# 1 Meet the Real Numbers - Welcome to Math 360

Question Prompt 1:

- Introduce yourself to your groupmates - share something fun or exciting!
- How would you define what it means to be a "number"? Are there different "types" of numbers? Are there any numbers that are somehow "special" or "fundamental"?

STOP

Question Prompt 2:

- Introduce yourself to your groupmates - share something fun or exciting!
- What are some things you can **do** with numbers? For example, what are somethings you could **do** with the numbers 3 and 7?

STOP

Question Prompt 3:

- Introduce yourself to your groupmates - share something fun or exciting!
- Is there a number that solves the equation $x^2 = 9$? Why or why not?
- Is there a number that solves the equation $x^2 = -10$? Why or why not?
- Is there a number that solves the equation $x^2 = 2$? Why or why not?

Below we will give definitions for the main number systems we will learn about in this class.

> "God created the integers, all else is the work of man"
>
> (Leopold Kronecker 1823 - 1891)

**What are the real numbers?**

It will take us about 5 classes before we are able to give a ________________________ of $\mathbb{R}$.

In fact, our definition will

- Be more detailed that the definitions on the previous page.
- Our definition will have ____________ and there will actually be a lot of them! (____________).
- In this class we are going to ____________ everything that we do. This class shows us how we can **build** everything we know about the **Real Numbers** from scratch.
- This will at times feel hard and confusing. That's ok! This is typically considered one of the hardest undergraduate math classes.
- (Jump scare coming) for instance, the book takes until page 23 to get to the definition of $\mathbb{R}$. There is good reason for this - because otherwise one of the axioms would look terrifying like this: (Please don't write this down!)
- Once we get used to the axioms, we will see that many "basic facts" like $0 < 1$ and $(-1)(-1) = 1$ are not on the list but we'll see how we can ________________________.
- Later we'll prove deeper results, like what it means to be ________________________ and why the Intermediate Value Theorem is true.
- Or why the number $e$ is ________________________.
- But most importantly - this class is **less about the results** and **more about the "why"**.

What to do for Friday:

- You have your first **Homework assignment** (with only 1 question) due on Friday at 5pm.
  - Homework will be submitted through a website called Gradescope.
- Also, check the **Schedule** and see there is a reading assignment for Friday.
  - For each writing assignment you must submit a Google form short reflection about the reading. This can be super short. I just want you to get into the habit of reading and reflecting.

## Questions

- Consider the **sequence** of numbers, defined by the **recursion**:

$$a_1 = 1, \quad a_{n+1} = 1 + \frac{1}{a_n + 1}.$$

Use this description to find $a_2$ and $a_3$. Are they **rational numbers?** Why or why not? Make a table below of the first few terms in this **sequence**. What are some things you notice? What are some things that you wonder?

A word of warning: We need to justify the "why".

- Consider the sequence of real numbers defined by:

$$a_1 = 1, \quad a_{n+1} = 4a_n + 2.$$

  What are the first few terms of this sequence? What do you think is going on? Will the methods we tried on the other problem work here? Why not?

# 2 The Natural Numbers and Rational Numbers

**Warmup:** Do you recognize the song that is playing? What does it have to do with $\mathbb{N}$?

**The Main Thing to know about the Natural Numbers:**

**Warning**: This does not work for $\mathbb{Q}$ or $\mathbb{R}$. What is the "next" real number after $\pi$ or the "next closest" rational number after 0.5.

**Example:** Consider the sum of the first $n$ odd natural numbers. Find a formula for this sum and prove your answer using mathematical induction.

**Example:** Find all $n \in \mathbb{N}$ such that $2^n > n^2$.

**Example:** Why is $\sqrt{2}$ not rational?

We will argue by ________________________.

Suppose that $\sqrt{2}$ **were** rational. **IF THIS WERE TRUE THEN**

(1) $\sqrt{2} = \_\_\_\_\_$ for some integers $a, b$ in lowest terms.

This means that $a$ and $b$ have no ____________ in common.

If we square both sides of the equation (1) then we get:

Multiplying both sides by $b^2$ we get the equation:

Now since $2b^2$ is a multiple of 2 it is ____________.

Then since $2b^2 = a^2$, we know that $a^2$ is also ____________.

Thus $a$ must be an ____________ number as well.

Since $a$ is even, this means that $a$ is a multiple of 2 and thus $a =$ ____________ for some integer $k$.

If we substitute this into equation (3) we get:

After doing some simplifying we obtain:

Which shows that $b^2$ is even, and thus that $b$ is also even. But then this means that $a$ and $b$ both have a factor of ____________ in common. This is a contradiction which means that it must NOT be true that $\sqrt{2}$ is rational.

(Fireside Chat)In this proof I had some blanks for you had to fill them in to make sure you understood what was going on. To do that you had to use **context** and prior information. When you are reading a complete proof in the book, you will use the same skills. Even though the proof is "complete" you should still read like a detective.

**Example:** Here is an example from Page 140 of the textbook that we will get to in November or December. For now, I want to model how to read such an intimidating statement.

**Example 1**

We verify (*) for the function $f(x) = \frac{1}{x^2}$ on $(0, \infty)$. Let $x_0 > 0$ and $\epsilon > 0$. We need to show $|f(x) - f(x_0)| < \epsilon$ for $|x - x_0|$ sufficiently small. Note that

$$f(x) - f(x_0) = \frac{1}{x^2} - \frac{1}{x_0^2} = \frac{x_0^2 - x^2}{x^2 x_0^2} = \frac{(x_0 - x)(x_0 + x)}{x^2 x_0^2}. \qquad (1)$$

If $|x - x_0| < \frac{x_0}{2}$, then we have $|x| > \frac{x_0}{2}$, $|x| < \frac{3x_0}{2}$ and $|x_0 + x| < \frac{5x_0}{2}$. These observations and (1) show that if $|x - x_0| < \frac{x_0}{2}$, then

$$|f(x) - f(x_0)| < \frac{|x_0 - x| \cdot \frac{5x_0}{2}}{(\frac{x_0}{2})^2 x_0^2} = \frac{10|x_0 - x|}{x_0^3}.$$

Thus if we set $\delta = \min\{\frac{x_0}{2}, \frac{x_0^3 \epsilon}{10}\}$, then

$$|x - x_0| < \delta \quad \text{implies} \quad |f(x) - f(x_0)| < \epsilon.$$

This establishes (*) for $f$ on $(0, \infty)$. Note that $\delta$ depends on both $\epsilon$ and $x_0$. Even if $\epsilon$ is fixed, $\delta$ gets small when $x_0$ is small. This shows that *our* choice of $\delta$ definitely depends on $x_0$ as well as $\epsilon$, though this may be because we obtained $\delta$ via sloppy estimates. As a matter of fact, in this case $\delta$ *must* depend on $x_0$ as well as $\epsilon$; see Example 3. Figure 19.1 shows how a fixed $\epsilon$ requires smaller and smaller $\delta$ as $x_0$ approaches 0. [In the figure, $\delta_1$ signifies a $\delta$ that works for $x_1$ and $\epsilon$, $\delta_2$ signifies a $\delta$ that works for $x_2$ and $\epsilon$, etc.] $\square$

# 3 What is an Ordered Field?

**Warmup:** Let's chat for 4 minutes with your neighbors about the following questions:

- We say that a number system $X$ is "closed under operation blah" if performing that operation on two things in the number system will return another element **in that same number system.**
- For example, $\mathbb{Z}$ is **not closed** under division, since although $2 \in \mathbb{Z}$ and $3 \in \mathbb{Z}$, $2/3 \notin \mathbb{Z}$. I've filled that box in for you.
- For each of the operations and each of the sets $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ decide whether it is "closed" with respect to the operations indicated.

| | $\mathbb{N}$ | $\mathbb{Z}$ | $\mathbb{Q}$ | | |
|---|---|---|---|---|---|
| addition | | | | | |
| subtraction | | | | | |
| multiplication | | | | | |
| division (by nonzero stuff) | | **No** | | | |
| | | | | | |

**The Field Axioms**

A field is a set with two ____________ denoted ____________

and two distinct (very special) elements called ____________ obeying the following 9 properties:

A1) for all $a, b, c$;

A2) or all $a, b$;

A3) for all $a$;

A4) for each $a$ there is an element ______ s.t.

M1) for all $a, b, c$;

M2) for all $a, b$;

M3) for all $a$;

M4) for each ____________ $a$ there is an

element ______ s.t.

DL) for all $a, b, c$.

In simple terms, a field is a number system where you can ____________________________________.

Examples of Fields Include:

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A **field** (meaning all the 9 properties above) that **also** satisfies the 5 axioms below

is called an ________________________

O1) Given $a$ and $b$ either

O2) If $a \leq b$ and $b \leq a$ then

O3) If $a \leq b$ and $b \leq c$ then

O4) If $a \leq b$ then for any $c$.

O5) If $a \leq b$ and ____________ then

The following are all examples of ordered fields:

**Example:**

Using the axioms, we can **prove** new statements, called Lemmas or Theorems. Here are some sample Theorems we can prove. **Let's prove some of these in class and then you'll prove four of them on your homework.**

**3.1 Theorem.**
*The following are consequences of the field properties:*

**(i)** $a + c = b + c$ *implies* $a = b$;
**(ii)** $a \cdot 0 = 0$ *for all* $a$;
**(iii)** $(-a)b = -ab$ *for all* $a, b$;
**(iv)** $(-a)(-b) = ab$ *for all* $a, b$;
**(v)** $ac = bc$ *and* $c \neq 0$ *imply* $a = b$;
**(vi)** $ab = 0$ *implies either* $a = 0$ *or* $b = 0$;

*for* $a, b, c \in \mathbb{R}$.

**3.2 Theorem.**
*The following are consequences of the properties of an ordered field:*

**(i)** *If* $a \leq b$, *then* $-b \leq -a$;
**(ii)** *If* $a \leq b$ *and* $c \leq 0$, *then* $bc \leq ac$;
**(iii)** *If* $0 \leq a$ *and* $0 \leq b$, *then* $0 \leq ab$;
**(iv)** $0 \leq a^2$ *for all* $a$;
**(v)** $0 < 1$;
**(vi)** *If* $0 < a$, *then* $0 < a^{-1}$;
**(vii)** *If* $0 < a < b$, *then* $0 < b^{-1} < a^{-1}$;

*for* $a, b, c \in \mathbb{R}$.

Note $a < b$ means $a \leq b$ and $a \neq b$.

**Definition:** We define the absolute value of a number $a$ to be:

**Extremely important pro-tips:**

$|x - b|$ means

$|x| < c$ means

For example:

Write the inequality $|\heartsuit - \Delta| < 3$ as a double inequality:

Let's draw some pictures:

**Theorem:** (The Triangle Inequality) if $x, y$ are elements of an ordered field, then

**Question:** What happens if you let $x = a - b,\ y = b - c$? What does the Triangle Inequality say?

**The Ordered Field Axioms**
A field is a set with two operations, denote $+, \cdot$ and two very special distinct elements $0, 1$ obeying the following 9 properties:

A1) $(a+b)+c = a+(b+c)$ for all $a, b, c$;

A2) $a+b = b+a$ or all $a, b$;

A3) $a+0 = a$ for all $a$;

A4) for each $a$ there is an element $-a$ such that

$$a + (-a) = 0.$$

M1) $a(bc) = (ab)c$ for all $a, b, c$;

M2) $ab = ba$ for all $a, b$;

M3) $a \cdot 1 = a$ for all $a$;

M4) for each **nonzero** $a$ there is an element $a^{-1}$ s.t.
$$aa^{-1} = 1.$$

DL) $a(b+c) = ab + ac$ for all $a, b, c$.

O1) Given $a$ and $b$ either $a \le b$ or $b \le a$

O2) If $a \le b$ and $b \le a$ then $a = b$

O3) If $a \le b$ and $b \le c$ then $a \le c$

O4) If $a \le b$ then $a + c \le b + c$ for any $c$.

O5) If $a \le b$ and $c \ge 0$ then $ac \le bc$.

# 4 Upper and Lower Bounds + Practice with Absolute Value

**Before the quiz:**

Write $|x^2 - x - 2| < 4$ as a **double inequality.**

The sentence |Barbie - Ken | < 5 means that the

"Distance between ________________________________________________"

Write |Barbie - Ken| < 5 as a double inequality and solve for Barbie by adding Ken to all three parts of the inequality. Your inequality should be something like: ?? < Barbie < ??. Use your inequality to shade in the region of the number line where Barbie could be.


Suppose that Ken > 0. What is the biggest that |Barbie| could be? What is the smallest?

Suppose that Ken < 0. What is the biggest that |Barbie| could be? What is the smallest?

Let's start our Quiz!

**Warmup:** After the quiz, let's review the concepts you read about for today:

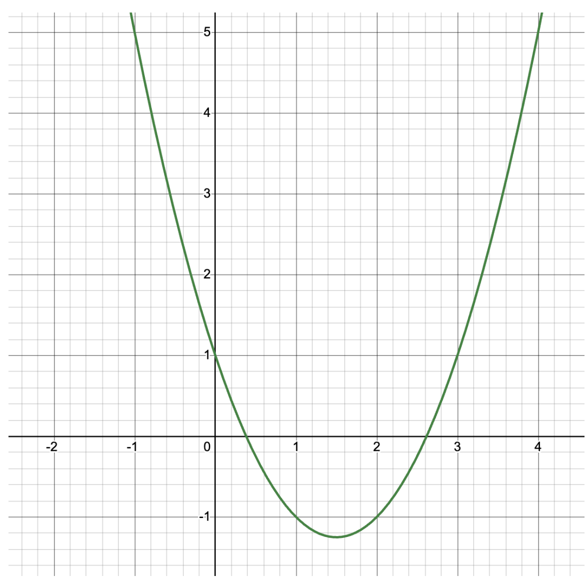
- 10 is an upper bound for the set $\{1,2,3,4,5,\pi,6,7,8,9\}$ (True / False)
- 10 is an upper bound for the set $\{1,2,3,4,5,\pi,6,7,8,9,10\}$ (True - it is big [K]enough/ False)
- 100 is an upper bound for the set $[1,3]$ (True / False)
- Ariana thinks that 5 is an upper bound for the set $(0,3)$. Is Ariana correct?
  Taylor thinks she can find an upper bound for the set that is smaller than Ariana's? Is Taylor correct? What would a possibility for Taylor's Version be? What is the least (meaning smallest) possible upper bound that Taylor could give?
- The set $\{1,2,3,4\}$ has a maximum (True/False)
- The set $\mathbb{Z}$ has an upper bound (True / False)
- The set $(-\infty,3)$ is bounded above (True / False). It is bounded below (True / False)
- If a set has a maximum, then it has an upper bound (True/False)
- If a set has an upper bound, then it has a maximum. (True / False)
- If a set $S$ has an upper bound $m$ and $m \in S$ then $S$ has a maximum.

