GOOD LUCK!!!!

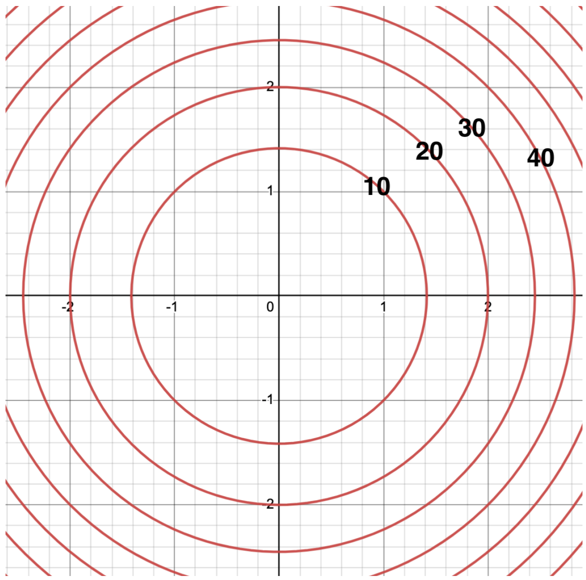
1. (24 points total) Complete the following multiple choice questions:

   (a) If two vectors in $\mathbb{R}^3$ are perpendicular then their dot product is [sometimes/always/never] equal to 0.

   (b) The line given by $\mathbf{r}(t) = \langle 3 - 2t, 4t, 1 - 8t \rangle$ intersects the xy plane [True / False]

   (c) The plane $3x - 4y + 10z = 4$ passes through the origin [True / False]

   (d)



   In the picture above, which is a contour map for a function $f(x, y)$, draw and label points with the following properties. If it is not possible to draw such a point, don't do so.

   - P where f_x is positive
   - Q where f_x is negative
   - R where f_y is zero
   - S where f_y is negative.

   (e) Consider The sphere

$$(x - 3)^2 + (y - 4)^2 + (z - 5)^2 = 25.$$

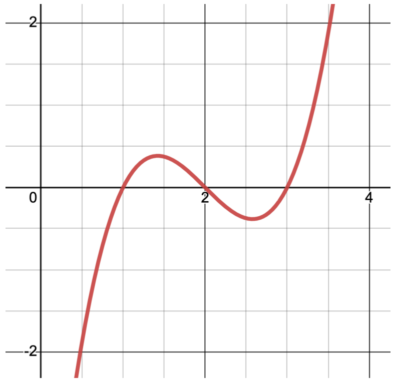
   For each of the following, circle the best answer. The sphere

   intersects the plane $z = 4$ in a [circle/point/line/set of two points/it will not intersect the plane].

   intersects the xy plane in a [circle/point/line/set of two points/it will not intersect the plane].

   intersects the x-axis in a [circle/point/line/set of two points/it will not intersect the x-axis].

(f) Antoine the ant is walking along a curve with parametrization: $\mathbf{c}(t) = \langle x(t), y(t)\rangle$. You are given a graph of $x(t)$ and a table of values for $y(t)$. Use this information to answer the questions below. You may need to make some estimations from the graph. Do your best!

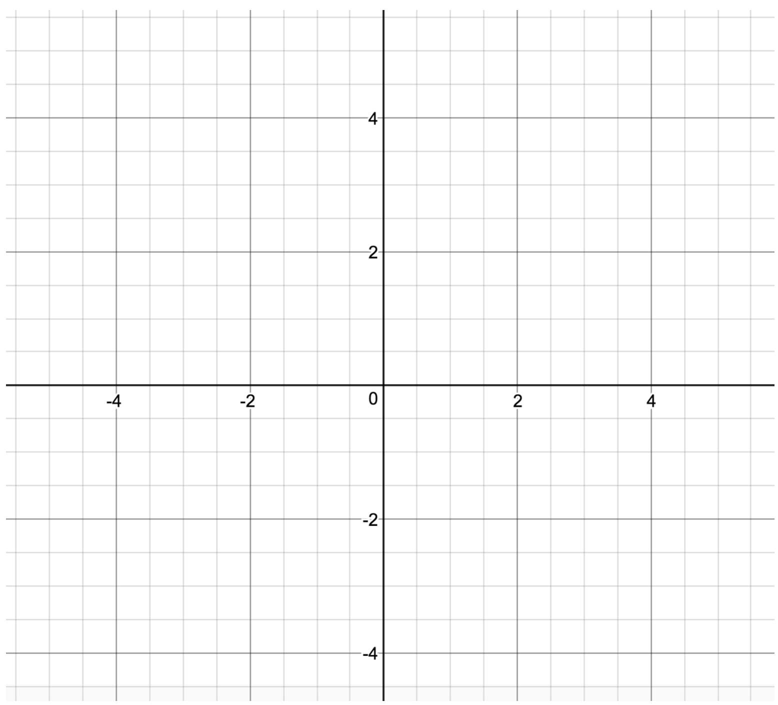


A graph of $x(t)$

t	0	1	2	3
$y(t)$	8	5	3	2
$y'(t)$	-1	-3	-2	-1

i. Below, label with the letter P where Antoine is at time $t = 2$.

