Math 250 Packets

## Contents

## Part I

# Daily Worksheets

# 1 Welcome to Calc 3

Today our **Goals** are to

- Get to know each other
- Learn a little bit about the course
- Review a bit from Calculus 1 and 2

Please stand up and find a group of people to talk to:

- Get to know each other. I'll ask you to introduce your group members
- Discuss: "What is something you remember about **derivatives**. What do they measure, why would we care, applications, etc." No worries if you've forgotten, but try to come up with some memories.

STOP

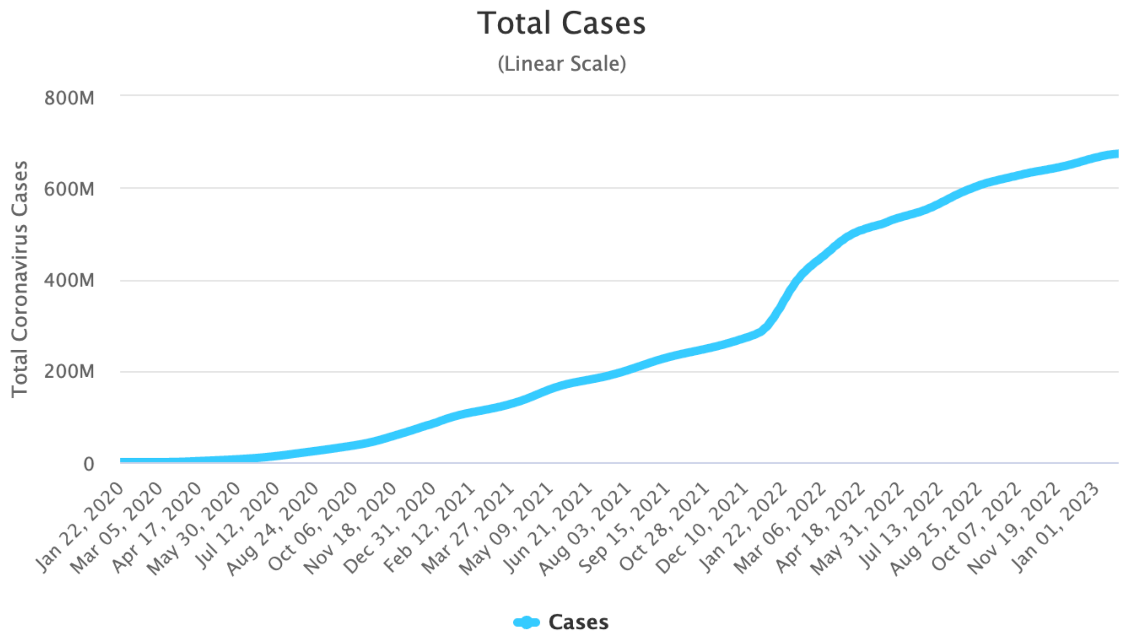
Now find a NEW group of people:

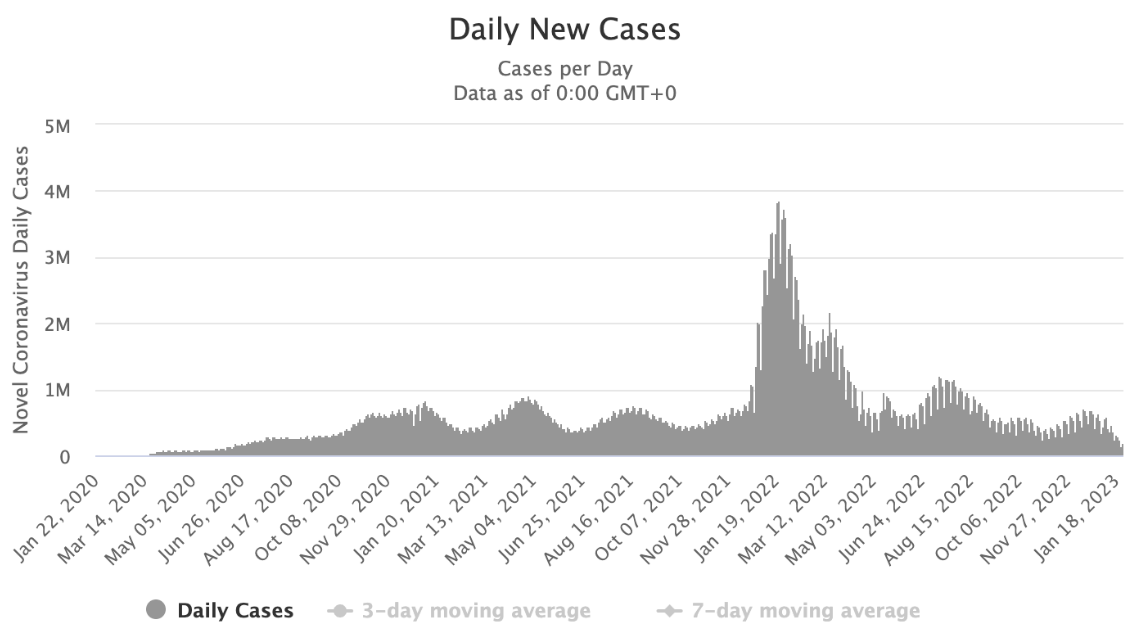
- Get to know each other. I'll ask you to introduce your group members
- Discuss: "What is something you remember about **integrals**. What do they measure, why would we care, applications, etc." No worries if you've forgotten, but try to come up with some memories.

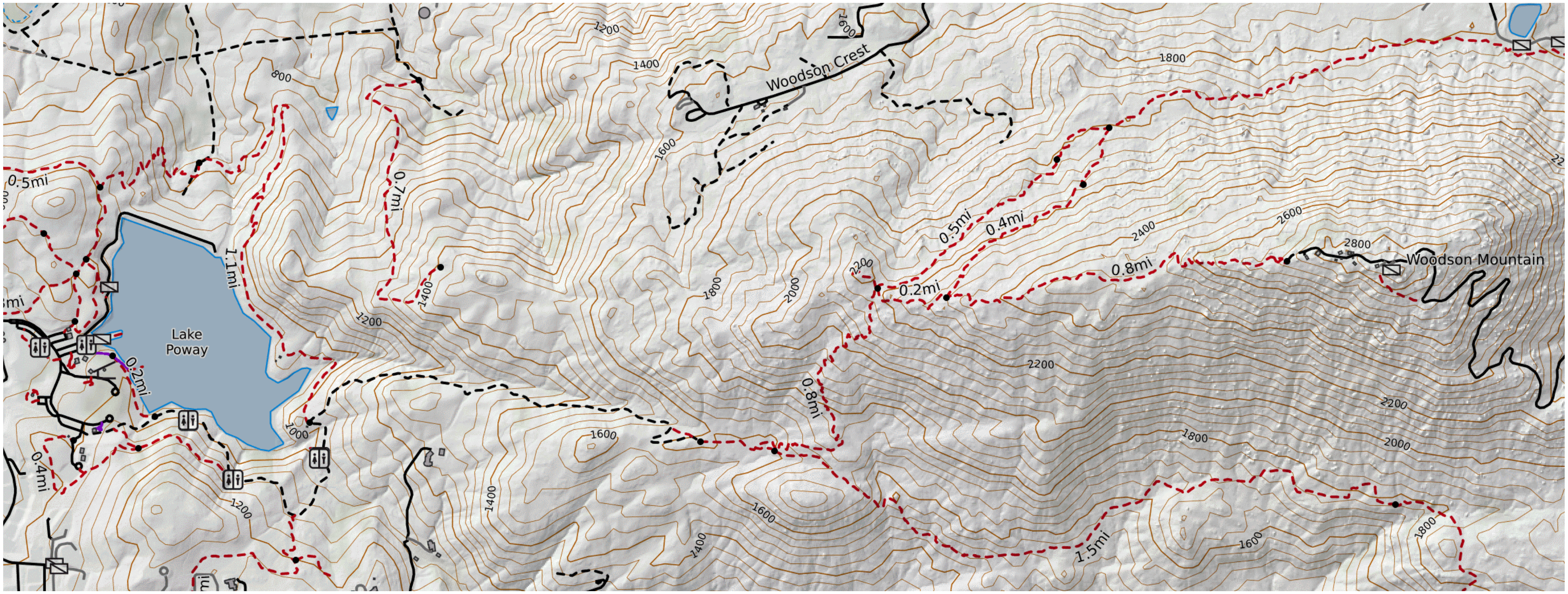
STOP

Now find a NEW group of people:

- Get to know each other. I'll ask you to introduce your group members
- Discuss: "What were some of the most challenging parts of Calc 1 and 2? What are you excited about / worried about in Calc 3?"


Total Cases
(Linear Scale)
800M
600M
400M
200M
0
Total Coronavirus Cases
Jan 22, 2020
Mar 05, 2020
Apr 17, 2020
May 30, 2020
Jul 12, 2020
Aug 24, 2020
Oct 06, 2020
Nov 18, 2020
Dec 31, 2020
Feb 12, 2021
Mar 27, 2021
May 09, 2021
Jun 21, 2021
Aug 03, 2021
Sep 15, 2021
Oct 28, 2021
Dec 10, 2021
Jan 22, 2022
Mar 06, 2022
Apr 18, 2022
May 31, 2022
Jul 13, 2022
Aug 25, 2022
Oct 07, 2022
Nov 19, 2022
Jan 01, 2023
Cases


Daily New Cases
Cases per Day
Data as of 0:00 GMT+0
5M
4M
3M
2M
1M
0
Novel Coronavirus Daily Cases
Jan 22, 2020
Mar 14, 2020
May 05, 2020
Jun 26, 2020
Aug 17, 2020
Oct 08, 2020
Nov 29, 2020
Jan 20, 2021
Mar 13, 2021
May 04, 2021
Jun 25, 2021
Aug 16, 2021
Oct 07, 2021
Nov 28, 2021
Jan 19, 2022
Mar 12, 2022
May 03, 2022
Jun 24, 2022
Aug 15, 2022
Oct 06, 2022
Nov 27, 2022
Jan 18, 2023
Daily Cases
3-day moving average
7-day moving average

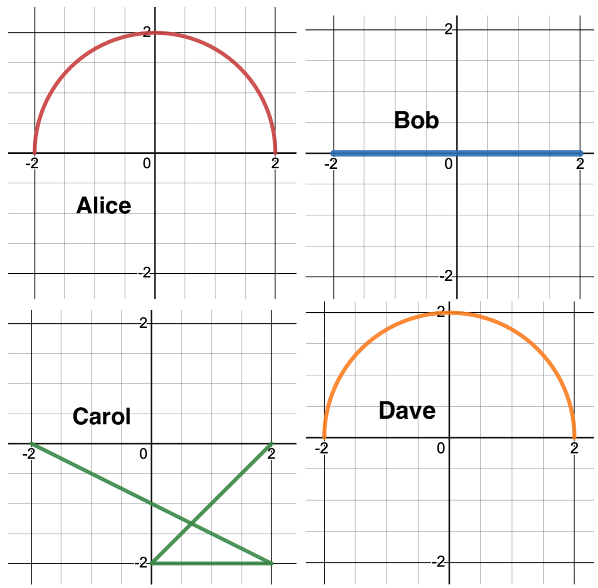



Reading for Monday's Class: Section 11.1 on Parametric Equations

## 2 What is a parametric equation?

**Warmup:**

Four friends, named Alice, Bob, Carol, Dave all decide to have a race from START (2,0) to the FINISH (-2,0). Their paths are below:




**Discuss the following questions with your neighbors:**

- Who won the race?
- Who ran the longest distance? The shortest?

STOP

**Have you got the time?**

When we ________________________ a curve in the plane,

we are describing its coordinates in terms of a ________________________.

It's helpful to think of $t$ as representing ____________.

**Example:** Let's have desmos animate the curve with parametrization $c(t) = (\cos t, \sin t)$. How can we modify this equation? What would it do to the curve? Does anyone see an implicit equation for the curve in terms only of $x$ and $y$? How do the values of the parameter $t$ matter in this?

Depending on the situation, we need to know if we're interested in the "curve" which is just the set of points, or "a parametrization of the curve" which is a "movie" of a particle moving along the curve.

**Example:**

A bug is flying around the $xy$ plane so that its position at time $t$ is given by

$$x(t) = 2t - 4, \quad y(t) = 3 + t^2.$$

Determine where the bug is at time $t = -2, 0, 2, 4$. Which way is the bug moving? Can you find a **Cartesian equation** relating the $x$ and $y$ coordinates, by **eliminating the parameter $t$?**

Is there another way we can parametrize this curve?

Important Points

There are always many ways to parametrize a curve.

- Sometimes it matters which one we choose.and Dave.
- Other times we'll see that it doesn't matter (Nice!)

In this class it will be crucially important to be able to:

**Parametrize the graph of a function** $y = f(x)$.

**Parametrize parts of a circle** (we'll keep coming back to this)

**Parametrize a line segment**

**Example:** Parametrize line of slope 2 through the point $(1, 3)$ in two different ways.

**Example:** Parametrize the left half of the circle of radius 3 centered at $(2, -1)$

**Tangent lines and speed:**

For parametrized curve $(x(t), y(t))$:

$x'(t)$ measures the $x$-coordinate's ______________________.

$y'(t)$ measures the $y$-coordinate's ______________________.

To describe the tangent to the curve at time $t$ its slope will be given by

and its speed $s(t)$ at time $t$ is given by

**Example:** Consider the following parametrized curves, all for

$a(t) = (t^2, 2t^2)$

$b(t) = (t, 2t)$

$c(t) = (3t, 6t)$

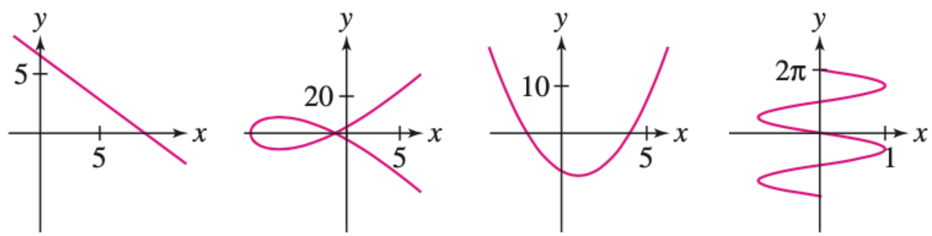
$d(t) = (\cos t, 2\cos t)$

Before we graph these on Desmos - what do we expect? What if we range $t$ from $-4$ to $4$? From $-\infty$ to $\infty$?

**Example:** Consider the curve parametrized as

$$c(t) = \left(t^2 - 9, 8t - t^3\right).$$

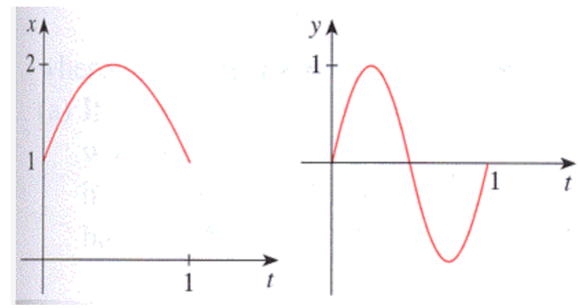
How many times will the $x$-coordinate be 0? How many times will the $y-$coordinate be 0? What is the smallest the $x$ coordinate can ever be? The $y$ coordinate? Which of the following graphs **could** be the graph of our curve? Could you find the equations of the tangent line at a given point?



(I) (II) (III) (IV)

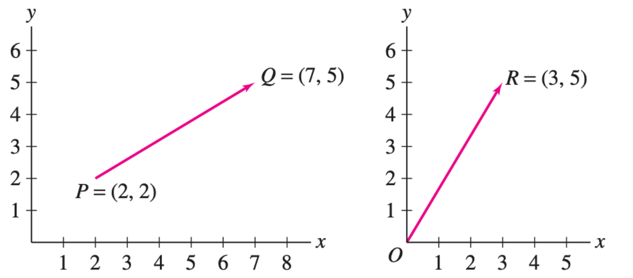
**Example:** Below you are given information about $x(t)$ and $y(t)$. Sketch a graph of $c(t) = (x(t), y(t))$ using this informaiton.

Hints: What happens when $t = 0$? $t = 1$? Can you add this to your graph? You can do it! This is an important skill that will feel hard.



# 3 What is a vector?

A ______________ in the plane is determined by ____________________________, called the head and tail.




(A) The vector $\overrightarrow{PQ}$ (B) The position vector $\overrightarrow{OR}$

If a vector starts at the ______________ and goes to

a point $R$, we call that vector

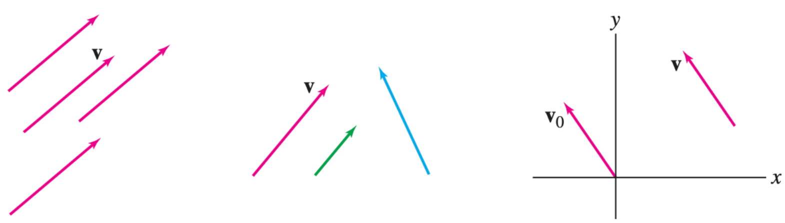
Note that every vector has a ____________________________

and a ____________________________.

Most important thing to know about vectors

We will consider two vectors **equivalent** if one can be translated (without stretching or rotating) to the other.

***Every vector* $\mathbf{v}$ *is equivalent to a unique vector* $\mathbf{v}_0$ *based at the origin.***




(A) Vectors equivalent to $\mathbf{v}$ (translates of $\mathbf{v}$) (B) Inequivalent vectors (C) $\mathbf{v}_0$ is the unique vector based at the origin and equivalent to $\mathbf{v}$.

FIGURE 3

**Example:**

Find the unique vector based at the origin that is equivalent to the vector $\overrightarrow{PQ}$ if $P = (3, 7)$ and $Q = (6, 5)$.

What is its length?

How can you tell if $\overrightarrow{PQ}$ is equivalent to the vector $\overrightarrow{AB}$ where $A = (-1, 4)$ and $B = (2, 1)$?

Pro-tip

It's always a good idea to translate vectors to the origin. If you do this, you can immediately check if they are equal by comparing their components.

**Notation:** The vector with tail at the origin and head at the point $(a, b)$ will be denoted ______________.

In the previous example, we saw that the length of the vector $\langle a, b \rangle$ is

**Why do we care about vectors?**

- It's helpful to think about vectors as "forces", perhaps like the **pull of a rope** or the **push of the wind.**
- Forces can be combined - imagine two ropes pulling in different directions and their forces being **added**.
- Imagine the strength of a force being **multipled** by a constant. (A constant is called a _____________. )
- If one vector is a scalar multiple of the other, we say that these vectors are _____________.

**Example:** Find the sum of the two vectors: $\mathbf{v} = \langle 2, 1\rangle$ and $\mathbf{w} = \langle 1, 3\rangle$.

**Example:** In general, let's draw pictures of $\mathbf{v}, \mathbf{w}$ and see how to "add" and "subtract" them.

Suppose that $\mathbf{v} = \langle 1, 2\rangle$ and $\mathbf{w} = \langle -3, 4\rangle$, find $||3\mathbf{v} - \mathbf{w}||$ (this denotes the **length** or **magnitude**.)

Multiplying vectors by scalars and adding them together is called a _____________ combination.
In many areas of engineering you might want to "break down a force/signal/data set" into a combination of "simpler" pieces. This is the subject of a **linear algebra** course.

**Example:** A plane is flying North East at a speed of 400 miles per hour. There is a wind that is blowing 50 mph due north. With these combined forces, in what direction (and at what speed) will the plane actually be flying?

A ____________ vector is a vector of length 1.

**Example:** Find a unit vector that is parallel to the vector $\langle 3,5\rangle$. Write it as a **linear combination** of the vectors

$$\mathbf{i} = \langle 1,0\rangle, \quad \mathbf{j} = \langle 0,1\rangle$$

Every vector in the plane can be broken down into a combination of the very special vectors $\mathbf{i}$ and $\mathbf{j}$.

■ **EXAMPLE 6** Find the forces on cables 1 and 2 in Figure 19(A).

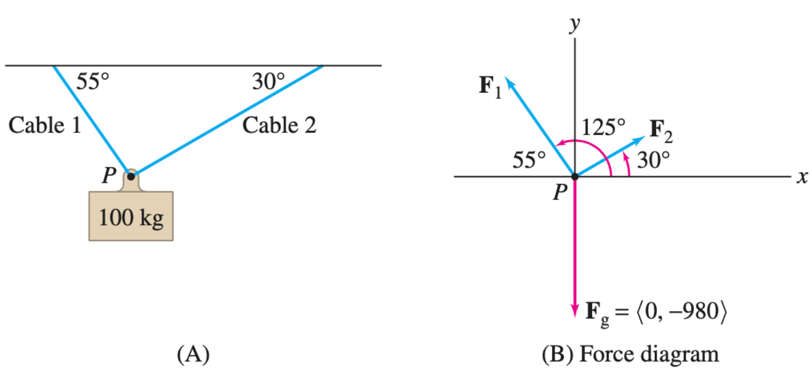



FIGURE 19

# 4 Group Work and More on Vectors

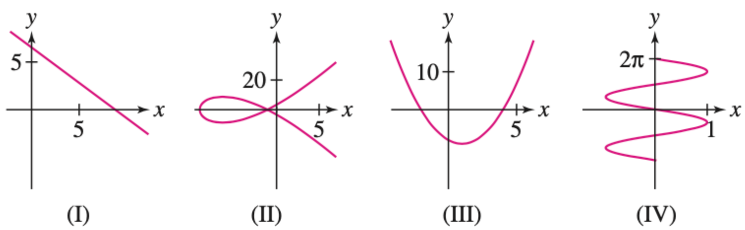
**Warmup (15 minutes)**

Let's warmup with some group activities

**Problem 1:** Consider the curve parametrized as

$$c(t) = \left(t^2 - 9, 8t - t^3\right).$$

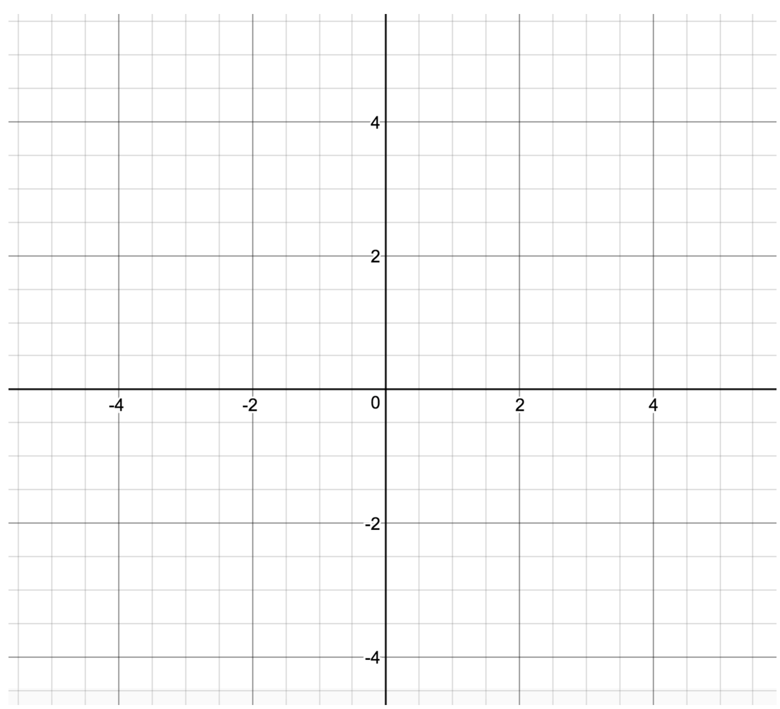
How many times will the $x$-coordinate be 0? How many times will the $y-$coordinate be 0? What is the smallest the $x$ coordinate can ever be? The $y$ coordinate? Can they both be zero at the same time? Which of the following graphs **could** be the graph of our curve? Could you find the equations of the tangent line at a given point?




**Problem 2:** On the graph paper at the right, draw in the position vectors $\mathbf{v} = \langle 2, 1\rangle$ and $\mathbf{w} = \langle 1, 3\rangle$. To "add" them together we just add the separate coordinates. Draw this vector in your picture as well and see if you can make sense of how what is going on geometrically.

$$\mathbf{v} + \mathbf{w} = \langle 2, 1\rangle + \langle 1, 3\rangle = \langle 3, 4\rangle$$

Make sense of the statement: "If I add two vectors it's like following one burst of wind followed by another."



The **plane** is called $\mathbb{R}^2$ because every point has **2** coordinates that are both real numbers (the symbol $\mathbb{R}$.)

In the **3 dimensional space** $\mathbb{R}^3$ every point has **3** coordinates, $(x, y, z)$.

When drawing pictures, we will use the ____________________________________.

**Example:** Plot the point $P = (3, 4, 12)$ and calculate the distance from the origin to $P$.

**The distance formula**

The distance between $(x, y, z)$ and $(a, b, c)$ is given by

$$D = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}.$$

This is basically the extended **Pythagorean Theorem.**

---

The set of all points in $\mathbb{R}^2$ of distance $r$ from a point $(a, b)$ is best described as a ____________.

The set of all points in $\mathbb{R}^3$ of distance $r$ from a point $(a, b)$ is best described as a ____________.

**Example:** Describe the sets of solutions to the following equations in $\mathbb{R}^3$:

$(x-3)^2 + (y+2)^2 + (z)^2 = 9$ $\qquad$ $x^2 + y^2 = 4,$ $\qquad$ $x^2 + y^2 = 25$

**Example:** What would be the difference between the equation $x = 3$ in

$\mathbb{R}^1$ $\qquad$ $\mathbb{R}^2$ $\qquad$ $\mathbb{R}^3$

What are **vectors** like in $\mathbb{R}^3$? They are pretty much the same as in $\mathbb{R}^2$ but with one extra coordinate.

$$\mathbf{i} = \langle 1, 0, 0 \rangle \quad \mathbf{j} = \langle 0, 1, 0 \rangle \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

**Example:** If $\mathbf{v} = \langle 3, 4, -2 \rangle$ and $\mathbf{w} = \mathbf{j} - \mathbf{k}$ then find $\mathbf{v} + 3\mathbf{w}$.

What is $||\mathbf{v}||$?

Find the unit vector in the direction of $\mathbf{w}$.

**Parametric equation of Lines - an easier way**

What does the set of all scalar multiples of a position vector $\mathbf{v}$ look like in $\mathbb{R}^2$? ____________________________.

**Example:** Describe the **line** in $\mathbb{R}^3$ that is parallel to $\langle 2, 1, 7 \rangle$ that passes through the point $(3, -1, 4)$.

Note that as $t$ goes from 0 to 1, the "bug" moves from _____________ to _____________.

**Example:** Use vectors to parametrize the line through $(1, 2)$ and $(3, 0)$.
Let's make a parametrization that has $c(0) = (1, 2)$ and $c(1) = (3, 0)$.

■ **EXAMPLE 6** **Intersection of Two Lines** Determine whether the following two lines intersect:

$$\mathbf{r}_1(t) = \langle 1, 0, 1\rangle + t\,\langle 3, 3, 5\rangle$$
$$\mathbf{r}_2(t) = \langle 3, 6, 1\rangle + t\,\langle 4, -2, 7\rangle$$

## 5 The Dot Product and Angles

**DEFINITION Dot Product** The **dot product** $\mathbf{v} \cdot \mathbf{w}$ of two vectors

$$\mathbf{v} = \langle a_1, b_1, c_1 \rangle, \qquad \mathbf{w} = \langle a_2, b_2, c_2 \rangle$$

is the scalar defined by

$$\boxed{\mathbf{v} \cdot \mathbf{w} = a_1 a_2 + b_1 b_2 + c_1 c_2}$$

(There is a similar definition for vectors in 2 dimensions)

**Example:** $\langle 2, 3, 1 \rangle \cdot \langle -4, 2, 5 \rangle =$

**Example:** $\langle 4, 2 \rangle \cdot \langle -1, 2 \rangle =$

**Example:** $\langle a, b, c \rangle \cdot \langle a, b, c \rangle =$

The dot product of $\mathbf{v}$ and $\mathbf{w}$ can be

- positive - this means the angle between the vectors is ________________________;
- zero - this means the angle between the vectors is ________________________;
- negative - this means the angle between the vectors is ________________________.

There are lots of properties of dot product.

**THEOREM 1 Properties of the Dot Product**

**(i)** $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

**(ii) Commutativity:** $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$

**(iii) Pulling out scalars:** $(\lambda \mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (\lambda \mathbf{w}) = \lambda (\mathbf{v} \cdot \mathbf{w})$

**(iv) Distributive Law:** $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

$(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$

**(v) Relation with length:** $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

tl;dr: The dot product is basically like "multiplication" you've known all your life.

**warning: in the next class we'll see a different type of product that does NOT have such nice properties.**

The Most Important Property:

Let's go back to the example from earlier and find the **angle** between the two vectors we say in the Day 4 packet.

**Example:** Can you draw a picture of all the vectors that are perpendicular to $\langle 2, 1\rangle$?

- How many are there?
- Are they related to one another in any way?

**Example:** Can you draw a picture of all the vectors that are perpendicular to $\langle 2, 0, -3\rangle$?

- How many are there?
- Are they related to one another in any way?
- Can you find two that are NOT multiples of each other?

# 6 The Cross Product

**Warmup:** Last time we learned about the **dot product** of two vectors. Let $\mathbf{v} = \langle -2, 1, 4 \rangle$ and $\mathbf{w} = \langle 3, 2, 5 \rangle$.

- What is $\mathbf{v} \cdot \mathbf{w}$?
- Since dot product is [positive/negative/zero] this means the angle between these vectors is [acute/obtuse/90].
- Draw a rough sketch of these two vectors - don't try to draw in 3D, just draw some arrows that form the type of angle you just discovered.

- Can you draw in any vectors that are perpendicular to BOTH of the vectors you drew? Will any of them live on the page? What about outside of the page?
- Write down algebraic equations using the dot product that represents the sentence: "$\langle x, y, z \rangle$ is perpendicular to $\langle -2, 1, 4 \rangle$." (and similarly for the other vector)

let's try to solve this system of equations:

**How many vectors were there that are perpendicular to both $\mathbf{v}, \mathbf{w}$?**

If $\mathbf{v}, \mathbf{w}$ are two vectors in $\mathbb{R}^3$ then there are always ________________ vectors that are perpendicular to BOTH $\mathbf{v}$ and $\mathbf{w}$.

- In general you could solve a system of equations like we just did.
- There is a useful method to find a particular perpendicular vector called ____________. That's the focus of today's lesson.

Warning: The way to calculate $\mathbf{v} \times \mathbf{w}$ looks intimidating.

- There is the "calculation" aspect, which is slightly tedious, but once you get some practice it is not too bad.
- There is also the conceptual part as well. This also requires some practice.
- We'll work on these separately.

**Method 1 for calculating cross product: (Using determinants)**

**DEFINITION The Cross Product** The cross product of vectors $\mathbf{v} = \langle a_1, b_1, c_1 \rangle$ and $\mathbf{w} = \langle a_2, b_2, c_2 \rangle$ is the vector

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \mathbf{k} \qquad \boxed{3}$$

This is an example of a $3 \times 3$ ____________________ that is being calculated in terms of smaller "____________" that are $2\times2$ determinants.

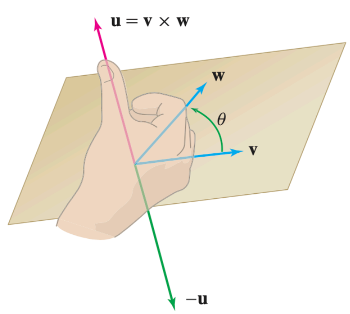
They can be calculated by the formula $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

**Example:** Let $\mathbf{v} = \langle -2, 1, 4 \rangle$ and $\mathbf{w} = \langle 3, 2, 5 \rangle$. Find $\mathbf{v} \times \mathbf{w}$.

**Method 2 for calculating cross product: (Slightly faster?) Example:** Let $\mathbf{v} = \langle 1, 2, 1 \rangle$ and $\mathbf{w} = \langle 3, 1, 1 \rangle$. Find $\mathbf{v} \times \mathbf{w}$.

**THEOREM 1 Geometric Description of the Cross Product** The cross product $\mathbf{v} \times \mathbf{w}$ is the unique vector with the following three properties:

**(i)** $\mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$.
**(ii)** $\mathbf{v} \times \mathbf{w}$ has length $\|\mathbf{v}\|\,\|\mathbf{w}\| \sin\theta$ ($\theta$ is the angle between $\mathbf{v}$ and $\mathbf{w}$, $0 \le \theta \le \pi$).
**(iii)** $\{\mathbf{v}, \mathbf{w}, \mathbf{v} \times \mathbf{w}\}$ forms a right-handed system.




**THEOREM 2 Basic Properties of the Cross Product**

**(i)** $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$
**(ii)** $\mathbf{v} \times \mathbf{v} = \mathbf{0}$
**(iii)** $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ if and only if $\mathbf{w} = \lambda\mathbf{v}$ for some scalar $\lambda$ or $\mathbf{v} = \mathbf{0}$.
**(iv)** $(\lambda\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (\lambda\mathbf{w}) = \lambda(\mathbf{v} \times \mathbf{w})$
**(v)** $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$
$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

## Examples:

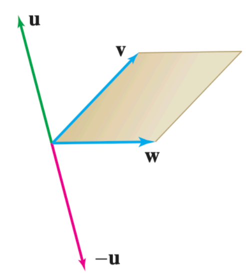



FIGURE 16

**24.** Find $\mathbf{v} \times \mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are vectors of length 3 in the $xz$-plane, oriented as in Figure 15, and $\theta = \frac{\pi}{6}$.

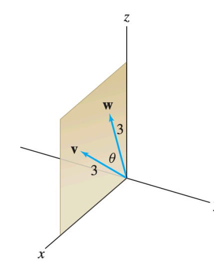



FIGURE 15

Use the properties of the cross product to calculate:

$$\mathbf{u} \times \mathbf{v} = \langle 1, 1, 0 \rangle, \quad \mathbf{u} \times \mathbf{w} = \langle 0, 3, 1 \rangle, \quad \mathbf{v} \times \mathbf{w} = \langle 2, -1, 1 \rangle$$

**17.** $\mathbf{v} \times \mathbf{u}$

**18.** $\mathbf{v} \times (\mathbf{u} + \mathbf{v})$

**19.** $\mathbf{w} \times (\mathbf{u} + \mathbf{v})$

**20.** $(3\mathbf{u} + 4\mathbf{w}) \times \mathbf{w}$

**21.** $(\mathbf{u} - 2\mathbf{v}) \times (\mathbf{u} + 2\mathbf{v})$

**22.** $(\mathbf{v} + \mathbf{w}) \times (3\mathbf{u} + 2\mathbf{v})$

## 7 Planes

**Warmup:** If you want to know the angle between two vectors $\mathbf{v}, \mathbf{w}$ the best way to find this angle is to use the [dot/cross]-product. This will give the [sine/cosine] of the angle between them according to the formula:

To quickly check if two vectors are perpendicular, you can take their dot product and see if it is ____________.

If you have two vectors $\mathbf{v}, \mathbf{w}$ and you want to find a vector perpendicular to **both** of them, then ____________ does the trick.

**Today** we are going to see an application of perpendicular vectors. To start, imagine that on the table in the center of the room is the center of the universe (well the **origin** $(0, 0, 0)$ of $\mathbb{R}^3$). What vectors are perpendicular to the "straight up" vertical vector $\langle 0, 0, 1\rangle$? How many are they? What shape do they form?

> Main Point:
> If we have a vector $\mathbf{n}$ then the equation:
>
> describe a ____________ through the origin that is ____________ to $\mathbf{n}$.

> Pro-tip: The words **perpendicular, normal, orthogonal** are all synonyms

**Example:** Find the equation of the plane through the origin that is normal to the vector $\langle 3, 4, 5\rangle$.

How would this answer change if we wanted a plane not through the origin, but through the point $(2, 1, 4)$? We'll draw a big picture.

> The way to find a plane through a point $(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c\rangle$:
>
> $$\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0.$$
>
> $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$
>
> $$ax + by + cz = d$$
>
> where $d = \mathbf{n} \cdot \langle x_0, y_0, z_0\rangle$

**Example:** A student writes the equation $3x - 7y = 3z + 5$. Find a normal vector to this plane.

**Example:** Parallel planes will have the same normal vector. Find a plane that is parallel to the plane

$$3x - 2y + z = 6.$$

General Strategy to find the equation of a plane:

- Find a point (any point) on the plane
- Find the normal vector to that plane. One good way to do this is by

**Example:** Find the plane through the three points $P = (1, 0, -1)$, $Q = (2, 2, 1)$, $R = (4, 1, 2)$.

**Example:** Where does the line $\mathbf{r}(t) = \langle 1, 2, 1\rangle + t\langle -2, 0, 1\rangle$ intersect the plane $3x - 9y + 2z = 7$?

**Example** Find the equation of the plane that contains the lines $\mathbf{r_1}(t) = \langle t, 2t, 3t\rangle$ and $\mathbf{r_1}(t) = \langle 3, t, 8t\rangle$.

**Example** Find the equation of the plane that contains the point $(-1, 0, 1)$ and the line $\mathbf{r}(t) = \langle t + 1, 2t, 3t - 1\rangle$.

## 8 Practice with Planes and Parametrization

**Example:** Find the plane through the three points $P = (1, 0, -1)$, $Q = (2, 2, 1)$, $R = (4, 1, 2)$.

Cool Fact:

The area of the **parallelogram** spanned by two vectors $\mathbf{v}, \mathbf{w}$ is equal to $||\mathbf{v} \times \mathbf{w}||$.

This means that the area of the **triangle** formed by two vectors is 1/2 of that.

**Use this to find the area of the triangle PQR in the previous problem:**

**Example:** Where does the line $\mathbf{r}(t) = \langle 1, 2, 1\rangle + t\langle -2, 0, 1\rangle$ intersect the plane $3x - 9y + 2z = 7$? Hint: what are the $x, y, z$ coordinates of the line $\mathbf{r}(t)$? Plug them into the equation of the plane and see what you get!

**Example** Find the equation of the plane that contains the lines $\mathbf{r_1}(t) = \langle t, 2t, 3t\rangle$ and $\mathbf{r_1}(t) = \langle 3, t, 8t\rangle$. Hint: Draw a picture to get started.

**Example** Find the equation of the plane that contains the point $(-1, 0, 1)$ and the line $\mathbf{r}(t) = \langle t+1, 2t, 3t-1\rangle$.

**Visualizing by looking at intersections:**

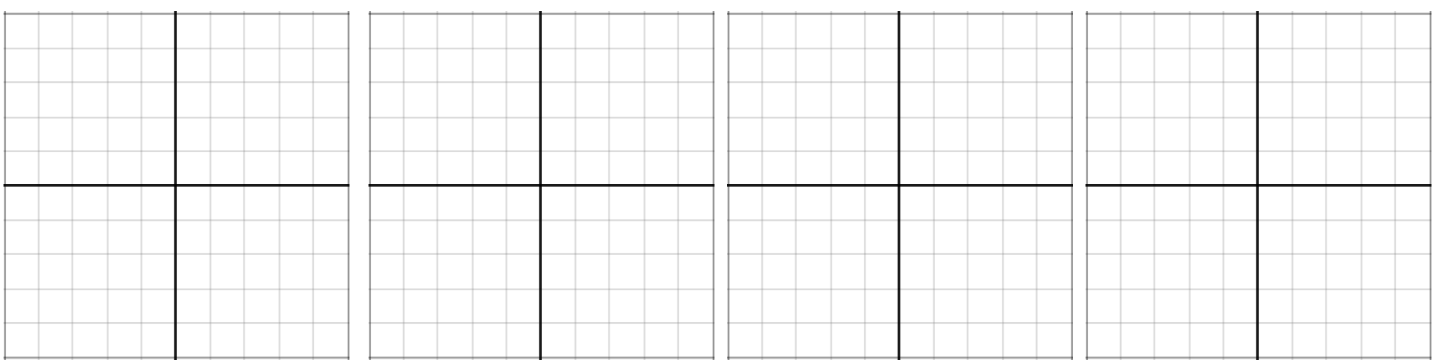
A) Where does the plane $2x - 6y + 3z = 7$ intersect the $xy$ plane? Hint: In the $xy$ plane, what is the $z$-coordinate? You should draw a picture and label your axes $x$ and $y$. This is called the **trace** of the plane in the $xy$ plane.

B) Now do the same thing but for the $yz$ trace.

**We can also take traces more generally:** For instance, what happens if you are looking at the equation: $x^2 + y^2 = z^2$. What does this look like in the plane where $z = 2$? What about when $z = 1$? $z = 0$? What about $z = -1$?

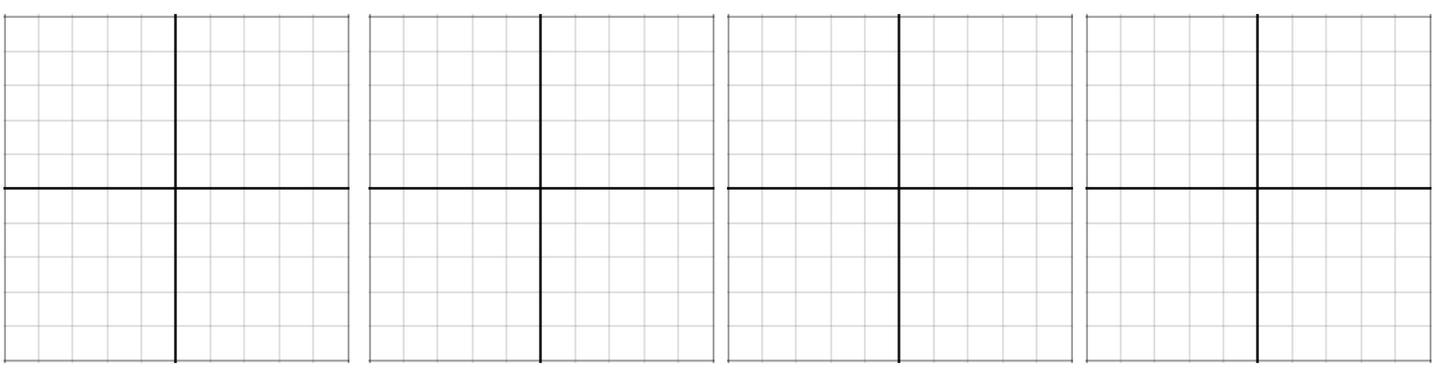
Draw your pictures below as **four separate** pictures. Now discuss with your group how to assemble these all together in 3D space to get a full description of the 3D surface. What do you think your surface looks like?

Different slices of $x^2 + y^2 = z^2$



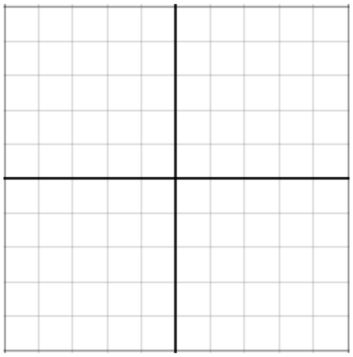
$z = 2$ $\qquad$ $z = 1$ $\qquad$ $z = 0$ $\qquad$ $z = -1$

Now do the same thing with the surface defined by $x^2 + y^2 = z^2 + 1$. What is different?

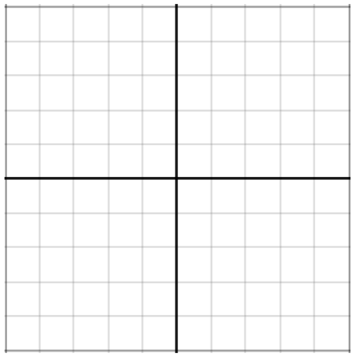


$z = 2$ $\qquad$ $z = 1$ $\qquad$ $z = 0$ $\qquad$ $z = -1$

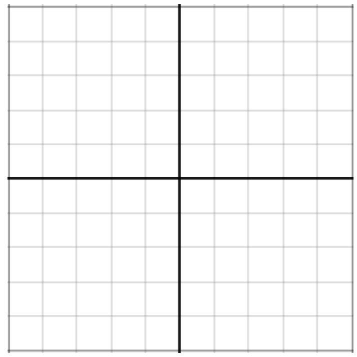
Now do the same thing with the surface defined by $x^2 + y^2 + z^2 = 1$. What is different? (you may want to think about how your traces change as you vary $z$, say to things like $z = 1/2$ or $z = 1/3$.



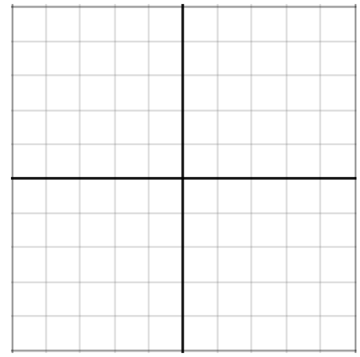
$z = 1$



$z = 0.5$



$z = 0$



$z = -1$

# 9 3D Parametrizations

**Warmup: (A cool application of vectors)**

Consider the plane with equation

$$ax + by + cz = d.$$

What would have to be true if this plane passed through $(0, 0, 0)$?

What is a normal vector to the plane? $\mathbf{n} =$

What down a parametrization for the line through the point $(1, 4, 8)$ that is parallel to this normal vector. (Use $P + t\mathbf{v}$.)

**3D parametrizations**

**Example:** Consider the curve in $3D$ with parametrization

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle.$$

What do you think is happening as $0 \leqslant t \leqslant 6\pi$? What is the curve doing? Think about what is happening in the $x$ and $y$ coordinates first, and then ponder what might be happening in the $z$-coordinate. Try to draw a picture of the curve as best you can.

If you had to guess, what do you think the $\mathbf{r}'(t)$ would be? Hint: it is a vector with three coordinates, each is the derivative of the corresponding coordinate of $\mathbf{r}(t)$. (ok that was more of an answer than a hint :) ) This is actually called the **tangent vector** at time $t$.

Let's consider three different parametrizations of the circle:

$\mathbf{r_1}(t) = \langle \cos t, \sin t \rangle$

$\mathbf{r_2}(t) = \langle \cos 2t, \sin 2t \rangle$

$\mathbf{r_3}(t) = \langle \cos(t^2), \sin(t^2) \rangle$

What would their tangent vectors be?

What is the tangent vector at the point $(0, 1)$?

Meaning

The tangent vector shows the **direction** of the particle's movement.

The **magnitude** of the tangent vector represents the **speed** of the particle.

**Example:** Consider the parametrized curve from the quiz this week:

$$\mathbf{r}(t) = (9 - t^2, 8t - t^3)$$

What are the tangent vectors? Where is the speed a minimum? (This is like the webwork problem we solved a week ago.)

**Example:** Viviani's Curve: Let's try to understand the intersection of the two surfaces:

$$x^2 + y^2 = z^2, \qquad y = z^2$$

## 10 Arc Length

**Warmup:**

If we have a parametrized **curve** $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ the letter $t$ represents ______________.

If we take the derivative, we get $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ which represents the ______________________________ to the curve. If take the length of this vector we get $||\mathbf{r}'(t)||$ which is the _______________ at time $t$.

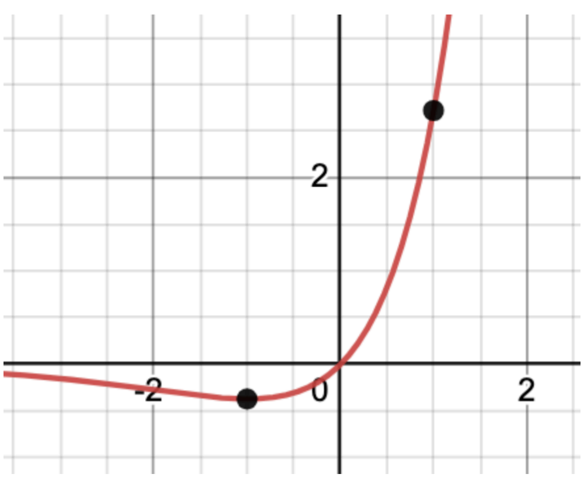
If you want to measure distance, then distance = speed$\cdot$ time.

This leads us to a formula for the ______________________________ of a parametrized curve $\mathbf{r}(t)$:

Find the length of the curve defined by $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$ for $0 \leqslant t \leqslant \pi$.

Find the length of the curve defined by $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t \rangle$ for $0 \leqslant t \leqslant 2\pi$.

Find the length of portion of the graph of $y = xe^x$ indicated below:



**Your Turn:**

Set up an integral that will calculate the length along the parabola $y = x^2$ from the point $(2, 4)$ to the point $(3, 9)$

Set up an integral that will calculate the length of one trip around the circle parametrized by $\langle 4\cos(5t), 4\sin(5t)\rangle$. Hint: I'd recommend starting $t = 0$ but where should $t$ go to?

Consider the **sphere** defined by $(x-3)^2 + (y-4)^2 + (z-5)^2 = 25$.

Its center is at _______ and has radius _______.

Where does it intersect the $yz$ plane? How would you set this up? Can you parametrize your answer?

How would we parametrize the intersection of the cylinder $x^2 + y^2 = 25$ with the plane $2x + 5y - 3z = 10$?

**Example:** Viviani's Curve: Let's try to understand the intersection of the two surfaces:

$$x^2 + y^2 = z^2, \qquad y = z^2$$

- Could you "easily" solve for $y$ and $z$ in terms of $x$?
- Could you "easily" solve for $x$ and $z$ in terms of $y$?
- Could you "easily" solve for $y$ and $x$ in terms of $z$?
- Show that the curve lies on the surface of a sphere of radius 1 with center $(0, 1, 0)$

1. Find the distance from the point $(3, -4, 5)$ to
   (a) The $xy$ plane
   (b) The $y$-axis.
2. Find the equation of the sphere with center $(2, -6, 4)$ and radius 5. Describe its intersection with each of the coordinate planes.
3. Find a vector that has the same direction as $\langle -2, 4, 2 \rangle$ but has length 6. (Hint: Can you scale to find a unit vector and then scale again?)
4. Diane's "are these meaningful for dot and cross products"

   Are the following True or False:
   (a) Two lines parallel to a third line are parallel.
   (b) Two lines perpendicular to a third line are parallel.
   (c) Two planes parallel to a third plane are parallel.
   (d) Two planes perpendicular to a third plane are parallel.
   (e) Two lines parallel to a plane are parallel.
   (f) Two lines perpendicular to a plane are parallel.
   (g) Two planes parallel to a line are parallel.
   (h) Two planes perpendicular to a line are parallel.
   (i) Two planes either intersect or are parallel.
   (j) Two lines either intersect or are parallel.
   (k) A plane and a line either intersect or are parallel.
5. Find a vector function that represents the curve of intersection of the two surfaces: The paraboloid z = 4x2+y2 and the parabolic cylinder y=x2
6.

# 11 Last Day of Chapter 14

**Warmup:**

$\mathbf{r}(t) = t\mathbf{i} + (2t - t^2)\mathbf{k} = \langle \qquad\qquad\qquad\qquad \rangle$ is describing a [curve / surface].

If we want to know where the particle is at time $t = 2$ we would just plug in and get: ______________.

The equation $z = x^2 + y^2$ is describing a **surface**. (More on this in the coming weeks!) This is similar to the equation of a plane, in that it gives a relationship between the $x, y, z$ coordinates. For example, fill in the blank for the the $z$-coordinates:

$$(1, 2, \qquad), \quad (0, 1, \qquad), \quad (1, -1, \qquad), \quad (a, b, \qquad\qquad) \quad (\heartsuit, \quad \square, \qquad\qquad)$$

How can we find where the **curve** intersects the **surface**? What could you set up and solve?

