Lecture Notes 7
Random Processes

- Definition
- IID Processes
- Bernoulli Process
  - Binomial Counting Process
  - Interarrival Time Process
- Markov Processes
- Markov Chains
  - Classification of States
  - Steady State Probabilities

Corresponding pages from B&T: 271–281, 313–340.

Random Processes

- A *random process* (also called *stochastic process*) $\{X(t) : t \in \mathcal{T}\}$ is an infinite collection of random variables, one for each value of time $t \in \mathcal{T}$ (or, in some cases distance)
- Random processes are used to model random experiments that evolve in time:
  - Received sequence/waveform at the output of a communication channel
  - Packet arrival times at a node in a communication network
  - Thermal noise in a resistor
  - Scores of an NBA team in consecutive games
  - Daily price of a stock
  - Winnings or losses of a gambler
  - Earth movement around a fault line

Questions Involving Random Processes

- Dependencies of the random variables of the process:
  - How do future received values depend on past received values?
  - How do future prices of a stock depend on its past values?
  - How well do past earth movements predict an earthquake?
- Long term averages:
  - What is the proportion of time a queue is empty?
  - What is the average noise power generated by a resistor?
- Extreme or boundary events:
  - What is the probability that a link in a communication network is congested?
  - What is the probability that the maximum power in a power distribution line is exceeded?
  - What is the probability that a gambler will lose all his capital?

Discrete vs. Continuous-Time Processes

- The random process $\{X(t) : t \in \mathcal{T}\}$ is said to be *discrete-time* if the index set $\mathcal{T}$ is countably infinite, e.g., $\{1, 2, \ldots\}$ or $\{\ldots, -2, -1, 0, +1, +2, \ldots\}$:
  - The process is simply an infinite sequence of r.v.s $X_1, X_2, \ldots$
  - An outcome of the process is simply a sequence of numbers
- The random process $\{X(t) : t \in \mathcal{T}\}$ is said to be *continuous-time* if the index set $\mathcal{T}$ is a continuous set, e.g., $(0, \infty)$ or $(-\infty, \infty)$
  - The outcomes are random *waveforms* or random occurances in continuous time
- We only discuss discrete-time random processes:
  - IID processes
  - Bernoulli process and associated processes
  - Markov processes
  - Markov chains

IID Processes

- A process $X_1, X_2, \ldots$ is said to be *independent and identically distributed* (IID, or i.i.d.) if it consists of an infinite sequence of independent and identically distributed random variables
- Two important examples:
  - Bernoulli process: $X_1, X_2, \ldots$ are i.i.d. $\mathrm{Bern}(p)$, $0 < p < 1$, r.v.s. Model for random phenomena with binary outcomes, such as:
    - Sequence of coin flips
    - Noise sequence in a binary symmetric channel
    - The occurrence of random events such as packets (1 corresponding to an event and 0 to a non-event) in discrete-time
    - Binary expansion of a random number between 0 and 1
  - Discrete-time white Gaussian noise (WGN) process: $X_1, X_2, \ldots$ are i.i.d. $\mathcal{N}(0, N)$ r.v.s. Model for:
    - Receiver noise in a communication system
    - Fluctuations in a stock price

- Useful properties of an IID process:
  - *Independence*: Since the r.v.s in an IID process are independent, any two events defined on sets of random variables with *non-overlapping* indices are independent
  - *Memorylessness*: The independence property implies that the IID process is memoryless in the sense that for any time n, the *future* $X_{n+1}, X_{n+2}, \ldots$ is independent of the *past* $X_1, X_2, \ldots, X_n$
  - *Fresh start*: Starting from any time n, the random process $X_n, X_{n+1}, \ldots$ behaves identically to the process $X_1, X_2, \ldots$, i.e., it is also an IID process with the same distribution. This property follows from the fact that the r.v.s are identically distributed (in addition to being independent)

The Bernoulli Process

- The Bernoulli process is an infinite sequence $X_1, X_2, \ldots$ of i.i.d. $\mathrm{Bern}(p)$ r.v.s
- The outcome from a Bernoulli process is an infinite sequence of 0s and 1s
- A Bernoulli process is often used to model occurrences of random events; $X_n = 1$ if an event occurs at time n, and 0, otherwise
- Three associated random processes of interest:
  - Binomial counting process: The number of events in the interval $[1, n]$
  - Arrival time process: The time of event arrivals
  - Interarrival time process: The time between consecutive event arrivals
- We discuss these processes and their relationships

