MATH1061/1002 Cheatsheet (Linear Algebra)

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1 Complex numbers

1.1 Imaginary unit

Imaginary unit
$i = \sqrt{-1}$

1.2 Cartesian, standard polar, exponential polar forms

Name	Form
Cartesian	$z = a + bi$
Standard Polar	$r(\cos\theta + i\sin\theta)$
Exponential Polar	$re^{i\theta}$

1.3 Complex conjugate

Complex conjugate
Complex conjugate of $z = a + bi$ is $\overline{z} = a - bi$

Complex conjugate properties	
Property	**Definition**
Conjugate of a Sum	$\overline{z+w} = \overline{z} + \overline{w}$
Conjugate of a Product	$\overline{zw} = \overline{z} \times \overline{w}$
Conjugate of a Power	$\overline{z^n} = (\overline{z})^n$

1.4 Modulus of a complex number

Modulus
The modulus of a complex number $z = a + bi$ is $\lvert z\rvert = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}}$

1.5 Principal argument

Principal argument
The principal argument of $z \in \mathbb{C}$, denoted Arg z, satisfies $-\pi <$ Arg $z \leq \pi$ So, to get principal argument, add or subtract multiples of 2π to/from θ.

Modulus properties	
Property	**Definition**
Multiplicative Property of Moduli	$\lvert zw\rvert = \lvert z\rvert\lvert w\rvert$
Division Property of Moduli	$\lvert\frac{z}{w}\rvert = \frac{\lvert z\rvert}{\lvert w\rvert}$
Triangle Inequality	$\lvert z + w\rvert \leq \lvert z\rvert + \lvert w\rvert$
Reverse Triangle Inequality	$\lvert z - w\rvert \geq \lvert z\rvert - \lvert w\rvert$

1.6 Arithmetic in cartesian form

Operation	**let** $z = a + bi$ **and** $w = c + di$
Addition	$z + w = (a + c) + (b + d)i$
Subtraction	$z - w = (a - c) + (b - d)i$
Multiplication	$z \times w = (a + bi)(c + di)$ $z \times w = ac + adi + bci + bdi^2$ $z \times w = (ac - bd) + (ad + bc)i$
Division	$z \div w = \frac{z}{w} \times \frac{\overline{w}}{\overline{w}}$ $z \div w = \frac{a + bi}{c + di} \times \frac{c - di}{c - di}$

1.7 Arithmetic in polar forms

let $z = re^{i\theta}$ **and** $w = se^{i\phi}$		
Operation	**Exponential polar**	**Standard polar**
Multiplication	$zw = rse^{i(\theta+\phi)}$	$zw = rs(\cos(\theta + \phi) + i\sin(\theta + \phi))$
Division	$\frac{z}{w} = (\frac{r}{s})e^{i(\theta-\phi)}$	$\frac{z}{w} = \frac{r}{s}(\cos(\theta - \phi) + i\sin(\theta - \phi))$
Power	$z^n = r^n e^{in\theta}$ where $n \in \mathbb{Z}$	$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$

1.8 Equality of complex numbers

Equality in Cartesian form
$(a + bi) = (c + di)$ if and only if $a = c$, $b = d$

Equality in polar forms
$re^{i\theta} = se^{i\phi}$ if and only if $r = s$ and $\theta = \phi + 2k\pi$ for $k \in \mathbb{Z}$

1.9 Roots of complex numbers

When asked to find the roots of $z^n = \alpha$
$z = r^{\frac{1}{n}} e^{i(\frac{\theta+2k\pi}{n})}$ satisfy $z^n = \alpha$ for each $k = 0, 1, 2, ..., n - 1$.

1.10 Complex exponential function

Complex exponential function
For $z = a + bi$, we define $e^z = e^a e^{bi}$, which means $e^z = e^a(\cos b + i\sin b)$ from which we know $\|e^z\| = e^a$ and $arg(e^z) = b$.

