MATH1061/1021 Cheatsheet (Calculus)

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1 Sets, numbers, functions

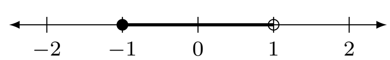
1.1 Union, intersection, difference, complement, subset

Operation	Notation	Description
Union	$A \cup B$	All elements that are in A or B or both
Intersection	$A \cap B$	All elements that are both in A and B
Difference	$A \setminus B$	All elements that are in A but not in B
Complement	$\overline{A}$	All elements in the universal set that are not in A
Subset	$A \subseteq B$	A is a subset of B; every element in A is also in B
Proper subset	$A \subset B$	A is a subset of B and $A \neq B$

1.2 Common number sets

Name (Symbol)	Set
Natural numbers ($\mathbb{N}$)	$\{0, 1, 2, 3, 4, ...\}$
Integers ($\mathbb{Z}$)	$\{... -2, -1, 0, 1, 2, ...\}$
Rational numbers ($\mathbb{Q}$)	$\left\{\frac{1}{2}, -\frac{4}{3}, \frac{17}{12}, \dots\right\}$
Irrational numbers ($\mathbb{R} \setminus \mathbb{Q}$)	$\{\sqrt{2}, \pi, e, \sqrt{7}, ...\}$
Real numbers ($\mathbb{R}$)	All numbers

1.3 Interval notation

Interval notation
$D = [-1, 1)$

Note: Infinity ∞ is always open-bracketed, i.e. (∞, ∞)

1.4 Modulus

Modulus
Distance on the number line $\lvert x - y \rvert$

1.5 Injective, surjective, bijective functions

Function type	Definition
Injective	A function $f : X \to Y$ where for all $x \in X$, x maps to a different $y \in Y$, and $\lvert X\rvert \leq \lvert Y\rvert$.
Surjective	A function $f : X \to Y$ where for all $y \in Y$, y is an image of some $x \in X$, and $\lvert X\rvert \geq \lvert Y\rvert$.
Bijective	A function that is both injective and surjective where $\lvert X\rvert = \lvert Y\rvert$.

1.6 Composite, inverse, hyperbolic functions

Composite functions
A function in the form $f \circ g(x) = f(g(x))$ where some function f takes another function g as input.

Inverse functions	
Definition	**How to invert a function**
There exists an inverse function f^{-1} if and only if f is injective. If $f(x) = y$, then $f^{-1}(y) = x$, which implies that if $f : A \to B$, then $f^{-1} : B \to A$.	1) Rewrite $f(x)$ as y 2) Swap x and y 3) Make y the subject 4) Rewrite y as $f^{-1}(x)$

Hyperbolic functions		
Function	**Derivative**	**Inverse**
$\cosh(x) = \frac{e^x + e^{-x}}{2}$	$\frac{\mathrm{d}}{\mathrm{dx}}\cosh(x) = \sinh(x)$	$\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$
$\sinh(x) = \frac{e^x - e^{-x}}{2}$		$\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$

2 Limits

2.1 Definition of a limit

Limit
$\lim_{x \to a} f(x) = L$ as x approaches a, but not reaching a, the output gets closer and closer to L.

2.2 Limit laws

if we have $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$	
No	**Law**
1	$\lim_{x\to c}[f(x) + g(x)] = L + M$
2	$\lim_{x\to c}[f(x) - g(x)] = L - M$
3	$\lim_{x\to c}[f(x)g(x)] = LM$
4	$\lim_{x\to c}[\frac{f(x)}{g(x)}] = \frac{L}{M}$
5	$\lim_{x\to c}[f(x)]^n = L^n$
6	$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$

2.3 One-sided limits, existence of limit

One-sided limits
Limit from below (left) is denoted as $\lim_{x\to c^-} f(x)$
Limit from above (right) is denoted as $\lim_{x\to c^+} f(x)$

Existence of a limit
A limit exists only if
$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$
in which case
$\lim_{x\to c} f(x) = L$

2.4 Infinite limits

Infinite limits
If we take x larger and larger, and $f(x)$ gets closer to L, we say
$\lim_{x\to\infty} f(x) = L$
You can algebraically solve infinite limits by
dividing by the highest power of x in the denominator
or to by using conjugates.

