Lower Bounds on Betti Numbers

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Dedicated to David Eisenbud on the occasion of his 75th birthday.

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1 Introduction

Consider a polynomial ring over a field k, say $R = k[x_1, \ldots, x_n]$. When studying finitely generated graded modules M over R, there are many important invariants we may consider, with the Betti numbers of M, denoted $\beta_i(M)$, being among some of the richest. The Betti numbers are defined in terms of generators and relations (see Sect. 2), with $\beta_0(M)$ being the number of minimal generators of M, $\beta_1(M)$ the number of minimal relations on these generators, and so on. Despite this simple definition, they encode a great deal of information. For instance, if one knows the Betti numbers[1] of M, one can determine the Hilbert series, dimension, multiplicity, projective dimension, and depth of M. Furthermore, the Betti numbers provide even finer data than this, and can often be used to detect subtle geometric differences (see Example 3.4 for an obligatory example concerning the twisted cubic curve).

There are many questions one can ask about Betti numbers. What sequences arise as the Betti numbers of some module? Must the sequence be unimodal? How small, or how large, can individual Betti numbers be? How large is the sum? Questions like

[1] Really, we mean the graded Betti numbers of M, to be defined in Sect. 3.

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this are but just a few examples of those that have been studied in the past decades, and of the flavor we will discuss in this survey. We will focus on perhaps one of the longest standing open questions in this area, which is due to Buchsbaum–Eisenbud, and independently Horrocks (BEH). Their conjecture proposes a lower bound for each $\beta_i(M)$ depending only on the codimension c of M: that $\beta_i(M) \geqslant \binom{c}{i}$. While the conjecture remains widely open in the general setting, there are some special cases that are known. Moreover, if the conjecture is true, then the total Betti number of M, $\beta(M) := \beta_0(M) + \cdots + \beta_n(M)$, must satisfy $\beta(M) \geqslant 2^c$. Recently, Mark Walker [69] proved this bound on the total Betti number—known as the Total Rank Conjecture—in all cases except when char $k = 2$. Walker also showed that equality holds if and only if M is isomorphic to R modulo a regular sequence—such modules are called complete intersections.

The Betti numbers of modules that are *not* complete intersections are quite interesting. For example, it follows from Walker's result that if our module M is not a complete intersection, then $\beta(M) \geqslant 2^c + 1$, but there is reason to believe that $\beta(M)$ might be much bigger than 2^c. Charalambous, Evans, and Miller [31] asked if in fact we must have $\beta(M) \geqslant 2^c + 2^{c-1}$, and proved that this holds when M is either a graded module small codimension ($c \leqslant 4$), or a multigraded module of finite length (meaning $c = n$) for arbitrary c [29, 30]. More evidence towards this larger bound for Betti numbers has recently been found, including [11, 12].

For example, Erman showed [41] that if M is a graded module of small regularity (in terms of the degrees of the first syzygies), then not only is the BEH Conjecture 4.1 true, but in fact $\beta_i(M) \geqslant \beta_0(M)\binom{c}{i}$. The first author and Wigglesworth [12] then extended Erman's work to say that under the same low regularity hypothesis, $\beta(M) \geqslant \beta_0(M)(2^c + 2^{c-1})$. This stronger bound asserts that on average, each Betti number $\beta_i(M)$ is at least 1.5 times $\beta_0(M)\binom{c}{i}$.

The main goal of this survey is to discuss these lower bounds on Betti numbers and present some of the motivation for these conjectures. We start with a short introduction to free resolutions and Betti numbers, why we care about them, and some of the very rich history surrounding these topics. We also collect some open questions, discuss some possible approaches, and present examples that explain why certain hypothesis are important.