How would we parametrize the intersection of the cylinder $x^2 + y^2 = 25$ with the plane $2x + 5y - 3z = 10$? Hint: What does it look like in the $xy$ plane?

**Example:** Viviani's Curve: Let's try to understand the intersection of the two surfaces:

$$x^2 + y^2 = z^2, \qquad y = z^2$$

- Could you "easily" solve for $y$ and $z$ in terms of $x$?

- Could you "easily" solve for $x$ and $z$ in terms of $y$?

- Could you "easily" solve for $y$ and $x$ in terms of $z$?

- Show that the curve lies on the surface of a sphere of radius 1 with center $(0, 1, 0)$

## 12 Functions of More than one Variable

**Warmup:**

The function $f(x) = \sqrt{x}$ is a function of ______________ variable. When we **graph** this function, we think of the variable $x$ on the horizontal axis, and think of the $y$ values as the ____________________________. This is why we write something like

$$y = f(x).$$

In **two variables**, we'll have something like this $f(x, y) = \sqrt{x + y}$.

$f(7, 2) =$ $\qquad$ $f(0, 0) =$ $\qquad$ $f(-5, 21) =$ $\qquad$ $f(4, -10) =$

Just like with functions of one variable, we have a ______________ of allowable inputs to our function. What is the **domain** of the two functions above?

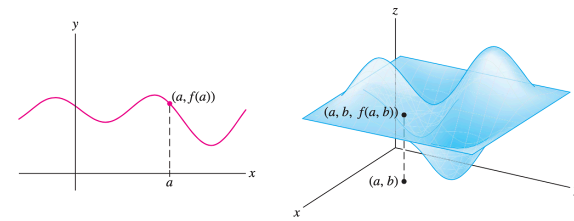
What we just draw is NOT the graph of the function $f(x, y)$. It is just the set of allowable inputs.

If we want to **graph** or visualize a function of two variables, we will think of the **outputs** as being $z$ values. This is why we will often write

$$z = f(x, y).$$

> Remember: If we have a function of two variables $f(x, y)$ then the **input** is an ordered pair $(x, y)$.

**Example:** What is the domain of the function $f(x, y) = \sqrt{9 - x^2 - y}$? If we swapped the order of the inputs would we still get the same output? How could we test this? What do you think the shape of the **graph** of $z = f(x, y)$ looks like? Over which $x, y$ values does it live?




(A) Graph of $y = f(x)$ (B) Graph of $z = f(x, y)$

**FIGURE 4**

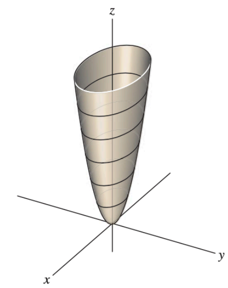



**FIGURE 5** Graph of $f(x, y) = 2x^2 + 5y^2$

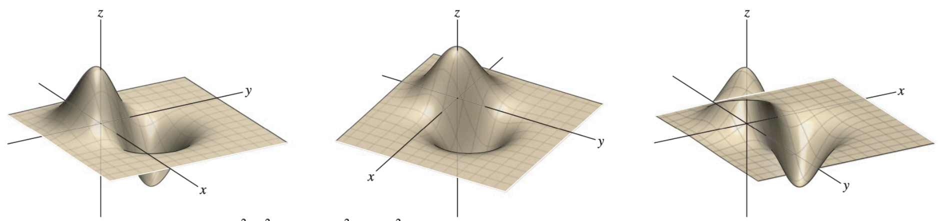



**FIGURE 6** Different views of $z = e^{-x^2-y^2} - e^{-(x-1)^2-(y-1)^2}$

## How can we graph?

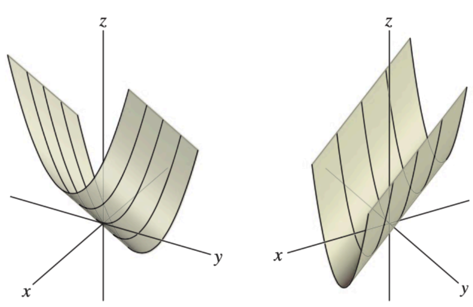
- We can slice vertically (setting $x =$ or setting $y =$ different constants)
- We can slice horizontally (setting $z =$ different constants)
- In both cases you can think about these as "movies"

Let's imagine the graph of $z = x \sin y$

- Slicing as $-10 \leqslant x \leqslant 10$:
- Slicing as $-10 \leqslant y \leqslant 10$:
- Slicing as $-10 \leqslant z \leqslant 10$:

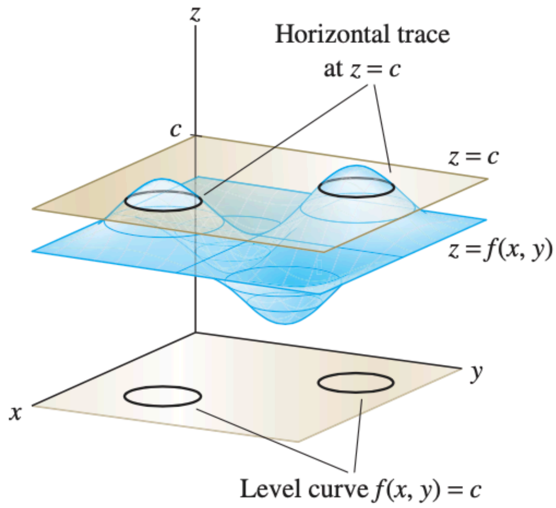
**17.** Match graphs (A) and (B) in Figure 21 with the functions

(i) $f(x, y) = -x + y^2$ (ii) $g(x, y) = x + y^2$




(A) (B)

**FIGURE 21**




**Horizontal slices** are called ________________________. They often come up in ____________________________.

**Example:** What are the level curves of the surface: (Remember: think about the movie:)

$$z = x^2 + 3y^2$$

.

Helpful facts:
Circle of radius $R$ centered at origin:

$$x^2 + y^2 = R^2$$

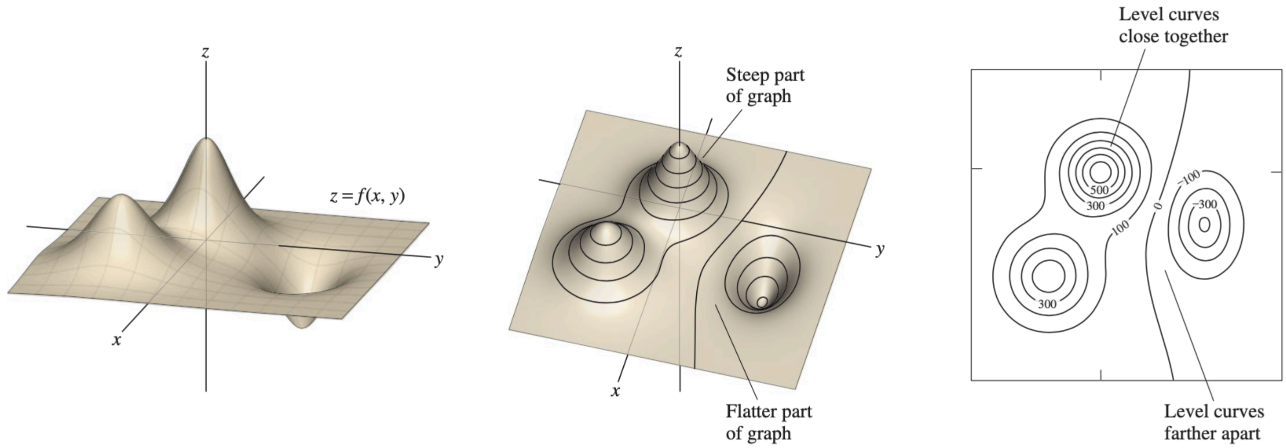
Ellipse with $x$ intercepts $\pm a$ and $y$ intercepts $\pm b$:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

When drawing a **contour map**:

we will draw it so that the level curves $f(x,y)=c$ for ________________________ values of $c$.

If the contours are **close** together then the graph is ________________________




(A) (B) Horizontal traces (C) Contour map

What do the level curves of $f(x,y)=x^2-3y^2$ look like? What does the surface look like?

**Example:** Find the average rate of change the function pictured below:

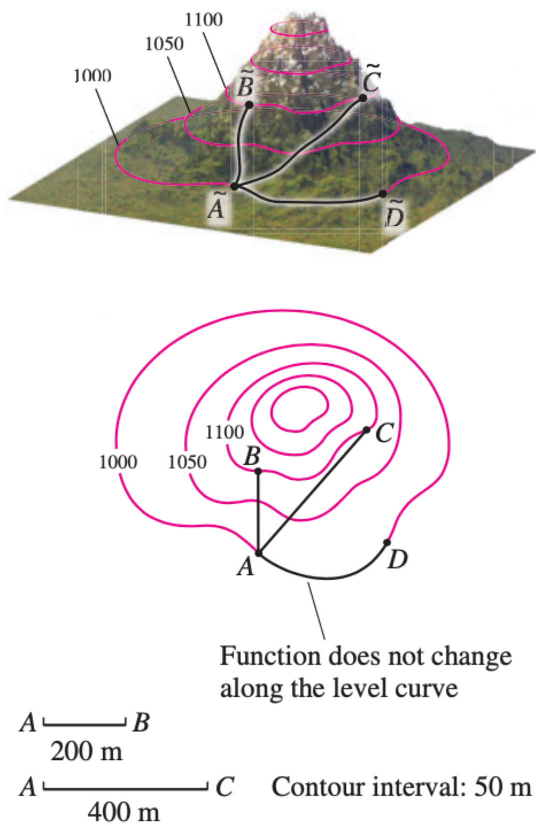
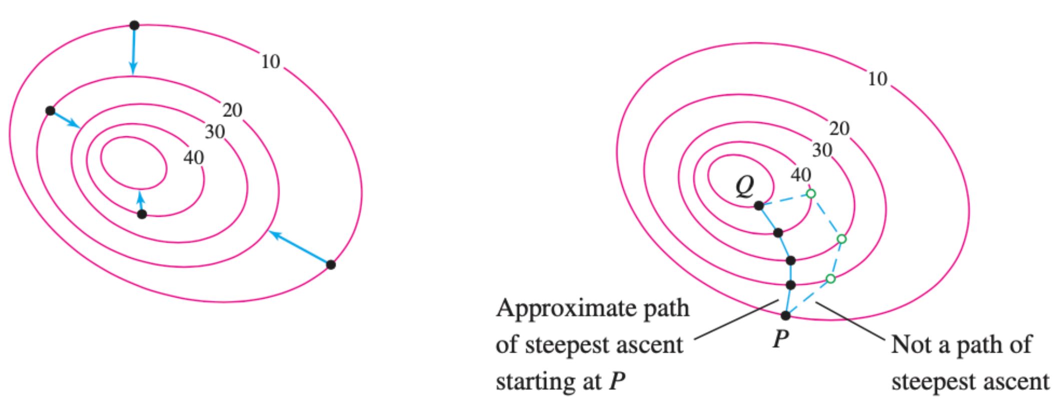



FIGURE 17

From $A$ to $B$

From $A$ to $C$

From $A$ to $D$




(A) Vectors pointing approximately in the direction of steepest ascent

(B)

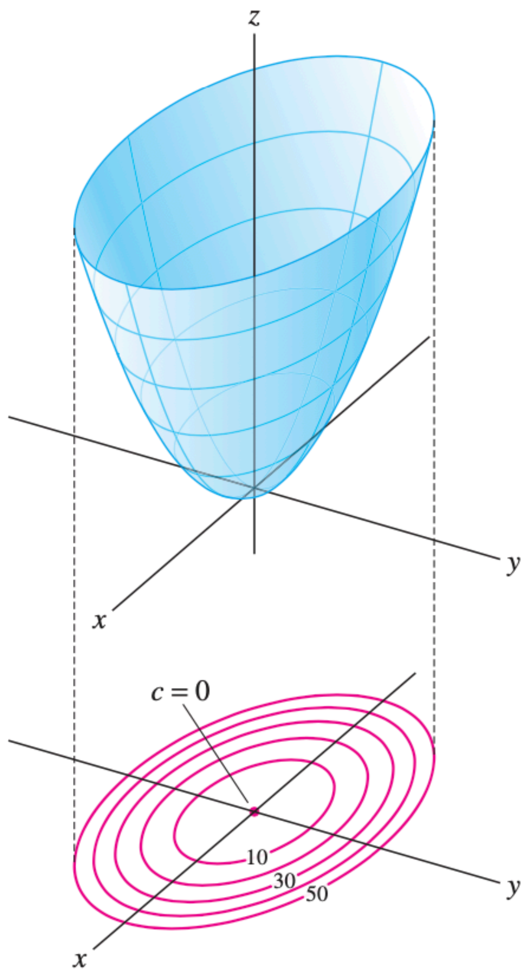



**FIGURE 12** $f(x, y) = x^2 + 3y^2$. Contour interval $m = 10$.

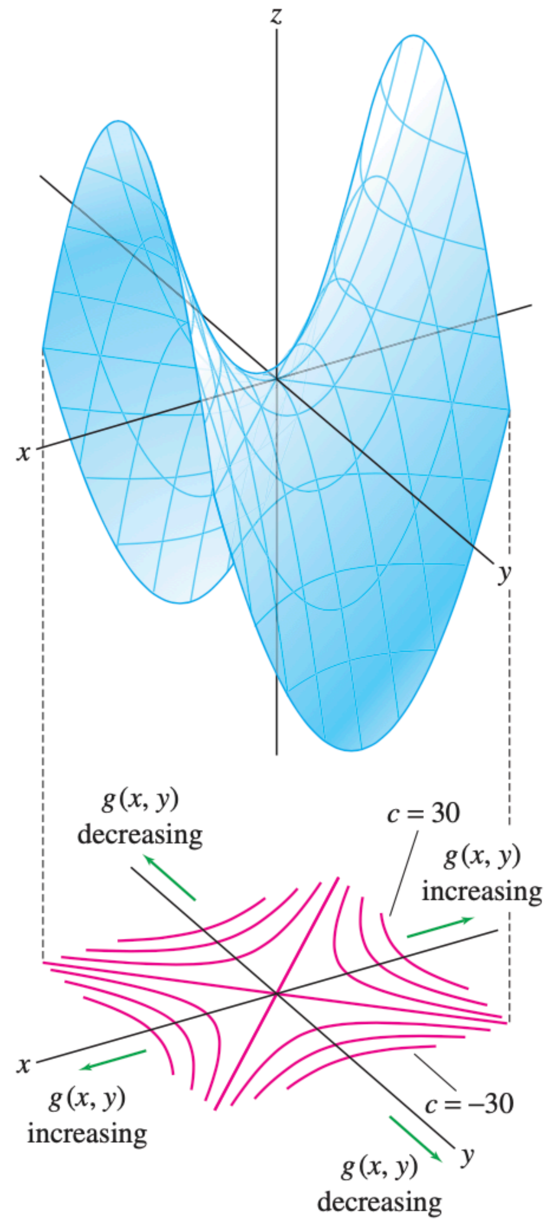
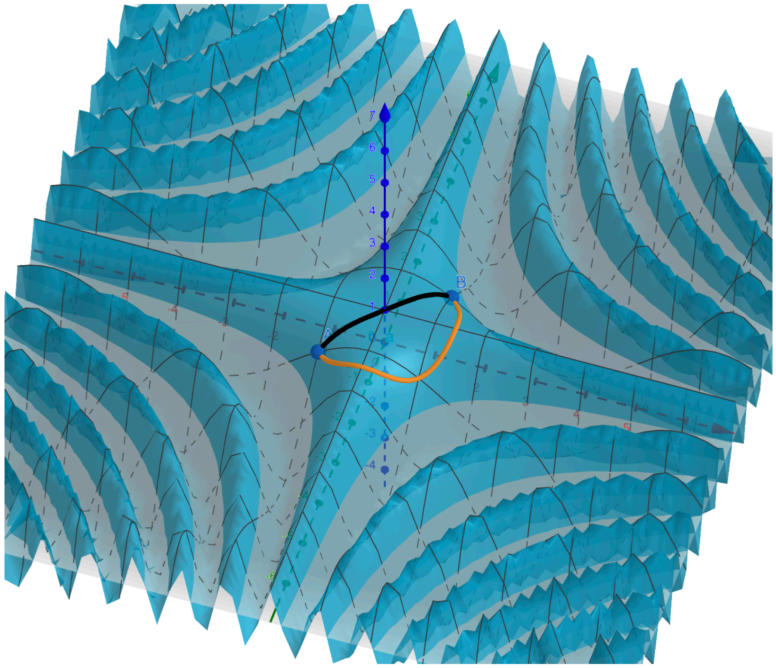



**FIGURE 13** $g(x, y) = x^2 - 3y^2$. Contour interval $m = 10$.

## 13 Some Practice + Limits and Continuity



Using the contour map below, notice the two small lines pointing out of point $P$. If you were to start at $P$ and walk along the one pointing to the east, what would your change in $x$ and change in $z$ be approximately? Write this down as a ratio $\frac{\Delta z}{\Delta x}$

If you were to start at $P$ and walk along the one pointing straight north, what would your change in $y$ and change in $z$ be approximately? Write this down as a ratio $\frac{\Delta z}{\Delta y}$

These are called **partial derivatives** and we'll learn more about these on Monday.

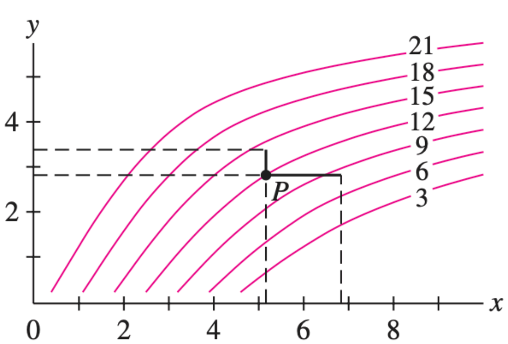
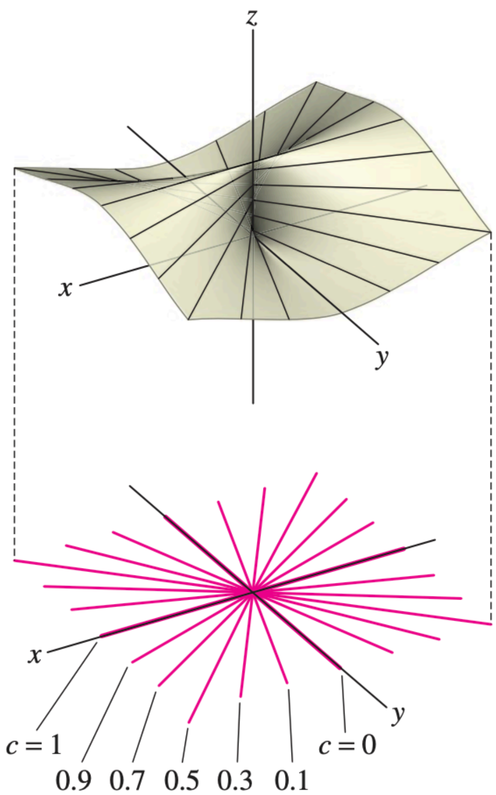



FIGURE 10

Here is a contour map for a pretty strange function $f(x, y) = \dfrac{x^2}{x^2 + y^2}$.




What would happen to your height ($z$ value) if you walked along the line labeled $c = 0.5$ toward the origin?

What would happen to your friend's height ($z$ value) if they walked along the line labeled $c = 0$ toward the origin?

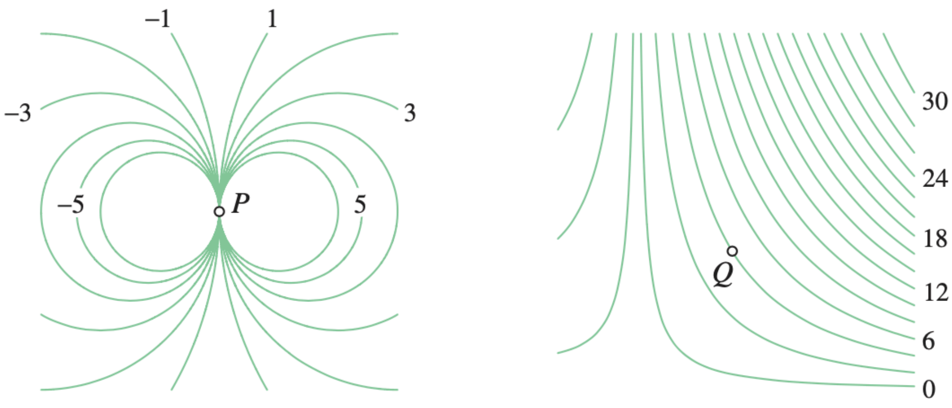
Would you meet at some "limit" point $(0, 0, ?)$?

This function [is/is not] continuous at $(0, 0)$ because the limit [does / does not exist.]

> Note: This is a strange behavior that often happens when we are near a "singularity". In most cases, say for trig functions, rational functions, polynomials, logs, exponentials and their inverses, everything will be continuous as long as we are at a point in the domain.

Here's another problem to discuss:

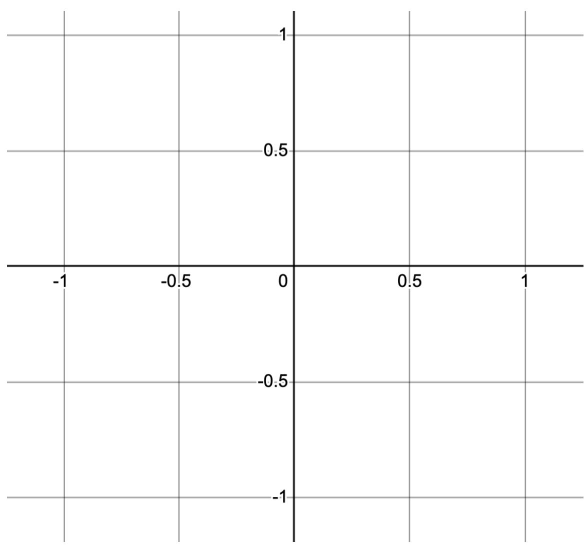
**35.** Figure 7 shows the contour maps of two functions. Explain why the limit $\lim_{(x,y)\to P} f(x, y)$ does not exist. Does $\lim_{(x,y)\to Q} g(x, y)$ appear to exist in (B)? If so, what is its limit?




(A) Contour map of $f(x, y)$ (B) Contour map of $g(x, y)$

**FIGURE 7**

1. A bug wants to move along the line segment from the point $(-1, -1)$ to the point $(1, 1)$. Draw a picture of this path below. I want you to parametrize this path by using $P + t\mathbf{v}$ BUT I want you to do set it up so that $0 \leqslant t \leqslant 2$. That means your bug should start at $P + 0\mathbf{v}$ and end at $P + 2\mathbf{v}$. In your picture, label what vector $\mathbf{v}$ will be. Just imagine your $\mathbf{v}$ as your "step size".



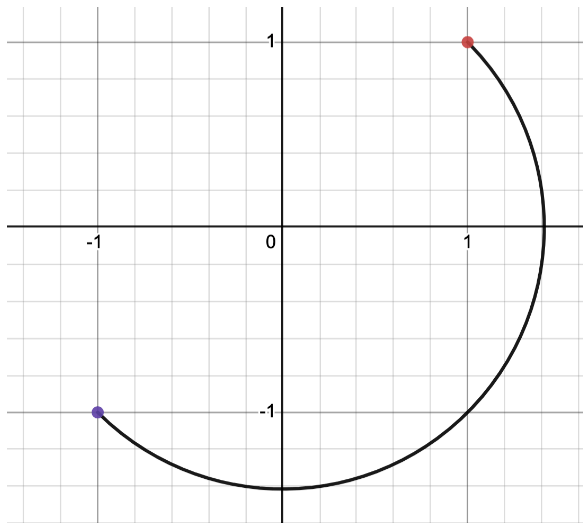

Using the setup $P + t\mathbf{v}$ write down a parametrization for this bug

2. Oh snap, it turns out that was just the SHADOW in the $xy$ plane of where the bug was walking. In truth the bug was walking along the surface $z = \cos(xy)$. (Check out the picture on the front page) Using the parametrization you found above, write down what the parametrization would be for the bug's position in 3D. (Hint: all you have to do is figure out what the $z$ coordinate is.)

3. What is the tangent vector to this bug's journey? Write it down carefully - you might want to write very large.

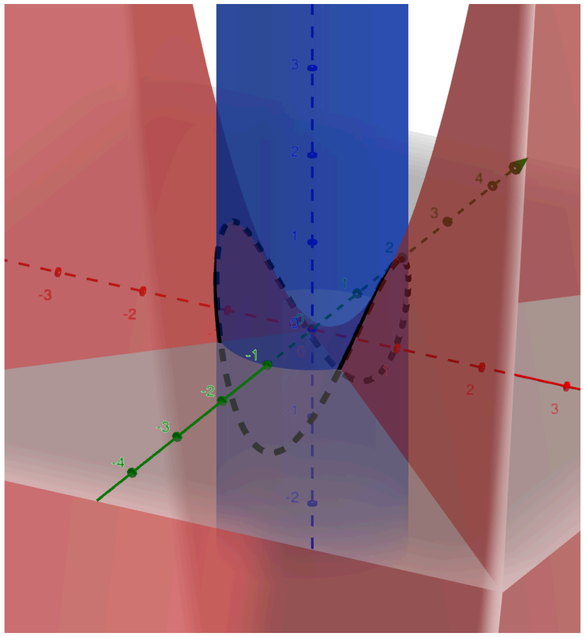
4. Write down an integral that calculates the length of the bug's journey. Use a computer to get a number and check with me.

5. Now repeat these steps for the curve below. This is another path to get from $(-1,-1)$ to $(1,1)$. Be careful finding the radius and the bounds. Set up an integral to find the length of this journey. Check your answer with me.

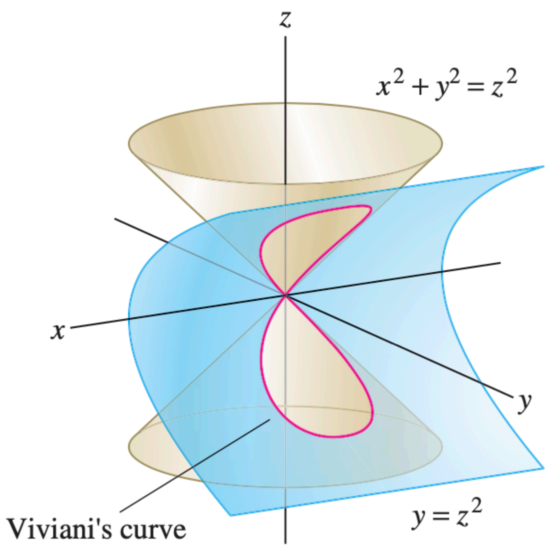


### Additional Practice

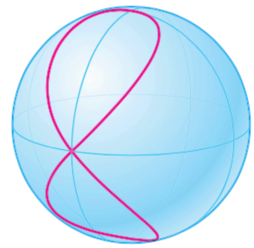
Parametrize the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the surface $z = x^2 - y^2$



Parametrize the curve pictured below:




(A)



(B) Viviani's curve viewed from the negative $y$-axis

# 14 Partial Derivatives

Last time I asked you work out the "rate of change" in the $x$ and $y$ directions. Let's do that again now:

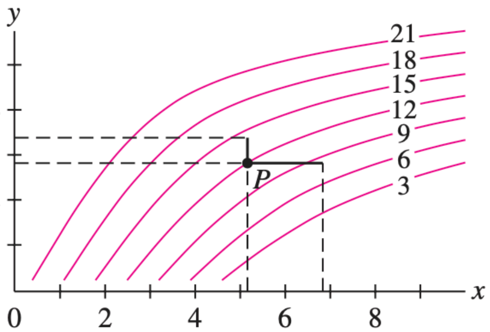



**FIGURE 10**

If we could "zoom in" we could get 'instantanous' rate of change in the $x$ and $y$ directions.

Say that $z = f(x, y)$. What would we want to calculate:

The instantaneous rate of change in the $x$ direction while $y$ is fixed:

The instantaneous rate of change in the $y$ direction while $x$ is fixed:

In general, these rates of change are called __________________________ and we will use symbols like this:

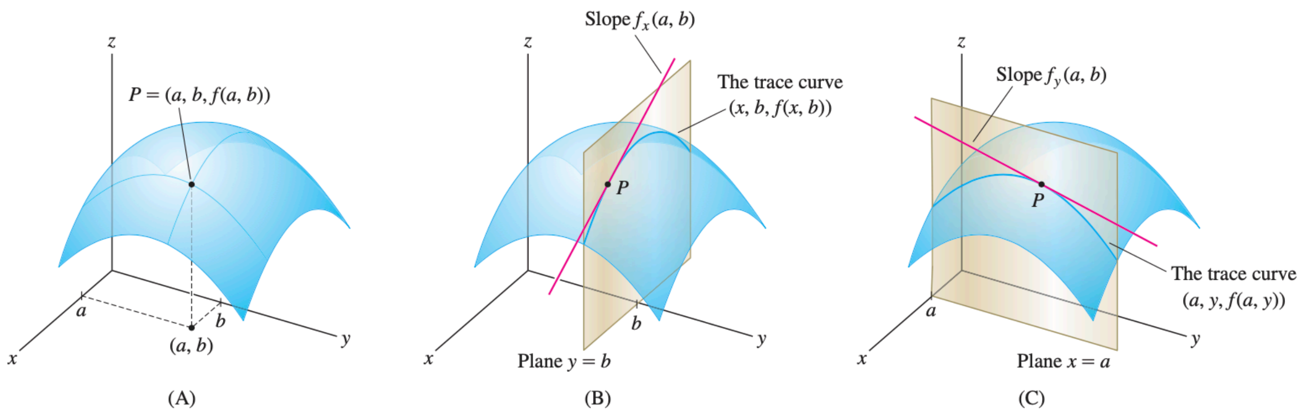
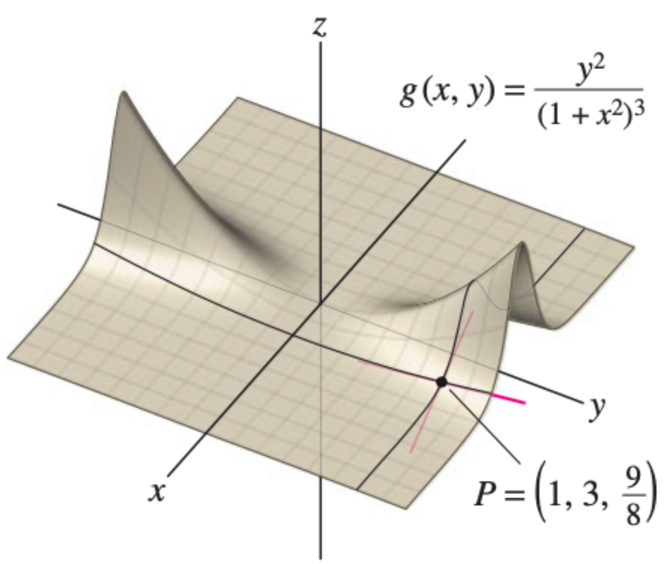



**FIGURE 1** The partial derivatives are the slopes of the vertical trace curves.

To calculate partial derivatives, just use the your normal derivative rules, but treat the other variable as a constant.

**Example:**

Find $g_x(1,3)$ and $g_y(1,3)$ for the function $g(x,y) = \dfrac{y^2}{(1+x^2)^3}$. Before doing this, let's check a graph and see what signs we expect the answer to be:




**Example:** Your turn, find the following partial derivatives:

$\dfrac{\partial}{\partial x}\left(x^2 + 2x^3y^2\right) =$

$\dfrac{\partial}{\partial y}\left(x^2 + 2x^3y^2\right) =$

$\dfrac{\partial}{\partial x}\left(x\sin(y)\right) =$

$\dfrac{\partial}{\partial y}\left(x\sin(y)\right) =$

$\dfrac{\partial}{\partial x}\left(\sin(x^2y^3)\right) =$

$\dfrac{\partial}{\partial y}\left(\sin(x^2y^3)\right) =$

**8.** Determine whether the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are positive or negative at the point $P$ on the graph in Figure 7.

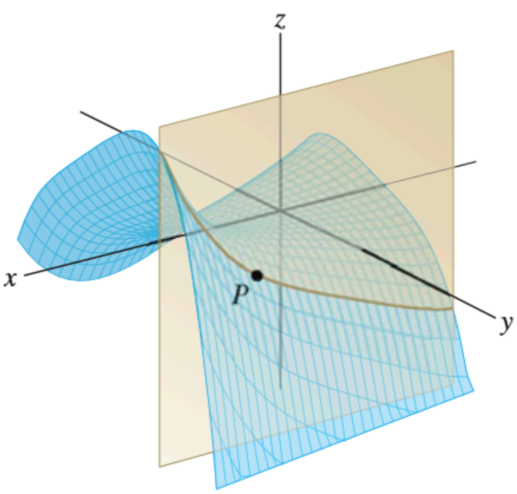



FIGURE 7

**Higher order derivatives:** We can take derivatives of derivative. Let's work out what the **2nd** derivatives of the following function are:

$f(x, y) = x^3 + y^2 e^x$.

(Clairaut's Theorem)

If $f_{xy}$ and $f_{yx}$ are both continuous functions near a point $(x_0, y_0)$ then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

for all points near $(x_0, y_0)$.

**Example:**

Find $g_{zzwx}$ where

$$g(x, y, z, w) = x^3 w^2 z^2 + \sin\left(\frac{xy}{z^2}\right)$$

**Example:**

Can you think of a function $u(x,t)$ that satisfies the equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}?$$

## 15 Tangent Planes and Linear Approximation

### Discussion

Back in Calc 1 you probably encountered a function that had $f'(2) = 3$. Which of the following would be the **vector** that points in the direct of the tangent line as we increase $x$? Before you answer, draw a quick picture of a function that has this property:

$\langle 2, 3\rangle$ $\langle 1, 2\rangle$ $\langle 1, 3\rangle$ $\langle -1, 3\rangle$ $\langle -1, -3\rangle$

Now suppose that you did the same think in $3D$. If you know that $f_x(5, 3) = 7$ then which of the following would represent this?

$\langle 5, 3, 7\rangle$ $\langle 0, 1, 7\rangle$ $\langle 5, -1, 7\rangle$ $\langle 1, 0, 7\rangle$ $\langle 5, 1, 7\rangle$

Now suppose that you did the same think in $3D$. If you know that $f_y(5, 3) = -2$ then which of the following would represent this?

$\langle 5, 3, -2\rangle$ $\langle 0, 1, -2\rangle$ $\langle 5, -2, 3\rangle$ $\langle -2, 0, 1\rangle$ $\langle -5, -3, 2\rangle$

Main Point:

The vectors __________________________ and __________________________

are both tangent to the surface $z = f(x, y)$ at the point $(a, b)$. Let's find the equation of the **tangent plane** to the surface.

There isn't always a tangent plane.

- For there to be a tangent plane, you **need** the partial derivatives to exist at the point in question.
- And the partials need to be continuous.

If the tangent plane exists at a point, we say that the function $f(x, y)$ is __________________________ at that point.

**Example:** Find the equation of the tangent plane to the function $f(x, y) = 5x + 4y^2$ at the point $(a, b) = (2, 1)$.

**Example:** Where is the function $f(x, y) = \sqrt{x^2 + y^2}$ differentiable?

**Example:** Find the tangent plane of the graph of $f(x, y) = xy^3 + x^2$ at the point $(2, -2)$.

**Example:** Use the linear approximation of $f(x, y) = e^{x^2+y}$ at $(0, 0)$ to esti- mate $f(0.01, 0.02)$. Compare with the value obtained using a calculator.

Do you think there's a "Taylor Series" for functions of 2 variables?

# 16 Gradient

The **gradient** of a function $f(x, y)$ is:

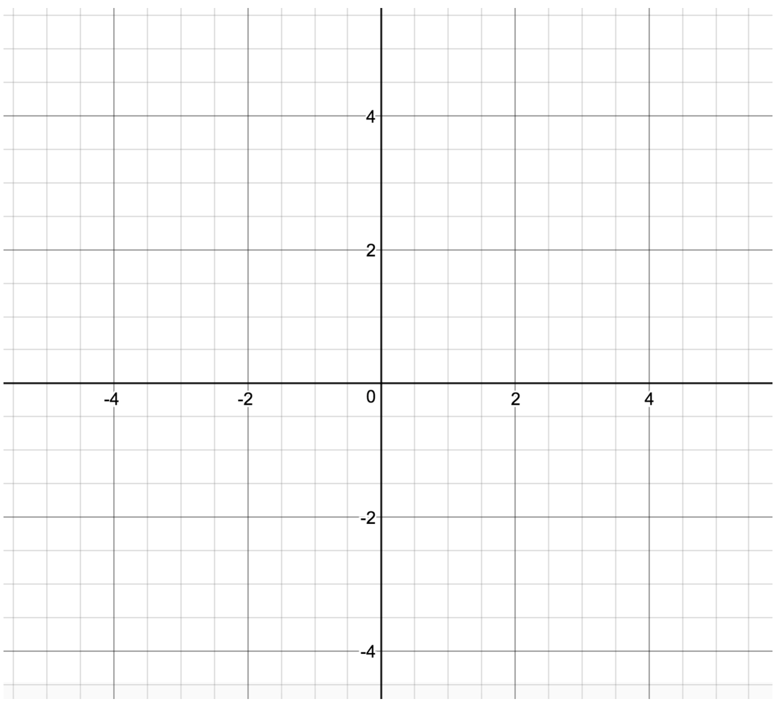
$$\nabla f =$$

The **gradient** of a function $f(x, y, z)$ is:

$$\nabla f =$$

Important Notation:
Sometimes we will write $\nabla f_P$ to mean "evaluate the partial derivatives at the point $P$.

**Example:** Let $f = x^2 + y^2$. Find $\nabla f$ and then find $\nabla f_P$ at the points $(1, 1)$, $(-3, -4)$, $(-0.5, -0.5)$, $(4, 0)$ Draw these vectors in the picture below. If you finish early, draw in some level curves for $c = 2, 4, 5$.




**THEOREM 1 Properties of the Gradient** If $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $c$ is a constant, then

**(i)** $\nabla(f + g) = \nabla f + \nabla g$

**(ii)** $\nabla(cf) = c\nabla f$

**(iii) Product Rule for Gradients:** $\nabla(fg) = f\nabla g + g\nabla f$

**(iv) Chain Rule for Gradients:** If $F(t)$ is a differentiable function of one variable, then

$$\nabla(F(f(x, y, z))) = F'(f(x, y, z))\nabla f \qquad \boxed{1}$$

**Example:** Find the gradient of $f(x, y, z) = (x^2 + y^2 + z^2)^8$.

**There are many applications:**

1. Finding rates of change of a function $f(x, y)$ if we walk along a curve $\mathbf{c}(\mathbf{t})$.

2. Find rates of change in a general direction vector $\mathbf{u}$.

3. Using the gradient to find the direction that is increasing most quickly.

4. How does the gradient relate to level curves of a function?

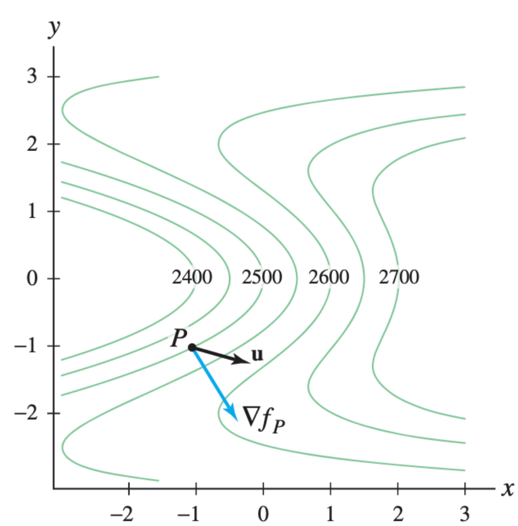



**FIGURE 11** Contour map of the function $f(x, y)$ in Example 9.

**3.** Figure 14 shows the level curves of a function $f(x, y)$ and a path $\mathbf{c}(t)$, traversed in the direction indicated. State whether the derivative $\dfrac{d}{dt} f(\mathbf{c}(t))$ is positive, negative, or zero at points $A$–$D$.

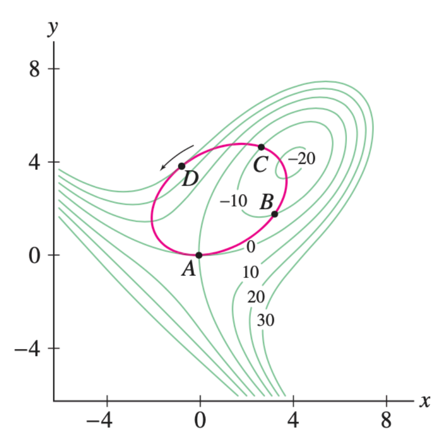



**FIGURE 14**

# 17 What can we do with the Gradient?

We will focus on two parts for today:

- How to use the gradient as a tool
- What the gradient means conceptually

Pro-tip: The box below has some properties of the gradient. We might use these from time to time when working through proofs or derivations.

- They can help keep things clean (not messy)
- These aren't super important though, since you can often get around them by just doing things "by hand".
- So don't fret too much about these.

**THEOREM 1 Properties of the Gradient** If $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $c$ is a constant, then

**(i)** $\nabla(f+g) = \nabla f + \nabla g$

**(ii)** $\nabla(cf) = c\nabla f$

**(iii) Product Rule for Gradients:** $\nabla(fg) = f\nabla g + g\nabla f$

**(iv) Chain Rule for Gradients:** If $F(t)$ is a differentiable function of one variable, then

$$\nabla(F(f(x, y, z))) = F'(f(x, y, z))\nabla f \qquad (1)$$

Pro-tip of a different sort: The stuff below this box is SUPER important and we will use all the time. It's important to understand it both in terms of the ______________________ and also what it means conceptually.

**The Chain Rule for Paths:**

Let's imagine that you are talking across a mountain terrain, and your path in the $xy$ plane is given by some curve $\mathbf{c}(t) = \langle x(t), y(t)\rangle$.

$\mathbf{c}(t)$ is a function that takes ____________ as input and spits out ______________________ as outputs.

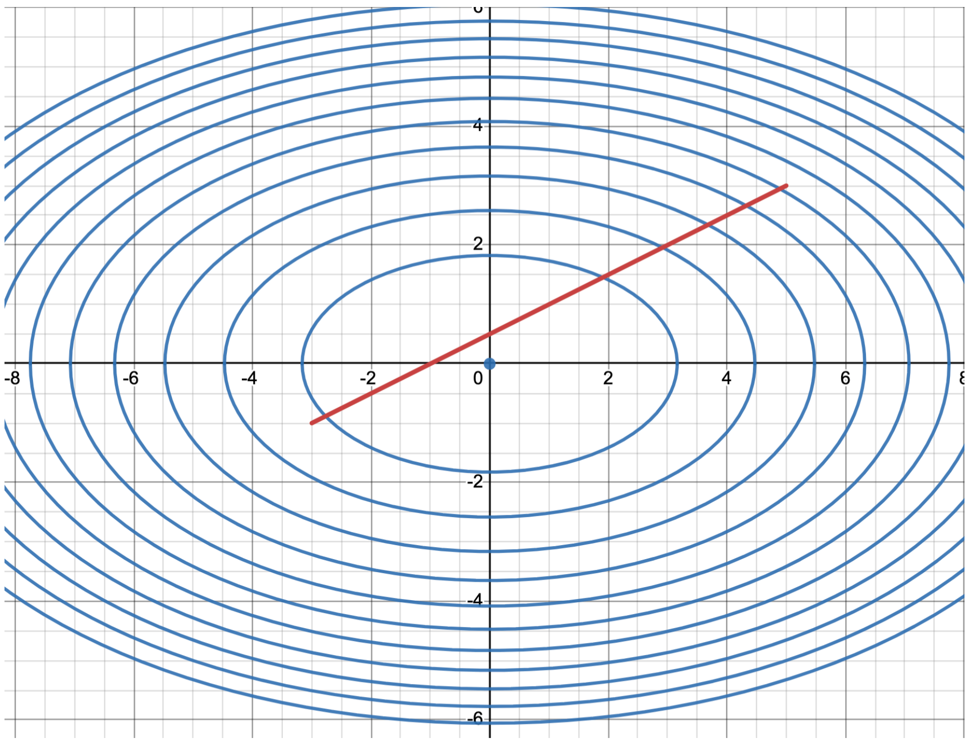
Now at the same time, you have a function $f(x, y)$ that gives the temperature at a point $xy$. So

$f(x, y)$ is a function that takes ________________ as inputs and gives ________________ as outputs.

The picture below can help us visualize what happens if we do $f(\mathbf{c}(t))$.

$f(\mathbf{c}(t))$ is a function that takes ________________ as inputs and gives ________________ as outputs.

$\frac{d}{dt}f(\mathbf{c}(t))$ will tell us the rate that ____________ is changing with respect to ____________.




**Example:** Suppose that the temperature function is given by $f(x, y) = 10 - x^2 - 3y^2$. And our curve is the line segment given by $(-3 + 4t, -1 + 2t)$ as $0 \leqslant t \leqslant 4$. These level curves are evenly spaced every 10 degrees Celsius.

- What is the temperature at the origin?
- Label some of the contours and indicate a direction on the curve $\mathbf{c}(t)$.
- How do you think the temperature is changing as we walk along the path?
- Simplify $f(\mathbf{c}(t))$ by plugging in the $x$ and $y$ coordinates. This will give the temperature as a function of $t$.
- What is the derivative of the function that you just found. How does that match what you found earlier?

When taking this derivative, by the **chain rule** the **partial dervatives of** $f$ were involved, but so were the **derivatives of the components of** $\mathbf{c}(t)$.

**Chain Rule for Paths**

$$\frac{d}{dt} f(\mathbf{c}(t)) = \nabla f_{\mathbf{c}(t)} \cdot \mathbf{c}'(t)$$

In other words, we take the gradient of $f$, plug in the point we want to measure the change at, and then take the **dot** product with the tangent vector.

**Example:** Find $\dfrac{d}{dt}f(\mathbf{c}(t))$ if $f(x,y) = x^2 - 3xy$ and $\mathbf{c}(t) = (\cos t, \sin t)$, at time $t = \pi/2$.

**Example:** In the picture below, will $\dfrac{d}{dt}f(\mathbf{c}(t))$ be positive negative or zero at the indicated points?

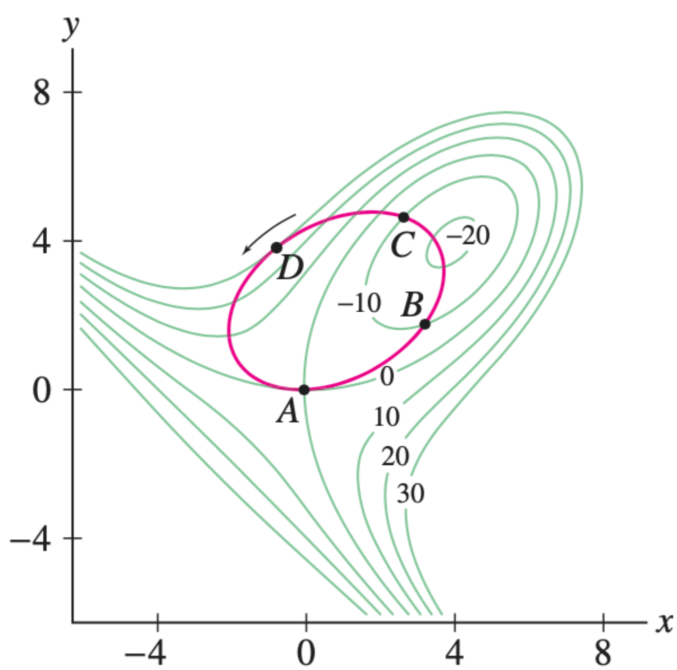
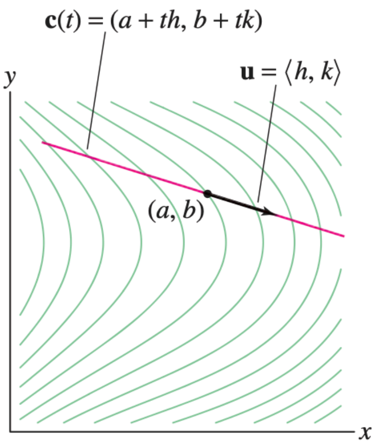


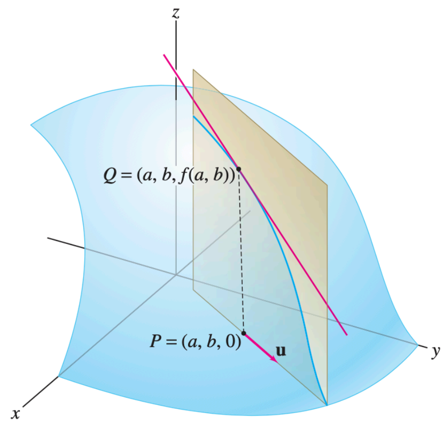

FIGURE 14

## Directional Derivatives

Another application of the gradient is to calculate directional derivatives.




Contour map of $f(x, y)$




$\dfrac{\partial f}{\partial x}(a,b)$ = instantaneous rate of change of $f$ ____________ as we move in the ____________ direction.

$\dfrac{\partial f}{\partial y}(a,b)$ = instantaneous rate of change of $f$ ____________ as we move in the ____________ direction.

$D_{\mathbf{u}}f(a,b)$ = instantaneous rate of change of $f$ ____________ as we move in the ____________ direction.

**How do we calculate it?**

**THEOREM 3 Computing the Directional Derivative** If $\mathbf{v} \neq \mathbf{0}$, then $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\|$ is the unit vector in the direction of $\mathbf{v}$, and the directional derivative is given by

$$\boxed{D_{\mathbf{u}} f(P) = \frac{1}{\|\mathbf{v}\|} \nabla f_P \cdot \mathbf{v}} \qquad (4)$$

**Example:** Find $\mathbf{D_u}f(P)$ for the function $f(x,y) = x^2 + y^3$, in the direction of the vector $\mathbf{v} = \langle 4, 3\rangle$, at the point $P = (1,2)$

**A 3D example:** Find the directional derivative of $f(x,y,z) = xy + z^3$ at $P = (3,2,1)$ in the direction pointing to the origin.

In this example, we might think of $f$ as representing the temperature of a point $(x,y,z)$ in three dimensions. This will tell us how the temperature is changing as we move in that direction.

**Discussion:** What would happen if we walk along a level curve of a function? How does the value of our function change?

Answer:

So if $\mathbf{c}(t)$ is a **level curve** for a function $f(x, y)$, then if we look at the function

$$f(\mathbf{c}(t))$$

then this function will always be ______________________________.

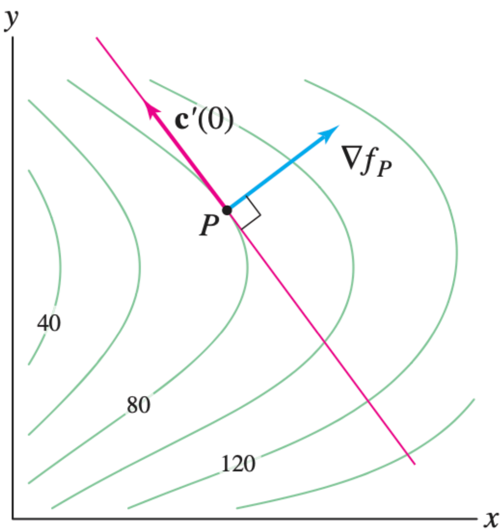
This means that

$$\frac{d}{dt} f(\mathbf{c}(t)) =$$

which means that:

Important Summary: The gradient $\nabla(f_P)$ is always ______________________________ to the level curve of $f$ at the point $P$.

