Complex Numbers Introduction and Cartesian Form

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1 Imaginary unit i

In the real number system, the square root of any negative number, such as $\sqrt{-3}$, $\sqrt{-\frac{1}{2}}$, and $\sqrt{-\pi}$, are not defined. Instead, mathematicians came up with the concept of an imaginary unit, denoted by i, which is defined by $\sqrt{-1}$.

$$\boxed{i = \sqrt{-1}}$$

2 Introduction to complex numbers

Complex numbers are numbers that consist of a combination of a real part (Re) and an imaginary part (Im). In general, complex numbers can be expressed in three forms:

- Cartesian form (also known as Rectangular form): $\boxed{a + bi}$
- Polar form (also known as Modulus-Argument form): $\boxed{r(\cos\theta + i\sin\theta)}$, often abbreviated as $\boxed{r = \text{cis }\theta}$
- Exponential form: $\boxed{re^{i\theta}}$

3 Cartesian/Rectengular form

The Cartesian/Rectangular form of complex numbers is $\boxed{a + bi}$, where a is the real part, while b is the imaginary part multiplied by the imaginary unit i.

So, if we have a complex number $z = 4 + 6i$, we can say that:

- $\text{Re}(z) = 4$
- $\text{Im}(z) = 6$

3.1 Performing arithmetic operations in Cartesian form

The Cartesian form $\boxed{a + bi}$ is the most effective form for performing arithmetic on complex numbers. To help illustrate how to do arithmetic on complex numbers, let's suppose that we have the complex numbers $z = 2 + 4i$ and $w = 3 + 5i$.

- **Addition and subtraction**

  To perform addition, simply add real parts together and imaginary parts together. Here's an example involving the complex numbers $z = 2 + 4i$ and $w = 3 + 5i$ we defined earlier.

  $z + w = (2 + 4i) + (3 + 5i)$
  $z + w = (2 + 3) + (4 + 5)i$
  $z + w = 5 - 9i$

  Likewise for subtraction; subtract real parts from real parts and imaginary parts from imaginary parts.

  $z - w = (2 + 4i) - (3 + 5i)$
  $z - w = (2 - 3) + (4 - 5)i$
  $z - w = -1 - 1i$

- **Multiplication**

  To perform multiplication, just use the FOIL (First, Outer Inner, Last) method, where given $(a + b)(c + d)$, you follow:

  1. F: perform $a \times c$
  2. O: perform $a \times d$
  3. I: perform $b \times c$
  4. L: perform $b \times d$

  So, let's now perform the FOIL method to multiply $z = 2 + 4i$ with $w = 3 + 5i$.

  $z \times w = (2 + 4i)(3 + 5i)$
  $z \times w = 6 + 10i + 12i + 20i^2$

  Remember that i^2 equates to $\sqrt{-1}$. Therefore, $20i^2 = 20(-1)$

  $z \times w = -14 + 22i$

- **Division**

  We cannot divide by an imaginary number; any fraction must have a real number denominator. So, in order to divide one complex number by another, we need to introduce "**complex conjugates**".

$\boxed{\text{The complex conjugate of } z = a + bi \text{ is defined as } \bar{z} = a - bi}$

To perform division between complex numbers $z = a + bi$ and $w = c + di$, you follow:

$$\frac{z}{w} = \frac{z}{w} \times \frac{\bar{w}}{\bar{w}}$$

$$\frac{z}{w} = \frac{(a + bi)(c - di)}{(c + di)(c - di)}$$

A **complex conjugate** of a complex number is the number with the imaginary part negated. Complex conjugates are useful for divison between complex numbers because a complex number multiplied by its conjugate always results in a real number.