ii. Draw the tangent vector to Antoine's path at the point you just drew. (NOTE the gridlines on the graph)



2. (a) (5 points) Find the equation of the plane through the origin that contains the vectors $\langle 1,2,4\rangle$ and $\langle 0,2,1\rangle$

(b) (5 points) Find the cosine of the angle that the line $\mathbf{r}(t) = \langle 3-2t, 4t, 1-t\rangle$ makes with the vector normal to the plane $3x - 4y = 4$.

(c) (7 points) Suppose that $f(x,y)$ is a differentiable function that gives the temperature at a point (x,y) in the plane. You are given that.

$$f(4,5) = 10, \qquad \frac{\partial f}{\partial x}(4,5) = -3, \qquad \frac{\partial f}{\partial y}(4,5) = 6.$$

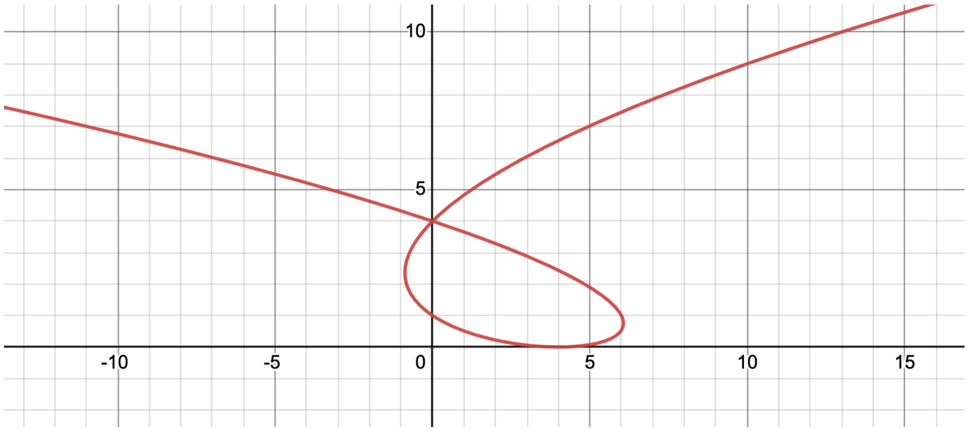
Use this information to answer the following:

i. If you were standing at the point $(4,5)$ and were heading EAST (in the x direction) then the temperature should [increase / decrease.]

ii. At the point $(4,5)$ in which direction would the temperature be increasing the most? Write down a position vector.

iii. Write down the linear approximation for $f(x,y)$ at the point $(4,5)$.

3. A particle is moving along the curve parametrized by $\mathbf{c}(t) = \left(t^3 - t^2 - 4t + 4, t^2\right)$ where t is measured in seconds. A picture of this curve is below.



(a) (5 points) This is a chance to redo a similar problem to last week's quiz. Calculate and **draw** the two tangent vectors to the curve at the point $(0, 4)$. Show your work carefully.

(b) (4 points) Write down an integral that represents the length of the "loop" at the bottom of the curve. Please do NOT simplify your expression or foil anything out. The work you did in part a should make this fairly simple.

(c) (This is still continuing the problem on the previous page)
Suppose the temperature of each point (x, y) in the plane is given by

$$F(x, y) = x^2y + x^3 - 2y$$

(in degrees F).