Let $S = \{\frac{1}{n} \ : \ n \in \mathbb{N}\}$. What numbers are in $S$, jot down 3 or 4 of them. Where would they appear on a number line?

1. Does $S$ have an upper bound? If so, how many?
2. Does $S$ have a lower bound? If so, how many?
3. Does $S$ have a max?
4. Does $S$ have a min?
5. Does $S$ have a least upper bound?
6. Does $S$ have a greatest lower bound?

Let $T = \{n^2 - 3n + 1 \ : \ n \in \mathbb{Z}\}$. (The graph of $y = x^2 - 3x + 1$ is below in case that helps!)

1. Does $S$ have an upper bound? If so, how many?
2. Does $S$ have a lower bound? If so, how many?
3. Does $S$ have a max?
4. Does $S$ have a min?
5. Does $S$ have a least upper bound?
6. Does $S$ have a greatest lower bound?



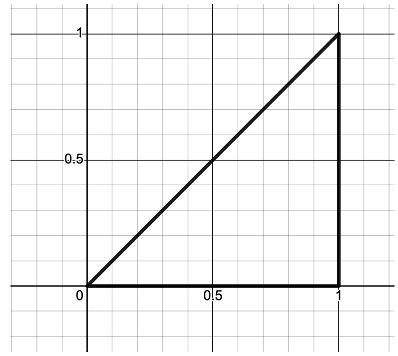
Let $S = \{x \in \mathbb{Q} : \ x <$ the length of the hypotenuse of a right triangle with base and side of length $1\}$.

1. Does $S$ have an upper bound? If so, how many?

2. Does $S$ have a min? a max?

3. Does $S$ have a least upper bound?

4. Does $S$ have a greatest lower bound?




(On Monday you will have a definition quiz)

**Definitions**

We say that a real number $M$ is an upper bound for a set $S$ if

We say that a real number $M$ is a lower bound for a set $S$ if

We say that a set $S$ is bounded above if

We say that a set $S$ is bounded below if

We say that a set $S$ is bounded if

A real number $M$ is called the supremum of $S$ (or the least upper bound of $S$) if

A real number $M$ is called the infimum of $S$ (or the greatest lower bound of $S$) if

**Theorem:** If $S$ is a bounded set, then there exists a number $B$ such that $|s| \leq B$ for all $s \in S$.

Proof:

# 5 Let's get unsettled - The Missing Axiom

Warning: This might give you nightmares!

Which of them can we **prove** using the axioms of any ordered field?

1. $0 < 1$
2. if $ac = bc$ and $c \neq 0$ then $a = b$
3. The equation $x^2 = 2$ has a solution
4. The equation $x^2 = 9$ has a solution
5. If $x$ is an element, then for a big enough $n$, $0 < \frac{1}{n} < x$.
6. The equation $x^2 = -1$ has no solutions
7. Given any natural number $x$ there is a natural number $n$ with $x < n$.
8. Given an element $x$ in our ordered field, there is a natural number $n$ such that $x < n$.

This **terrifying** realization is because the real numbers are ________________________

and we are ____________________________________.

**The missing axiom of the real numbers:**

Facts: There many many many very different ________________________

There is one and only one ____________________________________.

**Definition:**

We define the

So in practice this means that the Real Numbers are a system where we can use the 15 axioms:

A1 - A4, M1-M4, DL, O1 - O5, C

**Now the ________________________ begins.**

For the rest of the day I would like to just write some proofs and see how the technicalities come together.

**The Archimedean Principle:**

**The Denseness of the Rationals**

**Find (with proof) the supremum of the set**

$$S = \{\frac{2n-3}{n+1} \quad : \; n \in \mathbb{N}\}$$

# 6 Practice writing proofs with sup, inf, and Archimedean Principle

**Warmup: (6 minutes discussion + 10 as a class)** Let $S = \{\frac{1}{3n+1} : n \in \mathbb{N}\}$. What do you think the sup and inf of $S$ are? How might you write a proof? (This would be a reasonable problem for the quiz on Friday). Work with your team to

- Let's start with the supremum. What do you think the supremum is? Can you come up with any reasons why this is an upper bound for $S$? About why it is the LEAST upper bound for $S$? Try to spend your group time talking about strategies and ideas. Don't worry so much about the writing.
- What do you think the infimum of $S$ is? How might you show that your answer is a lower bound, and how might you show it is the GREATEST upper bound?

| Pro-tip: |
| --- |
| |

How do we use the Archimedean Principle? (7 minutes all group work)

As a model, consider the following solution to showing that there is some integer $N$ such that $N^3 > 5x$.

Proof: Let $x \in \mathbb{R}$. By the AP, I can choose an $N \in \mathbb{N}$ such that ________________________.
Now since $N \in \mathbb{N}$, I know that

$$N^3 \geq N > 5x.$$

Notice that in this proof, we made it clear how we were using the AP. Now it's your turn:

Prove that:

1. If $x \in \mathbb{R}$ then there is a natural number $N$ such that $3N > x$;

   Proof: Let $x \in \mathbb{R}$ then by the AP, I can choose an $N \in \mathbb{N}$ such that ________________________.
   Then [fill in some correct inequalities below, starting with 3N and ending with $x$]

   $$3N$$

2. If $x \in \mathbb{R}$ then there is a natural number $N$ such that $N^2 > x - 5$;

   Proof: Let $x \in \mathbb{R}$ then by the AP, I can choose an $N \in \mathbb{N}$ such that ________________________.
   Then [fill in some correct inequalities below, starting with $N^2$ and ending with $x - 5$]

   $$N^2$$

3. If $x \in \mathbb{R}$ then there is a natural number $N$ such that $N^2 + 2N > 3x$;

   Proof: Let $x \in \mathbb{R}$ then by the AP, I can choose an $N \in \mathbb{N}$ such that ________________________.
   Then [fill in some correct inequalities below, starting with $N^2 + 2N$ and ending with $3x$]

   $$N^2 + 2N$$

4. Suppose that $M < 4$. Prove there is a natural number $N$ such that $\dfrac{1}{3n+2} < \dfrac{4-M}{7}$;

   ( **Warning:** On this one, you will use AP2, the one that lets you pick $1/N < x$)

   Proof: Suppose $M < 4$, then by the AP I can choose $N \in \mathbb{N}$ such that ________________________.
   Then [fill in some correct inequalities below, starting with $\frac{1}{3N+2}$ and ending with $\frac{4-M}{7}$]

   $$\frac{1}{3N+2} \leq \frac{1}{N}$$

   (Make sure you know why this inequality is true! The numerators are equal but the bigger denominator goes with the smaller number.)

Let $S = \{4 - \frac{7}{3n+2}\}$

**Whatsup?** (that's a joke - I want you to find the supremum of $S$ with proof) (10 minutes)

**The Denseness of the Rationals**

# 7 The Infinity Symbol

First some new **definitions.**

Let $S$ be a subset of $\mathbb{R}$. Then $\sup S = \infty$ means

Let $S$ be a subset of $\mathbb{R}$. Then $\inf S = -\infty$ means

**Example:** On your homework you will prove that $\sup\{n^2 + 3n + 5 : n \in \mathbb{N}\} = \infty$.

You will do this by arguing by __________________________

and showing that if $M \in \mathbb{R}$ then $M$ is NOT an upper bound.

**The Denseness of the Rationals**

Let $A = \{1,\ \frac{1}{1},\ 1,\ \frac{1}{2},\ 1,\ \frac{1}{3},\ 1,\ \frac{1}{4},\ 1,\ \dots\}$. What is

$$\sup A, \quad \inf A?$$

Which of the following statements is true:

1. $$\forall N \in \mathbb{R}, \exists a \in A : a > N$$

2. $$1 \geq a \quad \forall a \in A$$

3. $$\forall a \in A \quad 1 \geq a$$

4. $$\forall r < 1 \quad \exists a \in A : a > r$$

5. $$a > 0 \quad \forall a \in A$$

6. $$\forall r > 0 \quad \exists a \in A : a > r$$

7. $$\forall r > 0 \quad \exists a \in A : a < r$$

8. $$\exists a \in A : \forall r > 0, \quad a < r$$

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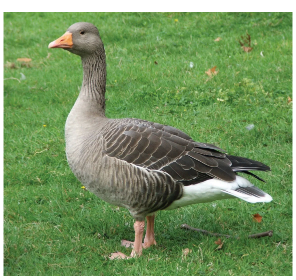

What is silly about the way that the set $A$ was written?

Rewrite it in a more sensible way:

$$A = \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$$

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Instead if we think about the ____________________

$$a_n : 1,\ \frac{1}{1},\ 1,\ \frac{1}{2},\ 1,\ \frac{1}{3},\ 1,\ \frac{1}{4},\ 1,\ \dots$$

Then we can start asking some richer questions, which we will do on Monday.

# 8 The Limit of a Sequence

Take a look at the following sequences. All of them converge to 5

(A) $s_n = 5$: $5,\ 5,\ 5,\ 5,\ 5,\ 5,\ 5,\ 5,\ 5,\ 5,\ \ldots$

(B) $s_n = 5 - 1/10^n$: $4.9,\ 4.99,\ 4.999,\ 4.9999,\ \ldots$

(C) $s_n = 5 + \frac{(-1)^n}{10^n}$: $4.9,\ 5.01,\ 4.999,\ 5.0001,\ \ldots$

(D) $4.9,\ 4.99,\ 5.1,\ 4.999,\ 4.9999,\ 5.01,\ 4.99999,\ 4.999999,\ 5.001,\ \ldots$

What do you notice? What do you wonder?

Our goal today is to motivate the **definition** of
what it means to converge to a number.

Which of these correctly captures the notion that the sequence $s_n$ is converging to 5?

If you think it does **not** capture the notion of convergence - write down a sequence that has the property, but does NOT converge.

| | If $s_n$ converges to 5 then this much be true: | If this is true then the sequence definitely converges to 5 |
|---|---|---|
| 1. At some point the sequence gets as close to 5 as we'd like. (i.e. if I want to get within any distance $d$ of 5 I know there will be a term of the sequence that is within that distance) | | |
| 2. The sequence gets as close as we'd like to 5 and this happens not just at some point, but at infinitely many points. | | |
| 3. If $s_n$ is within a certain distance of 5 then $s_{n+1}$ is even closer to 5 and this continues forever. | | |
| 4. For any distance $\epsilon$ (like .0001 or .0000001) there are infinitely many terms in the sequence that are within that distance from 5. | | |
| 5. The sequence at some point is equal to 5. | | |
| 6. The sequence is equal to 5 infinitely many times. | | |

If this is true then $s_n$ definitely converges to something

1. The distance between successive terms gets smaller at each step.
2. The distance between successive terms in the sequence gets smaller and smaller and this distance gets arbitrarily small.
3. No matter how small you want the distance between successive terms to be, at some point it will happen.
4. No matter how small you want the distance between successive terms to be, at some point it will happen and that will continue to happen forever.

The point is that defining the limit of a sequence is **very** tough. Historically it took a lot of time to develop.

The correct definition of $\lim s_n = 5$ is:

For **any** distance ____________ we choose,

there is a point in time such that ____________________________________

the distance between ____________________________________ is at most ____________.

**Definition:** Let $s_n$ be a sequence and let $L \in \mathbb{R}$. We say that $s_n$ converges to $L$ if

Let $s_n$ be a sequence. We say that $s_n$ converges if

The limit definition of $\lim s_n = s$ is:

"For any $\epsilon > 0$ there exists an $N$ such that for all $n > N$ we have that $|s_n - s| < \epsilon$."

We rephrased this as:

"For any positive distance ($\epsilon > 0$) no matter how small, there is some point in time ($N$) after which and forever afterwards ($\forall n > N$) the distance between $s_n$ and the limit $s$ is smaller than $\epsilon$."

Let's rephrase this mantra

"Math Student: Oh you don't believe that $s_n$ converges to $s$? Well how close would you like $s_n$ to get to $s$?
(Skeptic: Within some distance $\epsilon$, like .001 or .00001 but I won't be convinced unless you can do it for an arbitrary small distance.)
Math Student: Oh, well if I could prove to you that after some point in time $N$ your sequence would definitely be within a distance of $\epsilon$ from the limit $s$ would that convince you?
(Skeptic: You mean it'd be super close to $s$ after $N$ and forever after that? I don't want to it move away after awhile....)
Math Student: Indeed - every term $s_n$ will be within that threshold so long as you take $n > N$. I can prove it to you.
(Skeptic: I await your proof!)"

Today I'd like to get some practice writing some of these proofs on the board. I'll do a few on the board to get us started and then I'd like you to try some yourself. There are 5 online videos on our playlist. Since we're spending class time to work on these problems I'll outsource a few theorems to the videos. **you are still responsible for understanding those proofs** but of course I'll be happy to talk about them in office hours if you'd like!

The following template will be our guide throughout the class:

Proof: Let $\varepsilon > 0$ be arbitrary.
By the AP there is an $N \in \mathbb{N}$ such that __________________________.
Then if $n > N$:

$|s_n - s| = \qquad \cdots \qquad < \varepsilon.$

In the middle of these steps you'll need to work out some inequalities to see how big $N$ has to be.

**Example:**

Show that the sequence $s_n = 5 + \frac{(-1)^n}{n^2}$ converges. Prove your result.

What can we use? I know we have not yet proved that a positive real number has a square root. This is true, though. You can use without proof the fact that if $x > 1$ then $\sqrt{x} < x$.

1. Try working out proofs of the following limits: (Remember the first step is to figure out what the limit is)
   - $a_n = 1/(\sqrt{n+4})$
   - $a_n = (2n-1)/(3n+1)$
   - $a_n = \frac{\sin n}{n}$
2. Finally we will also prove that $(-1)^n$ does NOT converge to any number. You should be able to prove this - take some time to practice writing this proof, or variants of it. E.g. can you prove that $(-1)^n(5 + 1/n)$ has no limit?
3. Much harder is to try and compute the limit of the sequence:

$$a_n = \frac{1}{n^3 - 50n + 4}$$

   This one is a LOT harder, and will require some different techniques.

# 9 Writing Limit Proofs

Commentary:

Let's write the definition of the limit below:

In the future we will not require that $N \in \mathbb{N}$. Why is this ok?