Binomial Counting Process

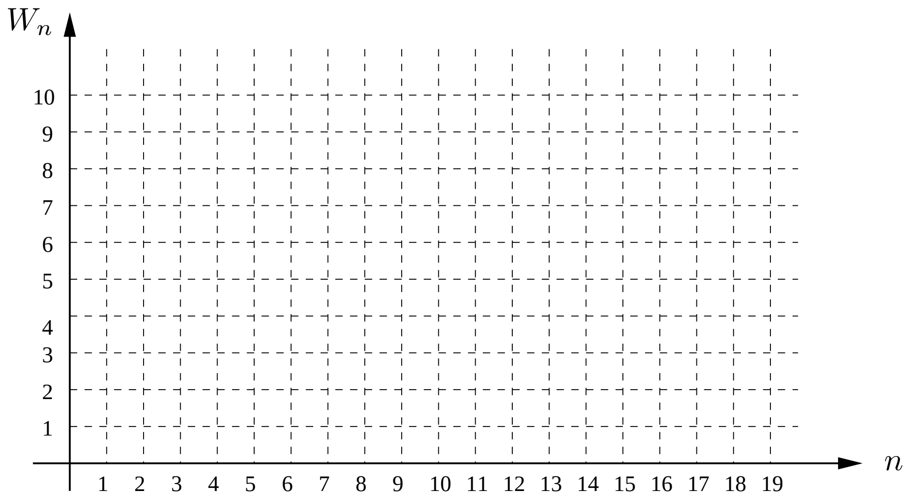
- Consider a Bernoulli process $X_1, X_2, \ldots$ with parameter p
- We are often interested in the number of events occurring in some time interval
- For the time interval $[1, n]$, i.e., $i = 1, 2, \ldots, n$, we know that the number of occurrences

$$W_n = \left(\sum_{i=1}^{n} X_i \right) \sim \mathrm{B}(n, p)$$

- The sequence of r.v.s $W_1, W_2, \ldots$ is referred to as a *Binomial counting process*
- The Bernoulli process can be obtained from the Binomial counting process as:

$$X_n = W_n - W_{n-1}, \text{ for } n = 1, 2, \ldots,$$

  where $W_0 = 0$
- Outcomes of a Binomial process are integer valued stair-case functions



X_n 1 0 0 1 1 0 1 1 0 1 1 0 0 0 0 1 0 1 0

- Note that the Binomial counting process is not IID
- By the fresh-start property of the Bernoulli process, for any $n \geq 1$ and $k \geq 1$, the distribution of the number of events in the interval $[k+1, n+k]$ is identical to that of $[1, n]$, i.e., W_n and $(W_{k+n} - W_k)$ are identically distributed

- Example: Packet arrivals at a node in a communication network can be modeled by a Bernoulli process with $p = 0.09$.
  1. What is the probability that 3 packets arrive in the interval $[1, 20]$, 6 packets arrive in $[1, 40]$ and 12 packets arrive in $[1, 80]$?
  2. The input queue at the node has a capacity of 10^3 packets. A packet is dropped if the queue is full. What is the probability that one or more packets are dropped in a time interval of length $n = 10^4$?

  Solution: Let W_n be the number of packets arriving in interval $[1, n]$.

  1. We want to find the following probability

$$\mathrm{P}\{W_{20} = 3, W_{40} = 6, W_{80} = 12\},$$

which is equal to

$$\mathrm{P}\{W_{20} = 3, W_{40} - W_{20} = 3, W_{80} - W_{40} = 6\}$$

By the independence property of the Bernoulli process this is equal to

$$\mathrm{P}\{W_{20} = 3\}\mathrm{P}\{W_{40} - W_{20} = 3\}\mathrm{P}\{W_{80} - W_{40} = 6\}$$

Now, by the fresh start property of the Bernoulli process

$$\mathrm{P}\{W_{40} - W_{20} = 3\} = \mathrm{P}\{W_{20} = 3\}, \text{ and}$$
$$\mathrm{P}\{W_{80} - W_{40} = 6\} = \mathrm{P}\{W_{40} = 6\}$$

Thus

$$\mathrm{P}\{W_{20} = 3, W_{40} = 6, W_{80} = 12\} = (\mathrm{P}\{W_{20} = 3\})^2 \times \mathrm{P}\{W_{40} = 6\}$$

Now, using the Poisson approximation of Binomial, we have

$$\mathrm{P}\{W_{20} = 3\} = \binom{20}{3}(0.09)^3(0.91)^{17} \approx \frac{(1.8)^3}{3!}e^{-1.8} = 0.1607$$
$$\mathrm{P}\{W_{40} = 6\} = \binom{40}{6}(0.09)^6(0.91)^{34} \approx \frac{(3.6)^6}{6!}e^{-3.6} = 0.0826$$

Thus

$$\mathrm{P}\{W_{20} = 3, W_{40} = 6, W_{80} = 12\} \approx (0.1607)^2 \times 0.0826 = 0.0021$$

2. The probability that one or more packets are dropped in a time interval of length $n = 10^4$ is

$$\mathrm{P}\{W_{10^4} > 10^3\} = \sum_{n=1001}^{10^4} \binom{10^4}{n}(0.09)^n(0.91)^{10^4-n}$$

Difficult to compute, but we can use the CLT!
Since $W_{10^4} = \sum_{i=1}^{10^4} X_i$ and $\mathrm{E}(X) = 0.09$ and $\sigma_X^2 = 0.09 \times 0.91 = 0.0819$, we have

