Reading Through Axioms and Set Theory

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According to legend, Pythagoras and his followers believed that all numbers are "harmonious ratios". This poetic statement can be expressed in mathematical language as: for any number x, there are two integers m and n, such that $x = \frac{m}{n}$. In other words, every number is rational. This was once believed as an axiom, but not for long. It was later disproved by the discovery of the irrational number $\sqrt{2}$. In proposition 9 of Book X of his *Elements*, Euclid provides a proof of this fact, which we will see again later in this text.

In any case, the existence of a number like $\sqrt{2}$ was enough to push us towards seeking a set that consists of "all numbers," namely, the set of all real numbers $\mathbb{R}$. When this set is equipped with the ordering $\leq$ that we had already been using before it is formally defined, we have what is called the real number line, i.e., the ordered set $(\mathbb{R}, \leq)$.

What the mathematicians of the Renaissance knew was that, each element of the real line (rational or irrational) can be approached by a series of rational numbers. For example, such a series can be used to approach $\sqrt{2}$:

$$s_n = \sum_{i=0}^{n} \left(\frac{1}{2}\right)^{k_i} = \left(\frac{1}{2}\right)^{k_0} + \left(\frac{1}{2}\right)^{k_1} + \cdots + \left(\frac{1}{2}\right)^{k_n},$$

where, for any $n \in \mathbb{N}$, k_n is the smallest natural number such that $s_n^2 < 2$. The larger the natural number n, the closer s_n is to $\sqrt{2}$.

The rigorous definition of real numbers was introduced in the mid-19th century and was developed by mathematicians Richard Dedekind and Georg Cantor. In this chapter, we will discuss the methods of defining the set of real numbers $\mathbb{R}$ and the ordered set $(\mathbb{R}, \leq)$, which is usually referred to as the real number line. We will also verify the validity of this definition to a certain extent.

3.1 Postulations of the Real Line

Before giving the formal definition of the set of real numbers, we assume the existence of the real number line $(\mathbb{R}, \leq)$, i.e., temporarily accept

$$\mathbb{R} \text{ and } \leq$$

without definitions (just as we have done for years). We will spend some time discussing the intuitive meaning of the properties of the real line. Then, we will proceed to formally describe these properties, within the context of set theory. These properties are what we expect the real line to have, so, temporarily, we will accept them as postulates. The formal definition of $(\mathbb{R}, \leq)$ and the proof of the postulations will be discussed in the next chapter.

3.1.1 Postulation of the Total Ordering

Intuitively, the order relation $\leq$ arranges the set of real numbers $\mathbb{R}$ into a string, or more precisely:

Postulation R1. $(\mathbb{R}, \leq)$ is totally ordered.

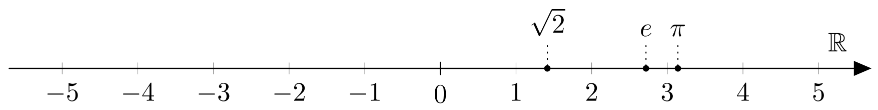

Figure 3.1: Real number line.

3.1.2 Postulation of Unboundedness

The "string" $(\mathbb{R}, \leq)$ extends infinitely in both directions. That is to say, for any $x \in \mathbb{R}$, there is always a $y \in \mathbb{R}$ satisfies $x < y$; and for any $a \in \mathbb{R}$, there is always a $b \in \mathbb{R}$ satisfies $b < a$.

I must mention this because $\mathbb{R}$ itself is just a set, and any set is unordered unless equipped with an ordering. However, different ordering $\preceq$, even if it is total, can also give us an ordered set $(\mathbb{R}, \preceq)$ that we do not want. Therefore, we must ensure that $\leq$ on $\mathbb{R}$ is not such a counterintuitive ordering.

Definition 3.1.1. Let $(A, \preceq)$ be an ordered set, and let $S \subseteq A$.