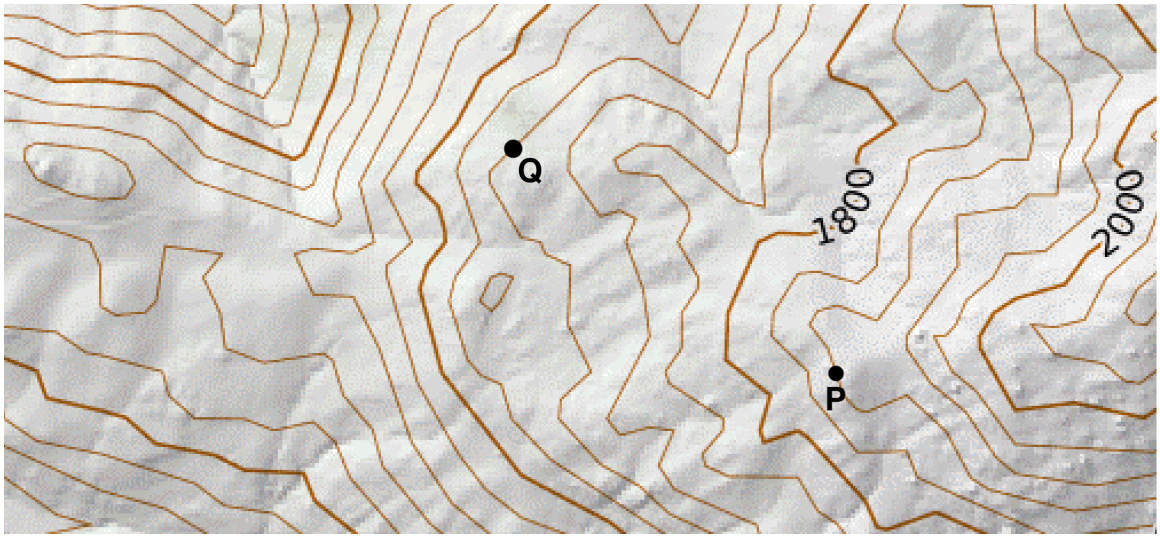
i. (4 points) Find $\nabla f(x, y)$

ii. (3 points) What is the temperature of the particle at time $t = 1$?

iii. (3 points) What is the speed of the particle at time $t = 1$?

iv. (5 points) At rate (in degrees per second) is the temperature of the particle increasing at the time it passes through the point $(0, 1)$?

4. Consider the contour map below of the popular hiking area known with the famous "potato chip rock." Suppose that $f(x,y)$ is the function that gives the vertical height (in feet) of the mountain over a point (x,y).



(a) (2 points) Assuming the lines are level curves that are equally spaced, what is the elevation at the point P? Include units!

(b) (2 points) At the indicated point Q draw a vector indicating the direction of the vector ∇f_Q.

(c) (4 points) Draw ONE curve $c(t)$ for $0 \leq t \leq 4$ on the graph with ALL of the following properties. Label time stamps for $t = 0, t = 1$, $t = 2$, $t = 3$ for your curve.

i. $f(c(0)) = 1600$

ii. At $t = 1$, $\dfrac{d}{dt}(f(c(t)) = 0$

iii. At $t = 2$, $\dfrac{d}{dt}(f(c(t)) > 0$

iv. At $t = 3$, $\dfrac{d}{dt}(f(c(t)) < 0$

v. ALSO at $t = 3$, $f(c(t)) = P$.

5. Let S be the surface given by

$$z = xy.$$

(a) (5 points) Parametrize the intersection of S with the cylinder $x^2 + y^2 = 25$. Include bounds for your parameter t.

(b) (7 points) Antoine the bug is flying along the parametrized curve given by $\mathbf{r}(t) = \langle t, t-1, 3t-4 \rangle$. At what point(s) does Antoine intersect the surface S? Make sure you read this problem carefully and you know what S is.

Extra Credit (4 points possible)

Let S be the surface defined by $x^2 + xy - y^3 + ax + z = 2$. Suppose that S contains a curve C whose tangent line at the point $(0, 0, -2)$ is parametrized by $(0, 0, -2) + t(1, 2, 3)$. Compute a. Explain your steps.

Math 250 Midterm 2

Your Name

- You have 55 minutes to do this exam.
- You may use a scientific calculator and a small note card.
- For justifications, please use complete sentences and make sure to explain any steps which are questionable.
- Good luck!

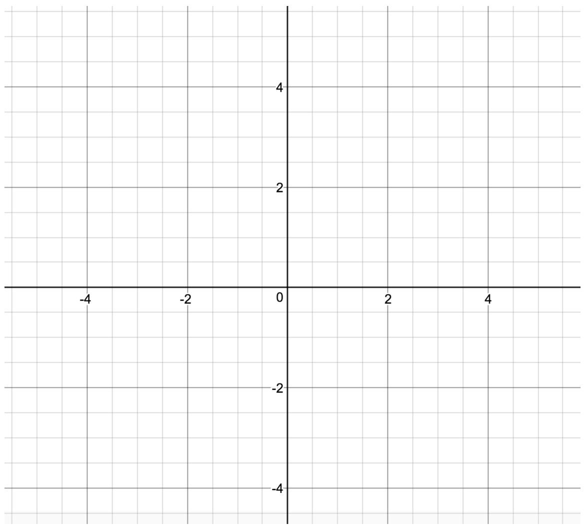
Problem	Total Points	Score
1	35	
2	18	
3	12	
4	10	
EC	4	
Total	75	

GOOD LUCK!!!!