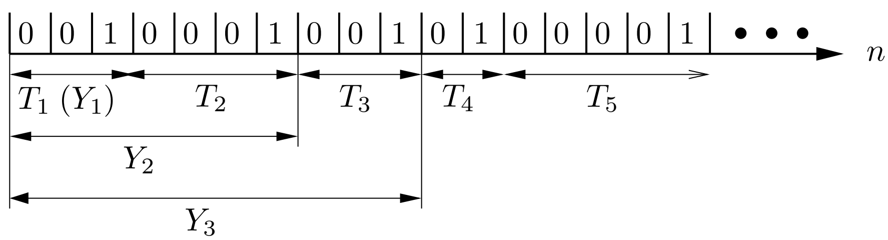
$$\begin{aligned}
\mathrm{P}\left\{\sum_{i=1}^{10^4} X_i > 10^3\right\} &= \mathrm{P}\left\{\frac{1}{100}\sum_{i=1}^{10^4}\frac{(X_i - 0.09)}{\sqrt{0.0819}} > \frac{10^3 - 900}{100\sqrt{0.0819}}\right\} \\
&= \mathrm{P}\left\{\frac{1}{100}\sum_{i=1}^{10^4}\frac{(X_i - 0.09)}{0.286} > 3.5\right\} \\
&\approx Q(3.5) = 2 \times 10^{-4}
\end{aligned}$$

Arrival and Interarrival Time Processes

- Again consider a Bernoulli process $X_1, X_2, \ldots$ as a model for random arrivals of events
- Let Y_k be the time index of the kth arrival, or the kth arrival time, i.e., smallest n such that $W_n = k$
- Define the *interarrival time* process associated with the Bernoulli process as

$$T_1 = Y_1 \text{ and } T_k = Y_k - Y_{k-1}, \text{ for } k = 2, 3, \ldots$$

Thus the kth arrival time is given by: $Y_k = T_1 + T_2 + \ldots + T_k$



- Let's find the pmf of T_k:

First, the pmf of T_1 is the same as the number of coin flips until a head (i.e, a 1) appears. We know that this is $\mathrm{Geom}(p)$. Thus $T_1 \sim \mathrm{Geom}(p)$

Now, having an event at time T_1, the future is a fresh starting Bernoulli process. Thus, the number of trials T_2 until the next event has the *same* pmf as T_1

Moreover, T_1 and T_2 are independent, since the trials from 1 to T_1 are independent of the trials from $T_1 + 1$ onward. Since T_2 is determined exclusively by what happens in these future trials, it's independent of T_1

Continuing similarly, we conclude that $T_1, T_2, \ldots$ are i.i.d., i.e., the interarrival process is an IID $\mathrm{Geom}(p)$ process

- The interarrival process gives us an alternate definition of a Bernoulli process:

Start with an IID $\mathrm{Geom}(p)$ process $T_1, T_2, \ldots$. Record the arrival of an event at time T_1, $T_1 + T_2$, $T_1 + T_2 + T_3$,. . .

- Arrival time process: The sequence of r.v.s $Y_1, Y_2, \ldots$ is denoted by the arrival time process. From its relationship to the interarrival time process $Y_1 = T_1$, $Y_k = \sum_{i=1}^k T_i$, we can easily find the mean and variance of Y_k for any k

$$\mathrm{E}(Y_k) = \mathrm{E}\left(\sum_{i=1}^k T_i\right) = \sum_{i=1}^k \mathrm{E}(T_i) = k \times \frac{1}{p}$$

$$\mathrm{Var}(Y_k) = \mathrm{Var}\left(\sum_{i=1}^k T_i\right) = \sum_{i=1}^k \mathrm{Var}(T_i) = k \times \frac{1-p}{p^2}$$

Note that, $Y_1, Y_2, \ldots$ is *not* an IID process

It is also not difficult to show that the pmf of Y_k is

$$p_{Y_k}(n) = \binom{n-1}{k-1} p^k (1-p)^{n-k} \text{ for } n = k, k+1, k+2, \ldots,$$

which is called the Pascal pmf of order k

- Example: In each minute of a basketball game, Alicia commits a foul independently with probability p and no foul with probability $1-p$. She stops playing if she commits her sixth foul or plays a total of 30 minutes. What is the pmf of of Alicia's playing time?

Solution: We model the foul events as a Bernoulli process with parameter p

Let Z be the time Alicia plays. Then

$$Z = \min\{Y_6, 30\}$$

The pmf of Y_6 is

$$p_{Y_6}(n) = \binom{n-1}{5} p^6 (1-p)^{n-6},\ n = 6, 7, \ldots$$

Thus the pmf of Z is

$$p_Z(z) = \begin{cases} \binom{z-1}{5} p^6 (1-p)^{z-6}, & \text{for } z = 6, 7, \ldots, 29 \\ 1 - \sum_{z=6}^{29} p_Z(z), & \text{for } z = 30 \\ 0, & \text{otherwise} \end{cases}$$

Markov Processes

- A discrete-time random process $X_0, X_1, X_2, \ldots$, where the X_ns are discrete-valued r.v.s, is said to be a *Markov process* if for all $n \geq 0$ and all $(x_0, x_1, x_2, \ldots, x_n, x_{n+1})$

$$\mathrm{P}\{X_{n+1} = x_{n+1} | X_n = x_n, \ldots, X_0 = x_0\} = \mathrm{P}\{X_{n+1} = x_{n+1} | X_n = x_n\},$$

  i.e., the *past*, $X_{n-1}, \ldots, X_0$, and the *future*, X_{n+1}, are conditionally independent given the *present* X_n

- A similar definition for continuous-valued Markov processes can be provided in terms of pdfs

- Examples:
  - Any IID process is Markov
  - The Binomial counting process is Markov