2 What Is a Free Resolution?

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. We will be primarily concerned with **finitely generated graded R-modules M**. One important invariant of such a module is the minimal number of elements needed to generate M. In fact, this number is the first in a sequence of **Betti numbers** that describe how far M

is from being a free module. Indeed, suppose that M is minimally generated by β_0 elements; this means there is a surjection from R^{β_0} to M, say

$$R^{\beta_0} \xrightarrow{\pi_0} M.$$

If π_0 is an isomorphism, then $M \cong R^{\beta_0}$ is a **free module** of **rank** β_0. Otherwise, it has a nonzero kernel, which will also be finitely generated and can be written as the surjective image of some free module R^{β_1}:

$$\begin{tikzcd} R^{\beta_1} \arrow[rr, dashed] \arrow[dr] & & R^{\beta_0} \arrow[r, two heads, "\pi_0"] & M. \\ & \ker(\pi_0) \arrow[ur, hook] & & \end{tikzcd}$$

Notice that if M is generated by $m_1, \ldots, m_{\beta_0}$, and π_0 is the map sending each canonical basis element e_i in R^{β_0} to m_i, then an element $(r_1, \ldots, r_{\beta_0})^T$ in the kernel of π_0 corresponds precisely to a **relation** among the m_i, meaning that

$$r_1 m_1 + \cdots + r_{\beta_0} m_{\beta_0} = 0.$$

Such relations are called **syzygies**[2] of M and the module $\ker \pi_0$ is called the first syzygy module of M.

Continuing this process we can *approximate* M by an exact sequence

$$\cdots \longrightarrow F_p \xrightarrow{\pi_p} \cdots \xrightarrow{\pi_2} F_1 \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} M \longrightarrow 0$$

where each F_i is free. Such an exact sequence is called a **free resolution of** M.

If at each step we have chosen F_i to have the minimal number of generators, then we say the resolution is **minimal**, and we set $\beta_i(M)$ to be the rank of F_i in any such minimal free resolution. This is well-defined, because it is true that two minimal free resolutions of M are isomorphic as complexes. Furthermore, one has the following,

$$\beta_i(M) = \operatorname{rk} F_i = \operatorname{rk}_k \operatorname{Tor}_i^R(M, k).$$

The i**th syzygy module of** M, denoted $\Omega_i(M)$, is defined to be the image of π_i, or equivalently the kernel of π_{i-1}. We note that $\Omega_i(M)$ is defined only up to isomorphism.

If at some point in the resolution we obtain an injective map of free modules, then its kernel is trivial, and we obtain a finite free resolution, in this case of length p:

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

[2] Fun fact: in astronomy, a syzygy is an alignment of three or more celestial objects.

If a module M has a finite minimal projective resolution, the length of such a resolution is called the projective dimension of M, and we write it $\operatorname{pdim} M$.

Remark 2.1 We will often implicitly apply the Rank-Nullity Theorem to conclude that

$$\beta_i(M) = \operatorname{rk}\Omega_i(M) + \operatorname{rk}\Omega_{i+1}(M).$$

Example 2.2 If $M = R/(f_1, \dots, f_c)$ where the f_i form a regular sequence, then the minimal free resolution of M is given by the **Koszul complex.** For instance if $c = 4$ then the minimal resolution has the form

$$0 \longrightarrow R^1 \xrightarrow{1} R^4 \xrightarrow{3} R^6 \xrightarrow{3} R^4 \xrightarrow{1} R^1 \xrightarrow{0} M.$$

Note that the numbers over the arrows represent the rank of the corresponding map, which is equal to the rank of the corresponding syzygy module $\Omega_i(M)$. We will discuss this in more detail in Sect. 3.2. We will also see that the ranks occurring in the Koszul complex are conjectured to be the smallest possible for modules of codimension c (see Conjecture 4.2).

Example 2.3 One of the strongest known bounds on ranks of syzygies is the Syzygy Theorem 3.13 which states that except for the last syzygy module, the rank of $\Omega_i(M)$ is always at least i. A typical use of such a result might be as follows. Suppose we had a rank zero module M with Betti numbers $\{1, 7, 8, 8, 7, 1\}$. Then we could calculate the ranks of the syzygy modules by using Remark 2.1 to obtain the ranks labeled in the diagram below:

$$0 \longrightarrow R^1 \xrightarrow{1} R^7 \xrightarrow{6} R^8 \xrightarrow{2} R^8 \xrightarrow{6} R^7 \xrightarrow{1} R^1 \xrightarrow{0} M.$$

We would also obtain from Remark 2.1 that $\operatorname{rk}\Omega_3(M) = 2$, which we will see violates Theorem 3.13. Therefore, such a module does not exist! See also Example 5.17.