1. (a) (6 points) John B. is looking for sunken treasure and while on the surface of the ocean discovers that the temperature at given point (x, y are measured in feet) is given by $f(x, y) = x^2 e^y + 72$ (in degrees F). Fill in the blanks below. There will be partial credit, so show your work:

   If John B. is at the point $(1, 0)$ then maximal rate of change of the temperature will be

   ____________________________________ (include units!) if he swims in the direction indicated below (I am looking for you to draw a vector)



   (b) (4 points) Meanwhile, Antoine the ant is walking along the surface of the earth along the ocean floor. This part of the ocean is a mountainous terrain whose height is given by $f(x, y)$. Antoine arrives at a saddle point with coordinates (a, b). Circle the best choices below:

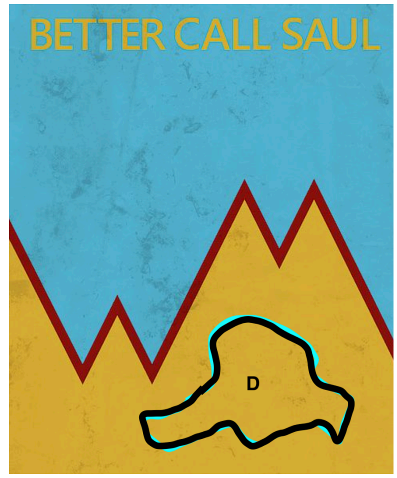
   $f_x(a, b)$ is [positive / negative / zero / can't tell]

   $f_y(a, b)$ is [positive / negative / zero / can't tell]

   Write down 3 numbers in the blanks below that would be possible for the second partial derivatives. (There will be many correct answers, you just need to make sure that the point is a saddle point.)

   $f_{xx}(a, b) =$ ____________, $f_{yy}(a, b) =$ ____________, $f_{xy}(a, b) =$ ____________

(c) (6 points) Excited by the potential discovery of sunken treasure, John's legal team, the law firm Wexler and McGill designs a logo that looks like the picture at the right. In your investigations, you discover that a part of their logo, called D has center of mass at the point $(5,4)$ and that $\text{Area}(D) = 8$. Using this information, fill in the blanks:



$$\iint_D 2\ dA = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

and

$$\iint_D 3x\ dA = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(Note: you will NOT be able to determine anything from the picture. This question is about using area and center of mass to solve these problems.)

(d) (4 points) To help the treasure seekers, Wednesday Adams volunteers to integrate the function $x^2 + y^2$ over a region in the plane and wants to use polar coordinates. You are not sure what the region looks like, it could be part of a circle, or perhaps the region between two circles. You don't know! However, only one of the following could possibly be a correct setup - circle the correct one and write a short 1-2 sentence explanation why.

$$\int_0^{\pi/2}\int_0^3 r^3 dr\ d\theta \qquad \int_{-\pi}^{\pi/2}\int_3^6 r\cos\theta dr\ d\theta$$

$$\int_0^{3\pi/2}\int_0^3 r^2 dr\ d\theta \qquad \int_0^{\pi/2}\int_0^3 r dr\ d\theta$$

Explanation:

(e) (4 points) The cone defined by $z = 7\sqrt{x^2 + y^2}$ makes what angle ϕ with the z axis? You may leave your answer in terms of inverse trig functions. (Partial Credit will be given)

(f) (2 point) In spherical coordinates, the plane $z = 1$ is given by which of the following?

- $\rho = \sin^2 \phi$
- $\rho = \dfrac{1}{\cos \phi}$
- $\rho = \cos \phi \sin \theta$
- $\rho = \phi + \theta$

(g) (4 points) Consider the region W described in spherical coordinates by

$$0 \leq \rho \leq 4, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq \pi/2$$

Then the x coordinates of points in W are [always ≥ 0, always ≤ 0, some mixture]

Then the y coordinates of points in W are [always ≥ 0, always ≤ 0, some mixture]

Then the z coordinates of points in W are [always ≥ 0, always ≤ 0, some mixture]

W is best described as a [whole sphere / hemisphere / quarter sphere / eighth of a sphere].

(h) (5 points) While swimming, John B. sees a strange sea creature whose shape is given by the region bounded between the two paraboloids:

$$z = 14 - x^2 - y^2, \qquad z = 2 + 2x^2 + 2y^2.$$

Draw a rough 3D picture of this sea creature. I just want you to draw the rough general shape. You don't have to calculate the shadow.

2. Consider the region W described in spherical coordinates by the following description:

$$0 \le \rho \le 4, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/4$$

(6 points) Draw a rough 3D picture of W **AND** draw a picture of the **shadow** of this region in the xy plane.

(12 points) In the boxes below, set up the integral of the function

$$f(x, y, z) = x$$

for this region in spherical and cylindrical coordinates. You just need to fill in the blanks. Notice that someone helped you fill in one of the blanks!