In fact, the gradient will point in the direction of the ______________________________ of $f$ at the point $P$.



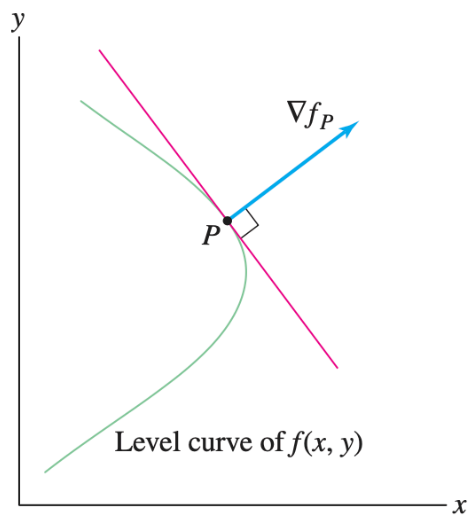




**FIGURE 10** The gradient points in the direction of maximum increase.

**THEOREM 4 Interpretation of the Gradient** Assume that $\nabla f_P \neq \mathbf{0}$. Let $\mathbf{u}$ be a unit vector making an angle $\theta$ with $\nabla f_P$. Then

$$\boxed{D_{\mathbf{u}} f(P) = \|\nabla f_P\| \cos\theta} \qquad (6)$$

- $\nabla f_P$ points in the direction of maximum rate of increase of $f$ at $P$.
- $-\nabla f_P$ points in the direction of maximum rate of decrease at $P$.
- $\nabla f_P$ is normal to the level curve (or surface) of $f$ at $P$.

**Example:** Consider the ellipse defined by $2x^2+y^2 = 17$. You can think of this as a level curve of the function

$$f(x, y) =$$

and as such the gradient of $f$ will be perpendicular to this curve. Let's try it out. Find the gradient at the point $(2, 3)$ and check that it is perpendicular to the ellipse.

**Example:** Consider the ellipsoid defined by $2x^2 + y^2 + 2z^2 = 19$. You can think of this as a level curve of the function

$$f(x, y, z) =$$

and as such the gradient of $f$ will be perpendicular to this surface. Let's try it out. Find the gradient at the point $(2, 3, 1)$ and check that it is perpendicular to the ellipsoid.

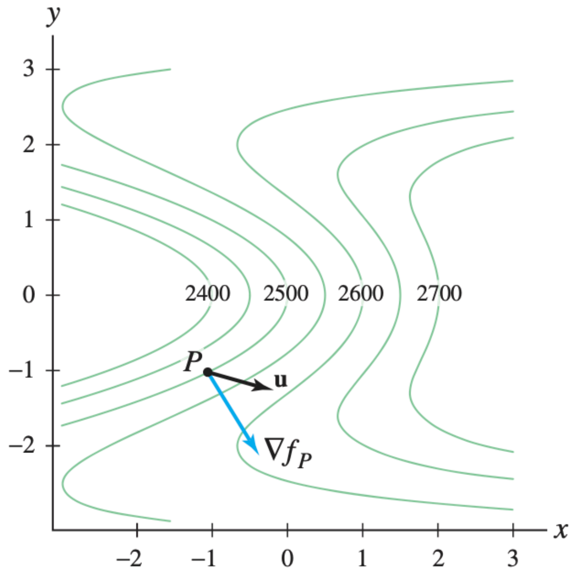



**FIGURE 11** Contour map of the function $f(x, y)$ in Example 9.

■ **EXAMPLE 9** The altitude of a mountain at $(x, y)$ is

$$f(x, y) = 2500 + 100(x + y^2)e^{-0.3y^2}$$

where $x, y$ are in units of 100 m.

**(a)** Find the directional derivative of $f$ at $P = (-1, -1)$ in the direction of unit vector $\mathbf{u}$ making an angle of $\theta = \frac{\pi}{4}$ with the gradient (Figure 11).
**(b)** What is the interpretation of this derivative?

There are more gems in this section, so please **read** this section as you are working on the homework problems.

# 18 Review for Exam

# 19 Exam Day

# 20 Last day on Gradient

**Warmup:** 1) By guessing and checking, can you find a function $f(x, y, z)$ such that

$$\nabla f = \langle 2x, 1, 2 \rangle?$$

Answer: $f(x, y, z) =$

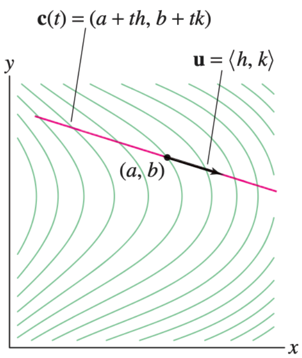
2) Can you find a function $f(x, y, z)$ such that $\nabla f = \langle z, 2y, x \rangle$

Answer: $f(x, y, z) =$

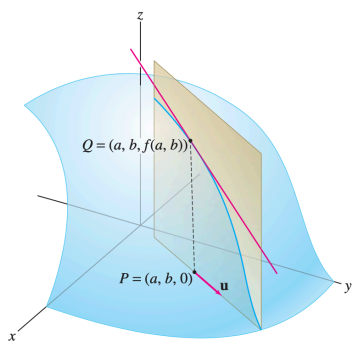
3) Explain why there is NOT a function $f(x, y)$ whose gradient is $\langle y^2, x \rangle$. Hint: We'll use Clairaut's Theorem that says that $f_{xy} = f_{yx}$.

**Directional Derivatives**

Another application of the gradient is to calculate directional derivatives.



Contour map of $f(x, y)$




$\dfrac{\partial f}{\partial x}(a, b) =$ instantaneous rate of change of $f$ ____________ as we move in the ____________ direction.

$\dfrac{\partial f}{\partial y}(a, b) =$ instantaneous rate of change of $f$ ____________ as we move in the ____________ direction.

$D_{\mathbf{u}}f(a, b) =$ instantaneous rate of change of $f$ ____________ as we move in the ____________ direction.

**How do we calculate it?**

If $\mathbf{u}$ is a unit vector, then

$$\mathbf{D_u}f(P) = \nabla f_P \cdot \mathbf{u}.$$

In other words just dot with the gradient. But remember that we need to use a ________________ vector.

**Example:** Find $\mathbf{D_u}f(P)$ for the function $f(x, y) = x^2 + y^3$, in the direction of the vector $\mathbf{v} = \langle 4, 3 \rangle$, at the point $P = (1, 2)$

Homework Question:

**33.** A bug located at $(3, 9, 4)$ begins walking in a straight line toward $(5, 7, 3)$. At what rate is the bug's temperature changing if the temperature is $T(x, y, z) = xe^{y-z}$? Units are in meters and degrees Celsius.

Hint: Draw a picture and figure out what **u** is. Check with your neighbors before proceeding.

The gradient $\nabla f_P$ points in the direction of maximum rate of increase of $f$ at the point $P$.

But what about the magnitude?

$||\nabla f_P||$ represents that maximum rate.

**Example:** You are located at the point $(-1, -1)$ on a mountain terrian whose height is given by

$$f(x, y) = 2500 + 100(x + y^2)e^{-0.3y^2}$$

where $x$ and $y$ are in units of 100m.

- What is the direction of maximum rate of increase?

- What is that maximum rate?

- What is the **angle of inclination** at this point?

- If you deviated from that direction by 10 degrees, what would your rate of increase be?

We saw that the gradient $\nabla f_P$ is always perpendicular to the level curve of $f$ through the point $P$. The same is true in higher dimensions. For instance, if we look at the function $f(x, y, z) = x^2 + y^2 + z^2$ then the "level surfaces"

$$f(x, y, z) = 1$$

$$f(x, y, z) = 5$$

$$f(x, y, z) = 9$$

are all spheres and $\nabla f_P$ will always be perpendicular to these surfaces.

**Example:** Consider the ellipsoid defined by $2x^2 + y^2 + 2z^2 = 19$. You can think of this as a level **surface** of the function

$$f(x, y, z) = 2x^2 + y^2 + 2z^2$$

and as such the gradient of $f$ will be perpendicular to this surface. Let's try it out. Find the gradient at the point $(2, 3, 1)$ and use this to find the equation of the tangent plane to the ellipsoid.

## 21 The Chain Rule and a Peek at Polar Coordinates

When we studied the **Chain Rule for Paths** we saw that:

$$\frac{d}{dt}\left(f(\mathbf{c}(t))\right) = \nabla f \cdot \mathbf{c}'(t)$$

If we are in two variables this means that

$$\frac{df}{dt} =$$

The same formula holds more generally if we have more than one variable, for instance, suppose that

$$f(x, y, z) = xy + z, \quad x = s^2, \quad y = st, \quad z = t^2.$$

Then we could think of $f$ as a function of $s$ and $t$. Let's work out explicitly what it is:

The General Chain Rule Says

$$\frac{\partial f}{\partial s} =$$

$$\frac{\partial f}{\partial t} =$$

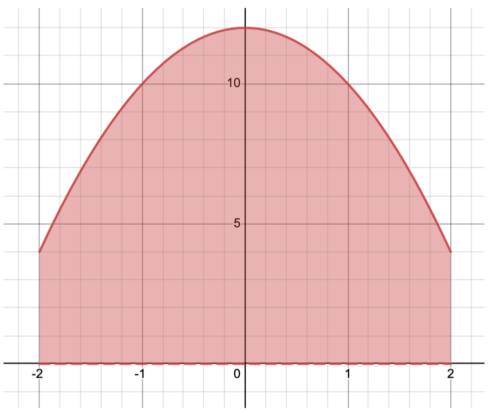
Check that the Chain Rule works in this example by finding these derivatives in two ways:

**Example:** Let $f(x, y) = e^{xy}$. Evaluate $\partial f/\partial t$ at the point $(s, t, u) = (2, 3, -1)$ where $x = st$, $y = s - ut^2$.

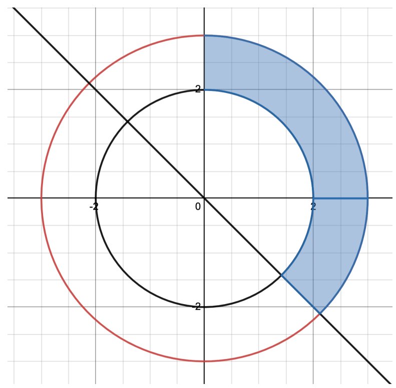
**Why on earth would anyone do this? What does it mean?**

**Polar Coordinates:** If we want to describe the shaded on the left, we can do so easily using $x$ and $y$. But the figure on the right would be a LOT more complicated. Instead, we can use **polar** coordinates.

Consider the regions below:




$y = 12 - 2x^2$

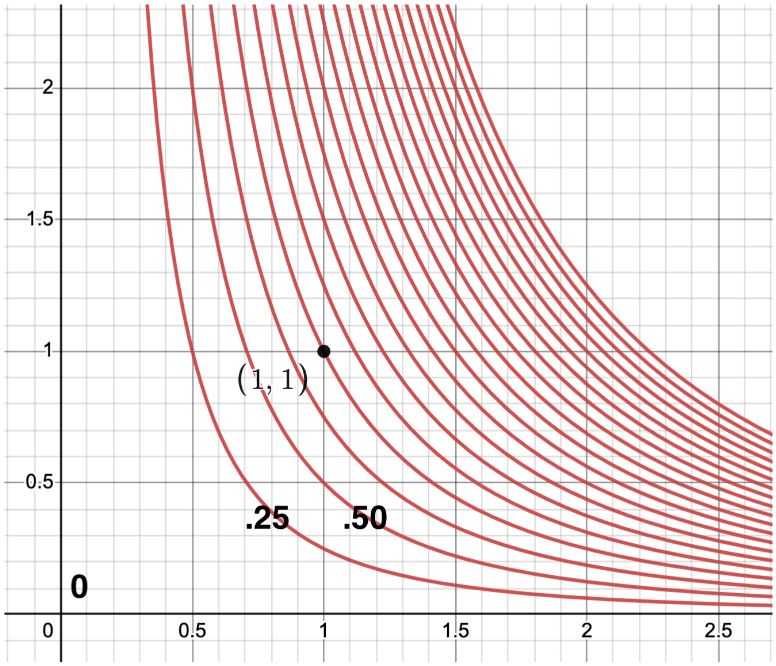



> It is sometimes ____________________________ to describe something in terms of $r$ and $\theta$
>
> rather than using $x$ and $y$.

**Relationship between** $x, y, r, \theta$**:**

In looking at the contour map of the function $f$ below, what signs would we expect the following to have at the point $(1,1)$:

$$\frac{\partial f}{\partial x} = \qquad \frac{\partial f}{\partial y} = \qquad \frac{\partial f}{\partial r} = \qquad \frac{\partial f}{\partial \theta} =$$



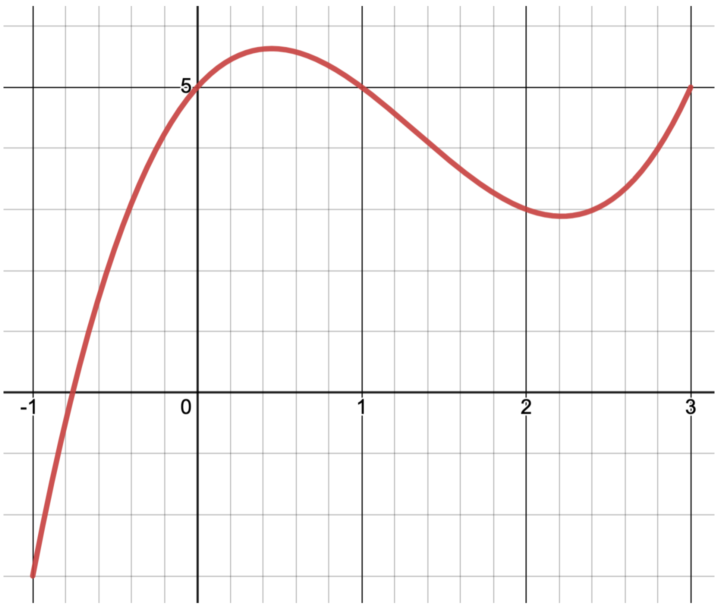

The function in question is $f(x,y) = x^2y$. Let's do the calculations!

## 22 Optimization Week - Day 1

Our goal is to find **local and global** extrema. These are things like "local min/max" and "global min/max".

How can we use calculus to help? Let's recall some ideas from Calc 1. Suppose you wanted to find the extrema of the function below. Write down what you would look for to find the extrema of the function below. Be specific about your steps. (2 minutes)

$$f(x) = \text{some formula and the graph is below}$$




**Ideas:**

General Strategy:

**How will this change in higher dimensions?**

We will focus on functions of 2 variables, $f(x, y)$ and our goal is to find **local and global** extrema.

- Suppose you have a point $P$ and $f_x(P) = 2$. Then if we move *EAST* the function will [increase/decrease/can't tell]

  and if we move *WEST* the function will [increase/decrease/can't tell].

  We conclude that if $f_x(P) = 2$ then $P$ is [a local max / a local min / neither / we cannot tell]

- What if $f_x(P) = 0$ and $f_y(P) = -3$ then $P$ is [a local max / a local min / neither / we cannot tell]

- What if $f_x(P) = 0$ and $f_y(P) = 0$ then $P$ is [a local max / a local min / neither / we cannot tell]

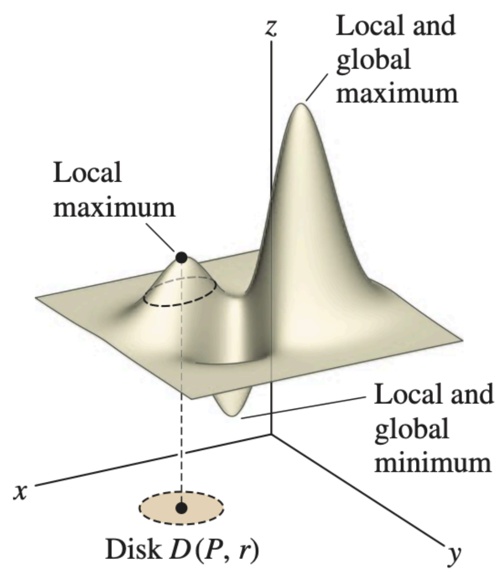



**FIGURE 1** $f(x, y)$ has a local maximum at $P$.

Important Theorem

If $f(x, y)$ has a ______________________ or a ______________________ at a point $P$ then

Points $P$ with these properties are called ______________________.

Note: Critical points require information about BOTH $f_x$ and $f_y$.

- One way to remember it is to say that $P$ is a critical point if and only if ____________ is ____________ or ____________.

**Example:** Find the critical points of the following function:

$$f(x, y) = 11x^2 - 2xy + 2y^2 + 3y$$

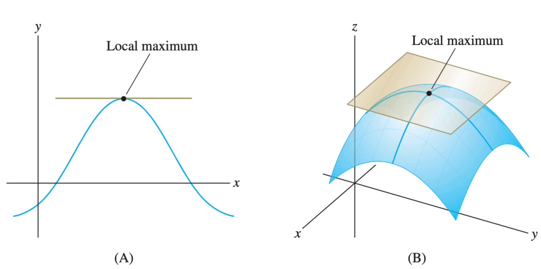
**Part 2:** How do we tell whether a critical point is a local max / local min or neither?

There is a toolkit.

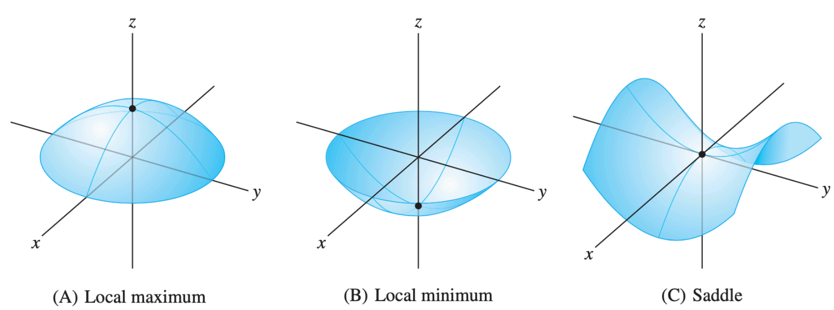
- It involves 2nd derivatives.
- It is related to thinking about ________________

Let's explore this on Geogebra:

`https://www.geogebra.org/3d/pjjhrxkg`



(A) (B)



(A) Local maximum (B) Local minimum (C) Saddle

**How to Find and Classify a Critical Point of a function $f(x, y)$**

1. Find critical points by looking where the partial derivatives are **both** (zero / undefined)
2. To test if a critical point $P$ is a local max / local min / saddle, you will want to look at the second derivatives: $f_{xx}, f_{xy}, f_{yy}$. It is helpful to put them in a matrix like this:

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \qquad D = \det H = \underline{\hspace{6cm}}.$$

   $D$ is called the __________________________ of $f(x, y)$.

3. If $D > 0$ then you will have a ______________________________________.

   You can tell which by looking at $f_{xx}$ or $f_{yy}$. They will have the same sign.

   If that sign is positive you have a __________________________

   If that sign is negative you have a

4. If $D < 0$ you have a __________________________.
5. If $D = 0$ the test is __________________________.

Often the hardest part of these problems is doing the **algebra**. Let's get some practice.

**Example:** Analyze the critical points of $f(x, y) = x^3 + y^3 - 12xy$.

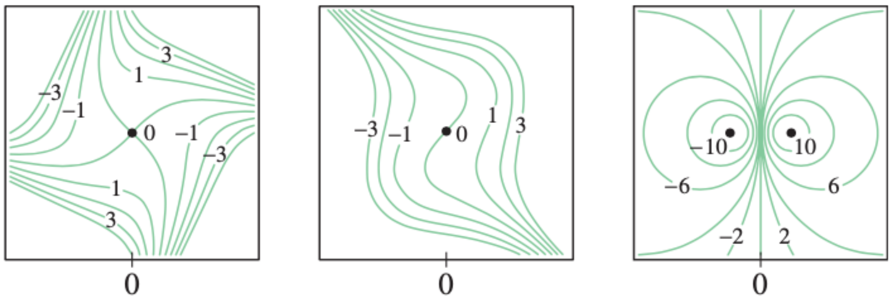
**Example:** Analyze the critical points of $f(x, y) = y^2x - yx^2 + xy$

**Example:** Suppose you want to build a box of volume 27 cubic inches. What would the dimensions be of a box whose **surface** area was minimal?

# 23 Optimization Week - Day 2

**Warmup:**

In the contour maps below, classify the points as local max / local min / saddle points.



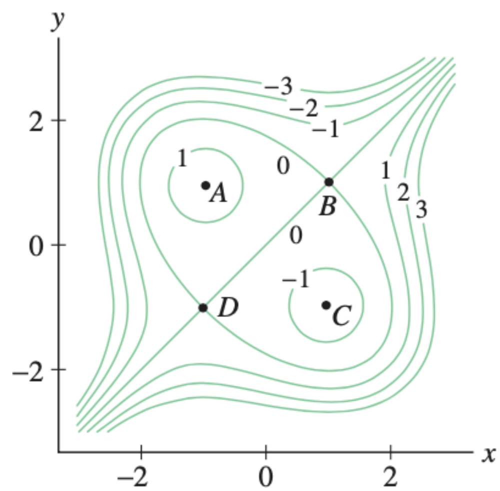




**FIGURE 19**

**Example:** Suppose that the temperature of a point $(x, y)$ is given by

$$T(x, y) = 2x^2 + y^2 - y.$$

But suppose that we are only concerned with the region inside the unit circle: $x^2 + y^2 = 1$.

Where would the temperature be **maximal**?

This problem is asking for a **global** extremum. It is a Theorem that any continuous function on a **closed and bounded** domain will achieve a max and a min somewhere on that region.

In practice, we find this by

- Finding the critical points (no need to classify them!)
- Checking what happens on the boundary (this is like checking the "endpoints" in Calc 1)
- To see what happens on the boundary, you will need to parametrize the boundary and then solve a 1-dimensional Calculus problem.
- You now have a bunch of candidates for max/min. You can now plug these into the original function and see which values are max/min.

**Example:** Suppose that the temperature of a point $(x, y)$ is given by

$$T(x, y) = 2x^2 + y^2 - y.$$

But suppose that we are only concerned with the region inside the unit circle: $x^2 + y^2 = 1$.

Find the global min and max of $T$ on this region.

## 24 Getting practice - We will work in groups on homework problems

# 25 Lagrange Multipliers and Constrained Optimization

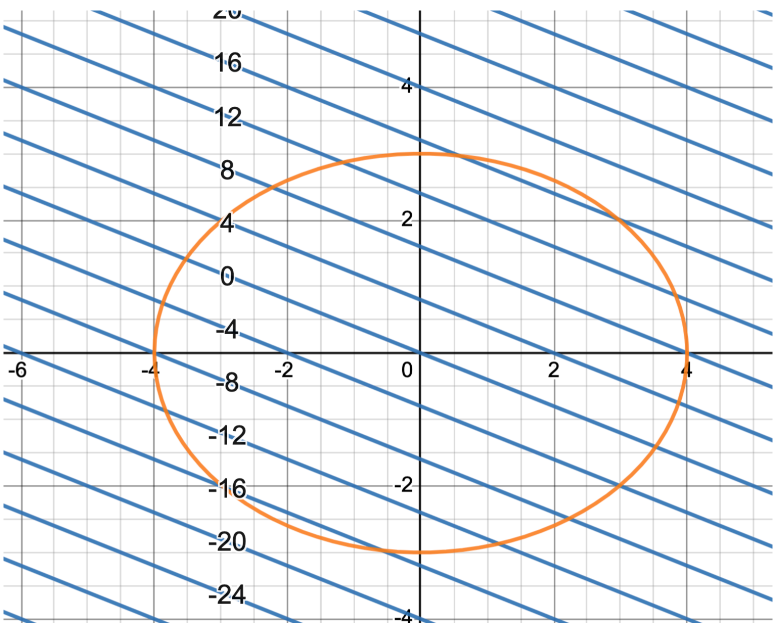
■ **EXAMPLE 1** Find the extreme values of $f(x, y) = 2x + 5y$ on the ellipse

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

What is this problem asking?

What is one way we could approach this?

Today we will learn a different (easier?) way to solve this. On the picture below, do you see where the maximum would be? What is happening geometrically?



**THEOREM 1 Lagrange Multipliers** Assume that $f(x, y)$ and $g(x, y)$ are differentiable functions. If $f(x, y)$ has a local minimum or a local maximum on the constraint curve $g(x, y) = 0$ at $P = (a, b)$, and if $\nabla g_P \neq \mathbf{0}$, then there is a scalar $\lambda$ such that

$$\boxed{\nabla f_P = \lambda \nabla g_P} \qquad (1)$$

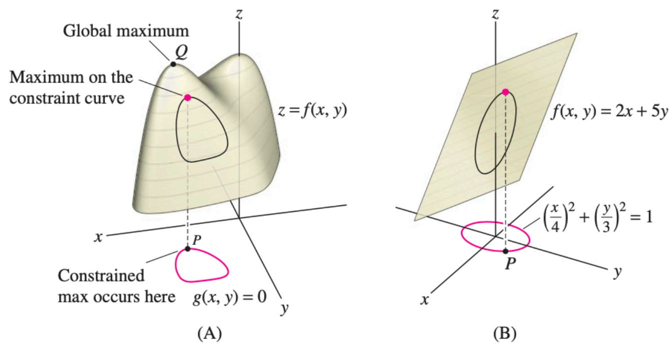
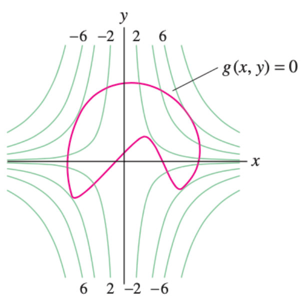



**FIGURE 6**

## Conceptual Practice

**(a)** Identify the points where $\nabla f = \lambda \nabla g$ for some scalar $\lambda$.

**(b)** Identify the minimum and maximum values of $f(x, y)$ subject to $g(x, y) = 0$.




Contour plot of $f(x, y)$
(contour interval 2)

**FIGURE 10** Contour map of $f(x, y)$; contour interval 2.

**2.** Figure 9 shows a constraint $g(x, y) = 0$ and the level curves of a function $f$. In each case, determine whether $f$ has a local minimum, a local maximum, or neither at the labeled point.

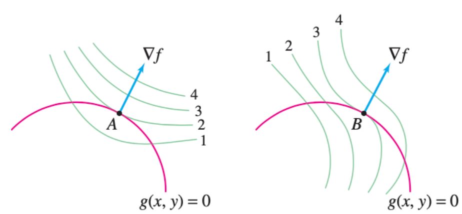



**FIGURE 9**

General Strategy

1. Set up your problem so you have your function $f(x, y)$ you want to optimize and your equation $g(x, y) = 0$ that gives your constraint.
2. Write down the **system** of equations given by $\nabla f = \lambda \nabla g$.
3. Do your best to solve this system for $x, y, \lambda$. (A variety of strategies will be required here.)
4. Remember you have your constraint equation $g(x, y) = 0$ as well.
5. Check that your solution $(x, y, \lambda)$ is actually a solution and that the gradient is **nonzero** at this point.

**Example** Find the maximum and minimum values of the function $f(x, y) = x^2 + y^2$ subject to the constraint that $2x + 3y = 6$.

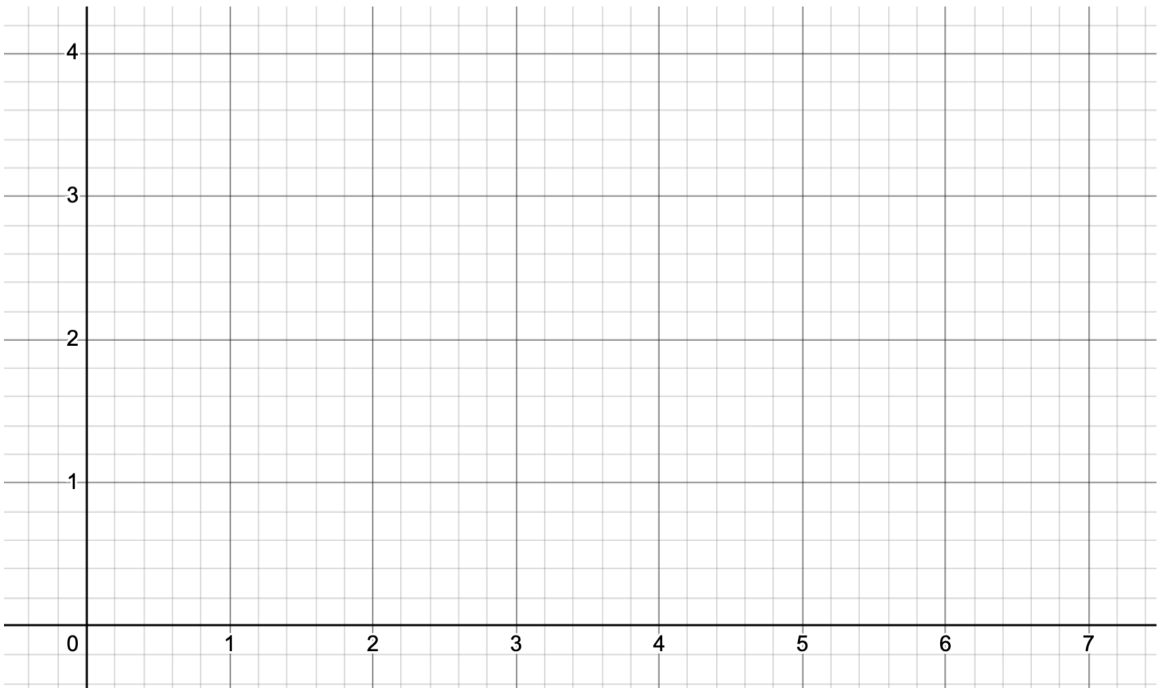
**EXAMPLE 3** **Lagrange Multipliers in Three Variables** Find the point on the plane $\frac{x}{2} + \frac{y}{4} + \frac{z}{4} = 1$ closest to the origin in $\mathbf{R}^3$.

# 26 Introduction to Integration

**Warmup:** You are working measuring the number of covid cases per day in a certain town and you record the following data:

| day number | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|
| number of cases (in thousands) | 4.2 | 3.1 | 1.7 | 2.0 | 3.5 |

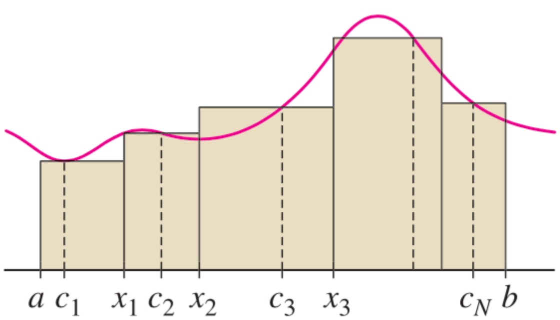
Plot these cases in the graph below and label the axes. What does the **area** under the shaded curve represent?




Here our **domain** was ____________ (days) and our function $f(x)$ was measuring the number of cases/day. The integral (area) had units ________________________

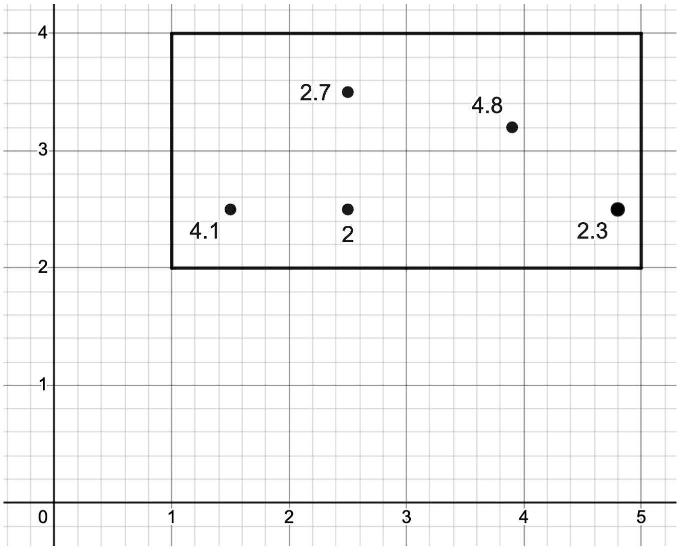
The general idea of integration:

You __________________ the areas of a bunch of ________________________




What do you think will happen if our function has two variables?

**Example:** In a certain part of the ocean, there is an oil spill. We collect some data and measure the thickness of oil in a few different spots. The data is represented in the picture below (thickness is in mm) The $x$ and $y$ axes are measured in $m$.




We want to estimate the total amount of oil in the rectangle.

Notation: This rectangle will be denoted by

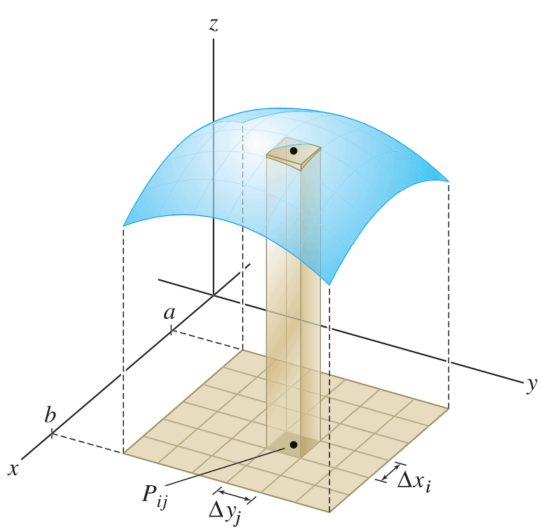
**Estimate:**

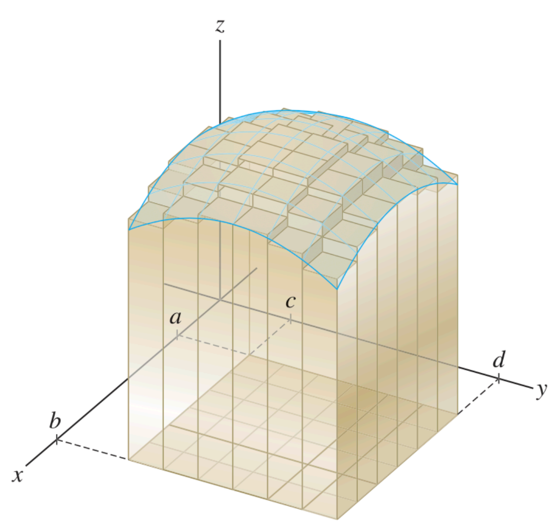
Total volume of oil $\approx$

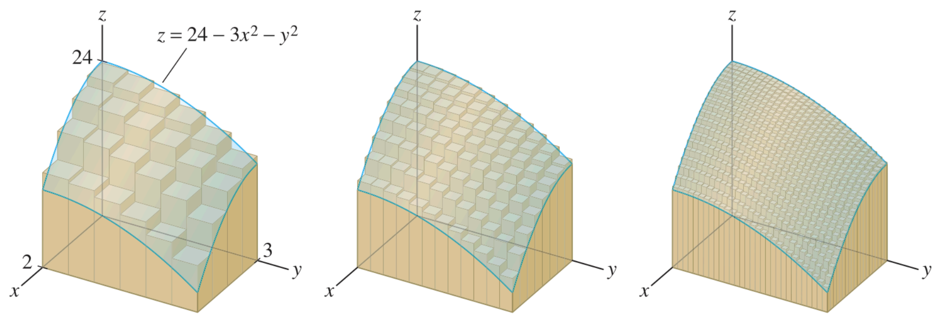
**How could we get a better estimate?**

If $f(x, y)$ represents the thickness of oil at a point $(x, y)$, then the total amount of oil on ____________________________ will be

We could calculate this, by using Riemann Sums and a large number of rectangular ________________.





(A) $M = 6, N = 4$ (B) $M = 12, N = 8$ (C) $M = 30, N = 20$

**Example:** Here is how we might visualize an integral. Fill in the blanks below:

The volume illustrated below is given by :

where $\mathcal{R} =$ ____________________________.

**Question for y'all**:

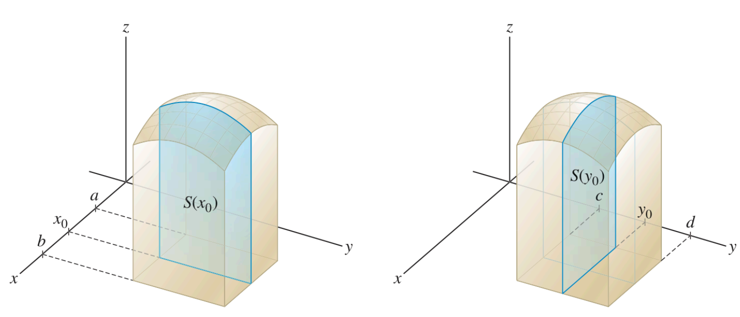
What is the area of the slice of in the "front" above. Write your answer down as an integral:

Now what about the slice in the "back"?

What about the slice in the middle where $x = 1$?

How could you "add up" all the areas of all the slices in the middle?

This is called an ____________________________



(A) (B)

Let's find the volume by calculating:

**Another query for you:** Do you think we could have instead, set it up the "other way" by adding up slices from "left to right" instead of "back to front?"

This is known as **Fubini's Theorem**. If $f(x, y)$ is a continuous function then over the rectangle $\mathcal{R} = [a, b] \times [c, d]$ then

and both of these iterated integrals will give you the integral $\displaystyle\iint\limits_R f(x, y)dA$.

■ **EXAMPLE 4** Evaluate $\displaystyle\int_2^4 \left( \int_1^9 y e^x \, dy \right) dx$.

Calculating these iterated integrals is kind of like a partial derivative.

- When doing the $y$ antiderivative we treat $x$ as a constant.
- When doing the $x$ antiderivative we treat $y$ as a constant.
- Remember - the answer is always a ______________. There shouldn't be anyvariables left at the end.

■ **EXAMPLE 8** Calculate $\displaystyle\iint_{\mathcal{R}} \frac{dA}{(x+y)^2}$, where $\mathcal{R} = [1, 2] \times [0, 1]$ (Figure 14).

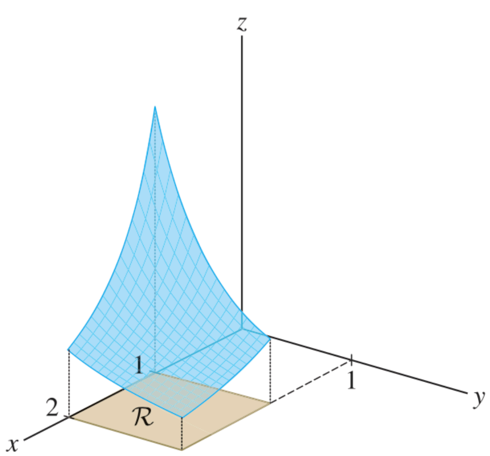



**FIGURE 14** Graph of $z = (x+y)^{-2}$ over $\mathcal{R} = [1, 2] \times [0, 1]$.

## 27 Integration over More General Regions (also Day 28)

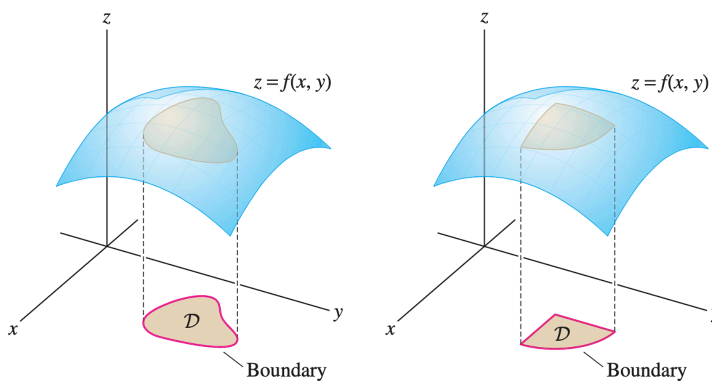
**Warmup:** Last time we learned how to calculate something like

$$\iint_{\mathcal{R}} 4 - x + y \; dA$$

In this expression, $\mathcal{R}$ represented __________________________.

We evaluated this by using something called an __________________________.

**Next Goal:** Try to understand how to calculate integrals that are not "over a rectangle"



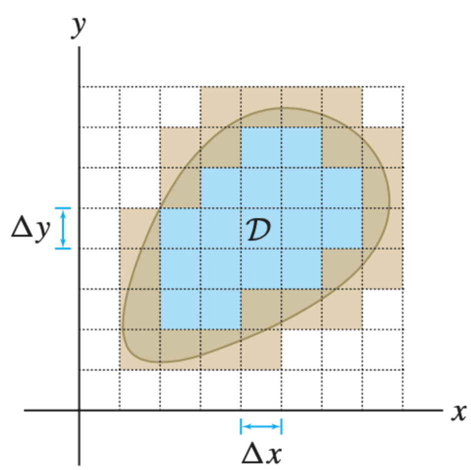

> Remember:
> $\mathcal{D}$ is just the domain we are integrating over. The function is living "above" it.

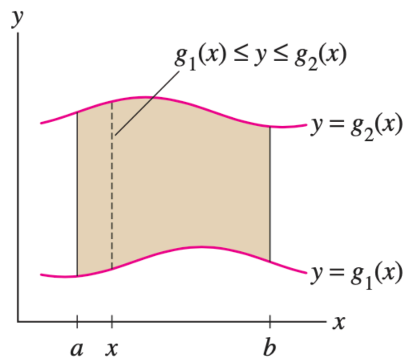
1 dimension

2 dimensions

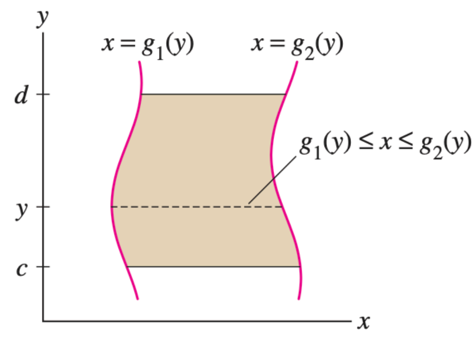
Our main strategy for calculating integrals over more complicated regions is to approximate with rectangles:

- We could use Riemann sums to get an approximation
- Smaller rectangles would give us a better approximation
- We will learn how to set up an **iterated integral** as well.

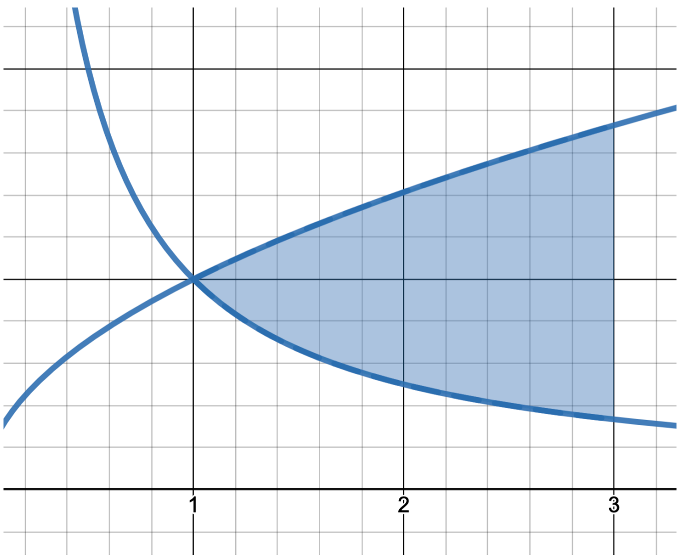





(A) Vertically simple region




(B) Horizontally simple region




**Example:** Describe the region between the two curves at the left.

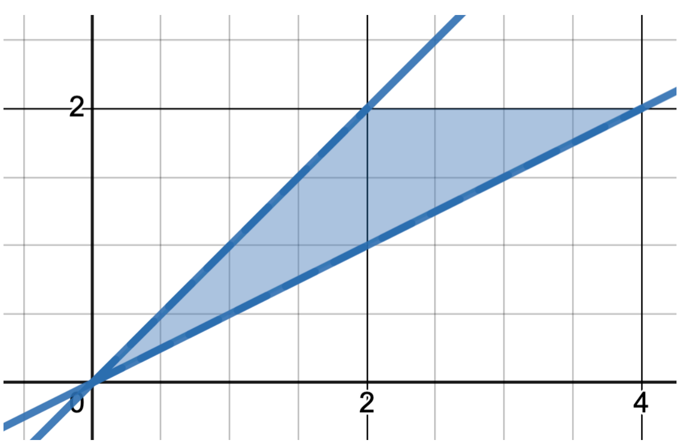
The two curves are $y = \sqrt{x}$ and $y = 1/x$.

Label the top and bottom curves.

This region is [vertically/horizontally] simple.

So I can describe this region as:

Set up an iterated integral that would calculate $\iint_{\mathcal{D}} x^2 y \, dA$.




**Example:** The region at left is given between the lines $y = x$ and $y = x/2$.

Label the top and bottom curves

This region is [vertically/horizontally] simple

Describe this region

Set up an iterated integral that would calculate $\iint_{\mathcal{D}} x^2 y \, dA$.

Sometimes finding an antiderivative in terms of elementary functions is literally impossible.

- $\int e^{x^2}\ dx$
- $\int \sin(x^2)\ dx$
- $\int \cos(x^2)\ dx$
- $\int \frac{e^x}{x}\ dx$
- For more, see `https://en.wikipedia.org/wiki/Nonelementary_integral`

**Sound the fanfare!**

Sometimes, even if an antiderivative is impossible, if we are in the context of a **double** integral, we can still calculate it.

We will use some ______________________________

aka _____________________________________________

**Example:**

Calculate the integral

$$\iint_{\mathcal{D}} e^{y^2}\ dA$$

over the region bounded between the $y$ axis, the line $y = x/2$ and the line $y = 2$.

Step 1: (Draw a picture!)

■ **EXAMPLE 5** **Changing the Order of Integration** Sketch the domain of integration $\mathcal{D}$ corresponding to

$$\int_{1}^{9}\int_{\sqrt{y}}^{3} xe^{y}\,dx\,dy$$

How can we calculate this integral? What happens if we do it $dxdy$? Can we change it to $dydx$?

If $\mathcal{D}$ is some region in the plane, what is $\displaystyle\iint_{\mathcal{D}} 1\ dA$?

Important Interpretation:

**Average Value**

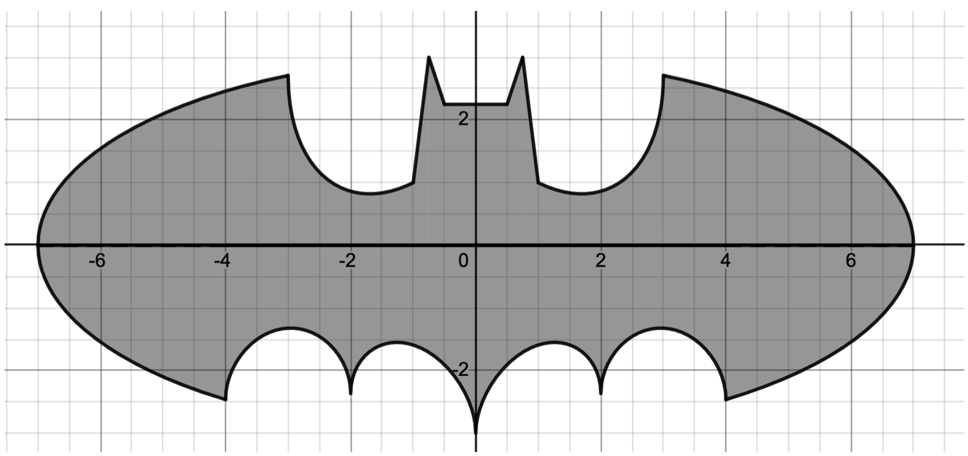
The average value of a function, $\overline{f}$ on a region $\mathcal{D}$ is defined to be:

$$\overline{f} = \frac{\iint_{\mathcal{D}} f(x,y)\ dA}{Area(\mathcal{D})}$$

this is the same as

$$\overline{f} = \frac{\iint_{\mathcal{D}} f(x,y)\ dA}{\iint_{\mathcal{D}} 1\ dA}$$

**Example:** Suppose that the region $\mathcal{D}$ is pictured below.




What is the average value of the function $f(x,y) = x$ on this region? Answer: ____________________

What is the average value of the function $f(x,y) = y$ on this region? Answer: ____________________

Which of these answers is exact? Which is an estimate?

Suppose that you know the area of this region is 24.2. Express the following integrals in terms of $A$:

$$\iint_{\mathcal{D}} 1\ dA$$

$$\iint_{\mathcal{D}} 2\ dA$$

$$\iint_{\mathcal{D}} x\ dA$$

$$\iint_{\mathcal{D}} 3x - 10 \; dA$$

$$\iint_{\mathcal{D}} 3x + y - 1 \; dA$$

## 28 Triple Integrals

Draw the **domains** for integrals and double integrals. Then we'll draw the domain for a **triple integral**.

The **simplest** domain for a triple integral is a __________________________. We'll write notation below:

■ **EXAMPLE 1** **Integration over a Box** Calculate the integral $\iiint_{\mathcal{B}} x^2 e^{y+3z}\, dV$, where $\mathcal{B} = [1, 4] \times [0, 3] \times [2, 6]$.

**More general regions?**

- Say you want to integrate a function $f(x, y, z)$ over a region $W$ in $\mathbb{R}^3$ (3-D domain)
- Pick a direction in which your **domain** is simple. E.g. maybe you can say

$$\text{lower function} \leqslant z \leqslant \text{upper function}$$

- This will be your ____________________________ integral
- Now find the ____________________________ of your region below. E.g. if you chose $z$ above, then your shadow will be in the $xy$ plane. Call this shadow $\mathcal{D}$.
- Now you just calculate:

$$\boxed{\iiint_{\mathcal{W}} f(x, y, z)\, dV = \iint_{\mathcal{D}} \left( \int_{z=z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz \right) dA}$$

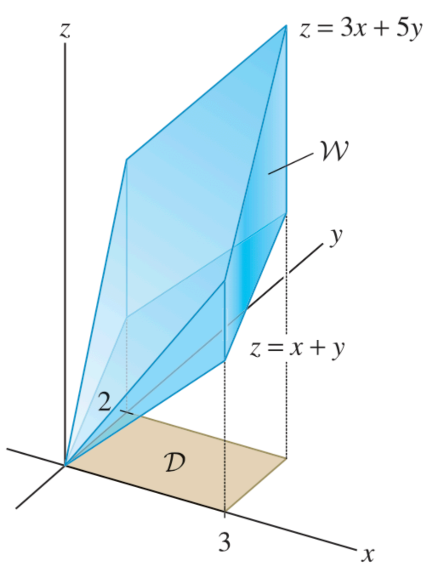
## What does it all mean?

In general a triple integral of a function $f(x, y, z)$ over a 3D region $\mathcal{W}$ is representing a 4-dimensional volume, so it's hard to visualize.