- Suppose that Taylor proves that $\forall n > 47,\ |s_n - 5| < \epsilon$
- Suppose that Justin proves that $\forall n > 42.5,\ |s_n - 5| < \epsilon$
- Suppose that Ariana proves that $\forall n \geq 52,\ |s_n - 5| < \epsilon$

If you want to play along: `https://tinyurl.com/usd360day9`

**Example:** Let
$$s_n = 5 - \frac{2(-1)^n}{n^2}.$$
Prove that $s_n$ converges to 5.

**Proof:**

**Warmup:**
Suppose you are in charge and you wanted to write an inequality $n + 5 \leq$ ____________
Which of these would be correct for all $n \in \mathbb{N}$?

$$2n, \quad n^2, \quad n^3,$$

Ok, well what if in your **analysis** you really really wanted it to be true that

$$n + 5 \leq 2n$$

But what if YOU were in charge of what $n$ was, meaning YOU could say

"Hey, let's agree that $n \geq$ ______."

Could you make it so that $n + 5 \leq 2n$? What would you put in the blank?


**Example:** Let

$$s_n = \frac{n+5}{n^2 + 2n + 4}$$

Prove that $s_n$ converges to 0.

**Example** Find, with proof the limit of

$$s_n = \frac{n+2}{3n-4}$$

To show that a sequence $s_n$ does NOT converge to $L$ we need to show:

To show that a sequence $s_n$ does NOT converge we need to show:

**Example:**

Prove that the sequence $(-1)^n$ does NOT converge.

# 10 Important Limit Theorems

**Some Main Limit Theorems:**

Suppose that $a_n \to a$ and $b_n \to b$. (Note this notation means that "$a_n$ converges to $a$"). Then

1. The sequence $a_n + b_n$ converges to $a + b$;
2. The sequence $a_n \cdot b_n$ converges to $ab$;
3. If $c \in \mathbb{R}$ then the sequence $c \cdot a_n$ converges to $ca$.
4. If $b_n$ is never zero, and $b \neq 0$ then the sequence $\dfrac{1}{b_n}$ converges to $\dfrac{1}{b}$
5. If $p \in \mathbb{N}$, the sequence $\dfrac{1}{n^p}$ converges to 0;
6. The constant sequence $s_n = L$ converges to $L$.

**Example:** Use the Theorems above to prove that the following sequence converges to 2/3:

$$s_n = \frac{4n^3 - 17n^2 + 3}{6n^3 - 45}.$$

Step 1: Multiply top and bottom by $\frac{1}{n^3}$. This is allowed since $n$ is never 0. (Why?)

Step 2: After simplifying, what does the denominator converge to?

What Theorems above did you use?

What about the numerator?

**In your Homework for Tuesday: please write up a very careful proof explaining how you applied the theorems.**

We will prove some of these statements, but to do so, we need a very important fact:

**Theorem:** If $s_n$ is converges then $s_n$ is ________________________

Before we present a proof, **discuss**. Is the converse true? Does every bounded sequence converge?

**In your homework:** You will modify this proof to prove the statement:

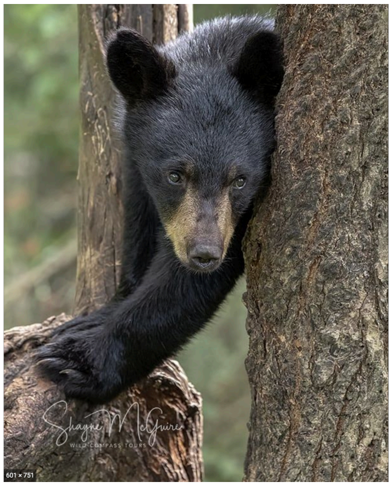
**Theorem:** Suppose that $s_n$ is a sequence of numbers that converge to 0.3. It is true that some of the terms of $s_n$ could be negative, but prove that only finitely many of them can be. More specifically, prove that there is an $N$ such that for all $n > N$, $s_n > 0$. Remember all you know is that $s_n \to 0.3$. **study the proof on this page to understand how to adapt.**

**Theorem:** If $a_n \to a$ and $b_n \to b$, then the sequence $a_n \cdot b_n$ converges to $ab$.

**Theorem:** If $b_n \to b$ and $b_n$ is never zero and $b \neq 0$, then the sequence $\dfrac{1}{b_n}$ converges to $\dfrac{1}{b}$.

# 11 Some More Important Theorems

### The Squeeze Theorem



### Some Follow Ups:

Suppose that $a_n < b_n$ for all $n$ and $a_n \to a$ and $b_n \to b$. Is it true that $a < b$? (Hint: it's not. Write down an example illustrating this).

### Theorem:

We are finally able to show that $\sqrt{2}$ exists!

**Theorem:** Let $a > 0$. Then $a$ has a square root. That is, there is a $y > 0$ (in $\mathbb{R}$) such that $y^2 = a$.

For extra credit, try to prove that if $p \in \mathbb{N}$ and $a > 0$ then there is a $y > 0$ such that $y^p = a$. In other words, every positive number has a $p$th root. **Much later in the class, we'll prove this in a very nice and elegant way.** But if you want to prove it now, you might want to study the Binomial Theorem below.

For these we'll need the Binomial Theorem which states that, e.g.

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

and that in general the coefficients for $(x+y)^n$ come from Pascal's Triangle. Namely we'll need that

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \cdots.$$

$$= 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots$$

# 12 When Limits are Infinite

**Warmup - (Something new for 10 minutes):**

$$a_n = 1, 2, 3, 4, 5, 6, 7, 8, 9, \dots$$
$$b_n = 1, 0, 3, 0, 5, 0, 7, 0, 9, \dots$$

Neither of these sequences converge to a real number, and both are unbounded. However, the first one seems like it is consistently going off "toward infinity", whereas the second one hasn't really made up its mind. We might write something like $\lim a_n = \infty$, but say $\lim b_n \neq \infty$.

**Yo doc, that paragraph was a little vague. We ♡ quantifiers show us what this means!**

Pshaw, ok!

$\lim a_n = \infty$ should mean that now matter how big you want $a_n$ to be, it will definitely be that big at some point - and forever afterwards will stay that big! That is,

$$\forall M, \exists N \text{ such that } \forall n > N, a_n > M.$$

Now you can try to write the definition for $\lim a_n = -\infty$.

These two definitions will be on the **definition quiz.**

**By the way:** All of our definitions are on a different tab on the spreadsheet that has our schedule.

**Remember:** When we say a sequence **converges**, it always means to a real number. (Not $\infty$).

If a sequence **diverges** it might still be true that it has a limit of $\pm\infty$. Don't worry about this too much though.

Here are some True / False questions:

If $s_n$ is unbounded, then $\lim s_n = \infty$ or $-\infty$.

If $\lim s_n = \infty$ or $-\infty$ then $s_n$ is unbounded.

If $\lim s_n = \infty$ or $-\infty$ then $s_n$ diverges.

If $\lim s_n = \infty$ or $-\infty$ then $s_n$ converges.

The sequence $s_n = n^2$ diverges.

$\lim n^2 = \infty$

$\lim(-n)^2 = \infty$

$\lim(-n)^3 = \infty$

$\lim(n)^3 = \infty$

In the book, you can read about some other properties of what it means for a limit to be infinite. For instance, it is true that

$$\lim a_n = \infty \iff \text{ eventually } a_n > 0 \text{ and } \lim(\frac{1}{a_n}) = 0.$$

(Note here, that to make sense of $1/a_n$ we may have to delete terms where $a_n$ is zero. This is messy business, don't worry about it too much.)

# 13 Monotone Convergence Theorem

**Where are we going next?**

1. Consider a sequence defined recursively as $a_1 = 1$ and $a_{n+1} = \frac{a_n^2+2}{2a_n}$.
   - On your homework you will investigate this sequence.
   - The first few terms are: 1, 1.5, 1.4166666666666667, 1.4142156862745099, 1.4142135623746899, 1.414213562373095
   - Do you think this sequence has a limit? What do you notice about this sequence? How could we prove such a thing?

2. Does the following sequence converge? (The terms are $2, 2.5, 2.66, 2.708, 2.7166, \ldots$)

$a_1 = 1 + 1$

$a_2 = 1 + 1 + 1/2$

$a_3 = 1 + 1 + 1/2 + 1/3!$

$a_4 = 1 + 1 + 1/2 + 1/3! + 1/4!$

$a_n = 1 + 1 + 1/2 + \cdots + 1/n!$

3. Consider another sequence, something like $a_n = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{n^3}$.
   - In Calc 2 you probably learned that this sequence (and thus its corresponding infinite series) converges. But mathematicians have no explicit formula for the limit.
   - Compare with the following:

$$\begin{aligned} 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots &= \frac{\pi^2}{6} \\ 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots &= \text{some mystery number.} \\ 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots &= \frac{\pi^4}{90}. \end{aligned}$$

Definition

We say that a sequence $s_n$ is **increasing** if

We say that a sequence $s_n$ is **decreasing** if

We say a sequence is **monotone** if

Note: Sometimes we will say something like **strictly increasing** which just means the inequalities are strict.

**Examples:**

1) The sequence $s_n = \frac{1}{n}$ is [increasing/decreasing/neither/both]. It is also [bounded/unbounded].

Does it converge?

2) The sequence $s_n = (-1)^n$ is [increasing/decreasing/neither/both]. It is also [bounded/unbounded].

Does it converge?

3) The sequence $s_n = n^2$ is [increasing/decreasing/neither/both]. It is also [bounded/unbounded].

Does it converge?

4) The sequence $s_n = 5$ is [increasing/decreasing/neither/both]. It is also [bounded/unbounded].

Does it converge?

---

Theorem (The Monotone Convergence Theorem)

Let $s_n$ be a sequence that is __________________ and __________________.

Then $s_n$ __________________ (to a finite limit).

**Proof:**

There are two cases. When $s_n$ is increasing or when $s_n$ is __________________. We will do the case when $s_n$ is increasing.

So suppose that $s_n$ is increasing and __________________. Let $L =$ ____________.

We will show that $s_n$ converges to $L$.

Let ____________.

# 14 Using the Monotone Convergence Theorem

**Tool from HW11 or Calc 2:** Let

$$b_n = 1 + 1/2 + 1/4 + 1/8 + 1/16 + \cdots + 1/2^n.$$

Then $b_n$ converges to 2.

**Awesome Theorem:**.

Let $a_n$ be the sequence defined by

$$a_n = 1 + 1/1! + 1/2! + \cdots 1/n!.$$

Then $a_n$ converges to some number $L$. We call this number $e$, after Leonhard Euler.

**Proof:**

**Example:**

Let

$$s_1 = 5, \text{ and } s_{n+1} = \frac{s_n^2 + 5}{2s_n} \text{ for } n \geq 1.$$

A) Prove that $s_n$ converges.

B) Prove that $\lim s_n = \sqrt{5}$.

**Proof:**

Let's do part B) first, assuming part A):

Proof of A)

# 15 What is a Cauchy Sequence?

Let's begin upstanding! Here are some questions to talk about:

1. Does every set have a least upper bound? Why or why not?
2. Does every increasing sequence converge? What about every bounded increasing sequence?
3. Does every bounded sequence converge?
4. If a sequence is bounded below, is it necessarily bounded?
5. If a sequence is decreasing and bounded below is it necessarily bounded?
6. If a sequence converges, does that mean it is bounded? Can you prove this if asked on a quiz? (Not saying I will, but thinking about things like this will help identify gaps in your understanding)
7. In the reading you say the definition of a Cauchy sequence:

   $$\forall \epsilon > 0, \quad \exists N, \quad \text{such that for all } m, n > N \ |a_m - a_n| < \epsilon.$$

   What the heck is it saying in common English?
8. Why might we care about the "monotone bounded" property or the Cauchy property? There is something that both of these have in common. (Or rather, something that they lack.).

The sequence $a_n$ is **Cauchy** if

$$\forall \epsilon > 0, \text{there is some point in time, after which}\underline{\hspace{4cm}}$$

Let's be **wild** and define our own property: We say a sequence $a_n$ is a **Torero** sequence if

$$\forall \epsilon > 0, \text{there is some point in time, after which}\underline{\hspace{4cm}}$$

What does it mean to be a Torero sequence? Do you think Torero sequences converge?

Below are lots of properties. Today we're going to fill in what we've proven so far and prove a few more directions:

$\boxed{a_n \text{ is constant}}$

$\boxed{a_n = \sqrt{n}}$

$\boxed{a_n \text{ is bounded}}$

$\boxed{a_n \text{ is Torero}}$

$\boxed{a_n \text{ converges (to a real number)}}$

$\boxed{a_n \text{ is Cauchy}}$

$\boxed{a_n \text{ is monotone and bounded}}$

$\boxed{a_n = \frac{1}{n}}$

**Let's record some helpful facts:**

**Question:** Suppose that $H_n$ is the sequence defined by

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Determine whether or not $H_n$ is Torero.

# 16 What is a Subsequence?

**Warmup:** Prove, using the definition that the sequence $a_n = \frac{1}{n}$ is a Cauchy sequence.

**Remember our big goal: We want to prove that if a sequence is Cauchy then it converges (to some real number).**

- So far we have proved the converse, that convergent sequences are Cauchy
- And we also showed that Cauchy sequences are ____________.

Today our **goal** is to investigate what a **new** property that sequences may or may not have.

**Warmup:** It's NOT true that every bounded sequence converges. Write down a bounded sequence that doesn't converge (simpler is better here). Now talk with your group about why this doesn't converge.

But does your sequence "kinda sorta sometimes halfway converge?" Write down some ideas that you have.

**Definition** If $a_n$ is a sequence then a **subsequence** of $a_n$ is a sequence obtained by selecting, in order, an infinite subset of the terms of $a_n$. (Notationally it's often cumbersome to denote - we'll see below how we denote them typically.)

Consider the sequence $a_n = n^{(-1)^n}$, $\{a_n\} = \{\frac{1}{1}, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, 8, \ldots\}$.

Write down the first five terms of each of the subsequences defined by

- $\{a_{2k-1}\}$

- $\{a_{k^2}\}$

- $\{a_{n_k}\}$ where $\{n_k\}$ has as first few terms $\{1, 2, 6, 7, 14, 35, 67, 68, 69, 100, 121, \ldots\}$.

What is the 4th term of each of the above subsequences? In the original sequence, does this term occur before or after the 4th term of the original sequence?

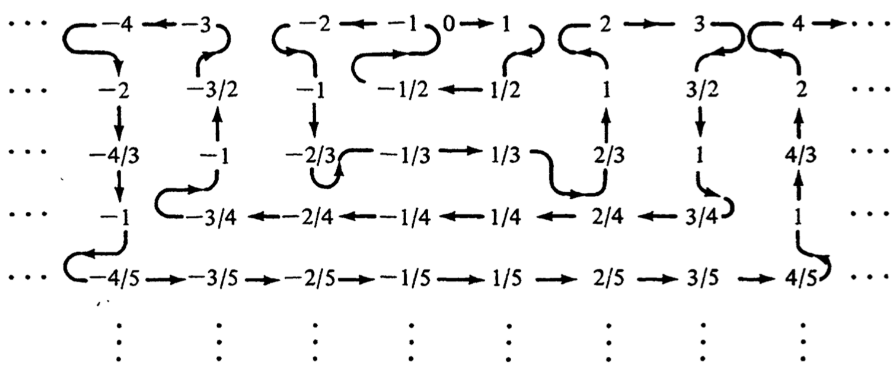
Does $a_n$ converge? Does $a_n$ have any subsequence that converges?