Example 2.4 In [36], Dugger discusses almost complete intersection ideals and the tantalizing fact that we currently do not know whether or not there is an ideal I of height 5 with minimal free resolution

$$0 \longrightarrow R^6 \xrightarrow{6} R^{12} \xrightarrow{6} R^{10} \xrightarrow{4} R^9 \xrightarrow{5} R^6 \xrightarrow{1} R^1 \xrightarrow{0} R/I.$$

David Hilbert, interested in studying minimal free resolutions as a way to count invariants, was able to prove that finitely generated modules over a polynomial ring always have finite projective dimension [49].

Theorem 2.5 (Hilbert's Syzygy Theorem, 1890) *Let $R = k[x_1, \dots, x_n]$ be a polynomial ring in n variables over a field k. If M is a finitely generated graded R-module, then M has a finite free resolution of length at most n.*

While we are primarily interested in studying polynomial rings over fields, Hilbert's Syzygy Theorem is true more generally for any Noetherian regular ring. In fact, if we focus our study on local rings instead, the condition that every finitely generated module has finite projective dimension characterizes regular local rings [3, 68]. While we will be working over polynomial rings throughout the rest of the paper, we point out that the theory of (infinite) free resolutions over non-regular rings is quite interesting and rich; [59] and [5] are excellent places to start learning about this.

The upshot of Hilbert's Syzygy Theorem is that to each finitely generated R-module M we attach a finite list of Betti numbers $\beta_0(M), \ldots, \beta_n(M)$. Note that while some of these might vanish, M has at most $n+1$ non-zero Betti numbers.

Our main goal in this paper is to discuss the following question:

Question A *If M is a finitely generated graded module over $R = k[x_1, \ldots, x_n]$, where k is a field, can we bound the Betti numbers of M, either from above or below?*

As we will see, there are many results and conjectures relevant to the answer to this question. Feel free to skip the next section if you can't handle the suspense!

3 Why Study Resolutions?

Before getting to the heart of the matter in Sect. 4, we would first like to offer some motivation as to why one might care about Betti numbers at all.

3.1 Betti Numbers Encode Geometry

In a sense, a minimal free resolution of M contains redundant information—after all, the first map $\pi_1\colon F_1 \to F_0$ is a presentation of M. However, suppose we do not know the maps in the resolution, but just the **numerical data** of the resolution, namely the numbers $\{\beta_i\}$. Surprisingly, this coarse invariant encodes much geometric and algebraic information about M. First of all, the Betti numbers β_i tell us that M has β_0 generators, that there are β_1 relations among those generators, and β_2 relations among those relations, and so on. But the Betti numbers also encode more sophisticated information about M. For instance, since rank is additive across exact sequences, we have

$$\operatorname{rk} M = \beta_0 - \beta_1 + \cdots + (-1)^n \beta_n.$$

Moreover, if we have a graded module M, we can take the resolution of M to be a graded resolution, and if among the β_i generators of $\Omega_i(M)$, exactly β_{ij} of them live in degree j, then the following formula gives the Hilbert series for M:

$$HS(M) = \frac{\sum_{i=0}^{d}(-1)^i \beta_{ij} t^j}{(1-t)^d}. \tag{3.1}$$

We recall that the **Hilbert series** of M is a power series that encodes the k-vector space dimension of each graded piece M_i of M, as follows:

$$HS(M) = \sum_{i=0}^{\infty} \dim_k(M_i) t^i.$$

This is a classical tool that contains important algebraic and geometric information about our module. For example, once we write $HS(M) = p(t)/(1-t)^m$ with $p(1) \neq 0$, we have $\dim(M) = m$ and $p(1)$ is equal to the degree of M. So just by knowing its (graded) Betti numbers, we can then determine the multiplicity (i.e. degree), dimension, projective dimension, Cohen-Macaulayness, and other properties and invariants of a module M.