However, here are some ways to help visualize:

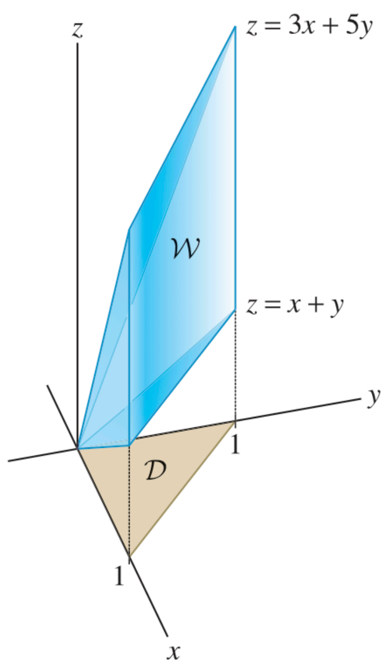
- If $f(x, y, z) = 1$ then $\displaystyle\iiint_{\mathcal{W}} 1dV$ = the volume of $\mathcal{W}$.
- If $f(x, y, z)$ represents the **density** (say in grams / L) of a goo at the point $(x, y, z)$ then $\displaystyle\iiint_{\mathcal{W}} f(x, y, z)dV$ will represent the mass of the goo inside of $\mathcal{W}$.
- There are also applications to center of mass, etc that we'll do next week!

**EXAMPLE 2** **Solid Region with a Rectangular Base** Evaluate $\iiint_{\mathcal{W}} z\, dV$, where $\mathcal{W}$ is the region between the planes $z = x + y$ and $z = 3x + 5y$ lying over the rectangle $\mathcal{D} = [0, 3] \times [0, 2]$ (Figure 3).



# 29 Triple Integrals Continued

**EXAMPLE 3** **Solid Region with a Triangular Base** Evaluate $\iiint_{\mathcal{W}} z\,dV$, where $\mathcal{W}$ is the region in Figure 4.



Advice: We will often see **paraboloids** which are of the form

$$f(x, y) = c + 3x^2 + 4y^2$$

$$f(x, y) = c - 1x^2 - 2y^2$$

What's important is that **both** the coefficients of the $x^2$ and $y^2$ terms have the same sign. In this case, these look like upward and downward facing bowls respectively.

**EXAMPLE 4** **Region between Intersecting Surfaces** Integrate $f(x, y, z) = x$ over the region $\mathcal{W}$ bounded above by $z = 4 - x^2 - y^2$ and below by $z = x^2 + 3y^2$ in the octant $x \geq 0$, $y \geq 0$, $z \geq 0$.

**EXAMPLE 5** **Writing a Triple Integral in Three Ways** The region $\mathcal{W}$ in Figure 7 is bounded by

$$z = 4 - y^2, \qquad y = 2x, \qquad z = 0, \qquad x = 0$$

Express $\iiint_{\mathcal{W}} xyz\, dV$ as an iterated integral in three ways, by projecting onto each of the three coordinate planes (but do not evaluate).

**28.** Let $\mathcal{W}$ be the region bounded by

$$y + z = 2, \quad 2x = y, \quad x = 0, \quad \text{and } z = 0$$

(Figure 14). Express and evaluate the triple integral of $f(x, y, z) = z$ by projecting $\mathcal{W}$ onto the:

**(a)** $xy$-plane **(b)** $yz$-plane **(c)** $xz$-plane

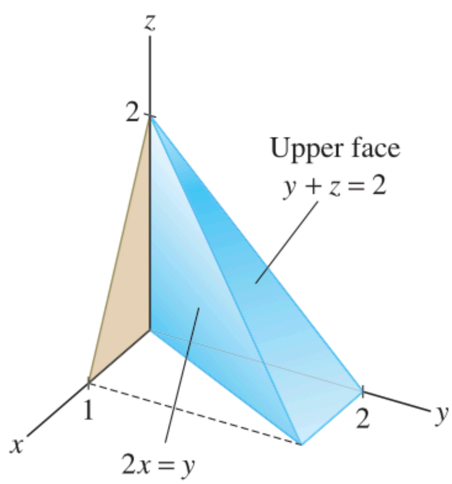



FIGURE 14

## 30 On to Polar Coordinates

**Warmup: (4 minutes)**

You want to set up an integral of the function $f(x, y, z) = e^{xyz}$ over the region in the **first octant** that is bounded by the plane defined by:

$$2x + y + 3z = 6.$$

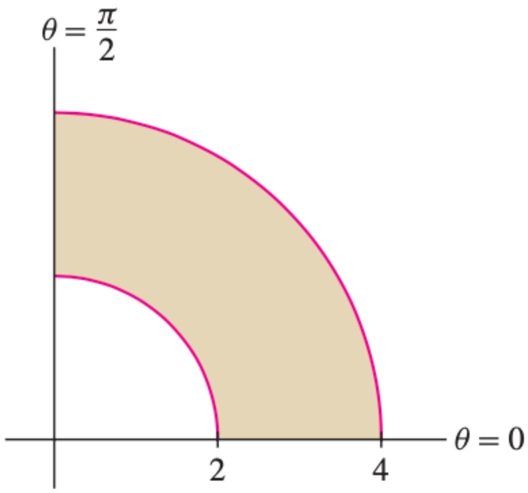
1. Always start with a picture! Draw a picture of this plane. Hint: At what points does the plane intersect the $x$-axis? The $y$-axis? The $z$-axis?

2. I'd like you to try and set up this integral in 3 different ways. One with $z$ as the inner integral, which involves looking at the shadow in the $xy$ plane. Below, draw a sketch of the shadow in the $xy$ plane.

3. Now repeat for the other two ways.

**Please practice doing problems like this - they unfortunately aren't the quickest of problems, but they are ones where practice makes perfect.**

**Using Polar Coordinates in Integrals**




This region at left would be tough to describe in terms of $x$ and $y$ coordinates:

But in **polar coordinates** it is easier:

Notice that the region in the $r, \theta$ plane is a __________________________, this will make setting up an integral a lot easier.

Remember that in polar coordinates:

| | |
|---|---|
| $x =$ | $y =$ |

**Example:**

Convert the following functions to polar coordinates:

$f(x, y) = x$

$g(x, y) = x^2 + y^2$

$h(x, y) = 4xy$

**Is there a catch?**

Kinda...

$$\iint_D f(x,y)\; dA$$

We have to figure out how $dA$ (change in area) changes as $r$ and $\theta$ change:

If you are integrating over a region $D$ that would be a lot nicer in polar coordinates,

- Convert $x$ and $y$ everywhere using $x = r\cos\theta$ and $y = r\sin\theta$.
- Add a factor of $r$ to compensate for the fact that $dA \approx r\; dr\; d\theta$

**Example:** Set up an integral that calculates the **area** of the region on the previous page and calculate it in polar coordinates.

**Find the center of mass of the region above:**

**Application:** Suppose you have a region $D$ in the plane. Suppose the object has uniform density. Then the coordinates of the center of mass of $D$ are given by:

$$\overline{x} = \frac{\iint_D x \, dA}{\iint_D 1 \, dA}, \qquad \overline{y} = \frac{\iint_D y \, dA}{\iint_D 1 \, dA}$$

Notice that the denominator is the ____________ of $D$.

**What if there's not a uniform density, but some changing density function $f(x,y)$?** Then you'll have:

$$\overline{x} = \frac{\iint_D x \cdot f(x,y) \, dA}{\iint_D f(x,y) \, dA}, \qquad \overline{y} = \frac{\iint_D y \cdot f(x,y) \, dA}{\iint_D f(x,y) \, dA}$$

Notice that the denominator is the ____________ of $D$.

**Is the same thing true for triple integrals?** Yes - you'll have similar formulas for the $x, y, z$ coordinates.

## 31 Quiz day

**28.** Let $\mathcal{W}$ be the region bounded by

$$y + z = 2, \quad 2x = y, \quad x = 0, \quad \text{and } z = 0$$

(Figure 14). Express and evaluate the triple integral of $f(x, y, z) = z$ by projecting $\mathcal{W}$ onto the:

**(a)** $xy$-plane **(b)** $yz$-plane **(c)** $xz$-plane

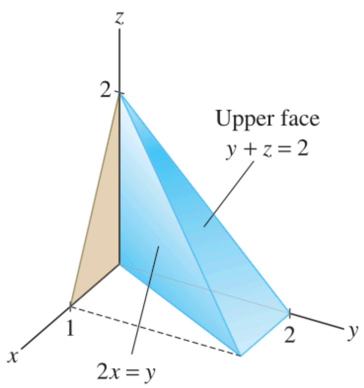



FIGURE 14

$$\int \quad \int \quad \left[\int \quad z \; d \quad \right] d \quad d$$

We're going to do part ($c$), so we will have a **shadow** on the $xz$ plane.

What will the middle integral be? [dx / dy / dz]

Looking at the picture, what will the $y$ coordinate range from:

1. between 0 and the "upper face"
2. between the plan $2x = y$ and the "upper face"
3. between the xy plane and the zy plane.
4. between the zy plane and the plane 2x = y

Write your answer from the previous problem as an inequality:

$$\leqslant y \leqslant$$

Now fill in your answer in the middle integral above.

Now let's think about the **shadow.** The shadow has three edges: one is the $z$ axis, one is the $x$ axis. The third comes from diagonal edge of the pyramid. Notice how the coordinates labelevd in the picture are so helpful. They should help you draw a picture of the shadow in the $xz$ plane. Please draw a picture of the shadow in the $xz$ plane.

## 32 Cylindrical Coordinates and Spherical Coordinates

**Warmup:**

If you have a **circular** type of region in the plane, then polar coordinates might be a good idea.

This can apply to shadows as well.

Set up the integral of the function $z$ over the region bounded by

$$x^2 + y^2 = 16, \qquad x + 2y + z = 6, \qquad z = -2$$

- Which shapes are these 3D objects?

  [circle / sphere / cylinder / bowl / saddle / salad / pastry / plane / silly goose / bagel shape ]
- Sketch a 3D picture roughly of these shapes.
- Draw a picture of the shadow in the $xy$ plane.
- Set up the integral using polar coordinates.
- What additional steps would we have to take to find the $z$ coordinate of the center of mass of this object?

We have just set up a **triple** integral using polar coordinates in the shadow, but we left $z$ alone. Doing this is called using **cylindrical coordinates**

**Spherical Coordinates**

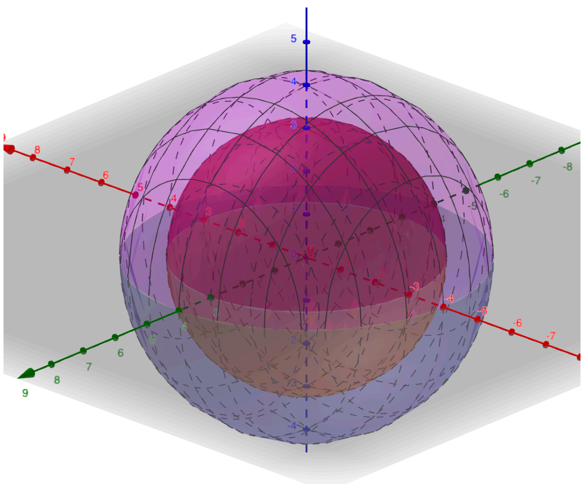
Here are two links:

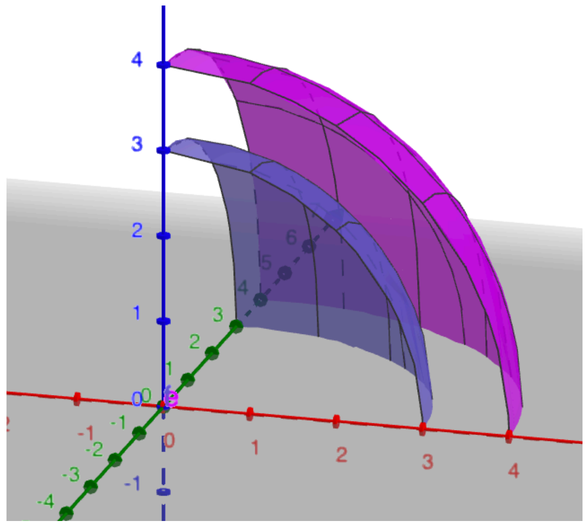
`https://www.geogebra.org/m/h9xS5ZZs`

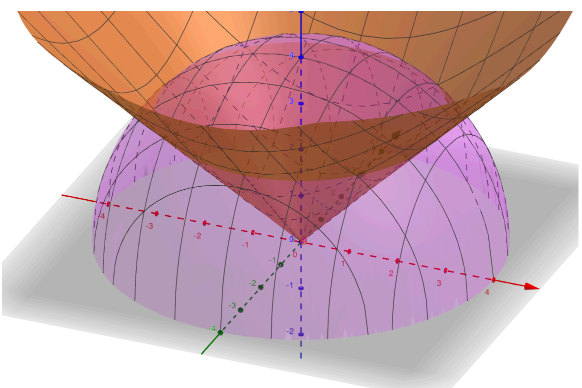
`https://www.geogebra.org/m/bstdbhsz`

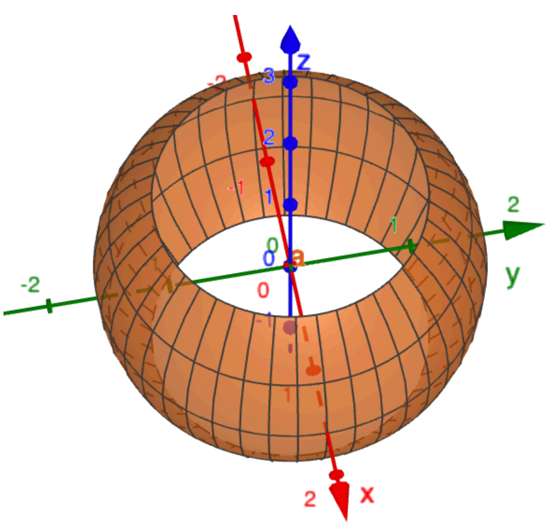
Two Lynx

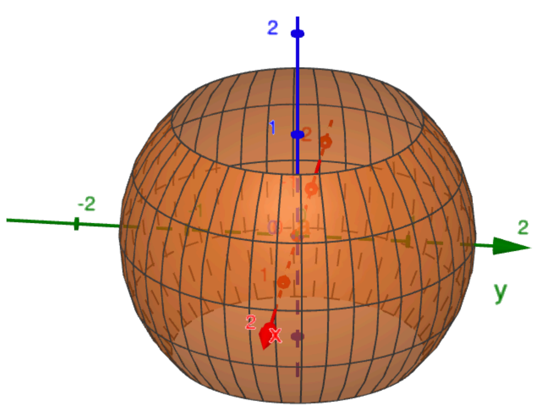
How could we describe the following regions? They are all parts of spheres.









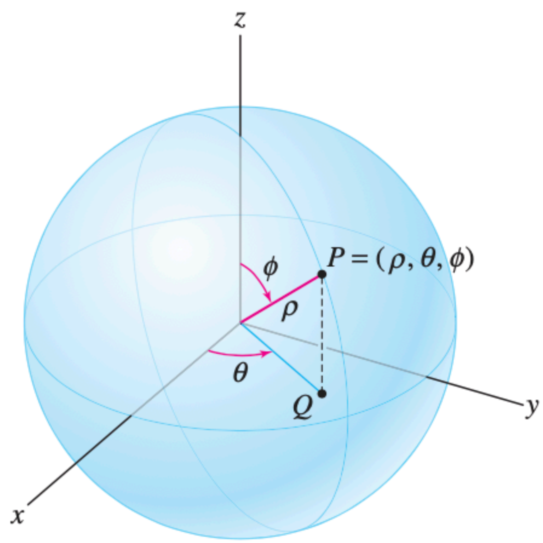



**FIGURE 6** Spherical coordinates $(\rho, \theta, \phi)$.

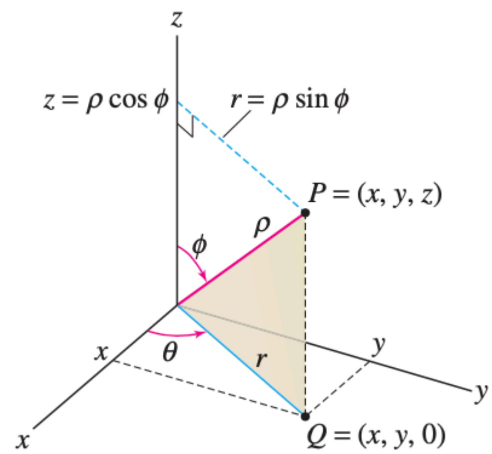


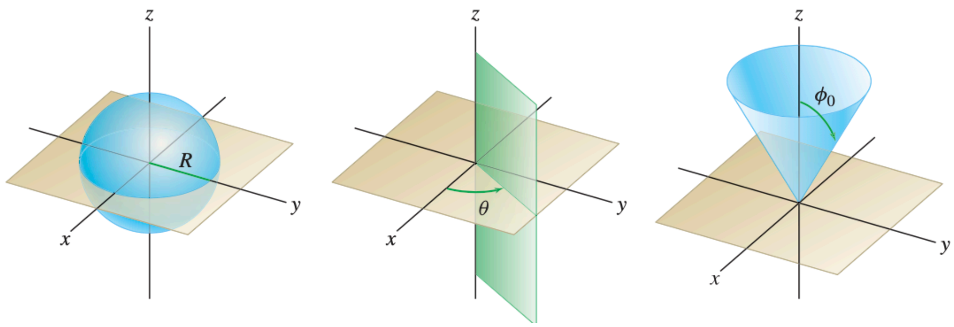

**FIGURE 7**

| Spherical to rectangular | Rectangular to spherical |
|---|---|
| $x = \rho\cos\theta\sin\phi$ | $\rho = \sqrt{x^2 + y^2 + z^2}$ |
| $y = \rho\sin\theta\sin\phi$ | $\tan\theta = \dfrac{y}{x}$ |
| $z = \rho\cos\phi$ | $\cos\phi = \dfrac{z}{\rho}$ |

■ **EXAMPLE 4** **From Spherical to Rectangular Coordinates** Find the rectangular coordinates of $P = (\rho, \theta, \phi) = \left(3, \frac{\pi}{3}, \frac{\pi}{4}\right)$, and find the radial coordinate $r$ of its projection $Q$ onto the $xy$-plane.

■ **EXAMPLE 5** **From Rectangular to Spherical Coordinates** Find the spherical coordinates of the point $P = (x, y, z) = (2, -2\sqrt{3}, 3)$.

**How do we describe these regions with spherical coordinates?**




**Example:** Describe the cone $z = \sqrt{x^2 + y^2}$ in spherical coordinates.

Describe the part of the solid sphere of radius 4 that lies above the cone $z = \sqrt{x^2 + y^2}$.

Describe the solid-hemisphere of radius 3 that is below the $xy$ plane.

Which region is described by $0 \leqslant \rho \leqslant 4, \qquad \pi/2 \leqslant \theta \leqslant \pi, \qquad \pi/2 \leqslant \phi \leqslant \pi$.

This is part of a [solid / hollow] sphere of radius ______________. It is a

Whole sphere | A hemisphere | A quarter sphere | An eighth of a sphere

Lying in:

the region where $x$ is [ $\geqslant 0, \quad \leqslant 0,$ some mixture]

the region where $y$ is [ $\geqslant 0, \quad \leqslant 0,$ some mixture]

the region where $z$ is [ $\geqslant 0, \quad \leqslant 0,$ some mixture]

# 33 Practice Setting up More Integrals

Remember, if we are using polar coordinates, then

$$dA = \underline{\hspace{5cm}}$$

And if we are using cylindrical coordinates (i.e. polar plus a $z$ coordinate), then

$$dV = \underline{\hspace{5cm}}$$

Finally, if we are using spherical coordinates, then

$$dV = \underline{\hspace{5cm}}$$

**Example:**
Set up and evaluate an integral in spherical coordinates that describes the volume of a sphere of radius $a$.

**Example:** The equation $z = \sqrt{3x^2 + 3y^2}$ describes a cone. Let's draw in some lengths and angles.

Imagine we have a sphere of radius 4 centered at the origin. Let $W$ be the region between the sphere and the cone $z = \sqrt{3x^2 + 3y^2}$. Set up an integral of the function $f(x, y, z)$ over the region $W$.

a) **Using Spherical Coordinates**

b) **Using Cylindrical Coordinates**

Let's work with the same region as before, but instead of a cone at the top, let's have a flat plane at $z = 5$. Set up the integral over this 3D region.

a) **Using Spherical Coordinates**

b) **Using Cylindrical Coordinates**

# 34 Catchup Day

# 35 Review Day

# 36 Exam Day

## 37 Finishing up with Integrals

**Warmup:** Which of the following steps are valid? (Be careful - I'm not trying to trick you, but I want you to think about the steps)

$$\begin{aligned}
\int_2^4 \left[\int_3^7 3x^2y^3\ dy\right]\ dx &= \int_2^4 \left[3 \cdot \int_3^7 x^2y^3\ dy\right]\ dx\quad [ok?] \\
&= 3\int_2^4 \left[\int_3^7 x^2y^3 dy\right] dx\quad [ok?] \\
&= 3\int_2^4 \left[x^2 \int_3^7 y^3 dy\right] dx\quad [ok?] \\
&= 3\left[\int_2^4 x^2 dx\right] \cdot \left[\int_3^7 y^3 dy\right]\quad [ok?]
\end{aligned}$$

**Calculation Tool**

If you are integrating an expression that is a ____________________________ of functions and the variables are completely separate, then you can split up the integrals.

**Example:** Quickly evaluate the following:

$$\int_0^{2\pi} \int_0^{\pi} \int_0^3 \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$$

**Much Cooler Example:**

Remember that finding an antiderivative for $e^{-x^2}$ in terms of elementary functions is

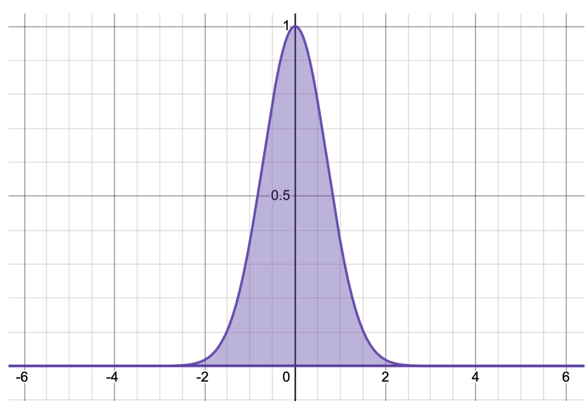
[ hard / extremely hard / something you'll learn in a later course / literally impossible - no one can do it]

However, things like the following still exist and they're just numbers.

$$\int_{-1}^{1} e^{-x^2} dx = 1.49364826562$$

$$\int_{-3}^{3} e^{-x^2} dx = 1.77241469652$$

$$\int_{-10}^{10} e^{-x^2} dx = 1.77245385091$$




**Question:** What do you think these numbers are converging too - in other words, what is

$$\int_{-\infty}^{\infty} e^{-x^2} dx?$$

**The answer might surprise you!**

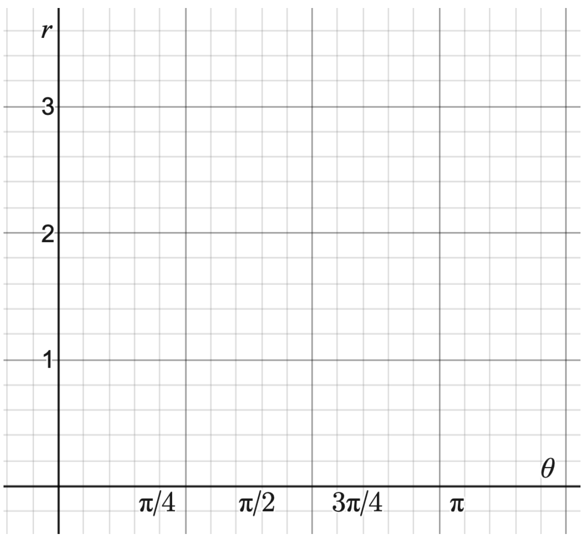
We will start by ______________________________.

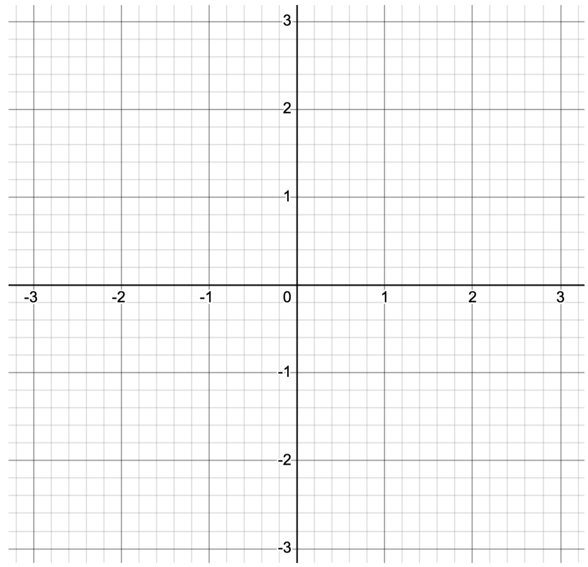
**Change of Coordinates**

Below are two axes. I want you to draw the rectangle $[\pi/4, \pi] \times [1, 3]$ in the plane at the left. Then I want you to draw where this would go under the **transformation** determined by

$$f(r, \theta) = (r\cos\theta, r\sin\theta).$$

(Hint: this is the familiar $x = r\cos\theta$ and $y = r\sin\theta$ you are used to with polar coordinates. Draw your picture carefully.)




This is an example of a **transformation** that takes two inputs ______________ and gives two outputs ______________.

Another word for transformation is **function**. We will write such a function as

$$f : \mathbb{R}^2 \to \mathbb{R}^2.$$

This means we have 2 inputs and two outputs.

**Example with this notation:**

Consider the parametrized curve: $r(t) = (\cos t, \sin t)$. This is an example that takes ______________ input and gives ______________ outputs, which we can think of as a point in the plane. We will write this function as

$$r : \mathbb{R}^1 \to \mathbb{R}^2.$$

This notation where we are identifying inputs and outputs will be helpful later on. It's also very important in computer programming, where you have different "types" of objects.

Summary of this page:

- We are visualizing polar coordinates as a transformation.
- The very simple rectangle on the left turned into a more complicated region on the right.
- We used this **all the time** in setting things up in polar coordinates.
- $\displaystyle\iint_D x + y \, dA = \int_{\pi/4}^{\pi}\int_1^3 (r\cos\theta + r\sin\theta)$______________
- Remember this **conversion factor.**

**The Big Secret your Calculus Teachers Won't Tell You**

$$\boxed{\text{In } xy \text{ coordinates } dA = dxdy.}$$

But what about in $r\theta$ coordinates? (We already know the answer - $dA =$ ______________ But let's pretend we didn't know. And we just had to guess. Let's manifest our dreams and see what happens.

$$x = r\cos\theta, \quad y = r\sin\theta.$$

What do you think $dx$ is?
A) The partial with respect to $r$: $\cos\theta$
B) The partial with respect to $\theta$: $-r\sin\theta$
C) I can't choose. $dx$ should involve both!

What do you think $dy$ is?
A) The partial with respect to $r$: $\sin\theta$
B) The partial with respect to $\theta$: $r\cos\theta$
C) I can't choose. $dy$ should involve both!

You're right that $dx$ and $dy$ have two different parts. So then how on Earth are we going to **multiply them together?**

Well (and of course there's tons of theory here) the ____________________________ is basically how we can multiply things together that have different parts. (Take a linear algebra course to learn more!)

$$dA = dx \cdot dy = \det \left| \qquad\qquad\qquad\qquad \right| drd\theta.$$

It really is that simple!

**Wait really?**

Well actually what's going on is best understood in terms of what are called "differential forms" and something called the "exterior algebra." Basically what we do is we create a new method of multiplication, denoted by $\wedge$ with the usual multiplication properties except for two new rules:

$$a \wedge a = 0 \quad \text{(Equation 1)}$$

$$a \wedge b = -b \wedge a \quad \text{(Equation 2)}$$

These can basically be thought of as saying:

- (Equation 1) "to get an area, you need two different directions. If you multiply the same direction by itself then you'll get no area."
- (Equation 2) "area is supposed to be base times height, but if you switch the order, then it's like you turned the paper upside down, so let's introduce a negative sign.
- (Actually Equation 2 is a consequence of Equation 1: try applying equation 1 with $a = s + t$ and see what you get!)

**Why the abstraction?** I think in many ways knowing that there's a "simpler but more abstract thing out there" can help our understanding.

For example what we did in the purple box above is:

$$dx = \cos\theta dr - r\sin\theta d\theta, \qquad dy = \sin\theta dr - r\cos\theta d\theta$$

And then

$$dx \wedge dy = (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr - r\cos\theta d\theta)$$

## What is true more generally?

If you express $x$ and $y$ in terms of other variables, say $u$ and $v$, then you have two pictures:

We can form the **matrix of partial derivatives** called the ______________________

$$\mathrm{J} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

then (under modest assumptions)

$$\iint\limits_{D} f(x,y)dA = \iint\limits_{D_0} f(\text{sub in } x \text{ and } y \text{ in terms of } uv) \cdot |J| \; du \; dv$$

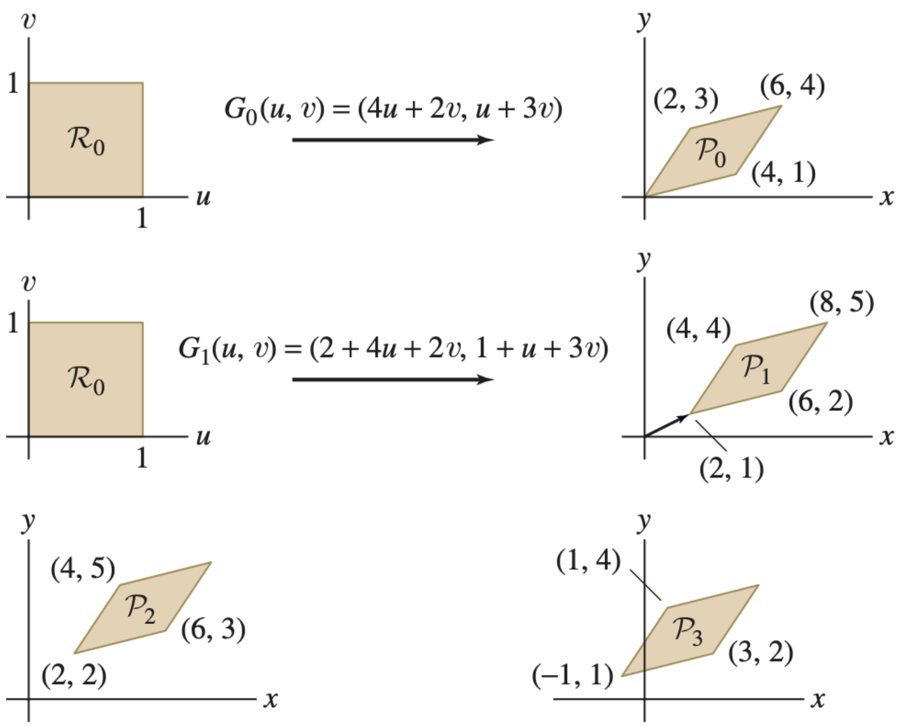



FIGURE 15

**21.** Let $\mathcal{D}$ be the parallelogram in Figure 13. Apply the Change of Variables Formula to the map $G(u, v) = (5u + 3v, u + 4v)$ to evaluate $\iint_{\mathcal{D}} xy\, dx\, dy$ as an integral over $\mathcal{D}_0 = [0, 1] \times [0, 1]$.

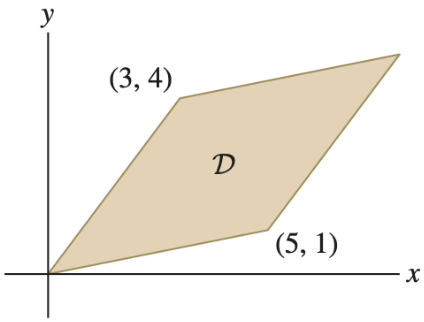
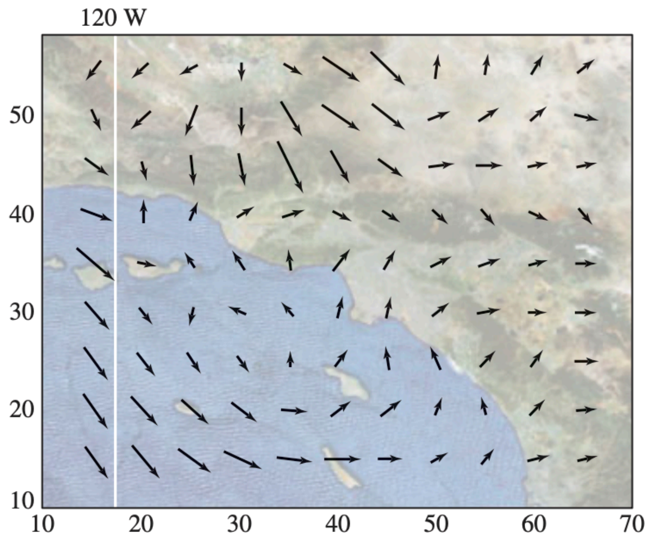



FIGURE 13

[1] Ans: 194.08

# Vector Fields

A **vector field** is a function that assigns to each point in $\mathbb{R}^n$ a ____________________________. For instance, in the picture below we have assigned each point a vector indicated the velocity of wind at that point.




Where does the wind seem the strongest?

The weakest?

How do you know?

If $\mathbf{F}(x, y)$ is this vector field, estimate

$\mathbf{F}(40, 50) =$

$\mathbf{F}(40, 15) =$

Be careful - is your answer a vector or a scalar? What is the scale in this picture?

**EXAMPLE 1** Which vector is attached to the point $P = (2, 4, 2)$ by the vector field $\mathbf{F} = \langle y - z, x, z - \sqrt{y} \rangle$?

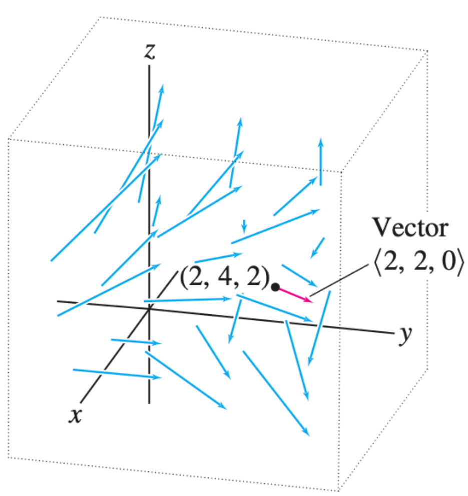



**FIGURE 3**

**Example:** Describe the vector field given by $\mathbf{F} = \mathbf{i} + x\mathbf{j}$

**Example:** Describe the Vector Field given by $\mathbf{F} = \langle -y, x \rangle$

**Example:** Describe the **unit vector field F** with the property that at each point, the vector is pointing radially away from the origin. This is called the **radial** vector field and is denoted $\mathbf{e}_r$.

**Conservative Vector Fields**

One of our best friends from the course has been the __________________________.

If we had a function $f(x, y)$, say $f(x, y) = x^2 + 3xy$ then we would have

$$\nabla f =$$

Let's fill in the following. In this example, $f$ has _____________ inputs and gives _____________ outputs.

The gradient $\nabla f$ takes _____________ inputs and gives _____________ outputs.

So $\nabla f$ is a _______________________________________ on _____________.

**The gradient of a function is ALWAYS a vector field.**

**But NOT every vector field comes is a gradient.**

**Definition.**
We say that a vector field $\mathbf{F}$ is ____________________________ if

and the function $V$ is called the ____________________________ of $\mathbf{F}$.

■ **EXAMPLE 3** Verify that $V(x, y, z) = xy + yz^2$ is a potential function for the vector field $\mathbf{F} = \langle y, x + z^2, 2yz \rangle$.

**THEOREM 1 Cross-Partial Property of a Conservative Vector Field** If the vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is conservative, then

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \qquad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$$

**Why is this true?**

Suppose that $\mathbf{F}$ is a **conservative** vector field.

Then this means that $\mathbf{F}$ has a ____________________________.

**Example:** Show that the vector field $\mathbf{F} = \langle y, x^2 \rangle$ is NOT conservative.

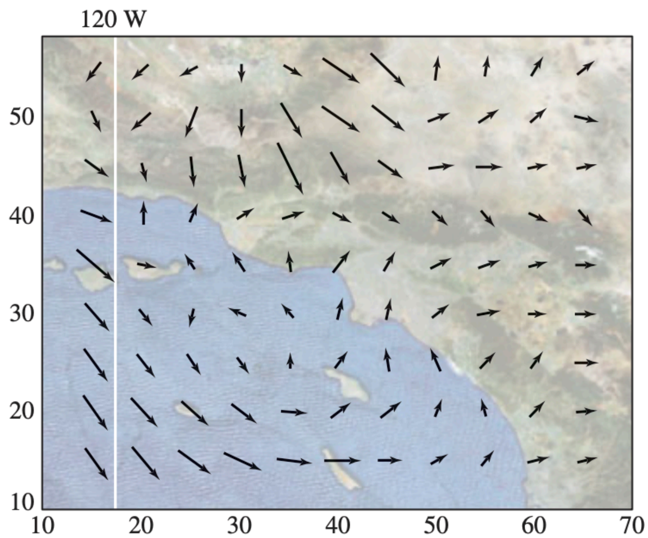
**Example:** Determine if the vector field $\mathbf{F} = \langle 3x^2y + z^2 + \cos x, x^3, 2xz \rangle$ is conservative. If your answer is YES, find a potential function.

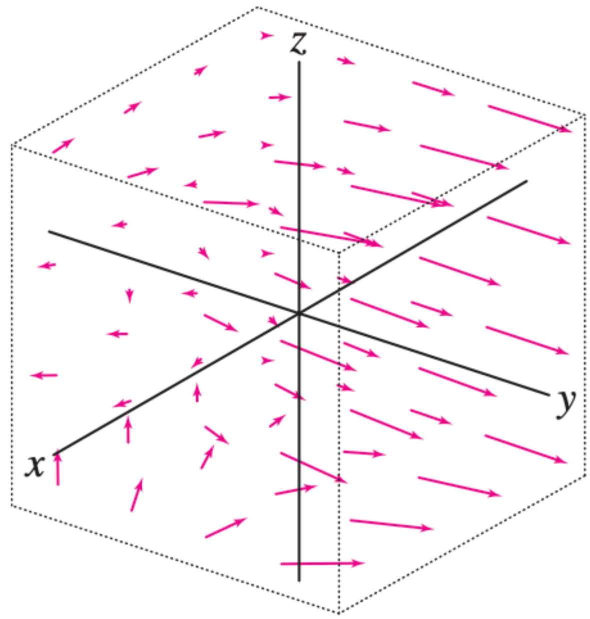
**Theorem**
A conservative vector field will have ______________________________ potential functions,

but they all differ by a ______________________.

## 39 What can we do with vector fields?

We will spend a good 3-4 weeks doing lots of awesome stuff with vector fields. For today let's just think about some of the big picture ideas that we'll do.




- What are some real-world questions you could ask about these vector fields?
- If $C$ is a **curve**, what are some things we could ask about the relationship between the vector field $\mathbf{F}$ and $C$?
- What about if you had a **surface**, like a balloon or John B's and Antoine the Ant's sea creature?

Our main focus will be on ________________________ and for problems about ________________________

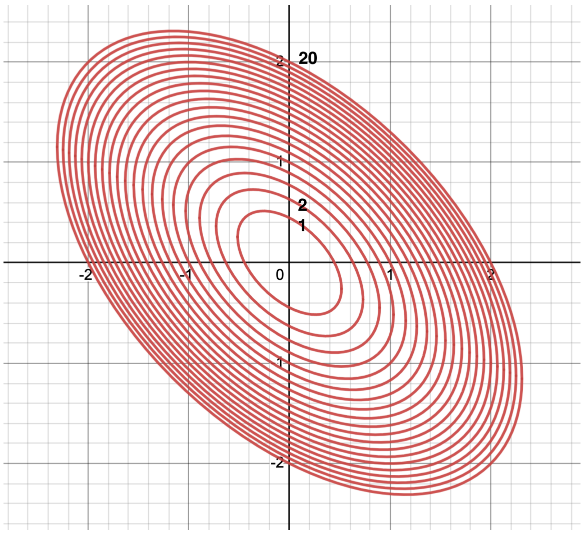
we will write things like:

These are measuring properties about the **forces** as we move along a curve, or as it passes through a surface.

**At the same time**

We will also think about what are called **scalar fields** which are something we've been used to for awhile.

For example the picture below does NOT show a vector field, but a scalar field. You can think of this as the height of a mountain or maybe the temperature at a point in the plane. Or maybe the density of oxygen.




What are some things we could study about a **scalar fields** $f(x, y)$?

- What are some real-world questions you could ask about these scalar fields?
- If $C$ is a **curve**, what are some things we could ask about the relationship between the scalar field $f(x, y)$ and $C$?
- What about if you had say a function measuring oil density $f(x, y, z)$ and a **surface**, like a balloon or John B's and Antoine the Ant's sea creature?

# 40 Line Integrals - Yes another type of integral

**Warmup:**

A vector field is called **conservative** if it the gradient of a function. That function is called a **potential.**

The vector field $\mathbf{F} = \langle 3x^2y + z^2 + \cos x, x^3, 2xz \rangle$ is conservative. Find a potential function.

**NOT every** vector field is conservative. In fact, if you just wrote down "random" functions, it probably would not be conservative. Let's see why.

Let's see what has to be true for a **conservative** vector field $\mathbf{F}$.

Let's say that $\mathbf{F} = \langle F_1, F_2 \rangle$. (these $F_i$ represent the first and second components).

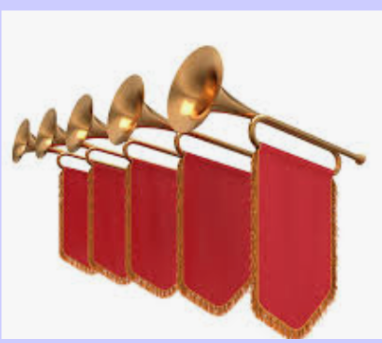
Now if $\mathbf{F}$ were a gradient, then there would be a potential function $V$.

So that means $\langle F_1, F_2 \rangle = \nabla V$.

So $F_1 =$ ____________ and $F_2 =$ ____________. (Fill in the blanks in terms of the function $V$.)

But then

$$\frac{\partial F_1}{\partial y} = \underline{\hspace{3cm}} \qquad \text{and} \qquad \frac{\partial F_2}{\partial x} = \underline{\hspace{3cm}}$$



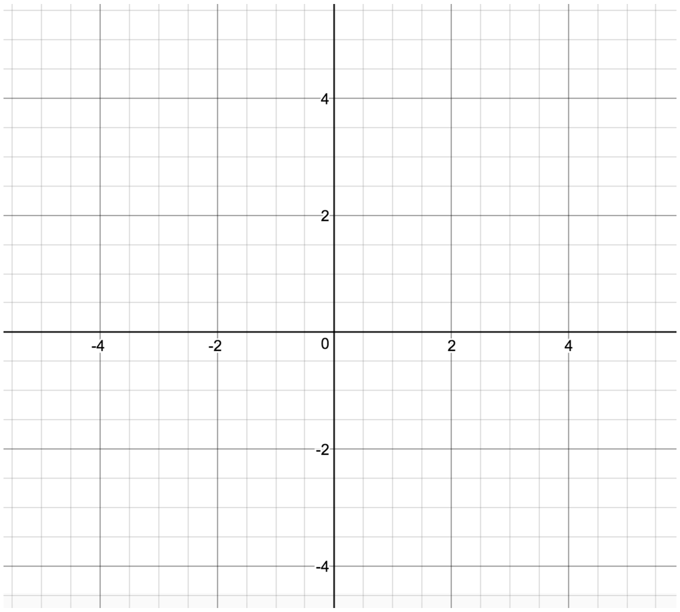
Sound the fanfare!
If $\langle F_1, F_2 \rangle$ is a conservative vector field, then it MUST be the case that

This gives us a very specific test to see if a vector field is conservative.

**Note:** Under mild assumptions, this test is actually all we need to tell if a vector field is conservative.

**Example:** Use this test to prove that $\langle x^2 + y^2, 3yx \rangle$ is not a conservative vector field.

You can use this test for vector fields with 3 components, you just have to check each pair of coordinates. See the boxed theorem from Day 28 for the detailed statement.




John B. wants to walk along a straight line from the point $(-1, 1)$ to the point $(2, 4)$. He notices that along the path there is gold scattered on the ground with density given by $f(x, y) = 2x^2 + y^2$ grams per foot.

How much gold did John B. just walk over?

**Warmup:** Using the $P + t\mathbf{v}$ setup, parametrize the line segment.
Make sure you include your parameters for $t$.

$\mathbf{c}(t) =$

To find this amount of gold we are going to integrate the [scalar field / vector field ] $f(x, y) = 2x^2 + y^2$ over the __________________________ that we parametrized by $\mathbf{c}(\mathbf{t})$. We will write this:

| What if the gold had constant density, say 1 gram per foot. How much gold would John B. walk past? |
|---|

**Answer:**

**Definition of the line integral of a scalar field**:

If $C$ is the curve parametrized by $\mathbf{c}(t)$ for $a \leqslant t \leqslant b$. And $f(x, y)$ is a scalar function. (Like density) Then

$$\int_C f(x, y)ds =$$

**Example:** How much gold did John B. walk past?

**Wait a minute...**

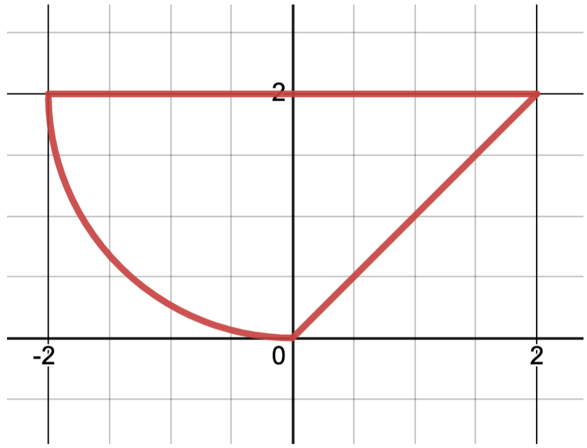
Doesn't our answer depend on how we parametrized the curve? Let's write down a different parametrization of the same line. When we do this we will get

[the same exact integral / a different integral] and [a different amount of gold / the same amount of gold].

I don't like parametrizing curves - is there a way to do this without parametrizing?

Not in general. It's very important you can parametrize the following quickly:

- Line segments - use $P + t\mathbf{v}$
- Parts of circles
- Graphs of functions, like $y = e^x - x^2$.
- Combinations of these. E.g. what if John B. walks along the bound of this shape:




If we were to walk along this curve starting at $(0, 0)$ and walking counterclockwise, we have three different paths:

Part 1: A straight line:

Part 2: A straight line:

Part 3: A part of a circle of radius ______________

where the angle $t$ goes is between ____________________________

and where the center has been shifted from the origin to ______________.

If we wanted to calculate the line integral over the whole curve, we would split this problem into three different parts.

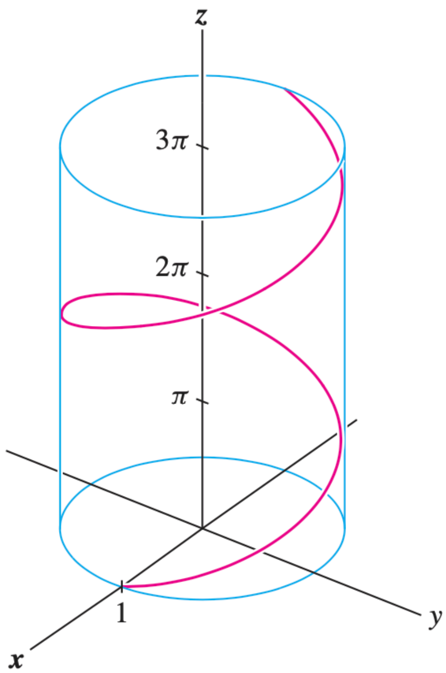



**FIGURE 3** The helix $\mathbf{c}(t) = (\cos t, \sin t, t)$.

■ **EXAMPLE 1** **Integrating along a Helix** Calculate

$$\int_{\mathcal{C}} (x + y + z)\, ds$$

where $\mathcal{C}$ is the helix $\mathbf{c}(t) = (\cos t, \sin t, t)$ for $0 \le t \le \pi$ (Figure 3).

That's it: to set up an integral of a **scalar field** you need to

- parametrize the curve
- substitute your parametrization into the function (think of this as density)
- use the fact that $ds = ||\mathbf{c}'(t)||dt$.

# 41 Line Integrals of Vector Fields

**Review:**

Identify the following parts of the following expression:

$$\int_C f(x,y)\ ds = \int_a^b f(\mathbf{c}(t))||\mathbf{c}'(t)||\ dt.$$

The $C$ represents a __________________________

The function $f(x,y)$ is a [scalar field / vector field]. We can tell this because [it is/ it is not] bold-faced.

A second reason we can tell this is because of the $ds$ which is also not bold-faced.

When calculating this, we have to __________________________ $C$ in terms of $t$.

That's what $\mathbf{c}(t)$ is. This is a [scalar-valued / vector valued] function.

$||\mathbf{c}'(t)||$ measures the __________________________ of the _____________ vector $\mathbf{c}'(t)$.

Yo doc,

I can't write bold-face on my paper. How would I notate a bold-face **vector-valued function** like $\mathbf{c}(t)$ or a vector field like $\mathbf{F}$?

Sincerely,

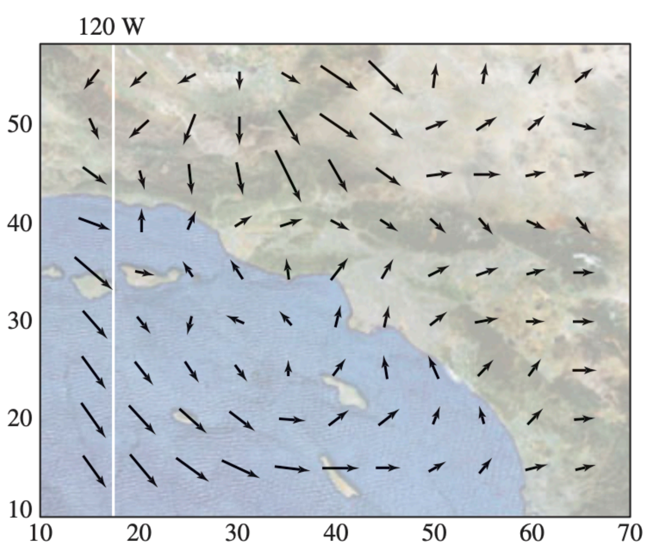
A concerned student

**Answer:**

Now we are going to learn about **another** type of **line integral**.

$$\int_C \mathbf{F}\cdot \mathbf{ds} = \qquad\qquad\qquad\qquad .$$

We will fill in the right hand side in a moment with a formula, but for now, what do you notice is different? What is the same?



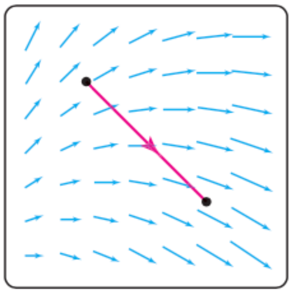

The **dot product** is capturing the

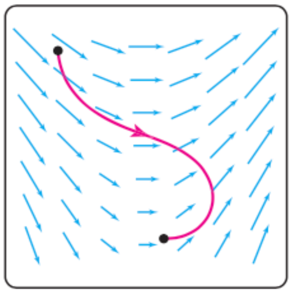
If you think of $\mathbf{F}$ as measuring a **force** then this integral measures

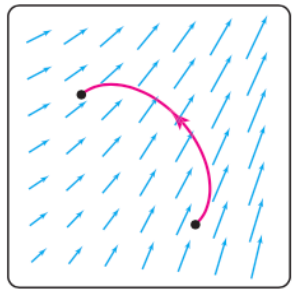
**Examples:** Would the **Force field F** do positive or negative work on an object moving along the curve?