$$(1 + 2i)(1 - 2i) = 1 - 2i + 2i - 4i^2 = \boxed{5}$$

Recall that a divisor must always be a real number, so in the context of complex number divison, we can theoretically multiply the divisor with a complex conjugate to turn it into a real number! So, let's now perform divison between $z = 2 + 4i$ and $w = 3 + 5i$,

$$\frac{z}{w} = \frac{(2 + 4i)(3 - 5i)}{(3 + 5i)(3 - 5i)}$$

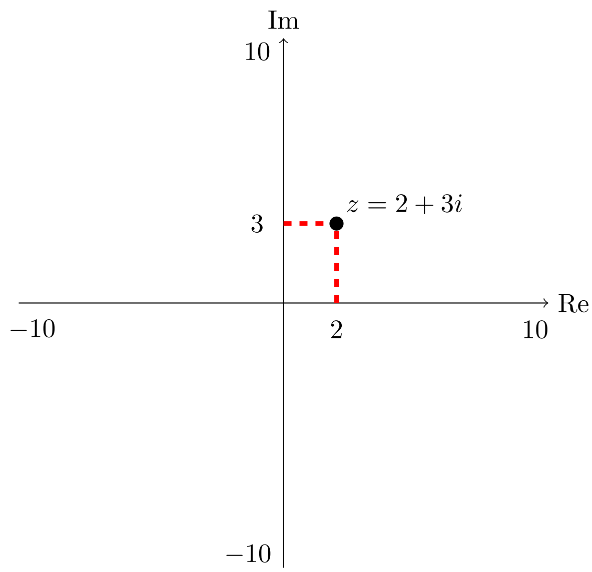
$$= \frac{6 - 10i + 12i - 20i^2}{9 - 15i + 15i - 25i^2}$$

$$= \frac{26 + 2i}{34}$$

$$\approx 0.765 + 0.0588i \text{ (3 sf.)}$$

3.2 Argand diagram (Complex plane)

The Argand diagram, otherwise known as the complex plane, consists of a horizontal real axis (Re) and a vertical imaginary axis (Im). It works like your usual Cartesian plane. Let's say that we want to plot a complex number $z = 2 + 3i$. To do that, just go 2 units to the right from the origin along the real axis, and then go 3 units up.



3.3 Modulus/absolute value

Modulus, also known as "absolute value", can be defined as:

- $|x|$, the distance from zero to a real number x
- or $|z|$, the distance from the origin to a complex number $|z = a + bi|$

x and z are generic variables and can be replaced by any other name.

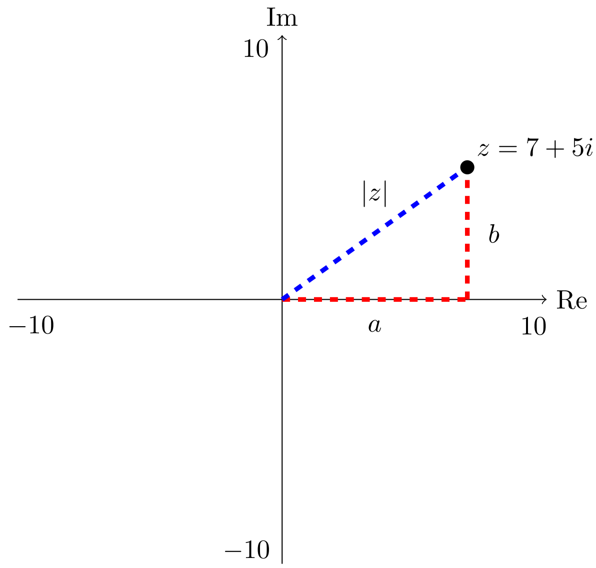
Modulus of a real number

For any real number x, the modulus is always x (eg. it takes 3 units to go from 0 to 3). Alternatively, on the "real number line", the modulus of any real number x is:

- x if x is positive
- $-x$ if x is negative

Modulus of a complex number

For any complex number $z = a + bi$, the modulus is $\boxed{|z| = \sqrt{a^2 + b^2}}$. You might realize that this is perfectly identical to Pythagoras' Theorem, and that's because it is (See the following argand diagram)



In this case, we can find the modulus of $z = 7 + 5i$ like so:

$$|z| = \sqrt{7^2 + 5^2}$$

$$|z| = \sqrt{74} \approx 8.60\text{units (3 sf.)}$$

Properties of the modulus of complex numbers

1. $|zw| = |z| \times |w|$
2. $|z \div w| = |z| \div |w|$, provided that $w \neq 0$
3. $|z + w| \leq |z| + |w|$, known as "triangle inequality"
4. $|z - w| \geq |z| - |w|$, known as "reverse triangle inequality"