Summer Workshop for Teachers
Sandy, Utah
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Mix-up at the Workshop

A hundred people at a workshop have mixed up their nametags and they wish to fix the problem. There are lots of ways they could do this, but since they want to get to know each other, they decide they'll try to do this by performing swaps - a move where just two people exchange nametags. They perform many swaps and hope that eventually order can be restored. Mathematicians call swaps **transpositions**.

Question 0.1.

- The focus of this workshop is symmetry. Do you see any way in which permutations may be described as symmetries of the set of people in this room?
- How might we define "symmetry"? On the board, throughout the week we will try to come up with our own definition of symmetry and see how it changes throughout the week. For now, talk with your group for a few minutes about what you think symmetry is. Jot down some ideas about what you talk about.

Now, 100 people is a lot! As if often the case in math, the number 100 isn't what's important in the problem - in fact, just understanding this problem with 10 people would be just as hard! In the following, it might be helpful to think instead about what would happen with a smaller number. When things are shuffled, we sometimes say that they have been **permuted** and call the resulting order a **permutation**.

Question 0.2. How can you write down a permutation? (There are lots of different ways - but see if you can come up with a few of your own. Pictures are suggested!)

Question 0.3.

- Is it always possible to get everyone their own nametag by this process? Why or why not?
- Can you describe an algorithm for doing this?
- Given an arrangement, how many swaps would you require? What arrangement of nametags would require the most number of swaps?

Now suppose that people aren't allowed to swap nametags, but people want to get to know each other in groups of threes and are only then allowed to rotate their nametags in a cyclic fashion. For example, Amanda, Evelyn and Gary can get together and pass their nametags around in a small circle. ($A \to E \to G \to A$.)

Question 0.4.

- Is it always possible to get everyone their own nametag by this process? Why or why not?

- What algorithm or ideas can you come up with this process?

- Can you think of a physical object that has "rules" like this - where swaps aren't allowed, but rotations are very natural?

Symmetries of a Square

Let's start by looking at the square you have in front of you. The goal of this page is to determine how many *symmetries* the square has. But first we need to define what that means. To get you started, we'll say that "rotating clockwise by 90 degrees" is a symmetry. Discuss with your group for a few minutes and write down what you think a "symmetry of the square" should mean and how many symmetries there are.

In coming up with your definition, consider the following questions:

- Is rotating by 90 degrees clockwise the same symmetry as rotating 270 degrees counter-clockwise?
- Is doing the same reflection 3 times the same as just doing it once?
- Is reflecting across a line the same as rotating 180 degrees?
- Is "rotation by 360 degrees" a symmetry? What about "doing nothing"?
- Would stretching or shrinking the square be a symmetry?

A symmetry of a square is

All of your definitions of symmetry are worthy of study - and will yield different properties. However this week we'll want to make sure we're all working with the same definition of symmetry. For us, our definition is on the next page - don't look until you've got something written in the box.

If we have a shape in the plane, a symmetry of that shape is a way of "picking it up, moving it around and putting it back down over the same outline." We consider two symmetries "the same" if they have the same ultimate effect on the points in the shape.

Can you fill in the blanks in this table?

Operation	Is the Same as
Rotating clockwise by 90 degrees	Rotating clockwise by 450 degrees
Flipping across an axis of symmetry	Flipping across that axis of symmetry 3 times
Flipping across the horizontal axis of symmetry and then rotating by 90 degrees clockwise	Flipping across the ______________________________ axis of symmetry

Can you write down all the different symmetries of the square? How many are there?

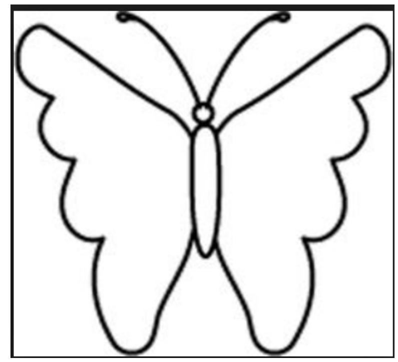
"Identity" - the "do nothing" symmetry	

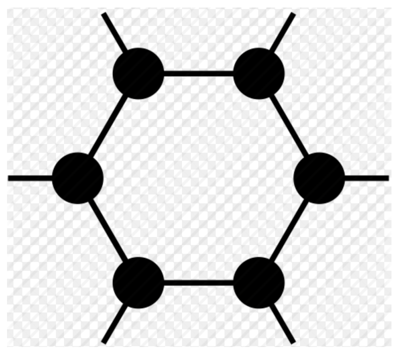
Symmetries by Example - Same or Different?