2 Vectors

2.1 Vector algebra

Let $\mathbf{u} = [u_1, ..., u_n], \mathbf{v} = [v_1, ..., v_n] \in \mathbb{R}^n, c \in \mathbb{R}$			
Operation	**What to do**	**Operation**	**What to do**
Addition	$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$	Scalar multiplication	$c\mathbf{u} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}$
Subtraction	$\mathbf{u}-\mathbf{v} = \begin{bmatrix} u_1 - v_1 \\ \vdots \\ u_n - v_n \end{bmatrix}$	Negation	$-\mathbf{u} = \begin{bmatrix} -u_1 \\ \vdots \\ -u_n \end{bmatrix}$

2.2 Magnitude of a vector

Magnitude of a vector
The magnitude (length) of a vector $\mathbf{u}$ is given by $\lvert\lvert\mathbf{u}\rvert\rvert = \sqrt{u_1^2 + u_2^2 + ... + u_n^2}$ Also note that $\lvert\lvert c\mathbf{u}\rvert\rvert = \lvert c\rvert\lvert\lvert\mathbf{u}\rvert\rvert$

2.3 Useful theorems for vector algebra

Let $\mathbf{u} = [u_1, ..., u_n], \mathbf{v} = [v_1, ..., v_n] \in \mathbb{R}^n, c, d \in \mathbb{R}$ and $\mathbf{o}$ be the zero vector	
No	**Theorem**
1	$\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$
2	$(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$
3	$\mathbf{u}+\mathbf{o} = \mathbf{u}$
4	$\mathbf{u}+(-\mathbf{u}) = \mathbf{o}$
5	$c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$
6	$\mathbf{u}(c+d) = c\mathbf{u}+d\mathbf{u}$
7	$c(d\mathbf{u}) = (cd)\mathbf{u}$
8	$1\mathbf{u} = \mathbf{u}$

2.4 Vector space

Vector Space

A *vector space* is a set V equipped with:

- **Addition:** For any $\mathbf{u}, \mathbf{v} \in V$, there exists $\mathbf{u}+\mathbf{v} \in V$.
- **Scalar multiplication:** For any $\mathbf{u} \in V$ and any scalar $c \in \mathbb{R}$, there exists $c\mathbf{u} \in V$.

These operations must satisfy properties (1)-(8) defined in Section 2.3.
Examples: $\mathbb{R}^n$ and $\mathbb{C}$ are both examples of vector spaces.

2.5 Unit vector

Unit vector
A "unit vector" is a vector of length 1. If $\mathbf{u} \in \mathbb{R}^n, \lvert\lvert \mathbf{u} \rvert\rvert \neq 0$ then $\frac{1}{\lvert\lvert \mathbf{u} \rvert\rvert}\mathbf{u}$ is a unit vector.

2.6 Linear combination

Linear combination
A "linear combination" of $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k} \in \mathbb{R}^n$ is a vector of the form $C_1\mathbf{v_1} + C_2\mathbf{v_2} + ... + C_k\mathbf{v_k}$. **Example:** $[2, 2]$ is a linear combination of $[-1, 2], [0, 6]$ because $[2, 2] = (-2)[-1, 2] + (1)[0.6]$

2.7 Parallel vectors

Parallel vectors
Vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u} = c\mathbf{v}$ or $\mathbf{v} = c\mathbf{u}$ for $c \in \mathbb{R}$.

2.8 Standard basis vectors

Standard basis vectors
$\mathbf{e_1} = [1, 0, 0], \mathbf{e_2} = [0, 1, 0], \mathbf{e_3} = [0, 0, 1]$ are the standard basis vectors in $\mathbf{R}^3$. Similarly, $\mathbf{e_1} = [1, 0, ..., 0], \mathbf{e_2} = [0, 1, 0, ..., 0], ..., \mathbf{e_n} = [0, 0, ..., 0, 1] \in \mathbb{R}^n$ are the standard basis vectors in $\mathbb{R}^n$

2.9 Dot product

Dot product
For $\mathbf{u} = [u_1, ..., u_n], \mathbf{v} = [v_1, ..., v_n] \in \mathbb{R}^n$, we define the dot product $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + ... + u_nv_n \in \mathbb{R}$

2.10 Theorems for dot products

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n, c \in \mathbb{R}$	
No	**Theorem**
1	$\mathbf{u} \cdot \mathbf{v}$
2	$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
3	$(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
4	$\mathbf{u} \cdot \mathbf{u} \geq 0$
5	$\lvert\lvert \mathbf{u} \rvert\rvert = \sqrt{\mathbf{u} \cdot \mathbf{u}}$
6	$\lvert \mathbf{u} \cdot \mathbf{v} \rvert \leq \lvert\lvert \mathbf{u} \rvert\rvert \lvert\lvert \mathbf{v} \rvert\rvert$ (Cauchy-Schwarz Inequality)
7	$\lvert\lvert \mathbf{u} + \mathbf{v} \rvert\rvert \leq \lvert\lvert \mathbf{u} \rvert\rvert + \lvert\lvert \mathbf{v} \rvert\rvert$ (Triangle Inequality)