S is **bounded above** if and only if

$$\exists p \in A : \forall s \in S : s \preceq p,$$

in which case p is an **upper bound** of S, denoted by $S \preceq p$.

S is **bounded below** if and only if

$$\exists q \in A : \forall s \in S : q \preceq s,$$

in which case q is a **lower bound** of S, denoted by $q \preceq S$.

S is **bounded** if and only if it is either bounded above or bounded below, otherwise, S is **unbounded**.

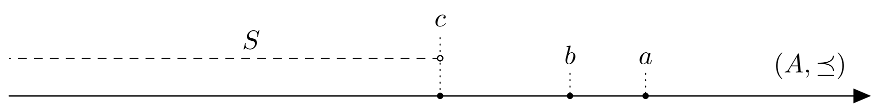

Figure 3.2: a, b, c are upper bounds of S.

Note 3.1.1. This definition also holds for $S = \emptyset$. However, in this case every element of A is an upper bound and a lower bound of $\emptyset$.

Postulation R2. $\mathbb{R}$ is unbounded $(\mathbb{R}, \leq)$.

3.1.3 Postulation of Completeness

We expect the real number line $(\mathbb{R}, \leq)$ to be complete. Informally speaking, if r is a number such that there are $x, y \in \mathbb{R}$ with $x < r < y$, then $\mathbb{R}$ can not be a subset of the union of the intervals,

$$(-\infty, r) \cup (r, \infty).$$

In this sense, we usually say that $\mathbb{R}$ has no "gap".

However, the existence of $\sqrt{2}$ tells us that we cannot expect $(\mathbb{Q}, \leq)$ to be our real number line, because it breaks at least at the "gap" of $\sqrt{2}$.

$$\mathbb{Q} \subseteq \left(-\infty, \sqrt{2}\right) \cup \left(\sqrt{2}, \infty\right)$$

In order to formally describe this completeness in the context of set theory, we need the following definitions.

Definition 3.1.2. Let $(A, \preceq)$ be an ordered set, and let $S \subseteq A$.

An element p is a **maximum** of S, if and only if

$$p \in S \wedge (\forall s \in S : x \preceq p).$$

An element q is a **minimum** of S, if and only if

$$q \in S \wedge (\forall s \in S : q \preceq x).$$

It is easy to see that not all bounded subsets have a maximum or minimum. For example, the real interval $(-\infty, 0)$ has neither a maximum nor a minimum.

On the real line $(\mathbb{R}, \leq)$, if a subset $S \subseteq \mathbb{R}$ has a maximum p, we write $\max S = p$. However, strictly speaking, to define the notation "$\max S = p$", we need to ensure that p is the *uniquely* maximum. This is because Definition 3.1.2 does not include any description of the uniqueness of the maximum. The same issue exists in the case of the minimum. Fortunately, if we adhere to the definition, this is not difficult to prove.

Lemma 3.1.1. *Let $(A, \preceq)$ be an ordered set, and let $S \subseteq A$. Set $p \in S$.*

If p is a maximum of S, then p is the unique maximum of S.

If p is a minimum of S, then p is the unique minimum of S.

Proof. Assume p is a maximum of S. Let p' also be a maximum of S.

For any $s \in S$, $s \preceq p$. So, as $p' \in S$, $p' \preceq p$.

Similarly, $p \preceq p'$. As $\preceq$ is antisymmetric,

$$p' \preceq p \wedge p \preceq p' \implies p = p'.$$

Thus, p is the unique maximum of S.

By a similar method, we can prove that p is the unique minimum of S if it is a minimum of S. □

Definition 3.1.3. Guaranteed by Lemma 3.1.1, we write

$$p = \max S = \max_{s \in S} s \iff p \text{ is } \textit{the} \text{ maximum of } S,$$
$$p = \min S = \min_{s \in S} s \iff p \text{ is } \textit{the} \text{ minimum of } S.$$

We call $\max S$ the **greatest** element of S, and called $\min S$ the **least** element of S.