The following example gives the spirit of these ideas:

Example 3.1 Suppose that $R = k[x, y, z]$ and that $M = R/(xy, xz, yz)$ corresponds to the affine variety defining the union of the three coordinate lines in k^3. This variety has dimension one and degree three. Let us illustrate how the (graded) Betti numbers communicate this. The minimal free resolution for M is

$$0 \longrightarrow R^2 \xrightarrow{\psi = \begin{bmatrix} z & 0 \\ -y & y \\ 0 & -x \end{bmatrix}} R^3 \xrightarrow{\phi = \begin{bmatrix} xy & xz & yz \end{bmatrix}} R \longrightarrow M.$$

From this minimal resolution, we can read the Betti numbers of M:

- $\beta_0 = 1$, since M is a cyclic module;
- $\beta_1 = 3$, and these three quadratic generators live in degree 2;
- $\beta_2 = 2$, and these represent linear (degree 1) syzygies on quadrics (degree 2), and thus live in degree 3 $(= 1 + 2)$.

We can include this *graded* information in our resolution, and write a *graded* free resolution of M:

$$0 \longrightarrow R(-3)^2 \xrightarrow{\psi = \begin{bmatrix} z & 0 \\ -y & y \\ 0 & -x \end{bmatrix}} R(-2)^3 \xrightarrow{\phi = \begin{bmatrix} xy & xz & yz \end{bmatrix}} R \longrightarrow M.$$

The $R(-2)^3$ indicates that we have three generators of degree 2. Formally, the R-module $R(-a)$ is one copy of R whose elements have their degrees shifted by a: the polynomial 1 lives in degree 0 in R and degree a in $R(-a)$, and in general the degree d piece of $R(-a)$ consists of the elements of R of degree $d-a$. With this convention, the map ϕ keeps degrees unchanged—we say it is a degree 0 map: for example, it takes the vector $[1,1,1]^T$, which lives in degree 2, to the element $xy+xz+yz$, which is an element of degree 2. When we move on to the next map, ψ, we only need to shift the degree of each generator by 1, but since ψ now lands on $R(-2)^3$, we write $R(-3)^2$.

The graded Betti number $\beta_{ij}(M)$ of M counts the number of copies of $R(-j)$ in homological degree i in our resolution. So we have

$$\beta_{00}=1,\ \beta_{12}=3,\ \text{and}\ \beta_{23}=2.$$

We can collect the graded Betti numbers of M in what is called a *Betti table*:

$$\begin{array}{r|ccc} \beta(M) & 0 & 1 & 2 \\ \hline 0 & \beta_{00} & \beta_{11} & \beta_{22} \\ 1 & \beta_{01} & \beta_{12} & \beta_{23} \\ & & & \end{array}, \qquad \begin{array}{r|ccc} \beta(M) & 0 & 1 & 2 \\ \hline 0 & 1 & - & - \\ 1 & - & 3 & 2 \\ & & & \end{array}.$$

Remark 3.2 To the reader who is seeing Betti tables for the first time, we point out that although we will write resolutions so that the maps go from left to right, and thus the Betti numbers appear from right to left $\{\ldots,\beta_2,\beta_1,\beta_0\}$ in a Betti table, the opposite order is used. Furthermore, by convention, the entry corresponding to (i,j) in the Betti table of M is $\beta_{i,i+j}(M)$, and *not* $\beta_{ij}(M)$.

Finally, we can use this information to calculate the Hilbert series of M:

$$HS(M)=\frac{1t^0-3t^2+2t^3}{(1-t)^3}=\frac{1+2t}{(1-t)^1},$$

and since this last fraction is in lowest terms, we see that the dimension of M is 1 (the degree of the denominator) and that the degree of M is equal to $p(1)=1+2\cdot 1=3$. Recall that M corresponded to the union of 3 lines. Notice that in this example, the projective dimension of M is 2, which is equal to the codimension $3-1=2$ of M. Hence, M is Cohen-Macaulay. In summary, we can get lots of information about M from its (graded) Betti numbers.

