MATH1061/1002 Cheatsheet (Linear Algebra)

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Abyan Majid

Source

1 Complex numbers

1.1 Imaginary unit

Imaginary unit
$i = \sqrt{-1}$

1.2 Cartesian, standard polar, exponential polar forms

Name	Form
Cartesian	$z = a + bi$
Standard Polar	$r(\cos\theta + i\sin\theta)$
Exponential Polar	$re^{i\theta}$

1.3 Complex conjugate

Complex conjugate
Complex conjugate of $z = a + bi$ is $\overline{z} = a - bi$

Complex conjugate properties	
Property	**Definition**
Conjugate of a Sum	$\overline{z+w} = \overline{z} + \overline{w}$
Conjugate of a Product	$\overline{zw} = \overline{z} \times \overline{w}$
Conjugate of a Power	$\overline{z^n} = (\overline{z})^n$

1.4 Modulus of a complex number

Modulus
The modulus of a complex number $z = a + bi$ is $\lvert z\rvert = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}}$

1.5 Principal argument

Principal argument
The principal argument of $z \in \mathbb{C}$, denoted Arg z, satisfies $-\pi <$ Arg $z \leq \pi$
So, to get principal argument, add or subtract multiples of 2π to/from θ.

Modulus properties	
Property	**Definition**
Multiplicative Property of Moduli	$\lvert zw\rvert = \lvert z\rvert\lvert w\rvert$
Division Property of Moduli	$\lvert\frac{z}{w}\rvert = \frac{\lvert z\rvert}{\lvert w\rvert}$
Triangle Inequality	$\lvert z+w\rvert \leq \lvert z\rvert + \lvert w\rvert$
Reverse Triangle Inequality	$\lvert z-w\rvert \geq \lvert z\rvert - \lvert w\rvert$

1.6 Arithmetic in cartesian form

Operation	**let** $z = a + bi$ **and** $w = c + di$
Addition	$z + w = (a + c) + (b + d)i$
Subtraction	$z - w = (a - c) + (b - d)i$
Multiplication	$z \times w = (a + bi)(c + di)$ $z \times w = ac + adi + bci + bdi^2$ $z \times w = (ac - bd) + (ad + bc)i$
Division	$z \div w = \frac{z}{w} \times \frac{\overline{w}}{\overline{w}}$ $z \div w = \frac{a + bi}{c + di} \times \frac{c - di}{c - di}$

1.7 Arithmetic in polar forms

let $z = re^{i\theta}$ **and** $w = se^{i\phi}$		
Operation	**Exponential polar**	**Standard polar**
Multiplication	$zw = rse^{i(\theta+\phi)}$	$zw = rs(\cos(\theta + \phi) + i\sin(\theta + \phi))$
Division	$\frac{z}{w} = (\frac{r}{s})e^{i(\theta-\phi)}$	$\frac{z}{w} = \frac{r}{s}(\cos(\theta - \phi) + i\sin(\theta - \phi))$
Power	$z^n = r^n e^{in\theta}$ where $n \in \mathbb{Z}$	$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$

1.8 Equality of complex numbers

Equality in Cartesian form
$(a + bi) = (c + di)$ if and only if $a = c$, $b = d$

Equality in polar forms
$re^{i\theta} = se^{i\phi}$ if and only if $r = s$ and $\theta = \phi + 2k\pi$ for $k \in \mathbb{Z}$

1.9 Roots of complex numbers

When asked to find the roots of $z^n = \alpha$
$z = r^{\frac{1}{n}} e^{i(\frac{\theta+2k\pi}{n})}$ satisfy $z^n = \alpha$ for each $k = 0, 1, 2, ..., n - 1$.

1.10 Complex exponential function

Complex exponential function
For $z = a + bi$, we define $e^z = e^a e^{bi}$, which means $e^z = e^a(\cos b + i\sin b)$ from which we know $\lvert e^z \rvert = e^a$ and $arg(e^z) = b$.

