Value of the Steinitz constant

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We sketch the proof that $p_a(\gamma) = 0$, where $\gamma = \gamma_\tau \otimes I$. It can be shown that there exists a partial isometry A in $C^U_{M_1}$ such that $\sigma_t^U(A) = \lambda^{it}A$ $(t \in \mathbf{R})$ and $\theta_1^U(A) = A$ (cf. the proof of [5, Theorem 4.4.2]). Then $A \in C^U_M$, $\sigma_t^U(A) = \lambda^{it}A$ $(t \in \mathbf{R})$, and $\gamma_\tau^U(A) = \lambda^{it}A$. On the other hand, $C^U_{R_\infty}$ contains a partial isometry B such that $\sigma_t^U(B) = \lambda^{it}B$ (cf. [5]). But then, $AB^* \in (C^U_{M \otimes R_\infty})_{\rho_U}$ and $\gamma_\tau^U(AB^*) = \lambda^{i\tau}AB^*$, and therefore $p_a(\gamma) = 0$ (cf. [5, Sec. 4.2]).

LITERATURE CITED

1. A. Connes, J. Funct. Anal., 16, 415-445 (1974).
2. H. Araki and E. J. Woods, Publ. RIMS, Kyoto Univ., Ser. A, 3, 51-130 (1968).
3. A. Connes, J. Funct. Anal., 18, 318-327 (1975).
4. V. Ya. Golodets, Izv. Akad. Nauk SSSR, Ser. Mat., 39, 635-656 (1975).
5. V. Ya. Golodets, Usp. Mat. Nauk, 33, No. 1, 43-94 (1978).
6. M. Takesaki, Acta Math., 131, Nos. 3-4, 249-310 (1973).

VALUE OF THE STEINITZ CONSTANT

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A well-known lemma of Steinitz [1] states that for every normal m-dimensional space there exists a constant B such that for every collection of vectors $x_1, x_2, \ldots, x_n$ satisfying $x_1 + x_2 + \ldots + x_n = 0$, $\|x_i\| \le 1$ $(i = 1, 2, \ldots, n)$ there exists a permutation π such that for all positive integers $k \le n$ we have $\left\| \sum_{i=1}^{k} x_{\pi(i)} \right\| \leqslant B$. The smallest possible value of B is denoted by C and will be called the Steinitz constant. C depends only on m and the given norm.

It is shown in [2] that for a Euclidean space,

$$C \leqslant \sqrt{(4^m - 1)/3}. \tag{1}$$

A discussion of the Steinitz lemma and its applications is given in [3-5] and [6], where estimate (1) was rediscovered.

Up to now, only upper estimates exponential with respect to m have been known for C. We show that $C \le m$ for every norm (not necessarily even symmetric).

LEMMA. Let K be a polyhedron in $\mathbf{R}^n$ defined by a system

$$\begin{cases} f_i(x) = a_i, & i = 1, 2, \ldots, p, \\ g_j(x) \leqslant b_j, & j = 1, 2, \ldots, q, \end{cases}$$

where the f_i and g_j are linear functions. Let x_0 be a vertex of K and $A = \{j : g_j(x_0) = b_j\}$.

Then $|A| \ge n - p$, where $|A|$ is the cardinality of the set A.

Proof. Assume the contrary. Then the system

$$\begin{cases} f_i(x) = 0, & i = 1, 2, \ldots, p, \\ g_j(x) = 0, & j \in A, \end{cases}$$

has a nontrivial solution x_1. The vectors $x_0 - \varepsilon x_1$ and $x_0 + \varepsilon x_1$ belong to K for sufficiently small $\varepsilon > 0$, which contradicts the fact that x_0 is a vertex.

THEOREM 1. Let an arbitrary norm be given in $\mathbf{R}^m$ and assume that $\|x_i\| \le 1$ $(i = 1, 2, \ldots, n)$ and $x_1 + \ldots + x_n = x$.

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Then there exists a permutation π such that for all positive integers $k \leq n$

$$\left\| \sum_{i=1}^{k} x_{\pi(i)} - \frac{k-m}{n} x \right\| \leqslant m. \tag{2}$$

Proof. By induction, we construct a chain of sets

$$A_m \subset A_{m+1} \subset \cdots \subset A_n = \{1, 2, \ldots, n\}$$

and numbers λ_k^i ($k = m, m+1, \ldots, n$; $i \in A_k$) with the properties:

$$|A_k| = k, \quad 0 \leqslant \lambda_k^i \leqslant 1, \quad \sum_{i \in A_k} \lambda_k^i = k - m, \quad \sum_{i \in A_k} \lambda_k^i x_i = \frac{k-m}{n} x.$$

$$k = n. \quad A_n = \{1, 2, \ldots, n\}, \quad \lambda_n^i = \frac{n-m}{n}.$$

Inductive Step $k+1 \to k$. Consider the set K of collections $\{\mu_i, i \in A_{k+1}\}$ with the properties:

$$0 \leqslant \mu_i \leqslant 1, \quad \sum_{i \in A_{k+1}} \mu_i = k - m, \quad \sum_{i \in A_{k+1}} \mu_i x_i = \frac{k-m}{n} x. \tag{3}$$