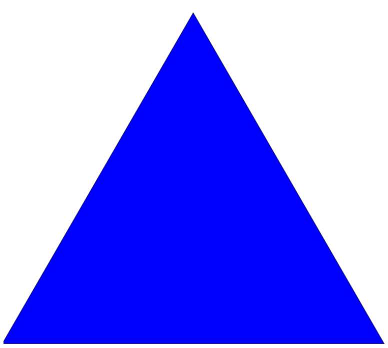
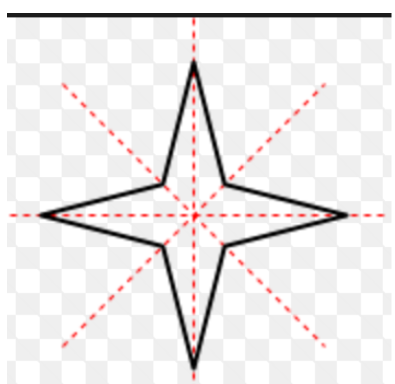
Below are some pictures of shapes that have different types of symmetries. With your groups, discuss

- How many symmetries each shape has;
- Whether any two of the shapes have **the same** symmetries; (what do you think this means?)








Composing Symmetries

Yesterday, one of the main things we started the approach that a symmetry is an action or **transformation**.

- Mixing up objects (permutations) is an action. We can perform successive permutations and the end result is another permutation.
- Symmetries of the square are those transformations that return the square to its original shape.

Today our goal is to understand what happens when we compose symmetries. We'll start with the square - whose 8 symmetries we found yesterday:

Symmetries of the Square

"Identity" - the "do nothing" symmetry	Rotate clockwise by 90 degrees
Rotate clockwise by 180 degrees	Rotate clockwise by 270 degrees
Flip across vertical axis	Flip across horizontal axis
Flip across diagonal line "$y = x$"	Flip across diagonal line "$y = -x$"

One of the beautiful things about symmetries is that we can do one after the other, in any combination. Using your square - work out the following:

Question 0.5. Which symmetry is obtained upon:

- Rotating 270 degrees clockwise after rotating 180 degrees clockwise
- Rotating 180 degrees clockwise after rotating 270 degrees clockwise
- Rotating 90 degrees clockwise 10 times
- Rotating 90 degrees clockwise after flipping across the vertical axis
- Flipping across the vertical axis of symmetry after rotating 90 degrees

Why do you think I've insisted on writing everything in the form "do X **after** Y"? Does this resemble anything we teach our students about functions?

Activity 0.6. Can you come up with a multiplication table for the symmetries? In other words, a table that says how to combine any symmetries. Feel free to come up with letters / nicknames for the symmetries.

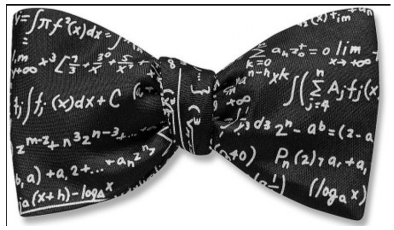
The Triangle Now let's move on to the equilateral triangle. Can you describe the symmetries of the triangle? In your group can you come up with:

- How many symmetries are there?
- How can you describe them? Can you put give them names?

"Identity" - the "do nothing" symmetry	id		

- What labels do you want to use for these operations?
- Is composition of symmetries commutative?
- Can you write down any formulas for some of the composition rules?