Definition 3.1.4. Let $(A, \preceq)$ be an ordered set, and let $S \subseteq A$.

The **supremum** of S is the least upper bound of S, i.e.,

$$\sup S := \sup_{s \in S} s = \min\{x \in A : \forall s \in S : s \preceq x\}.$$

The **infimum** of S is the greatest upper bound of S, i.e.,

$$\inf S := \inf_{s \in S} = \max\{x \in A : \forall s \in S : x \preceq s\}.$$

Example 3.1.1. (Figure 3.3) Let $s : \mathbb{N} \to \mathbb{R}$ be defined as

$$s_n := \sum_{i=1}^{n} \frac{1}{n!},$$

and define $S = \{s_n : n \in \mathbb{N}\}$. The element $e = \sup S$ is the famous irrational number, Euler's number. Since e is an irrational number, $\sup S \notin \mathbb{Q}$.

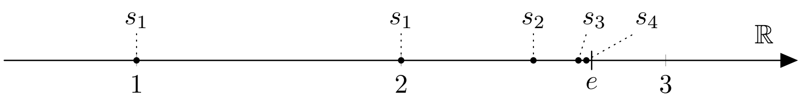

Figure 3.3: The series $(s_n)_{n\in\mathbb{N}}$ approaching Euler's number e.

Definition 3.1.5. Let $(A, \preceq)$ be an ordered set.

$(A, \preceq)$ is **complete** if and only if for every non-empty $S \subseteq A$ which is bounded above,

$$(S \neq \emptyset \wedge (\exists a \in A : S \preceq a)) \implies \sup S \in A.$$

Note 3.1.2. In this definition, $(A, \preceq)$ is not necessarily a total ordered set.

Imagine a particle's motion in two-dimensional space $\mathbb{R}^2$. Suppose the displacement of the particle can be represented by the equation

$$\mathbf{r}(t) = (-t, -\sin(t)),$$

In the context of set theory, $\mathbf{r}(t)$ defines a function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$. In this way, the function $\mathbf{r}$ maps the entire real number line $(\mathbb{R}, \leq)$, which serves as the temporal axis, into the two-dimensional space $\mathbb{R}^2$.

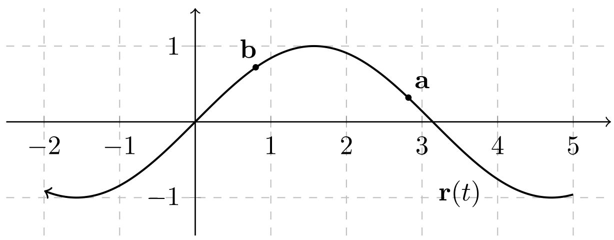

Figure 3.4: $\mathbf{a} \preceq \mathbf{b}$.

In this case, if we define $\preceq$ as an ordered relation on $\mathbb{R}^2$: for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$, $\mathbf{a} \preceq \mathbf{b}$ if and only if

$$\exists t_1, t_2 \in \mathbb{R} : t_1 \leq t_2 \wedge \mathbf{a} = s(t_1) \wedge \mathbf{b} = s(t_2),$$

then $(\mathbb{R}^2, \preceq)$ is completely but not totally ordered.

Lemma 3.1.2. *The rational line $(\mathbb{Q}, \leq_{\mathbb{Q}})$ is not complete.*

Proof. Let

$$L := \left\{x \in \mathbb{Q} : x^2 <_{\mathbb{Q}} 2 \vee x <_{\mathbb{Q}} 0\right\}.$$

(Thus, if $\sqrt{2}$ is a rational number, then it is equivalent to $\sup L \in \mathbb{Q}$. However, we have not defined the set of real numbers $\mathbb{R}$, and we have not defined the meaning of $x^{\frac{1}{n}}$ for any $x \in \mathbb{Q}$ and $n \in \mathbb{Z} \setminus \{0\}$. Therefore, we can only use L to replace the concept of $\sqrt{2}$.)