2.11 Angle between vectors using dot product

Angle between vectors using dot product
We define angle between vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ to be a unique value $\theta \in [0, \pi]$ such that: $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\lvert\lvert\mathbf{u}\rvert\rvert\lvert\lvert\mathbf{v}\rvert\rvert}$ which means we can also find $\mathbf{u} \cdot \mathbf{v}$ if we know the the magnitude of u and v, and the angle in between: $\mathbf{u} \cdot \mathbf{v} = \lvert\lvert\mathbf{u}\rvert\rvert\lvert\lvert\mathbf{v}\rvert\rvert \cos\theta$

2.12 Orthogonal vectors

Orthogonal vectors
Vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are orthogonal iff $\mathbf{u} \cdot \mathbf{v} = 0$ Note: Orthogonal vectors are perpendicular $\mathbf{u} \cdot \mathbf{v} = 0 \Longrightarrow \cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\lvert\lvert\mathbf{u}\rvert\rvert\lvert\lvert\mathbf{v}\rvert\rvert} = 0$ $\theta = \cos^{-1}(0) = \frac{\pi}{2}$, i.e. perpendicular.

2.13 Projections

Projections
The projection of $\mathbf{u}$ onto $\mathbf{v}$ is defined by $\mathrm{proj}_{\mathbf{v}}(\mathbf{u}) = (\dfrac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}})\mathbf{v}$

2.14 Cross products

Cross products
Cross product of $\mathbf{u} = [u_1, u_2, u_3], \mathbf{v} = [v_1, v_2, v_3] \in \mathbb{R}^3$ is the vector $\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$

2.15 Properties for cross products

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3, c \in \mathbb{R}$	
No	**Property**
1	$\mathbf{u} \times \mathbf{v} \neq \mathbf{v} \times \mathbf{u}$ (not commutative)
2	$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ (not associative)
3	$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ (anti-commutative)
4	$\mathbf{u} \times \mathbf{u} = 0$
5	$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0 = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$ i.e. given 2 vectors, their cross product is orthogonal to both vectors.
6	$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
7	$c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$

2.16 Area inscribed by vectors in $\mathbb{R}^3$

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$	
Shape	**Area**
Parallelogram	The area of the parallelogram inscribed by $\mathbf{u}, \mathbf{v}$ is $\|\|\mathbf{u} \times \mathbf{v}\|\|$
Triangle	The area of the triangle inscribed by $\mathbf{u}, \mathbf{v}$ is $\frac{1}{2}\|\|\mathbf{u} \times \mathbf{v}\|\|$

2.17 Angle between vectors using cross product

Angle between vectors using cross product
Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and θ the angle between them. Then, $\|\|\mathbf{u} \times \mathbf{v}\|\| = \|\|\mathbf{u}\|\|\|\|\mathbf{v}\|\| \sin\theta$ which means $\theta = \sin^{-1} \frac{\|\|\mathbf{u} \times \mathbf{v}\|\|}{\|\|\mathbf{u}\|\|\|\|\mathbf{v}\|\|}$

2.18 Forms of lines in $\mathbb{R}^2$

Let
$\mathbf{p}$: position vector pointing to the line
$\mathbf{n}$: normal vector (vector orthogonal to the line)
$\mathbf{d}$: direction vector (vector parallel to the line)

Name	**Form**	**How to obtain**	**Survives $\mathbb{R}^n$**
Normal form	$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$	1) Find $\mathbf{p}, \mathbf{n} \in \mathbb{R}^2$ 2) Write $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$	No
General form	$ax + by = c$	1) Find the normal form 2) Let $\mathbf{x} = [x, y], \mathbf{n} = [a, b]$, and write $c = \mathbf{n} \cdot \mathbf{p}$ 3) Simplify as follows $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ $\implies [a, b] \cdot ([x, y] - \mathbf{p}) = 0$ $\implies [a, b] \cdot [x, y] - c = 0$ $\implies ax + by = c$	No
Vector form	$\mathbf{x} = \mathbf{p} + t\mathbf{d}, t \in \mathbb{R}$	1) Find $\mathbf{p}, \mathbf{d} \in \mathbb{R}^2$ 2) Write $\mathbf{x} = \mathbf{p} + t\mathbf{d}, t \in \mathbb{R}$	Yes
Parametric equations	$x = p_1 + td_1$ $y = p_2 + td_2$ for $t \in \mathbb{R}$	1) Find the vector form 2) Write equations using each corresponding components of $\mathbf{x}, \mathbf{p}$ and $\mathbf{d}$	Yes