2.5 Squeeze law

Squeeze law
Suppose $g(x) \leq f(x) \leq h(x)$ for x near a
if $\lim_{x\to a} h(x) = L$ and $\lim_{x\to a} g(x) = L$, then $\lim_{x\to a} f(x) = L$
we say "$f(x)$ is squeezed between $g(x)$ and $h(x)$"

2.6 Continuous functions

Continuous functions
A function is continuous at a point if the limit exists at
the point, and is equal to the value at that point.
ie. $f(x)$ is continuous at $x = c$ if and only if:
1) $f(c)$ is defined
2) $\lim_{x\to c} f(x)$ exists and is finite
3) $\lim_{x\to c} f(x) = f(c)$

3 Differential Calculus

3.1 Limit definition of derivatives

Limit definition of derivatives
The derivative of a function f at a point a is given by
$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

3.2 Common derivatives

Common derivatives	
$\mathbf{f(x)}$	$\mathbf{f'(x)}$
x^n where $n \neq 0$	nx^{n-1}
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
$\tan(x)$	$\sec^2(x)$
c^x	$c^x \ln(c)$
$\ln y$	$\frac{1}{y}\frac{dy}{dx}$

3.3 Differential rules

Differential rules		
No	**Name**	**What to do**
1	Power rule	$\frac{d}{dx}x^n = nx^{n-1}$
2	Constant rule	$\frac{d}{dx}c = 0$, where $c \in \mathbb{R}$
3	Sum rule	$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
4	Difference rule	$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$
5	Product rule	$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$
6	Quotient rule	$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
7	Chain rule	We are to find $\frac{d}{dx}[f \circ g(x)]$ **Method 1:** Let $u = g(x)$, $y = f(u)$, and so $y'(x)$ is given by $\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$ **Method 2:** If you can see f and g clearly, then $\frac{d}{dx}[f \circ g(x)] = f'(g(x))g'(x)$

3.4 Implicit differentiation

Implicit differentiation
Functions that can't (or aren't easy) to be written as $y = f(x)$ can instead be written implicitly in the form $F(x, y) = 0$. In which case, to get the derivative, find $\frac{d}{dx}[F(x, y)]$ applying chain rule for terms involving y, where $\frac{d}{dx}y \Longrightarrow \frac{dy}{dx}$ and finally, make $\frac{dy}{dx}$ the subject.

3.5 Useful algebra for logarithmic differentiation

No	Useful algebra
1	$a^x \Leftrightarrow e^{x \ln a}$
2	$\ln(e^{x \ln(a)}) = x \ln a$
3	$y \ln(a) = \ln(x) \Leftrightarrow y = \dfrac{\ln(x)}{\ln(a)}$
4	$\ln(\sqrt{x}) \Leftrightarrow \dfrac{1}{2} \ln(x)$
5	$\log_b(a) \Leftrightarrow \dfrac{\ln(a)}{\ln(b)}$

3.6 L'Hopital's Rule

L'Hopital's Rule
If $\lim_{x \to c} \dfrac{f(x)}{g(x)}$ is an indeterminate form, i.e. can be written as $\dfrac{0}{0}$ or $\pm\dfrac{\infty}{\infty}$, then $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \lim_{x \to c} \dfrac{f'(x)}{g'(x)}$ where c can be infinity. NOTE: You can keep applying L'Hopital's Rule until you get a definite form.

3.7 Extrema

Extrema An "extremum" is a maximum or a minimum.	
Extremum	**Definition**
Local max	A local max at $x = c$ means $f(c) \geq f(x)$ for any x near c
Local min	A local min at $x = c$ means $f(c) \leq f(x)$ for any x near c
Global max	A global max on some domain A where $f : A \to \mathbb{R}$ means $f(c) \geq f(x)$ for all $x \in A$
Global min	A global min on some domain A where $f : A \to \mathbb{R}$ means $f(c) \leq f(x)$ for all $x \in A$

3.8 First-derivative test

First-derivative test If f is continuous and $f'(c) = 0$		
Case	**Observation**	**Conclusion**
1	• $f'(x)$ is positive for $x < c$ • $f'(x)$ is negative for $x > c$	Local max
2	• $f'(x)$ is negative for $x < c$ • $f'(x)$ is positive for $x > c$	Local min
3	$f'(x)$ doesn't change sign	Not a local extremum (neither max or min)

3.9 Critical points

No	**Critical point type**
1	$f'(c)$ does NOT exist
2	$f'(c) = 0$
NOTES: • If $f(x)$ is differentiable, then any local extrema must occur at points $x = c$ where $f'(c) = 0$ • $f'(c) = 0$ does NOT guarantee a local extremum! You must apply first-derivative test!	

3.10 Finding max & min on a closed interval

Extreme value theorem
If f is a continuous function from a closed interval A to $\mathbb{R}$, then f attains a global max & global min value in A.

Finding max & min on a closed interval
To find max & min values on a closed interval, simply check: 1) Critical points 2) End points of the interval

3.11 Concavity

Concavity
• Concave up: $f''(x) > 0$ • Concave down: $f''(x) < 0$ • Point of inflection: if $f''(x)$ changes sign at c then $x = c$ is a point of inflection.