Example 3.3 (The Hilbert Series Doesn't Determine the Betti Numbers) Let k be a field, $R=k[x,y]$, and consider the two ideals

$$I=(x^2,xy,y^3) \quad \text{and} \quad J=(x^2,xy+y^2).$$

One can check that both R/I and R/J have the same Hilbert series:

$$HS(R/I) = HS(R/J) = 1 + 2t + 1t^2.$$

However, these modules have different Betti numbers. We work out the minimal free resolution and Betti numbers for R/I. Since I has two generators of degree 2 and one of degree 3, there are graded Betti numbers β_{12} and β_{13}. Similarly, the two minimal syzygies of R/I correspond to the relations

$$y(x^2) - x(xy) = 0 \text{ which has degree 3, so } \beta_{23} = 1$$

and

$$y^2(xy) - x(y^3) = 0 \text{ which has degree 4, so } \beta_{24} = 1.$$

Continuing this process, we find the following minimal free resolutions and graded Betti numbers for R/I and R/J, respectively:

$$\begin{matrix} R(-3)^1 \\ \bigoplus \\ R(-4)^1 \end{matrix} \xrightarrow{\begin{bmatrix} y & 0 \\ -x & y^2 \\ 0 & -x \end{bmatrix}} \begin{matrix} R(-2)^2 \\ \bigoplus \\ R(-3)^1 \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} R$$

$\beta_{23}(R/I) = 1$
$\beta_{24}(R/I) = 1$

$\beta_{12}(R/I) = 2$
$\beta_{13}(R/I) = 1$

$\beta(R/I)$	0	1	2
0	1	–	–
1	–	2	1
2	–	1	1

$$R(-4)^1 \xrightarrow{\begin{bmatrix} xy + y^2 \\ -x^2 \end{bmatrix}} R(-2)^2 \xrightarrow{\begin{bmatrix} x^2 & xy + y^2 \end{bmatrix}} R.$$

$\beta_{24}(R/J) = 1$

$\beta_{12}(R/J) = 2$

$\beta(R/J)$	0	1	2
0	1	–	–
1	–	2	–
2	–	–	1

$\beta(R/I)$	0	1	2
0	1	–	–
1	–	2	1
2	–	1	1

$\beta(R/J)$	0	1	2
0	1	–	–
1	–	2	–
2	–	–	1

Finally, if we calculate the Hilbert series from Eq. 3.1, we notice that the calculation is the same for R/I and R/J:

$$HS(R/I) = \frac{1 - 2t^2 - t^3 + t^3 + t^4}{(1-t)^3} = \frac{1 - 2t^2 + t^4}{(1-t)^3} = HS(R/J).$$

The cancellation of the t^3 terms is known as a **consecutive cancellation**, and one can see the two 1s on the diagonal in the Betti table for R/I. For the reader who knows about Gröbner degenerations, I is the initial ideal of J coming from a Lex term-order. Any such degeneration will preserve the Hilbert series, but not necessarily the

Betti numbers. For results concerning the relationship between the Betti numbers of ideals and those of their initial ideals, see [2, 10, 32–34, 61].

Example 3.4 We would be remiss if, in this article dedicated to David Eisenbud on his birthday, we didn't also mention that the connection between graded Betti numbers and geometry is a rich and beautiful story. In his book [37], he paints a story that begins with the following surprising fact from geometry. If X is a set consisting of seven general[3] points in $\mathbb{P}^3$, then the Hilbert series of the coordinate ring for X is completely determined by this data. However, this is not sufficient to determine the Betti numbers of the coordinate ring of X. Indeed, these numbers are either $\{1, 4, 6, 3\}$ or $\{1, 6, 8, 3\}$ depending on whether or not the points lie on a curve of degree 3.

3.2 *Resolutions for Ideals with Few Generators*

Over a polynomial ring $R = k[x_1, \ldots, x_n]$, calculating a free resolution is tantamount to producing the sets of dependence relations among the generators of a module. In simple cases this is straightforward, as the following example shows:

Example 3.5 Consider the module $M = R/(f)$, where f is a homogeneous polynomial in R. Then

$$0 \longrightarrow R \xrightarrow{[f]} R \longrightarrow M$$

is a minimal free resolution of length 1, since over our polynomial ring R, f is a *regular* element and cannot be killed by multiplication by any nonzero element.