Markov Chains

- A discrete-time Markov process $X_0, X_1, X_2, \ldots$ is called a *Markov chain* if
  - For all $n \geq 0$, $X_n \in \mathcal{S}$, where $\mathcal{S}$ is a finite set called the *state space*. We often assume that $\mathcal{S} \in \{1, 2, \ldots, m\}$
  - For $n \geq 0$ and $i,\, j \in \mathcal{S}$

$$\mathrm{P}\{X_{n+1} = j | X_n = i\} = p_{ij}, \text{ independent of } n$$

  So, a Markov chain is specified by a *transition probability matrix*

$$\mathcal{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

  Clearly $\sum_{j=1}^{m} p_{ij} = 1,$ for all i, i.e., the sum of any row is 1

- By the Markov property, for all $n \geq 0$ and all states

$$\mathrm{P}\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} = \mathrm{P}\{X_{n+1} = j | X_n = i\} = p_{ij}$$

- Markov chains arise in *many* real world applications:
  - Computer networks
  - Computer system reliability
  - Machine learning
  - Pattern recognition
  - Physics
  - Biology
  - Economics
  - Linguistics

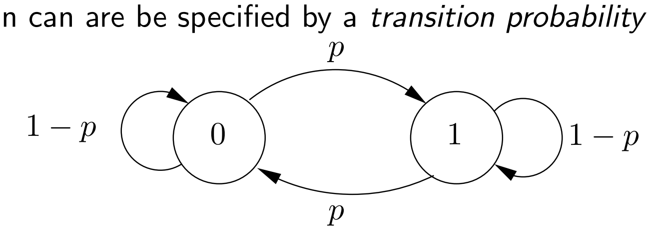
Examples

- Any IID process with discrete and finite-valued r.v.s is a Markov chain
- *Binary Symmetric Markov Chain*: Consider a sequence of coin flips, where each flip has probability of $1 - p$ of having the same outcome as the previous coin flip, regardless of all previous flips

  The probability transition matrix (head is 1 and tail is 0) is

$$\mathcal{P} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$

  A Markov chain can are be specified by a *transition probability graph*



  Nodes are states, arcs are state transitions; (i, j) from state i to state j (only draw transitions with $p_{ij} > 0$)

Can construct this process from an IID process: Let $Z_1, Z_2, \ldots$ be a Bernoulli process with parameter p

The Binary symmetric Markov chain X_n can be defined as

$$X_{n+1} = X_n + Z_n \mod 2, \text{ for } n = 1, 2, \ldots,$$

So, each transition corresponds to passing the r.v. X_n through a binary symmetric channel with additive noise $Z_n \sim \mathrm{Bern}(p)$

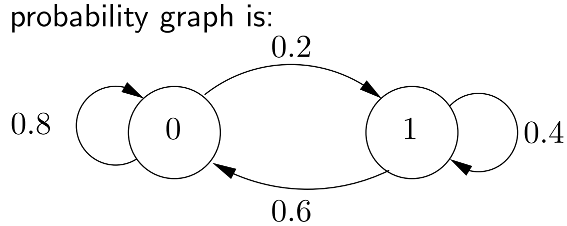
- Example: *Asymmetric binary Markov chain*

  State 0 = Machine is working, State 1 = Machine is broken down

  The probability transition matrix is

$$\mathcal{P} = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

  and the transition probability graph is:



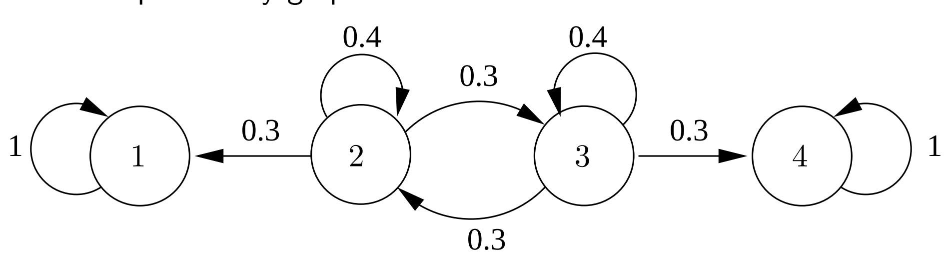
- Example: *Two spiders and a fly*
  A fly's possible positions are represented by four states

  States 2, 3 : safely flying in the left or right half of a room
  State 1: A spider's web on the left wall
  State 4: A spider's web on the right wall

  The probability transition matrix is:

$$\mathcal{P} = \begin{bmatrix} 1.0 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0 & 0.3 & 0.4 & 0.3 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

  The transition probability graph is:



- Given a Markov chain model, we can compute the probability of a sequence of states given an initial state $X_0 = i_0$ using the chain rule as:

$$\mathrm{P}\{X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n | X_0 = i_0\} = p_{i_0 i_1} p_{i_1 i_2} \ldots p_{i_{n-1} i_n}$$

Example: For the spider and fly example,

$$\begin{aligned}\mathrm{P}\{X_1 = 2, X_2 = 2, X_3 = 3 | X_0 = 2\} &= p_{22} p_{22} p_{23} \\ &= 0.4 \times 0.4 \times 0.3 \\ &= 0.048\end{aligned}$$

- There are many other questions of interest, including:
  - n-state transition probabilities: Beginning from some state i what is the probability that in n steps we end up in state j
  - Steady state probabilities: What is the expected fraction of time spent in state i as $n \to \infty$?

