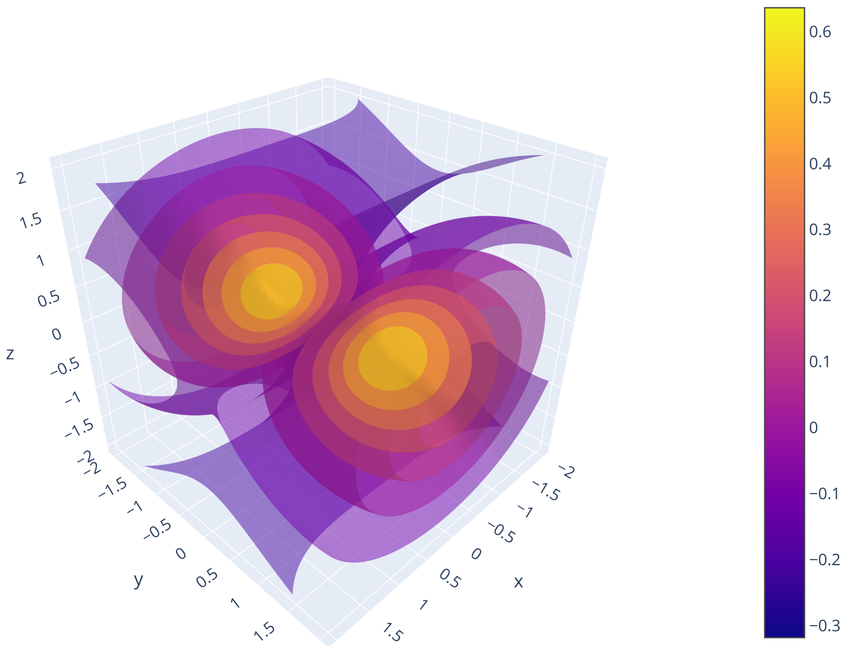

Figure C.4: Stereographic projection of surfaces of constant value of the hyperspherical harmonic $H_{2,2,0}$

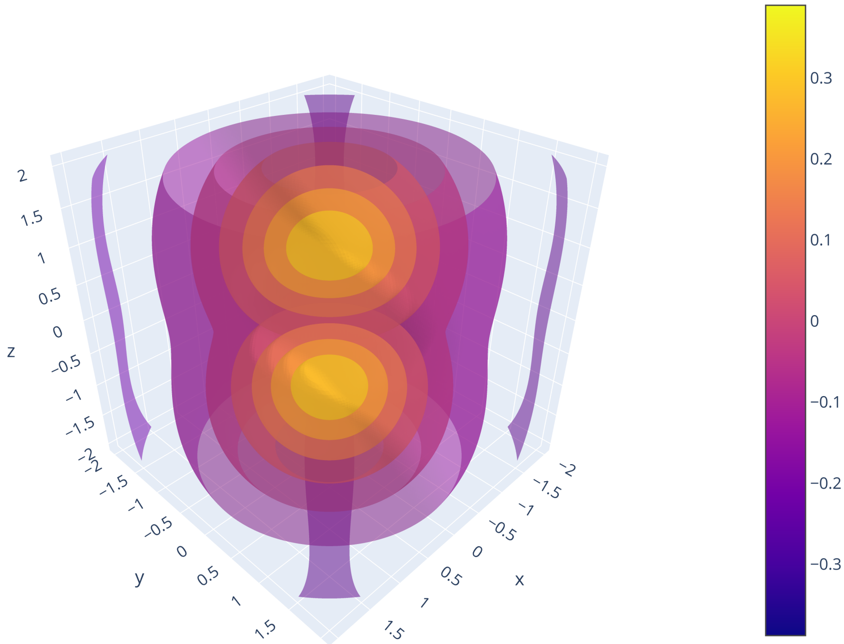

Figure C.5: Stereographic projection of surfaces of constant value of the hyperspherical harmonic $H_{2,2,2}$