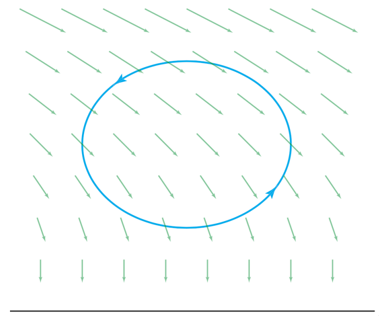
To actually find the integral $\int_C \mathbf{F} \cdot \mathbf{ds}$ we need to:

- Parametrize the curve $C$. This might be given to you, but you might need to do it yourself.







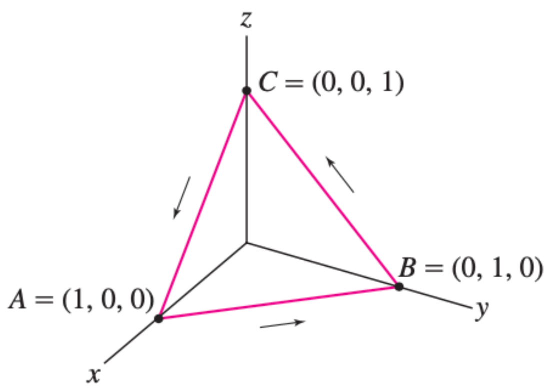



■ **EXAMPLE 8** **Calculating Work** Calculate the work performed moving a particle from $P = (0, 0, 0)$ to $Q = (4, 8, 1)$ along the path

$$\mathbf{c}(t) = (t^2, t^3, t) \text{ (in meters)} \qquad \text{for } 1 \le t \le 2$$

in the presence of a force field $\mathbf{F} = \langle x^2, -z, -yz^{-1} \rangle$ in newtons.

**EXAMPLE 7** Compute $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F} = \langle e^z, e^y, x + y \rangle$ and $\mathcal{C}$ is the triangle joining $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ oriented in the counterclockwise direction when viewed from above (Figure 10).



## 42 Our First "Fundamental Theorem"

**Pro-tip:**

Sometimes a line integral of a **vector field** will be written of the form:

$$\int_C F_1\ dx + F_2\ dy$$

Which is $\int_C \langle F_1, F_2\rangle \cdot \mathbf{ds}$. The $dx$ and $dy$ juts represent the derivatives of the $x$ and $y$ coordinates of $\mathbf{c}(t)$.

**Example A:** Calculate the line integral $\int_C (2x+y)\ dx + x\ dy$ where $C$ is the oriented line from $(1,2)$ to $(5,7)$.

---

Here's a Calculus 1 question:

Suppose that $f(t)$ measures the temperature of the north pole at time $t$. You collect the following data:

$$f(0) = 3, \quad f(1) = 1, \quad f(2) = 3, \quad f(3) = 7, \quad f(4) = 6$$

Use this information to find the following:

$$\int_1^4 f'(t)\ dt =$$

The ______________________________ says that integrals and derivatives are related. Really it is saying:

If you integrate the ____________________ of a function $f(x)$, then you are essentially adding up all the small __________ and in total you get the ____________________.

$$\int_a^b f'(x)dx = f(b) - f(a).$$

**Question:**

How many different types of derivatives have we seen in this class?

How many different types of integrals? (And yes there are more types coming!)

Do you think there is a Fundamental Theorem relating some of these 2D and 3D derivatives and integrals?

**Do you remember** the Chain Rule for Paths:

Suppose that $f(x, y)$ is measuring the temperature of a point in the plane and $\mathbf{c}(t)$ is a parametrized curve that John B. is walking along. Then

$\mathbf{c}(t)$ gives John B's ______________________ at ______________________.

$f(\mathbf{c}(t))$ gives John B's ______________________ at ______________________.

$\frac{d}{dt}f(\mathbf{c}(t))$ gives the ____________________ of John B's ______________________ as a function of t.

And the formula we learned was that:

$$\frac{d}{dt}f(\mathbf{c}(t)) =$$

Does that right hand side look like anything we've done recently?

**First Fundamental Theorem for Line Integrals:**

If we want to calculate $\int_C \mathbf{F} \cdot \mathbf{ds}$ and $\mathbf{F}$ is a ______________________ vector field with potential $V$, then

$$\int_C \mathbf{F} \cdot \mathbf{ds} =$$

Why is this true?

**Who cares?** This is saying something pretty phenomenal. It is saying that if $\mathbf{F}$ is conservative then if you want to find $\int_C \mathbf{F} \cdot \mathbf{ds}$ then all that matters are the __________________________________ of the curve.

In other words, the actual path you take doesn't matter.

Wait, but what if the start and endpoint were the same? Well in that case Under the hypothesis that $\mathbf{F}$ is then

$$\oint_C \mathbf{F} \cdot \mathbf{ds} =$$

**Example A revisited:** Let's re-do Example A above but use the first fundamental theorem. Which way was easier?

■ **EXAMPLE 3** **Integral around a Closed Path** Let $V(x, y, z) = xy\sin(yz)$. Evaluate $\oint_{\mathcal{C}} \nabla V \cdot d\mathbf{s}$, where $\mathcal{C}$ is the closed curve in Figure 5.

**2.** Which of the following statements are true for all vector fields, and which are true only for conservative vector fields?

**(a)** The line integral along a path from $P$ to $Q$ does not depend on which path is chosen.
**(b)** The line integral over an oriented curve $C$ does not depend on how $C$ is parametrized.
**(c)** The line integral around a closed curve is zero.
**(d)** The line integral changes sign if the orientation is reversed.
**(e)** The line integral is equal to the difference of a potential function at the two endpoints.
**(f)** The line integral is equal to the integral of the tangential component along the curve.
**(g)** The cross-partials of the components are equal.

**(a)** If $\mathbf{F}$ has a potential function, then $\mathbf{F}$ is conservative.
**(b)** If $\mathbf{F}$ is conservative, then the cross-partials of $\mathbf{F}$ are equal.
**(c)** If the cross-partials of $\mathbf{F}$ are equal, then $\mathbf{F}$ is conservative.

**Example:** Find the line integral

$$\int_C \cos y \; dx \; - x \sin y \; dy$$

over the upper half of the unit circle centered at the origin, oriented counterclockwise.

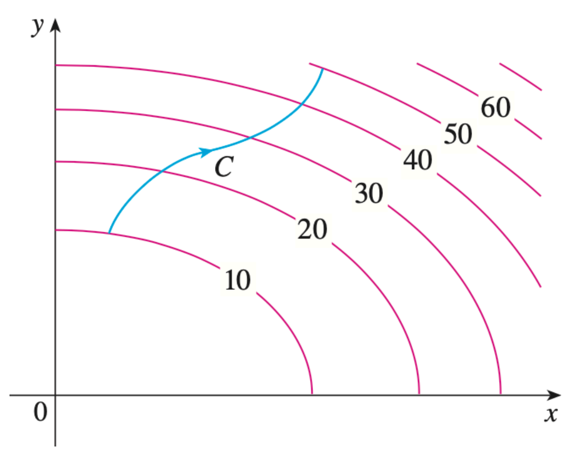
- How would you set up this line integral explicitly. Write it down carefully. It will be a very complicated integral. Draw a picture.
- Does the "First Fundamental Theorem" apply? Why or why not?
- Describe in words two different ways you could evaluate this integral.

## 43 Practice with Line Integrals

Notation:

1. How would you parametrize the upper half of the circle of radius 2 going "clockwise" instead of counter-clockwise?

2. The figure below shows a curve $C$ and a contour map of a function $f$ whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{s}$.



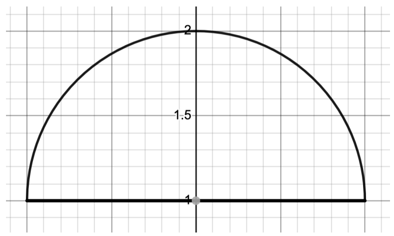

3. Remember: it is essential to be able to parametrize. Parametrize the curve $C$ that is the part of the parabola $y = 2x^2$ going from $(-1, 2)$ to $(2, 8)$. Use that to set up the integral of $\int_C x^2\ dx + y^2\ dy$ and calculate it fully.

Now redo this problem, using the Fundamental Theorem for Gradients.

4. Is the vector field $\mathbf{F} = \langle y, -x \rangle$ conservative? [Yes / No]

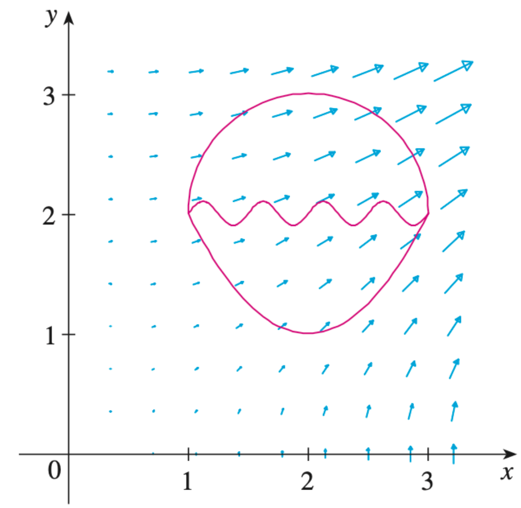
How do you know?

Would you be able to use the Fundamental Theorem for Gradients to evaluate this line integral: $\int_C \mathbf{F} \cdot \mathbf{ds}$ where $C$ is the curve below oriented counterclockwise.




Work with your team to calculate this line integral.

5. The figure shows the vector field $\mathbf{F} = \langle 2xy, x^2 \rangle$ and three curves that start at $(1, 2)$ and end at $(3, 2)$.




Explain why $\int_C \mathbf{F} \cdot \mathbf{ds}$ has the same value for all three curves.

What is that common value?

# 44 What if the vector field isn't conservative? (Green's Theorem)

## Warmup:

**2.** Which of the following statements are true for all vector fields, and which are true only for conservative vector fields?

**(a)** The line integral along a path from $P$ to $Q$ does not depend on which path is chosen.

**(b)** The line integral over an oriented curve $\mathcal{C}$ does not depend on how $\mathcal{C}$ is parametrized.

**(c)** The line integral around a closed curve is zero.

**(d)** The line integral changes sign if the orientation is reversed.

**(e)** The line integral is equal to the difference of a potential function at the two endpoints.

**(f)** The line integral is equal to the integral of the tangential component along the curve.

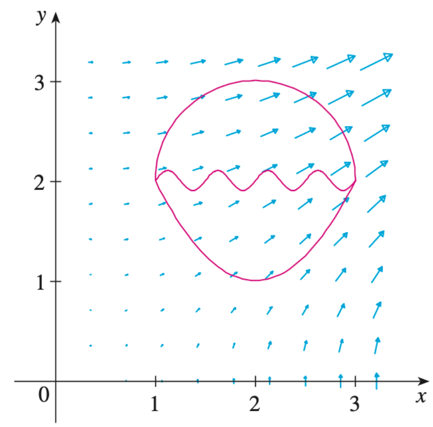
**(g)** The cross-partials of the components are equal.

**(a)** If $\mathbf{F}$ has a potential function, then $\mathbf{F}$ is conservative.

**(b)** If $\mathbf{F}$ is conservative, then the cross-partials of $\mathbf{F}$ are equal.

**(c)** If the cross-partials of $\mathbf{F}$ are equal, then $\mathbf{F}$ is conservative.

The figure shows the vector field $\mathbf{F} = \langle 2xy, x^2 \rangle$ and three curves that start at $(1, 2)$ and end at $(3, 2)$.




Explain why $\displaystyle\int_C \mathbf{F} \cdot \mathbf{ds}$ has the same value for all three curves.

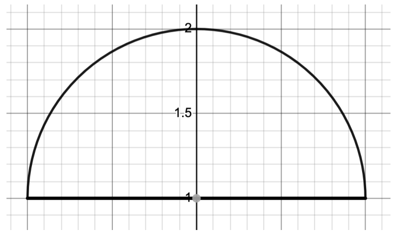
What is that common value?

## We're going to draw a flowchart:

**Example:** Is the vector field $\mathbf{F} = \langle y, -4x \rangle$ conservative? [Yes / No]

How do you know?

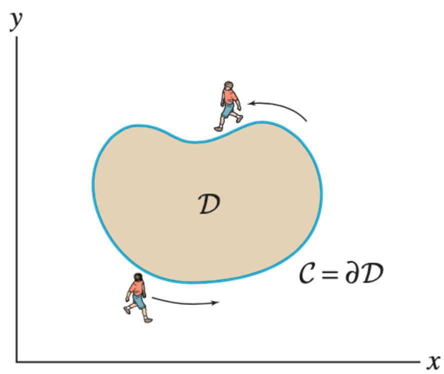
Would you be able to use the Fundamental Theorem for Gradients to evaluate this line integral: $\int_C \mathbf{F} \cdot \mathbf{ds}$ where $C$ is the curve below oriented counterclockwise.




Work with your team to calculate this line integral. **This is one of the rare cases where I want you to calculate it fully to the end. I might ask you to do something like this on the quiz tomorrow or on the Final Exam.** Be methodical and set up the integral along each piece.

We are about to state a Theorem that can help us find the line integral over a ____________ closed curve.

This means that the curve does not ____________________________________.




When applying this theorem we will talk about a "simple region" and its "boundary".

Since the direction we walk about a curve matters for line integrals of vector fields, we will ALWAYS choose the orientation where the region is on the if a person were walking around $D$.

**Are there any other assumptions?** We need to assume that our partial derivative appearing are all continuous functions.

**(Green's Theorem)**

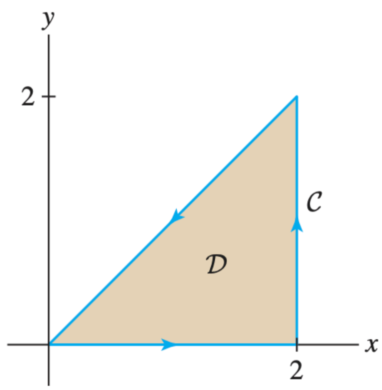
Let $D$ be a ________________________ and suppose $C$ is its ________________________.

Let $\mathbf{F}$ be a ________________________. If we orient $C$ so that $D$ is always on the ____________ then

This only applies when the partial derivatives are continuous functions. (So almost all the of the time in our class)

**Example:** Let's see what Green's Thoerem says about the previous example

■ **EXAMPLE 2** **Computing a Line Integral Using Green's Theorem** Compute the circulation of $\mathbf{F} = \left\langle \sin x, x^2y^3 \right\rangle$ around the triangular path $\mathcal{C}$ in Figure 5.




**Example:** Let $I = \oint_C \mathbf{F} \cdot \mathbf{ds}$ where $\mathbf{F} = \langle y + \sin(x^2), x^2 + e^{y^2} \rangle$ and $C$ is the circle of radius 4 centered at the origin.

Could we parametrize this curve? Would that lead to an integral we'd be able to calculate by hand?

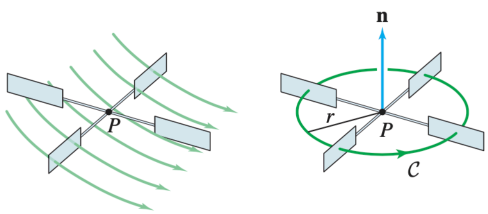
Would it be easier to use Green's Theorem?

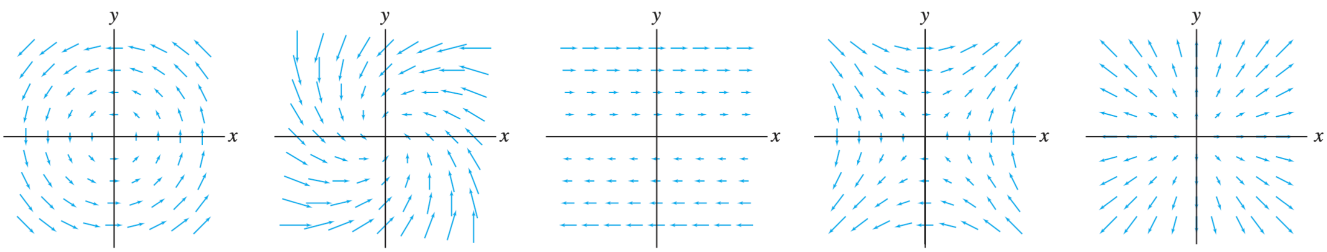
**Remark:** When we work with a vector field over a closed curve $C$, we sometimes call the line integral over $C$ the of $\mathbf{F}$ along $C$.

Green's Theorem is actually saying that this **line integral** which is only measuring stuff ____________________ somehow can be calculated by measuring this weird thing

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

**What does this measure?**




(A) $\mathbf{F} = \langle -y, x\rangle$ $\mathrm{curl}_z(\mathbf{F}) = 2$

(B) $\mathbf{F} = \langle -x-y, x-y\rangle$ $\mathrm{curl}_z(\mathbf{F}) = 2$

(C) $\mathbf{F} = \langle y, 0\rangle$ $\mathrm{curl}_z(\mathbf{F}) = -1$

(D) $\mathbf{F} = \langle y, x\rangle$ $\mathrm{curl}_z(\mathbf{F}) = 0$

(E) $\mathbf{F} = \langle x, y\rangle$ $\mathrm{curl}_z(\mathbf{F}) = 0$

FIGURE 10

Cool Idea: What would Green's Theorem say if we integrated the vector field $\langle -y, x\rangle$ over the boundary of a region like this:

Spooky Szn:

This is saying that somehow, just measuring the vector field along the boundary, the line integral "knows" the area of the whole region.

# 45 More Practice with Line Integrals

**12.** Compute the line integral of $\mathbf{F} = \langle x^3, 4x \rangle$ along the path from $A$ to $B$ in Figure 18. To save work, use Green's Theorem to relate this line integral to the line integral along the vertical path from $B$ to $A$.

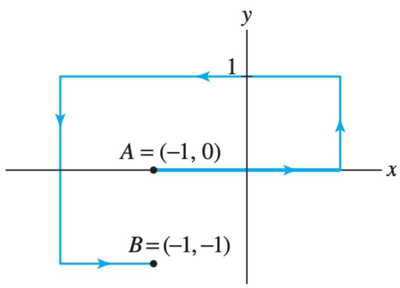



FIGURE 18

## 46 We worked on HW in class

# 47 What does it mean to parametrize a surface?

A **curve** is a ____________ dimensional object.

We could travel along a curve by parametrizing it in terms of ____________ parameter.

We think of this parameter sometimes as representing ____________.

Here are some pictures of curves:

A **surface** is a ____________ dimensional object.

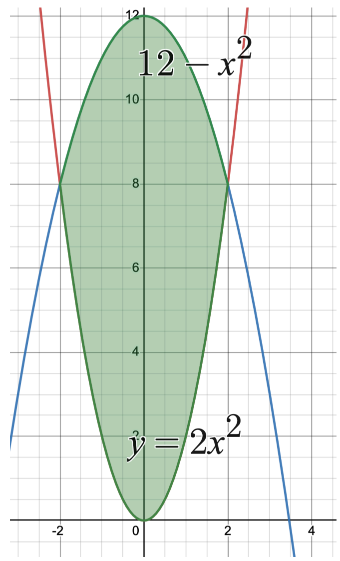
We could travel along a surface by parametrizing it in terms of ____________ parameters.

The parameters can be thought of as the "two types of directions".

Here are some pictures of surfaces:

We want to be able to **describe** a surface with equations.

Let's start with parts of the $xy$ plane. How could you describe the surface below:




Notice that it was very natural to use the parameters:

If we were to think about this as a subset of $\mathbb{R}^3$, what would the $z$ coordinate be?

**Example:** How could we parametrize the part of the surface $z = 9 - x^2 - y^2$ that lies above the $xy$ plane?

This example was an example of parametrizing the ________________________________.

To do this it is easiest to:

- Find the shadow (usually in $xy$ plane, but others are possible) and describe that shadow.
- Then use the function (usually $z = f(x, y)$) to get the $z$-coordinate.
- Your parametrization will be
$$G(x, y) = (x, y, f(x, y))$$
- But remember you need to give a description of what $x, y$ are in terms of the shadow.

**Example:** How would you parametrize the cylinder $x^2 + y^2 = 16$?

**Example:** How would you parametrize the part of the sphere of radius 5 that lies above the cone $z = \sqrt{x^2 + y^2}$.

**Example:** How would you parametrize the plane through the point $(1, 2, 3)$ with normal vector $\langle -1, 1, 4 \rangle$?

**Example:** How do you parametrize the cone $z^2 = x^2 + y^2$ that lies above (or below) the disk $x^2 + y^2 \leqslant 4$ in the $xy$ plane?

**What will we do with surfaces?**

A surface will generally have a whole ____________ of tangent vectors. Here is a quick way to find two different tangent vectors:

**Example:** Consider the parametrized surface $G(\theta, z) = (2\cos\theta, 2\sin\theta, z)$.

- What is this surface? Draw a picture
- Find a tangent vector that reflects adjusting the parameter $\theta$

  $\mathbf{T}_\theta =$

- Find a tangent vector that reflects adjusting the parameter $z$

  $\mathbf{T}_z =$

  To find these tangent vectors you just take the derivative in each component.

- Find a normal vector to the surface at a point $(\theta, z)$:

  $\mathbf{n}(\theta, z)$:

  The normal vector should be ________________________ to the tangent vectors. That's

  why we will use the ____________________________________.

Find the equation of the tangent plane to this surface at the point where $\theta = \pi/4$ and $z = 5$.

**Practice:** Find the tangent vectors and normal vector for the surface of a sphere using spherical coordinates:

**What does it all mean?**

You might remember ....

This is essentially saying that on our surface, that the ________________________ of the normal vector is helping measure the ____________ of that patch of the surface.

Putting it all together:

- If $S$ is a parametrized surface
- with parameters $uv$ ranging over some domain $D$ in the parameter domain
- and $\mathbf{n}(u, v)$ is the normal vector at a point $G(u, v)$

then the following integral measures the ________________________ of $S$:

**Surface Integrals of Scalar Valued Functions**

If $S$ is a surface, and $f(x, y, z)$ is a **scalar field** then we can define the integral:

$$\iint\limits_S f(x, y, z)dS =$$

If $f(x, y, z)$ is measuring the density of something (the density of the cloth of a sail on a sailboat, of density of gold on a metal plate, etc) in $g/m^2$
then this integral is measuring ____________.

Notice how this is similar to the what we saw for **line integrals**.

**Example:** Find the total charge (in coulombs) on a sphere of radius 5 whose charge density in spherical coordinates is $f(\theta, \phi) = .003 \cos^2(\phi)$ C/cm$^2$.

**Some useful formulas**

It will be useful to have formulas for $||\mathbf{n}||$ when setting up surface integrals. The main surfaces we'll need are:

- Planes $ax + by + cz = d$.

- Graphs of functions: $z = g(x, y)$

- Spherical Coordinates (remember $R$ is a constant now)

**Example:** Evaluate $\iint\limits_S z dS$ for the surface $S$ whose sides $S_1$ are given by the cylinder $x^2 + y^2 = 1$, whose bottom side $S_2$ is the circle $x^2 + y^2 \leqslant 1$ in the $xy$ plane $z = 0$ and whose top is the part of the plane $z = 1 + x$ that lies above $S_2$.

## 48 Continuing With Previous Packet

## 49 The surface integral of a Vector Field

The only remaining definition we need this week is how to calculate the **surface integral** of a **vector field**. Luckily, the definition is very simple.

Suppose that $G(u, v)$ is an __________________________ parametrization of a surface $S$.

- (This means that we are told which way the normal vector points, and that works out with our formulas.

Suppose that $\mathbf{F}$ is a vector field

- (think of it as representing the velocity of water flowing through a river)

Then there are two formulas you can use to find the surfrace integral of $F$ over $S$:

$$\iint_S \mathbf{F} \cdot \mathbf{dS} =$$

$$=$$

Note! The first one involves a unit normal vector, $\mathbf{e_n}$. The second involve the $\mathbf{n}$ you get from $\mathbf{T_u} \times \mathbf{T_v}$

**What does it measure?:**
This is surface integral is called the **flux** of $\mathbf{F}$ through $S$. You can think of this as measuring the flow rate of a fluid through the surface.

**Example:** Suppose that $S$ is a rectangle in the $yz$ plane of area 8, and suppose that the normal vector of $S$ points in the positive $x$ direction. Suppose that $\mathbf{F} = \langle 3, 4, 5 \rangle$ is a constant vector field. What is the flux of $\mathbf{F}$ through the surface $S$?

**Would it be easier to use the first formula or the second one? Why?**

**Example:** Compute the flux of $\mathbf{F} = (x+z)\mathbf{i} + 3\mathbf{j} + z\mathbf{k}$ through the surface $S$ given by $y = x^2 + z^2$ with $0 \leqslant y \leqslant 25$ $x \geqslant 0, z \geqslant 0$ oriented toward the $xz$ plane.

**Steps:** First sketch a picture: Then find a parametrization

Calculate $\mathbf{n}$ and double check that your setup is correct. If not, just change the sign of your final answer.

Set up your integral as a double integral of your **domain** that you used in your parametrization.

# 50 Some Practice Problems

1. How would we parametrize the surface $S$ which is the part of graph of the function $z = 2 - 4x + 2y$ over the region where $x, y, z \geqslant 0$. Assume the normal vector is pointed upwards.

   $G(\qquad) = (\qquad , \qquad , \qquad).$

   What is the picture of your parameter domain $D$?

   Now find $\mathbf{n}$:

   Now use this to find the surface area of $S$:

   Find the flux of the vector field $\mathbf{F} = z\mathbf{i} - x\mathbf{k}$ through the surface $S$.

2. How can you find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 16$ that is contained inside of the paraboloid $z = 2x^2 + 2y^2 + 4$. Draw a careful picture.

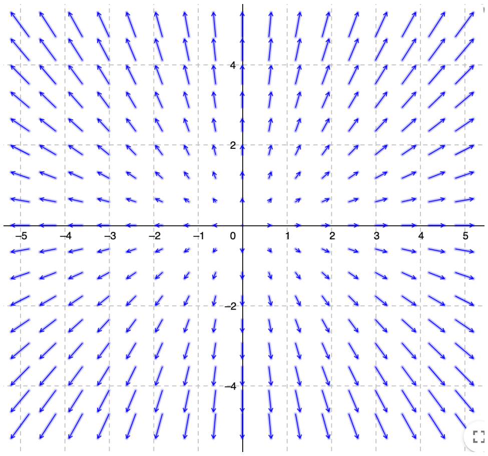
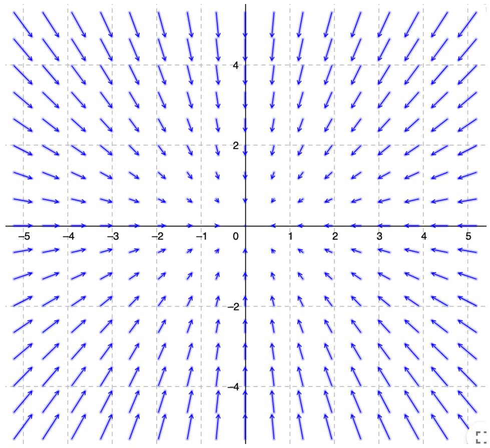
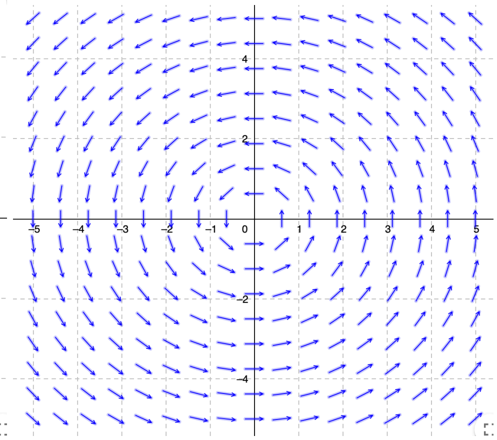
| |
|---|
| Useful tool:<br>You can use the fact that in **spherical coordinates**, $\|\|\mathbf{n}\|\| = R^2 \sin\phi$. |

| |
|---|
| In your homework you will work out what happens for a general angle $\phi = \alpha$. You will see that the surface area is $2\pi R^2(1 - \cos\alpha)$. You can check now and see that if $\alpha = \pi$ we get the area of the whole sphere! |

## 51 More Practice

## 52 What are Curl and Divergence?

**Warmup:** Look at the following graphs - in words, how could you describe this vector field?

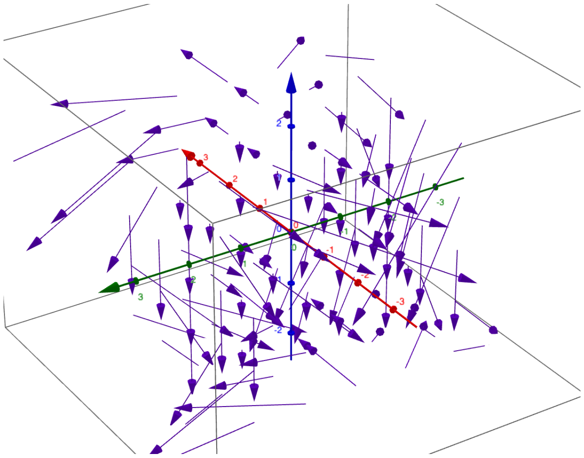
Today we are going to study how to measure how much a vector field is

- Being ________________________ or ________________________. This is called ________________________.
- ________________________ around a given point. This is called ________________________.

**The definitions:**

Let **F** be a vector field. Then we define:

**Example:**



Find the divergence of the vector field $\mathbf{F} = \langle xz, xyz, -y^2\rangle$. Use your answer to find the divergence at the point $(1, 1, 1)$. Is the vector field stretching or compressing at this point? What about at the point $(0, 0, -1)$?

What is the curl of $\mathbf{F}$? What is the curl at the point $(1, 1, 1)$? What does this represent?

**Reality Check:**

1. If $\mathbf{F}$ is a vector field then $\operatorname{div}\mathbf{F}$ is a [scalar field/vector field/not defined]
2. If $\mathbf{F}$ is a vector field then $\operatorname{curl}\mathbf{F}$ is a [scalar field/vector field/not defined]
3. If $\mathbf{F}$ is a vector field then $\nabla\mathbf{F}$ is a [scalar field/vector field/not defined]
4. If $f$ is a scalar field then $\operatorname{div} f$ is a [scalar field/vector field/not defined]
5. If $f$ is a scalar field then $\operatorname{curl} f$ is a [scalar field/vector field/not defined]
6. If $f$ is a scalar field then $\nabla f$ is a [scalar field/vector field/not defined]

**Example:**

Let $f$ be a scalar field and $\mathbf{F}$ a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.

(a) $\operatorname{curl} f$

(b) $\operatorname{grad} f$

(c) $\operatorname{div}\mathbf{F}$

(d) $\operatorname{curl}(\operatorname{grad} f)$

(e) $\operatorname{grad}\mathbf{F}$

(f) $\operatorname{grad}(\operatorname{div}\mathbf{F})$

(g) $\operatorname{div}(\operatorname{grad} f)$

(h) $\operatorname{grad}(\operatorname{div} f)$

(i) $\operatorname{curl}(\operatorname{curl}\mathbf{F})$

(j) $\operatorname{div}(\operatorname{div}\mathbf{F})$

(k) $(\operatorname{grad} f) \times (\operatorname{div}\mathbf{F})$

(l) $\operatorname{div}(\operatorname{curl}(\operatorname{grad} f))$

The following cartoon shows us how all these players interact:

**Big Questions:** (One of these we have already studied - which one?)

1. How do these different operations interact with one another?
2. How can you tell if a vector field $F$ is equal to $\nabla f$ for some $f$?
3. Do any of these things relate to to what we've been doing with integration?

**Some answers:**

**Theorem:** If $f$ is a function of three variables that has continuous second order partial derivatives then

This means that

**IF** ________________________ **THEN** ________________________

The converse is also true[2]. In other words, you can use this to test if a vector field is conservative.

**Theorem:** To check if a vector field $\mathbf{F}$ is conservative:

**Example:** Is the vector field $\mathbf{F} = \langle xz, xyz, -y^2 \rangle$ conservative? Why or why not?

**Another interaction:**

(What is the divergence of a curl?)

**Theorem:**

[2] So long as your vector field is defined on all of $\mathbb{R}^3$, or on a set that doesn't have any "holes" in it

**Example:** Show that the vector field $\mathbf{F} = \langle xz, xyz, -y^2 \rangle$ CANNOT be written as the curl of some other vector field $\mathbf{G}$.

**Example:** What is the curl of the vector field $\langle F_1, F_2, 0 \rangle$? What does the answer remind you of?

(Green's Theorem Can be stated using Curl)

## Some Practice Problems:

**13–18** Determine whether or not the vector field is conservative. If it is conservative, find a function $f$ such that $\mathbf{F} = \nabla f$.

**13.** $\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$

**14.** $\mathbf{F}(x, y, z) = xyz^2\,\mathbf{i} + x^2yz^2\,\mathbf{j} + x^2y^2z\,\mathbf{k}$

**15.** $\mathbf{F}(x, y, z) = 3xy^2z^2\,\mathbf{i} + 2x^2yz^3\,\mathbf{j} + 3x^2y^2z^2\,\mathbf{k}$

**16.** $\mathbf{F}(x, y, z) = \mathbf{i} + \sin z\,\mathbf{j} + y\cos z\,\mathbf{k}$

**17.** $\mathbf{F}(x, y, z) = e^{yz}\,\mathbf{i} + xze^{yz}\,\mathbf{j} + xye^{yz}\,\mathbf{k}$

**18.** $\mathbf{F}(x, y, z) = e^x\sin yz\,\mathbf{i} + ze^x\cos yz\,\mathbf{j} + ye^x\cos yz\,\mathbf{k}$

**19.** Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that $\operatorname{curl}\mathbf{G} = \langle x\sin y, \cos y, z - xy\rangle$? Explain.

**21.** Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}$$

where $f$, $g$, $h$ are differentiable functions, is irrotational.

**22.** Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(y, z)\,\mathbf{i} + g(x, z)\,\mathbf{j} + h(x, y)\,\mathbf{k}$$

is incompressible.

## 53 Stokes' Theorem and the Divergence Theorem

This is the last new material for the semester!

- We will use the symbol $\partial R$ to refer to the ________________________________________.
- NOT every region has a boundary. Let's look at some picture below and decide which have boundaries and what they are:

**Warmup:** The Fundamental Theorem of Calculus says that

$$\int_a^b f'(t)dt = f(b) - f(a).$$

This is basically saying:

$$\int_I \mathrm{d}(\text{stuff}) = \text{ formula in involving } f \text{ on } \partial I.$$

**Fundamental Theorem for Gradients**

**Green's Theorem**

**Our Last Two Theorems:**

---

**The Divergence Theorem**
Hypotheses:

- Let $E$ be a simple solid region and let $S$ be the boundary surface of $E$.
- Suppose $S$ is oriented so the normal vector points outwards.
- Let $\mathbf{F}$ be a vector field with continuous partial derivatives.

Then

**Example:**

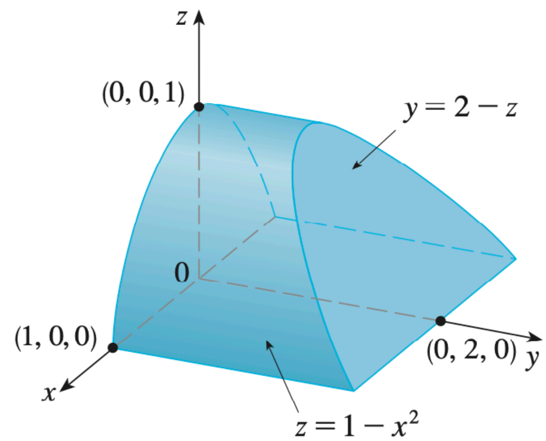
Use the divergence theorem to calculate the surface integral $\iint\limits_S \mathbf{F} \cdot \mathbf{dS}$ where $\mathbf{F} = \langle 3z, 2y, 3x + y \rangle$ where $S$ is the sphere $x^2 + y^2 + z^2 = 25$ with outward pointing normal vector.

**Example:**

Evaluate $\iint\limits_S \mathbf{F} \cdot \mathbf{dS}$ where

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + (y^2 + e^{xz^2})\mathbf{j} + \sin(xy)\mathbf{k}$$

$S$ is the surface of the region $E$ bounded by he parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$ and $y + z = 2$.




**If we wanted to calculate this by parametrizing** $S$ we would need to do ____________ surface integrals. And we would have to evaluate integrals involving things like $e^{xz^2}$ and $\sin(xy)$. ____________.

Let's use the Divergence Theorem:

**Suppose that $E$ is the solid picture we'll draw below:**

The boundary of $E$ is the sea monster whose top is the circle $x^2+y^2=16$ that lies in the plane $z=7$ and whose rest of the surface is a squiggly surface $S$.

You are given that $E$ has volume 18.

Find the value of

$$\iint_S \mathbf{F} \cdot \mathbf{dS}$$

where $\mathbf{F} = \langle 2x, -2yz, x^2+y^2+z^2 \rangle$.

**This problem has multiple steps - focus on the strategy first**.

**Stokes' Theorem**
Hypotheses:

- Let $S$ be an oriented piecewise-smooth surface
- Suppose $S$ is bounded by a simple closed piecewise-smooth boundary curve $C$
- Suppose $C$ is oriented so that if a person is walking along $C$ with their head pointing in the direction of $\mathbf{n}$ the surface will always be on their left.
- Let $\mathbf{F}$ be a vector field with continuous partial derivatives.

Then

**Reality Check:** Let $C$ be the unit circle in the $xy$ plane, oriented counterclockwise. Can you think of a surface $S$ whose boundary is $C$? In what direction would the normal vectors have to be pointed so that the hypotheses of Stokes' Theorem applied.

**Reality Check 2:** Can you think of another surface that has the same boundary? What would Stokes' Theorem say?

**Reality Check 3:** Take one of your surfaces from above. If $\mathbf{F}$ is conservative, then what is $\operatorname{curl}\mathbf{F}$? What would Stokes' Theorem say in this case?

**Spooky Szn:** If you want to calculate $\int_C \mathbf{F}\cdot\mathbf{ds}$ and $\mathbf{F}$ is any vector field (conservative or not) you can find this by applying Stokes' Theorem to any surface you want!

**Example:** Let $C$ be the intersection of the surfaces:

$$y + z = 2, \text{ and the cylinder } x^2 + y^2 = 1.$$

Suppose that $C$ is oriented to be counterclockwise when viewed from above. Calculate $\displaystyle\int_C \mathbf{F} \cdot \mathbf{ds}$

- By using Stokes' Theorem,
- By parametrizing $C$.

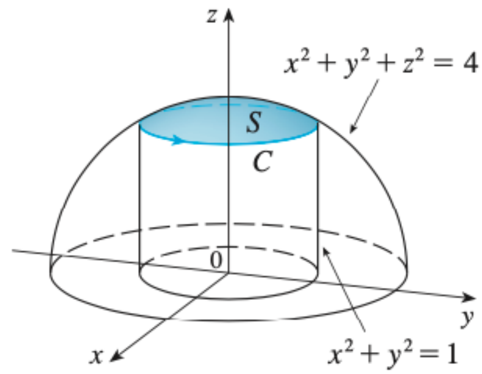
**Example:** Use Stokes' Theorem to compute the integral

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{dS}$$

where

$$\mathbf{F} = \langle xz, yz, xy \rangle$$

and $S$ is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the $xy$ plane.



Fundamental Theorem of Calculus

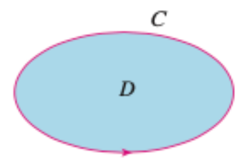
$$\int_a^b F'(x)\,dx = F(b) - F(a)$$


Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

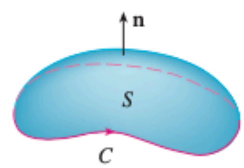

Green's Theorem

$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy$$



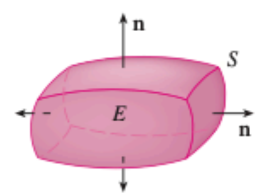

Stokes' Theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$




Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$



## Additional Practice

**1.** Indicate with an arrow the boundary orientation of the boundary curves of the surfaces in Figure 14, oriented by the outward-pointing normal vectors.

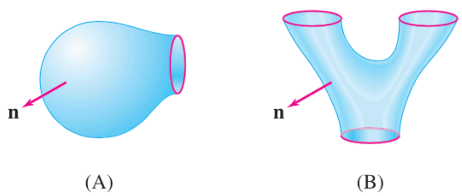



**FIGURE 14**

**2.** Let $\mathbf{F} = \text{curl}(\mathbf{A})$. Which of the following are related by Stokes' Theorem?

**(a)** The circulation of $\mathbf{A}$ and flux of $\mathbf{F}$.

**(b)** The circulation of $\mathbf{F}$ and flux of $\mathbf{A}$.

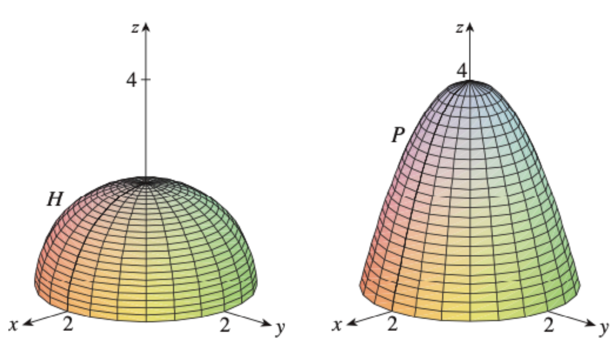
**4.** Which of the following statements is correct?

**(a)** The flux of $\text{curl}(\mathbf{A})$ through every oriented surface is zero.

**(b)** The flux of $\text{curl}(\mathbf{A})$ through every closed, oriented surface is zero.

**1.** A hemisphere $H$ and a portion $P$ of a paraboloid are shown. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\iint_H \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint_P \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$$




**13–15** Verify that Stokes' Theorem is true for the given vector field $\mathbf{F}$ and surface $S$.

**13.** $\mathbf{F}(x, y, z) = -y\,\mathbf{i} + x\,\mathbf{j} - 2\,\mathbf{k}$,
$S$ is the cone $z^2 = x^2 + y^2$, $0 \leqslant z \leqslant 4$, oriented downward

**14.** $\mathbf{F}(x, y, z) = -2yz\,\mathbf{i} + y\,\mathbf{j} + 3x\,\mathbf{k}$,
$S$ is the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upward

**15.** $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$,
$S$ is the hemisphere $x^2 + y^2 + z^2 = 1$, $y \geqslant 0$, oriented in the direction of the positive $y$-axis

**16.** Let $C$ be a simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z\,dx - 2x\,dy + 3y\,dz$$

depends only on the area of the region enclosed by $C$ and not on the shape of $C$ or its location in the plane.

**17.** A particle moves along line segments from the origin to the points $(1, 0, 0)$, $(1, 2, 1)$, $(0, 2, 1)$, and back to the origin under the influence of the force field

$$\mathbf{F}(x, y, z) = z^2\,\mathbf{i} + 2xy\,\mathbf{j} + 4y^2\,\mathbf{k}$$

Find the work done.

**1–4** Verify that the Divergence Theorem is true for the vector field $\mathbf{F}$ on the region $E$.

**1.** $\mathbf{F}(x, y, z) = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2xz\,\mathbf{k}$,
$E$ is the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$

**2.** $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z\,\mathbf{k}$,
$E$ is the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the $xy$-plane

**3.** $\mathbf{F}(x, y, z) = \langle z, y, x\rangle$,
$E$ is the solid ball $x^2 + y^2 + z^2 \leqslant 16$

**4.** $\mathbf{F}(x, y, z) = \langle x^2, -y, z\rangle$,
$E$ is the solid cylinder $y^2 + z^2 \leqslant 9$, $0 \leqslant x \leqslant 2$

**5–15** Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across $S$.

**5.** $\mathbf{F}(x, y, z) = xye^z\,\mathbf{i} + xy^2z^3\,\mathbf{j} - ye^z\,\mathbf{k}$,
$S$ is the surface of the box bounded by the coordinate planes and the planes $x = 3$, $y = 2$, and $z = 1$

**6.** $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + xy^2z\,\mathbf{j} + xyz^2\,\mathbf{k}$,
$S$ is the surface of the box enclosed by the planes $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, and $z = c$, where $a$, $b$, and $c$ are positive numbers

7. $\mathbf{F}(x, y, z) = 3xy^2\,\mathbf{i} + xe^z\,\mathbf{j} + z^3\,\mathbf{k}$,
$S$ is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$

8. $\mathbf{F}(x, y, z) = (x^3 + y^3)\,\mathbf{i} + (y^3 + z^3)\,\mathbf{j} + (z^3 + x^3)\,\mathbf{k}$,
$S$ is the sphere with center the origin and radius 2

9. $\mathbf{F}(x, y, z) = x^2 \sin y\,\mathbf{i} + x \cos y\,\mathbf{j} - xz \sin y\,\mathbf{k}$,
$S$ is the "fat sphere" $x^8 + y^8 + z^8 = 8$

10. $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + zx\,\mathbf{k}$,
$S$ is the surface of the tetrahedron enclosed by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where $a$, $b$, and $c$ are positive numbers

11. $\mathbf{F}(x, y, z) = (\cos z + xy^2)\,\mathbf{i} + xe^{-z}\,\mathbf{j} + (\sin y + x^2 z)\,\mathbf{k}$,
$S$ is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$

12. $\mathbf{F}(x, y, z) = x^4\,\mathbf{i} - x^3 z^2\,\mathbf{j} + 4xy^2 z\,\mathbf{k}$,
$S$ is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = x + 2$ and $z = 0$

13. $\mathbf{F} = |\mathbf{r}|\,\mathbf{r}$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,
$S$ consists of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and the disk $x^2 + y^2 \leqslant 1$ in the $xy$-plane

# Part II

# Homework Assignments

# Homework Assignments for Math 250

**Calc 1 and 2 Review:** (Due In Class on Tuesday January 31st)

In this class it will be very important to be able to accurately and confidently take **derivatives** of functions. Calculating derivatives is more or less straight-forward, you'll just need things like the chain rule, quotient rule, product rule, etc. Since very often **taking a derivative** will be the **first** step in solving a problem, it's important that we be able to do this.

Less frequently we will have to compute **integrals**. This is because calculating integrals can be pretty tough. I promise not to ask you to calculate super complicated integrals. That's something that you did in Calc2 with all of the integration techniques you learned. However, it **is** important to be able to perform basic integrals, including integration by substitution, and integration by parts. We will review this as it comes up.

Practice makes perfect, so for this first homework assignment, please check out the file "Computational Practice.pdf" this document contains lots of practice problems for you to look at, including some algebra review problems. These problems are all highly recommended problems for you to solve (and answers are included). For instance you might see something like

$$\int \frac{x^3}{7x^6}dx \qquad\qquad \text{Answer}: -\frac{1}{14x^2}+C$$

and forget how to do that. That's ok. You can:

- check out your old calc textbook (or earlier chapters of ours)
- look for videos online
- (My Favorite) go to symbolab.com and ask the website to do the integral. It will show the steps and explain what is happening. Then when you write down the steps you can ask yourself "ok, do I see why simplifying makes sense here." or "wait, what does the power rule for integrals mean?"

I think that **using technology to help perform calculations** can be a great way to help reinforce learning. Ultimately, though, it's important to remember that on a quiz I might ask you to calculate something like

$$\int (1-\frac{1}{x^2})\frac{x^3}{7x^6}dx$$

and it will be your responsibility to make sure you will be comfortable using the **skills you practiced** on the homework to solve the quiz problems.

1. Please choose 5 of the derivative problems and 5 of the integral problems from the ComputationalPractice.pdf up and turn in as part of this homework. I encourage you to choose ones that you think might be challenging for you.
2. None of the problems in the ComputationalPractice.pdf involved integration by substitution. Here are a few problems to help you review those concepts. These problems represent the types of problems that occur frequently in Calc 3, so that's why we're practicing them. You should make sure you are comfortable solving all of these, but you only need to write up solutions to **five** of them.

   (a) $\int \cos(3x)dx$

   (b) $\int e^{x/3}dx$

   (c) $\int \sqrt{3x+4}dx$

   (d) $\int xe^{x^2}dx$

   (e) $\int \cos^5 x \sin x \; dx$

   (f) $\int \frac{\ln x}{x}dx$

(g) $\int \sin^2 x \, dx$

(Hint: use the trig identity $\sin^2 x = \frac{1-\cos(2x)}{2}$)

(h) $\int e^x\sqrt{1+e^x}dx$

(i) $\int x\sin(x^2)dx$

(j) $\int \frac{dx}{5-3x}$

(k) $\int \frac{(\ln x)^2}{x}dx$

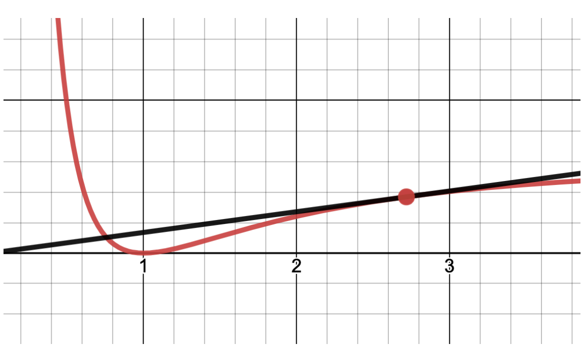
3. **Integration by parts** might come up from time to time. Recall that it says:

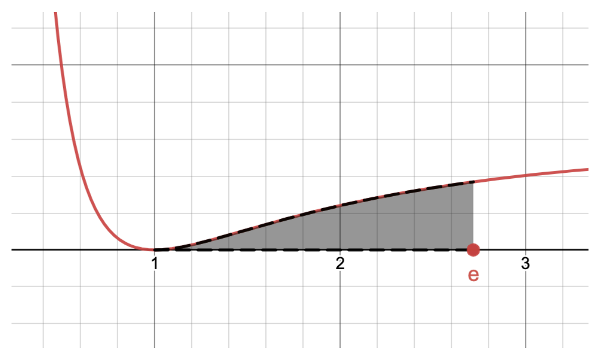
$$\int udv = uv - \int vdu$$

Use this formula to find $\int xe^x \, dx$ by letting $u = x$ and letting $dv = e^x dx$. Then $du =?$ and $v =?$. Fill these in and use the formula above to calculate the integral.

(Optional) For additional practice, find $\int x\ln x dx$ and $\int x\cos x dx$.

4. **What does it all mean?** The function $f(x) = \frac{(\ln(x))^2}{x}$ is graphed below.




A) On the graph on the left, you see a tangent line drawn in at the point where $x = e$. Using what you know about calculus and tangent lines, find the equation of the tangent line. (Hint: remember you should first find the $y$ coordinate at that point.) Your answer will involve the number $e$. It shouldn't be too messy though.

B) Find the area of the shaded region. Your answer should be a number that doesn't involve $e$. You should find this by using an integral.

C) Check your answers to both A and B by using Desmos. To check A) graph the function and the line you found. Make sure that it appears to be tangent. To check part B), you can type int and then put the bounds of your integral in.

For full credit, please write your work neatly and clearly. This will mean that someone reading your work should be able to understand what you are doing. For instance, if the problem were to find the derivative of $f(x) = x^2 + x \sin x$.

**Correct Work 1:** (Note how the solution clearly says what $f(x)$ and $f'(x)$ are. There doesn't need to be too much work since using the product rule can be done mentally.)

$f(x) = x^2 + x \sin x$

$f'(x) = 2x + \sin x + x \cos(x)$.

