Predicates and Quantifiers

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July 20, 2023

1 Predicates

Statements with a variable, such as "x is greater than 10", has two parts:

- A subject: "x"
- A predicate: "is greater than 10"

That said, a "**predicate**" is defined as the property that a subject(s) have. In this case, the subject x has the property of being "greater than 10". We can denote such subject-predicate statements as a propositional function $P(x)$. Of course, a statement may have more than one variable; $P(x_1, x_2, ..., x_Z)$.

When you input values to the variables in any propositional function, such as inputting $\{a = 2, b = 4, c = 6\}$ to $T(a, b, c) = a^2 + b^2 = c^2$, you will end up with a proposition,

$$T(a, b, c) = 2^2 + 4^2 = 6^2$$

and you know it's a proposition because it has a truth value.

$$4 + 16 \neq 36, \text{ False.}$$

2 Quantifiers

Quantifiers, or quantification, are/is used to express the extent to which a predicate is true over a range of inputs for its variables. There exists virtually an unlimited number of quantifiers because there is an unlimited ways in which you can perform quantification on a propositional function. This note will ONLY cover arguably the most important ones:

- Universal quantifier ($\forall$)
- Existential quantifier ($\exists$)
- Uniqueness quantifier ($\exists!$)

2.1 Universal quantifier ($\forall$)

The universal quantifier "$\forall$" (which can be read as "for all [element]") is used to express that a propositional function $P(x)$ holds true for any subject/input x in the given domain.

So the statement $\forall x P(x)$ posits that "$P(x)$ is true for every x", and its truth value is false when there exists an input(s) x for which $P(x)$ is false. That said, you should realize a statement such as $\forall x P(x)$ is the same as

$$P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge ... \wedge P(x_n)$$

2.2 Existential quantifier ($\exists$)

The existential quantifier "$\exists$" (which can be read as "There exists [an element]"), is used to express that there is (at least one) an element x that makes the propositional function $P(x)$ hold true.

So the statement "$\exists x P(x)$" posits that "There is an x for which $P(x)$ is true", and its truth value is false when there exists NO inputs x at all for which $P(x)$ is true. That said, you should realize that a statement such as $\exists x P(x)$ is the same as

$$P(x_1) \vee P(x_2) \vee P(x_3) \vee ... \vee P(x_n)$$

2.3 Uniqueness quantifier ($\exists!$)

The uniqueness quantifier "$\exists!$" (which can be read as "There exists a unique [element]"), is used to express that there is ONE and ONLY ONE element x that makes the propositional function $P(x)$ hold true.

So, the statement "$\exists! x P(x)$" posists that "There is EXACTLY ONE x for which $P(x)$ is true," and its truth value is false when there:

- exists no input for which $P(x)$ is true
- is more than one input for which $P(x)$ is true

2.4 Domain restriction for quantifiers

When you use quantifiers, you generally express something like $\forall x P(x)$ or in another form with the expression which $P(x)$ is equated to, such as $\forall x(x + 3 > 5)$. Now, you can also perform a **domain restriction**, wherein you would restrict the possible values of the input x which the quantifier plug into the propositional function $P(x)$ and turn it into a number of propositions. Below is an example:

$$\exists x > 5(x + 22 = 30), \text{where } x \text{ is an element of the set of natural numbers } \mathbb{N}.$$

Here, we're inputting all natural numbers greater than 5 into x in the equation $x + 22 = 30$, and we are proposing that there is an input x for which $x + 22 = 30$. This proposition is true because there exists $x = 8$; $8 + 22 = 30$.

2.5 Precedence of quantifiers

Quantifiers have a higher precedence than all logical operators. For example, given $\forall x P(x) \vee Q(x)$, you perform $(\forall x P(x)) \vee Q(x)$ instead of $\forall x(P(x) \vee Q(x))$.

2.6 Negation of quantified expressions

Suppose we have the proposition "Every employee at Google has a bachelor's degree," which can be quantified expressed in universal quantification as $\forall x P(x)$ where $P(x)$ is the proposition "Google employee x has a bachelor's degree."

If you negate $\forall x P(x)$, you get $\neg \forall x P(x)$, which read "It is not the case that every employee at Google has a bachelor's degree."

Or, even more concisely, we can negate $\forall x P(x)$ as $\exists x \neg P(x)$, which read "There exists an employee x at Google who does not have a bachelor's degree."

That said, we have the logical equivalence

$$\boxed{\neg \forall x P(x) \equiv \exists x \neg P(x)}$$

Which is valid and is one of the two De Morgan's Laws for Quantifiers

2.6.1 De Morgan's Laws for Quantifiers

Negation 1	**Negation 2**	**when True**	**when False**
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is False.	There is an x for which $P(x)$ is True.
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is False	For every x, $P(x)$ is True.

Given the statements $\exists x P(x)$ and $\forall x P(x)$, their negations are defined respectively, based on De Morgan's Laws, as:

- $\boxed{\neg \exists x P(x) \equiv \forall x \neg P(x)}$
  Because recall that $\neg \exists x P(x)$ is the same as the statement $\neg(P(x_1) \vee P(x_2) \vee ... \vee P(x_n))$, and we know based on De Morgan's Laws that there exists an equivalent statement $\neg P(x_1) \wedge \neg P(x_2) \wedge ... \wedge \neg P(x_n)$, and that's exactly what $\forall x \neg P(x)$ represent.

$$\neg(P(x_1) \vee P(x_2) \vee ... \vee P(x_n)) \equiv \neg P(x_1) \wedge \neg P(x_2) \wedge ... \wedge \neg P(x_n)$$

- $\boxed{\neg \forall x P(x) \equiv \exists x \neg P(x)}$
  Because recall that $\neg \forall x P(x)$ is the same as the statement $\neg(P(x_1) \wedge P(x_2) \wedge ... \wedge P(x_n))$, and we know based on De Morgan's Laws that there exists an equivalent statement $\neg P(x_1) \vee \neg P(x_2) \vee ... \vee \neg P(x_n)$, and that's exactly what $\exists x \neg P(x)$ represent.

$$\neg(P(x_1) \wedge P(x_2) \wedge ... \wedge P(x_n)) \equiv \neg P(x_1) \vee \neg P(x_2) \vee ... \vee \neg P(x_n)$$