Spherical Coordinates

$$\int \int \int \underline{\hspace{6cm}} \, d\rho \, d\phi \, d\theta$$

Cylindrical Coordinates

$$\int \int \int_{z=r} \underline{\hspace{6cm}} \, dz \, dr \, d\theta$$

3. (12 points) John B. has journeyed down to the ocean floor and is now working with Antoine the ant to find treasure. They encounter a pesky math professor who asks them to classify the critical points of the function

$$g(x, y) = x^3 - 3x^2 + y^2 - 4y.$$

Find all critical points of this function and classify them as local min / local max / saddle in order to appease the professor.

4. (10 points) On the ocean floor they discover a metal plate whose shape is the region D.

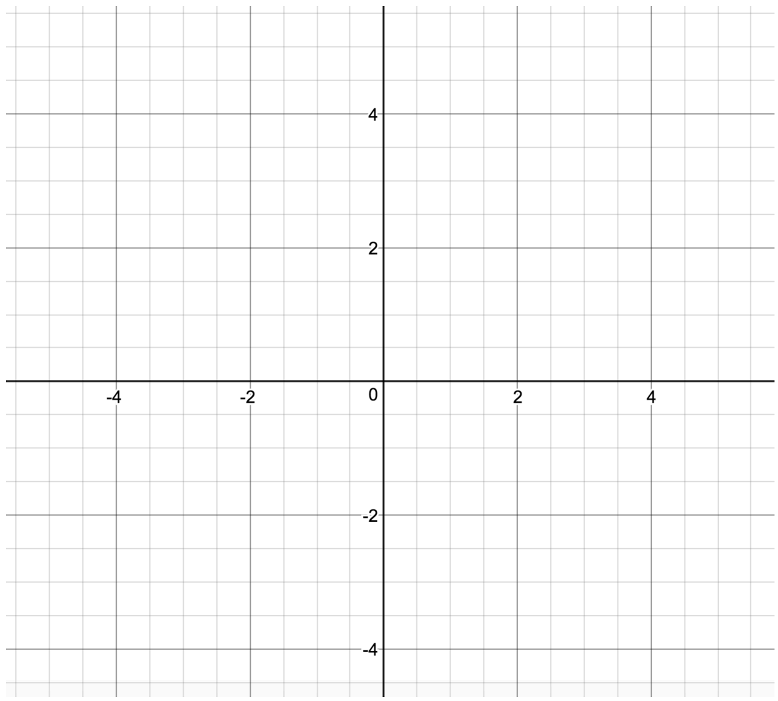
If $f(x,y)$ measures the density (in grams per m^2) of the plate at each point, then $\iint\limits_D f(x,y)\,dA$ gives the ____________________________ of the whole plate.

(This is a four letter word that starts with m and has 2 s's).

If

$$\iint\limits_D f(x,y)\,dA = \int_{-4}^{2}\int_{\frac{1}{2}x-2}^{3} f(x,y)\,dy\,dx.$$

then sketch the region D **and** change the order of integration by setting up an integral in the order $dx\,dy$.



Extra Credit (4 points)

Find the volume of the region contained in both of the following cylinders:

$$x^2 + y^2 = 1, \qquad x^2 + z^2 = 1$$

Math 250 Final Exam

Your Name

- You have 120 minutes to do this exam.
- You may use a scientific calculator and a small note card.
- For justifications, please use complete sentences and make sure to explain any steps which are questionable.
- Good luck!

Problem	Total Points	Score
1	20	
2	25	
3	8	
4	15	
5	10	
6	5	
7	14	
8	10	
9	8	
EC	4	
Total	115	

1. (20 points) The following are True/False Questions. You don't need to write any justification.

(1) $\boxed{\textbf{T} \mid \textbf{F}}$ The dot product $\langle 1,1,2\rangle \cdot \langle 2,3,4\rangle = 13$.

(2) $\boxed{\textbf{T} \mid \textbf{F}}$ If $\mathbf{c}(t)$ is a curve in $\mathbb{R}^3$ then then $\mathbf{c}'(t_0)$ gives a vector perpendicular to the curve at time t_0.

(3) $\boxed{\textbf{T} \mid \textbf{F}}$ If $\nabla f_{(3,1)} = \langle 0,0\rangle$ and $f_{xx}(3,1) > 0$ then $(3,1)$ is a local minimum of f.

(4) $\boxed{\textbf{T} \mid \textbf{F}}$ If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ and S is the boundary of a $3D$ solid E then the triple integral $\iiint\limits_E \text{div}\mathbf{F}\ dV$ is always zero since the flux of $\mathbf{F}$ through the boundary surface is zero.

(5) $\boxed{\textbf{T} \mid \textbf{F}}$ If D is a bounded region in the plane, then $\iint\limits_D x\ dA$ gives the x coordinate of the center of mass of D.