The Bowtie



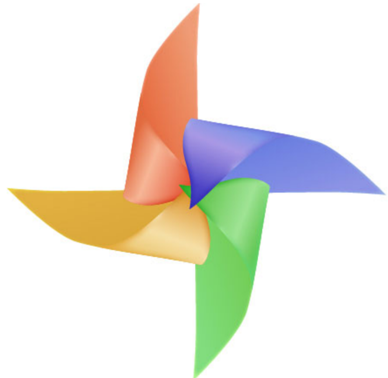
Can you describe the symmetries of the bowtie? In your group can you come up with:

- How many symmetries are there?
- How can you describe them? Can you put give them names?

"Identity" - the "do nothing" symmetry	id		

- What labels do you want to use for these operations?
- Is composition of symmetries commutative?
- Can you write down any formulas for some of the composition rules?

The Pinwheel



Can you describe the symmetries of the pinwheel? In your group can you come up with:

- How many symmetries are there?
- How can you describe them? Can you put give them names?

"Identity" - the "do nothing" symmetry	id		

- What labels do you want to use for these operations?
- Is composition of symmetries commutative?
- Can you write down any formulas for some of the composition rules?

The Circle Now let's move on to the circle. Can you describe the symmetries of the circle? In your group can you come up with:

- How many symmetries are there?
- How can you describe them? Can you put give them names?

"Identity" - the "do nothing" symmetry	id		

- What labels do you want to use for these operations?
- Is composition of symmetries commutative?
- Can you write down any formulas for some of the composition rules?

The n-gon Now let's move on to the n-gon. Can you describe the symmetries of the n-gon? In your group can you come up with:

- How many symmetries are there?
- How can you describe them? Can you put give them names?

"Identity" - the "do nothing" symmetry	id		

- What labels do you want to use for these operations?
- Is composition of symmetries commutative?
- Can you write down any formulas for some of the composition rules?

Platonic Solids

The Platonic solids, first studied by the ancients, are 3D objects that include the cube and the triangle-based pyramid (tetrahedron). In some sense, these are just the 3D objects that are "highly symmetric". However, as we've learned this week, there are different types of symmetry and we'll need a more precise definition to move forward. Indeed, non-Platonic solids like the sphere and square-based pyramid are also highly symmetric, but in different ways. Since I expect people in your group have seen these objects before, let's start with:

Activity 1.1. Write a definition of **Platonic Solid** on the board. As your group discusses some properties, please add them to the soon-to-be growing definition on the board.

We'll discuss the definition together as a group, but if you want to move on to the next activity once you're satisfied with the definition, feel free!

Activity 1.2. In the room we have a wide variety of resources: Some paper polygons - some Zome tools, some plastic 3D solids. Using these tools, with your group, try to build some platonic solids and see what you notice. Some things to consider:

- Try to pretend that you don't already know what the solids are, and instead think "what can I build from triangles? or squares? pentagons, ... 10-gons? etc".
- What shapes will you use to build the solid?
- How do they come together?
- What's going on with the glass one with a cube on the inside (Go check it out!)
- How many edges, faces, vertices do you use in each example?
- Are angles helpful to think about?

Finally, continuing on with our definition of symmetry from this week, let's try to understand what the symmetries of these shapes look like:

Activity 1.3. What are the symmetries of the Platonic Solids? (Remember that symmetries are ways of transforming the object in space)

- How many symmetries does each object have?
- What sort of relationships are there between the symmetries?
- What are the possible relationships to permutations?

Here's a video summarizing some of these ideas `https://www.youtube.com/watch?v=voUVDAgFtho`

Derangements

Today let's do something that's totally deranged! Imagine that you arrived at a workshop with 1000 people, and all of your nametags had been randomly permuted! Restoring order was our task for day one; today's let's instead consider the following question:

Question 1.1. What do you think the probability is that no one (not a single person!) actually got the correct nametag?

After writing down your guess, share it with a neighbor and discuss why you made your guess. Talk about whether you think this probability would change if instead of 1000 people we had only 10 people. We'll come back together at this point to share our intuitions among groups.

We'll now do some experiments with cards to see roughly what the probability would be if we had 13 people. (Audible instructions - just put down cards A - K in one suit, and then shuffle another suit and lay them down beneath and see if you have any matches.) We'll all perform these shuffles and we'll report our findings to a counter at the board who will keep track.

