The Foundation of Mathematical Analysis
(In progress)

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Contents

Part I

Set Theory

Chapter 1.

Constructions of Sets

§1.1 Fundamental concepts of set theory

In this book, Zermelo-Fraenkel set theory with the axiom of choice (ZFC) is invoked as the fundamental theory of mathematical analysis.

In ZFC, there are only two primitive notions, *sets* and *membership*. Sets is a specific collections of objects, and membership, denoted by $\in$, is the relation between specific collections and their objects. If A is such a collection, an object x is an *element* of A, if and only if $x \in A$.

Definition 1.1.1 (Classes)**.** A collection A is a *class* if and only if it satisfies the following condition.

1. For any $x \in A$, x is a set.
2. There is an unambiguous predicate $P(\cdot)$ on sets, such that

$$A = \{x : P(x)\}.$$

A is a *proper class* if and only if it is not a set.

The concept of classes is not formally introduced in ZFC. But, as it can encompass the "set-like" mathematical objects without proving it is a set or not, Definition 1.1.1 is involved in this book, and, the notation $\mathscr{S}$ is adopted to denotes the *universal class*, the class of all sets, in class-builder notation,

$$\mathscr{S} = \{S : S \text{ is a set}\}.$$

Indeed, there are some classes which are not sets. In 1901, Bertrand Russell published his discovery of paradox in naive set theory.

Theorem 1.1.1 (Russell's paradox)**.** Let $R := \{x : P(x)\}$, where $P(x) :=$ "$x \notin x$". R is not a set.

Proof. Assume R is a set. Then, either $R \in R$ or $R \notin R$. If $R \in R$, then $\neg P(R)$ implies $R \notin R$; if, however, $R \notin R$, then $P(R)$ implies $R \in R$. ■

By Definition 1.1.1, R here is a proper class. In ZFC, however, we actually obtain $R = \mathscr{S}$. (See Section 1.3).

In ZFC, urelements, the elements which are not sets, is not included. Every set, in this context, is considered a set of sets, and it hence is a class.

ZFC is build upon the primitive notions, sets and elements, with nine axioms. Each ZFC axiom is a proposition with the universe of sets as its domain. So, in our discussion, every letter, A, B, x, y, etc., indicates a set (of sets), unless it is declared to be something else.

In this book, each axiom is introduced separately within the close context of definitions derived from the primitive notions.

Axiom 1 (Extensionality)**.**

$$\forall A \forall B (A = B \iff \forall x (x \in A \Leftrightarrow x \in B)).$$

The name of the axiom may comes from the philosophical terminology "extensionality", which indicates the identity of an object. In this sense, the axiom of extensionality states that (the identity of) every set is uniquely defined by its element.

Another simple facts follow from the axiom. Firstly, the elements in a set is unordered. For example, $\{a, b\} = \{b, a\}$. Secondly, repeating any element of a set does not change the extensionality of the set. For example, $\{a, a\} = \{a\}$.

§1.2 The closure of set operations

Axiom 2 (Subsets)**.** Let $P(\cdot)$ be a predicate defined on sets.

$$\forall A \exists B \forall x (x \in B \iff (x \in A \wedge P(x))).$$

Definition 1.2.1. Let A be a set. For any predicate $P(\cdot)$ defined on sets, the set

$$B = \{x \in A : P(x)\}$$

is a *subset* of A, in which case, we write $B \subseteq A$.

B is a *proper subset* of A, denoted $B \subset A$, if and only if there are some $x \in A$ such that $\neg P(x)$, i.e., $x \notin B$.

Note 1.2.1. This definition also make sense if we replace "set A" by "class A", in which case B is a *subclass* of A. In this sense, the axiom schema of specification can be re-expressed as: *every subclass of a set is a set.*

Theorem 1.2.1. The universal class, $\mathscr{S}$, is not a set.

Proof. By Russell's paradox (Theorem 1.1.1), $R = \{x : x \notin x\}$ is a proper class, and for any $x \in R$, $x \in \mathscr{S}$. Suppose $\mathscr{S}$ is a set, $R \subseteq \mathscr{S}$ is also a set, by Axiom 2. This contradicts that R is a proper class. ∎

Corollary 1.2.1. By the axiom schema of specification, there is a set $\emptyset = \{\}$, called *emptyset*, i.e., it contains no elements. By the axiom of extensionality, if sets A and B are both empty, then

$$(x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A)$$

implies $A = B$. Thus, $\emptyset$ is a unique set.

Definition 1.2.2. Let $\mathcal{A}$ be a set. The *intersection* of $\mathcal{A}$-elements is defined as

$$\bigcap \mathcal{A} := \{x : S \in \mathcal{A} \Rightarrow x \in S\} = \bigcap_{S \in \mathcal{A}} S.$$

In particular, if $\mathcal{A} = \{S, T\}$, we also write $S \cap T$ for $\bigcap \mathcal{A}$.

Corollary 1.2.2. In Definition 1.2.2, $\bigcap \mathcal{A} \subseteq S$ whereas $S \in \mathcal{A}$. Thus, $\bigcap \mathcal{A}$ is a set.