2 Vectors

2.1 Vector algebra

Let $\mathbf{u} = [u_1, ..., u_n], \mathbf{v} = [v_1, ..., v_n] \in \mathbb{R}^n, c \in \mathbb{R}$			
Operation	**What to do**	**Operation**	**What to do**
Addition	$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$	Scalar multiplication	$c\mathbf{u} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}$
Subtraction	$\mathbf{u} - \mathbf{v} = \begin{bmatrix} u_1 - v_1 \\ \vdots \\ u_n - v_n \end{bmatrix}$	Negation	$-\mathbf{u} = \begin{bmatrix} -u_1 \\ \vdots \\ -u_n \end{bmatrix}$

2.2 Magnitude of a vector

Magnitude of a vector
The magnitude (length) of a vector $\mathbf{u}$ is given by $\vert\vert\mathbf{u}\vert\vert = \sqrt{u_1^2 + u_2^2 + ... + u_n^2}$ Also note that $\vert\vert c\mathbf{u}\vert\vert = \vert c\vert\vert\vert\mathbf{u}\vert\vert$

2.3 Useful theorems for vector algebra

Let $\mathbf{u} = [u_1, ..., u_n], \mathbf{v} = [v_1, ..., v_n] \in \mathbb{R}^n, c, d \in \mathbb{R}$ and $\mathbf{o}$ be the zero vector	
No	**Theorem**
1	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2	$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3	$\mathbf{u} + \mathbf{o} = \mathbf{u}$
4	$\mathbf{u} + (-\mathbf{u}) = \mathbf{o}$
5	$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6	$\mathbf{u}(c + d) = c\mathbf{u} + d\mathbf{u}$
7	$c(d\mathbf{u}) = (cd)\mathbf{u}$
8	$1\mathbf{u} = \mathbf{u}$

2.4 Vector space

Vector Space

A *vector space* is a set V equipped with:

- **Addition:** For any $\mathbf{u}, \mathbf{v} \in V$, there exists $\mathbf{u} + \mathbf{v} \in V$.
- **Scalar multiplication:** For any $\mathbf{u} \in V$ and any scalar $c \in \mathbb{R}$, there exists $c\mathbf{u} \in V$.

These operations must satisfy properties (1)-(8) defined in Section 2.3.
Examples: $\mathbb{R}^n$ and $\mathbb{C}$ are both examples of vector spaces.

2.5 Unit vector

Unit vector
A "unit vector" is a vector of length 1. If $\mathbf{u} \in \mathbb{R}^n, \|\|\mathbf{u}\|\| \neq 0$ then $\frac{1}{\|\|\mathbf{u}\|\|}\mathbf{u}$ is a unit vector.

2.6 Linear combination

Linear combination
A "linear combination" of $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k} \in \mathbb{R}^n$ is a vector of the form $C_1\mathbf{v_1} + C_2\mathbf{v_2} + ... + C_k\mathbf{v_k}$. **Example:** $[2, 2]$ is a linear combination of $[-1, 2], [0, 6]$ because $[2, 2] = (-2)[-1, 2] + (1)[0.6]$

2.7 Parallel vectors

Parallel vectors
Vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u} = c\mathbf{v}$ or $\mathbf{v} = c\mathbf{u}$ for $c \in \mathbb{R}$.

2.8 Standard basis vectors

Standard basis vectors
$\mathbf{e_1} = [1, 0, 0], \mathbf{e_2} = [0, 1, 0], \mathbf{e_3} = [0, 0, 1]$ are the standard basis vectors in $\mathbf{R}^3$. Similarly, $\mathbf{e_1} = [1, 0, ..., 0], \mathbf{e_2} = [0, 1, 0, ..., 0], ..., \mathbf{e_n} = [0, 0, ..., 0, 1] \in \mathbb{R}^n$ are the standard basis vectors in $\mathbb{R}^n$

2.9 Dot product

Dot product
For $\mathbf{u} = [u_1, ..., u_n], \mathbf{v} = [v_1, ..., v_n] \in \mathbb{R}^n$, we define the dot product $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + ... + u_nv_n \in \mathbb{R}$