**Discuss**: Let's now think of an arbitrary sequence $a_n$. Maybe it converges, maybe it doesn't. And we'll have $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$.

- Is it possible that the subsequence converges? Might different subsequences converge to different limits? Could there be three different limits? 10? Infinitely many? Is it possible that no subsequence converges?
- If the original sequence $a_n$ converges, is it possible that a subsequence doesn't converge?
- If some property holds **eventually** for every term in $a_n$ then will the same be true for each term of $a_{n_k}$? (Say the property is that, maybe "this term" is within $\epsilon$ of 5.)
- What about the converse? Suppose that eventually, every term of the subsequence has some property. Does that need to hold for the original sequence?

**Snakes**

I promised that there would be a snake! What's going on in the following picture?




How many times does the number 3 occur in this sequence? Does the number 3.14 appear in this sequence? What about the number 3.1415? Does this sequence converge? Does it have a subsequence that converges?

The sequence defined by the snake map above has what amazing property?

**Theorem** If $a_n$ converges to $L$ then EVERY subsequence of $a_n$ converges to $L$.

# 17 The Bolzano Weierstrass Theorem

**The Bolzano Weierstrass Theorem:**

**Warmup:**

$a_n = (-1)^n$, $b_n = n^2$, $c_n = \sin n$

$d_n : 1, 2, 1, 3, 1, 4, 1, 5, 1, 6, 1, 7, 1, 8, 1, 9, 1, 10, 1, 11, \ldots$

- Which of these sequences does BW say has a convergent subsequence?
- Which of the sequences actually have convergent subsequences.

**We will prove this in two different ways.** (everybody's so creative!)

**Proof 1:** (The one you read about)

We will actually prove something much ________________________.

**Theorem:** Every bounded sequence contains a ________________________ subsequence.

(Do you see why this will imply the BW Theorem? What ingredient helps us do so?

Answer: ________________________________________________ )

**Proof:**

**Proof 2:**

Let $a_n$ be a bounded sequence, say it's between two numbers $s_1$ and $t_1$.

$$s_1 \le a_n \le t_1 \quad \text{for all } n.$$

Draw a line segment below and label $s_1$ and $t_1$ (space them far apart)

**Warmup: How many terms are there in our sequence? How many of them are in the interval $I_1 = [s_1, t_1]$?**

Let's draw a line in the middle of our interval $I_1$ to split it into two intervals. Draw this split in your picture. How many terms of $a_n$ are in the first interval? The second? Do we know? Is it possible that each contains only finitely many terms?

Conclusion: One of these two intervals ____________________________. Call this interval $I_2$.

We will continue this constructing $I_3$ as either the left or right half of $I_2$.

E.g. $I_{k+1}$ will be either the left half or right half of ______ (depending on which half contains __________________________.)

Let's label the endpoints:

$I_1 = [s_1, t_1]$. This interval contains infinitely many terms of $a_n$.

$I_2 = [s_2, t_2]$. This interval contains infinitely many terms of $a_n$.

...

$I_k = [s_k, t_k]$. This interval contains infinitely many terms of $a_n$.

We will now construct a subsequence $c_n$ of $a_n$:

Take $c_1 = a_1$.

Then take $c_2$_____________ (and be sure to pick a term after $c_1$)

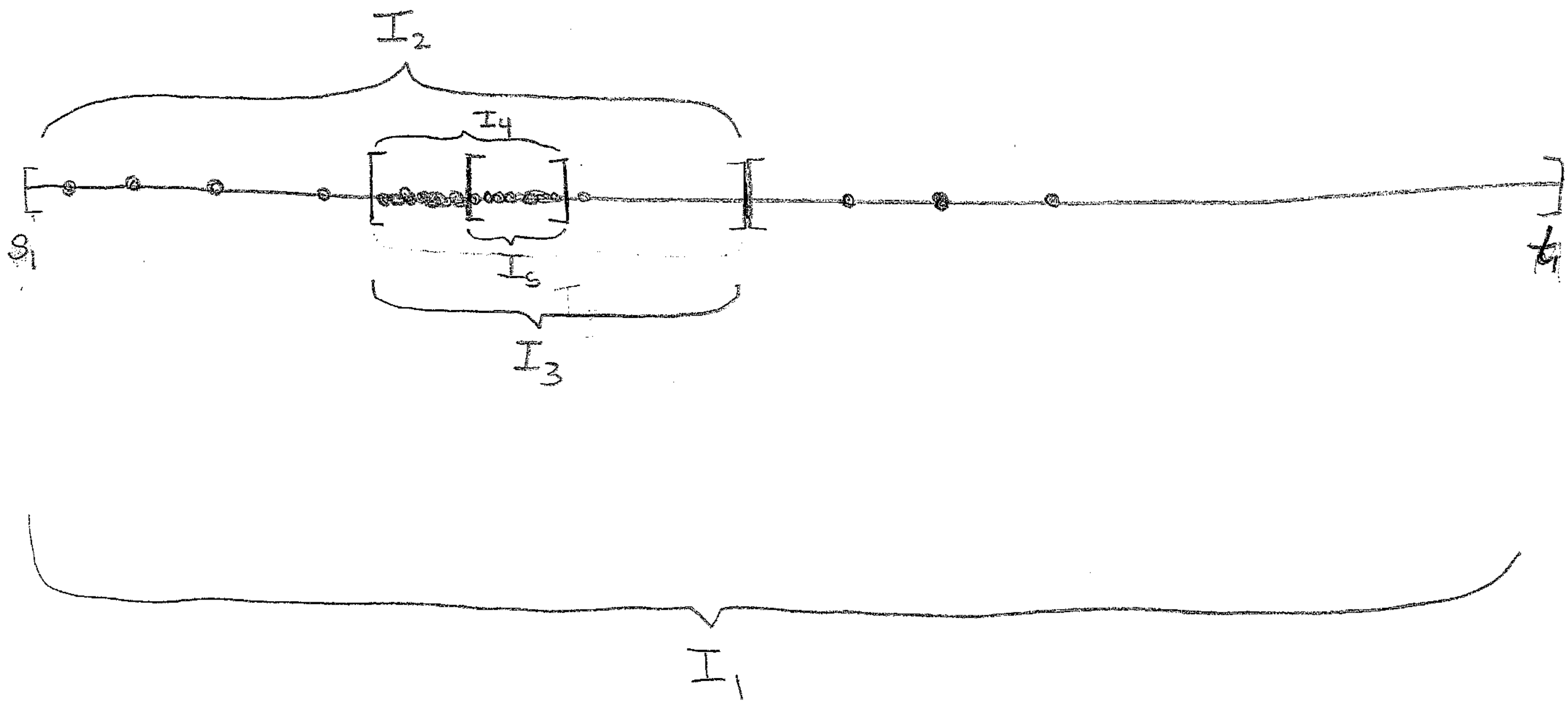
Then take $c_3$_____________ (and be sure to pick a term after $c_2$)

...

This defines a subsequence $c_n$ such that $\qquad \le \qquad \le \qquad$.

What do we know about the sequences $s_n$ and $t_n$?

a sequence $a_n$:



# 18 Using the Bolzano Weierstrass Theorem

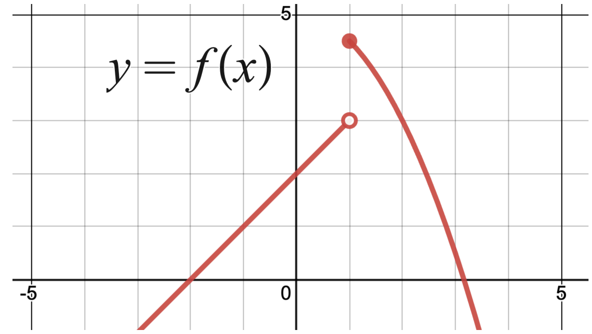
We're going to use BW to prove the following theorem:

**Big Theorem:** If $a_n$ is a Cauchy sequence then it converges (to some real number $L$).

**If time permits:** What's a limsup? liminf?

# 19 Introduction to Continuity

Today we will learn a **new** and important definition, of what it means for a function $f$ to be continuous at a point.




Our definition of **continuity** will use:

This is a good thing because we are all ______________________________ at studying them!

Imagine the picture at the right were a mountain terrain and you told your friend one of the following. Are you SURE what their $y$-value would be?

"Meet me at the point where $x = 0$." Their $y$-value will be at _______________.

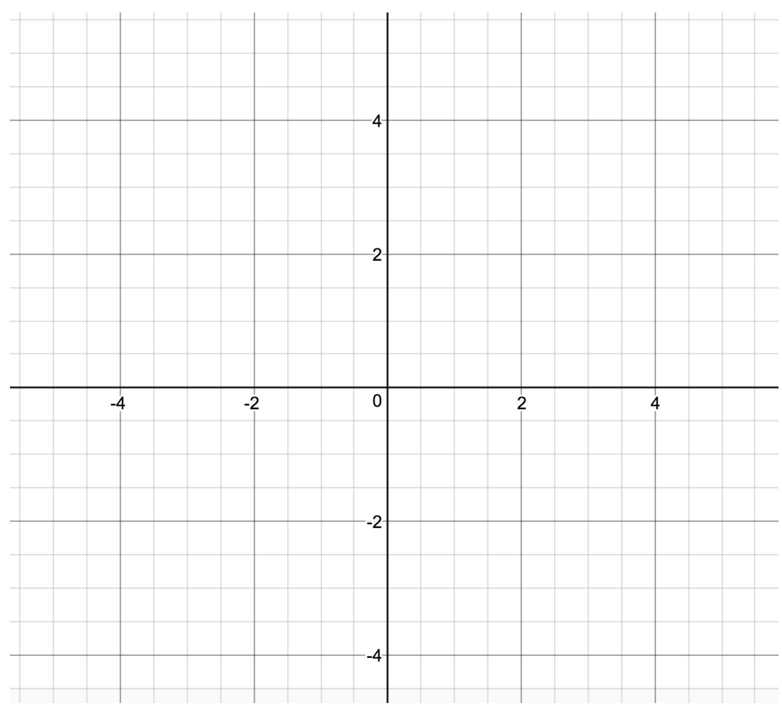
"Meet me near the point where $x = 0$." Their $y$-value will be near _______________.

"Meet me at the point where $x = 1$." Their $y$-value will be at _______________.

"Meet me near the point where $x = 1$." Their $y$-value will be near _______________.

> Every time we talk about a function, we will specify its domain of allowable inputs.

**Example: Sketch the graphs of the functions below:**




Let $f : [0, \infty) \to \mathbb{R}$ be the function $f(x) = \sqrt{x}$.

Let $g : [-1, 1) \to \mathbb{R}$ be the function $g(x) = x^2$.

Let $h : [0, \infty) \to \mathbb{R}$ be the function

$$h(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 4 & \text{if } x = 0 \end{cases}$$

> Don't overthink it. The domain $D$ will ALWAYS be given to you. It's the thing before the arrow in
>
> $$f : D \to \mathbb{R}.$$

**Discussion:** To say that a function $f : D \to \mathbb{R}$ is $\boxed{\textbf{continuous} \text{ at } 5}$ means that

"if the inputs are really close to 5, then the outputs should be really close to $f(5)$"

How can you rephrase this using words in the wordbank below:

[$f(x_n)$ / then / sequence / take / converge / to / 5 / $f$ / if / $x_n$ ]

**Formal Definition**
We say that the function $f : D \to \mathbb{R}$ is continuous at $a$ if

We say that $f$ is continuous if

Notice that this is a $\forall$ statement.

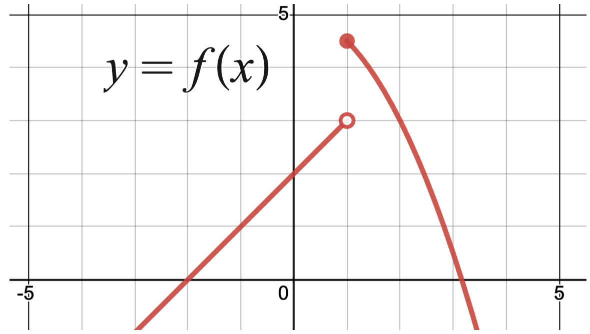
1. So to prove that it is true, we would want to START with

   and END with

2. To show something is NOT continuous, we need to find just ONE example of:

**Example:** The function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} x+2 & \text{if } x < 1 \\ 5 - \frac{1}{2}x^2 & \text{if } x \geq 1 \end{cases}$$




Prove this function is NOT continuous at 1.

To do this we will need an example of a sequence $x_n$

that DOES ________________________

but such that $f(x_n)$ DOESN'T ________________________.

**Proof:**

Let ________________________. This is a sequence that is ________________________.

Notice that ________________________.

We will show that the sequence $f(x_n)$ ________________________ converge to ____________.

**Example:** Show that the function $f(x) = x^2 + 4x - 3$ is continuous at 3.

| |
|---|
| To do this we will need to show for an _____________ sequence $x_n$ __________________________. <br> The sequence _____________ converges to _____________. |

**Proof:**

# 20 Getting Practice With Continuous Functions

General Theorems about Continuous Functions:

Suppose that $f, g$ are continuous functions and $c \in \mathbb{R}$. Then

1. $f + g$ is continuous
2. $fg$ is continuous
3. $f - g$ is continuous
4. $c \cdot f$ is continuous
5. $f \circ g$ (the composition) is continuous
6. $f/g$ is continuous.
7. $|f|$ is continuous.
8. The function $h(x) = c$ is continuous
9. The function $h(x) = x$ is continuous
10. The function $h(x) = \sqrt{x}$ is continuous
11. so are the functions $e^x, \sin x, \cos x, \ln x$ (this is harder, because e.g. the easiest way to define $e^x$ is as an infinite series.)

All of this is on the domain of these functions.

**Discuss:**

How could you use these theorems to prove that $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = x^2 + 4x - 7$ is continuous?

Does this mean that $f(x) = 1/x$ is continuous? Why or why not?

Remember that continuity is measured at each point. We say a function is continuous if it is continuous at every point in its domain.

**Proof of 5:**

Suppose that $f$ and $g$ are ________________________ on their respective domains $D_f$ and $D_g$.