Definition 1.2.3. Let A be a set. The *complement* of A is defined as

$$A^{\mathsf{C}} := \{x \notin A\}.$$

Let B be another set. The *complement of A in B* is defined as

$$B \setminus A := \{x \in B : x \notin A\}.$$

Corollary 1.2.3. In Definition 1.2.3, $B \setminus A = B \cap A^{\complement} \subseteq B$. By the axiom schema of specification, it is a set.

Axiom 3 (Unions)**.**

$$\forall \mathcal{A} \exists U \forall x (x \in U \iff \exists S (S \in \mathcal{A} \wedge x \in S)).$$

Definition 1.2.4. Let $\mathcal{A}$ be a set. The *union* of $\mathcal{A}$-sets is defined as the set

$$\bigcup \mathcal{A} := \{x : S \in \mathcal{A} \wedge x \in S\} = \bigcup_{S \in \mathcal{A}} S.$$

In particular, if $\mathcal{A} = \{S, T\}$, we also write $S \cup T$ for $\bigcup \mathcal{A}$.

Note 1.2.2. As Definition 1.2.4 also make sense if we replace "set $\mathcal{A}$" by "class $\mathcal{A}$", Axiom 3 can be re-expressed as: *every union of set-elements is a set.* However, if class $\mathcal{A}$ is not a set, at least we know, the union $\bigcup \mathcal{A}$ is not necessarily a set, although every $\mathcal{A}$-element is a set. For example, we have proved in Theorem 1.2.1 that the universal class, $\mathscr{S}$, is not a set. So $\bigcup \mathscr{S}$ is not a set either, because this union still contains all sets in $\mathscr{S}$, i.e., $\bigcup \mathscr{S} = \mathscr{S}$, since, for any $x \in \mathscr{S}$, we can find a $y \in \mathscr{S}$, such that $x \in y$.

Corollary 1.2.4. In Definition 1.2.4, $S \subseteq \bigcup \mathcal{A}$ where as $S \in \mathcal{A}$.

Corollary 1.2.5. Let A, B and C be sets.

(i) $A \cap A = A \cup A = A$.

(ii) $A \cup B = B \cup A$; $A \cap B = B \cap A$.

(iii) $A \cup B \cup C = A \cup (B \cup C)$; $A \cap B \cap C = A \cap (B \cap C)$.

(iv) $A \cup B \cap C = (A \cap C) \cup (B \cap C)$; $A \cap B \cup C = A \cap (B \cup C)$.

Theorem 1.2.2 (De Morgan's laws)**.** Let $\mathcal{A}$ be a class. Then,

$$\text{(i)} \bigcap_{S \in \mathcal{A}} S^{\complement} = \left(\bigcup \mathcal{A}\right)^{\complement}; \quad \text{(ii)} \bigcup_{S \in \mathcal{A}} S^{\complement} = \left(\bigcap \mathcal{A}\right)^{\complement}.$$

Proof. We prove (i). For any x,

$$\begin{aligned} x \in \bigcap_{S \in \mathcal{A}} S^{\complement} &\iff \forall S(S \in \mathcal{A} \Rightarrow\ x \notin S) \\ &\iff \neg(\exists S(S \in A \wedge x \in S)) \\ &\iff \neg\left(x \in \bigcup \mathcal{A}\right) \\ &\iff x \in \left(\bigcup \mathcal{A}\right)^{\complement}. \end{aligned}$$

□

We prove (ii). For any x,

$$\begin{aligned} x \in \bigcup_{S \in \mathcal{A}} S^{\complement} &\iff \exists S(S \in A \wedge x \notin S) \\ &\iff \neg(\forall S(S \in A \Rightarrow x \in S)) \\ &\iff \neg\left(x \in \bigcap \mathcal{A}\right) \\ &\iff x \in \left(\bigcap \mathcal{A}\right)^{\complement}. \end{aligned}$$

■

§1.3 Sets are not self-contained

Axiom 4 (Power sets)**.**

$$\forall A \exists \mathcal{M} \forall S(S \in \mathcal{M} \iff \forall x(x \in S \Rightarrow x \in A)).$$

Definition 1.3.1. Let A be a set. The *power* of A is defined as the class

$$\mathcal{P}(A) := \{S \subseteq A\}.$$

Note 1.3.1. Note that, "set A" cannot be replaced by by "class A in this definition. Because, if A is a proper class, then $A \in \mathcal{P}(A)$ contradicts Definition 1.1.1.

Axiom 5 (Foundation)**.**

$$\forall A \exists x(x \in A \wedge \neg(\exists y(y \in x \wedge y \in A))).$$

Theorem 1.3.1. For any set x, $x \notin x$.

Proof. Suppose there is a set x with $x \in x$. By Axiom 2 and 4, $\{x\} \subseteq \mathcal{P}(x)$ is a set, and hence $x \cap \{x\} \subseteq x$ is a set.

$$\begin{aligned} x \in x \cap \{x\} &\iff x \in x \wedge x \in \{x\} \\ &\iff x \in x \wedge x = x \\ &\iff x \in x. \end{aligned}$$

This implies for any $y \in \{x\}$, $y \cap \{x\} \neq \emptyset$, although $\{x\}$ is a set. This contradicts Axiom 5. ■

Note 1.3.2. Theorem 1.2.1 can be proved by Theorem 1.3.1, as if $\mathscr{S}$ was a set, then $\mathscr{S} \in \mathscr{S}$, contradicts Axiom 5.