Roots and Complex Roots

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Recall that for a positive real number a and a positive integer n, the nth root of a, denoted as $\sqrt[n]{a}$ or $a^{1/n}$, is the positive real number x such that $x^n = a$.

For example,
$\sqrt{9} = 3$
$3^2 = 9$

1 Roots of complex numbers

Any complex number except 0 has exactly n distinct nth roots.

To find the nth roots of a complex number, you first add $2k\pi$ to θ in order to take account of all possible arguments, where k is an integer multiplier of 2π (this is all given the fact that any complex number has infinitely many arguments of integer multiples 2π).

So, we now have $z = r[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]$, or $z = re^{i(\theta+2k\pi)}$.

From here, we can actually use the De Moivre's Theorem in REVERSE in order to find roots! So, if in the usual case of z^n, we would do,

$$z^n = r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta}$$

in order to find the roots, i.e. $z^{1/n}$, we can instead do:

$$z^{1/n} = r^{1/n}[\cos(\frac{\theta + 2k\pi}{n}) + i\sin(\frac{\theta + 2k\pi}{n})] = r^{1/n}e^{i(\frac{\theta+2k\pi}{n})}$$

This is the form of De Moivre's theorem you would use to find the nth roots of complex numbers.

So when raising a complex number to a power using De Moivre's theorem, we raise the modulus r to the power of n and multiply the argument θ by n. ON THE OTHER HAND, when finding the ROOTS of a complex number, we can take the nth root of the modulus r and divide all arguments by n. You know, the opposite of raising to a power is taking a root, and the opposite of multiplying is dividing.

2 Complex roots of polynomials

Polynomials can have complex roots, and you should give these complex roots in the event that you're asked to give the roots "over the set of complex numbers". If you're asked to provide complex roots, there is no need to do so.

To get the complex roots of quadratics, you may use the quadratic formula. Otherwise, you can do factorization to get the complex roots of polynomials of degree greater than 2.

Example: Find the roots of $x^3 - x^2 + x - 1$

$$x^3 - x^2 + x - 1 = 0$$

We can factor out $x - 1$,

$$(x-1)(x^2+1) = 0$$

Using the zero factor principle, ie. if $ab = 0$ then $a = 0$, $b = 0$,

$$x - 1 = 0 \qquad\qquad x^2 + 1 = 0$$

$$x = 1 \qquad\qquad x = \pm\sqrt{-1} = \pm i$$

$$\boxed{x = 1, i, -i}$$

The roots of $x^3 - x^2 + x - 1$ are $1, i$, and $-i$

2.1 Complex conjugate roots theorem

Notice that in the above example, the complex roots of $x^3 - x^2 + x - 1$ are i and $-i$, which are complex conjugates of one another.

For any polynomial with **real coefficients**, their complex roots will always come in **complex conjugate pairs**.

So if you're given a polynomial and one of its complex roots, you can easily identify the other complex root because it will always be the conjugate. Suppose that I have a polynomial $f(x)$ of degree 5, and I was told that two of its roots are $2 + i$ and $-3i$. I already know that the complex conjugates of these numbers are also roots of the same polynomial! So, I know the roots are:

1. $2 + i$
2. $2 - i$
3. $-3i$
4. $3i$

5. R_5

where R_5 is the fifth root which has to be real (the polynomial has to be of degree 6 in order for R_5 to be complex, because then, its conjugate can exist!).