Aiming for a contradiction, suppose $(\mathbb{Q}, \leq)$ is complete, then, as L is bounded above, $\sup L \in \mathbb{Q}$. Thus, there must be a $q \in \mathbb{Q}$ with $q^2 = 2$. Thus, for any $x \in L$,

$$x^2 <_{\mathbb{Q}} q^2 \implies x <_{\mathbb{Q}} q.$$

As $q \in \mathbb{Q}$, there are $m, n \in \mathbb{Z}$ such that $q = \frac{m}{n}$ and m and n has no common divider, i.e.,

$$\forall \varphi \in \mathbb{N}_{>_{\mathbb{Q}} 1} : \frac{m^2}{\varphi^2}, \frac{n^2}{\varphi^2} \notin \mathbb{Z} \subseteq \mathbb{Q}.$$

(The details of this step can all be derived from the construction of $\mathbb{Q} = \mathbb{Z} \times \mathbb{Z} / \sim_{\mathbb{Q}}$ and the definitions of $+_{\mathbb{Q}}$ and $\times_{\mathbb{Q}}$ on it.)

But, by our assumption, then we have

$$\frac{m^2}{n^2} = 2 \implies \frac{m^2}{2}, \frac{n^2}{2} \in \mathbb{Z}.$$

This contradicts the conclusion that m and n has no common divider.

Thus, $\sup L = q \notin \mathbb{Q}$, and $(\mathbb{Q}, \leq)$ is thereby incomplete. $\square$

Postulation R3. The real line $(\mathbb{R}, \leq)$ is complete.

3.1.4 Postulation of Rational Density

The reason why the rational line $(\mathbb{Q}, \leq_{\mathbb{Q}})$ was once believed to be "complete" may have been largely due to its density. That is to say, there are always rational numbers between any two distinct rational numbers.

In order theory, density is defined as a more general concept.

Definition 3.1.6. An ordered set $(A, \preceq)$ is said to be **densely ordered** if and only if, for any $x, y \in A$,

$$x \prec y \implies \exists z \in A : x \prec z \prec y.$$

Lemma 3.1.3. *The rational line* $(\mathbb{Q}, \leq)$ *is densely ordered.*

Proof. Let $x, y \in \mathbb{Q}$ with $x <_{\mathbb{Q}} y$. Since the rational field $(\mathbb{Q}, +_{\mathbb{Q}}, \times_{\mathbb{Q}})$ is Archimedean under $\leq_{\mathbb{Q}}$, there is an $n \in \mathbb{N}$ such that

$$\begin{aligned} y - x < n(y - x) &\implies \frac{y-x}{n} < y - x \\ &\implies x < x + \frac{y-x}{n} < y. \end{aligned}$$

So the positive part of $(\mathbb{Q}, \leq)$ is densely ordered. Similarly, its negative part is densely ordered. Thus, so is the union $(\mathbb{Q}, \leq)$. $\square$

Definition 3.1.7. Let $(A, \preceq)$ be an ordered set.

A subset $S \subseteq A$ is said to be **everywhere dense**, or **dense**, in $(A, \preceq)$ if and only if for any $x, y \in A$,

$$x \prec y \implies \exists s \in S : x \prec s \prec y.$$

Postulation R4. The rational part of $(\mathbb{R}, \leq)$ is everywhere dense.

3.1.5 Postulation of the Ordering Embedding

Since we believe that the real line $(\mathbb{R}, \leq)$ contains all rational numbers, we certainly hope that the ordering $\leq$ on $\mathbb{R}$ preserves the ordering of $\leq_{\mathbb{Q}}$ completely. Strictly speaking:

Postulation R5. There is an order embedding $f : (\mathbb{Q}, \leq_{\mathbb{Q}}) \to (\mathbb{R}, \leq)$ being an order embedding. That is, for any $x, y \in \mathbb{Q}$,

$$x \leq_{\mathbb{Q}} y \implies f(x) \leq f(y).$$

3.1.6 Postulation of Irrational Density

Finally, the proposition that we will need to verify is as follows.