We want to show that ____________ is continuous. Let $x_0$ be a point in the domain of ____________ and let $x_n$ be a sequence (________________________) that converges to ____________. Now since

**Remember:**

- If you want to prove that a function $f$ IS continuous at a point $a$, then you should write "Let $x_n$ be an arbitrary sequence in the domain of $f$ that converges to $a$. And then **prove** that $f(x_n)$ converges to $f(a)$. (You likely have a formula for $f(x)$ so use this!)
- If you want to prove that a function $f$ is NOT continuous at a point $a$, then you should find one **specific** sequence $x_n$ that converges to $a$ with the property that $f(x_n)$ does NOT converge to $f(a)$. (You might need to be creative here!)

**5 - 10 minutes:** Start by sharing your proof portfolio proof with a neighbor. After taking some time to carefully read the proof, talk with each other about what you noticed.

With your feedback, try to find at least 2 positive things to say about your partner's proof, and at least 1 piece of constructive feedback. Remember you want to help each other write as clearly and precisely as possible.

After you finish your discussion: Work on the following. I want you to **write your proofs carefully**. These are due on Tuesday as part of your homework.

1. Prove that
$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$
is not continuous at 0. (Hint: Can you find a sequence of numbers $x_n \to 0$ such that $f(x_n)$ does not converge to 0? Be specific. Your answer should use the number $\pi$.)

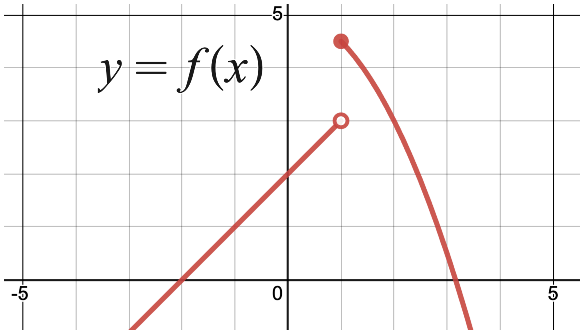
2. Prove that
$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$
is continuous at 0. (Hint: Begin your proof by saying "Suppose that $x_n \to 0$, then I will show that $f(x_n) \to f(0)$. ..." and then work out what you know about $f(x_n)$. Hint: you might need the squeeze theorem.)

3. Prove that the function
$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$
is not continuous at any point $a$. (Hint: there are two cases, one when $a$ is rational and one when $a$ is irrational. You can use without proof that there is a sequence of rationals/irrationals that converge to a rational/irrational (mix/match))

## 21 Epsilon gets a Friend!

An equivalent way of defining continuity at a point $a$ is to say:

"You know, if $x$ is really close to $a$ then $f(x)$ should be really close to $f(a)$"




**Explain:** Is it true that inputs "close" to 1 have outputs "close" to 4.5? Why or why not?

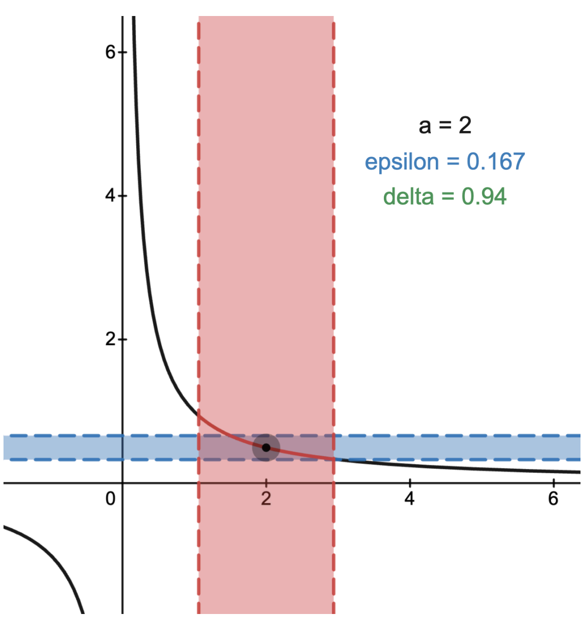
Does this depend on your notion of what "close" means?

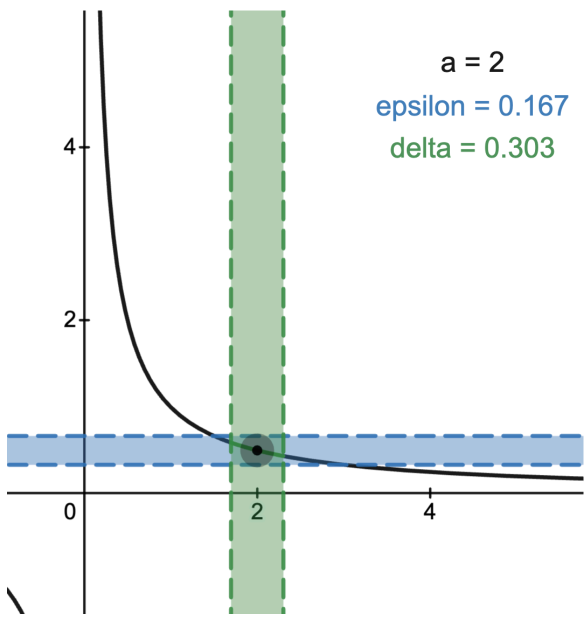
**A Different Definition of Continuity:**

We say that a function $f : D \to \mathbb{R}$ is continuous at $a$ if

Later, we will show that this definition is **equivalent** to the one we saw before. For now, let's play around with Desmos:

`https://www.desmos.com/calculator/v55iltr7ms`




**Important Question:**

If a given value of $\delta$ "works", then a smaller value of delta will [sometimes/always/never] work.

We will investigate the following functions:

1. Why is $f(x) = 3x - 7$ continuous at $a = 4$? Given an $\epsilon$, how do we find a $\delta$?
2. Why is $f(x) = x^2 - 7x$ continuous at $a = 2$? Given an $\epsilon$, how do we find a $\delta$?
3. Why is $f(x) = x^3$ continuous at $a = 1$?
4. Why is the function $f(x) = 2x^2 + 1$ continuous?
5. Why is the function below continuous at $a = 0$?

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

## 22 Why are the two definitions of continuity equivalent?

Today we will prove that the two definitions we have seen for continuity are **equivalent.** Buckle in for a big day!

Let's recall the two definitions.

For both of these, we fix a function

$$f : D \to \mathbb{R}$$

and an input $a \in D$.

| **I. Sequential Definition** | **II. $\delta, \epsilon$ Definition** |
|---|---|

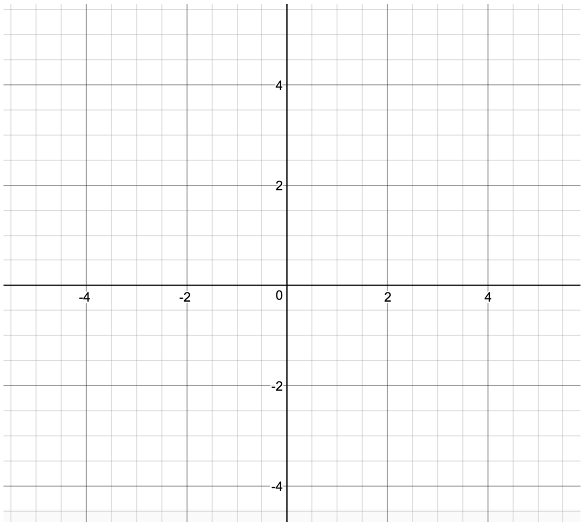
**Claim 1:** If $f$ satisfies property I then it satisfies property II.

**Claim 2:** If $f$ satisfies property I then it satisfies property II.

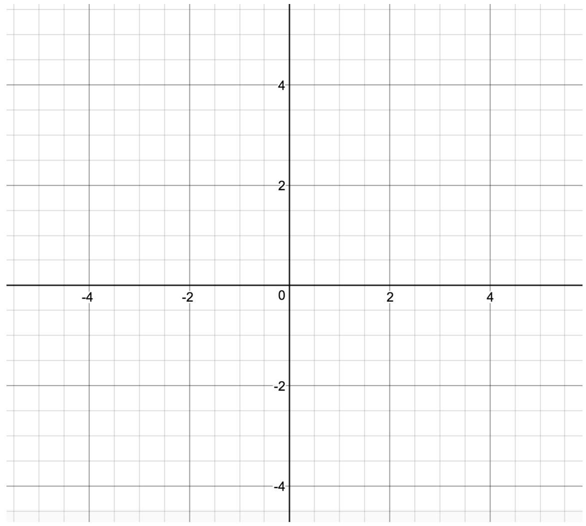
# 23 The Extreme Value Theorem

On the graphs below, see if you can draw a **continuous** function $f : D \to \mathbb{R}$ with or without the following properties.

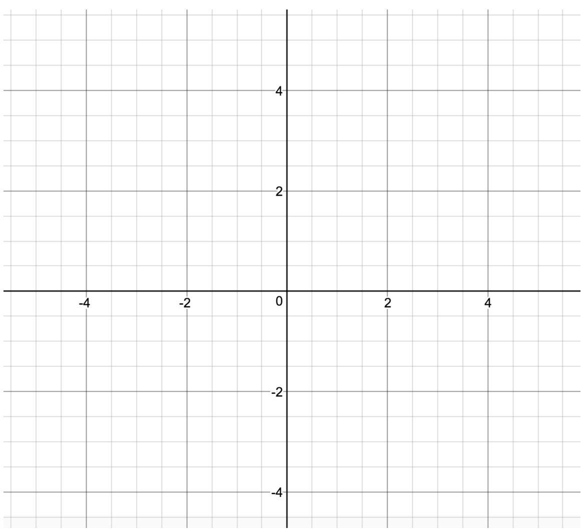
$D = (-2, 2)$ and
$f$ achieves a maximum on $D$.



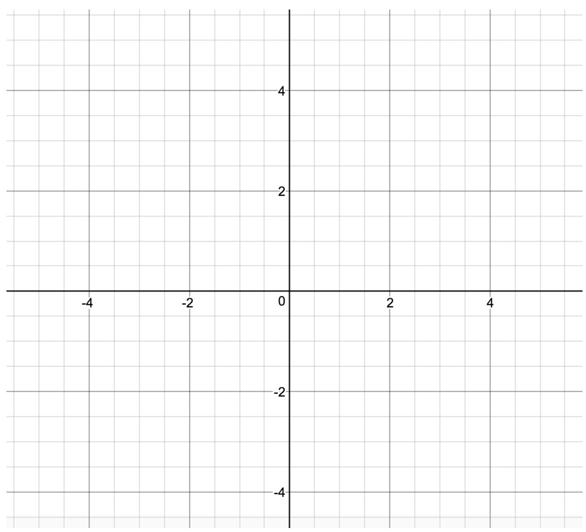

$D = [-2, 2]$ and
$f$ achieves a maximum on $D$.




$D = (-2, 2)$ and
$f$ does not achieve a maximum on $D$.




$D = [-2, 2]$ and
$f$ does not achieve a maximum on $D$.




**Theorem:** (____________________________________)

**Remember:** The functions we draw are ones we are likely familiar with, but in general functions might be crazy, and who knows - maybe some crazy function is actually continuous? For example, here's a function:

$$f : (0,2) \to \mathbb{R}, \qquad f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational.} \\ n & \text{if } x \text{ is rational and } x = m/n \text{ in lowest terms} \end{cases}$$

Now it turns out that $f(x)$ is NOT continuous, but this is an example of a crazy function.

What about this function here:

$$f : (0,2) \to \mathbb{R}, \qquad f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational.} \\ 1/n & \text{if } x \text{ is rational and } x = m/n \text{ in lowest terms} \end{cases}$$

Which of these functions is continuous at 1? at $\sqrt{2}$? (It's hard to tell!)

**You will explore some of this on your homework.**

Before we prove **The Extreme Value Theorem** let's talk about some implications:

1. On your homework, you'll show that a continuous function on a closed interval

   also achieves a _____________.

2. This shows that a continuous function on a closed interval must be ____________________,

   namely between:

3. Is this true for continuous functions on an open interval?

**Proof of the Extreme Value Theorem**

# 24 In the Middle

**During the last class:** We proved that a continuous function $f : [a, b] \to \mathbb{R}$ had to be ____________.

We didn't complete the proof and explain why this means that $f$ actually achieves a ____________.

With your group, fill in the missing details of that proof below:

**Proof**: Let $D = [a, b]$. Let $S = \{f(x) \ : \ x \in D\}$, the set of all [inputs/outputs].

Since $f$ is ____________, this means that $S$ is ____________.

Let $M =$ ________________________ which exists by the ________________________ axiom.

We will show that $M$ is the ____________ of $f$ on $D$. Since $M$ is an ______________________ for $S$, this means that $M$ ____ $f(x)$ for all $x \in D$.

All we have to do is show that $M =$ __________ for some $x_0 \in D$. This will show that $f$ achieves its maximum on $D$.

By the definition of least upper bound,

we know that $M - 1/n$ is ________________________ for $S$.

This means that for each $n$ there is some element $y_n$ of $S$ with

$$M - 1/n < y_n \leq M.$$

[Pause here and make sure your group understands *why*]

$\star$ Note that $y_n \to$ ____________. [Why?]

Now since $y_n \in S$, $y_n =$ __________ for some $x_n$ in $D$. [Go back and see what $S$ is, if you forgot].

So restating line $\star$ becomes:

____________ $\to$ ____________

Note that for all $n$, $x_n$ is between ____ and ____ because it is an input to the function $f$. Thus the sequence $x_n$ is ____________.

Now by the ________________________ Theorem, there is a convergent subsequence $x_{n_k}$ of $x_n$ that converges to some $x_0$. This limit $x_0$ must be in $D$ because:

Earlier in class we proved a Theorem that says if a sequence converges to a limit, then [some / all] subsequences converges to that same limit. Since we know:

$f(x_n) \to M$ and that $x_{n_k} \to x_0$. The Theorem then means that: ____________ $\to$ ____________.

But now by continuity, since $x_{n_k} \to x_0$ we know that the $f(x_{n_k}) \to$ __________.

Finally, we conclude that ____________ = ____________

CONGRATULATIONS!

You may turn the page for more goodies

The picture shows some data about a function $f$.