n-State Transition Probabilities

- Consider an m-state Markov Chain. Define the n-step transition probabilities as

$$r_{ij}(n) = \mathrm{P}\{X_n = j | X_0 = i\} \text{ for } 1 \le i,\ j \le m$$

- The n-step transition probabilities can be computed using the *Chapman-Kolmogorov* recursive equation:

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj} \text{ for } n > 1, \text{ and all } 1 \le i,\ j \le m,$$

starting with $r_{ij}(1) = p_{ij}$

This can be readily verified using the law of total probability

- We can view the $r_{ij}(n)$, $1 \leq i,\, j \leq m$, as the elements of a matrix $R(n)$, called the n-*step transition probability matrix*, then we can view the Kolmogorov-Chapman equations as a sequence of matrix multiplications:

$$\begin{aligned}
R(1) &= \mathcal{P} \\
R(2) &= R(1)\mathcal{P} = \mathcal{P}\mathcal{P} = \mathcal{P}^2 \\
R(3) &= R(2)\mathcal{P} = \mathcal{P}^3 \\
&\vdots \\
R(n) &= R(n-1)\mathcal{P} = \mathcal{P}^n
\end{aligned}$$

- Example: For the binary asymmetric Markov chain

$$\begin{aligned}
R(1) = \mathcal{P} &= \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \\
R(2) = \mathcal{P}^2 &= \begin{bmatrix} 0.76 & 0.24 \\ 0.72 & 0.28 \end{bmatrix} \\
R(3) = \mathcal{P}^3 &= \begin{bmatrix} 0.752 & 0.248 \\ 0.744 & 0.256 \end{bmatrix} \\
R(4) = \mathcal{P}^4 &= \begin{bmatrix} 0.7504 & 0.2496 \\ 0.7488 & 0.2512 \end{bmatrix} \\
R(5) = \mathcal{P}^5 &= \begin{bmatrix} 0.7501 & 0.2499 \\ 0.7498 & 0.2502 \end{bmatrix}
\end{aligned}$$

  In this example, each r_{ij} seems to converge to a non-zero limit independent of the initial state, i.e., each state has a *steady state* probability of being occupied as $n \to \infty$

• Example: Consider the spiders-and-fly example

$$\mathcal{P} = \begin{bmatrix} 1.0 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0 & 0.3 & 0.4 & 0.3 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

$$\mathcal{P}^2 = \begin{bmatrix} 1.0 & 0 & 0 & 0 \\ 0.42 & 0.25 & 0.24 & 0.09 \\ 0.09 & 0.24 & 0.25 & 0.42 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

$$\mathcal{P}^3 = \begin{bmatrix} 1.0 & 0 & 0 & 0 \\ 0.50 & 0.17 & 0.17 & 0.16 \\ 0.16 & 0.17 & 0.17 & 0.50 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

$$\mathcal{P}^4 = \begin{bmatrix} 1.0 & 0 & 0 & 0 \\ 0.55 & 0.12 & 0.12 & 0.21 \\ 0.21 & 0.12 & 0.12 & 0.55 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

As $n \to \infty$, we obtain

$$\mathcal{P}^\infty = \begin{bmatrix} 1.0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 1/3 \\ 1/3 & 0 & 0 & 2/3 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

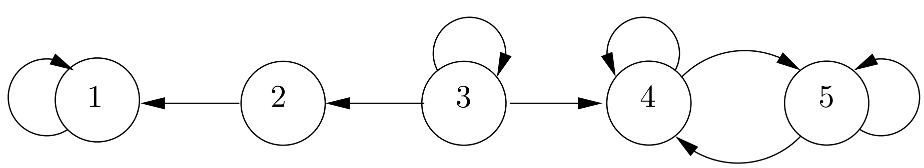
Here r_{ij} converges, but the limit depends on the initial state and can be 0 for some states

Note that states 1 and 4 corresponding to capturing the fly by one of the spiders are *absorbing* states, i.e., they are infinitely repeated once visited
The probability of being in non-absorbing states 2 and 3 diminishes as time increases

Classification of States

- As we have seen, various states of a Markov chain can have different characteristics
- We wish to classify the states by the long-term frequency with which they are visited
- Let $A(i)$ be the set of states that are *accessible* from state i (may include i itself), i.e., can be reached from i in n steps, for some n
- State i is said to be *recurrent* if starting from i, any accessible state j must be such that i is accessible from j, i.e., $j \in A(i)$ iff $i \in A(j)$. Clearly, this implies that if i is recurrent then it must be in $A(i)$
- A state is said to be *transient* if it is not recurrent
- Note that recurrence/transcience is determined by the arcs (transitions with nonzero probability), not by actual values of probabilities

- Example: Classify the states of the following Markov chain



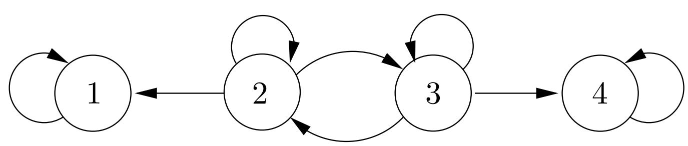
- The set of accessible states $A(i)$ from some recurrent state i is called a *recurrent class*
- Every state k in a recurrent class $A(i)$ is recurrent and $A(k) = A(i)$