K is convex and compact in R^{k+1} and nonempty (we can take $\mu_i = \frac{k-m}{k+1-m} \lambda_{k+1}^i$). Let $\{\bar{\mu}_i, i \in A_{k+1}\}$ be a vertex of K. By the lemma $|\{i: 0 < \bar{\mu}_i < 1\}| \leq m + 1$. Using (3), we get that $\{i: \bar{\mu}_i = 0\} \neq \phi$. Let $\bar{\mu}_j = 0$. We put $A_k = A_{k+1} \setminus \{j\}$, $\lambda_k^i = \bar{\mu}_i$ ($i \in A_k$), which completes the construction.

We put $\{\pi(i)\} = A_i \setminus A_{i-1}$ ($i = m+1, \ldots, n$), with π otherwise arbitrary. For $k \leq m$, inequality (2) to be proved is obvious. For $k \geq m+1$,

$$\left\| \sum_{i=1}^{k} x_{\pi(i)} - \frac{k-m}{n} x \right\| = \left\| \sum_{i \in A_k} (1 - \lambda_k^i) x_i \right\| \leqslant \sum_{i \in A_k} (1 - \lambda_k^i) = m.$$

Remark. Symmetry of the norm has not been used anywhere.

For the case $x = 0$ and a symmetric norm (Steinitz' lemma), a more cumbersome proof is given in [7].

The following example shows that the estimate obtained is best possible. Take the unit ball to be a regular (unsymmetric) simplex with center at zero. Put $n = m + 1$, and let x_i be the i-th vertex of the simplex. Then $\Sigma x_i = 0$ and $\|x_{\pi(1)} + \ldots + x_{\pi(n-1)}\| = m$ for every permutation π.

The maximal lower estimate for the Steinitz constant known to the author for the case of a symmetric norm ($(m+1)/2$ in the space l_1, $C \geq (m+3)^{1/2}/2$ for a Euclidean space) is attained on the collection of vectors $B_k = \{e_i\ (i = 1, \ldots, m-1);\ a_i^k\ (i = 1, \ldots, k);\ b_i^k\ (i = 1, \ldots, k)\}$, $k \geq (m-1)/2$ as $k \to \infty$, where the e_i are unit vectors; $a_i^k(j) = b_i^k(j) = -1/2k$ ($j = 1, \ldots, m-1$, $i = 1, \ldots, k$), $a_i^k(m) = -b_i^k(m) = 1 - (m-1)/2k$ ($i = 1, \ldots, k$).

We state without proof another assertion relevant to the topic considered.

THEOREM 2. Let $x_i \in R^m$, $\|x_i\| \leqslant 1$ $(i = 1, 2, \ldots, n)$, $\Sigma x_i = x$. Then $\forall \rho\ (0 \leqslant \rho \leqslant 1)$ $\exists A \subseteq \{1, 2, \ldots, n\}$, for which $\left\| \sum_{i \in A} x_i - \rho x \right\| < m$, and in the case of a symmetric norm $\left\| \sum_{i \in A} x_i - \rho x \right\| \leqslant m/2$.

Both the estimates are exact. For the proof put $n = m$ and let $\{x_i\}$ be the standard basis in R^m. In the symmetric case, we define an l_1-norm and set $\rho = 1/2$. For the unsymmetric case, let the unit ball be the convex hull of the vectors $\{x_1, \ldots, x_n, -\varepsilon x_1, \ldots, -\varepsilon x_n\}$ and $\rho = \varepsilon$, where ε is a small positive number.

LITERATURE CITED

1. E. Steinitz, J. Reine Angew. Math., 143, 128-175 (1913); 144, 1-40 (1914).
2. V. Bergström, Abhandl. Math. Sem. Hamb. Univ., 8, 148-152 (1931).
3. H. Hadwiger, Math. Z., 41, No. 5, 732-738 (1936).
4. H. Hadwiger, Math. Z., 46, No. 1, 70-79 (1940).
5. H. Hadwiger, Math. Z., 47, No. 4, 663-668 (1942).
6. M. I. Kadets, Usp. Mat. Nauk, 8, No. 1, 139-143 (1953).
7. S. V. Sevast'yanov, Discrete Analysis [in Russian], No. 32, Novosibirsk (1978).