**Correct Work 2:** (Note here the student didn't want to do the product rule mentally, so instead they wrote it out in multiple steps. But they definitely took care that every time they used an equals sign, things were really equal. Note that they didn't refer to $f(x)$. That's ok though, because their work clearly shows that they took a function and found its derivative.)

$(x^2 + x \sin x)' = 2x + (x)'(\sin x) + (x)(\sin x)' = 2x + \sin x + x \cos x$.

**Incorrect Work:** $x^2 + x \sin x = 2x + \sin x + x \cos x$ (This is incorrect because these quantities are NOT equal. One is the function $f(x)$ and the other is its derivative.)

**Incorrect Work:** $x^2 + x \sin x \rightarrow 2x + \sin x + x \cos x$ (This is incorrect because it's not clear what the arrow means.)

**Why is this important?** Later in this class, we're going to solve some pretty gnarly questions, where you'll have to go back and refer to your work from previous parts, and you'll want to make sure you have clearly labeled and referred to everything. I know with simple problems like the one above, you might be thinking "oh come on, it's not important how we write." (and I'm sympathetic to that) but as the problems get more and more complicated, having clearly **organized** work will be **extremely helpful**.

**Homework Due Friday February 3rd in class**

Online Part: Our first "Online" Webwork homework assignment "01" is due Friday at 6:00pm as well. In general, I will try to alternate the days of the online and written homework. But this Friday we have both due.

- On the online homework you get unlimited attempts.
- If you get stuck, hit the "email instructor button and I can help!"
- After the deadline passes you will have until May 19th to complete the assignment for half-credit. For instance, if you complete 16/20 problems on time you have an 80%. If you later complete the remaining 4 problems, you'll get half the credit, so you final score will go to 90%.

Written Part:

**Homework from Rogawski 12.1 "Parametric Equations": (from Monday's class)**

7, 11, 13,

19, (for these, include a brief explanation of how you might be able to figure these ones out "by hand". It's absolutely ok to use Desmos to check, but give it a start by saying "hmm, the $x$-coordinate is $x = \sin t$ so that means the $x$ coordinate can only ever be between....)

23, 24 (note when you are finding parametric equations for a function $y = f(x)$ the easiest way to parametrize is just to let $x = t$ and $y = f(t)$.)

27, 28, 33, 34 (You might want to keep track on a notecard **how to parametrize line segments and circles** this will come up a lot throughout the course.)

39, 40, 41 (some trial and error might be needed here. When you write these three problems up, please explain your reasoning, and include a check by hand that your parameterization has the desired properties. Of course Desmos can help you check as well.)

45, 46, Make sure you explain your reasoning. Explain what it is that you notice about the graphs that confirms your answer? In this class I am very likely to ask conceptual questions like 45 and 46 so make sure you can give good explanations explaining your answers.

Remember that the answers to the odd problems are in the back of the book. Please check your answers and see if you have made any mistakes. Remember, if you're stuck, I'm here to help!

**Homework from Rogawski 13.1 "Vectors in the Plane": (from Tuesday's class)**

2, 3, 5, 11, 13, 16, 17, 29, 31, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 47, 57, 59.

I know there are a lot of problems here, but these should go quickly and will help you get familiar with vectors. Pay careful attention to **16,17, 45**. Drawing vectors graphically can often help solve problems. There are similar practice questions on the webwork.

Finally, at the end of the homework, please write how many hours you spent working on the material that comprises this assignment. For example you can just say "I spent X hours working on Friday's homework, including the online part." (Don't include the time for Tuesday's homework.)

**Homework 3: Due in class on Friday Feb. 11**

**Rogawski 13.2 "Vectors in three Dimensions":** (From Wednesday's Class)

29, 31, 33, 35, 37, 38

**Rogawski 13.3 "Dot Product and the Angle Between Vectors":** (From Friday's Class)

34, 35, 39, 40, 41, 43, 49

**Rogawski 13.4 "The Cross Product"** (From Monday's Class)

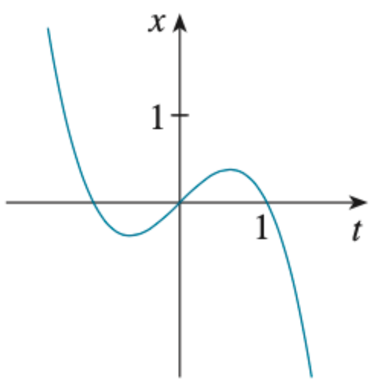
1, 3, 5, 9, 10, 13, 15, 17, 19, 21, 23, 25, 26, 30

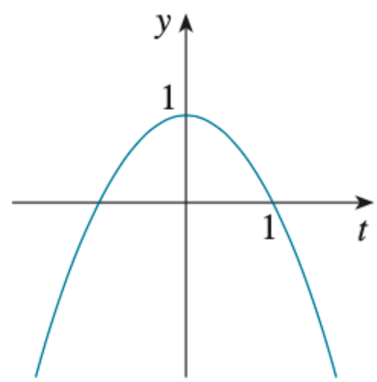
**Rogawski 13.5 "Planes in Three Space"** (From Tuesday's Class)

1, 3, 5, 9, 11, 13, 15, 17, 21, 26, 27, 28, 29, 31, 33, 37, 39, 47, (optional: 19, 23, 25)

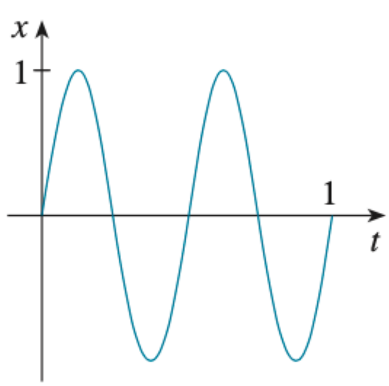
**Additional Problems:**

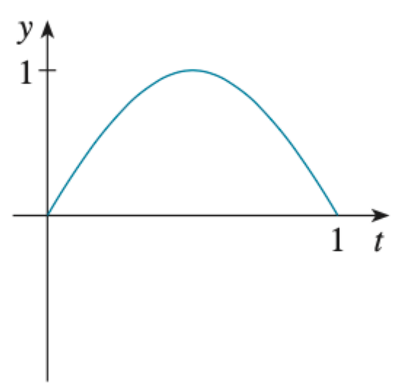
1. Below are graphs of $x(t)$ and $y(t)$. Use these to draw a sketch of the parametric curve defined by $c(t) = (x(t), y(t))$. Label the points on the curve corresponding to $t = 0$ and $t = 1$.




2. (Optional) If you want additional practice, here is another



**Homework 4: Due in class on Friday Feb. 18**

1. First, here are some helpful hints for the Webwork this week:

   (a) If you need to find the angle between a plane and a line, just take the angle between the normal vector and the line. As practice, if your plane was $3x-4y+7z=4$ and your line was $\langle(1+2t, 3-4t, 7+17t\rangle$ then you would be taking the angle between the vectors $\langle 3, -4, 7\rangle$ and $\langle 2, -4, 17\rangle$. Make sure you understand where these are coming from. To easily find the angle between two vectors, use the [dot / cross] product which will help you find the [cosine / sine] of the angle. (Hint: the answer is dot product and cosine).

   (b) Suppose you have parametrized something like this:

   $$\mathbf{r}(t) = \langle 3+t, 4+t^2+2t, \cos t\rangle$$

   Where is the particle at time $t=0$? What is the tangent vector at time $t=0$? what is the speed of the particle at time $t=0$. (Answers: $\langle 3,4,1\rangle, \langle 1,2,0\rangle$ and $\sqrt{5}$.)

   (c) Now suppose you wanted to take the parametrization from the previous problem and have the particle pass through the point you just found at time $t=13$ instead of time $t=0$. Here is an easy way to do it:

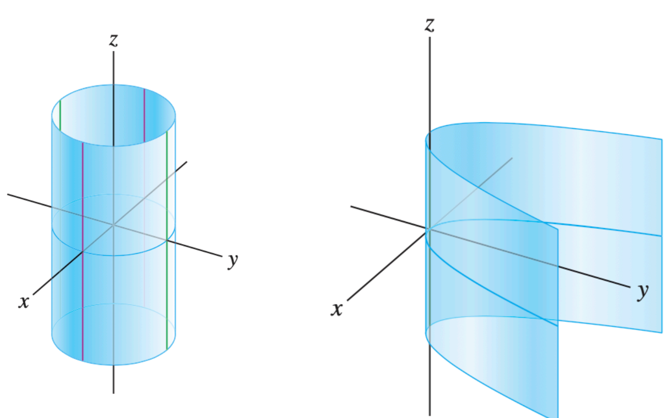
      i. Everywhere you see $t$ replace that $t$ with $t-13$.

      ii. That's it! Write this down.

      iii. Think about why this works - now when you plug in $t=13$, it's like you were plugging in $t=0$ into the "original" parametrization.

      iv. Make sure you understand this - it will come up on the webwork.

2. Here two two examples of a "cylinder". The first one is a cylinder over the circle $x^2+y^2=r^2$ in the $xy$ plane. The second is a cylinder over the equation $y=ax^2$ in the $xy$ plane. "Cylinder" just means we take a shape in one plane and then stack a bunch of copies of it on top of itself.

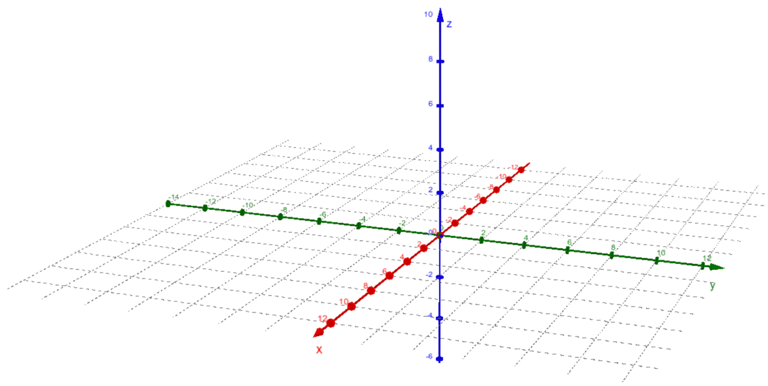



$x^2+y^2=r^2$
Right-Circular cylinder of radius $r$
**FIGURE 11**

$y=ax^2$
Parabolic cylinder

Notice that these equations only involve $x, y$ so the $z$-coordinate is free to be anything. For example, in the equation $x^2+y^2=4$ the point $(2,0,5)$ and $(2,0,-1000)$ both satisfy this equation, so they both live on the surface.

For this problem, please draw pictures of the following curves. They will all have the same shape as these pictures, but will be in different orientations. Please make sure you label your axes in the same way as the pictures. Do the best you can - I know graphing is hard. If you want to have a "blank" set of axes, you can use this. **Your homework should have 8 different graphs for this problem.**



(a) $x = z^2$

(b) $z = x^2$

(c) $z = y^2$

(d) $z = -x^2$

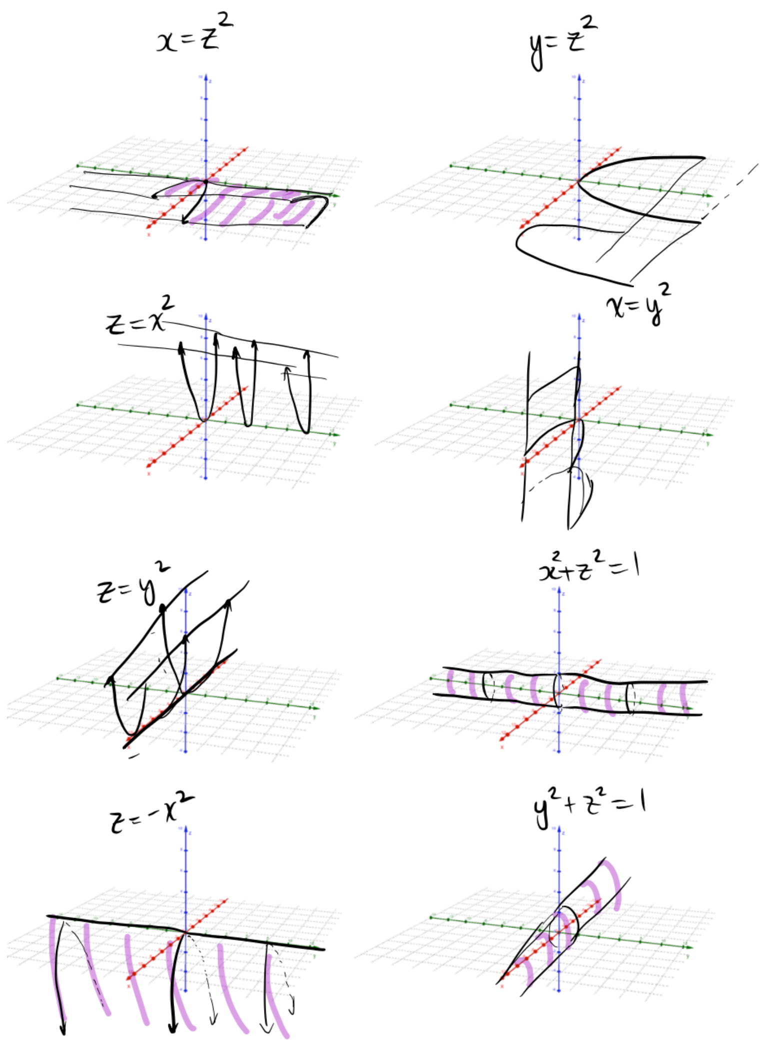
(e) $y = z^2$

(f) $x = y^2$

(g) $x^2 + z^2 = 1$

(h) $y^2 + z^2 = 1$

Here is my attempt. Notice that they're not so great, and that if we wanted a really good picture, we'd use a computer. What's important is getting practice drawing in 3D.




3. Solve these problems from Rogawski 14.1

**7.** Match the space curves in Figure 8 with their projections onto the $xy$-plane in Figure 9.

**8.** Match the space curves in Figure 8 with the following vector-valued functions:

**(a)** $\mathbf{r}_1(t) = \langle \cos 2t, \cos t, \sin t\rangle$ **(b)** $\mathbf{r}_2(t) = \langle t, \cos 2t, \sin 2t\rangle$
**(c)** $\mathbf{r}_3(t) = \langle 1, t, t\rangle$

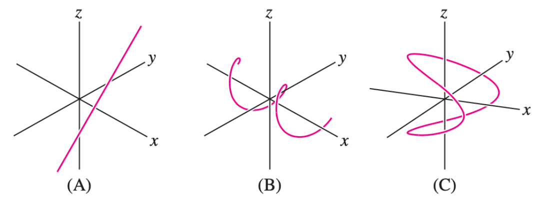


FIGURE 8

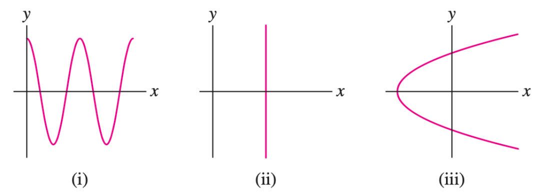


FIGURE 9

A and (ii), (B) and (i) and (C) and (iii). Let me know if you have questions on these and we can talk. For matching the other ones, $r_3$ is the line, since we can see it's of the form $P + t\mathbf{v}$ (what is $\mathbf{v}$? - make sure you can answer this.) Notice that $r_1$ will never have $x, y, z$ greater than 1 or less than -1, so that can't be graph B. So $r_1$ will match with $C$. So that means $r_2$ matches with graph B.

Note that $r_1$ lies on the surface of $x = y^2 - z^2$ because of the trig identity $\cos(2t) = \cos^2 t - \sin^2 t$.

4. Parametrizations of circles will come up a lot. The basic building blocks are (as always) $\cos t$ and $\sin t$. Use these ingredients to parametrize the circles:

   (a) A circle in the $xz$ plane with center at the origin and radius 4.

   (b) A circle in the $yz$ plane with center at the origin and radius 2.

   (c) A horizontal circle (meaning it's parallel to the $xy$ plane) of radius 3 and center (2,5,7)

   a) The building blocks are going to be $4\cos t$ and $4\sin t$. We put this together in the $x$ and $z$ coordinates: $\langle 4\cos t, 0, 4\sin t\rangle$.

   b) $\langle 0, 2\cos t, 2\sin t\rangle$.

   c) Here we want to emphasize that the $xy$ coordinates will have the trig functions, and the $z$ coordinate will always be 7. So we get: $\langle 2 + 3\cos t, 5 + 3\sin t, 7\rangle$

5. What is a normal vector to the plane defined by $z = ax + by + c$? (Hint: there should be a minus sign or two in your answer.) Remember to first move everything to one side: Answer: $\langle a, b, -1\rangle$. Also valid is $\langle -a, -b, 1\rangle$, or any nonzero multiple of either of these.

6. You are given that $\mathbf{r}'(t) = \langle 2t, 3\rangle$ and that $\mathbf{r}(1) = \langle 4, 2\rangle$. Find $\mathbf{r}(t)$. Hint: you should integrate each component and use a $+C$ in each component. These constants can be different. Then use the given condition to solve for the constants.

   By integrating we see that $\mathbf{r}(t) = \langle t^2 + C_1, 3t + C_2\rangle$. After you plug in that $\mathbf{r}(1) = \langle 4, 2\rangle$ you see that $C_1$ must be 3 and $C_2$ is $-1$. So $\mathbf{r}(t) = \langle t^2 + 3, 3t - 1\rangle$

7. Using Desmos, let's study the curve: $c(t) = \langle \cos 2t, \sin 3t\rangle$.

   (a) Start by typing $(\cos 2a, \sin 3a)$ in and animate from $a = 0$ to $a = 6\pi$. You should see the point moving back and forth along a curve.

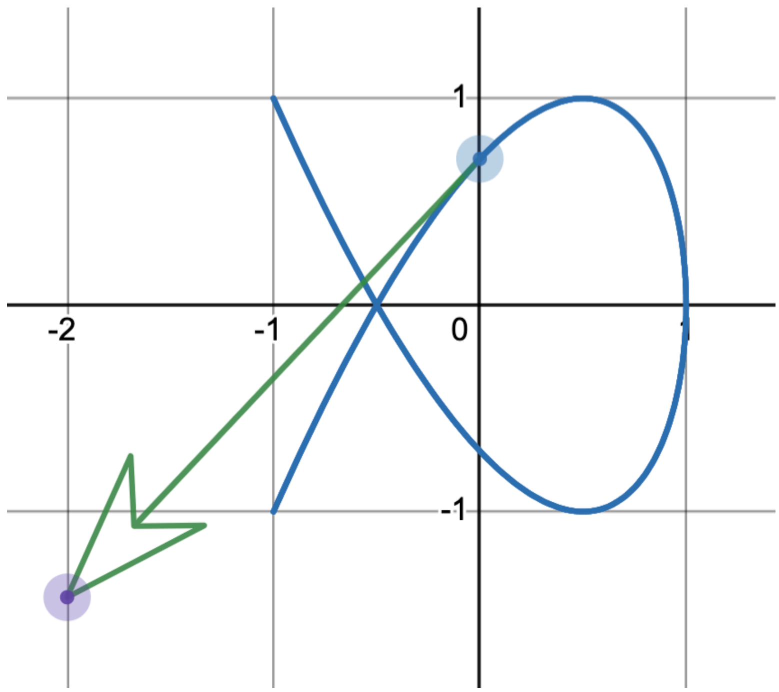
   (b) Find the tangent vector at time $t = \pi/4$. Draw a graph of this curve and carefully draw your tangent vector. **The tangent vector can be thought of a velocity vector. Its length represents speed.**

   (c) Now, using your well-practiced skill for parametrizing lines, write down the parametrization of the **tangent line** at this point. You will just be using $P + t\mathbf{v}$ as always. The square root of 2 should be involved.

(d) Type this into Desmos to confirm that you got the correct tangent line.

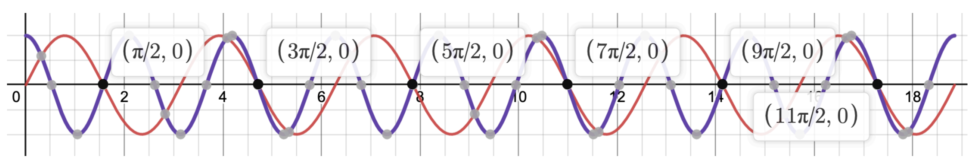
(e) You should notice a spot where the particle seems to "pause" and switch direction. What time does it look like this happens? Confirm your guess by seeing when the velocity vector is equal to $\langle 0, 0\rangle$. In your work you should find yourself thinking about something along the lines of "hmm when are $\sin(2a)$ and $\cos(3a)$ both going to be zero?

b) The tangent vector at time $t$ is given by: $\mathbf{c}'(t) = \langle -2\sin(2t), 3\cos(3t)\rangle$. So at time $t = \pi/4$ we get $\mathbf{c}'(t) = \langle -2, -3\sqrt{2}/2\rangle$. Here is a picture. Note how long the vector is:




c) At $t = \pi/4$ the point on the graph is $(0, \sqrt{2}/2)$ (we get that by plugging into $\mathbf{r}(t)$. So when we do $P + t\mathbf{v}$ we get $\langle 0 + -2t, \sqrt{2}/2 - 3t\sqrt{2}/2\rangle$.

e) The particle will switch directions when the speed is zero, which happens when the tangent vector is the zero vector $\langle 0, 0\rangle$. For this to happen we need that $-2\sin(2t) = 0$ AND $3\cos(3t) = 0$ at the same time. Perhaps the easiest way to visualize this is to graph $\sin(2t)$ and $\cos(3t)$ at the same time and see when they are both 0. I went and did this on the interval from 0 to $6\pi$ and got this:




and those are the times highlighted there.

8. Below are three problems from the book that are more involved. **Choose one** of these and write your solution out carefully. For extra credit, do all three. I will be looking for a very careful writeup. These are challenging, but give them a try. Try to think about what it would mean for a laser to be be able to hit the origin / x-axis. Even if you don't succeed at first with this problem, a productive struggling with trying to figure out what's going on will help build some 3D intuition. You can do it!

**57.** A fighter plane, which can shoot a laser beam straight ahead, travels along the path $\mathbf{r}(t) = \langle 5 - t, 21 - t^2, 3 - t^3/27 \rangle$. Show that there is precisely one time $t$ at which the pilot can hit a target located at the origin.

**58.** The fighter plane of Exercise 57 travels along the path $\mathbf{r}(t) = \langle t - t^3, 12 - t^2, 3 - t \rangle$. Show that the pilot cannot hit any target on the $x$-axis.

**63.** Prove that the **Bernoulli spiral** (Figure 9) with parametrization $\mathbf{r}(t) = \langle e^t \cos 4t, e^t \sin 4t \rangle$ has the property that the angle $\psi$ between the position vector and the tangent vector is constant. Find the angle $\psi$ in degrees.

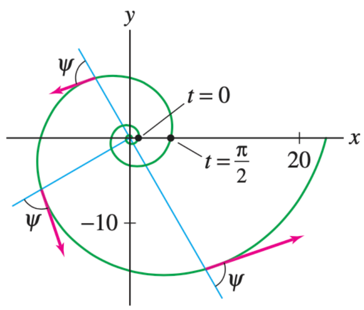



**FIGURE 9** Bernoulli spiral.

57. Notice that if the plane can only shoot "straight ahead" then this means we are looking at tangent vectors. You might want to graph this on Geogebra and see what the curve looks like: `https://www.geogebra.org/3d/fevupbdu`.

If we are at a time $a$ then we are at the point $(5 - a, 21 - a^2, 3 - a^3/27)$ and our tangent vector is $\langle -1, -2a, -a^2/9 \rangle$.What this question is asking is whether we can hit the origin $(0,0,0)$ from the point going along in the direction of the tangent vector. That means we want to know that if we do $P + t\mathbf{v}$ for our point and tangent vector (and a positive time $t$, since we are shooting forward) we should get $(0,0,0)$:

$$(5 - a, 21 - a^2, 3 - a^3/27) + t\langle -1, -2a, -a^2/9 \rangle = (0,0,0)$$

This means that all three of these equations must be true:
$5 - a - t = 0$ $\quad$ $21 - a^2 - 2at = 0,$ $\quad$ $3 - a^3/27 - a^2t/9 = 0$
Phew, this is tough, we will need to figure out how to solve this. Luckily we can solve for $t$ in the first equation and see that $t = 5 - a$. Then we can substitute in the second equation to get:

$$21 - a^2 - 2a(5 - a) = 0 \implies a^2 - 10a + 21 = 0 \implies (a - 7)(a - 3) = 0.$$

So $a = 7$ or $a = 3$. Now if $a = 7$ then $t = -2$, and if we plug that into the third equation we see this is NOT a solution to all three equations. However, if $a = 3$ then $t = 2$ and when we plug this into the third equation we indeed see it is a solution to all three. Thus the plane can shoot the origin at the point when $a = 3$, which is $(2, 12, 2)$

58) Let's set this up in the same way as before, at time $a$ we are at the point $\mathrm{P} = (a - a^3, 12 - a^2, 3 - a)$ and our tangent vector is $\mathbf{v} = \langle 1 - 3a^2, -2a, -1 \rangle$. So to hit the $x$ axis, we would need $P + t\mathbf{v} = \langle blah, 0, 0 \rangle$ where blah can be anything. We will focus on what this means for the last two coordinates to be zero. For the middle coordinate it means:

$$12 - a^2 - 2at = 0$$

For the last coordinate it means:

$$3 - a - t = 0.$$

This last one seems reasonable to solve, so we get $t = 3 - a$ and substitute that in to the middle:

$$12 - a^2 - 2a(3 - a) = 0$$

$$12 + a^2 - 6a = 0$$

which has no real solutions. (the discriminant "$b^2 - 4ac$" is negative) Thus there is not a solution.

63) Remember, to find angles we should use the dot product. The position vector at time $t$ is going to be $\langle e^t \cos 4t, e^t \sin 4t\rangle$ and its tangent vector at time $t$ will be $\langle e^t \cos 4t - 4e^t \sin 4t, e^t \sin 4t + 4e^t \cos 4t\rangle$. Taking the dot product, we get:

$$\begin{aligned}\mathbf{r}(\mathbf{t}) \cdot \mathbf{r}'(t) &= \langle e^t \cos 4t, e^t \sin 4t\rangle \cdot \langle e^t \cos 4t - 4e^t \sin 4t, e^t \sin 4t + 4e^t \cos 4t\rangle \\ &= e^{2t}\cos^2 4t - 4e^{2t}\cos(4t)\sin(4t) + e^{2t}\sin^2(4t) + 4e^{2t}\sin 4t \cos 4t \\ &= e^{2t}(\cos^2 4t + \sin^2 4t) \\ &= e^{2t}.\end{aligned}$$

We will also need the lengths

$$||r(t)|| = \sqrt{e^{2t}\cos^2 4t + e^{2t}\sin^2 4t} = \sqrt{e^{2t}} = e^t.$$

$$||r'(t)|| = \sqrt{(e^t \cos 4t - 4e^t \sin 4t)^2 + (e^t \sin 4t + 4e^t \cos 4t)^2}$$

$$= \sqrt{e^{2t}(\cos^2 4t + 16\sin^2 4t - 8\cos 4t \sin 4t + \sin^2 4t + 16\cos^2 4t + 8\sin 4t \cos 4t)}$$

$$= \sqrt{e^{2t} \cdot 17} = e^t\sqrt{17}.$$

So to find the angle,

$$\cos\psi = \frac{\mathbf{r}(\mathbf{t}) \cdot \mathbf{r}'(t)}{||\mathbf{r}(\mathbf{t})||||\mathbf{r}'(t)||} = \frac{e^{2t}}{e^t e^t\sqrt{17}} = \frac{1}{\sqrt{17}}$$

so the angle is $\psi = \arccos\dfrac{1}{\sqrt{17}}$ which is about 76 degrees.

[Optional for Extra Practice] Make sure you're able to parametrize lines using the $P + t\mathbf{v}$ setup. For example:

Parametrize the line through the point $(3, 1, 4)$ that is parallel to the $z$ axis.

$(3, 1, 4) + t\langle 0, 0, 1\rangle$

Can you parametrize the line through $(2, 1, 0)$ that is perpendicular to the plane $z = 3x - y + 5$.

$(2, 1, 0) + t\langle 3, -1, 5\rangle$

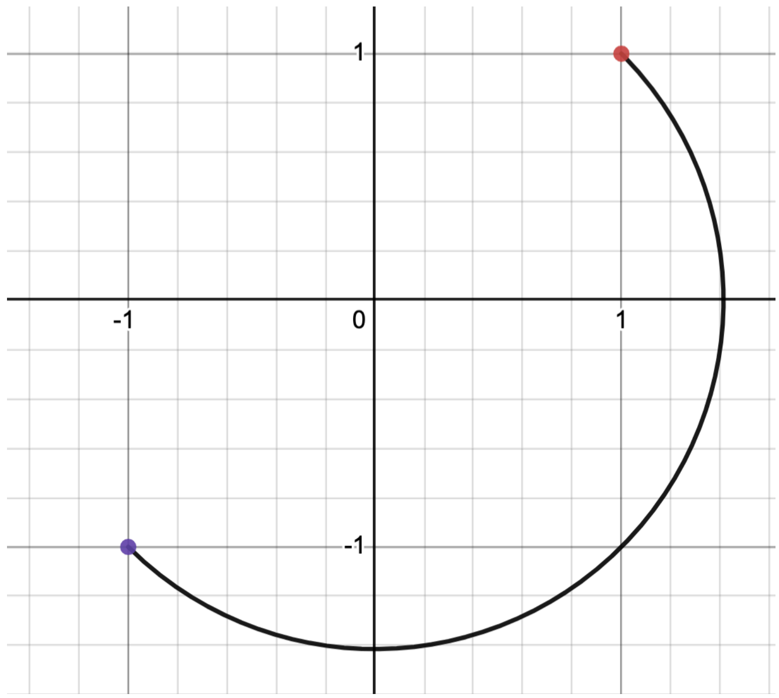
**Homework 5: Due in class on Friday Feb. 25**

1. (a) A bug wants to move along the line segment from the point $(-1,-1)$ to the point $(1,1)$. Using the setup $P + t\mathbf{v}$ write down a parametrization for this bug.

   (b) Oh snap, it turns out that was just the SHADOW in the $xy$ plane of where the bug was walking. In truth the bug was walking along the surface $z = \cos(xy)$. Using the parametrization you found above, write down what the parametrization would be for the bugs position in 3D. (Hint: all you h ave to do is figure out what the $z$ coordinate is.

   (c) Write down an integral that calculates the length of the bug's journey.

   (d) Now repeat these steps for the curve below. This is another path to get from $(-1,-1)$ to $(1,1)$. Be careful finding the radius and the bounds. Set up an integral to find the length of this journey.




   (e) The answers for the lengths should work out to be: 3.11 and 4.88

2.

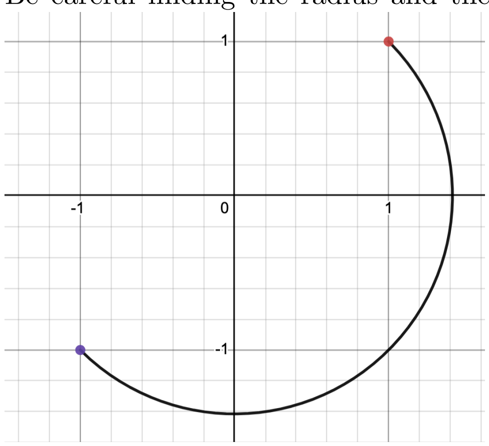
**Homework 5: Due in class on Friday Feb. 25**

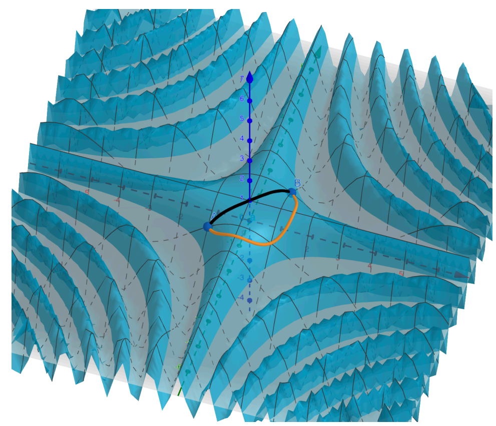
1. (a) A bug wants to move along the line segment from the point $(-1,-1)$ to the point $(1,1)$. Using the setup $P + t\mathbf{v}$ write down a parametrization for this bug.

   (b) Oh snap, it turns out that was just the SHADOW in the $xy$ plane of where the bug was walking. In truth the bug was walking along the surface $z = \cos(xy)$. Using the parametrization you found above, write down what the parametrization would be for the bugs position in 3D. (Hint: all you h ave to do is figure out what the $z$ coordinate is.

   (c) Write down an integral that calculates the length of the bug's journey.

   (d) Now repeat these steps for the curve below. This is another path to get from $(-1,-1)$ to $(1,1)$. Be careful finding the radius and the bounds. Set up an integral to find the length of this journey.



   The answers for the lengths should work out to be: 3.11 and 4.88

2. Find and sketch the domain of the function $f(x, y) = \dfrac{\sqrt{y-x}}{x}$. Hint: your domain should involve a dotted line for part, and a solid line for part.

3. Carefully draw a contour map for the function $f(x, y) = x^2 + y^2$ with level curves at $c = 0, 4, 8, 12$. Label your picture carefully.

4. Carefully draw a contour map for the function $f(x, y) = x + 2y - 1$ with level curves at $c = -4, -2, 0, 2, 4$. Label your picture carefully.

5. Let $f(x, y) = \dfrac{xy}{x^2 + y^2}$

   (a) Show that as $f(x, y)$ approaches 0 as $(x, y) \to (0, 0)$ along the $x$-axis.

   (b) Show $f(x, y)$ approaches 0 as $(x, y) \to (0, 0)$ along the $y$-axis.

   (c) Show $f(x, y)$ approaches something different as $(x, y) \to (0, 0)$ along the line $y = x$.

   (d) Does $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ exist? Explain your answer using complete sentences.

6. **Book Problems**

   Rogawski 15.1 (Functions of Two or More Variables) 38 and 39

   Optional: I recommend looking at Rogawski 44 - 47 for extra practice.

   Rogawski 15.2: (None)

   Rogawski 15.3 (Partial Derivatives): 3, 5, 7, 9, 10, 11, 12, 13, 14, 15, 19, 29, 31, 41, 42, 56a, 57, 59, 63,

   Optional, but recommended: 16, 17, 19, 22, 25, 37, 56, 64,

   Rogawski 15.4 (Differentiability and Tangent Planes): 1, 3, 4, 5, 11, 13, 19

**Homework 6: Due in class on Friday March 3rd**

Rogawski 15.5 (The Gradient):

I'll split these up according to the types:

Calculations, and using formulas like chain rule: (do extra of these if you want practice for the midterm)

2, 5, 9, 11, 13, 38,

Finding directional derivatives (not on exam). Again for these there is a formula (do extra if you want practice)

21, 23, 31, 33, 35

Problems about the meaning and interpretation of the gradient:

39, 40, 43

Finding a "potential function". I recommend solving these using the "guess and check method". Make sure you understand what 53 is asking.

49, 51, 53

## Homework 7: Due in class on Friday March 17th

Here are a few **required** problems from Rogawski 15.7:

**2.** Find the critical points of the functions

$$f(x, y) = x^2 + 2y^2 - 4y + 6x, \qquad g(x, y) = x^2 - 12xy + y$$

Use the Second Derivative Test to determine the local minimum, local maximum, and saddle points. Match $f(x, y)$ and $g(x, y)$ with their graphs in Figure 17.

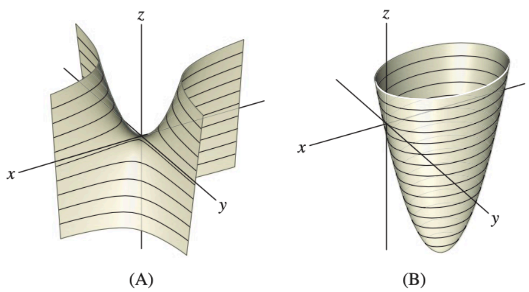



FIGURE 17

Find the critical points for the functions below. Classify them as local maxima, local minima or saddle points. Write your work very neatly and clearly.

**7.** $f(x, y) = x^2 + y^2 - xy + x$

**9.** $f(x, y) = x^3 + 2xy - 2y^2 - 10x$

### Additional Required Problems

1. Find the absolute minimum and absolute maximum of $f(x, y) = 9 - 6x + 8y$ on the closed triangular region with vertices $(0, 0)$, $(8, 0)$ and $(8, 12)$. List the minimum value as well as the coordinates where it occurs. Do the same for the maximum value.

2. Here's a problem you can solve **without** using any Calculus. Let $f(x, y) = \sin x + \cos y$. Determine the absolute maximum and minimum values of $f(x, y)$ at what points do these extreme values occur?

3. For some mystery function $f(x, y)$ you know that $f_x = x^2 - y - 4$ and $f_y = -x + y - 2$. Find the critical points of $F$ and classify each as a local max, local min, or saddle point.

4. The problems on the following page are all taken from the online book:
   `https://activecalculus.org/multi/S-10-7-Optimization.html`

   Please choose TWO of these problems to solve. For extra credit, please solve all four.

**Activity 10.7.6.** Let $f(x,y) = x^2 - 3y^2 - 4x + 6y$ with triangular domain $R$ whose vertices are at $(0,0)$, $(4,0)$, and $(0,4)$. The domain $R$ and a graph of $f$ on the domain appear in Figure 10.7.10.

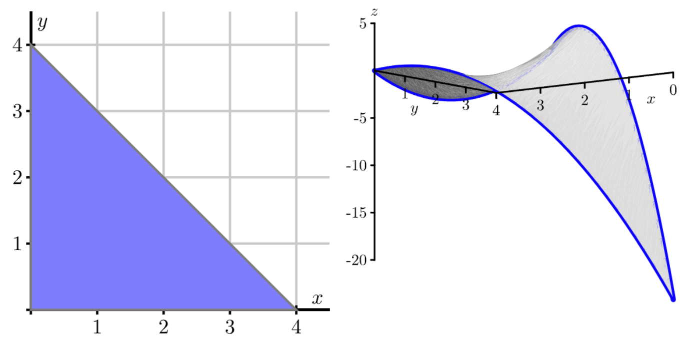



**Figure 10.7.10.** The domain of $f(x,y) = x^2 - 3y^2 - 4x + 6y$ and its graph.

a. Find all of the critical points of $f$ in $R$.
b. Parameterize the horizontal leg of the triangular domain, and find the critical points of $f$ on that leg. (Hint: You may need to consider endpoints.)
c. Parameterize the vertical leg of the triangular domain, and find the critical points of $f$ on that leg. (Hint: You may need to consider endpoints.)
d. Parameterize the hypotenuse of the triangular domain, and find the critical points of $f$ on the hypotenuse. (Hint: You may need to consider endpoints.)
e. Find the absolute maximum and absolute minimum values of $f$ on $R$.

**20.** A manufacturer wants to procure rectangular boxes to ship its product. The boxes must contain 20 cubic feet of space. To be durable enough to ensure the safety of the product, the material for the sides of the boxes will cost \$0.10 per square foot, while the material for the top and bottom will cost \$0.25 per square foot. In this activity we will help the manufacturer determine the box of minimal cost.

a. What quantities are constant in this problem? What are the variables in this problem? Provide appropriate variable labels. What, if any, restrictions are there on the variables?
b. Using your variables from (a), determine a formula for the total cost $C$ of a box.
c. Your formula in part (b) might be in terms of three variables. If so, find a relationship between the variables, and then use this relationship to write $C$ as a function of only two independent variables.
d. Find the dimensions that minimize the cost of a box. Be sure to verify that you have a minimum cost.

**22.** The airlines place restrictions on luggage that can be carried onto planes.

- A carry-on bag can weigh no more than 40 lbs.
- The length plus width plus height of a bag cannot exceed 45 inches.
- The bag must fit in an overhead bin.

Let $x$, $y$, and $z$ be the length, width, and height (in inches) of a carry on bag. In this problem we find the dimensions of the bag of largest volume, $V = xyz$, that satisfies the second restriction. Assume that we use all 45 inches to get a maximum volume. (Note that this bag of maximum volume might not satisfy the third restriction.)

a. Write the volume $V = V(x,y)$ as a function of just the two variables $x$ and $y$.
b. Explain why the domain over which $V$ is defined is the triangular region $R$ with vertices (0,0), (45,0), and (0,45).
c. Find the critical points, if any, of $V$ in the interior of the region $R$.
d. Find the maximum value of $V$ on the boundary of the region $R$, and then determine the dimensions of a bag with maximum volume on the entire region $R$. (Note that most carry-on bags sold today measure $22$ by $14$ by $9$ inches with a volume of $2772$ cubic inches, so that the bags will fit into the overhead bins.)

**21.** A rectangular box with length $x$, width $y$, and height $z$ is being built. The box is positioned so that one corner is stationed at the origin and the box lies in the first octant where $x$, $y$, and $z$ are all positive. There is an added constraint on how the box is constructed: it must fit underneath the plane with equation $x + 2y + 3z = 6$. In fact, we will assume that the corner of the box “opposite” the origin must actually lie on this plane. The basic problem is to find the maximum volume of the box.

a. Sketch the plane $x + 2y + 3z = 6$, as well as a picture of a potential box. Label everything appropriately.
b. Explain how you can use the fact that one corner of the box lies on the plane to write the volume of the box as a function of $x$ and $y$ only. Do so, and clearly show the formula you find for $V(x,y)$.
c. Find all critical points of $V$. (Note that when finding the critical points, it is essential that you factor first to make the algebra easier.)
d. Without considering the current applied nature of the function $V$, classify each critical point you found above as a local maximum, local minimum, or saddle point of $V$.
e. Determine the maximum volume of the box, justifying your answer completely with an appropriate discussion of the critical points of the function.

# Additional Optional Questions

**35.** Find the maximum of

$$f(x, y) = x + y - x^2 - y^2 - xy$$

on the square, $0 \leq x \leq 2, 0 \leq y \leq 2$ (Figure 21).

**(a)** First, locate the critical point of $f$ in the square, and evaluate $f$ at this point.

**(b)** On the bottom edge of the square, $y = 0$ and $f(x, 0) = x - x^2$. Find the extreme values of $f$ on the bottom edge.

**(c)** Find the extreme values of $f$ on the remaining edges.

**(d)** Find the largest among the values computed in (a), (b), and (c).

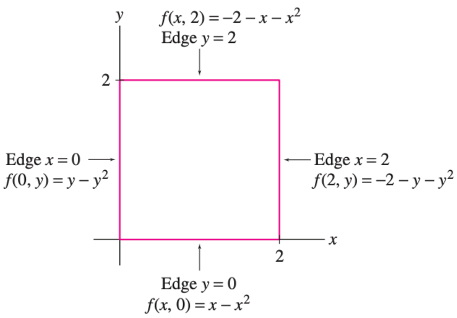



**FIGURE 21** The function $f(x, y) = x + y - x^2 - y^2 - xy$ on the boundary segments of the square $0 \leq x \leq 2, 0 \leq y \leq 2$.

**17.** Respond to each of the following prompts to solve the given optimization problem.

a. Let $f(x, y) = \sin(x) + \cos(y)$. Determine the absolute maximum and minimum values of $f$. At what points do these extreme values occur?

b. For a certain differentiable function $F$ of two variables $x$ and $y$, its partial derivatives are

$$F_x(x, y) = x^2 - y - 4 \text{ and } F_y(x, y) = -x + y - 2.$$

Find each of the critical points of $F$, and classify each as a local maximum, local minimum, or a saddle point.

c. Determine all critical points of $T(x, y) = 48 + 3xy - x^2y - xy^2$ and classify each as a local maximum, local minimum, or saddle point.

d. Find and classify all critical points of $g(x, y) = \frac{x^2}{2} + 3y^3 + 9y^2 - 3xy + 9y - 9x$

e. Find and classify all critical points of $z = f(x, y) = ye^{-x^2-2y^2}$.

f. Determine the absolute maximum and absolute minimum of $f(x, y) = 2 + 2x + 2y - x^2 - y^2$ on the triangular plate in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.

g. Determine the absolute maximum and absolute minimum of $f(x, y) = 2 + 2x + 2y - x^2 - y^2$ over the closed disk of points $(x, y)$ such that $(x - 1)^2 + (y - 1)^2 \leq 1$.

h. Find the point on the plane $z = 6 - 3x - 2y$ that lies closest to the origin.

## Homework 8: Due in class on Friday March 24th

From Rogawski Chapter 16.1

For this first one, think about the values of the function - will they be positive/negative? Is there symmetry?

**6.** For which of the following functions is the double integral over the rectangle in Figure 15 equal to zero? Explain your reasoning.

**(a)** $f(x, y) = x^2 y$ **(b)** $f(x, y) = xy^2$

**(c)** $f(x, y) = \sin x$ **(d)** $f(x, y) = e^x$

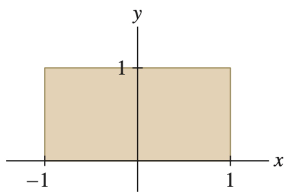



FIGURE 15

**9.** Evaluate $\iint_{\mathcal{R}} (15 - 3x)\, dA$, where $\mathcal{R} = [0, 5] \times [0, 3]$, and sketch the corresponding solid region (see Example 2).

Please carefully write down your steps as you solve these problems:

- These problems are mainly meant to help review anti-derivatives.
- Make sure your answers match those in the back of the book.
- You don't have to do them all, but do at least 5 of the 9 problems.

**19.** $\displaystyle\int_1^3 \int_0^2 x^3 y\, dy\, dx$

**21.** $\displaystyle\int_4^9 \int_{-3}^8 1\, dx\, dy$

**23.** $\displaystyle\int_{-1}^1 \int_0^\pi x^2 \sin y\, dy\, dx$

**25.** $\displaystyle\int_2^6 \int_1^4 x^2\, dx\, dy$

**27.** $\displaystyle\int_0^1 \int_0^2 (x + 4y^3)\, dx\, dy$

**29.** $\displaystyle\int_0^4 \int_0^9 \sqrt{x + 4y}\, dx\, dy$

**31.** $\displaystyle\int_1^2 \int_0^4 \frac{dy\, dx}{x + y}$

**33.** $\displaystyle\int_0^4 \int_0^5 \frac{dy\, dx}{\sqrt{x + y}}$

**45.** Evaluate $\displaystyle\int_0^1 \int_0^1 \frac{y}{1 + xy}\, dy\, dx$. *Hint:* Change the order of integration.

For extra credit: Please solve this problem and present your solution during office hours:

**48.** Prove the following extension of the Fundamental Theorem of Calculus to two variables: If $\dfrac{\partial^2 F}{\partial x\, \partial y} = f(x, y)$, then

$$\iint_{\mathcal{R}} f(x, y)\, dA = F(b, d) - F(a, d) - F(b, c) + F(a, c)$$

where $\mathcal{R} = [a, b] \times [c, d]$.

From Rogawski Chapter 16.2

**3.** Express the domain $\mathcal{D}$ in Figure 21 as both a vertically simple region and a horizontally simple region, and evaluate the integral of $f(x, y) = xy$ over $\mathcal{D}$ as an iterated integral in two ways.

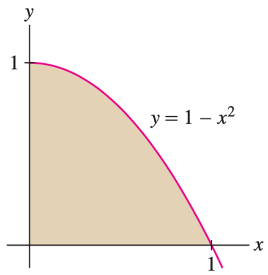



FIGURE 21

**9.** Integrate $f(x, y) = x$ over the region bounded by $y = x^2$ and $y = x + 2$.

**11.** Evaluate $\iint_{\mathcal{D}} \frac{y}{x}\, dA$, where $\mathcal{D}$ is the shaded part of the semicircle of radius 2 in Figure 23.

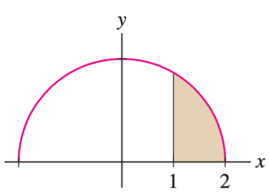



FIGURE 23 $y = \sqrt{4 - x^2}$

Set up, **but do not evaluate** the following three integrals. You can check your answers in the back of the book.

**17.** $f(x, y) = x^2 y; \quad 1 \le x \le 3, \quad x \le y \le 2x + 1$

**18.** $f(x, y) = 1; \quad 0 \le x \le 1, \quad 1 \le y \le e^x$

**19.** $f(x, y) = x; \quad 0 \le x \le 1, \quad 1 \le y \le e^{x^2}$

In Exercises 25-28, sketch the domain of integration and express as an iterated integral in the opposite order. 26 and 27 are optional.

**25.** $\displaystyle\int_0^4 \int_x^4 f(x, y)\, dy\, dx$

**26.** $\displaystyle\int_4^9 \int_{\sqrt{y}}^3 f(x, y)\, dx\, dy$

**27.** $\displaystyle\int_4^9 \int_2^{\sqrt{y}} f(x, y)\, dx\, dy$

**28.** $\displaystyle\int_0^1 \int_{e^x}^e f(x, y)\, dy\, dx$

In the following problems: Please calculate the integral of the given function over the pictured domain. You may have to set up the order of integration wisely. Numbers 40 and 42 are optional. Please calculate these integrals fully and check your answers in the back of the book.

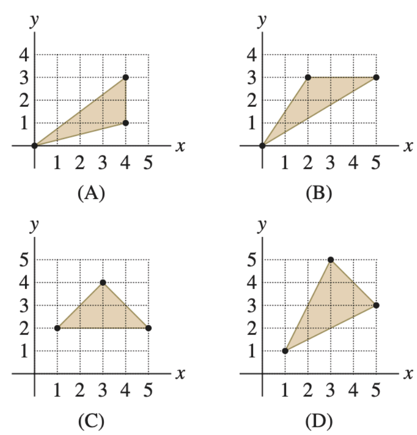



FIGURE 25

**39.** $f(x, y) = e^{x^2}$, (A)

**40.** $f(x, y) = 1 - 2x$, (B)

**41.** $f(x, y) = \dfrac{x}{y^2}$, (C)

**42.** $f(x, y) = x + 1$, (D)

**Homework 9 - Due Friday March 31st in Class**

The goal of this homework is to help you practice setting up integrals.

- There are a few problems where I want you to actually carry out the integration. Please do this carefully and show all your steps.
- If a problem asks you to draw a picture, please draw a **careful** picture. Please make your picture large, at least 3 inches by 3 inches. This might mean you have to use a bit more paper than you are used to.

1. A student wants to calculute the integral of the function $f(x, y, z) = z^4$ over the region $W = [2, 8] \times [0, 5] \times [0, 1]$.

   Set up this integral as an iterated integral and draw a careful picture of the region $W$. Use a calculator to check your setup. (Ans: 6)

2. Set up the integral of the function $x + y$ over the region $W$ described below. Set up the integral so that the shadow is in the $xy$ plane. Draw a picture of the shadow. You do not need to draw a 3D picture. (Ans: 1/6)

   $y \leqslant z \leqslant x$, $0 \leqslant y \leqslant x$, $0 \leqslant x \leqslant 1$.

3. Set up the integral of the function $xyz$ over the region $W$ described below. Set up the integral so that the shadow is in the $xy$ plane. Draw a picture of the shadow. Now draw a 3D picture of this region. Remember, draw your pictures clearly and carefully. Use Desmos to find the value of this integral.

   $0 \leqslant z \leqslant 1$, $0 \leqslant y \leqslant \sqrt{1 - x^2}$, $0 \leqslant x \leqslant 1$.