Postulation R6. The irrational part of $(\mathbb{R}, \leq)$ is everywhere dense.

3.2 Dedekind Cuts and Real Numbers

In the proof of Lemma 3.1.2, we mentioned a set,

$$L = \{x \in \mathbb{Q} : x^2 <_{\mathbb{Q}} 2 \vee x_{\mathbb{Q}} < 0\}.$$

whose supremum, $\sqrt{2}$, is not an element of $\mathbb{Q}$.

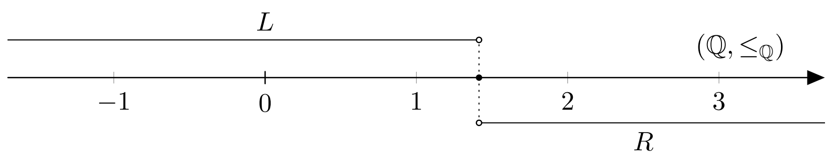

Figure 3.5: A "cut" of $(\mathbb{Q}, \leq_{\mathbb{Q}})$

In Figure 3.5, $R = \mathbb{Q} \setminus L$. Then $\mathbb{Q}$ is "cut" into two parts, L and R. Both L and R look like "open intervals" on the rational number line $(\mathbb{Q}, \leq_{\mathbb{Q}})$ with their endpoints, $\sqrt{2}$, removed. In the work of the German mathematician Richard Dedekind, the ordered pair (L, R) defines the irrational number $\sqrt{2}$.

Definition 3.2.1. Let $(A, \preceq)$ be an ordered set.

An ordered pair (L, R) is a **Dedekind cut** of $(A, \preceq)$ if and only it satisfies the following conditions.

1. $\{L, R\}$ is a partition of A.
2. $L \neq \emptyset$.
3. L is not bounded below.
4. $\max L \notin L$.

If $\mathscr{D}$ is the set of all Dedekind cut of $(A, \preceq)$,[1] then the following set is called the A**-part** of $\mathscr{D}$:

$$\{(L, R) \in \mathscr{D} : \sup L \in A\}.$$

Note 3.2.1. Having Dedekind cuts does not require A to be densely ordered, but it implies that A is "somewhere dense". For example, let $f, g : \mathbb{N} \to \mathbb{Q}$ be defined as

$$f(n) = \frac{1}{2^n} \text{ and } g(n) = -\frac{1}{2^n},$$

and let $S = f[\mathbb{N}] \cup g[\mathbb{N}]$.

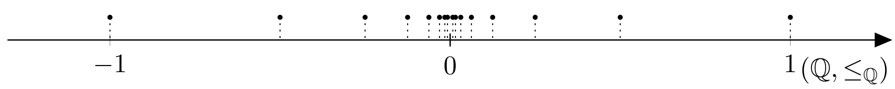

Figure 3.6: S represented as dots on $(\mathbb{Q}, \leq_{\mathbb{Q}})$.

Then, as $(S, \leq_{\mathbb{Q}})$ is densely ordered only at 0, and it has only one Dedekind cut, $(\{x \in \mathbb{Q} : x <_{\mathbb{Q}} 0\}, \{x \in \mathbb{Q} : x \geq_{\mathbb{Q}} 0\})$.

Here is a quick answer: the set of all real numbers $\mathbb{R}$ is defined as the the set of all Dedekind cuts of $(\mathbb{Q}, \leq_{\mathbb{Q}})$. The ordering $\leq$ can be thereby unambiguously defined. But before the formal definition is given, we need to verify that if, defined in this way, $(\mathbb{R}, \leq)$ satisfies the postulations we have.

Theorem 3.2.1. *Let $\mathscr{D}$ be the set of all Dedekind cuts of $(\mathbb{Q}, \leq_{\mathbb{Q}})$. Let $\preceq$ be an ordering on $\mathscr{D}$ defined as*

$$x \preceq y \iff x_L \subseteq x_R,$$

whereas $x = (x_L, x_R), y = (y_L, y_R) \in \mathscr{D}$. Then, the following propositions are true.