3.12 Second-derivative test

Second-derivative test		
If f, f' are differentiable and $f'(c) = 0$		
Case	**Observation**	**Conclusion**
1	$f''(c) > 0$	Local minimum
2	$f''(c) < 0$	Local maximum
3	$f''(c) = 0$	No conclusion (might be local min, max, or neither.) What to do from here: Do first-derivative test

3.13 Sensible order to curve-sketching

Sensible order to curve-sketching	
Step	**To work out**
1	What is the domain?
2	What is the y-intercept?
3	What happens as $x \to \pm\infty$ (Limits)
4	Is there anywhere that $f(x) \to \pm\infty$ (is denominator zero?)
5	Where are the critical points? • $f'(x) = 0$ • $f'(x)$ does NOT exist
6	Where is $f(x)$ increasing/decreasing? • $f'(x) > 0$: increasing • $f'(x) < 0$: decreasing
7	Where is $f(x)$ concave up or down? • $f''(x) > 0$: concave up (slope decreasing) • $f''(x) < 0$: concave down (slope increasing)

3.14 Tangent line (local linear approximation)

Tangent line
The tangent line of $f(x)$ at point c is given by $y = f(c) + f'(c)(x - c)$ This is also called the local linear approximation.

3.15 Taylor polynomials

Taylor polynomials

The Taylor polynomial for $f(x)$ around $x = a$ is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

or, using sigma notation,

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

3.16 Lagrange's form of remainder

Lagrange's form of remainder

If $p_n(x)$ is the Taylor polynomial of $f(x)$ around $x = a$,
then the n-th order remainder is given by

$$R_n(x) = f(x) - p_n(x),$$

which satisfies the limit condition

$$\lim_{x \to a} \frac{R_n(x)}{(x-a)^n} = 0.$$

If $f(x)$ has continuous derivatives up to order $n+1$, then $R_n(x)$ can be expressed in **Lagrange's Form:** There exists a number c, strictly between a and x, for each x, such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

3.17 Partial sum, infinite series, geometric series

Term	Definition
Partial sum	$S_n = a_0 + a_1 + a_2 + \ldots + a_n = \sum_{k=0}^{n} a_k$
Infinite series	$S = \sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k$ If this limit exists and is finite, the series converges; otherwise, it diverges.
Geometric series	$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ for $\lvert r\rvert < 1$

3.18 Taylor series

Taylor Series
The Taylor series of $f(x)$ near $x = a$ is given by $p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$ or, in short, $\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ If the limit as $n \to \infty$ converges for x, then $f(x) = \lim_{n\to\infty} p_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

4 Integral calculus

4.1 Riemann sums

Riemann sums
Given an interval split into n subdivisions from a to b, each of width $\triangle x = \frac{b-a}{n}$, the lower and upper Riemann sums L_n and U_n are given by $L_n = \sum_{i=1}^{n} m_i \triangle x$ and $U_n = \sum_{i=1}^{n} M_i \triangle x$ where m_i is the minimum value of $f(x)$ on $[x_{i-1}, x_i]$ and M_i is the maximum value of $f(x)$ on $[x_{i-1}, x_i]$.

4.2 Riemann integral

Riemann integral
If we take $\triangle x$ smaller and smaller, then m_i and M_i will get closer and closer. Hence, if U_n and L_n converge to the same limit, then Area $= \lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n = \int_a^b f(x)dx$

4.3 Properties of Riemann integral

Properties of Riemann integral	
No	**Property**
1	$\int_a^b cf(x)dx = c\int_a^b f(x)dx$
2	$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
3	$\int_a^b f(x)dx = -\int_b^a f(x)dx$

4.4 Fundamental Theorem of Calculus I

Fundamental Theorem of Calculus I
If $F(x) = \int_a^x f(t)dt,$ then $F'(x) = f(x)$ i.e. with area, we can find antiderivative.

4.5 Fundamental Theorem of Calculus II

Fundamental Theorem of Calculus II
$\int_a^b f(x)dx = \int_a^b F'(x)dx = F(b) - F(a)$ i.e. with antiderivative, we can find area.

4.6 Leibniz integral rule

Leibniz integral rule
$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(t)\,dt\right) = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x)$

4.7 Logarithms

Logarithms
If $y = a^x$ then $x = \log_a(y)$ which results in two main rules: $\log_a(x) = \frac{\ln(x)}{\ln(a)}$ and $\frac{d}{dx}[\log_a(x)] = \frac{1}{x\ln(a)}$

4.8 Properties of logarithms

Properties of logarithms	
No	**Property**
1	$\ln(ax) = \ln(a) + \ln(x)$
2	$\ln(\frac{x}{a}) = \ln(x) - \ln(a)$
3	$\ln(x^n) = \ln(x \cdot x \cdot ... \cdot x) = n\ln(x)$