Observed Experimental Probability That No Card is in the Correct Position: ______________

Now let's get to work sorting out what's going on! A permutation in which no one gets their own nametag is called a **derangement**. In the following activity we'll work out how many derangements there are if we have n people.

Activity 1.2. Today we're going investigate what this probability is when we have n people, by first working through some cases with small n. With your group, try to fill in the following table. You might want to write down all the permutations of ABCD, or use the playing cards.

number of people	total number of permutations	number of **derangements**	probability of getting a **derangement**
1			
2			
3			
4			

Question 1.3. How is the probability changing as we add an additional person? You should see a nice pattern :) Use this to fill in the table below:

number of people	probability of getting a **derangement**
5	
6	
7	
8	

Using the "..." (dot dot dot) notation, can you come up with a conjecture for the probability with $n = 1000$ people?

Conjecture: The probability that no card is in the correct position when n cards are permuted is equal

to ____

Activity 1.4. At this point, I'd like to give groups some possible options of how to proceed, because there are two interesting things going on here:

- Investigate why this conjecture is true. How could you count the permutations that are derangements, and why might these $+$ and $-$ signs come up? When do we have to add / subtract when counting?
- What number is this probability approaching? How could we explore whether this number is something we have already seen? Out of curiosity, what happens if we only use plus signs?

After talking with your group, how do you think you could move forward with these? I can offer some hints along the way :)

I was first shown this beautiful "probleme de rencontres" by Prof. Liviu Nicolaescu at U. Notre Dame when I was a high school student attending a math circle he was organizing. For a document with more details, including the "answers" to some of these questions, see `http://www.geometer.org/mathcircles/derange.pdf` For the video we watched in the lesson, see `https://www.youtube.com/watch?v=DoAbA6rXrwA`

Frontiers For Young Minds

Frontiers for Young Minds (http://kids.frontiersin.org/) is a non-profit journal written by scientists and reviewed by kids. They are going to soon start publishing some papers in mathematics and I thought we would look at some articles in other fields and think about about what will be the same and different in mathematics. Their blurb is:

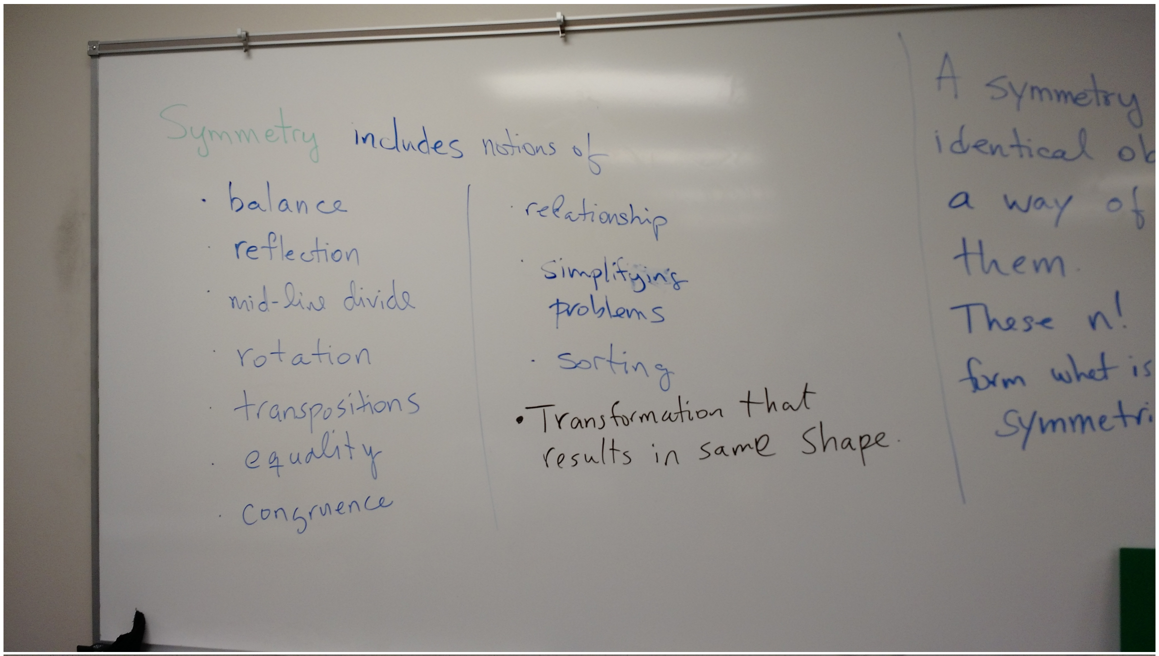
"Math is the language of the universe. It is how we use logical reasoning to explain big ideas like symmetry, chaos, infinity, change, and truth. Some mathematicians study patterns for their own sake: often even the simplest problems can reveal beautiful and unexpected structures in the universe. Other mathematicians study how these patterns enable us to better understand other fields, ranging from physics to economics, neuroscience to astronomy, meteorology to music. This section of Frontiers for Young Minds will include articles from all areas of pure and applied mathematics, covering fundamental ideas, cutting-edge advances, and a broad range of applications. Understanding Mathematics wants to communicate to the next generation that mathematics is not only essential for describing the real world, but is also accessible to everyone and is an amazing natural source of beauty in its own right."