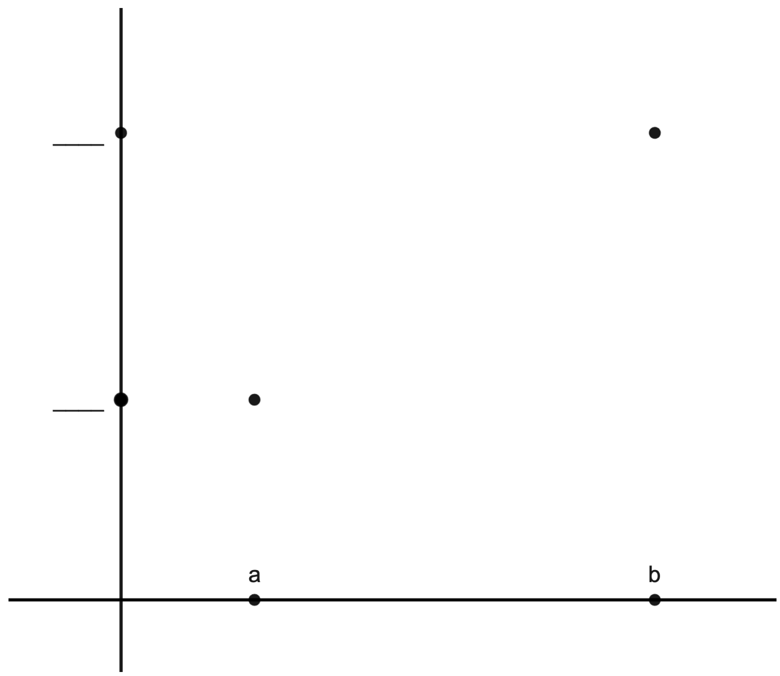
First things first, fill in the blanks on the $y$ axis.

Now I want you to try your best and write down a careful statement of the Intermediate Value Theorem that you read about. It's ok if it's not correct - but try really hard to capture what you think it is saying.

**So that we all use consistent letters:**

If you need to talk about an input, on the $x$ axis, use the letter $c$.

On the $y$-axis, use the letter $W$. (Hint: $W$ stands for "want")




**The Intermediate Value Theorem (~~Taylor's~~ My Version)**

Don't turn the page too early - no peeking! After you have written your IVT statement, please call me over to have a look. Then I'll let you know if you can turn the page.

**The Intermediate Value Theorem (Version for the Definition Quiz)**

If $f$ is a continuous on some interval $I$ and $a, b \in I$ with $a < b$, then if $W$ is any real number in between $f(a)$ and $f(b)$ then there exists at least one $c \in (a, b)$ such that $f(c) = W$.

We will **prove** the IVT in the next class, but for now, let's practice **using** it.

I want you to be very specific with these. You must say:

- What is your function $f$? Why is $f$ continuous?
- What are your values of $a$, $b$, $W$.
- And then when you use the IVT you get a number $c$. Make sure you give a conclusion.

**Example 1:** Show that there is a solution $x \in \mathbb{R}$ to the equation $x^5 - 10x = 2$.

**Example 2:** Show that there is a solution $x \in \mathbb{R}$ to the equation $\cos x = x$.

Hint: This problem is about 2 functions, but what trick (using subtraction) did you learn on the homework last week to turn two functions into 1?

**Example 3:** Let $a \in \mathbb{R}$ with $a > 0$ and let $m \in \mathbb{N}$. Prove that $a$ has an $m$th root, i.e. that there is a positive number $r$ with the property that $r^m = a$.

# 25 A Proof of the Intermediate Value Theorem

**The Intermediate Value Theorem**

If $f$ is a continuous on some interval $I$ and $a, b \in I$ with $a < b$, then if $W$ is any real number in between $f(a)$ and $f(b)$ then there exists at least one $c \in (a, b)$ such that $f(c) = W$.

**Proof:**

(Divide and Conquer) This is similar to a "binary search" algorithm that you might see in computer science. Here's a rough outline of how the proof would go.

For simplicity let's assume that $f(a) < f(b)$ and so that $f(a) < W < f(b)$.

Algorithm:

- You know that $W$ is in between $f(a)$ and $f(b)$.
- Try the midpoint $t = (a + b)/2$.
  - If $f(t) = W$ you win!
  - Otherwise, $W$ is either between $f(a)$ and $f(t)$ or between $f(t)$ and $f(b)$
- Repeat this algorithm successively dividing the interval in half.
- Eventually you'll have a sequence of intervals:

  $$[a_1, b_1], [a_2, b_2], \ldots,$$

  where $W$ is between $f(a_n)$ and $f(b_n)$ at each step. But now since these intervals are getting smaller and smaller, (cut in half each time). Then these intervals are approaching one limit, call it $c$.

  $$a_n \to c, \qquad b_n \to c.$$

  And hence by continuity:

  But now we know that $f(a_n) < W < f(b_n)$, and so by the squeeze theorem ________________________.

# 26 Introduction to Uniform Continuity

Today we are going to learn about **uniform continuity**. Below are many many examples of functions on a set $S$:

- Which ones are continuous on the set $S$? Answer: ______________________________.

$f(x) = 4x - 3$ on $(\infty, \infty)$

$f(x) = 4x - 3$ on $S$ (some $S \subset \mathbb{R}$).

$f(x) = x^2$ on $[0, 4]$

$f(x) = x^2$ on $(0, 4)$

$f(x) = x^2$ on $[0, \infty)$

$f(x) = \sin x$ on $[0, \pi]$

$f(x) = \sin x$ on $(0, 4)$

$f(x) = \frac{1}{x}$ on $(0, 4)$

$f(x) = \frac{1}{x}$ on $(0, \infty)$

$f(x) = \frac{1}{x}$ on $(3, \infty)$

$f(x) = \frac{1}{x}$ on $[3, 6]$

$f(x) = \frac{1}{x^2}$ on $[3, \infty)$

$f(x) = \sqrt{x}$ on $[0, 1]$

$f(x) = \sqrt{x}$ on $[0, \infty)$

$f(x)$ a continuous bounded function on $[a, b]$

$f(x)$ a continuous bounded function on $(a, b)$

$f(x)$ a Lipschitz function on $(a, b)$

$f(x)$ a continuous function with bounded slope on $(a, b)$

$f(x)$ is continuous and unbounded on $[0, \infty)$.

$f(x)$ is continuous and unbounded on $[3, 10]$.

$f(x) = \sin(\frac{1}{x})$ on $(0, 1)$

$f(x) = x \sin(\frac{1}{x})$ on $(0, 1)$

$f(x) = x^3$ on $(0, 1)$

$f(x)$ a continuous function with unbounded slope on $[a, b]$

$f(x)$ a continuous function with unbounded slope on $(0, 1)$

This means that for any of these functions, if we picked any point $a$ in their domain, and any $\epsilon > 0$ we could find a $\delta$ that ____________,

Meaning that:

**Question:** Do you think that the $\delta$ can depend on $\epsilon$ [yes/no] can it also depend on the point $a$ [yes/no]?

Intuitively, what would have to be happening with the graph to force us to pick a smaller $\delta$?

**Definition:**

### What is this definition saying?

It is saying that if $f$ is uniformly continuous on some set $S$ then

for a given $\epsilon$, there is ______________________________________.

Let's go back to the front page and see
which functions we think might be uniformly continuous.

Note that $S$ is commonly going to be some sort of interval, but it doesn't have to be. In theory it could be some random set of points. But most of the time, $S$ will be one of two things:

- A bounded interval: $(a,b),[a,b],(a,b],[a,b)$ where $a$ and $b$ are real numbers.
- An unbounded interval: $[a,\infty),(a,\infty),(-\infty,a],(-\infty,a)$.

**Example:** Show that $f(x) = 4x - 3$ is uniformly continuous on $(-\infty,\infty)$.

**Pro-tip**: If you know a function is uniformly continuous on $S$ then it will automatically be uniformly continuous on any smaller set!

**Example:** Using the definition, show that $f(x) = 1/x^2$ is uniformly continuous on $[3, \infty)$.

**Example:** Show that $\sqrt{x}$ is uniformly continuous on $[3, \infty)$.

**Big Theorems about Uniform Continuity**

1. A **continuous** function on a **closed** bounded interval is **always** uniformly continuous on that interval.
2. A Lipschitz function is uniformly continuous on its domain.
3. A continuous function $f(x)$ is uniformly continuous on a bounded open interval if and only if it can be extended to the include the end points so that the extension is continuous. (Think about "filling in values for $f(a)$ and $f(b)$ that make the whole function continuous on the closed interval).
4. A uniformly continuous function on any bounded interval $(a,b)$ or $[a,b]$ must be bounded.
5. If $f$ is differentiable (has a derivative) on its domain and this derivative is **bounded**, then $f$ is uniformly continuous. This works even on infinite intervals.

**Extending:** We will talk about "extending" a function to a larger domain. This means we will add new points to the function.

**Example:** Extend the function $f:(0,\infty)$ given by $f(x) = x^2 + 1$ to a function $F:[0,\infty)$ in two different ways. Can you extend it in a way that is continuous?

**Example:** Extend the function $f:(0,\infty)$ given by $f(x) = 1/x$ to a function $F:[0,\infty)$ in two different ways. Can you extend it in a way that is continuous?

Let's go through the examples on the front page again and see
if we can determine whether any of these are uniformly continuous.

# 27 Proving Some Theorems about Uniform Continuity

**Theorem:** If $f : [a, b] \to \mathbb{R}$ is continuous then $f$ is uniformly continuous.

**Example**

Use this theorem to explain why $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

**Theorem:** If $f : S \to \mathbb{R}$ is uniformly continuous, and $x_n$ is a Cauchy (i.e. convergent) sequence in $S$, then $f(x_n)$ is a Cauchy (i.e. convergent) sequence.

Wait a minute? Why do we need uniform continuity?

Isn't it true if $f$ is just plain old regular continuous, then if $x_n \to x_0$ then $f(x_n) \to f(x_0)$?

Not quite - the issue is:

In fact here's an explicit example of a **continuous** function that doesn't have this property:

The Theorem on this page shows us that the function ____________

is NOT uniformly continuous on ____________.

**Proof of theorem:**

A similar method can be used to show that uniformly continuous functions must be bounded.

**Theorem** A continuous function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on an open interval if and only if it can be extended to the include the end points in a way so that the extension is continuous.

**Proof:** (Sketch)

## 28 Relationship Between Derivatives and Uniform Continuity

On your homework you proved that if a function is ________________________ then it is uniformly continuous.

Remember that Lipschitz is the same as ________________________.

**Theorem** Suppose $I$ is a (possibly infinite) interval. If $f : I \to \mathbb{R}$ is differentiable (has a derivative) on its domain and this derivative is **bounded**, then $f$ is uniformly continuous.

**Proof:**

# 29 Something Totally Deranged

This day and the next we are going to briefly talk about infinite series. As a warmup, let's consider the sum:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

What can we do to make **precise** sense of what this means?

We can use __________________________, meaning: let $s_n =$ ______________________________________.

Then we will have a __________________________ $s_n$ and we can test whether or not this converges.

On your homework earlier in the semester, you proved that the **series** above converged to:

And more generally that

(We'll sumtimes use $\sum$ notation. My professional advice is actually to avoid using it unless necesseary. It will be necessary for us sometimes, but in general it can get in the way).

There are lots of sums that you've seen before: For instance in Calc 2 you saw the following:

$$\frac{1}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

Which ones do you remember? Do some of these converge / diverge? Does it matter what $x$ is?

There's a whole treasure trove of cool stuff you could study about these things!

# Derangements

Today let's do something that's totally deranged! Imagine that you arrived at a workshop with 1000 people, and all of your nametags had been randomly permuted!

**Question** What do you think the probability is that no one (not a single person!) actually got the correct nametag?

After writing down your guess, share it with a neighbor and discuss why you made your guess. Talk about whether you think this probability would change if instead of 1000 people we had only 10 people.

We'll now do some experiments to see what happens. Imagine our names are ABC. Below are the 6 permutations of these letters:

$$ABC, \quad ACB, \quad BAC, \quad BCA, \quad CAB, \quad CBA$$

Under which of these permutations would NO ONE get the correct nametag? Circle them. These are called **derangements.**

Determine how many derangements there would be among the 24 permutations of $ABCD$ below:

$$ABCD, ABDC, ACBD, ACDB, ADBC, ADCB,$$

$$BACD, BADC, BCAD, BCDA, BDAC, BDCA,$$

$$CABD, CADB, CBAD, CBDA, CDAB, CDBA,$$

$$DABC, DACB, DBAC, DBCA, DCAB, DCBA$$

Fill in your information in the table below:

| number of people $n$ | total number of permutations | number of **derangements** $T(n)$ | probability of getting a **derangement** $P(n)$ |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 2 | 2 | 1 | $\frac{1}{2}$ |
| 3 | | | |
| 4 | | | |
| 5 | | | |
| 6 | | | |
| 7 | | | |

On your homework, you will learn more about this problem, but for now, see if you can write down a guess for a formula for $P(n)$, and $T(n)$. What do you notice?

I was first shown this beautiful "probleme de rencontres" by Prof. Liviu Nicolaescu at U. Notre Dame when I was a high school student attending a math circle he was organizing. For a document with more details, including the "answers" to some of these questions, see `http://www.geometer.org/mathcircles/derange.pdf` For the video we watched in the lesson, see `https://www.youtube.com/watch?v=DoAbA6rXrwA`

# 30 Does it Converge?

**Example:** The harmonic series diverges.

**Theorem: The Ratio Test** A series $\sum a_n$ of **nonzero** terms. Let $L = \lim |\frac{a_{n+1}}{a_n}|$. Then $\sum a_n$

1. converges if $L < 1$
2. diverges if $L > 1$
3. if $L = 1$ the test gives no information.

**Example**

Use the ratio test to prove that the series

$$\frac{1}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

converges.

**Definition:**

This definition makes sense because ________________________________________

**Proof of the Ratio Test**

# 31 Limits of Functions and Introduction to Derivatives

We will learn about "Limits of Functions." Below is a schematic for thinking about what it means to say something like

$$\lim_{x \to a} f(x) = -\infty.$$

To make sure we're on the same page let's draw a picture of what this looks like

## Definitions

To understand all the definitions of limits of functions, we need to understand the building blocks of these definitions. Here is a dictionary of key concepts that we need to be able to translate into precise mathematical language:

1. $x$ is close but not equal to

| | | |
|---|---|---|
| (a) $a \in \mathbb{R}$ from the right: | $a < x < a + \delta$ | $x \to a^+$ |
| (b) $a \in \mathbb{R}$ from the left: | $a - \delta < x < a$ | $x \to a^-$ |
| (c) $a \in \mathbb{R}$: | $0 < \lvert x - a \rvert < \delta$ | $x \to a$ |
| (d) $\infty$: | $x > m$ | $x \to \infty$ |
| (e) $-\infty$: | $x < m$ | $x \to -\infty$ |

2. $f(x)$ is close to

| | | |
|---|---|---|
| (a) $L \in \mathbb{R}$: | $\lvert f(x) - L \rvert < \epsilon$ | $\lim_{x \to \star} f(x) = L$ |
| (b) $\infty$: | $f(x) > M$ | $\lim_{x \to \star} f(x) = \infty$ |
| (c) $-\infty$: | $f(x) < M$ | $\lim_{x \to \star} f(x) = -\infty$ |

By combining these options you will get a total of 15 options for limits! Rather than memorize all of these, let's practice putting them together.