If I is an ideal minimally generated by two polynomials f and g, then the minimal free resolution of R/I has length two. Indeed, if $c = \gcd(f, g)$, then the following is a minimal free resolution:

$$0 \longrightarrow R \xrightarrow{\begin{bmatrix} g/c \\ -f/c \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} f & g \end{bmatrix}} R \longrightarrow R/I \ .$$

This example can be summarized by the following result:

Proposition 3.6 *If I is an ideal in a polynomial ring R that is minimally generated by one or two homogeneous polynomials, then the projective dimension of R/I is equal to the minimal number of generators, and the Betti numbers are either* $\{1, 1\}$ *or* $\{1, 2, 1\}$.

[3] This means that no more than 3 lie on a plane and no more than 5 on a conic.

Whatever optimistic generalization of this proposition one might have in mind for ideals with 3 or more generators will certainly fail to be true, as we have the following astonishing results of Burch and Bruns:

Theorem 3.7 (Burch [20]) *For each $N \geqslant 2$, there exists a three-generated ideal I in a polynomial ring $R = k[x_1, \ldots, x_N]$ such that* $\mathrm{pdim}(R/I) = N$.

So we can always find free resolutions of maximal length by simply using 3 generated ideals. In fact, in some sense "every" free resolution is the free resolution of a 3-generated ideal:

Theorem 3.8 (Bruns [15]) *Let $R = k[x_1, \ldots, x_n]$ and*

$$0 \longrightarrow F_n \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M$$

be a minimal free resolution of a finitely generated graded R-module M. Then there exists a 3-generated ideal I in R with minimal free resolution

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_3 \longrightarrow F_2' \longrightarrow R^3 \longrightarrow R \longrightarrow R/I\ .$$

Remark 3.9 Note that the rank of F_2' may be different than that of F_2, but a rank calculation yields that

$$\mathrm{rk}\, F_2' = 3 - 1 + \mathrm{rk}\, F_3 - \mathrm{rk}\, F_4 + \cdots \pm \mathrm{rk}\, F_n = 2 + \mathrm{rk}\, F_2 - \mathrm{rk}\, F_1 + \mathrm{rk}\, F_0.$$

From this, it follows that β_2 can be arbitrarily large for 3-generated ideals.

Our point in presenting these results is to make plain that free resolutions are complicated—even for ideals with 3 generators! However, if in Example 3.5 we add a further restriction for the ideal $I = (f, g)$ and require that f and g have no common factors (meaning that g is a *regular* element modulo f), then the only relations between f and g are given by the "obvious" relation that $gf - fg = 0$. This fact *does* generalize nicely to any set $\{f_1, \ldots, f_c\}$ of homogeneous polynomials provided f_i is a regular element modulo the previous f_j. Such elements form what is called a **regular sequence**, and the ideal they generate is resolved by the **Koszul complex**. Rather than introducing the topic here, we point the reader to some of the many nice references for learning about the Koszul complex, such as [37, Chapter 17], [16, Section 1.6], or [5, Example 1.1.1].

The most important fact we will need about the Koszul complex is that it is a resolution (of $R/(f_1, \ldots, f_c)$) if and only if the $f_1, \ldots, f_c$ form a regular sequence, and that the Betti numbers (and ranks of syzygy modules) of the Koszul complex are given by binomial coefficients.

Theorem 3.10 *If I is an ideal generated by a regular sequence of c homogeneous polynomials, then*

$$\operatorname{rk}\Omega_i(R/I) = \binom{c-1}{i-1},$$

and therefore

$$\beta_i(R/I) = \binom{c}{i}.$$

Remark 3.11 To the reader not familiar with Koszul complexes, it might be instructive to carefully write out the maps involved to get a feel for how resolutions are constructed. Essentially, the point is that the generating ith syzygies are built from using i generators and the fact that $f_j f_i = f_i f_j$. Alternatively, perhaps the quickest way to define the Koszul complex is just to take the tensor product of the c minimal free resolutions of $R/(f_i)$:

$$0 \longrightarrow R \xrightarrow{f_i} R \longrightarrow 0.$$

Since multiplication by f_i has rank one, if one calculates the ranks in the tensor product inductively, one will see Pascal's Triangle appearing, providing a justification of the claims in Theorem 3.10.