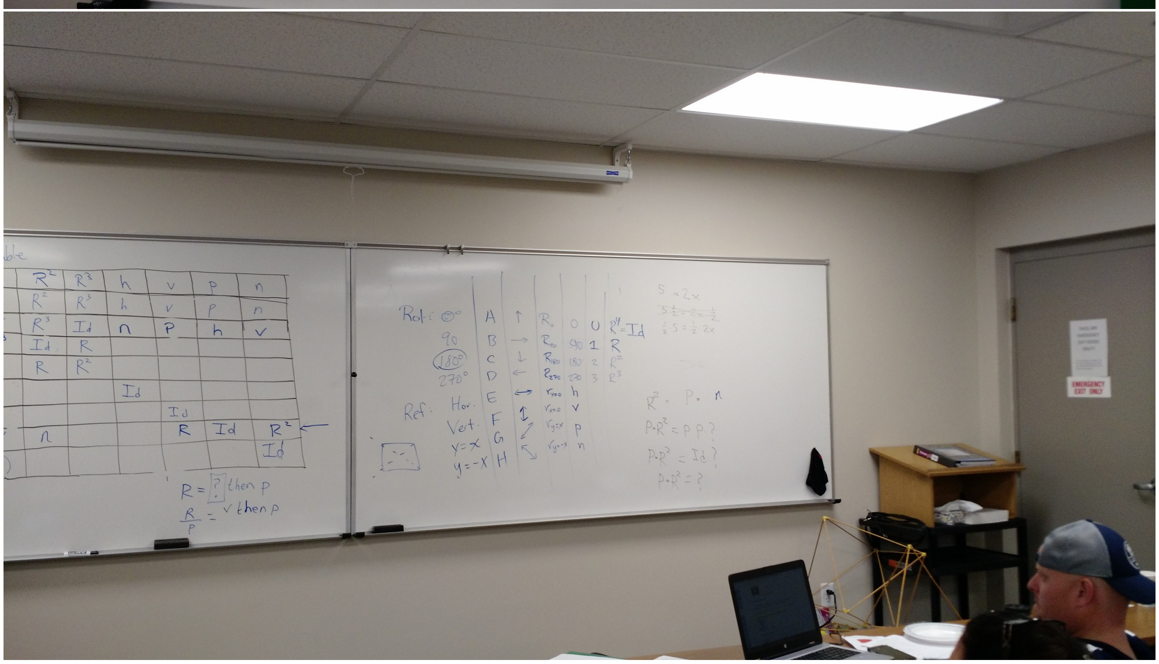
In this session, we'll discuss the three articles linked here (please try to read at least one before we meet):
http://kids.frontiersin.org/article/10.3389/frym.2017.00030
http://kids.frontiersin.org/article/10.3389/frym.2017.00032
http://kids.frontiersin.org/article/10.3389/frym.2017.00024