**Example 1a,2a**: $\lim_{x \to a+} f(x) = L$ should mean that for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x$ such that $a < x < a + \delta$ then $|f(x) - L| < \epsilon$.

**Example 1c,2b**: $\lim_{x \to a} f(x) = \infty$ should mean that for all $M \in \mathbb{R}$ there is some $\delta$ such that if $0 < |x - a| < \delta$ then $f(x) > M$.

**Example 1e,2c**: $\lim_{x \to -\infty} f(x) = -\infty$ should mean that for all $M \in \mathbb{R}$ there is some $m \in \mathbb{R}$ such that if $x < m$ then $f(x) < M$.

**Practice** Carefully state definitions for the following combinations:

- 1a, 2c
- 1b, 2a
- 1c, 2a
- 1d, 2b
- 1e, 2b

Here's some more practice

1. Suppose that you know that a function $f(x)$ satisfies the following:

   $\boxed{\text{For all } M \in \mathbb{R},\ \exists \delta \text{ such that if } 0 < |x-3| < \delta \text{ then } f(x) > M.}$

   Which of the following does this mean:

   - $\lim_{x\to 3} f(x) = 0$
   - $\lim_{x\to 3^+} f(x) = \infty$
   - $\lim_{x\to 3} f(x) = \infty$

2. Suppose that you know that a function $f(x)$ satisfies the following:

   $\boxed{\text{For all } M \in \mathbb{R},\ \exists \delta \text{ such that if } x > 3 \text{ and } x - 3 < \delta \text{ then } f(x) < M.}$

   Write down a statement like $\lim_{blah} blah = blah$ as in the previous problem.

3. Last one! Suppose you know that:

   $\boxed{\text{For all } M \in \mathbb{R},\ \exists N \text{ such that if } x < N \text{ and then } f(x) < M.}$

   Write down a statement like $\lim_{blah} blah = blah$ as in the previous problem.

4. If these seem like a foreign language - (they kinda are!) come talk to me in office hours this week!

Another way to define limits of functions is using sequences. For instance:

We say that $\lim_{x \to a} f(x) = L$:

if

for any sequence $x_n$ with:

- $x_n$ not including the point $a$, and $x_n \to a$

we have $f(x_n) \to L$.

- This definition is very flexible because it works generalizes nicely for left and right hand limits, and with infinity. For instance: $\lim_{x \to 3^+} f(x) = -\infty$ means:

  for any sequence $x_n$ with:
  - $x_n$ not including the point 3,
  - $x_n > 3$ for all $n$ (i.e. the sequence is to the right)
  - and $x_n \to a$

  we have $f(x_n) \to \infty$.

- For practice make sure you can write down similar statements for the following:
  1. $\lim_{x \to -1^-} f(x) = -2$
  2. $\lim_{x \to -\infty} f(x) = -3$
  3. $\lim_{x \to \infty} f(x) = -\infty$

- Remember that sequences are **always** sequences of real numbers and infinity is not a real number. So we don't have to require that a number not be infinite - it's automatic, so please don't write something like "$x_n$ not including $\infty$."

# 32 Introduction to Derivatives

**Definition of Derivative:**

Suppose that $f$ be a function on an ____________________ containing the number __________. Then

If this limit exists, then we say $f$ is ______________________________. If a function $f$ is

____________________ at all points in its domain, then we say it is ____________________.

**Note:** Sometimes we will use the equivalent definition:

**Example:** Find a formula for $f'(c)$ if $f(x) = x^2$. Draw a picture to represent what we are measuring.

The number $f'(c)$ (if it exists) measures the ______________________________

**Example:**

Earlier on a homework you discovered the factorization:

$$x^n - c^n = (x - c) \cdot ( \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ).$$

Use this to find a formula for $f'(c)$

**Here are some derivative rules:**

Let $I$ be an open interval and let

Let $f, g : I \to \mathbb{R}$ be two functions that are each differentiable at $c$, and let $k$ be a constant. Then

1. $(f+g)'(c) = f'(c) + g'(c)$,
2. $(f-g)'(c) = f'(c) - g'(c)$,
3. $(kf)'(c) = k \cdot f'(c)$.
4. $(fg)'(c) =$
5. $(f/g)'(c) =$

How might we prove them? Let's take a look at a proof of the Product Rule from your reading: `https://drive.google.com/open?id=1Kgeg1GCgWVa6Du7vN6JroUcBf0KBc_bX&usp=drive_fs`

**What about compositions of functions?**

**From Calc 1:** Informally: if $h(x) = f(g(x))$ then $h'(c) =$ __________________________.

**Theorem:** Suppose $I_1$ and $I_2$ are open intervals and $f : I_1 \to I_2$ and $g : I_2 \to \mathbb{R}$ are functions.

Let $h(x) =$

Suppose that ____________ is differentiable at ____________ and ____________ is differentiable at ____________.

Then $h'(c) =$

**Example:** Using these rules and others developed in Calculus, we can e.g. find $f'(2)$ or $f'(0)$ for any of the following functions:

$x^3 - 7x^2 + 14$ $\qquad$ $e^{3x} - 2023\tan(x)$ $\qquad$ $\sqrt{3x+1}$

But what about functions like those below:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \qquad f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \qquad f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$f(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \qquad f(x) = \begin{cases} x^3\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

**Can you find a function so that $f'(c)$ does not exist at some point?**

**Theorem:** IF $f$ is differentiable at $c$ THEN ________________________________________.

Before we do the proof, is the converse true?

**Proof:**

# 33 Properties of Differentiable Functions

**Discuss with your neighbor:** What did the Extreme Value Theorem say? Write down a brief summary of it.

We are going to turbo charge it now:

**Fermat's Theorem:** Suppose that $f : (a, b) \to \mathbb{R}$ is a function.

Suppose that $f$ ________________________ at some point $x_0$. Then

Points $x_0$ such that ____________________________________ are called ________________________.

This gives us a very good way to solve optimization problems:

So you wanna find the maximum of some function $f$ on some closed interval $[a, b]$?

Well the Extreme Value Theorem says there IS a max somewhere.

Let's look on the open interval $(a, b)$. If there were a max there it would have to be at a critical point.

Oh ok, so let's find the critical points and test them to see the values there.

Did we forget anything? Oh yes - we gotta check ____________________________________.

**Proof of Fermat's Theorem:**

**Theorem** (Rolle's Theorem)

**Proof:**

**Theorem** (The Mean Value Theorem)

Why is Rolle's Theorem a special case of the MVT?

How can we use this to our advantage?

**Proof:**

## 34 Antiderivatives

**Warmup:** Find all functions $f(x)$ such that $f'(x) = 2x$.

How do you know you have found them all?

**Theorem:** (Why we have that $+C$)

Using that really great idea we saw when working with the IVT and MVT proof:

**Corollary:**

**Examples:**

Find $f'(0)$ for the functions below or prove that they do not exist.

- You may use that $\lim_{x\to 0} \sin(1/x)$ does not exist.

| | Continuous? | Differentiable? | Is Derivative Continuous? | Is Derivative Differentiable? |
|---|---|---|---|---|
| $f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ | | | | |
| $f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ | | | | |
| $f(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ | | | | |
| $f(x) = \begin{cases} x^3\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ | | | | |

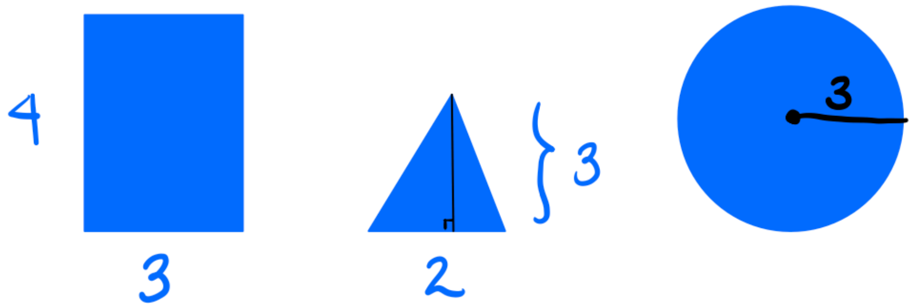
Note if we want to find the second derivative, we need to find a formula for the derivative in an open interval. It's not enough to just find it at one point.

## 35 Fun Day

I talked about primes and the fact that the sum of the reciprocals of the primes converges.

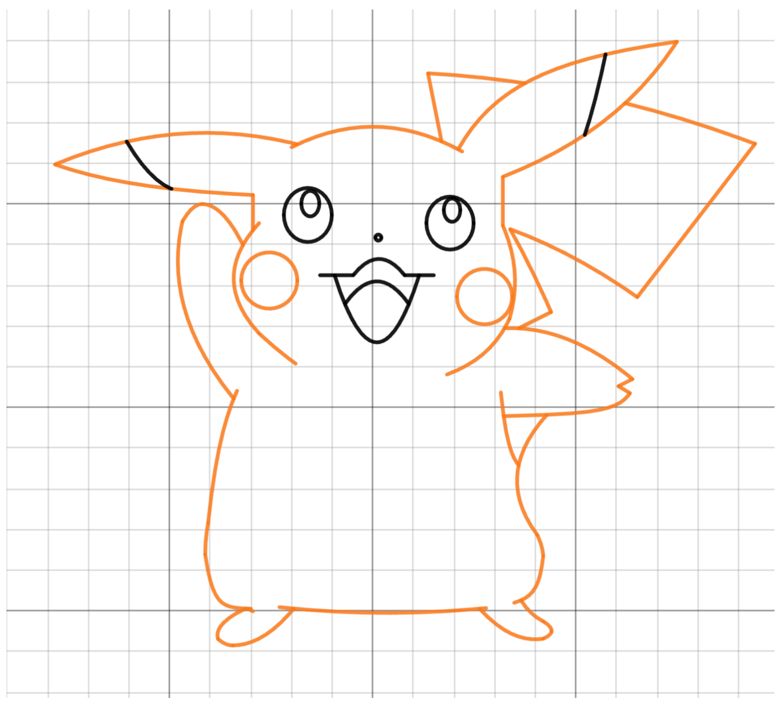
# 36 Introduction to Integration

**Warmup:** What are the areas of the shaded regions below:



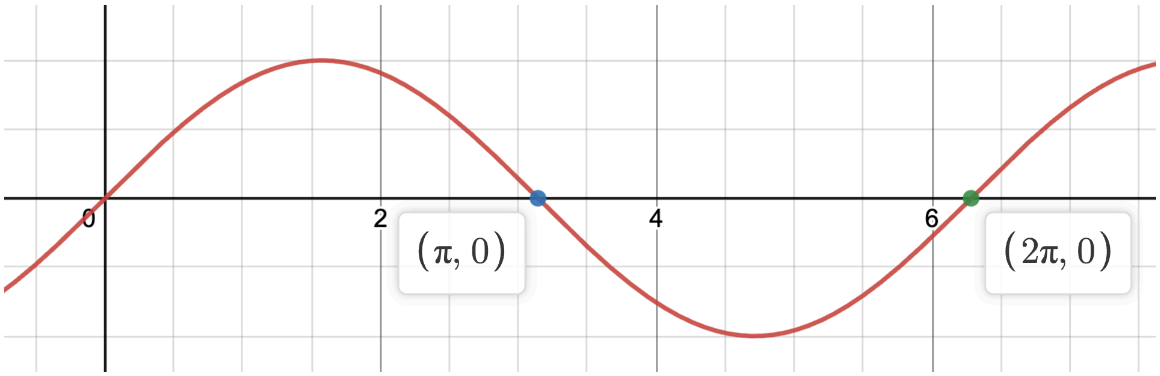

You probably used **formulas** to find these areas. That's good!

How would you find the area of the region below?:



Write down 3 key words that came up in your discussion.

**Example:**




A student calculates the following ________________________________ [two words]:

$$\int_{\pi}^{2\pi} \sin x \; dx = -\cos x\Big|_{\pi}^{2\pi} = (-\cos(2\pi)) - (-\cos(\pi)) = (-1) - (1) = -2.$$

Explain what the student has calculated. Be careful to explain why the answer is negative.

We are going to define what it means to calculate

$$\int_a^b f(x)dx.$$

The answer might be:

- It is not defined / DNE. In this case, we say $f$ is ______________________.
- It exists and is a number. In this case we say that $f$ is ______________________.
  This number measures the ______________________.

The definition is going to ________________________________,

but the basic idea is that we will use

**Info:** There are actually different ways to define the integral. We are going to learn __________,

These are ______________________, meaning that if you use either one, you'll get the same answer,

which remember is either ____________________________________________.

In graduate school, you would learn of even **more** ways to define an integral, and some of these will be "more general", meaning that these more ______________________ integrals can give numbers, when our integrals might just give DNE. If you're curious, let me know!

When we are talking about integrals, we will ALWAYS talk about functions

$$f : \qquad\qquad\qquad \to \mathbb{R}$$

(i.e. on a ____________ interval).

And we will ALWAYS assume that the function is bounded. This means the ______-values are bounded.

**So:** If we have a function that is unbounded on $[a, b]$, then $\int_a^b f(x)dx$ ________________________.

Wait a minute, you silly goose...

Isn't this impossible?

How could a function be unbounded on a closed interval?

**Example:** Here is an example of a function $F : [0, 1] \to \mathbb{R}$ that is unbounded.

$$F(x) = \begin{cases} \\ \\ \end{cases}$$

This function [is / is not] integrable. Why?

Note: You might think this function is silly, but it's still a function and we have to be able to answer questions about it. For instance, is it continuous? **Being able to come up with examples is important!**

Important point: If a function is unbounded, it is [sometimes/always/never] integrable.

**Example:** Here is an example of a function that IS integrable. Even though we haven't defined what the integral is, draw the graph of the following functions and see if you can "guess" what you think $\int_1^4 s(x)dx$ should be.

$$s : [1,4] \to \mathbb{R}, \qquad s(x) = \begin{cases} 3 & \text{if } 1 \le x \le 3 \\ 5 & \text{if } 3 < x \le 4 \end{cases}.$$

**Example:** Here are more examples of functions on $[0,1]$:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}, \qquad g(x) = \begin{cases} 1 & \text{if } x = 1/n \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

We will later see whether or not these functions are integrable. (In general it can be hard to tell).