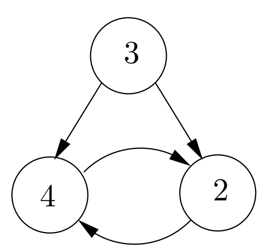
  Proof: Suppose i is recurrent and $k \in A(i)$. Then k is accessible from i and hence, since i is recurrent, k can access i and hence, through i, any state in $A(i)$. Thus $A(i) \subseteq A(k)$

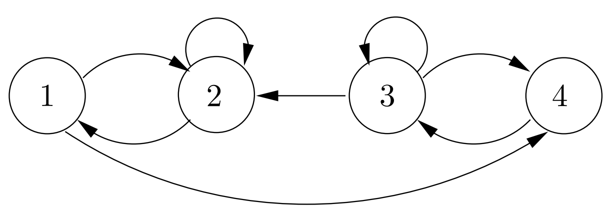
  Since i can access k and hence any state in $A(k)$, $A(k) \subseteq A(i)$. Thus $A(i) = A(k)$. This argument also proves that any $k \in A(i)$ is recurrent

- Two recurrent classes are either identical or disjoint
- Summary:
  - A Markov chain can be decomposed into one or more recurrent classes plus possibly some transient states
  - A recurrent state is accessible from all states in its class, but it is *not* accessible from states in other recurrent classes
  - A transient state is not accessible from any recurrent state
  - At least one recurrent state must be accessible from a given transient state

- Example: Find the recurrent classes in the following Markov chains

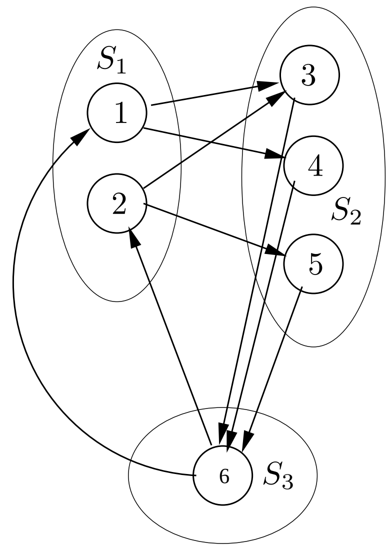




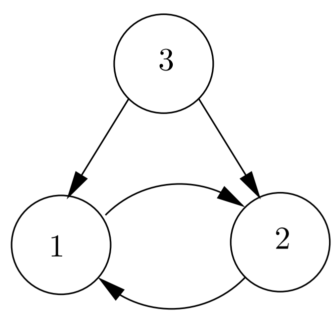


Periodic Classes

- A recurrent class A is called *periodic* if its states can be grouped into $d > 1$ disjoint subsets $S_1, S_2, \ldots, S_d$, $\cup_{i=1}^{d} S_i = A$, such that all transitions from one subset lead to the next subset



- Example: Consider a Markov chain with probability transition graph



Note that the recurrent class $\{1, 2\}$ is periodic and for $i = 1, 2$

$$r_{ii}(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

- Note that if the class is periodic, $r_{ii}(n)$ never converges to a steady-state

Steady State Probabilities

- *Steady-state convergence theorem*: If a Markov chain has only one recurrent class and it is not periodic, then $r_{ij}(n)$ tends to a steady-state π_j *independent* of i, i.e.,
$$\lim_{n\to\infty} r_{ij}(n) = \pi_j \text{ for all } i$$
- *Steady-state equations*: Taking the limit as $n \to \infty$ of the Chapman-Kolmogorov equations
$$r_{ij}(n+1) = \sum_{k=1}^{m} r_{ik}(n) p_{kj} \text{ for } 1 \le i, j \le m,$$
we obtain the set of linear equations, called the *balance equations*:
$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj} \text{ for } j = 1, 2, \ldots, m$$
The balance equations together with the *normalization* equation
$$\sum_{j=1}^{m} \pi_j = 1,$$

uniquely determine the steady state probabilities $\pi_1, \pi_2, \ldots, \pi_m$

- The balance equations can be expressed in a matrix form as:
$$\Pi \mathcal{P} = \Pi, \text{ where } \Pi = [\pi_1\ \pi_2\ \ldots,\ \pi_m]$$
- In general there are $m-1$ linearly independent balance equations
- The steady state probabilities form a probability distribution over the state space, called the *stationary distribution* of the chain. If we set $\mathrm{P}\{X_0 = j\} = \pi_j$ for $j = 1, 2, \ldots, m$, we have $\mathrm{P}\{X_n = j\} = \pi_j$ for all $n \ge 1$ and $j = 1, 2, \ldots, m$
- Example: *Binary Symmetric Markov Chain*
$$\mathcal{P} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$
Find the steady-state probabilities

Solution: We need to solve the balance and normalization equations
$$\begin{aligned}\pi_1 &= p_{11}\pi_1 + p_{21}\pi_2 = (1-p)\pi_1 + p\pi_2 \\ \pi_2 &= p_{12}\pi_1 + p_{22}\pi_2 = p\pi_1 + (1-p)\pi_2 \\ 1 &= \pi_1 + \pi_2\end{aligned}$$

Note that the first two equations are linearly dependent. Both yield $\pi_1 = \pi_2$. Substituting in the last equation, we obtain $\pi_1 = \pi_2 = 1/2$