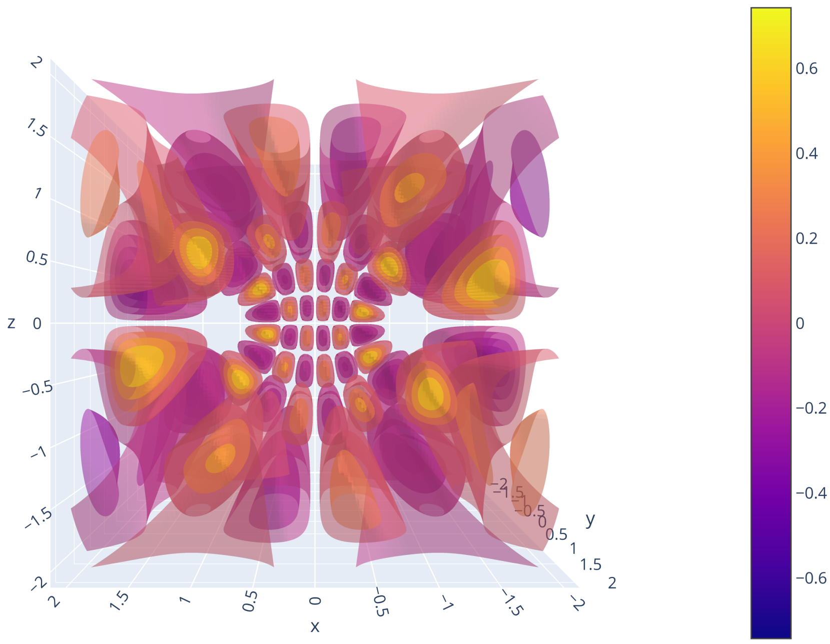

Figure C.6: Stereographic projection of surfaces of constant value of the hyperspherical harmonic $H_{10,5,5}$

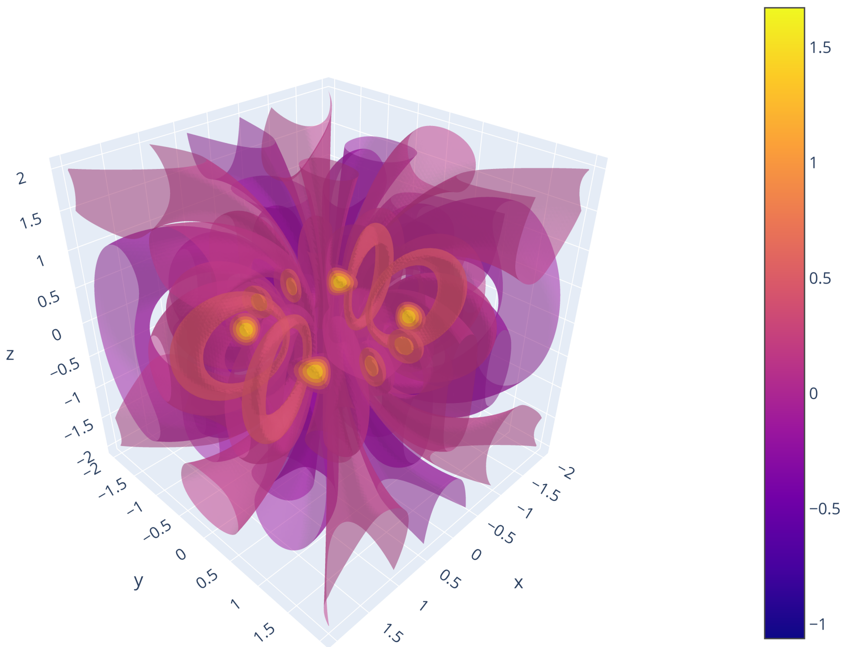

Figure C.7: Stereographic projection of surfaces of constant value of the hyperspherical harmonic $H_{12,6,0}$

