MATH1061/1021 Cheatsheet (Calculus)

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Source

1 Sets, numbers, functions

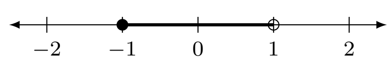
1.1 Union, intersection, difference, complement, subset

Operation	**Notation**	**Description**
Union	$A \cup B$	All elements that are in A or B or both
Intersection	$A \cap B$	All elements that are both in A and B
Difference	$A \setminus B$	All elements that are in A but not in B
Complement	$\overline{A}$	All elements in the universal set that are not in A
Subset	$A \subseteq B$	A is a subset of B; every element in A is also in B
Proper subset	$A \subset B$	A is a subset of B and $A \neq B$

1.2 Common number sets

Name (Symbol)	**Set**
Natural numbers ($\mathbb{N}$)	$\{0, 1, 2, 3, 4, ...\}$
Integers ($\mathbb{Z}$)	$\{... -2, -1, 0, 1, 2, ...\}$
Rational numbers ($\mathbb{Q}$)	$\left\{\frac{1}{2}, -\frac{4}{3}, \frac{17}{12}, \dots\right\}$
Irrational numbers ($\mathbb{R} \setminus \mathbb{Q}$)	$\{\sqrt{2}, \pi, e, \sqrt{7}, ...\}$
Real numbers ($\mathbb{R}$)	All numbers

1.3 Interval notation

Interval notation
$D = [-1, 1)$

Note: Infinity ∞ is always open-bracketed, i.e. (∞, ∞)

1.4 Modulus

Modulus
Distance on the number line
$\lvert x - y \rvert$

1.5 Injective, surjective, bijective functions

Function type	**Definition**
Injective	A function $f : X \to Y$ where for all $x \in X$, x maps to a different $y \in Y$, and $\lvert X\rvert \leq \lvert Y\rvert$.
Surjective	A function $f : X \to Y$ where for all $y \in Y$, y is an image of some $x \in X$, and $\lvert X\rvert \geq \lvert Y\rvert$.
Bijective	A function that is both injective and surjective where $\lvert X\rvert = \lvert Y\rvert$.

1.6 Composite, inverse, hyperbolic functions

Composite functions
A function in the form $f \circ g(x) = f(g(x))$ where some function f takes another function g as input.

Inverse functions	
Definition	**How to invert a function**
There exists an inverse function f^{-1} if and only if f is injective. If $f(x) = y$, then $f^{-1}(y) = x$, which implies that if $f : A \to B$, then $f^{-1} : B \to A$.	1) Rewrite $f(x)$ as y 2) Swap x and y 3) Make y the subject 4) Rewrite y as $f^{-1}(x)$

Hyperbolic functions		
Function	**Derivative**	**Inverse**
$\cosh(x) = \dfrac{e^x + e^{-x}}{2}$	$\dfrac{\mathrm{d}}{\mathrm{dx}}\cosh(x) = \sinh(x)$	$\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$
$\sinh(x) = \dfrac{e^x - e^{-x}}{2}$		$\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$

2 Limits

2.1 Definition of a limit

Limit
$\lim_{x \to a} f(x) = L$ as x approaches a, but not reaching a, the output gets closer and closer to L.

2.2 Limit laws

if we have $\lim_{x \to c} f(x) = L, \lim_{x \to c} g(x) = M$	
No	**Law**
1	$\lim_{x \to c}[f(x) + g(x)] = L + M$
2	$\lim_{x \to c}[f(x) - g(x)] = L - M$
3	$\lim_{x \to c}[f(x)g(x)] = LM$
4	$\lim_{x \to c}[\dfrac{f(x)}{g(x)}] = \dfrac{L}{M}$
5	$\lim_{x \to c}[f(x)]^n = L^n$
6	$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$

2.3 One-sided limits, existence of limit

One-sided limits
Limit from below (left) is denoted as $\lim_{x \to c^-} f(x)$ Limit from above (right) is denoted as $\lim_{x \to c^+} f(x)$

Existence of a limit
A limit exists only if
$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$
in which case
$\lim_{x\to c} f(x) = L$

2.4 Infinite limits

Infinite limits
If we take x larger and larger, and $f(x)$ gets closer to L, we say
$\lim_{x\to\infty} f(x) = L$
You can algebraically solve infinite limits by dividing by the highest power of x in the denominator or to by using conjugates.

2.5 Squeeze law

Squeeze law
Suppose $g(x) \le f(x) \le h(x)$ for x near a
if $\lim_{x\to a} h(x) = L$ and $\lim_{x\to a} g(x) = L$, then $\lim_{x\to a} f(x) = L$
we say "$f(x)$ is squeezed between $g(x)$ and $h(x)$"

2.6 Continuous functions

Continuous functions
A function is continuous at a point if the limit exists at the point, and is equal to the value at that point.
ie. $f(x)$ is continuous at $x = c$ if and only if:
1) $f(c)$ is defined
2) $\lim_{x\to c} f(x)$ exists and is finite
3) $\lim_{x\to c} f(x) = f(c)$

3 Differential Calculus

3.1 Limit definition of derivatives

Limit definition of derivatives
The derivative of a function f at a point a is given by
$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

