Rules of Inference

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1 Rules of Inference

In discrete maths, "rules of inference" are used to draw valid conclusions from given premises. There are 8 fundamental rules of inferences (Note: $\therefore$ means "therefore"):

Rule of Inference	***Tautology***	***Name***
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Figure 1: Summary of rules of inference
Source: "Discrete Mathematics and Its Applications" by Kenneth H. Rosen

- **Modus ponens**

  If we have the statement $p \to q$, and we know that p is true, we can conclude that q is also true.

$$\frac{\substack{p \\ p \to q}}{\therefore q}$$

  Example:

  Let p represent "It's raining", and q represent "The ground is wet"

  Premise 1 (p): It's raining.

  Premise 2 ($p \to q$): If it's raining, then the ground is wet.

  Conclusion ($\therefore q$): Therefore, the ground is wet.

- **Modus tollens**

  If we have the statement $p \to q$, and we know that q is false, we can conclude that p is also false.

$$\frac{\substack{\neg q \\ p \to q}}{\therefore \neg p}$$

  Example:

  Let p represent "It's raining", and q represent "The ground is wet"

  Premise 1 ($\neg q$): The ground isn't wet.

  Premise 2 ($p \to q$): If it's raining, then the ground is wet.

  Conclusion ($\therefore \neg p$): Therefore, it's not raining.

- **Hypothetical syllogism**

  If we have two statements $p \to q$, and $q \to r$, we can conclude $p \to r$.

$$\frac{\substack{p \to q \\ q \to r}}{\therefore p \to r}$$

  Example:

  Let p represent "It's raining", q represent "The ground is wet", and r represent "People wear boots."

  Premise 1 ($p \to q$): If it's raining, then the ground is wet.

  Premise 2 ($q \to r$): If the ground is wet, then people wear boots.

  Conclusion ($\therefore p \to r$): Therefore, if it's raining, then people wear boots.

- **Disjunctive syllogism**

  If we have two statements $p \vee q$, and we know that one of them is false (ie. either p is false or q

is false), we can conclude the other option is true.

$$\frac{\substack{p \vee q \\ \neg p}}{\therefore q}$$

$$\frac{\substack{p \vee q \\ \neg q}}{\therefore p}$$

Example:

Let p represent "It's raining", and q represent "The ground is wet".

Premise 1 ($p \vee q$): If it's raining, then the ground is wet.

Premise 2 ($\neg p$): If the ground is wet, then people wear boots.

Conclusion ($\therefore q$): Therefore, if it's raining, then people wear boots.

- **Addition**

If we have a statement p, we can add any other statement to it to form a new statement $p \vee q$

$$\frac{p}{\therefore p \vee q}$$

Example:

Let p represent "Tommorow will be rainy", and q represent "Tommorow will be sunny".

Premise 1 (p): Tommorow will be rainy.

Conclusion ($\therefore q$): Therefore, tommorow will be rainy or tommorow will be sunny.

- **Simplification**

If we have a statement $p \wedge q$, we can conclude either p or q separately

$$\frac{p \wedge q}{\substack{\therefore p \\ \therefore q}}$$

Example:

Let p represent "Tommorow will be sunny", and q represent "Tommorow will be hot".

Premise 1 (p): Tommorow will be sunny and hot.

Conclusion ($\therefore p, \therefore q$): Therefore, tommorow will be sunny (p) and tommorow will be hot (q)

- **Conjunction**

If we have two statements p and q, and they are both true, we can conclude $p \wedge q$.

$$\frac{\substack{p \\ q}}{\therefore p \wedge q}$$

Example:

Let p represent "Tommorow will be sunny", and q represent "Tommorow will be hot".
Premise 1 (p): Tommorow will be sunny.
Premise 2 (q): Tommorow will be hot.
Conclusion ($\therefore p \wedge q$): Therefore, tommorow will be sunny and hot.

- **Resolution**

Two statements that contradict one another cannot be both true at the same time. In such cases, you can use the resolution rule of inference to see if these statements can be combined to form a new statement.

$$\frac{\substack{p \vee q \\ \neg p \vee r}}{\therefore q \vee r}$$

Example:

Let p represent "It's sunny", q represent "Johan is playing volleyball", and r represent "Tenma is jogging."
Premise 1 ($p \vee q$): It's sunny or Johan is playing volleyball.
Premise 2 ($\neg p \vee r$): It's not sunny or Tenma is jogging.
Resolution ($\therefore q \vee r$): Therefore, Johan is playing volleyball or Tenma is jogging.

Explanation:

SUPPOSE IT'S SUNNY: If we suppose that it's sunny, premise 1 ($p \vee q$) must be true regardless of the truth value of q. That said, because we have $\neg p$ in premise 2 which contradicts the fact that it's sunny, r must be true in order for premise 2 ($\neg p \vee r$) to be true. Now, since both premises ($p \vee q$ and $\neg p \vee r$) are true, the resolution $q \vee r$ must also be true.

SUPPOSE IT'S NOT SUNNY: If we suppose that it's sunny, q must be true in order for premise 1 ($p \vee q$) to be true, because p contradicts the fact that it isn't sunny. That said, premise 2 ($\neg p \vee r$) must be true since $\neg p$ is true. Now, if both premises ($p \vee q$ and $\neg p \vee r$) are both true, then the resolution $q \vee r$ must also be true.

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2 Rules of inference for quantified statements

TABLE 2 **Rules of Inference for Quantified Statements.**

Rule of Inference	***Name***
$\dfrac{\forall x\, P(x)}{\therefore\; P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\; \forall x\, P(x)}$	Universal generalization
$\dfrac{\exists x\, P(x)}{\therefore\; P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\; \exists x\, P(x)}$	Existential generalization

Figure 2: Summary of rules of inference for quantified statements
Source: "Discrete Mathematics and Its Applications" by Kenneth H. Rosen

- **Universal instantiation** ($\frac{\forall x P(x)}{\therefore P(c)}$) states that $P(c)$ is true where c is an element in the specified domain, given the premise that $\forall x P(x)$ is true.
- **Universal generalization** ($\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$) states that $\forall x P(x)$ is true, given the premise that $P(c)$ is true - this shows that if you take any arbitrary element c in the specified domain, $P(c)$ will always be true, therefore proving $\forall x P(x)$
- **Existential instantiation** ($\frac{\forall x P(x)}{\therefore P(c)}$) states that there is an element c in the domain for which $P(c)$ is true, given the premise that $\exists x P(x)$ is true.
- **Existential generalization** ($\frac{\forall x P(x)}{\therefore P(c)}$) states that $\exists x P(x)$ is true, given the premise that there really does exist an element c for which $P(c)$ is true.