[1] If A is a set, then, guaranteed by the axiom of power sets and the axiom of replacement (which derives the schema of specification), $\mathscr{D} \subseteq \mathcal{P}(A) \times \mathcal{P}(A)$ is also a set.

(i). *$(\mathscr{D}, \preceq)$ is totally ordered.*

(ii). *$\mathscr{D}$ is unbounded in $(\mathscr{D}, \preceq)$.*

(iii). *There is an $f : (\mathbb{Q}, \leq_{\mathbb{Q}}) \to (\mathscr{D}, \preceq)$ being an ordered embedding.*

(iv). *$(\mathscr{D}, \preceq)$ is complete.*

(v). *The rational part of $(\mathscr{D}, \preceq)$ is everywhere dense.*

(vi). *The irrational part of $(\mathscr{D}, \preceq)$ is everywhere dense.*

Proof of (i). Let $x = (x_L, x_R), y = (y_L, y_R) \in \mathscr{D}$ with $\neg(y \preceq x)$.

As $\neg(y \subseteq x)$, there is a $u \in y_L \setminus x_L$. Fix u, and then we have

$$\forall v \in x_L : v \prec_A u \text{ and } \forall w \preceq u : w \in y_L.$$

This implies $x_L \subseteq y_L$. Thus, $x \preceq y$.

Thus, $(\mathscr{D}, \preceq)$ is totally ordered. □

Proof of (ii). Aiming for a contradiction, suppose $\mathscr{D}$ is bounded below. Then there must be a $r = (r_L, r_R) \in \mathscr{D}$, such that for any $x \in \mathscr{D}$, $l \preceq x$.

Let $x = (x_L, x_R) \in \mathscr{D}$. Then, $r \preceq x$ implies $r_L \subseteq r_L$. As x is arbitrary, r_L has to be empty. (Why?) This contradicts the assumption that p is a Dedekind cut. Thus, $\mathscr{D}$ is not bounded below.

Using a similar method, we can prove that $\mathscr{D}$ is not bounded above. □

Proof of (iii). Let $f : \mathbb{Q} \to \mathcal{P}(\mathbb{Q}) \times \mathcal{P}(\mathbb{Q})$ be defined as

$$f(x) := (L = \{y \in \mathbb{Q} : y \prec x\}, \mathbb{Q} \setminus L) = (f_L(x), f_R(x))$$

Let $a, b \in \mathbb{Q}$ with $a \leq_{\mathbb{Q}} b$. Then, $f(a), f(b) \in \mathscr{D}$, and $f_L(x) \subseteq f_R(x)$. (Why?) Thus, $f(a) \preceq f(b)$.

Thus, $f : (\mathbb{Q}, \leq_{\mathbb{Q}}) \to (\mathscr{D}, \preceq)$ is an order embedding. □

Proof of (iv). Let $S \subseteq \mathscr{D}$ be bounded above. Let $\mathcal{L}$ be the set of all left terms of elements of S, i.e.,

$$\mathcal{L} := \{x_L \subseteq \mathbb{Q} : (x_L, \mathbb{Q} \setminus x_L) \in S\}.$$

Then, for any $x = (x_L, x_R) \in S$, $x_L \subseteq \bigcup \mathcal{L}$. For any such x, x_L is non-empty, and so is $\bigcup \mathcal{L}$; x_L is not bounded below, and so is $\bigcup \mathcal{L}$; $\max x_L \notin x_L$, and then $\max (\bigcup \mathcal{L}) \notin \bigcup \mathcal{L}$. (Why?) So, let

$$y = \left(\bigcup \mathcal{L}, \mathbb{Q} \setminus \bigcup \mathcal{L}\right) = (y_L, y_R),$$

then $y \in \mathscr{D}$, and y is an upper bound of S.