3.3 How Small Can the Ranks of Syzygies Be?

If I is an ideal that is generated by a regular sequence then as we saw in the previous section, the minimal free resolution for R/I is given by the Koszul complex. For instance, if I has height 8, then $\beta_4(R/I)$ will be equal to $\binom{8}{4} = 70$, and the syzygy module $\Omega_4(R/I)$ will have rank $\binom{7}{3} = 35$. We will see in the next section (Conjectures 4.1 and 4.2) that among all ideals of height 8 these numbers are conjectured to be the smallest possible values for β_4 and $\operatorname{rk}\Omega_4$ respectively. In short, these conjectures assert a relationship between the ranks of syzygies and the height (or codimension) of the ideal. Before we present these conjectures, which will occupy the remainder of the paper, we close with an example and theorem that give the sharpest possible bound for ranks of syzygies if one does not refer to codimension.

Example 3.12 (Bruns [15]) Let $R = k[x_1, \dots, x_n]$. There is a finitely generated module M over R with the following resolution:

$$0 \longrightarrow R \longrightarrow R^n \longrightarrow R^{2n-3} \longrightarrow \cdots \longrightarrow R^5 \longrightarrow R^3 \longrightarrow R \longrightarrow M \longrightarrow 0.$$

In other words, the ith Betti number is $2i+1$ except for the last two Betti numbers. This is the case for an even nicer reason: if one calculates the ranks of each syzygy module (which can be read off as the rank of the ith map π_i in the resolution) one sees that the ranks are:

$$0 \longrightarrow R \xrightarrow{1} R^n \xrightarrow{n-1} R^{2n-3} \xrightarrow{n-2} \cdots \xrightarrow{3} R^5 \xrightarrow{2} R^3 \xrightarrow{1} R \xrightarrow{0} M \longrightarrow 0.$$

In other words, in this example the ith syzygy module has rank equal to i, except for the last one. This bound holds for any module, which is the content of the great Syzygy Theorem.

Theorem 3.13 (Syzygy Theorem, Evans–Griffith [44]) *Let M be a finitely generated module over a polynomial ring R. If $\Omega_i(M)$ is not free, then* $\operatorname{rk}\Omega \geqslant i$. *Hence, if* $\operatorname{pdim} M = p$, *then*

$$\operatorname{rk}\Omega_i(M) \geqslant i, \text{ for } i < p.$$

Moreover,

$$\beta_i(M) = \operatorname{rk}\Omega_i(M) + \operatorname{rk}\Omega_{i+1}(M) \geqslant \begin{cases} 2i+1 & \text{if } i < p-1 \\ p & \text{if } i = p-1 \\ 1 & \text{if } i = p \end{cases}$$

where $\Omega_i(M)$ denotes the ith syzygy module of M.

The Syzygy Theorem together with Bruns' example provides a sharp lower bound for $\beta_i(M)$. Without further conditions on M, there is not much more we can say. However, if we add additional hypotheses on M—for instance, requiring M to be Cohen-Macaulay, or of a fixed codimension c—then the bounds above appear to be far from sharp. Indeed, we will discuss a conjecture that states that in fact $\beta_i(M) \geqslant \binom{c}{i}$; when c is large, this conjecture is much stronger than the Syzygy Theorem's bound of $2i+1$. Note that the ideal in Example 3.12 is of codimension 2.

3.4 Other Possible Directions

Before we begin to focus on codimension, we want to say that there are many distinct and interesting alternative questions on bounds for Betti numbers that have been considered. We present some possibilities below.

One could decide to study ideals and then fix the number of generators of I; for example, one could study the sets of Betti numbers of ideals defined by 5 homogeneous polynomials. Theorem 3.8 shows that this approach will not allow for any upper bounds, except in trivial cases.