- Example: *Asymmetric binary Markov chain*

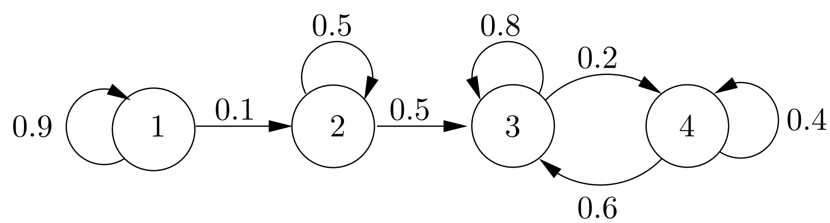
$$\mathcal{P} = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

The steady state equations are:

$$\pi_1 = p_{11}\pi_1 + p_{21}\pi_2 = 0.8\pi_1 + 0.6\pi_2, \text{ and } 1 = \pi_1 + \pi_2$$

Solving the two equations yields $\pi_1 = 3/4$ and $\pi_2 = 1/4$

- Example: Consider the Markov chain defined by the following probability transition graph



- Note: The solution of the steady state equations yields $\pi_j = 0$ if a state is transient, and $\pi_j > 0$ if a state is recurrent

Long Term Frequency Interpretations

- Let $v_{ij}(n)$ be the # of times state j is visited beginning from state i in n steps
- For a Markov chain with a single aperiodic recurrent class, can show that

$$\lim_{n\to\infty} \frac{v_{ij}(n)}{n} = \pi_j$$

  Since this result doesn't depend on the starting state i, π_j can be interpreted as the long-term frequency of visiting state j

- Since each time state j is visited, there is a probability p_{jk} that the next transition is to state k, $\pi_j p_{jk}$ can be interpreted as the long-term frequency of *transitions* from j to k
- These frequency interpretations allow for a simple interpretation of the balance equations, that is, the long-term frequency π_j is the sum of the long-term frequencies $\pi_k p_{kj}$ of transitions that lead to j

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj}$$

- Another interpretation of the balance equations: Rewrite the LHS of the balance equation as

$$\begin{aligned}\pi_j &= \pi_j \sum_{k=1}^{m} p_{jk} \\ &= \sum_{k=1}^{m} \pi_j p_{jk} = \pi_j p_{jj} + \sum_{k=1, k\neq j}^{m} \pi_j p_{jk}\end{aligned}$$

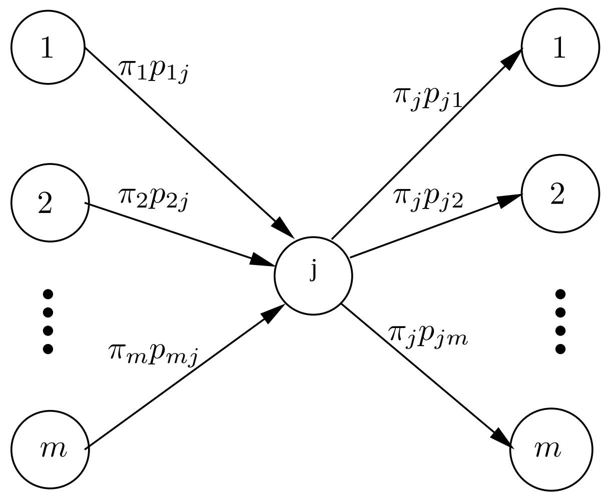
  The RHS can be written as

$$\sum_{k=1}^{m} \pi_k p_{kj} = \pi_j p_{jj} + \sum_{k=1, k\neq j}^{m} \pi_k p_{kj}$$

  Subtracting the $p_{jj}\pi_j$ from both sides yields

$$\sum_{k=1, k\neq j}^{m} \pi_k p_{kj} = \sum_{k=1, k\neq j}^{m} \pi_j p_{jk},\ j = 1, 2, \ldots, m$$

The long-term frequency of transitions into j is equal to the long-term frequency of transitions out of j

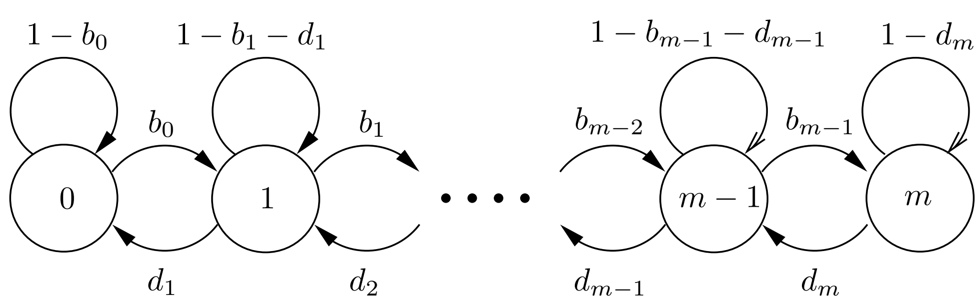


This interpretation is similar to Kirkoff's current law. In general, if we partition a chain (with single aperiodic recurrent class) into two sets of states, the long-term frequency of transitions from the first set to the second is equal to the long-term frequency of transitions from the second to the first