**Example:** But we will eventually also show some powerful theorems like:

- If a function on $[a,b]$ is continuous, then it is integrable;
- If a function on $[a,b]$ is monotone, then it is integrable;
- If a function on $[a,b]$ is continuous except at finitely many points, then it is integrable

Do these theorems tell us information about $s(x)$? [yes/no] $f(x)$ [yes/no] $g(x)$ [yes/no]

Which of the following exist according to these powerful theorems?

$$\int_1^4 \sin x \; dx, \quad \int_0^3 e^x \; dx, \quad \int_1^5 \frac{1}{x} \; dx, \quad \int_1^4 3x^2 + 8x \; dx, \quad \int_0^\pi \begin{cases} \sin(1/x) & \text{if } x > 0 \\ 32 & \text{else} \end{cases} dx$$

**How will we actually calculate them?**

1. In practice, you will use the ________________________________ which says that to evaluate

   these definite integrals, you can find an ______________________.
2. If you aren't able to find an antiderivative, then you can use lots of ______________________ to get an approximation.
3. Proving the Fundamental Theorem of Calculus is therefore really ________________.

**Is there anything cool we're going to learn?**

Yes! Here's something very cool. Remember how we built up a lot of things by using the axioms of the real numbers. We defined things like

$$e^x, \quad \sin x, \cos x$$

using ____________ sums, that we proved converged using the ____________ test.

Well, using integrals we are going to prove that $\ln x$ exists. We are going to define a function:

$$L(x) := \int_1^x \frac{1}{t}\, dt$$

On your homework you will workout some properties of this function by using $u$-substitution.

What does the fundamental theorem say anyways?

- There are ____________ different parts.

**Theorem:** (The Fundamental Theorem of Calculus)

1) Suppose

$$\int_a^b g'(x)dx =$$

2) Let $f : [a, b] \to \mathbb{R}$ be an ________________________ function. Then is we define the function

$$A(x) =$$

then $A(x)$ is ________________________.

3) If in addition, $f : [a, b] \to \mathbb{R}$ is ________________________

then $A(x)$ is ________________________ and $A'(x) =$ ____________.

The letter $A$ stands for either ____________ or ________________________.

**Example:** What is the derivative of the function $L(x)$ defined above? Why? (Be precise!)

**Example:** Suppose that John B. is running between 1pm and 3pm and his velocity at time $t$ in miles/hour is given by

$$s : [1, 4] \to \mathbb{R}, \qquad s(x) = \begin{cases} 3 & \text{if } 1 \le x \le 3 \\ 5 & \text{if } 3 < x \le 4 \end{cases}.$$

Notice that at 3pm he picks up the pace because Rafe is chasing him.

Draw a graph of $A(x) = \int_a^x s(x)dx$.

Where is this graph differentiable?

Is it continuous?

What would be the units on this graph?

# 37 Let's Get to Work!

The way we will define integrals is by using **rectangles** and finding their (signed) areas.

The area of a rectangle is given by ____________ × ____________.

We are working on an interval $[a, b]$ and we will split this domain up into little pieces.

Doing this is called ____________________________________.

Here is a partition of the interval $[2, 9]$:

**Definition:** A partition of the interval $[a, b]$ is a finite set:

$$P = \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$$

such that

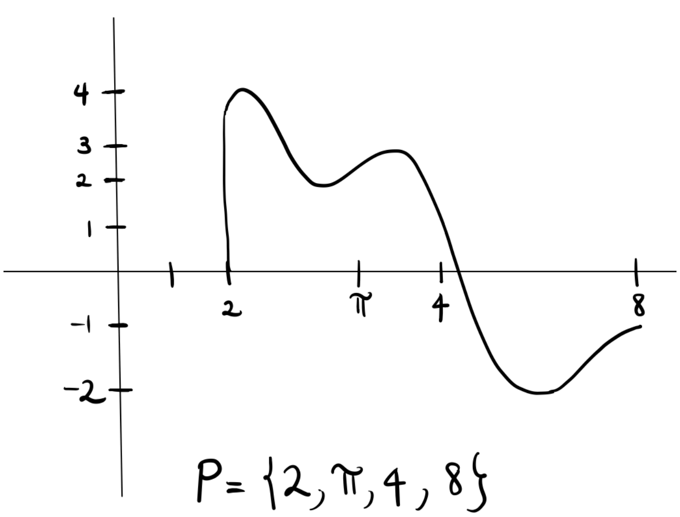
The partition will give us the __________________ of our rectangles.

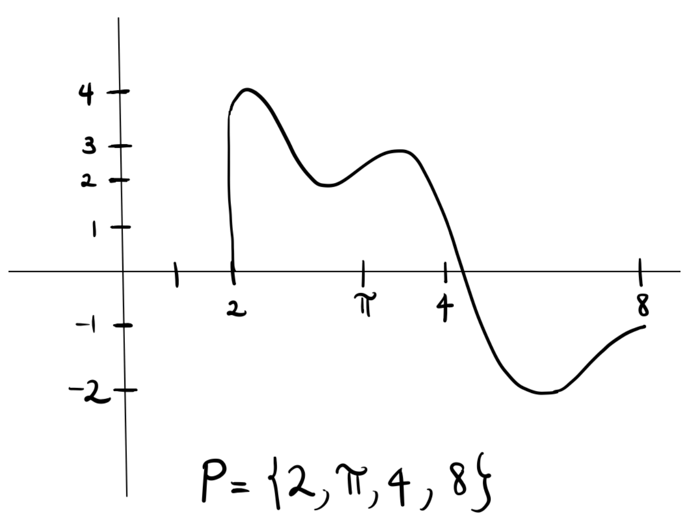
To get our heights:

**Notation:** Given a partition $P$ of $[a, b]$, there are ____________ subintervals, starting with $[x_0, x_1]$. If $f$ is a bounded function on $[a, b]$ then we will write:

- $m_i = \inf\{f(x) \ : \ x \in [x_{i-1}, x_i]\}$ (The height of the lower rectangle)
- $M_i = \sup\{f(x) \ : \ x \in [x_{i-1}, x_i]\}$ (The height of the upper rectangle)

**Example:** Here are two pictures of a function $f(x)$ on $[2, 8]$. We will label the $M_i$ and $m_i$ and draw in some rectangles.



**Definition:** Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function and that $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$. Then we will define:

The upper sum

$$U(f, P) =$$

The lower sum

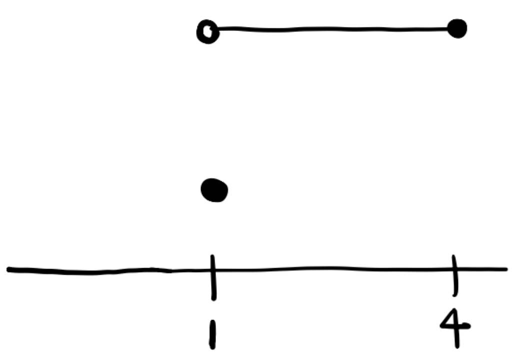
$$L(f, p) =$$

These represent the ________________________ of the upper and lower ________________________.

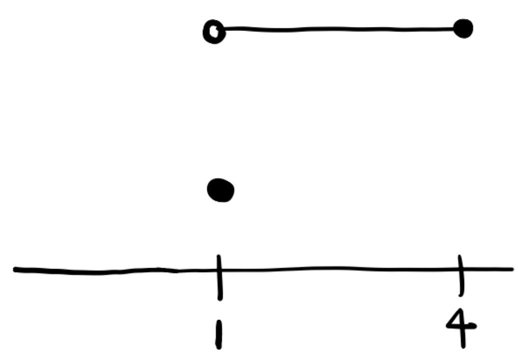
The actual value of the integral (if it exists!) will be between these two values. Desmos Link: `https://www.desmos.com/calculator/2bqcde6rzm`

Desmos Link: `https://www.desmos.com/calculator/2bqcde6rzm`

**Example:** Here is a graph of the function $f : [1, 4] \to \mathbb{R}$ given by $f(x) = 3$ if $x > 1$ and $f(1) = 1$. We will later see that this function IS integrable. What do you think the value of $\int_1^4 f(x)dx$ is? ____________.




$P = \{1, 1.1, 4\}$




$P' = \{1, 2, 3, 4\}$

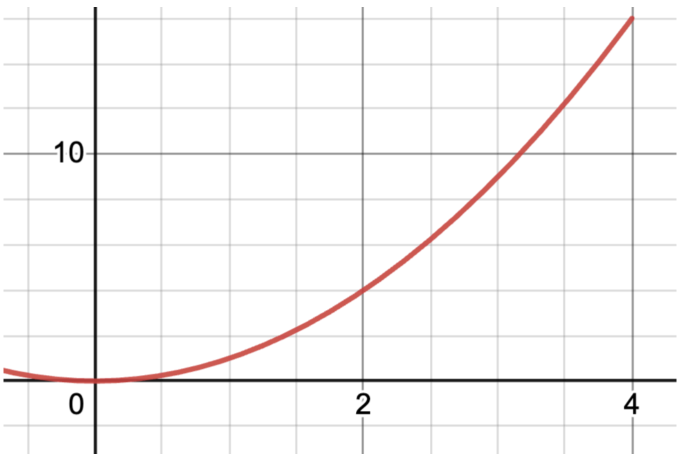
$L(f, P) =$ $\qquad$ $U(f, P) =$

and

$L(f, P') =$ $\qquad$ $U(f, P') =$

**If we have more points to our partition do these estimates get better?**

- Answer:
- In general just having more points doesn't mean it's a better approximation!




Here is a graph of $f(x) = x^2$. Suppose we want to calculate some upper and lower approximations over $[1, 4]$.

Using $P = \{1, 2, 3, 4\}$, draw in the upper rectangles.

Now add in some extra points, to get

the **refinement** $Q = \{1, 1.5, 2, 3, \pi, 4\}$.

Is your upper sum a better or worse approximation? (We'll use Desmos)

**Definition:** Suppose that $P$ is a partition of $[a, b]$. Then we say that a partiaion $Q$ is a refinement of $P$ if:

We have seen that if we ____________ a partition, then the approximations get "better". What does that mean?

For upper approximations it should mean:

For lower approximations it should mean:

**Theorem: (Refining a Partition will improve our bounds)**
Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function and $P$ is a partition of $[a, b]$.

If $Q$ is a refinement of $P$ then

**Proof: We will not have time to prove this** (sad day)

**Theorem** Suppose that $P$ is a partition of $[a, b]$ and $f : [a, b] \to \mathbb{R}$ is a bounded function. Then:

**We have seen** that if we take a partition and we refine it, then our approximations get better. But what if we take two random partitions $P_1, P_2$? What can we say about the four numbers below?

$$U(f, P_1), \quad U(f, P_2), \quad L(f, P_1), \quad U(f, P_2)$$

Intuitions: If there is an ________________________ then it would be ________________________

the ________________________ and ________________________ sums.

**Theorem:** Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

a.)

b.)

**Proof:**

**How can we get better approximations?** (Brainstorm)

**Definition:** ($U(f)$ and $L(f)$)

Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function.

**Definition**

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then we say that $f$ is ________________________ if

# 38 Some Examples of Integrals

**Example:** Let $f : [a,b] \to \mathbb{R}$ be defined by $f(x) = C$ for some constant $C$. Then $f$ is integrable on $[a,b]$. And

$$\int_a^b C \; dx =$$

**Example:** Show that the following function $f : [0,1] \to \mathbb{R}$ is NOT integrable.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

**Theorem:**

Suppose that $f : [a, b] \to \mathbb{R}$ is a ____________________ function.

Then __________________________________.

**Example:**

For example, this shows that the function $x$ and $x^2$ are both integrable on, say $[0,4]$.

**Question:**

How do you think we could use try and prove, say that the function $f(x) = |x|$ is integrable on $[-2,3]$.

**Other tools we could develop:**

- If $f$ is integrable on $[a,b]$ and on $[c,d]$ then $f$ is integrable on and
- If $f$ and $g$ are both integrable on $[a,b]$ then
- If $f$ is integrable on $[a,b]$ and $k \in \mathbb{R}$. then
- If $f:[a,b] \to \mathbb{R}$ is integrable and if $A \le f(x) \le B$ for all $x \in [a,b]$ then

# 39 What is sufficient to check that something is integrable?

Remember that we defined a function $f$ to be integrable if

But this means that we have to calculate something about ALL partitions of $[a, b]$ every single one! That's ridiculous, and probably impossible to check in practice. Here's a condition that we can actually check:

**Theorem:** Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f$ is integrable $\iff$ for all $\epsilon > 0$ there is a partition $P$ such that

**Corollary:** Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose there is a sequence of partitions $P_n$ that

$$\lim_{n \to \infty} (U(f, P_n) - L(f, P_n)) = 0.$$

Then $f$ is integrable on $[a, b]$.

In other words, if the upper sums and the lower sums for your partitions $P_n$

are converging to the ________________________ then your function is integrable.

**Proof of Theorem:**

**Example:** Use this function to show that the function below is integrable:

$$f : [0,4] \to \mathbb{R}, \qquad \begin{cases} 1 & \text{if } x \in [0,1] \\ 3 & \text{if } x \in (1,4] \end{cases}.$$

# 40 Continuous Functions are Integrable!

**Theorem:**

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then ______________________________.

**Proof:**

Let $\epsilon > 0$. We will find a ____________________ such that

$$< \epsilon.$$

Because $f$ is continuous on $[a, b]$ we know that $f$ is ____________________. This means that there exists a $\delta$ that ____________________ all points. That is:

IF __________ $< \delta$ THEN ____________________ $< \epsilon$.

So let's choose a ____________________ so that all the subintervals have length less than __________.

For simplicity let's assume all the intervals have the same width $\Delta$. So if we have $N$ rectangles, then

$$\Delta =$$

Let's pause to draw a picture:

On each subinterval, the difference between the sup and the inf will be at most _______________.

(By the way, since we have a continuous function, we can replace sup and inf with ____________________.)

So over each interval, the

But then

$$U(f, P) - L(f, P) \leq$$

**Note:** Check out the proof in the reading for more details!

Why do we care?

The previous theorem tells us that LOTS and LOTS of functions are integrable, so we can study integrals.

Even though finding derivatives is EASIER than finding antiderivatives, being integrable is an EASIER condition to satisfy than being differentiable.

This is because for example $|x|$ is integrable on any closed interval (why? because it is ____________) but it will not be differentiable at 0.

**Example:** Let

$$g(x) = \begin{cases} 1 & \text{if } x = 1/n \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Prove that $g(x)$ [is / is not] integrable on $[0, 1]$.

# 41 Proof of the Fundamental Theorem of Calculus

**Theorem:** (The Fundamental Theorem of Calculus)

1) Suppose that $g$ is differentiable on $(a, b)$ and that $g'(x)$ is integrable on $[a, b]$, then

$$\int_a^b g'(x)dx = g(b) - g(a).$$

2) Let $f : [a, b] \to \mathbb{R}$ be an integrable. Then is we define the function

$$A(x) = \int_a^x f(t)dt$$

then $A(x)$ is continuous.

3) If in addition, $f : [a, b] \to \mathbb{R}$ is continuous at a point $c$ then $A(x)$ is differentiable at $c$ and

$$A'(c) = f(c).$$

**Proof:**