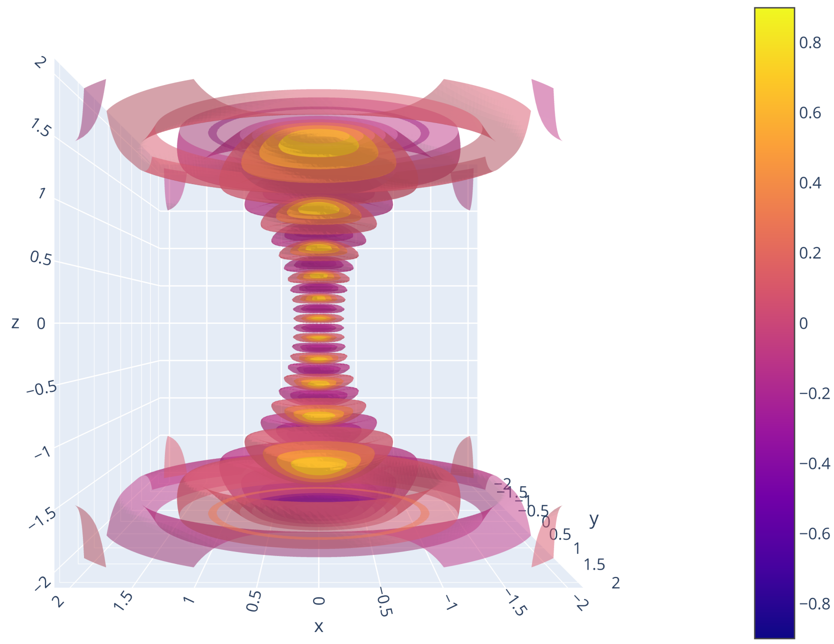

Figure C.8: Stereographic projection of surfaces of constant value of the hyperspherical harmonic $H_{15,15,15}$

D Brief description of the Python code *sas.py*

Part of this thesis is the Python code *sas.py* (sas stands for stereographic projection, approximation, solution) that generated all of the figures in this work. The code has three main functions.

1. Uses the stereographical projection on the input function and plots the iso-surface of the stereographic projection,

2. calculates the generalized Fourier coefficients of the input function with the hyperspherical harmonics H_{nlm} as the basis for the generalized Fourier series. This function of the code was not explicitly used in the thesis but helped with visual confirmation that we were using enough terms of the generalized Fourier series to be a good approximation of the input function,[1]

3. uses the Fourier coefficients $\langle H_{nlm}, \rho\rangle$ to solve the Poisson equation, as seen in the relation (3.44).

Because we did not explicitly use the approximation of the input function by the Fourier series in the thesis, we provide the reader with an illustration of this approximation. We approximated the function[2]

$$f(\varphi, \vartheta, \zeta) = (\cos\varphi + 2\cos\vartheta)\sin 2\varphi \sin\zeta$$

by using the 204 terms of the Fourier series ($n \leq 7$). Figure D.1 shows the stereographic projection of the function f and Figure D.2 depicts the stereographic projection of the approximation of the function f.

Python 3.9.12 was used to run the script. The code uses numpy (version 1.26.1), scipy (version 1.11.3) and plotly (version 5.6.0) Python packages. The code may be compatible with newer (or older) versions of the Python or the packages but the compatibility was not tested.

1. We found that setting the upper bound on the first coefficient of the hyperspherical harmonic H_{nlm} as $n \leq 7$ and summing all terms satisfying this condition (a total of 204 terms) is sufficient to consider this generalized Fourier series as a good visual approximation of the input function.
2. We already used this function in the third example of solving the Poisson equation as ρ_3.

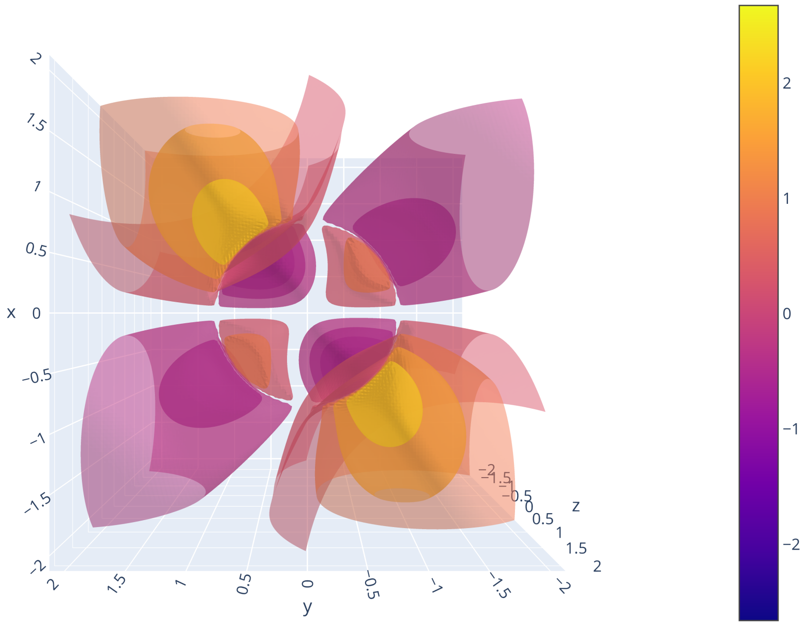

Figure D.1: The stereographic projection of the function $f = (\cos\varphi + 2\cos\vartheta)\sin 2\varphi \sin\zeta$