2.19 Skewness of lines

Skewness of lines
Two lines are "skew" if they are NOT parallel and do NOT intersect. (applicable only to $\mathbb{R}^3$ and above since all non-parallel lines in $\mathbb{R}^2$ has to intersect)

2.20 Forms of planes in $\mathbb{R}^3$

Let $\mathbf{p}$: position vector pointing to the line $\mathbf{n}$: normal vector (vector orthogonal to the line) $\mathbf{u}, \mathbf{v}$: direction vector (vector parallel to the line)			
Name	**Form**	**How to obtain**	**Survives $\mathbb{R}^n$**
Normal form	$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$	1) Find $\mathbf{p}, \mathbf{n} \in \mathbb{R}^3$ 2) Write $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$	No
General form	$ax + by + cz = d$	1) Find the normal form 2) Let $\mathbf{x} = [x, y, z], \mathbf{n} = [a, b, c]$, and write $d = \mathbf{n} \cdot \mathbf{p}$ 3) Simplify as follows $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p} = 0)$ $\implies [a, b, c] \cdot ([x, y, z] - \mathbf{p}) = 0$ $\implies ax + by + cz = d$	No
Vector form	$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$ for $s, t \in \mathbb{R}$	1) Find $\mathbf{p}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ 2) Write $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$, for $s, t \in \mathbb{R}$	Yes
Parametric equations	$x = p_1 + su_1 + tv_1$ $y = p_2 + su_2 + tv_2$ $z = p_3 + su_3 + tv_3$ for $s, t \in \mathbb{R}$	1) Find the vector form 2) Write equations using each corresponding components of $\mathbf{x}, \mathbf{p}, \mathbf{u}$ and $\mathbf{v}$	Yes

3 Matrices

3.1 Homogeneous system of equations

Homogeneous system of equations
Recall that a system of m linear equations in n variables $x_1, x_2, ..., x_n$ is $$\sum_{j=1}^{n} a_{ij}x_j = b_i \text{ for } 1 \leq i \leq m$$ The system is homogeneous if for all i, $b_i = 0$.

3.2 Augmented matrix of a system

Augmented matrix of a system
The augmented matrix of the system $$\sum_{j=1}^{n} a_{ij}x_j = b_i \text{ for } 1 \leq i \leq m \text{ is}$$ $$\left[\begin{array}{cccc\|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

3.3 Elementary row operations (EROs)

Elementary row operations (EROs)	
No	**Operation**
1	Swapping two rows ($R_i \leftrightarrow R_j$)
2	Multiplying (each entry of) a row by a nonzero constant ($R_i \to cR_i$)
3	Adding a multiple of one row to another ($R_i \to R_i + cR_j$)

3.4 Gaussian elimination (Mat $\to$ REF)

Gaussian Elimination

The process of applying EROs (Elementary Row Operations) to transform a matrix into Row Echelon Form (REF)

$$\begin{bmatrix} * & & & & \\ 0 & * & & & \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Once in REF, we can solve the system by back substituting, starting from the bottom row.

No	REF Characteristic
1	All rows of 0s are at the bottom
2	In each non-zero row, the leading entry (*) is in a column to the left of any leading entries below it.

3.5 Gauss-Jordan elimination (Mat $\to$ RREF)

Gauss-Jordan elimination

The process of using EROs to transform a matrix into reduced row echelon form (RREF)

Example:

$$\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

No	RREF Characteristic
1	It is in REF
2	Every leading entry is a 1
3	Each column with a leading 1 has zeros everywhere else in the column

3.6 Summarised procedure for solving systems

Step	What to do
1	Form the augmented matrix
2	Perform EROs to REF or RREF
3	Set a parameter (t, u, etc) for every column without a leading entry
4	Use nonzero rows to solve for the other variables.