Birth-Death Processes

- A *birth-death* process is a Markov chain in which the states are linearly arranged and transitions can only occur to a neighboring state, or else leave the state unchanged
- For a birth-death Markov chain use the notation:

$$b_i = \mathrm{P}\{X_{n+1} = i+1 | X_n = i\}, \text{ birth probability at state } i$$
$$d_i = \mathrm{P}\{X_{n+1} = i-1 | X_n = i\}, \text{ death probability at state } i$$



- For a birth-death process the balance equations can be greatly simplified: Cut the chain between states $i-1$ and i. The long-term frequency of transitions from right to left must be equal to the long-term frequency of transitions from left to right, thus:

$$\pi_i d_i = \pi_{i-1} b_{i-1}, \text{ or } \pi_i = \pi_{i-1} \frac{b_{i-1}}{d_i}$$

- By recursive substitution, we obtain

$$\pi_i = \pi_0 \frac{b_0 b_1 \ldots b_{i-1}}{d_1 d_2 \ldots d_i}, \; i = 1, 2, \ldots, m$$

To obtain the steady state probabilities we use these equations together with the normalization equation

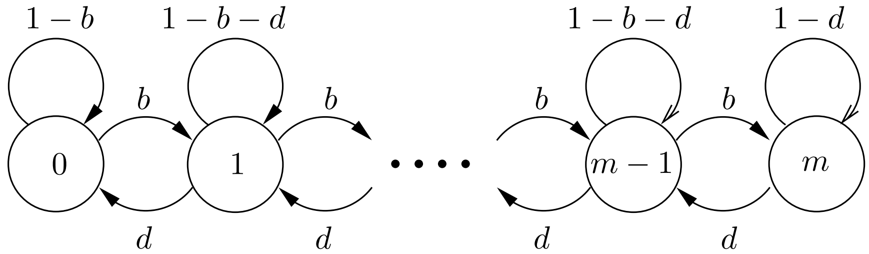
$$1 = \sum_{j=0}^{m} \pi_j$$

Examples

- Queuing: Packets arrive at a communication node with buffer size m packets. Time is discretized in small periods. At each period:
  - If the buffer has less than m packets, the probability of 1 packet added to it is b, and if it has m packets, the probability of adding another packet is 0
  - If there is at least 1 packet in the buffer, the probability of 1 packet leaving it is $d > b$, and if it has 0 packets, this probability is 0
  - If the number of packets in the buffer is from 1 to $m-1$, the probability of no change in the state of the buffer is $1-b-d$. If the buffer has no packets, the probability of no change in the state is $1-b$, and if there are m packets in the buffer, this probability is $1-d$

  We wish to find the long-term frequency of having i packets in the queue

We introduce a birth-death Markov chain with states $0, 1, \ldots, m$, corresponding to the number of packets in the buffer



The local balance equations are

$$\pi_i d = \pi_{i-1} b, \ i = 1, \ldots, m$$

Define $\rho = b/d < 1$, then $\pi_i = \rho \pi_{i-1}$, which leads to

$$\pi_i = \rho^i \pi_0$$

Using the normalizing equation: $\sum_{i=0}^{m} \pi_i = 1$, we obtain

$$\pi_0 (1 + \rho + \rho^2 + \cdots + \rho^m) = 1$$

Hence for $i = 1, \ldots, m$

$$\pi_i = \frac{\rho^i}{1 + \rho + \rho^2 + \cdots + \rho^m}$$

Using the geometric progression formula, we obtain

$$\pi_i = \rho^i \frac{1 - \rho}{1 - \rho^{m+1}}$$

Since $\rho < 1$, $\pi_i \to \rho^i (1 - \rho)$ as $m \to \infty$, i.e., $\{\pi_i\}$ converges to Geometric pmf

- *The Ehrenfest model*: This is a Markov chain arising in statistical physics. It models the diffusion through a membrane between two containers. Assume that the two containers have a total of $2a$ molecules. At each step a molecule is selected at random and moved to the other container (so a molecule diffuses at random through the membrane). Let Y_n be the number of molecules in container 1 at time n and $X_n = Y_n - a$. Then X_n is a birth-death Markov chain with $2a + 1$ states; $i = -a, -a + 1, \ldots, -1, 0, 1, 2, \ldots, a$ and probability transitions

$$p_{ij} = \begin{cases} b_i = (a - i)/2a, & \text{if } j = i + 1 \\ d_i = (a + i)/2a, & \text{if } j = i - 1 \\ 0, & \text{otherwise} \end{cases}$$

The steady state probabilities are given by:

$$
\begin{aligned}
\pi_i &= \pi_{-a}\frac{b_{-a}b_{-a+1}\dots b_{i-1}}{d_{-a+1}d_{-a+2}\dots d_i},\ i=-a,-a+1,\dots,-1,0,1,2,\dots,a \\
&= \pi_{-a}\frac{2a(2a-1)\dots(a-i+1)}{1\times 2\times\dots\times(a+i)} \\
&= \pi_{-a}\frac{2a!}{(a+i)!(2a-(a+i))!} = \binom{2a}{a+i}\pi_{-a}
\end{aligned}
$$

Now the normalization equation gives

$$\sum_{i=-a}^{a}\binom{2a}{a+i}\pi_{-a} = 1$$

Thus, $\pi_{-a} = 2^{-2a}$

Substituting, we obtain

$$\pi_i = \binom{2a}{a+i}2^{-2a}$$