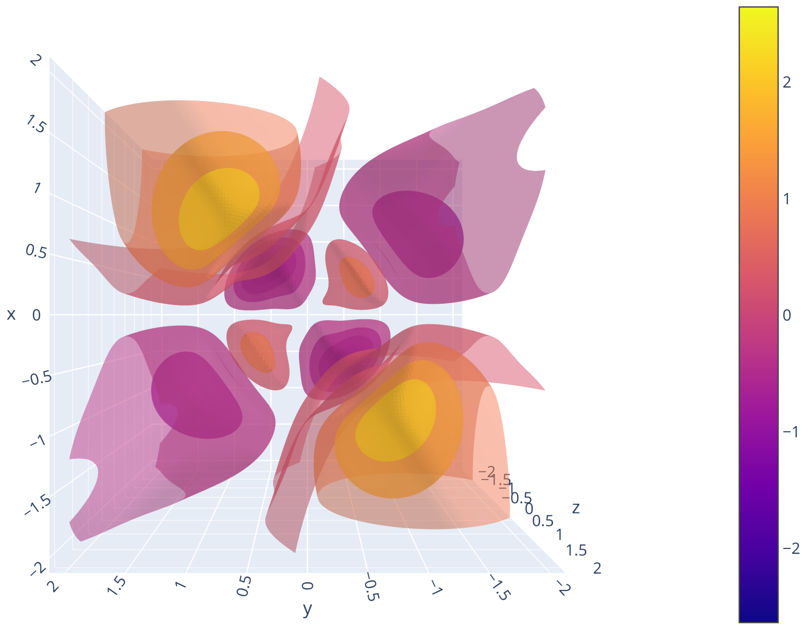

Figure D.2: The stereographic projection of the approximation by Fourier series of the function $f = (\cos\varphi + 2\cos\vartheta)\sin 2\varphi \sin\zeta$

Bibliography

1. GAJDŮŠEK, Josef. *Differential Forms and Electrodynamics*. 2020. Available also from: http://hdl.handle.net/10467/89901. BA thesis. Czech Technical University in Prague, Faculty of Electrical Engineering.

2. *Differential Forms*. GUILLEMIN, Victor; HAINE, Peter J. Available also from: https://math.mit.edu/classes/18.952/2018SP/files/18.952_book.pdf.

3. FRANKEL, Theodore. *The Geometry of Physics: An Introduction*. Revised. Cambridge University Press, 1999. ISBN 9780521387538.

4. NAKAHARA, Mikio. *Geometry, Topology and Physics*. 1st ed. Taylor Francis, 1990. ISBN 978-0852740958.

5. ABRAMOWITZ, Milton; STEGUN, Irene Ann. *Handbook of Mathematical Functions: with Formulas Graphs, and Mathematical Tables*. 10th ed. Mineola, USA: Dover Publications, 1972. ISBN 0-486-61272-4.

6. *LINEAR MATHEMATICS IN INFINITE DIMENSIONS: Signals Boundary Value Problems and Special Functions*. GERLACH, Urlich. Available also from: https://people.math.osu.edu/gerlach.1/math/BVtypset.

7. TALMAN, James. *Special Functions: A Group Theoretic Approach Based on Lectures by Eugene P. Wigner*. 1st ed. W. A. Benjamin, 1968. ISBN 978-1114630185.

8. FOLLAND, Gerald Budge. Harmonic analysis of the de Rham complex on the sphere. *Journal für die reine und angewandte Mathematik*. 1989, vol. 1989, no. 398, pp. 130–143. Available from DOI: doi:10.1515/crll.1989.398.130.

9. FEYNMAN, Richard Phillips; LEIGHTON, Robert Benjamin; SANDS, Matthew. *Feynmanovy přednášky z fyziky s řešenými příklady 2/3*. 1st ed. Havlíčkův Brod, Czech Republic: Fragment, 2001. ISBN 80-7200-420-4.