3.2 Common derivatives

Common derivatives	
$\mathbf{f(x)}$	$\mathbf{f'(x)}$
x^n where $n \neq 0$	nx^{n-1}
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
$\tan(x)$	$\sec^2(x)$
c^x	$c^x \ln(c)$
$\ln y$	$\frac{1}{y}\frac{dy}{dx}$

3.3 Differential rules

Differential rules		
No	**Name**	**What to do**
1	Power rule	$\frac{d}{dx}x^n = nx^{n-1}$
2	Constant rule	$\frac{d}{dx}c = 0$, where $c \in \mathbb{R}$
3	Sum rule	$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
4	Difference rule	$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$
5	Product rule	$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$
6	Quotient rule	$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
7	Chain rule	We are to find $\frac{d}{dx}[f \circ g(x)]$ **Method 1:** Let $u = g(x)$, $y = f(u)$, and so $y'(x)$ is given by $\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$ **Method 2:** If you can see f and g clearly, then $\frac{d}{dx}[f \circ g(x)] = f'(g(x))g'(x)$

3.4 Implicit differentiation

Implicit differentiation
Functions that can't (or aren't easy) to be written as $y = f(x)$ can instead be written implicitly in the form $F(x, y) = 0$. In which case, to get the derivative, find $\frac{d}{dx}[F(x, y)]$ applying chain rule for terms involving y, where $\frac{d}{dx}y \Longrightarrow \frac{dy}{dx}$ and finally, make $\frac{dy}{dx}$ the subject.

3.5 Useful algebra for logarithmic differentiation

No	Useful algebra
1	$a^x \Leftrightarrow e^{x \ln a}$
2	$\ln(e^{x \ln(a)}) = x \ln a$
3	$y \ln(a) = \ln(x) \Leftrightarrow y = \dfrac{\ln(x)}{\ln(a)}$
4	$\ln(\sqrt{x}) \Leftrightarrow \dfrac{1}{2} \ln(x)$
5	$\log_b(a) \Leftrightarrow \dfrac{\ln(a)}{\ln(b)}$

3.6 L'Hopital's Rule

L'Hopital's Rule
If $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)}$ is an indeterminate form, i.e. can be written as $\dfrac{0}{0}$ or $\pm\dfrac{\infty}{\infty}$, then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)}$ where c can be infinity. NOTE: You can keep applying L'Hopital's Rule until you get a definite form.

3.7 Extrema

Extrema An "extremum" is a maximum or a minimum.	
Extremum	**Definition**
Local max	A local max at $x = c$ means $f(c) \geq f(x)$ for any x near c
Local min	A local min at $x = c$ means $f(c) \leq f(x)$ for any x near c
Global max	A global max on some domain A where $f : A \to \mathbb{R}$ means $f(c) \geq f(x)$ for all $x \in A$
Global min	A global min on some domain A where $f : A \to \mathbb{R}$ means $f(c) \leq f(x)$ for all $x \in A$

3.8 First-derivative test

First-derivative test If f is continuous and $f'(c) = 0$		
Case	**Observation**	**Conclusion**
1	• $f'(x)$ is positive for $x < c$ • $f'(x)$ is negative for $x > c$	Local max
2	• $f'(x)$ is negative for $x < c$ • $f'(x)$ is positive for $x > c$	Local min
3	$f'(x)$ doesn't change sign	Not a local extremum (neither max or min)

3.9 Critical points

No	**Critical point type**
1	$f'(c)$ does NOT exist
2	$f'(c) = 0$

NOTES:
- If $f(x)$ is differentiable, then any local extrema must occur at points $x = c$ where $f'(c) = 0$
- $f'(c) = 0$ does NOT guarantee a local extremum! You must apply first-derivative test!

3.10 Finding max & min on a closed interval

Extreme value theorem
If f is a continuous function from a closed interval A to $\mathbb{R}$, then f attains a global max & global min value in A.

Finding max & min on a closed interval
To find max & min values on a closed interval, simply check: 1) Critical points 2) End points of the interval

3.11 Concavity

Concavity
• Concave up: $f''(x) > 0$ • Concave down: $f''(x) < 0$ • Point of inflection: if $f''(x)$ changes sign at c then $x = c$ is a point of inflection.

3.12 Second-derivative test

Second-derivative test

If f, f' are differentiable and $f'(c) = 0$

Case	**Observation**	**Conclusion**
1	$f''(c) > 0$	Local minimum
2	$f''(c) < 0$	Local maximum
3	$f''(c) = 0$	No conclusion (might be local min, max, or neither.) What to do from here: Do first-derivative test

3.13 Sensible order to curve-sketching

Sensible order to curve-sketching	
Step	**To work out**
1	What is the domain?
2	What is the y-intercept?
3	What happens as $x \to \pm\infty$ (Limits)
4	Is there anywhere that $f(x) \to \pm\infty$ (is denominator zero?)
5	Where are the critical points? • $f'(x) = 0$ • $f'(x)$ does NOT exist
6	Where is $f(x)$ increasing/decreasing? • $f'(x) > 0$: increasing • $f'(x) < 0$: decreasing
7	Where is $f(x)$ concave up or down? • $f''(x) > 0$: concave up (slope decreasing) • $f''(x) < 0$: concave down (slope increasing)