4.9 Properties of exponentials

Properties of exponentials	
No	**Property**
1	$\ln(e^x) = x$
2	$e^{a+x} = e^a e^x$
3	$(e^x)^n e^{nx}$
4	$\frac{d}{dx}[e^x] = e^x$

4.10 Integration

Integration		
No	$f(x)$	$\int f(x)dx$
1	$x^n \ (n \neq 1)$	$\frac{x^{n+1}}{n+1} + C$
2	e^x	$e^x + C$
3	$\frac{1}{x}$	$\ln(x) + C$
4	$\sin(x)$	$-\cos(x) + C$
5	$\cos(x)$	$\sin(x) + C$
6	$\sinh(x)$	$\cosh(x) + C$
7	$\cosh(x)$	$\sinh(x) + C$
8	$\tan(x)$	$\ln\|\sec(x)\| + C$

4.11 Linearity property of integration

Linearity property of integration
$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$
$\int kf(x)dx = k \int f(x)dx$

4.12 Integration by substitution

Integration by substitution

$$\int h'(g(x))g'(x)dx = h(g(x)) + C$$

which can be made more neat up by letting $u = g(x)$

$$\int h'(u)\frac{du}{dx}dx = h(u) + C$$

and even neater by letting $f(u) = h'(u)$

$$\int f(u)\frac{du}{dx}dx = \int f(u)du$$

(this is usually easier to integrate)

Note: For definite integrals, remember to change the bounds

$$\int_{x=a}^{x=b} f(u)\frac{du}{dx}dx = \int_{u(a)}^{u(b)} f(u)du$$

4.13 Integration by parts

Integration by parts
$\int u'vdx = uv - \int uv'dx$
or, for definite integrals:
$\int_a^b u'vdx = [uv]_a^b - \int_a^b uv'dx$

4.14 Integration by partial fractions

Integration by partial fractions
For a function f that can be written in the form
$f(x) = \frac{a+bx}{(x-\lambda)(x-\mu)}$
for distinct constants λ and μ, we can express $f(x)$ as a sum of simpler fractions
$f(x) = \frac{A}{x-\lambda} + \frac{B}{x-\mu}$
where A, B can be found by equating $A(x-\mu) + B(x-\lambda) = ax + b$
and choosing $x = \lambda$ and $x = \mu$ respectively
$A(\lambda - \mu) = a\lambda + b \Rightarrow A = \frac{a\lambda + b}{\lambda - \mu}$
$B(\mu - \lambda) = a\mu + b \Rightarrow B = \frac{a\mu + b}{\mu - \lambda}$
The decomposed fractions should be easier to integrate
$\int f(x)dx = \int (\frac{A}{x-\lambda} + \frac{B}{x-\mu})dx = \int \frac{A}{x-\lambda}dx + \int \frac{B}{x-\mu}dx$
which, using natural logarithms, results to
$A\ln\lvert x-\lambda\rvert + B\ln\lvert x-\mu\rvert + C$

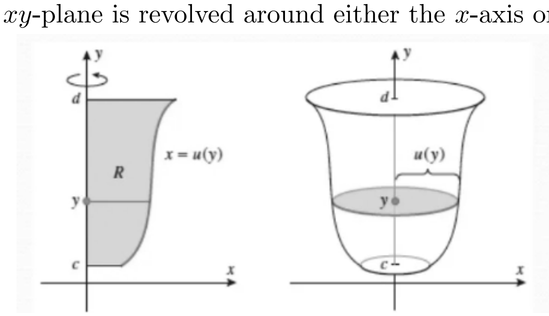
4.15 Length of a curve

Length of a curve
If $f : [a,b] \to \mathbb{R}$ is differentiable, then the length of its curve is given by
$\int_a^b \sqrt{1+[f(x)]^2}dx$

4.16 Solid of revolution

Solid of revolution

A solid of revolution is the solid obtained when a region R in the xy-plane is revolved around either the x-axis or y-axis.



4.17 Methods for finding solid of revolution

Disk method
Let R be the region bounded by $y = f(x) \geq 0, y = 0, x_1 = a, x_2 = b$.   The volume V of the solid S formed by revolving R around   The x-axis is given by $V = \pi \int_a^b (f(x))^2 dx$

Washer method (generalized disk method)
Let region $R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq f(x) \leq y \leq g(x), a \leq x \leq b\}$.   The volume V of the solid S formed by revolving R around the x-axis is given by $V = \pi \int_a^b [g(x)]^2 - [f(x)]^2 dx$

Shell method
Let R be the region with $0 \leq a \leq x \leq b$ and $0 \leq y \leq f(x)$ The volume V of the solid S formed by revolving R around the y-axis is given by $V = 2\pi \int_a^b x f(x) dx$