3.7 Matrices

Matrix definition
Let $n_r, n_c \in \mathbb{N}$. An $n_r \times n_c$ matrix is just a $n_r \times n_c$ grid of numbers where n_r is the no. of rows and n_c is the no. of columns. $$M = \begin{bmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,n_c} \\ m_{2,1} & m_{2,2} & \dots & m_{2,n_c} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n_r,1} & m_{n_r,2} & \dots & m_{n_r,n_c} \end{bmatrix}$$ Given a $n_r \times n_c$ matrix M we write $M = (m_{i,j})$ where for each $i = 1, 2, ..., n$ and $j = 1, 2, ..., n$ the symbol $m_{i,j}$ denotes the entry in row i, column j

3.8 Square matrix, diagonal entries

Term	Definition
Diagonal entries	The diagonal entries of a square matrix A are $a_{11}, a_{22}, ..a_{nn}$.
Square matrix	A matrix of size $n \times n$
Diagonal square matrix	A square matrix is "diagonal" if $a_{ij} = 0$ if $i \neq j$.

3.9 Identity matrix, zero matrix

Term	Definition
Identity matrix	The diagonal matrix whose diagonal entries are all 1. $$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
Zero matrix	The $m \times n$ matrix with entries all equal to 0. $$O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3.10 Row matrix, column matrix

Term	Definition
Row matrix	A matrix of size $1 \times n$ $$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$
Column matrix	A matrix of size $n \times 1$ $$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

3.11 Matrix addition, scalar multiple

Matrix addition, scalar multiple
Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times r}$, $c \in \mathbb{R}$ If $m = p, n = r$, then the **sum of A and B** is given by $A + B = (a_{ij} + b_{ij})_{m\times n}$ and we define the **scalar multiple** of A as $cA = (ca_{ij})_{m\times n}$

3.12 Matrix multiplication

Matrix multiplication
Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times r}$. If $n = p$, then we define the **product** AB to be the $m \times r$ matrix whose ij-entry is $\sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + \ldots + a_{in}b_{nj}$ Example: $A = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}, B = \begin{bmatrix} 7 & 3 \\ 2 & 8 \end{bmatrix}$ $AB = \begin{bmatrix} (1)(7)+(2)(2) & (1)(3)+(2)(8) \\ (4)(7)+(6)(2) & (4)(3)+(6)(8) \end{bmatrix} = \begin{bmatrix} 11 & 19 \\ 40 & 60 \end{bmatrix}$

3.13 Matrix multiplication properties

Matrix multiplication properties

Let A, B, C be matrices such that all the following operations are defined, and let $k \in \mathbb{R}$

No	Property
1	$A(BC) = (AB)C$
2	$A(B + C) = AB + AC$
3	$(A + B)C = AC + BC$
4	$k(AB) = (kA)B = A(kB)$

3.14 Function composition with matrix multiplication

Function composition with matrix multiplication
Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be given by $f(\mathbf{x}) = A\mathbf{x}$ Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be given by $g(\mathbf{x}) = B\mathbf{x}$ We have $(f \circ g)(\mathbf{x}) = (AB)\mathbf{x}$

3.15 Matrix transpose

Matrix transpose
For $A = (a_{ij})_{m\times n}$, we define the transpose of A to be the $n \times m$ matrix $A^T = (b_{ij})_{n\times m}$ where $b_{ij} = a_{ij}$ Meaning, rows of A become the columns of A^T and vice versa.

3.16 Operations with matrix transpose

Operations with matrix transpose	
No	**Operation**
1	$(A^T)^T = A$
2	$(A+B)^T = A^T + B^T$
3	$(kA)^T = kA^T$
4	$(AB)^T = B^T A^T$
5	$(A^n)^T = (A^T)^n$

3.17 Matrix inversion

Matrix inversion'
Let A be an $n \times n$ matrix. An inverse of A is an $n \times n$ matrix B such that $AB = BA = I_n$. If there exists such matrix B, then we say: A is invertible

Inverting a 2×2 matrix
$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Inverting an $n \times n$ matrix
Let A be an $n \times n$ matrix. This means A is invertible if and only if the matrix $[A\|I_n]$ can be row reduced to a matrix of the form $[I_n\|B]$. In which case, $B = A^{-1}$. Note: If you get a row of zeros on the left, then A is not invertible.

3.18 Inverses and systems of equations

Inverses and systems of equations
If $A\mathbf{x} = \mathbf{b}$ is a system of equations and A is invertible, then there is a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$

3.19 Determinant

Determinant of a 2×2 matrix
If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then det($A$)= $ad - bc$

Determinant of an $n \times n$ matrix
For an $n \times n$ matrix with $n \geq 2$, we denote A_{ij} to be the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j for $1 \leq i, j \leq n$. In which case, $\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{ij} \det(A_{ij})$ Note: The determinant of a 1×1 matrix $[a]$ is a.