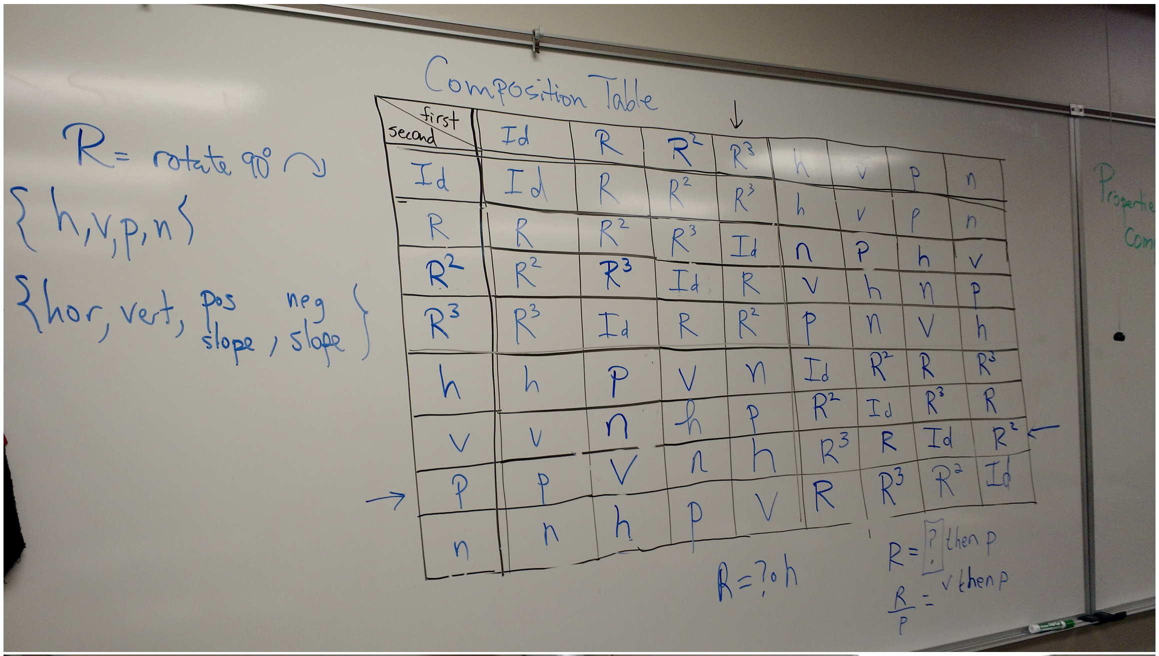
Some Items for Discussion

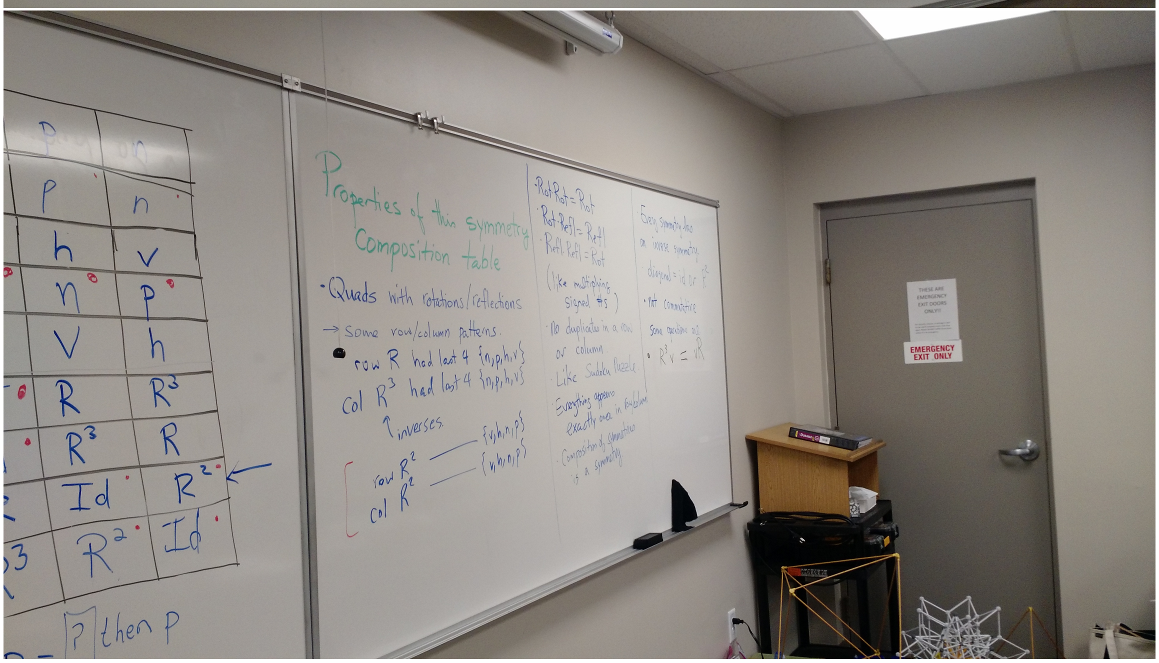
- What was your experience reading the articles? What were some things you liked?
- In what ways will math articles be different than scientific ones?
- How can we incorporate these articles into our teaching?
- Would your class be interested in reviewing a paper?
- Is there feedback we can offer to the publisher?