   B) Now set up this integral using polar coordinates. Convert $x$ and $y$ into polar coordinates. Now evaluate your integral again and verify that you get the same answer. (Ans: 1/16)

4. A) Using polar coordinates, set up the integral of the function $x$ over the region described below. Calculate your integral by hand and check it on Desmos. The region is the left half of the circle centered at the origin of radius 3. Draw a careful picture of the region. (Ans: -18)

   B) Recall that the center of mass of the region is given by the integral you just found, divided by the area of the region. Find the area of this region (just use the usual area formula for a circle) and use this to find the $x$ coordinate of the center of mass. Explain why the $y$-center of mass must be equal to zero. Add this to your picture. Does it look to be in the right spot?

5. (Optional) Set up the integral of the function $e^z$ over the region $W$ described below. Set up the integral so that the shadow is in the $xy$ plane. Now draw a 3D picture of this region. Draw a picture of the shadow. Remember, draw your pictures clearly and carefully. (Ans: $e - 2.5$)

   $x + y + z \leqslant 1$ and $x, y, z \geqslant 0$.

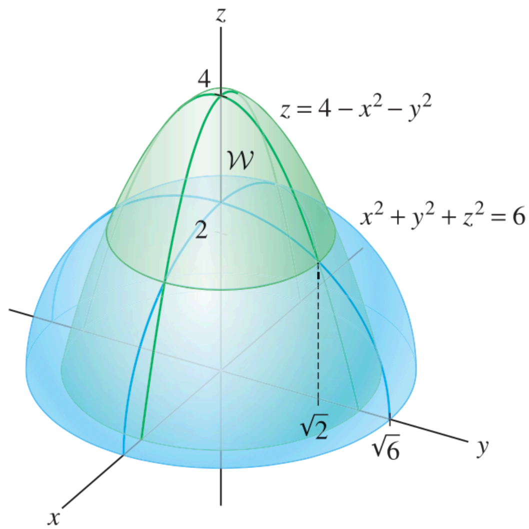
6. Set up the integral of the function $z$ over the region $W$ described below. Set up the integral so that the shadow is in the $xy$ plane. Draw a picture of the shadow. You do not need to draw a 3D picture.
   (Ans: 11/6)

   $x^2 \leqslant y \leqslant 2$, $0 \leqslant x \leqslant 1$, $x - y \leqslant z \leqslant x + y$.

7. Set up the integral that would calculate the volume of the region pictured below. This is the region **between** the parabaloid and the sphere. (Ans: 4.824)

   - Use algebra to find the equation of the curve of intersection of these surfaces. This will help you find the $xy$ shadow. (Hint: it is easier to solve for $x^2 + y^2$ in the equation $z = 4 - x^2 - y^2$ and then substitute that in the other equation.
   - Please draw your shadow in the $xy$ plane and describe it with polar coordinates.
   - Set up your integral in cylindrical coordinates (i.e. polar for $xy$) Don't forget to change your bounds, which might be in terms of $x, y$ to polar.

- Evaluate it on Desmos and check that it matches the answer.

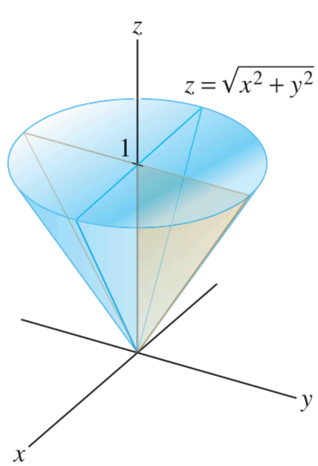



8. Let $W$ be the region inside of the cylinder $x^2 + y^2 = 4$ where $0 \leqslant z \leqslant y$. Draw a picture of this 3D solid and draw the shadow in the $xy$ plane. Set up the integral of $y^2$ over this region. (Ans: 128/15)

9. Find the volume of the solid in the octant $x, y, z \geqslant 0$ bounded by the planes $x + y + z = 1$ and $x + y + 2z = 1$. Do this by setting up the integral of the function 1. You must draw a careful 3D picture and also include a picture of the $xy$ shadow. (Ans: 1/12)

10. Recall that the equation of a sphere of radius $R$ centered at the origin is
$$x^2 + y^2 + z^2 = R^2.$$
Using this information, consider the two iterated integrals below:
$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} xy\, dz\, dy\, dx \qquad \int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_1^{\sqrt{5-x^2-z^2}} f(x, y, z)\, dy\, dx\, dz$$
For each of them, draw a 3D picture of the domain of integration. It will be some part of a sphere. Be careful with your pictures - is it all of the sphere? Part of it? Is it the upper part? The lower part? The left part?

11. One surface that will come up often in this class is the cone:
$$z = \sqrt{x^2 + y^2}$$
A picture is below:



a) The part of the cone that is pictured is given by $\sqrt{x^2+y^2} \leqslant z \leqslant 1$. Draw a picture of the shadow in the $xy$ plane of this region. Use your picture to set up the triple integral of $z$ over this cone and use Desmos to calculate this. Your setup should be in polar coordinates. (Ans: $\pi/4$)

b) Notice that in the $yz$ plane we have $x = 0$. Use this information and the equation $z = \sqrt{x^2+y^2}$ to see how this cone intersects the $yz$ plane. You should get a plus/minus in your answer.

c) Now (with the help of this picture) draw the shadow in the $yz$ plane. (Remember draw large careful pictures) and use this to set up the same integral but using the integration order $dxdydz$. Check that you get the same answer as you did in part a).

**Homework 10 - Due Friday April 14th in Class**

1. Consider the **transformation** $G : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$G(u, v) = (3u + v, u + 2v)$$

We can think of this as saying:

$$x = 3u + v, \quad y = u + 2v.$$

(a) Draw a careful picture, like we did in class that shows where the 4 corners of the unit square in the $uv$ plane will go to in the $xy$ plane. Connect the dots to form a parallelogram.

(b) What is the Jacobian of this transformation $G$? Is the Jacobian a constant or a function? In this case, the geometric meaning is that the parallelogram has area 5.

(c) Let's call the parallelogram $P$. Suppose you wanted to calculate

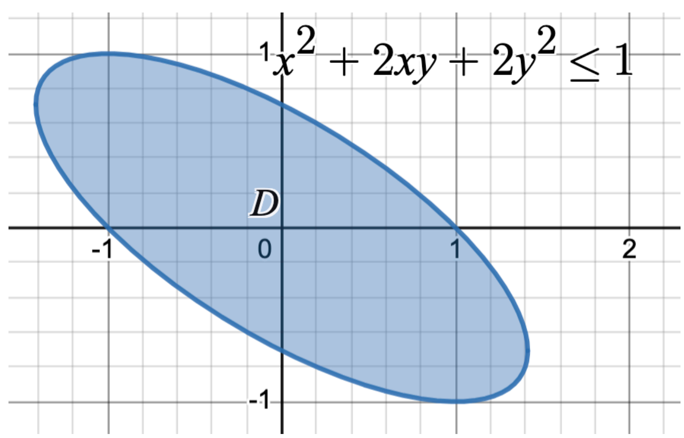
$$\iint_P xy dA$$

use the change of coordinates formula to carefully set this up. You should end up with an integral with the variables $uv$ and you should have a correction factor. Remember, we will use our picture in the $uv$ plane to set things up. Your bounds should be very simple. You can evaluate this by hand or by using Desmos. The answer should be 17.08.

2. Set up the Jacobian for the transformation of polar coordinates: $x = r\cos\theta, y = r\sin\theta$ and verify that the determinant is $r$. This is why we use $rdrd\theta$ as our correction factor.

3. Find the Jacobian for the transformation $G : \mathbb{R}^3 \to \mathbb{R}^3$ that we use for spherical coordinates. This will be a 3 by 3 determinant where the top row are the partial derivatives of $x$, the middle row for $y$ and the bottom row for $z$. This is a good review problem for calculating derivatives and using trig rules. Just take this slow and you'll get it.

4. Take a look at the shape below:




The **goal** in this problem is going to be to calculate the area of this ellipse, which is

$$A = \iint_D 1 dA.$$

If you tried to solve for $y$ in this equation in order to get our top and bottom curve, you would have to use the quadratic formula. Then the integrals would involve square roots and we'd have to use trig substitution. Yuck!

(a) Instead we're going to use a change of coordinates to make our life better. Draw two pictures, the left one being the unit circle, and the right one being this "tilted" ellipse. Label the axes of the left unit circle so it is the $uv$ plane. Shade it in and write an inequality (in terms of $u$ and $v$) that describe your region. This equation should be very simple since you have a circle!

(b) We are going to show that the transformation:

$$G(u, v) = (u - v, v)$$

will transform the unit circle into this tilted ellipse. Check where each of the 4 points $(1, 0), (0, 1)(-1, 0), (0, -1)$ on the unit circle go in your picture. You can draw arrows or color code them. You should be able to "see" the tilting.

(c) But just because we checked 4 points, doesn't mean that ALL the points will go to the blue region. To check that we're going to have to look at the inequality. Take the inequality $x^2 + 2xy + 2y^2 \leqslant 1$ and make the substitutions $x = u - v$ and $y = v$. Simplify and verify that you indeed get the interior of the unit circle in the $uv$ plane.

(d) Write down the Jacobian of the transformation $G$. (It should be a simple number)

(e) Finally, this means that you can use the change of coordinate formula to calculate the area of the ellipse. Write your steps carefully. You should get that the area is $\pi$, and the integral should be very easy to calculate since it will just be the area of the unit circle.

(f) What's the point? Sometimes a simple change of coordinates, can make things very nice. E.g. here, you had to do a bit of algebra, calculate a determinant, and got to avoid calculating this determinant here:

$$\int_{-\sqrt{2}}^{\sqrt{2}} \left( \int_{\frac{-2x-\sqrt{4x^2-8(x^2-1)}}{4}}^{\frac{-2x+\sqrt{4x^2-8(x^2-1)}}{4}} (1)\, dy \right) dx$$

$$= 3.14159265359$$

## Homework 11 - Due Friday April 21st in Class

### Some problems from 17.1

*In Exercises 13–16, match each of the following planar vector fields with the corresponding plot in Figure 10.*

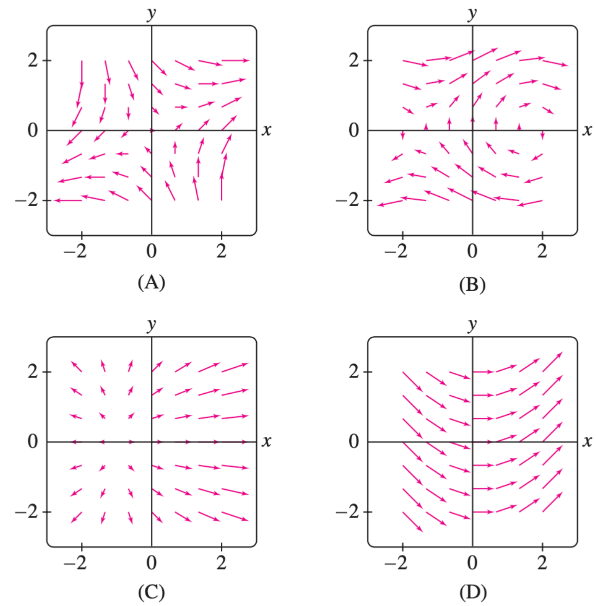



FIGURE 10

**13.** $\mathbf{F} = \langle 2, x \rangle$

**14.** $\mathbf{F} = \langle 2x + 2, y \rangle$

**15.** $\mathbf{F} = \langle y, \cos x \rangle$

**16.** $\mathbf{F} = \langle x + y, x - y \rangle$

**22.** Prove that $\mathbf{F} = \langle yz, xz, y \rangle$ is not conservative.

Note that by "prove" this means to write complete sentences explaining why e.g. it violates the Theorem we learned in class. Be specific about which part of the theorem it violates and include your conclusion.

For the vector fields below, show that they are conservative by finding a potential function. Your "reason" why these are conservative will be that you have found a potential. For each of these, after your work, please write "The potential function is $V = \ldots$ and I checked that $\nabla V = \langle \quad \rangle$.

Number 26 is optional

**23.** $\mathbf{F} = \langle x, y \rangle$

**24.** $\mathbf{F} = \langle ye^{xy}, xe^{xy} \rangle$

**25.** $\mathbf{F} = \langle yz^2, xz^2, 2xyz \rangle$

**26.** $\mathbf{F} = \langle 2xze^{x^2}, 0, e^{x^2} \rangle$

**28.** Show that $\mathbf{F} = \langle 3, 1, 2 \rangle$ is conservative. Then prove more generally that any constant vector field $\mathbf{F} = \langle a, b, c \rangle$ is conservative.

In addition, please write down the most general potential function for $\langle a, b, c \rangle$.

---

### Some problems from 17.2

The point of these first two exercise is that you will get the same answer, because what matters is the curve $C$ not how we parametrize it. In these problems you had two different parametrizations of the same **line**. You don't need to calculate the integrals, but please use Desmos or photomath to confirm you get the same value for both.

**1.** Let $f(x, y, z) = x + yz$, and let $\mathcal{C}$ be the line segment from $P = (0, 0, 0)$ to $(6, 2, 2)$.

**(a)** Calculate $f(\mathbf{c}(t))$ and $ds = \|\mathbf{c}'(t)\|\, dt$ for the parametrization $\mathbf{c}(t) = (6t, 2t, 2t)$ for $0 \le t \le 1$.

**(b)** Evaluate $\displaystyle\int_{\mathcal{C}} f(x, y, z)\, ds$.

**2.** Repeat Exercise 1 with the parametrization $\mathbf{c}(t) = (3t^2, t^2, t^2)$ for $0 \le t \le \sqrt{2}$.

This next problem is different. Remember, there are two different types of line integrals. Line integrals of scalar functions, like you did in the previous two problems, and this one where we have a **vector field** we are dotting with $ds$. Remember this means we need to dot with the derivative of the parametrization.

**3.** Let $\mathbf{F} = \langle y^2, x^2 \rangle$, and let $\mathcal{C}$ be the curve $y = x^{-1}$ for $1 \le x \le 2$, oriented from left to right.

**(a)** Calculate $\mathbf{F}(\mathbf{c}(t))$ and $d\mathbf{s} = \mathbf{c}'(t)\,dt$ for the parametrization of $\mathcal{C}$ given by $\mathbf{c}(t) = (t, t^{-1})$.

**(b)** Calculate the dot product $\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\,dt$ and evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$.

Here is a mixture of some problems where you have to calculate integrals either of a scalar function or of a vector field. **You only need to set these up as integrals, no need to evaluate them. The point is to get practice setting these up.** Please do **5, 7, 11, 13, 17, 19, 21, 27, 29, 35, 37, 38, 41, 42,44**. The others are optional.

*In Exercises 5–8, compute the integral of the scalar function or vector field over $\mathbf{c}(t) = (\cos t, \sin t, t)$ for $0 \le t \le \pi$.*

**5.** $f(x, y, z) = x^2 + y^2 + z^2$

**6.** $f(x, y, z) = xy + z$

**7.** $\mathbf{F} = \langle x, y, z \rangle$

**8.** $\mathbf{F} = \langle xy, 2, z^3 \rangle$

*In Exercises 9–16, compute $\int_{\mathcal{C}} f\,ds$ for the curve specified.*

**9.** $f(x, y) = \sqrt{1 + 9xy}$, $\quad y = x^3$ for $0 \le x \le 1$

**10.** $f(x, y) = \dfrac{y^3}{x^7}$, $\quad y = \frac{1}{4}x^4$ for $1 \le x \le 2$

**11.** $f(x, y, z) = z^2$, $\quad \mathbf{c}(t) = (2t, 3t, 4t)$ for $0 \le t \le 2$

**12.** $f(x, y, z) = 3x - 2y + z$, $\quad \mathbf{c}(t) = (2 + t, 2 - t, 2t)$ for $-2 \le t \le 1$

**13.** $f(x, y, z) = xe^{z^2}$, piecewise linear path from $(0, 0, 1)$ to $(0, 2, 0)$ to $(1, 1, 1)$

**17.** Calculate $\int_{\mathcal{C}} 1\,ds$, where the curve $\mathcal{C}$ is parametrized by $\mathbf{c}(t) = (4t, -3t, 12t)$ for $2 \le t \le 5$. What does this integral represent?

**19.** $\mathbf{F} = \langle x^2, xy \rangle$, $\quad$ line segment from $(0, 0)$ to $(2, 2)$

**20.** $\mathbf{F} = \langle 4, y \rangle$, $\quad$ quarter circle $x^2 + y^2 = 1$ with $x \le 0$, $y \le 0$, oriented counterclockwise

**21.** $\mathbf{F} = \langle x^2, xy \rangle$, $\quad$ part of circle $x^2 + y^2 = 9$ with $x \le 0$, $y \ge 0$, oriented clockwise

**22.** $\mathbf{F} = \langle e^{y-x}, e^{2x} \rangle$, $\quad$ piecewise linear path from $(1, 1)$ to $(2, 2)$ to $(0, 2)$

**23.** $\mathbf{F} = \langle 3zy^{-1}, 4x, -y \rangle$, $\quad \mathbf{c}(t) = (e^t, e^t, t)$ for $-1 \le t \le 1$

**24.** $\mathbf{F} = \left\langle \dfrac{-y}{(x^2 + y^2)^2}, \dfrac{x}{(x^2 + y^2)^2} \right\rangle$, $\quad$ circle of radius $R$ with center at the origin oriented counterclockwise

**25.** $\mathbf{F} = \left\langle \dfrac{1}{y^3 + 1}, \dfrac{1}{z + 1}, 1 \right\rangle$, $\quad \mathbf{c}(t) = (t^3, 2, t^2)$ for $0 \le t \le 1$

*In Exercises 27–32, evaluate the line integral.*

**27.** $\int_{\mathcal{C}} y\,dx - x\,dy$, $\quad$ parabola $y = x^2$ for $0 \le x \le 2$

**28.** $\int_{\mathcal{C}} y\,dx + z\,dy + x\,dz$, $\quad \mathbf{c}(t) = (2 + t^{-1}, t^3, t^2)$ for $0 \le t \le 1$

**29.** $\int_{\mathcal{C}} (x - y)\,dx + (y - z)\,dy + z\,dz$, $\quad$ line segment from $(0, 0, 0)$ to $(1, 4, 4)$

**30.** $\int_{\mathcal{C}} z\,dx + x^2\,dy + y\,dz$, $\quad \mathbf{c}(t) = (\cos t, \tan t, t)$ for $0 \le t \le \frac{\pi}{4}$

**31.** $\int_{\mathcal{C}} \dfrac{-y\,dx + x\,dy}{x^2 + y^2}$, $\quad$ segment from $(1, 0)$ to $(0, 1)$.

**32.** $\int_{\mathcal{C}} y^2\,dx + z^2\,dy + (1 - x^2)\,dz$, $\quad$ quarter of the circle of radius 1

*In Exercises 35 and 36, calculate the line integral of $\mathbf{F} = \langle e^z, e^{x-y}, e^y \rangle$ over the given path.*

**35.** The blue path from $P$ to $Q$ in Figure 14

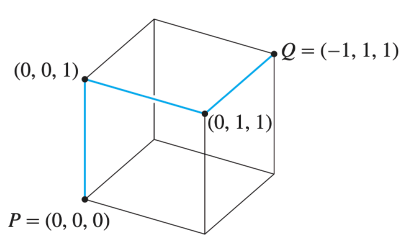



**FIGURE 14**

*In Exercises 37 and 38, $\mathcal{C}$ is the path from $P$ to $Q$ in Figure 16 that traces $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ in the orientation indicated, and* **F** *is a vector field such that*

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 5, \qquad \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = 8, \qquad \int_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s} = 8$$

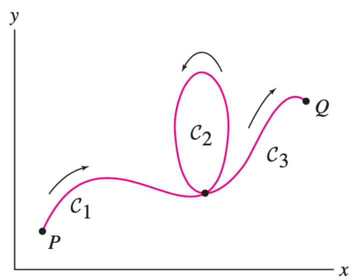



FIGURE 16

**37.** Determine:

**(a)** $\displaystyle\int_{-\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s}$ **(b)** $\displaystyle\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s}$ **(c)** $\displaystyle\int_{-\mathcal{C}_1-\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s}$

**38.** Find the value of $\displaystyle\int_{\mathcal{C}'} \mathbf{F} \cdot d\mathbf{s}$, where $\mathcal{C}'$ is the path that traverses the loop $\mathcal{C}_2$ four times in the clockwise direction.

**41.** Determine whether the line integrals of the vector fields around the circle (oriented counterclockwise) in Figure 19 are positive, negative, or zero.

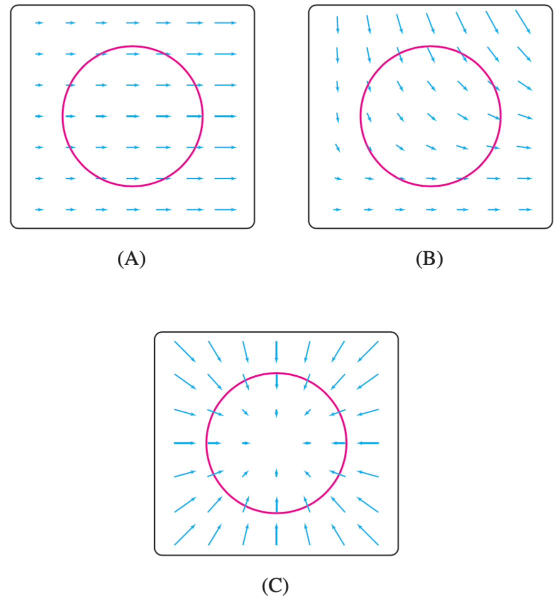


FIGURE 19

**42.** Determine whether the line integrals of the vector fields along the oriented curves in Figure 20 are positive or negative.

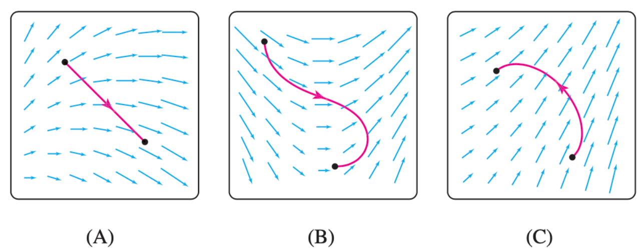


FIGURE 20

**44.** Calculate the total mass of a metal tube in the helical shape $\mathbf{c}(t) = (\cos t, \sin t, t^2)$ (distance in centimeters) for $0 \le t \le 2\pi$ if the mass density is $\rho(x, y, z) = \sqrt{z}$ g/cm.

**Homework 12 - Due Friday April 28th in Class**

1. The problems at the right are from Rogawski section 18.1.

   **Please solve 3,5,7**

   To solve these problems, please clearly write your steps and when you set up the double integral, use either rectangular or polar coordinates. Please check that your setup gives the answer in the back of the book.

*In Exercises 3–10, use Green's Theorem to evaluate the line integral. Orient the curve counterclockwise unless otherwise indicated.*

**3.** $\oint_C y^2\,dx + x^2\,dy$, where $C$ is the boundary of the unit square $0 \le x \le 1, 0 \le y \le 1$

**4.** $\oint_C e^{2x+y}\,dx + e^{-y}\,dy$, where $C$ is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$

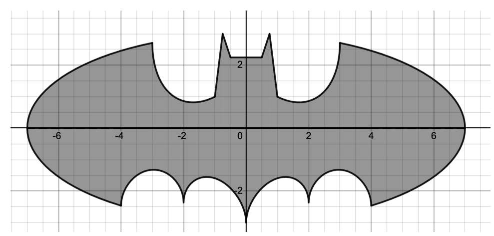
**5.** $\oint_C x^2\,y\,dx$, where $C$ is the unit circle centered at the origin

**6.** $\oint_C \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F} = \langle x + y, x^2 - y\rangle$ and $C$ is the boundary of the region enclosed by $y = x^2$ and $y = \sqrt{x}$ for $0 \le x \le 1$

**7.** $\oint_C \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F} = \langle x^2, x^2\rangle$ and $C$ consists of the arcs $y = x^2$ and $y = x$ for $0 \le x \le 1$

2. Parametrize the right side of the rectangle $R$ described below: $R$ has lower left vertex $(1, 3)$ and upper right corner $(c, d)$.

3. Let $D$ be the region shaded below and suppose that $C$ is the boundary of $D$ oriented counterclockwise. Suppose that you know that this region has area 20 and its center of mass is at the point $(0, -0.3)$.




   (a) Find the following double integrals:

$$\iint_D dA, \qquad \iint_D x\ dA, \qquad \iint_D y\ dA.$$

   Hint: All of your answers should be whole numbers. Make sure you understand how to do these sorts of problems.

   (b) Now use the information to calculate the following line integral using **Green's Theorem**.

$$\oint_C \langle \cos(x^7) - 13y^2, e^{y^3 \cos y} + 2x - 4x^2\rangle \cdot \mathbf{ds}.$$

4. Consider the integral: $\int_C \cos y\ dx\ - x \sin y\ dy$

   where $C$ is the upper half of the unit circle centered at the origin, oriented counterclockwise.

   (a) Would using Green's Theorem or The "Fundamental Theorem for Gradients" apply? (Only one of them does - which one and why)? Use your method to find the answer.

   (b) In part a, you should have discovered that the vector field was conservative, which means that the path from START to END didn't matter, just the endpoints. Verify this, by calculating the line integral from $(1, 0)$ to $(-1, 0)$ along the straight line. You should notice that your integral is very simple. Confirm you get the same answer as in part a).

5. Find $\oint_C (e^{3x} + 2y)\, dx + (x^2 + \sin y)\, dy$ where $C$ is rectangle with vertices $(2,1), (6,1), (6,4)$, and $(2,4)$. You may use any method you like.

6. Suppose that $D$ is a region in the $xy$ plane. You know that it is symmetric about the $x$ axis. You are given the following information:

$$\text{Area}(D) = 12, \qquad \iint_D 3 + 5x + 4y = 60.$$

Find the coordinates of the center of mass of $D$.

7. A blob $D$ whose boundary is a simple closed curve $C$ has center of mass $(4,5)$ and area 9. Use this information to calculate the following integral, assuming that $C$ is oriented counterclockwise:

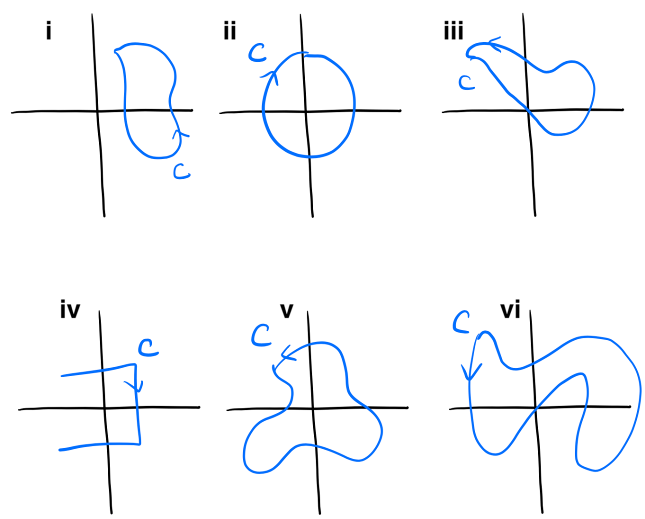
$$\oint_C \langle x^3 - 3x + 6y^2, 3x^2 + 4y^2 \rangle \cdot \mathbf{ds}.$$

This is similar to the Batman problem. Please do this carefully and make sure you understand how everything fits together. You can email me with your answer and I will tell if you if it is correct.

8. One important hypothesis that is necessary for Green's Theorem is that the partial derivatives of $F_1$ and $F_2$ be continuous at every point in $D$. In this example we will see what can happen if this isn't the case.

   (a) Let $\mathbf{F} = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$. There is one point in the plane where this vector field is undefined. Which one?

   (b) So in terms of the this vector field, there is only one point where we can't talk about it. This means that it's possible that $\int_C \mathbf{F} \cdot \mathbf{ds}$ might NOT be defined for some curves. For which of the following curves, would this integral NOT be defined. Explain your reasoning.



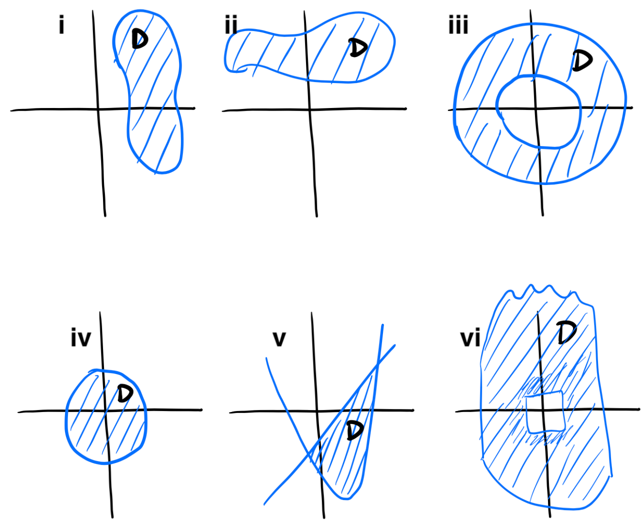

   (c) If $F_1$ and $F_2$ are the components of the vector field above, calculate

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

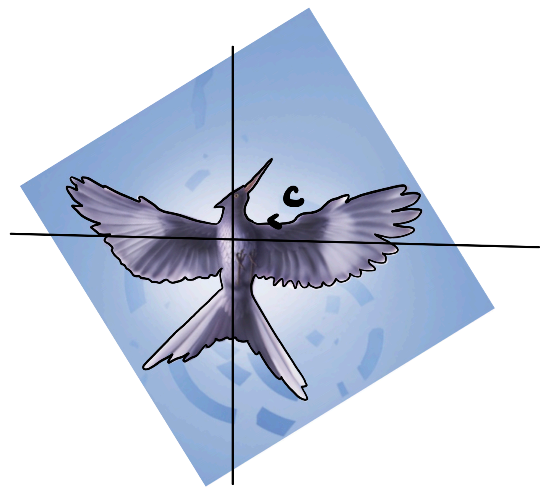
   and verify that it is zero at every point where it is defined. After your calculation, please write the sentence: "I just calculated and simplified and see that whenever $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ is defined, it is zero. But it is not defined at the origin. There is some 'singularity' happening at the origin."

   (d)

Now if we wanted to calculate $\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$ over some region, this will either be undefined (if there's a point in $D$ where we have trouble) or else if it's defined everywhere, the integral will just be $\iint_D 0 \, dA$ which is zero. For each of the regions $D$ below, say whether the integral would be undefined or zero. In the cases where the integral is defined, notice that you **could** use Green's Theorem on this region.

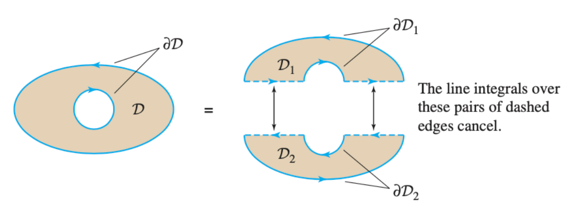



(e) Now consider the curve $C$ pictured below, (the boundary of the bird) which was on the cover of the 2010 Scifi novel Mockingjay from the Hunger Games series:




The integral $\int_C \mathbf{F} \cdot \mathbf{ds}$ IS defined for this curve $C$. Why? (This connects with part b) but we cannot use Green's Theorem to calculate it. Why? (This connects with part c).

(f) (Extra Credit) Calculate the integral of the vector field from this problem over the Mockingjay symbol. Hint: There's no way you can parametrize the Mockingjay symbol. However, there is a way to solve this problem using what can be called "surgery" or "cut and paste". You will have to introduce a new small circle into the picture and then use the facts you developed in parts a) -d). Please read the section "More General Form of Green's Theorem" in section 18.1 for hints, including this picture:




You can work on this problem until the last day of class. Please present your solution in my office sometime to receive the extra credit.

**Homework 13 - Due Friday May 5th in Class**

Setting up surface integrals can be a LOT of work. The purpose of this homework is to help give you practice. For each of these problems your goal is to set up the surface integral. For some of these it is for a vector field and for others it is for a scalar field. There are also a lot of webwork problems this week **also due Friday.** I encourage you to think of these as one combined assignment.

1. Suppose that $D$ is a region in the $xy$ plane that has area 5. What is the area of the portion of the plane $2x + 3y + 4z = 28$ that lies directly above $D$? Set this up carefully by parametrizing $S$. Write your parametrization $G$ and use it to calculate a surface integral that will give you your answer.
2. The surface area of a sphere of radius $R$ is $4\pi R^2$. Use this to answer the following questions:
   (a) Let $S$ be the sphere $x^2 + y^2 + z^2 = 25$. What is $\iint_S 1\, dS$? (Your answer should be a number)
   (b) A very large ball of radius 4 meters is dusted irregularly with gold. If $f(x, y, z)$ represents the density of gold (in $g/m^2$), then $\iint_S f(x, y, z)\, dS$ represents what property of this object?
   (c) (Continuing part b) If we take 8 samples, one from each octant of the sphere, and measure the densities to be: $4, 1, 3, 6, 2, 2, 1, 5$ $(g/m^2)$. Estimate the total amount of gold on the ball.
   (d) By using spherical coordinates calculate the surface area of a sphere of radius $R$. I want you to fully calculate the integral and confirm you get $4\pi R^2$.
   (e) Find the area of the small sliver of the sphere that is around the north pole that is determined by the angle $\phi = \alpha$. Show that the area of this region has area $2\pi R^2(1 - \cos\alpha)$. Yes again, I want you to fully evaluate this integral. **You will use this in the next problem, which is about goats.**
3. (Goats!) This problem is about goats, which are an important part of our world. As a kid, I had two goats, named Brady and Bo. They liked to eat all of the grass/weeds they could reach. Sometimes we'd put them on a leash and let them eat all the weeds under our deck. They also would eat the pine needles off of our Christmas trees - this seemed to be their favorite food. In these problems goats will be given four different regions, creatively called $A, B, C, D$, to roam.
   - Your goal is to figure out the areas of these regions. Clearly indicate the areas and then put them in order from smallest to largest.
   - For each description, draw the best picture you can and work out the area by using $\iint_S dS$. **Your answers will be in terms of $b$ and $a$.**
   - This is a challenging problem, but you can do it!
   - Region $A$ is the part of the flat plane that a goat attached to a leash of radius $b$ can reach. Draw a picture (note, you shouldn't need calculus to answer this one!)
   - In Region $B$, the goat is attached to the same leash of radius $b$, but in this case, the goat is walking on the outside of a sphere of radius $a$. You can assume that $b < a$. Draw a careful picture with your goat stationed at the north pole and see how this can help you figure out how to set up this integral.
   - In Region $C$, the goat has found its way inside of that sphere of radius $a$ and is still on that leash of radius $b$.
   - The goat in region $D$ has decided to break free from its leash and has instead been placed inside of a small ring of radius $b$ that has been placed on top of the sphere of radius $a$. (Imagine a crown of radius $b$ being placed on top of the sphere.)
4. (Optional - Extra Practice)

(a) Set up a double integral that calculates the surface area of the part of the cone $x^2 + y^2 = z^2$ between the planes $z = 2$ and $z = 5$.

(b) Use cylindrical coordinates to find the area of a sphere of radius $R$.

# Part III

# Quizzes

**Quiz 1 - Good Luck Everyone!** **Name:** ____________________

1. Find the derivatives: $\boxed{\text{You do NOT need to simplify your answers}}$

   (a) $\frac{d}{dx}(\sin(x \ln x)) =$

   (b) $\frac{d}{dx}(\frac{x+\pi}{x^2+e^x}) =$

   (c) $\frac{d}{dx}(\sin^3(x)\cos(5x)) =$

2. Find the following anti-derivatives. Don't forget the $+C$

   (a) $\int (3x+2x^2)(1-\frac{1}{x^4})\ dx =$

   (b) $\int \frac{1}{\sqrt{2x+3}}\ dx =$

   (c) $\int \frac{1}{x \ln x} dx =$

Please turn over the page for problem 3:

3. A student graphs a function $y = f(x)$ and draws the tangent line at the point $(2, 1)$ and gets the line

$$y - 1 = -3(x - 2).$$

(a) This means that $f'(2) =$ ______________.

(b) Set up (but do not evaluate) a definite integral that calculates the area in the first quadrant (that is, $x, y \geqslant 0$) bounded between the $x$-axis and the line given above. A picture will probably help!

**Makeup Quiz 1 - Good Luck Everyone!** **Name:** ____________________

1. Find the derivatives: $\boxed{\text{You do NOT need to simplify your answers}}$

   (a) $\dfrac{d}{dx}(\sqrt{2x+5}) =$

   (b) $\dfrac{d}{dx}(\sin(\sqrt{x^2+7})) =$

   (c) $\dfrac{d}{dx}(\tan^5(x)\cos(x)) =$

   (d) $\dfrac{d}{dx}(x^{-5/2}) =$

2. Find the following anti-derivatives. Don't forget the $+C$

   (a) $\displaystyle\int x^2\left(\frac{3}{x^4}+\frac{2}{x^3}\right)\,dx =$

   (b) $\displaystyle\int \frac{1}{\sqrt{2x+3}}\,dx =$

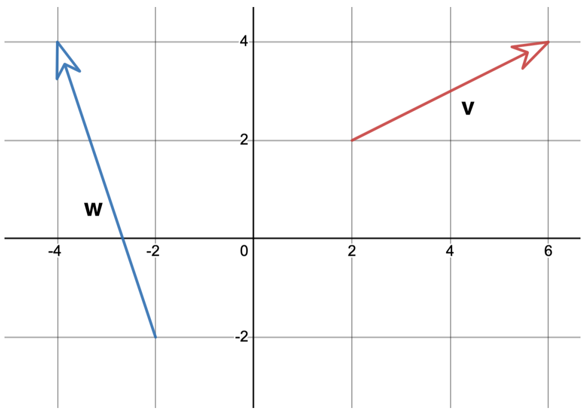
   (c) $\displaystyle\int xe^{4x^2}\,dx =$

**Quiz 2 - Good Luck Everyone!** **Name:** ______________________________

1. (1 point) In class, we learned that one of the simplest things to parametrize is the graph of a function. Fill in the blanks to write down a parametrization of the curve given by $f(x) = e^x + x^2$

   Answer: $c(t) =$ (________, ______________) with $-\infty < t < \infty$.

2. (4 points) Consider the two vectors $\mathbf{v}, \mathbf{w}$ in the picture below.




   (a) Find the position vectors (with the $\langle \ , \ \rangle$ brackets) for these vectors and write them down.

   $\mathbf{v} = \langle \qquad , \qquad \rangle$

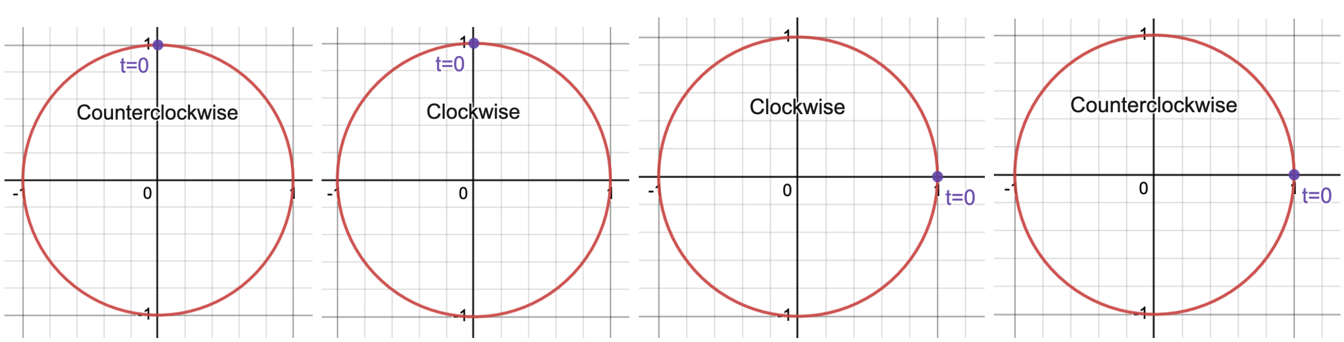
   $\mathbf{w} = \langle \qquad , \qquad \rangle$

   (b) Find $\mathbf{v} \cdot \mathbf{w}$.

3. (2 points) Let $\mathbf{v} = \langle 3, 4 \rangle$. If $\mathbf{w}$ is a vector of length 3 such that the angle between $\mathbf{w}$ and $\mathbf{v}$ is 60 degrees, what is $\mathbf{w} \cdot \mathbf{v}$?

4. (1 point) A student wants to use the formula $P + t\mathbf{v}$ to parametrize the line through the point $(1, 3, 2)$ that is parallel to the line through the points $(3, 1, 4)$ and $(5, 7, 8)$. What would $\mathbf{v}$ be in this problem? (There are multiple correct answers)

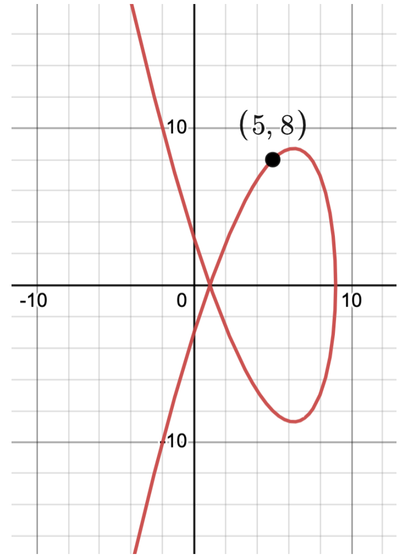
5. (1 point) The parametric curve $(\sin t, \cos t)$ corresponds to which of the curves below? (The word in the center describes the way the curve is traversed as $0 \leqslant t \leqslant 2\pi$.



6. Consider the parametric curve described by

$$c(t) = \left(9 - t^2, 8t - t^3\right).$$

A graph is given at the right




(a) (4 points) Find the equation of the tangent line at the point $(5, 8)$. **Show your work clearly.**

(b) (1 point) On the graph, if we were to animate this graph, say starting with $t = -1000$ and going to $t = 1000$ in which direction would this curve be traced? (Please draw an arrow to indicate your answer).

7. (Extra Credit) Determine (with explanation) whether or not the line parametrized by $(1 - t, 2 - t, 8 - 2t)$ intersects the $x$-axis.

**Quiz 3 - Good Luck Everyone!** **Name:** ______________________________

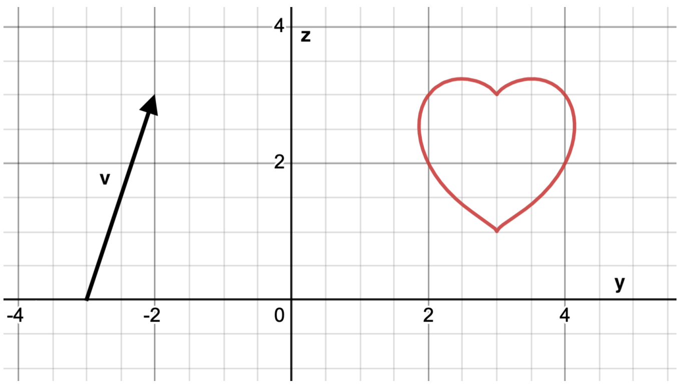
Extra Credit:

This quiz has 16 possible points but will be graded out of a total of 15.

1. (1 points)

   The normal vector to the plane $3x - 2y + 5z = 1$ is $\langle \qquad\qquad \rangle$.

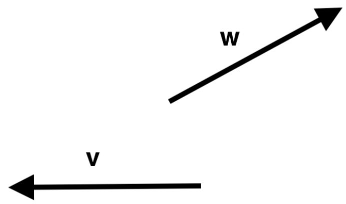
2. (3 points)




Cupid shoots an arrow in the direction of the vector **v** which lies in the $yz$ plane. Find the equation of the plane through the the **origin** that is perpendicular to the vector **v**. No need to simplify your answer, just fill in the blanks below:

$$(x- \quad) + \quad (y- \quad) + \quad (z- \quad) =$$

3. (6 points) Find the equation of the plane that contains the point $(1, 4, 5)$ and the line given by $\mathbf{r}(t) = \langle 3t, 4t, 5t \rangle$. Hint: Draw a picture.




4. (6 points) Suppose that $\mathbf{v}, \mathbf{w}$ are the two vectors in the picture above.

Then $\mathbf{v} \times \mathbf{w}$ points [out of the page / into the page] (Circle one)

How many vectors are perpendicular to both $\mathbf{v}, \mathbf{w}$? [One/Two/ Three/More than Three] (Circle one)

How many **unit** vectors are perpendicular to both $\mathbf{v}, \mathbf{w}$? [One/Two/ Three/More than Three] (Circle one)

Which of the following would necessarily be a unit vector perpendicular to both $\mathbf{v}, \mathbf{w}$: (Circle all that apply):

(a) $\dfrac{\mathbf{w} \times \mathbf{v}}{||\mathbf{v}||||\mathbf{w}||}$

(b) $-\dfrac{\mathbf{w} \times \mathbf{v}}{||\mathbf{v}||||\mathbf{w}||}$

(c) $\dfrac{\mathbf{w} \times \mathbf{v}}{||\mathbf{w} \times \mathbf{v}||}$

(d) $-\dfrac{\mathbf{w} \times \mathbf{v}}{||\mathbf{w} \times \mathbf{v}||}$

(e) $\dfrac{\mathbf{v} \times \mathbf{w}}{||\mathbf{v} \times \mathbf{w}||}$

(f) $\langle 0, 0, 0\rangle$.

**Quiz 4 - Good Luck Everyone!** **Name:** ____________________

1. A particle is moving along the path with parametrization:

$$\mathbf{r}(t) = \langle 3 - t^2 + 4t, 3t - 12, t^3 - 18t \rangle, \qquad -\infty < x < \infty$$

   (a) (3 points) At what time(s) is the particle at rest? If none, explain why not.

   (b) (3 points) At what point(s) does the particle cross the plane $xz$ plane?

2. Find parametrizations for the following:

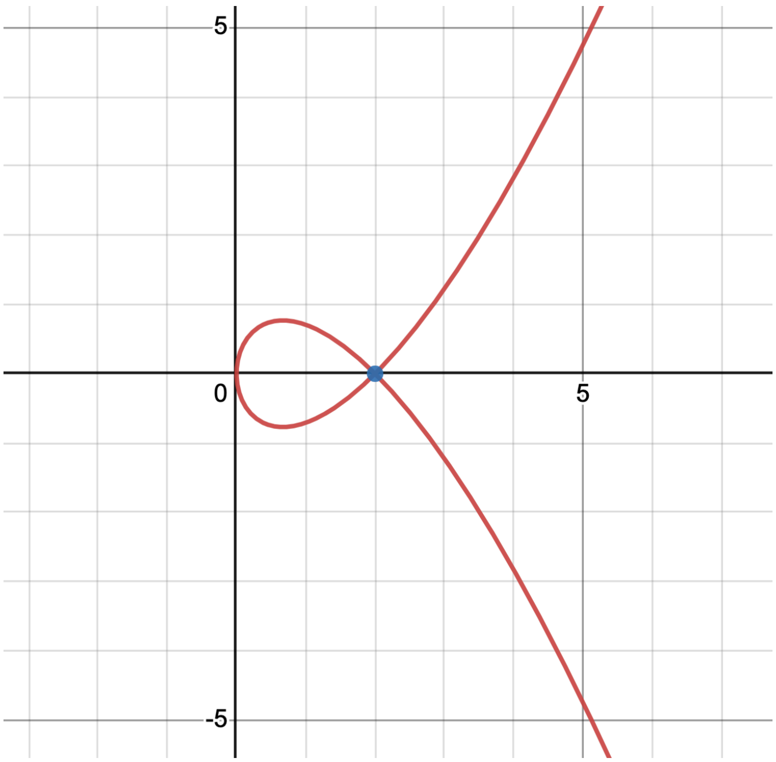
   (a) (3 points) The circle that is parallel to the $yz$ plane that has radius 4 and center at the point $(1, 1, 1)$? (Include your range for $t$)

   (b) (3 points) The intersection of the cylinder $x^2 + y^2 = 25$ with the plane $x + y + z = 4$. (Include your range for $t$.)

3. (6 points) Below is the graph of the curve with parametrization

$$\mathbf{r}(t) = (2t^2, 2t^3 - 2t), \quad -\infty < x < \infty$$

Showing your steps carefully, find the tangent vectors at the point $(2, 0)$ and draw them in as vectors starting at $(2, 0)$. Hint: there are two tangent vectors that happen at two different times.



**Quiz 5 - Good Luck Everyone!** **Name:** ______________________

This is a group quiz!

1. A student is trying to classify the critical points of a function $f(x, y)$. The student works out that

$$\frac{\partial f}{\partial x} = 3x^2 - 3 + 3y^2, \qquad \frac{\partial f}{\partial y} = 6xy.$$

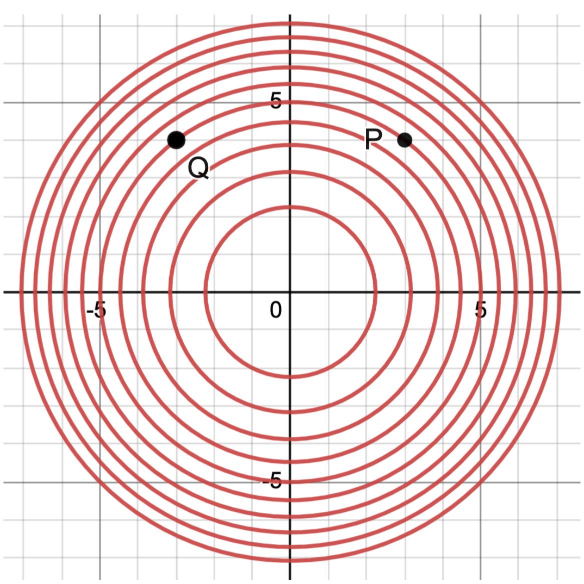
And finds four critical points, $(0,1), (0,-1), (1,0), (-1,0)$.

(a) (3 points) Write down the Hessian matrix $H$ in terms of $x$ and $y$.

(b) (2 points) Determine whether the point $(-1, 0)$ is a local max, local min, saddle point. Explain your answer briefly.

(c) (2 points) Determine whether the point $(0, -1)$ is a local max, local min, saddle point. Explain your answer briefly.

(d) (2 points) What is $\nabla f$ at the point $(1, 1)$?



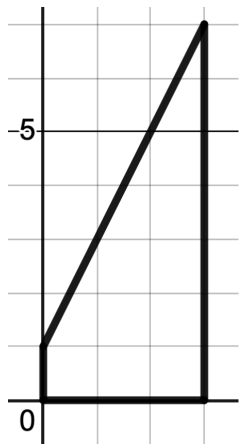

2. (4 points) The contour map of a function $f(x, y)$ is given in the picture to the left, you are given that the point $(0, 0)$ is a local maximum. Then

$f_x(P)$ is positive/negative/zero

$f_y(P)$ is positive/negative/zero

$f_x(Q)$ is positive/negative/zero

$f_y(Q)$ is positive/negative/zero




3. (11 points) A student is trying to find the global maximum and global minimum temperature on the region indicated at the left.

The equation for temperature is given by

$$T(x, y) = 12x - y^2$$

In order to find the global min and max of the temperature on this region,

you would start by finding the ________________________ which are obtained by setting the partial derivatives equal to 0. You would then find the temperature at these points.

Then you would study the ________________________ (8 letter word with a y in it) of this region, which has 4 pieces.

