Boolean Algebra

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Boolean algebra is simply algebra done with boolean values, ie. 0 and 1, where **0 represents "False" and 1 represents "True"**.

1 Boolean algebra operations

There are three basic boolean algebra operations which are (1) conjunction, (2) disjunction, and (3) negation.

Conjunction (AND)	**Disjunction (OR)**	**Negation (NOT)**
$a \wedge b$	$a \vee b$	$\neg a$
Output is 1 if both a and b are 1	Output is 1 if any of a and b is 1	Opposite value of a

You can also write:

- $a \wedge b$ as $a * b$
- $a \vee b$ as $a + b$
- $\neg a$ as a'

But I don't like the latter notations, because it's easy to mistake them for their functionality in terms of the decimal system. For instance, $1 + 1 \neq 10$ (you know, binary of 2 is supposedly 10). The "+" operator works like an "AND" logic instead of serving as an addition operator - which is kinda confusing $^=$w$^=$.

1.1 Properties of boolean algebra

- Closure: $a \wedge b$, $a \vee b$ are always in the set $\{0, 1\}$.
- Identity: $a \wedge 1 = a$, $a \vee 0 = a$.
- Domination: $a \vee 1 = 1$, $a \wedge 0 = 0$.
- Idempotence: $a \vee a = a$, $a \wedge a = a$.
- Commutativity: $a \vee b = b \vee a$, $a \wedge b = b \wedge a$.

- Associativity: $(a \lor b) \lor c = a \lor (b \lor c)$, $(a \land b) \land c = a \land (b \land c)$.
- Distributivity: $a \lor (b \land c) = (a \lor b) \land (a \lor c)$, $a \land (b \lor c) = (a \land b) \lor (a \land c)$.
- Complementation: $a \lor \neg a = 1$, $a \land \neg a = 0$.
- Absorption: $a \lor (a \land b) = a$, $a \land (a \lor b) = a$.
- De Morgan's Laws: $\neg(a \lor b) = \neg a \land \neg b$, $\neg(a \land b) = \neg a \lor \neg b$.

2 Truth table

A "truth table" is a systematic way of determining all possible outputs of a boolean expression where there are uninitialized variables. It's comprised of columns for all unique variable as well as an extra column denoted as "Result".

To help illustrate how truth tables work, suppose that we want to find all possible outputs of the boolean expression: $\boxed{a \land b \lor \neg c}$

We can first construct a truth table like so:

a	b	c	Result
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

Now, for every combination, assign the values for a, b, and c respectively.
For instance, for the combination $\{a = 0, b = 0, c = 0\}$, we have:

$$0 \land 0 \lor \neg 0$$

We know that $0 \land 0 = 0$, because neither operand is 1. So, we have

$$0 \lor \neg 0$$

We also know that $\neg 0 = 1$ because the opposite of 0 is 1. Therefore,

$$0 \lor 1 = 1$$

a	b	c	Result
0	0	0	1

If you do the math for the rest of the combinations, you will end up with:

a	**b**	**c**	**Result**
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

So, this table gives all posible outcomes for $\boxed{a \wedge b \vee \neg c}$.

3 Equivalent boolean expressions

For any boolean expression E_1, you can find two other "equivalent" expressions E_2 (sum of products) and E_3 (product of sums). We say that they are "equivalent", because they all should result in identical truth tables.

$$\boxed{E_1 \equiv E_2 \equiv E_3}$$

Suppose we have a boolean expression $E_1 = a \wedge b \vee \neg c$, which results in the following truth table:

a	**b**	**c**	**Result**
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Its equivalent expressions E_2 and E_3 is derivable by getting the sum of products and product of sums representations.

Equivalent Boolean Expressions ($\equiv$)

E_1	$\neg a \wedge \neg b \wedge \neg c$
E_2	$(\neg a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge \neg c) \vee (a \wedge \neg b \wedge \neg c) \vee (a \wedge b \wedge \neg c) \vee (a \wedge b \wedge c)$
E_3	$(a \vee b \vee \neg c) \wedge (a \vee \neg b \vee \neg c) \wedge (\neg a \vee b \vee \neg c)$

3.1 Sum of Products

To get the sum of products representation of E_1, you follow these steps:

1. Work only with rows in the truth table of E_1 which results in 1.

a	**b**	**c**	**Result**
0	0	0	1
0	1	0	1
1	0	0	1
1	1	0	1
1	1	1	1

2. Get the conjunctions for each row and negate the variables with a value 0.

$$\begin{array}{|c|}
\neg a \land \neg b \land \neg c \\
\neg a \land b \land \neg c \\
a \land \neg b \land \neg c \\
a \land b \land \neg c \\
a \land b \land c
\end{array}$$

3. All conjunctions are in a disjunction.

$$E_2 = (\neg a \land \neg b \land \neg c) \lor (\neg a \land b \land \neg c) \lor (a \land \neg b \land \neg c) \lor (a \land b \land \neg c) \lor (a \land b \land c)$$

3.2 Product of Sums

To get the product of sums representation of E_1, you basically do the opposite of whatever you did to get the sum of products. Specifically, you should follow these steps:

1. Work only with rows in the truth table of E_1 which results in 0.

a	**b**	**c**	**Result**
0	0	1	0
0	1	1	0
1	0	1	0

2. Get the disjunctions for each row and negate the variables with a value 1.

$$\begin{array}{|c|}
a \lor b \lor \neg c \\
a \lor \neg b \lor \neg c \\
\neg a \lor b \lor \neg c
\end{array}$$

3. All disjunctions are in a conjunction.

$$E_3 = (a \lor b \lor \neg c) \land (a \lor \neg b \lor \neg c) \land (\neg a \lor b \lor \neg c)$$