(6) $\boxed{\textbf{T} \mid \textbf{F}}$ Suppose that P is a critical point of a function $f(x,y)$. Then $|\nabla f_P| \geq 1$.

(7) $\boxed{\textbf{T} \mid \textbf{F}}$ The curl of a conservative vector field in $\mathbb{R}^3$ is zero.

(8) $\boxed{\textbf{T} \mid \textbf{F}}$ A constant vector field is always conservative.

(9) $\boxed{\textbf{T} \mid \textbf{F}}$ Let L be a line in $\mathbb{R}^3$. The paraboloid $z = x^2 + y^2$ will definitely intersect L.

(10) $\boxed{\textbf{T} \mid \textbf{F}}$ If $\nabla f_P = \langle 2,3\rangle$ then $f_x(P) = 2$.

(11) $\boxed{\textbf{T} \mid \textbf{F}}$ The volume of a 3D solid E is equal to the flux of the vector field $\langle 0,y,0\rangle$ through the boundary surface.

(12) $\boxed{\textbf{T} \mid \textbf{F}}$ If S is part of the surface of a sphere then $\iint\limits_S dS$ is the surface area of S.

(13) $\boxed{\textbf{T} \mid \textbf{F}}$ The vector $\langle 1,1,1\rangle$ is perpendicular to the plane $z = x + y + 1$.

(14) $\boxed{\textbf{T} \mid \textbf{F}}$ If S is a sphere of radius R with outward pointing normal vector and $\mathbf{F}$ is a constant vector field then $\iint\limits_S \mathbf{F} \cdot \mathbf{dS} = 4\pi R^2$.

(15) $\boxed{\textbf{T} \mid \textbf{F}}$ If E is a solid with volume 1, then $\iiint\limits_E x\ dV$ is equal to the x coordinate of the center of mass of E.

(16) $\boxed{\textbf{T} \mid \textbf{F}}$ The work done by a conservative vector field along a path only depends on the endpoints of the path.

(17) $\boxed{\textbf{T} \mid \textbf{F}}$ If a vector field $\mathbf{F}(x,y)$ is a gradient of some potential function $f(x,y)$ then the line integral $\int_C \mathbf{F} \cdot \mathbf{ds}$ along any closed curve C will be zero.

(18) $\boxed{\textbf{T} \mid \textbf{F}}$ On a contour map, the gradient vector ∇f is tangent to the level curves of f.

(19) | **T** | **F** | If $\mathbf{u}, \mathbf{v}$ are two perpendicular vectors in $\mathbb{R}^3$ then $\mathbf{u} \times \mathbf{v}$ is the zero vector.

(20) | **T** | **F** | Suppose that $f_x(a,b) = 2$ and $f_y(a,b) = 3$, then there is a direction in which the rate of change of f at (a,b) is zero.

2. (Describing Geometric Regions)

A) (5 points) Draw a careful 3D picture of the region in $\mathbb{R}^3$ described in spherical coordinates by:

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/3, \quad 0 \le \rho \le 4.$$

make sure you include any lengths or angles in your picture.

B) (5 points) Draw a 3D picture of the the region described by:

$$x^2 + y^2 \le z \le 12 - 2x^2 - 2y^2$$

C) (5 points) Find the intersection of the surfaces $z = x^2 + y^2$ and $z = 12 - 2x^2 - 2y^2$. Your answer should be a curve and you should describe your answer by giving a parametrization $\mathbf{c}(t)$. Remember your answer should have three components since we are in $\mathbb{R}^3$. **Hint: This is related to the previous problem - you might want to update your picture.**

D) (10 points) **There are TWO parts to this problem.**

Suppose that S is the part the part of the upper half (meaning $z \geq 0$) of the sphere $x^2 + y^2 + z^2 = 16$ that is inside the cylinder $x^2 + y^2 = 4$. The surface of S is gilded with gold such that the density of gold at the point is given by $f(x, y, z) = z$ (in g/m^2).