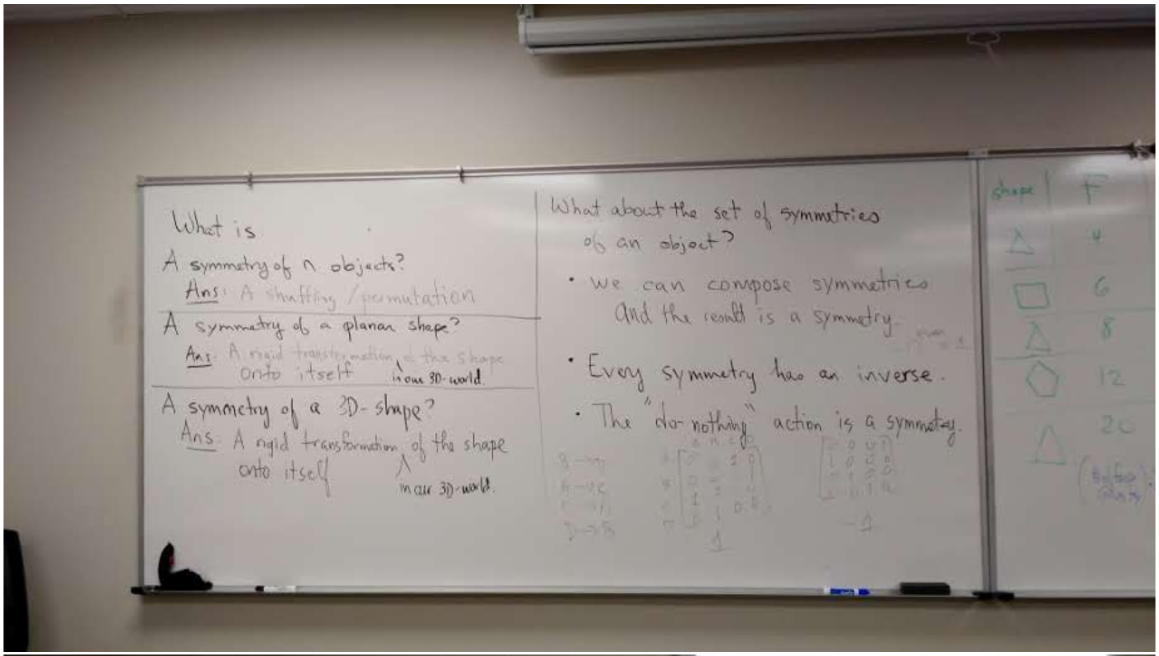
Our evolving views on Symmetry

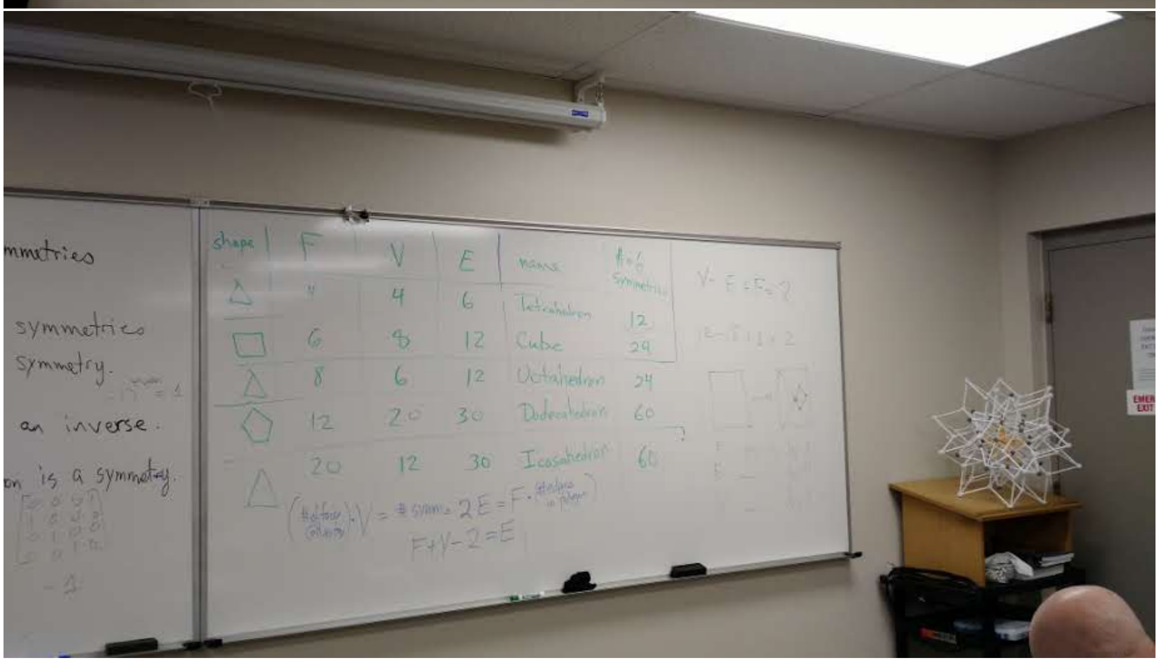





Composition Table
R = rotate 90°
{h, v, p, n}
{hor, vert, pos slope, neg slope}
first
second
Id R R² R³ h v p n
Id Id R R² R³ h v p n
R R R² R³ Id n p h v
R² R² R³ Id R v h n p
R³ R³ Id R R² p n v h
h h p v n Id R² R R³
v v n h p R² Id R³ R
p p v n h R³ R Id R²
n n h p v R R³ R² Id
R = ?∘h
R = ? then p
R/p = v then p


Properties of this symmetry Composition table
·Quads with rotations/reflections
→ some row/column patterns.
row R had last 4 {n, p, h, v}
col R³ had last 4 {n, p, h, v}
inverses.
row R² {v, h, n, p}
col R² {v, h, n, p}
·Rot·Rot = Rot
·Rot·Refl = Refl
·Refl·Refl = Rot
(like multiplying signed #s)
·No duplicates in a row or column
·Like Sudoku Puzzle
·Everything appears exactly once in row/column
·Composition of symmetries is a symmetry
·Every symmetry has an inverse symmetry.
diagonal = id or R²
·not commutative
Some operations are
· R³v = vR
THESE ARE EMERGENCY EXIT DOORS ONLY!
EMERGENCY EXIT ONLY


What is
A symmetry of n objects?
Ans: A shuffling / permutation
A symmetry of a planar shape?
Ans: A rigid transformation of the shape onto itself in our 3D-world.
A symmetry of a 3D-shape?
Ans: A rigid transformation of the shape onto itself in our 3D-world.
What about the set of symmetries of an object?
• We can compose symmetries and the result is a symmetry.
• Every symmetry has an inverse.
• The "do-nothing" action is a symmetry.


shape
F
V
E
name
# of symmetries
4
4
6
Tetrahedron
12
6
8
12
Cube
24
8
6
12
Octahedron
24
12
20
30
Dodecahedron
60
20
12
30
Icosahedron
60
V - E + F = 2
F + V - 2 = E