Let $y' = (y'_L, y'_R) \in \mathscr{D}$ also be an upper bound of S. Then, for any $x = (x_L, x_R) \in S$, $x_L \subseteq y'_L$. As y_L is the union of such x_L, $y_L \subseteq y'_L$, which implies $y \preceq y'$. Thus, y is the least upper bound of S, i.e., $y = \sup S$.

As S is arbitrary, $(\mathscr{D}, \preceq)$ is complete. □

Proof of (v). Let $x = (x_L, x_R), y = (y_L, y_R) \in \mathscr{D}$ with $x \prec y$.

$x_L \subset y_L$, thus there is a $u \in y_L \setminus x_L$.

Let $z_L = \{a \in \mathbb{Q} : a \leq_{\mathbb{Q}} u\}$ and $\mathbb{Q} \setminus z_R$, then $z = (z_L, z_R) \in \mathscr{D}$ is an element of the rational part of $\mathscr{D}$ and $x \prec z \prec y$.

Thus, the rational part of $\mathscr{D}$ is everywhere dense in $(\mathscr{D}, \preceq)$. □

Proof of (vi). Let $x = (x_L, x_R), y = (y_L, y_R) \in \mathscr{D}$ with $x \prec y$.

As the rational part of $\mathscr{D}$ is everywhere dense in $(\mathscr{D}, \preceq)$, there are rational $v = (v_L, v_R), w = (w_L, w_R) \in \mathscr{D}$ with $x \prec v \prec w \prec y$.

Let

$$I = \{q \in \mathbb{Q} : q^2 <_{\mathbb{Q}} 2 \text{ and } q >_{\mathbb{Q}} 0\}.$$

Then $\sup I \notin I$. Let $q \in I$.

As $(\mathbb{Q}, +_{\mathbb{Q}}, \times_{\mathbb{Q}})$ is an Archimedean field, there is an $n \in \mathbb{N}$, such that for any $r, s \in w_L \setminus v_L$ with $r \leq_{\mathbb{Q}} s$, we have

$$q < n(r - s) \implies \frac{q}{n} < r - s.$$

Fix n and q, and let

$$z = (z_L, z_R) : z_L = \left\{a +_{\mathbb{Q}} \frac{q}{n} : a \in v_L\right\}, z_R = \mathbb{Q} \setminus z_L.$$

Then, $z \in \mathscr{D}$ is irrational and $x \prec z \prec y$. (Why?)

Thus, the irrational part of $(\mathscr{D}, \preceq)$ is everywhere dense. □

Definition 3.2.2. The **set of all real numbers** $\mathbb{R}$ is defined as the set of all Dedekind cuts of $(\mathbb{Q}, \leq_{\mathbb{Q}})$. The **real line** is defined as the ordered set $(\mathbb{R}, \leq)$ where the ordering $\leq$ is defined as

$$\forall x = (x_L, x_R), y = (y_L, y_R) \in \mathbb{R} : x \preceq y \iff x_L \subseteq y_L.$$

Since any real number $r = (r_L, r_R)$ is a Dedekind cut of $(\mathbb{Q}, \leq)$, it becomes very clear how to determine whether r belongs to the rational or irrational part of $\mathbb{R}$:

$$r \in \mathbb{Q} \iff \sup r_L \in \mathbb{Q}.$$

However, we still need to keep in mind that $\mathbb{R} \subseteq \mathcal{P}(\mathbb{Q}) \times \mathcal{P}(\mathbb{Q})$, so strictly speaking, we can say that $\mathbb{Q} \hookrightarrow \mathbb{R}$, but not that $\mathbb{Q} \subseteq \mathbb{R}$. Nevertheless, for the sake of convenience, we often use $\mathbb{Q}$ to represent the rational part of $\mathbb{R}$ and simply call it the "set of rational numbers", without distinguishing it from the left component of $(\mathbb{Q}, \leq_{\mathbb{Q}})$, unless it causes confusion in context.