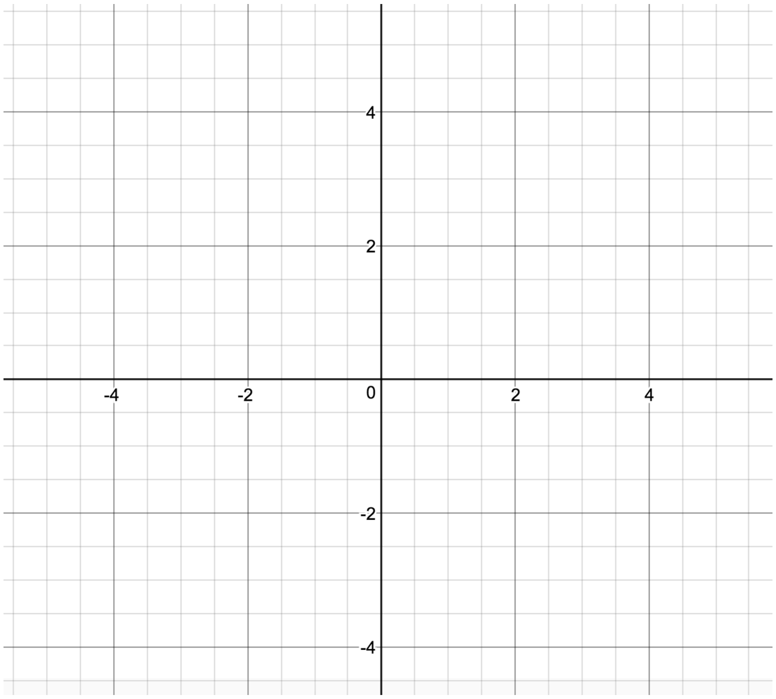
i) Draw a careful picture of S, labeling any important information.

ii) Set up an iterated integral using ϕ, θ that will calculate the total mass of gold on the surface S.

3. (8 points) A student is setting up an integral of a function $f(x,y)$ over a region D and writes down:

$$\iint_D f(x,y)\ dA = \int_{-3}^{-1}\int_{-2}^{2y+4} f(x,y)\ dx\ dy + \int_{-1}^{3} + \int_{-2}^{1-y} f(x,y)\ dx\ dy.$$

Draw a picture of the region of integration AND change the order of integration by setting up the integral $\iint_D f(x,y)\ dA$ as a $dydx$ integral.



4. (15 points) **Some shorter answer questions**:

A) Consider the region W described in spherical coordinates by

$$0 \le \rho \le 4, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le \pi/2$$

Then the x coordinates of points in W are [always ≥ 0, always ≤ 0, some mixture]

Then the y coordinates of points in W are [always ≥ 0, always ≤ 0, some mixture]

Then the z coordinates of points in W are [always ≥ 0, always ≤ 0, some mixture]

W is best described as a [whole sphere / hemisphere / quarter sphere / eighth of a sphere].

B) Consider the sphere:

$$(x-5)^2 + (y-4)^2 + (z-3)^2 = 25.$$

For each of the following, circle the best answer:

(a) The sphere intersects the plane $z = 4$ in a [line / circle / point /set of two points / it does not intersect]

(b) The sphere intersects the yz plane in a [line / circle / point / set of two points / it does not intersect]

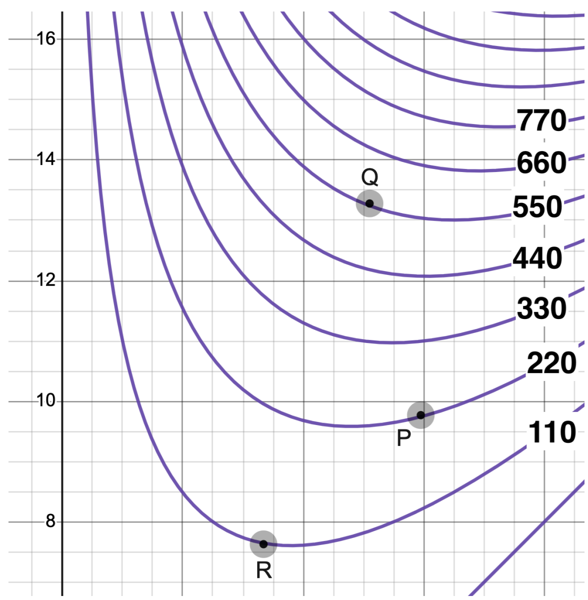
(c) The sphere intersects the y axis in a [line / circle / point / set of two points / it does not intersect]

C) If $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^3$ then

a) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u}$ is [undefined / 0 the scalar / $\mathbf{0}$ the zero vector / something else]

b) $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{u}$ is [undefined / 0 the scalar / $\mathbf{0}$ the zero vector/something else.]

D) The contour map of a function $f(x, y)$ is given in the figure at right.



(a) If you were to draw in gradient vectors at R, and Q which one would be longer? Why?

(b) If C is the curve consisting of a straight line path from R to Q followed by a path from Q to P, what is $\int_C \nabla \mathbf{f} \cdot \mathbf{ds}$? Explain your answer.