More facts about determinants	
No	**Fact**
1	If $A \xrightarrow{R_i \leftrightarrow R_j} B$, then det($B$)= $-$det(A)
2	If $A \xrightarrow{R_i \to cR_j} B$, then det($B$)= $c$det(A)
3	If $A \xrightarrow{R_i \to R_i + cR_j} B$, then det($B$)=det($A$)
4	A is invertible if and only if det(A)$\neq 0$
5	Let A, B be $n \times n$ matrices. 5.1. $\det(cA) = c^n \det(A)$ 5.2. $\det(AB) = \det(A)\det(B)$ 5.3. $\det(A^{-1}) = \frac{1}{\det(A)}$ 5.4. $\det(A) = \det(A^T)$

3.20 Laplace expansion theorem

Laplace expansion theorem
Let A be an $n \times n$ matrix where $n \geq 2$. For any $1 \leq i \leq n$, we have $\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$

3.21 Matrix minor and cofactor

Minor and cofactor
Let A be an $n \times n$ matrix. For $1 \leq i, j \leq n$, we define: 1. The (i, j)-minor to be $\det(A_{ij})$ 2. The (i, j)-cofactor to be $C_{ij} = (-1)^{i+j} \det(A_{ij})$

3.22 Eigenvalues, eigenvectors, eigenspace

Eigenvalues, eigenvectors, eigenspace
Let A be an $n \times n$ matrix. A scalar λ is an eigenvalue of A if $A\mathbf{x} = \lambda\mathbf{x}, \mathbf{x} \neq \mathbf{o}$ Such a vector $\mathbf{x}$ is called an eigenvector of A for λ. The eigenspace for λ is $E_\lambda(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \lambda\mathbf{x}\}$ You can think of the eigenspace as $E_\lambda(A) = (\{\text{all eigenvectors for } \lambda\} \cup \{\mathbf{o}\})$

3.23 How to find eigenvalues

How to find eigenvalues
The eigenvalues of A are the values of λ satisfying det$(A - \lambda \mathrm{I})$= 0
This is based on the following manipulation
$A\mathbf{x} = \lambda\mathbf{x}, \mathbf{x} \neq \mathbf{o}$
$\Rightarrow A\mathbf{x} = \lambda I_{\mathbf{x}}, \mathbf{x} \neq 0$
$\Rightarrow A\mathbf{x} - \lambda I\mathbf{x} = \mathbf{o}, \mathbf{x} = \mathbf{o}$
$\Rightarrow (A - \lambda I)\mathbf{x} = \mathbf{o}, \mathbf{x} \neq \mathbf{o}$
$\Rightarrow A - \lambda I$ is not invertible
$\Rightarrow$ det$(A - \lambda I)$= 0

3.24 How to find eigenspace

How to find eigenspace
Since $A\mathbf{x} = \lambda\mathbf{x} \Leftrightarrow (A - \lambda I)\mathbf{x} = \mathbf{o}$, we have
$E_\lambda(A) = \{\mathbf{x} : (A - \lambda I)\mathbf{x} = \mathbf{o}\}$
$= \{$solutions to the system $(A - \lambda I)\mathbf{x} = \mathbf{o}\}$

3.25 Characteristic polynomial and equation

Characteristic polynomial and equation
If A is an $n \times n$ matrix, then det$(A - \lambda I)$ is a
polynomial in λ of degree n called the characteristic polynomial.
det$(A - \lambda I)$= 0 is the characteristic equation.

3.26 Algebraic and geometric multiplicities

Algebraic and geometric multiplicities Let A be an $n \times n$ matrix, and let $\lambda = b$ be an eigenvalue of A	
Multiplicity	**Definition**
Algebraic	If when factored, the characteristic polynomial det$(A - \lambda I)$ has factor $(\lambda - b)^m$, then the algebraic multiplicity of $\lambda = b$ is m
Geometric	The geometric multiplicity of $\lambda = b$ is the number of parameters appearing in $E_\lambda(A)$

3.27 Diagonalization

Diagonalization
An $n \times n$ matrix A is diagonalizable if there is an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$
Another way to put it: An $n \times n$ matrix A is diagonalizable if and only if for all eigenvalues of A, the algebraic multiplcity is equal to the geometric multiplicity.