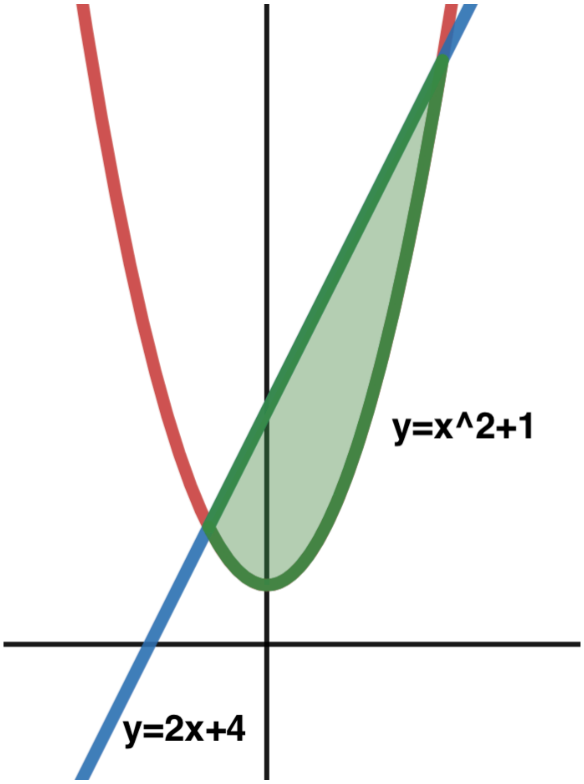
($\star$) On each piece you would parametrize the edge and then optimize the function, meaning you would note its min and max. You would record where these happen.

All together this problem would then have 5 steps, plus a conclusion.

On this problem I am only asking you to do step ($\star$) for the top left edge. To receive full credit you must include a sketch of a parabola like we did in class. Please write complete sentences including a conclusion: "On this edge the minimum of .... occurs at the point .... " (and similarly for the maximum). You might want to label where the min and max occur on the picture above, though that is not necessary.

**Quiz 6 - Good Luck Everyone!** **Name:** ______________________________

1. Consider the region graphed below:




Set up **but do not evaluate** a double integral of the function $f(x, y) = x + 2y$ over the region $D$. You may use $dy\ dx$ or $dx\ dy$ setup.

2. In the following integral, please

   **A) sketch the domain of integration**

   **B) reverse the order of integration.** By this I mean write an iterated integral of the form $dy\ dx$.

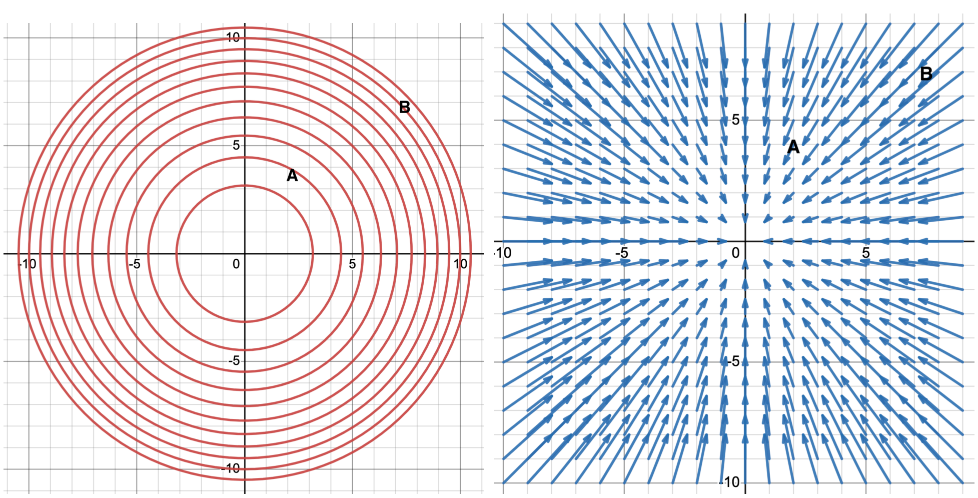
$$\int_1^4 \int_{1-y}^{y-1} f(x,y)\ dx\ dy.$$

**Quiz 7 - Good Luck Everyone!** **Name:** ____________________

1. (6 points) One of the following two vectors fields is conservative and the other is not. Determine which is which. For the conservative one, I want you to find the potential. For the other one, you should explain why you know it cannot be conservative like we did in class.

$$\mathbf{F} = \langle xy^2, x^2 \rangle, \qquad \mathbf{G} = \langle 16x + 4y, 4x + 6y \rangle$$

2. (2 points) Below are two pictures. One is the contour map of a function $T(x, y)$ that gives the temperature. The other is the vector field given by its gradient $\nabla T$. Using complete sentences, explain whether the temperature at point $A$ or point $B$ must be greater (or whether we cannot tell).



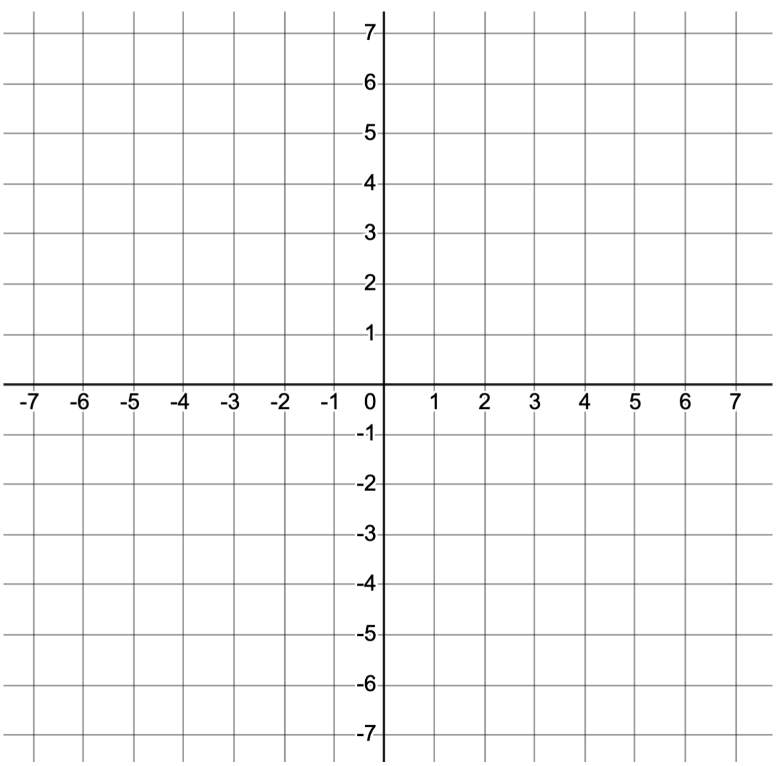
3. (10 points) A researcher is walking around the rainforest. At each point $(x, y)$, she measures an oxygen density given by $f(x, y) = 2 + x^2 + y^2$ and a temperature $T(x, y) = 30 + \sqrt{x^2 + 2y^2}$ in degrees Celsius. She also measures the wind, and determines that it is given by the vector field $\mathbf{W} = \langle -y, x - 1 \rangle$. One day the researcher is walking along the path with parametrization $\mathbf{c}(t) = \left(t^2, 3t - t^3\right)$. Use the above information to answer the following:

(a) What is the wind vector at the point $(4, 2)$? ______________________.

(b) Draw this vector in the picture (start the vector at the point $(4, 2)$ and label it carefully as $\vec{W}$. You must use the vector notation to get full credit.

(c) Find the velocity vector for the researcher's path at the point $(4, 2)$. Circle your answer and add it to the picture with an appropriate label.

(d) At time $t = 1$ what is the temperature where the researcher is? (Indicate this by putting a point at the appropriate part of the graph below and labeling it with the "Temp ="



**Quiz 8 - Good Luck Everyone!** **Name:** ________________________________

1. (2 points) Antoine the Ant is walking along the curve described below:

   $\mathbf{c}(t) = (t^2 + 5, t^3 - 6), \quad 1 \leqslant t \leqslant 2.$

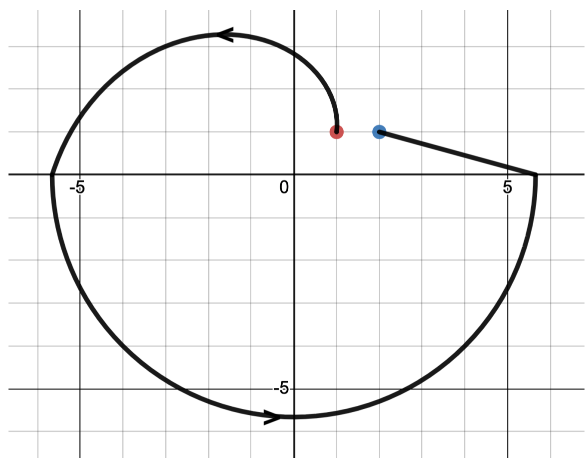
   $\mathbf{c}'(t) = \langle 2t, 3t^2 \rangle$

   Find the coordinates of the start and end of Antoine's path. Write your answers on the blanks:

   START: END :

---

2. (5 points) Evaluate the following line integral by finding a potential and using the Fundamental Theorem for Gradients. Simplify all answers completely.



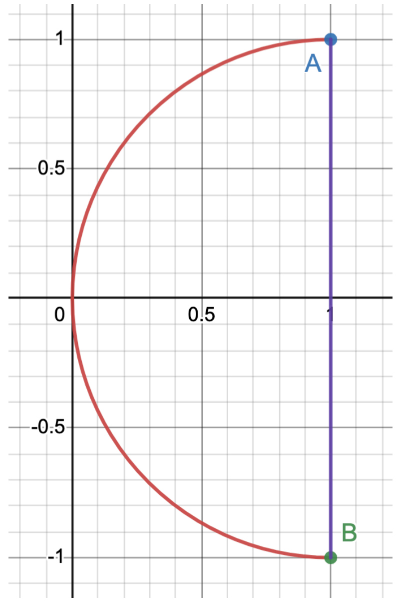

$$\int_C (2x + y + y^2)\ dx + (x + 2xy)\ dy$$

where $C$ is the curve pictured at right:

3. John B. is walking in the wind, which is given by the vector field:

$$\mathbf{W} = \langle y, 2x \rangle.$$

He is walking along the closed path (walking counterclockwise) in the figure to the right. This path consists of two different curves, a straight line and an arc of a circle. Please set up and **fully evaluate** the line integral of $\mathbf{W}$ along the **straight line** with the orientation described.



**Quiz 9 - Good Luck Everyone!** **Name:** ______________________________

1. (12 points) Let $C$ be the boundary of the circle $x^2 + y^2 = 9$ oriented **clockwise.**

   - Use Green's Theorem to evaluate $\oint_C (e^{\cos x} - 5y)\, dx + \ (3x - e^{y^3})dy$. You must simplify your answer completely. You may evaluate your integral or appeal to area / center of mass reasons, but you must show all your work and write in complete sentences. Any symbols you use should be clearly defined. Defining with a picture is ok.

2. (12 points) Short Answer

(a) If $D$ is a region with area 6, then $\iint_D 2\ dA = 12$ [True/False]

(b) If $D$ is a region with area 6, the $\iint_D 2x\ dA = 12x$ [True/ False]

(c) If $D$ is a region with area 6 whose center of mass is at $(4, 5)$ then $\iint_D x\ dA =$ ____________________

(fill in a number)

(d) Suppose $D$ is a region in the plane and $\iint_D x\ dA = 10$. Is this enough information to know the value of $\iint_D x^2\ dA$? [Yes / No]

(e) Suppose $D$ is a region in the plane and $\iint_D x\ dA = 10$ and $D$ has area 5. Is this enough information to know the value of $\iint_D x^2\ dA$? [Yes / No]

(f) Suppose that $D$ is the unit circle and $H$ is the right half of the unit circle. (I recommend drawing a quick picture to help!) Which of the following are true? Circle all that apply:

i. $\iint_D dA = 2 \iint_H dA.$

ii. $\iint_D x dA = 2 \iint_H x\ dA.$

iii. $\iint_D y dA = 2 \iint_H y\ dA.$

iv. $\iint_D x^2 dA = 2 \iint_H x^2\ dA.$

(g) Choose one of the items from the previous part, (part (f)) and explain why you made your choice. You can explain "I circled XXX because ..." or you can explain "I didn't circle XXX because."'

**Quiz 10 - Good Luck Everyone!**

**This quiz is take home. It is due Friday May 12th in Class**

All parts of this problem are about the same surface $S$. Since this is a take-home quiz, I will be paying more attention to your writeups and presentation, so please **write very neatly.**

This quiz score (if it is greater) will replace your lowest quiz score.

You **may** work with classmates on this quiz, but you must write up your own solutions.

I'm happy to help in office hours.

Asking for help from tutors, or anyone else outside of our class is forbidden. This includes posting these questions online asking for help.

For what it's worth, this question is something that could appear on the final, so I highly recommend everyone try it! Good luck!

Let $S$ be the surface consisting of the boundary of the 3D region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. Suppose that $S$ is oriented outwards.

1. Draw a very careful picture of $S$.
2. Your picture should show you that your surface has two different pieces. Describe them in words, and parametrize each of them. For each picture, carefully say what your parameter domain $D$ is. You will use these in the next problem.
3. Find the flux of the vector field $\mathbf{F} = \langle xz, z, 2 \rangle$ through the surface $S$. You should have two different parts, since your surface has two different pieces. Please write your work very neatly. You can use a calculator/Desmos to calculate your double integrals, but I will pay careful attention to your writeup. In lowest terms your answer should be a fraction with 3 in the denominator and $\pi$ in the numerator.

   Since your surface has 2 parts, you will need to calculate 2 integrals. Make sure your final answer makes it clear how you are combining your answers. This Quiz might take up more space than you're used to - that's ok, give yourself enough space to present everything clearly.
4. Find the divergence of the vector field from part 3. Calculate that the triple integral of div $\mathbf{F}$ over the **solid region bounded by** $S$ is the same answer you got in part $a$. This is a special case of the **divergence** theorem.

   (Optional) Which method was faster - the one in part 3 or in part 4?

   (Optional) This problem was mostly about setting up surface integrals and triple integrals. You might want to add some advice to your note card (which can be a full sheet of notes, front and back) for the final.

## Part IV

# Exams

Math 250 Midterm 1

Your Name

- You have 55 minutes to do this exam.
- You may use a scientific calculator and a small note card.
- For justifications, please use complete sentences and make sure to explain any steps which are questionable.
- Good luck!

| Problem | Total Points | Score |
|---|---|---|
| 1 | 24 | |
| 2 | 17 | |
| 3 | 24 | |
| 4 | 8 | |
| 5 | 12 | |
| EC | 4 | |
| Total | 85 | |

GOOD LUCK!!!!

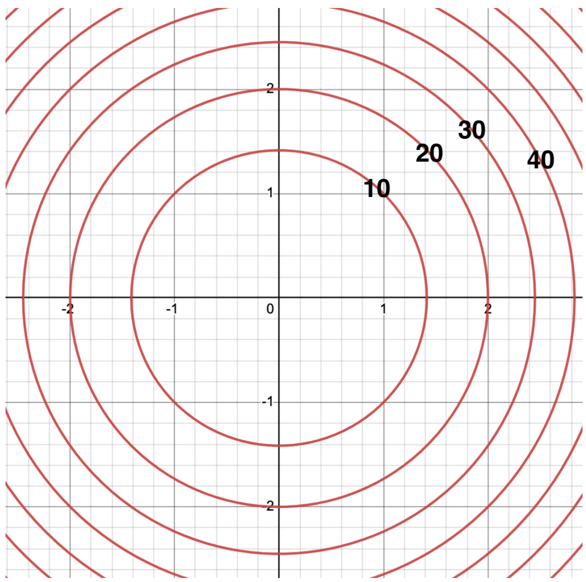
1. (24 points total) Complete the following multiple choice questions:

   (a) If two vectors in $\mathbb{R}^3$ are perpendicular then their dot product is [sometimes/always/never] equal to 0.

   (b) The line given by $\mathbf{r}(t) = \langle 3-2t, 4t, 1-8t \rangle$ intersects the $xy$ plane [True / False]

   (c) The plane $3x - 4y + 10z = 4$ passes through the origin [True / False]

   (d)




   In the picture above, which is a contour map for a function $f(x, y)$, draw and label points with the following properties. If it is not possible to draw such a point, don't do so.

   - $P$ where $f_x$ is positive
   - $Q$ where $f_x$ is negative
   - $R$ where $f_y$ is zero
   - $S$ where $f_y$ is negative.

   (e) Consider The sphere

$$(x-3)^2 + (y-4)^2 + (z-5)^2 = 25.$$

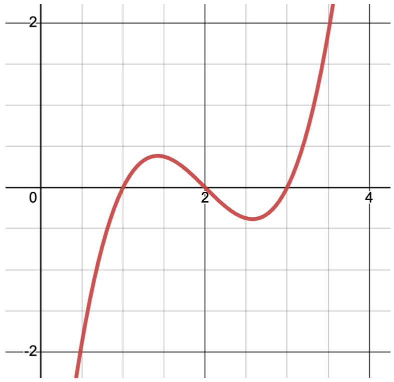
   For each of the following, circle the best answer. The sphere

   intersects the plane $z = 4$ in a [circle/point/line/set of two points/it will not intersect the plane].

   intersects the $xy$ plane in a [circle/point/line/set of two points/it will not intersect the plane].

   intersects the $x$-axis in a [circle/point/line/set of two points/it will not intersect the $x$-axis].

(f) Antoine the ant is walking along a curve with parametrization: $\mathbf{c}(t) = \langle x(t), y(t) \rangle$. You are given a graph of $x(t)$ and a table of values for $y(t)$. Use this information to answer the questions below. You may need to make some estimations from the graph. Do your best!

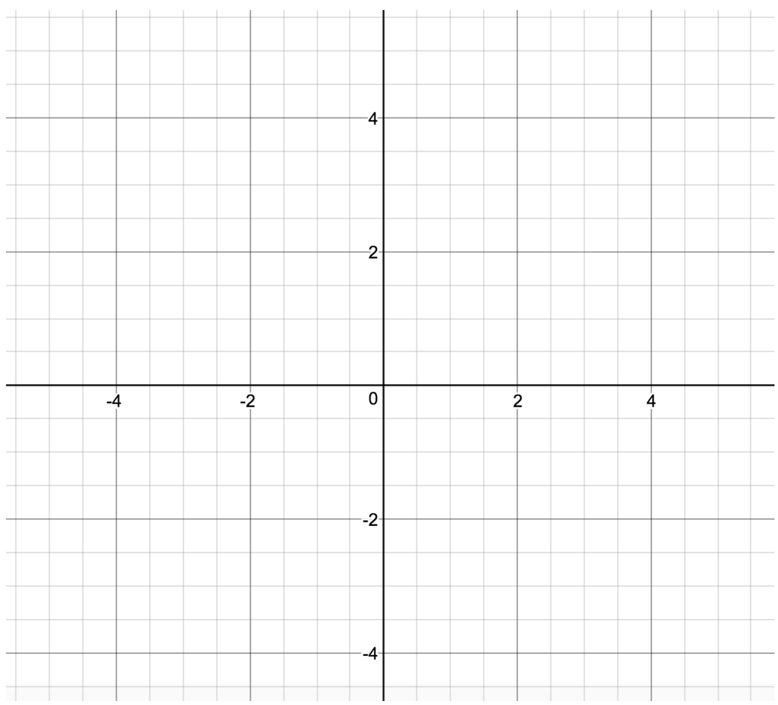



A graph of $x(t)$

| $t$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $y(t)$ | 8 | 5 | 3 | 2 |
| $y'(t)$ | $-1$ | $-3$ | $-2$ | $-1$ |

i. Below, label with the letter $P$ where Antoine is at time $t = 2$.

ii. Draw the tangent vector to Antoine's path at the point you just drew. (NOTE the gridlines on the graph)



2. (a) (5 points) Find the equation of the plane through the origin that contains the vectors $\langle 1, 2, 4\rangle$ and $\langle 0, 2, 1\rangle$

(b) (5 points) Find the cosine of the angle that the line $\mathbf{r}(t) = \langle 3 - 2t, 4t, 1 - t\rangle$ makes with the vector normal to the plane $3x - 4y = 4$.

(c) (7 points) Suppose that $f(x, y)$ is a differentiable function that gives the temperature at a point $(x, y)$ in the plane. You are given that.

$$f(4, 5) = 10, \qquad \frac{\partial f}{\partial x}(4, 5) = -3, \qquad \frac{\partial f}{\partial y}(4, 5) = 6.$$

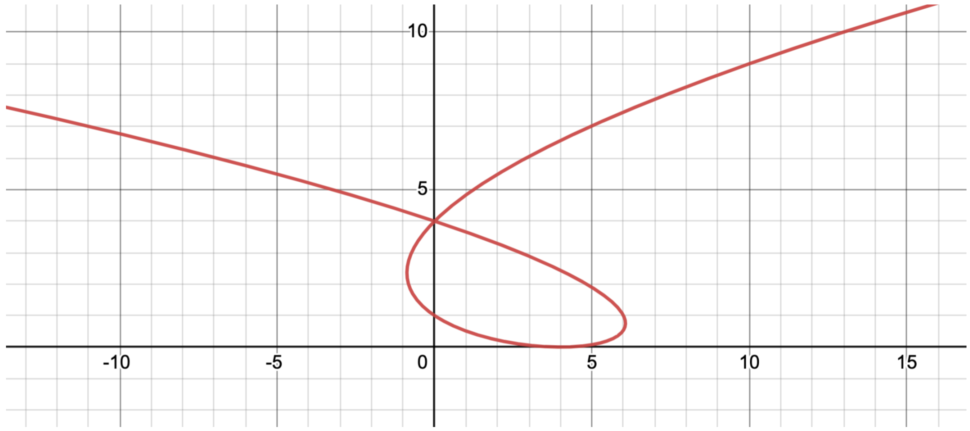
Use this information to answer the following:

i. If you were standing at the point $(4, 5)$ and were heading EAST (in the $x$ direction) then the temperature should [increase / decrease.]

ii. At the point $(4, 5)$ in which direction would the temperature be increasing the most? Write down a position vector.

iii. Write down the linear approximation for $f(x, y)$ at the point $(4, 5)$.

3. A particle is moving along the curve parametrized by $\mathbf{c}(t) = \left(t^3 - t^2 - 4t + 4, t^2\right)$ where $t$ is measured in seconds. A picture of this curve is below.




(a) (5 points) This is a chance to redo a similar problem to last week's quiz. Calculate and **draw** the two tangent vectors to the curve at the point $(0, 4)$. Show your work carefully.

(b) (4 points) Write down an integral that represents the length of the "loop" at the bottom of the curve. Please do NOT simplify your expression or foil anything out. The work you did in part a should make this fairly simple.

(c) (This is still continuing the problem on the previous page)
Suppose the temperature of each point $(x, y)$ in the plane is given by

$$F(x, y) = x^2y + x^3 - 2y$$

(in degrees $F$).

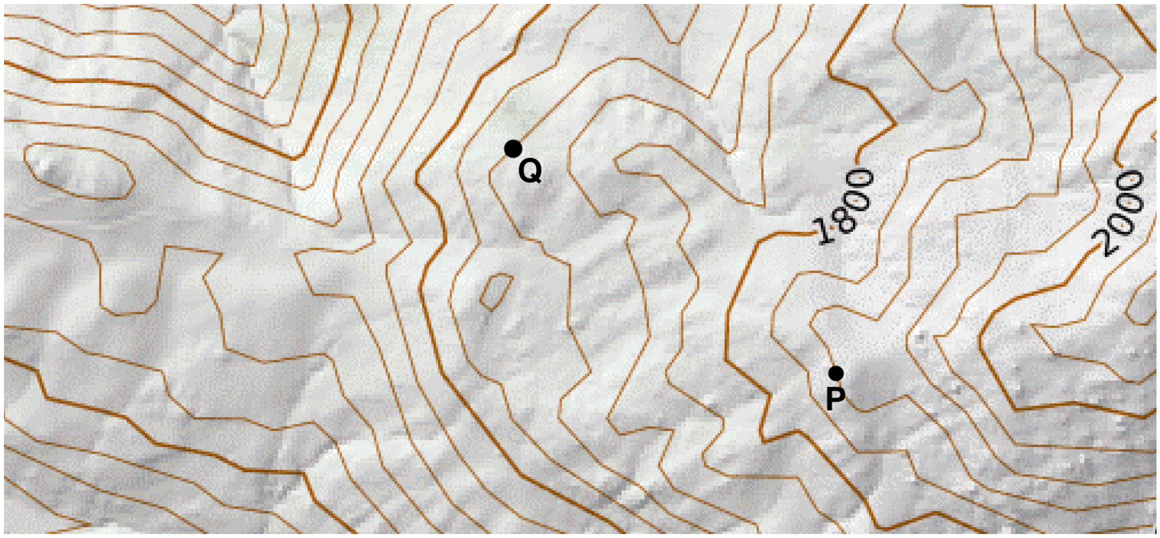
i. (4 points) Find $\nabla f(x, y)$

ii. (3 points) What is the temperature of the particle at time $t = 1$?

iii. (3 points) What is the speed of the particle at time $t = 1$?

iv. (5 points) At rate (in degrees per second) is the temperature of the particle increasing at the time it passes through the point $(0, 1)$?

4. Consider the contour map below of the popular hiking area known with the famous "potato chip rock." Suppose that $f(x, y)$ is the function that gives the vertical height (in feet) of the mountain over a point $(x, y)$.




(a) (2 points) Assuming the lines are level curves that are equally spaced, what is the elevation at the point $P$? Include units!

(b) (2 points) At the indicated point $Q$ draw a vector indicating the direction of the vector $\nabla f_Q$.

(c) (4 points) Draw ONE curve $c(t)$ for $0 \leqslant t \leqslant 4$ on the graph with ALL of the following properties. Label time stamps for $t = 0, t = 1$, $t = 2$, $t = 3$ for your curve.

i. $f(c(0)) = 1600$

ii. At $t = 1$, $\frac{d}{dt}(f(c(t)) = 0$

iii. At $t = 2$, $\frac{d}{dt}(f(c(t)) > 0$

iv. At $t = 3$, $\frac{d}{dt}(f(c(t)) < 0$

v. ALSO at $t = 3$, $f(c(t)) = P$.

5. Let $S$ be the surface given by

$$z = xy.$$

(a) (5 points) Parametrize the intersection of $S$ with the cylinder $x^2 + y^2 = 25$. Include bounds for your parameter $t$.

(b) (7 points) Antoine the bug is flying along the parametrized curve given by $\mathbf{r}(t) = \langle t, t-1, 3t-4 \rangle$. At what point(s) does Antoine intersect the surface $S$? Make sure you read this problem carefully and you know what $S$ is.

**Extra Credit** (4 points possible)

Let $S$ be the surface defined by $x^2 + xy - y^3 + ax + z = 2$. Suppose that $S$ contains a curve $C$ whose tangent line at the point $(0, 0, -2)$ is parametrized by $(0, 0, -2) + t(1, 2, 3)$. Compute $a$. Explain your steps.

Math 250 Midterm 2

Your Name

- You have 55 minutes to do this exam.
- You may use a scientific calculator and a small note card.
- For justifications, please use complete sentences and make sure to explain any steps which are questionable.
- Good luck!

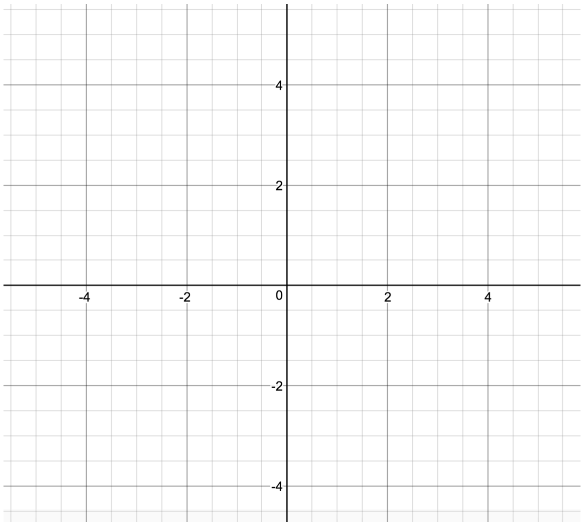
| Problem | Total Points | Score |
|---|---|---|
| 1 | 35 | |
| 2 | 18 | |
| 3 | 12 | |
| 4 | 10 | |
| EC | 4 | |
| Total | 75 | |

GOOD LUCK!!!!

1. (a) (6 points) John B. is looking for sunken treasure and while on the surface of the ocean discovers that the temperature at given point ($x, y$ are measured in feet) is given by $f(x, y) = x^2e^y + 72$ (in degrees $F$). Fill in the blanks below. There will be partial credit, so show your work:

   If John B. is at the point $(1, 0)$ then maximal rate of change of the temperature will be

   ______________________________ (include units!) if he swims in the direction indicated below (I am looking for you to draw a vector)




   (b) (4 points) Meanwhile, Antoine the ant is walking along the surface of the earth along the ocean floor. This part of the ocean is a mountainous terrain whose height is given by $f(x, y)$. Antoine arrives at a saddle point with coordinates $(a, b)$. Circle the best choices below:

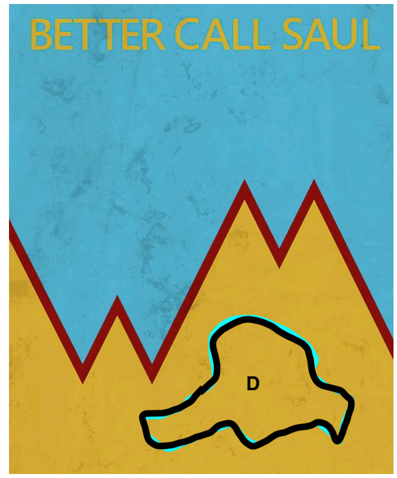
   $f_x(a, b)$ is [positive / negative / zero / can't tell]

   $f_y(a, b)$ is [positive / negative / zero / can't tell]

   Write down 3 numbers in the blanks below that would be possible for the second partial derivatives. (There will be many correct answers, you just need to make sure that the point is a saddle point.)

   $f_{xx}(a, b) =$ ____________, $f_{yy}(a, b) =$ ____________, $f_{xy}(a, b) =$ ____________

(c) (6 points) Excited by the potential discovery of sunken treasure, John's legal team, the law firm Wexler and McGill designs a logo that looks like the picture at the right. In your investigations, you discover that a part of their logo, called $D$ has center of mass at the point $(5, 4)$ and that $\text{Area}(D) = 8$. Using this information, fill in the blanks:




$$\iint_D 2 \; dA = \underline{\hspace{3cm}}$$

and

$$\iint_D 3x \; dA = \underline{\hspace{3cm}}$$

(Note: you will NOT be able to determine anything from the picture. This question is about using area and center of mass to solve these problems.)

(d) (4 points) To help the treasure seekers, Wednesday Adams volunteers to integrate the function $x^2 + y^2$ over a region in the plane and wants to use polar coordinates. You are not sure what the region looks like, it could be part of a circle, or perhaps the region between two circles. You don't know! However, only one of the following could possibly be a correct setup - circle the correct one and write a short 1-2 sentence explanation why.

$$\int_0^{\pi/2} \int_0^3 r^3 dr \; d\theta \qquad \int_{-\pi}^{\pi/2} \int_3^6 r \cos\theta dr \; d\theta$$

$$\int_0^{3\pi/2} \int_0^3 r^2 dr \; d\theta \qquad \int_0^{\pi/2} \int_0^3 r dr \; d\theta$$

Explanation:

(e) (4 points) The cone defined by $z = 7\sqrt{x^2 + y^2}$ makes what angle $\phi$ with the $z$ axis? You may leave your answer in terms of inverse trig functions. (Partial Credit will be given)

(f) (2 point) In spherical coordinates, the plane $z = 1$ is given by which of the following?

- $\rho = \sin^2 \phi$
- $\rho = \dfrac{1}{\cos \phi}$
- $\rho = \cos \phi \sin \theta$
- $\rho = \phi + \theta$

(g) (4 points) Consider the region $W$ described in spherical coordinates by

$$0 \leqslant \rho \leqslant 4, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi \leqslant \pi/2$$

Then the $x$ coordinates of points in $W$ are [always $\geqslant 0$, always $\leqslant 0$, some mixture]

Then the $y$ coordinates of points in $W$ are [always $\geqslant 0$, always $\leqslant 0$, some mixture]

Then the $z$ coordinates of points in $W$ are [always $\geqslant 0$, always $\leqslant 0$, some mixture]

$W$ is best described as a [whole sphere / hemisphere / quarter sphere / eighth of a sphere].

(h) (5 points) While swimming, John B. sees a strange sea creature whose shape is given by the region bounded between the two paraboloids:

$$z = 14 - x^2 - y^2, \qquad z = 2 + 2x^2 + 2y^2.$$

Draw a rough 3D picture of this sea creature. I just want you to draw the rough general shape. You don't have to calculate the shadow.

2. Consider the region $W$ described in spherical coordinates by the following description:

$$0 \leqslant \rho \leqslant 4, \quad 0 \leqslant \theta \leqslant 2\pi, \quad 0 \leqslant \phi \leqslant \pi/4$$

(6 points) Draw a rough 3D picture of $W$ **AND** draw a picture of the **shadow** of this region in the $xy$ plane.

(12 points) In the boxes below, set up the integral of the function

$$f(x, y, z) = x$$

for this region in spherical and cylindrical coordinates. You just need to fill in the blanks. Notice that someone helped you fill in one of the blanks!

**Spherical Coordinates**

$$\int \int \int \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \, d\rho \, d\phi \, d\theta$$

**Cylindrical Coordinates**

$$\int \int \int_{z=r} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \, dz \, dr \, d\theta$$

3. (12 points) John B. has journeyed down to the ocean floor and is now working with Antoine the ant to find treasure. They encounter a pesky math professor who asks them to classify the critical points of the function

$$g(x, y) = x^3 - 3x^2 + y^2 - 4y.$$

Find all critical points of this function and classify them as local min / local max / saddle in order to appease the professor.

4. (10 points) On the ocean floor they discover a metal plate whose shape is the region $D$.

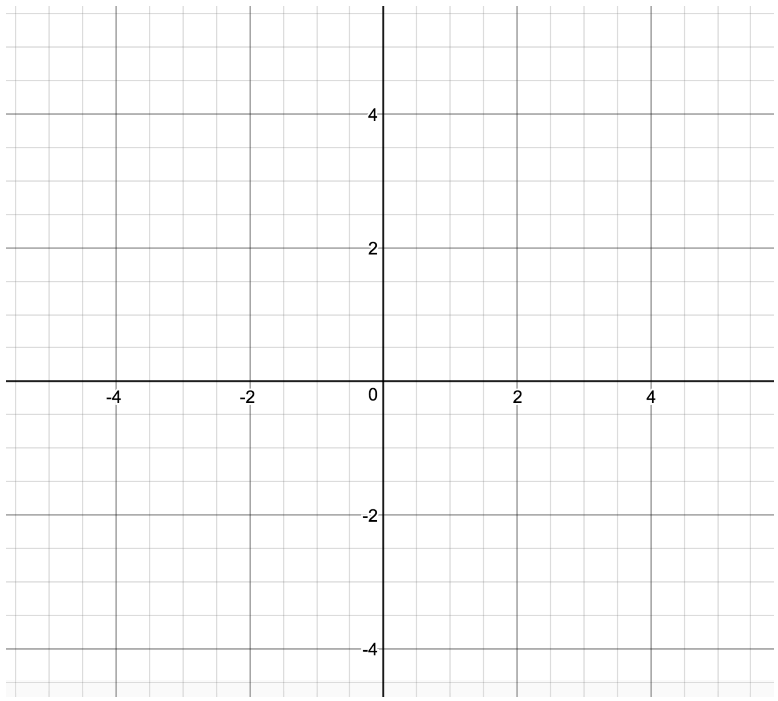
If $f(x,y)$ measures the density (in grams per $m^2$) of the plate at each point, then $\iint_D f(x,y)\,dA$ gives the ______________________________ of the whole plate.

(This is a four letter word that starts with $m$ and has 2 $s$'s).

If

$$\iint_D f(x,y)\,dA = \int_{-4}^{2}\int_{\frac{1}{2}x-2}^{3} f(x,y)\,dy\,dx.$$

then sketch the region $D$ **and** change the order of integration by setting up an integral in the order $dx\,dy$.



**Extra Credit (4 points)**

Find the volume of the region contained in both of the following cylinders:

$$x^2 + y^2 = 1, \quad x^2 + z^2 = 1$$

Math 250 Final Exam

Your Name

- You have 120 minutes to do this exam.
- You may use a scientific calculator and a small note card.
- For justifications, please use complete sentences and make sure to explain any steps which are questionable.
- Good luck!

| Problem | Total Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 25 | |
| 3 | 8 | |
| 4 | 15 | |
| 5 | 10 | |
| 6 | 5 | |
| 7 | 14 | |
| 8 | 10 | |
| 9 | 8 | |
| EC | 4 | |
| Total | 115 | |

1. (20 points) The following are True/False Questions. You don't need to write any justification.

**(1)** | **T** | **F** | The dot product $\langle 1,1,2\rangle \cdot \langle 2,3,4\rangle = 13$.

**(2)** | **T** | **F** | If $\mathbf{c}(t)$ is a curve in $\mathbb{R}^3$ then then $\mathbf{c}'(t_0)$ gives a vector perpendicular to the curve at time $t_0$.

**(3)** | **T** | **F** | If $\nabla f_{(3,1)} = \langle 0,0\rangle$ and $f_{xx}(3,1) > 0$ then $(3,1)$ is a local minimum of $f$.

**(4)** | **T** | **F** | If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ and $S$ is the boundary of a $3D$ solid $E$ then the triple integral $\iiint\limits_E \text{div}\mathbf{F}\, dV$ is always zero since the flux of $\mathbf{F}$ through the boundary surface is zero.

**(5)** | **T** | **F** | If $D$ is a bounded region in the plane, then $\iint\limits_D x\, dA$ gives the $x$ coordinate of the center of mass of $D$.

**(6)** | **T** | **F** | Suppose that $P$ is a critical point of a function $f(x,y)$. Then $|\nabla f_P| \geqslant 1$.

**(7)** | **T** | **F** | The curl of a conservative vector field in $\mathbb{R}^3$ is zero.

**(8)** | **T** | **F** | A constant vector field is always conservative.

**(9)** | **T** | **F** | Let $L$ be a line in $\mathbb{R}^3$. The paraboloid $z = x^2 + y^2$ will definitely intersect $L$.

**(10)** | **T** | **F** | If $\nabla f_P = \langle 2,3\rangle$ then $f_x(P) = 2$.

**(11)** | **T** | **F** | The volume of a 3D solid $E$ is equal to the flux of the vector field $\langle 0,y,0\rangle$ through the boundary surface.

**(12)** | **T** | **F** | If $S$ is part of the surface of a sphere then $\iint\limits_S dS$ is the surface area of $S$.

**(13)** | **T** | **F** | The vector $\langle 1,1,1\rangle$ is perpendicular to the plane $z = x + y + 1$.

**(14)** | **T** | **F** | If $S$ is a sphere of radius $R$ with outward pointing normal vector and $\mathbf{F}$ is a constant vector field then $\iint\limits_S \mathbf{F}\cdot d\mathbf{S} = 4\pi R^2$.

**(15)** | **T** | **F** | If $E$ is a solid with volume 1, then $\iiint\limits_E x\, dV$ is equal to the $x$ coordinate of the center of mass of $E$.

**(16)** | **T** | **F** | The work done by a conservative vector field along a path only depends on the endpoints of the path.

**(17)** | **T** | **F** | If a vector field $\mathbf{F}(x,y)$ is a gradient of some potential function $f(x,y)$ then the line integral $\int_C \mathbf{F}\cdot d\mathbf{s}$ along any closed curve $C$ will be zero.

**(18)** | **T** | **F** | On a contour map, the gradient vector $\nabla f$ is tangent to the level curves of $f$.

**(19)** | **T** | **F** | If $\mathbf{u}, \mathbf{v}$ are two perpendicular vectors in $\mathbb{R}^3$ then $\mathbf{u} \times \mathbf{v}$ is the zero vector.

**(20)** | **T** | **F** | Suppose that $f_x(a, b) = 2$ and $f_y(a, b) = 3$, then there is a direction in which the rate of change of $f$ at $(a, b)$ is zero.

2. (Describing Geometric Regions)

A) (5 points) Draw a careful 3D picture of the region in $\mathbb{R}^3$ described in spherical coordinates by:

$$0 \leqslant \theta \leqslant 2\pi, \quad 0 \leqslant \phi \leqslant \pi/3, \quad 0 \leqslant \rho \leqslant 4.$$

**make sure you include any lengths or angles in your picture.**

B) (5 points) Draw a 3D picture of the the region described by:

$$x^2 + y^2 \leqslant z \leqslant 12 - 2x^2 - 2y^2$$

C) (5 points) Find the intersection of the surfaces $z = x^2 + y^2$ and $z = 12 - 2x^2 - 2y^2$. Your answer should be a curve and you should describe your answer by giving a parametrization $\mathbf{c}(t)$. Remember your answer should have three components since we are in $\mathbb{R}^3$. **Hint: This is related to the previous problem - you might want to update your picture.**

D) (10 points) **There are TWO parts to this problem.**

Suppose that $S$ is the part the part of the upper half (meaning $z \geqslant 0$) of the sphere $x^2 + y^2 + z^2 = 16$ that is inside the cylinder $x^2 + y^2 = 4$. The surface of $S$ is gilded with gold such that the density of gold at the point is given by $f(x, y, z) = z$ (in $g/m^2$).

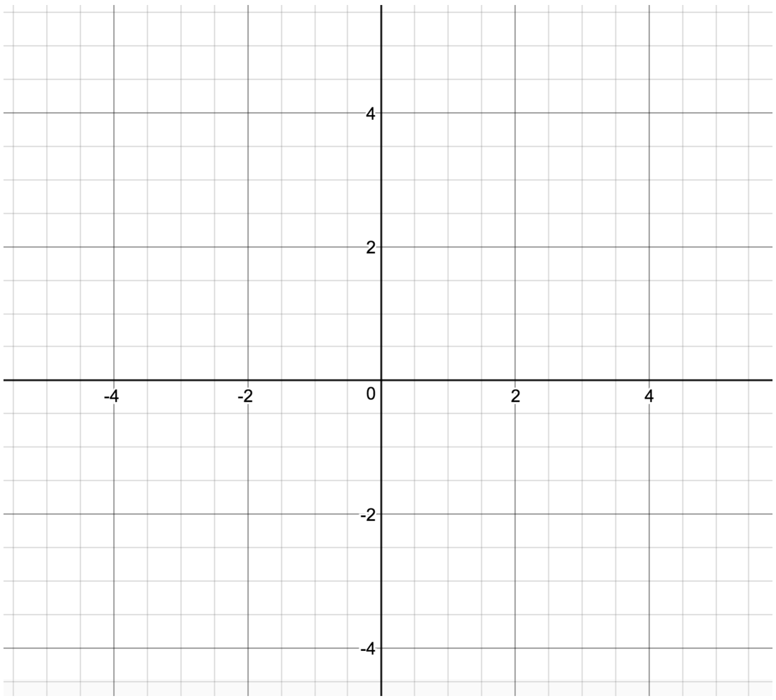
i) Draw a careful picture of $S$, labeling any important information.

ii) Set up an iterated integral using $\phi, \theta$ that will calculate the total mass of gold on the surface $S$.

3. (8 points) A student is setting up an integral of a function $f(x, y)$ over a region $D$ and writes down:

$$\iint\limits_D f(x,y)\ dA = \int_{-3}^{-1} \int_{-2}^{2y+4} f(x,y)\ dx\ dy + \int_{-1}^{3} + \int_{-2}^{1-y} f(x,y)\ dx\ dy.$$

Draw a picture of the region of integration AND change the order of integration by setting up the integral $\iint\limits_D f(x,y)\ dA$ as a $dydx$ integral.



4. (15 points) **Some shorter answer questions**:

A) Consider the region $W$ described in spherical coordinates by

$$0 \leqslant \rho \leqslant 4, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi \leqslant \pi/2$$

Then the $x$ coordinates of points in $W$ are [always $\geqslant 0$, always $\leqslant 0$, some mixture]

Then the $y$ coordinates of points in $W$ are [always $\geqslant 0$, always $\leqslant 0$, some mixture]

Then the $z$ coordinates of points in $W$ are [always $\geqslant 0$, always $\leqslant 0$, some mixture]

$W$ is best described as a [whole sphere / hemisphere / quarter sphere / eighth of a sphere].

B) Consider the sphere:

$$(x-5)^2 + (y-4)^2 + (z-3)^2 = 25.$$

For each of the following, circle the best answer:

(a) The sphere intersects the plane $z = 4$ in a [line / circle / point /set of two points / it does not intersect]

(b) The sphere intersects the $yz$ plane in a [line / circle / point / set of two points / it does not intersect]

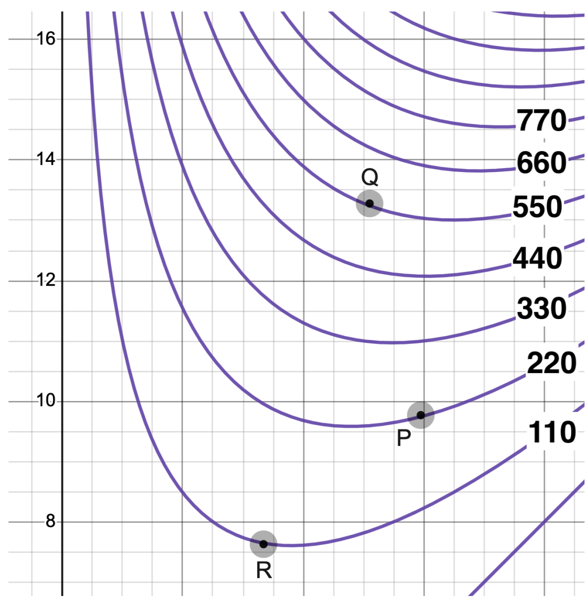
(c) The sphere intersects the $y$ axis in a [line / circle / point / set of two points / it does not intersect]

C) If $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^3$ then

a) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u}$ is [undefined / 0 the scalar / $\mathbf{0}$ the zero vector / something else]

b) $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{u}$ is [undefined / 0 the scalar / $\mathbf{0}$ the zero vector/something else.]

D) The contour map of a function $f(x, y)$ is given in the figure at right.




(a) If you were to draw in gradient vectors at $R$, and $Q$ which one would be longer? Why?

(b) If $C$ is the curve consisting of a straight line path from $R$ to $Q$ followed by a path from $Q$ to $P$, what is $\int_C \nabla \mathbf{f} \cdot \mathbf{ds}$? Explain your answer.

5. (10 points) Suppose that $C$ is the straight line starting at $(0, 2)$ and ending at the point $(-1, 4)$. A student wants to find the following line integral:

$$\int_C xy\ dx + x\ dy$$

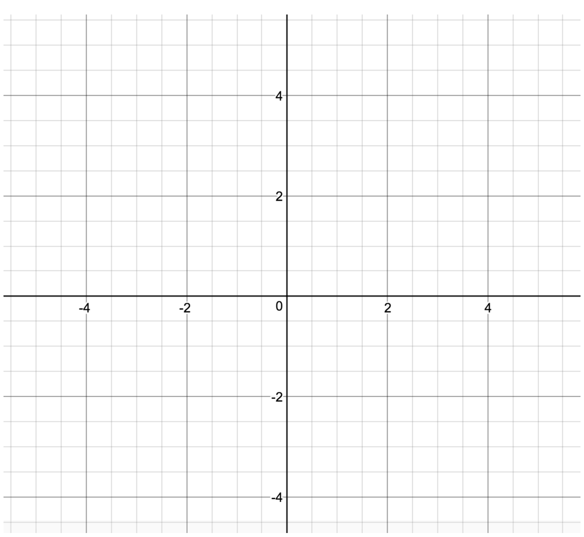
(a) Explain why the student cannot use The Fundamental Theorem for Gradients to solve this problem.

(b) Explain why the student cannot use Green's Theorem to solve this problem.

(c) By parametrizing the curve, set up a definite integral that will calculate this line integral. You do NOT have to evaluate the integral.

6. (5 points) John B. is looking for gold in a swamp and while walking discovers that the temperature at a given point ($x, y$ are measured in feet) is given by $T(x, y) = x^2 - 3y^2 + 72$ (in degrees F). Fill in the blanks below:

If John B. is at the point $(1, 1)$ then the maximal rate of change of temperature will be ____________________________ (include units!) if he swims in the direction indicated below (I am looking for you to draw a vector).



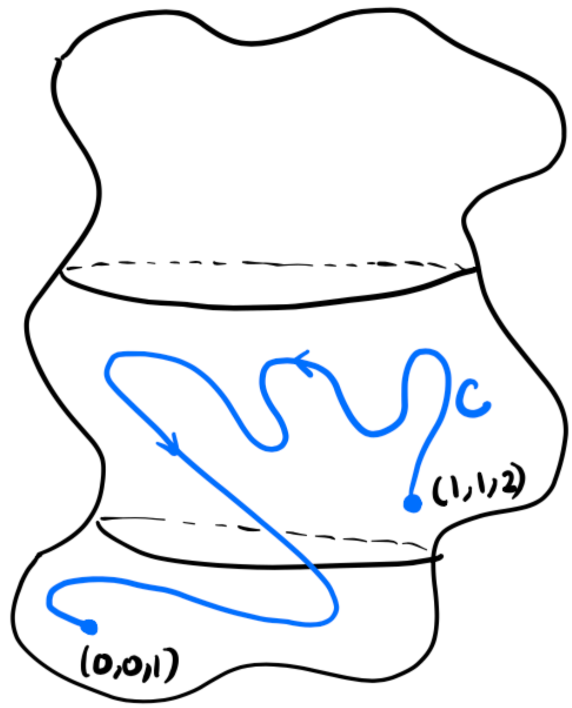
7. (14 points) A Let $S$ be the surface of the 3D figure pictured at right

Suppose that $S$ is oriented with outward pointing normal vector;

$S$ has surface area 12 square meters and encloses a volume of 4 cubic meters;

The center of mass of that 3D solid is at the point $(1, 2, 5)$;

Antoine the ant has crawled along the indicated curve $C$ on the surface that starts at the point $(1, 1, 2)$ and ends at the point $(0, 0, 1)$.




$$\mathbf{F} = \langle 3x, 3y + \sin(xz), z^2 \rangle, \qquad \mathbf{G} = \langle yz, xz, xy + 2z \rangle.$$

Use this information to find the following three integrals:

A) $\displaystyle\iint_S 4 \, dS$

B) $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{dS}$

C) $\displaystyle\int_C \mathbf{G} \cdot \mathbf{ds}$

8. (10 points) Let $S$ be the surface described by

$$z = 25 - x^2 - y^2, \qquad z \geqslant 0.$$

Setup fully the integral $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{dS}$ where $\mathbf{F}$ is the vector field $\langle x, z, 2 \rangle$. You may leave your answer as an iterated integral in whatever coordinate system you want. Please expand any dot products.

9. (8 points) Suppose that $C$ is an ellipse in the $xy$ plane that bounds a region whose center of mass is at the point $(4, 3)$. Suppose that $C$ bounds a region of area $4\pi$. Find

$$\int_C (4x^2 + y^2)\, dx + (xy + x)\, dy$$

where $C$ is oriented counterclockwise. **Show your work neatly.**

**Extra Credit**

Let

$$\mathbf{F} = \langle y^3, x - x^3 \rangle.$$

Find the closed curve $C$ in the plane such that the integral $\displaystyle\int_C \mathbf{F} \cdot \mathbf{ds}$ has the largest possible value. Explain your reasoning and find this largest possible value. Hint: A theorem we learned will help you!