5. (10 points) Suppose that C is the straight line starting at $(0,2)$ and ending at the point $(-1,4)$. A student wants to find the following line integral:

$$\int_C xy\ dx + x\ dy$$

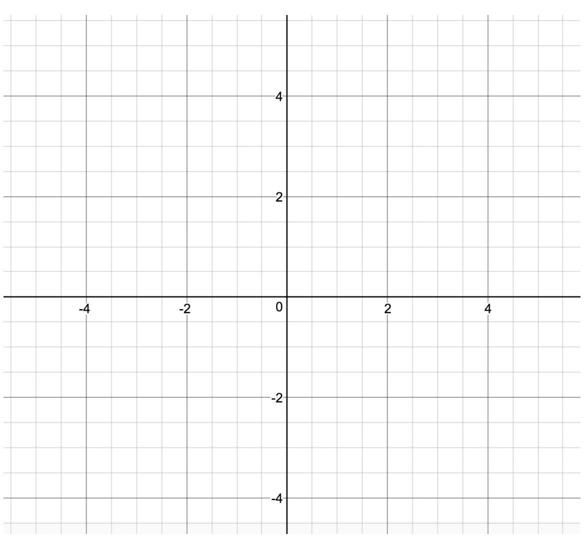
(a) Explain why the student cannot use The Fundamental Theorem for Gradients to solve this problem.

(b) Explain why the student cannot use Green's Theorem to solve this problem.

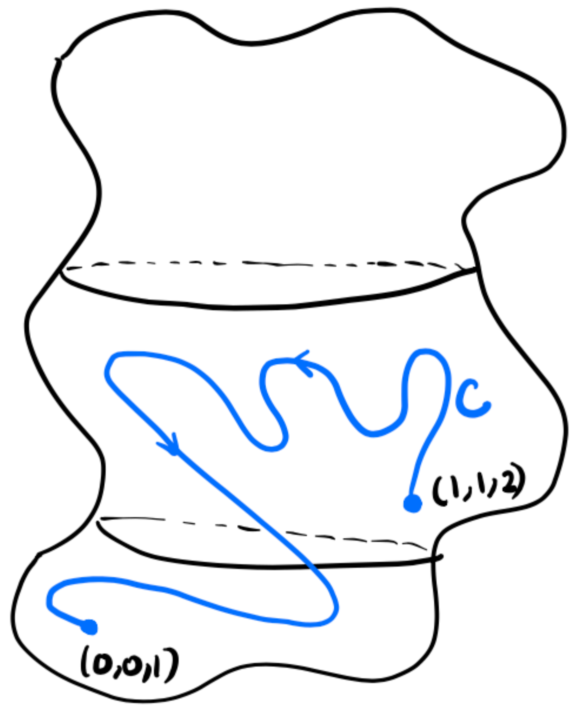
(c) By parametrizing the curve, set up a definite integral that will calculate this line integral. You do NOT have to evaluate the integral.

6. (5 points) John B. is looking for gold in a swamp and while walking discovers that the temperature at a given point (x, y are measured in feet) is given by $T(x,y) = x^2 - 3y^2 + 72$ (in degrees F). Fill in the blanks below:

If John B. is at the point $(1,1)$ then the maximal rate of change of temperature will be ______________________________ (include units!) if he swims in the direction indicated below (I am looking for you to draw a vector).



7. (14 points) A Let S be the surface of the 3D figure pictured at right



Suppose that S is oriented with outward pointing normal vector;

S has surface area 12 square meters and encloses a volume of 4 cubic meters;

The center of mass of that 3D solid is at the point $(1, 2, 5)$;

Antoine the ant has crawled along the indicated curve C on the surface that starts at the point $(1, 1, 2)$ and ends at the point $(0, 0, 1)$.

$$\mathbf{F} = \langle 3x, 3y + \sin(xz), z^2 \rangle, \qquad \mathbf{G} = \langle yz, xz, xy + 2z \rangle.$$

Use this information to find the following three integrals:

A) $\iint\limits_S 4 \, dS$

B) $\iint\limits_S \mathbf{F} \cdot \mathbf{dS}$

C) $\int_C \mathbf{G} \cdot \mathbf{ds}$

8. (10 points) Let S be the surface described by

$$z = 25 - x^2 - y^2, \qquad z \geq 0.$$

Setup fully the integral $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{dS}$ where $\mathbf{F}$ is the vector field $\langle x, z, 2 \rangle$. You may leave your answer as an iterated integral in whatever coordinate system you want. Please expand any dot products.

9. (8 points) Suppose that C is an ellipse in the xy plane that bounds a region whose center of mass is at the point $(4, 3)$. Suppose that C bounds a region of area 4π. Find

$$\int_C (4x^2 + y^2)\ dx + (xy + x)\ dy$$

where C is oriented counterclockwise. **Show your work neatly.**

Extra Credit

Let

$$\mathbf{F} = \langle y^3, x - x^3 \rangle.$$

Find the closed curve C in the plane such that the integral $\int_C \mathbf{F} \cdot \mathbf{ds}$ has the largest possible value. Explain your reasoning and find this largest possible value. Hint: A theorem we learned will help you!