2.10 Theorems for dot products

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n, c \in \mathbb{R}$	
No	**Theorem**
1	$\mathbf{u} \cdot \mathbf{v}$
2	$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
3	$(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
4	$\mathbf{u} \cdot \mathbf{u} \geq 0$
5	$\|\|\mathbf{u}\|\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$
6	$\|\mathbf{u} \cdot \mathbf{v}\| \leq \|\|\mathbf{u}\|\|\|\|\mathbf{v}\|\|$ (Cauchy-Schwarz Inequality)
7	$\|\|\mathbf{u} + \mathbf{v}\|\| \leq \|\|\mathbf{u}\|\| + \|\|\mathbf{v}\|\|$ (Triangle Inequality)

2.11 Angle between vectors using dot product

Angle between vectors using dot product
We define angle between vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ to be a unique value $\theta \in [0, \pi]$ such that: $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\|\mathbf{u}\|\|\|\|\mathbf{v}\|\|}$ which means we can also find $\mathbf{u} \cdot \mathbf{v}$ if we know the the magnitude of u and v, and the angle in between: $\mathbf{u} \cdot \mathbf{v} = \|\|\mathbf{u}\|\|\|\|\mathbf{v}\|\| \cos\theta$

2.12 Orthogonal vectors

Orthogonal vectors
Vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are orthogonal iff $\mathbf{u} \cdot \mathbf{v} = 0$ Note: Orthogonal vectors are perpendicular $\mathbf{u} \cdot \mathbf{v} = 0 \implies \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\|\mathbf{u}\|\|\|\|\mathbf{v}\|\|} = 0$ $\theta = \cos^{-1}(0) = \frac{\pi}{2}$, i.e. perpendicular.

2.13 Projections

Projections
The projection of $\mathbf{u}$ onto $\mathbf{v}$ is defined by $\text{proj}_{\mathbf{v}}(\mathbf{u}) = (\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}})\mathbf{v}$

2.14 Cross products

Cross products
Cross product of $\mathbf{u} = [u_1, u_2, u_3], \mathbf{v} = [v_1, v_2, v_3] \in \mathbb{R}^3$ is the vector $\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$

2.15 Properties for cross products

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3, c \in \mathbb{R}$	
No	**Property**
1	$\mathbf{u} \times \mathbf{v} \neq \mathbf{v} \times \mathbf{u}$ (not commutative)
2	$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ (not associative)
3	$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ (anti-commutative)
4	$\mathbf{u} \times \mathbf{u} = 0$
5	$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0 = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$ i.e. given 2 vectors, their cross product is orthogonal to both vectors.
6	$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
7	$c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$

2.16 Area inscribed by vectors in $\mathbb{R}^3$

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$	
Shape	**Area**
Parallelogram	The area of the parallelogram inscribed by $\mathbf{u}, \mathbf{v}$ is $\lvert\lvert \mathbf{u} \times \mathbf{v} \rvert\rvert$
Triangle	The area of the triangle inscribed by $\mathbf{u}, \mathbf{v}$ is $\frac{1}{2}\lvert\lvert \mathbf{u} \times \mathbf{v} \rvert\rvert$

2.17 Angle between vectors using cross product

Angle between vectors using cross product
Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and θ the angle between them. Then, $\lvert\lvert \mathbf{u} \times \mathbf{v} \rvert\rvert = \lvert\lvert \mathbf{u} \rvert\rvert \lvert\lvert \mathbf{v} \rvert\rvert \sin\theta$ which means $\theta = \sin^{-1} \frac{\lvert\lvert \mathbf{u} \times \mathbf{v} \rvert\rvert}{\lvert\lvert \mathbf{u} \rvert\rvert \lvert\lvert \mathbf{v} \rvert\rvert}$

2.18 Forms of lines in $\mathbb{R}^2$

Let $\mathbf{p}$: position vector pointing to the line $\mathbf{n}$: normal vector (vector orthogonal to the line) $\mathbf{d}$: direction vector (vector parallel to the line)			
Name	**Form**	**How to obtain**	**Survives $\mathbb{R}^n$**
Normal form	$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$	1) Find $\mathbf{p}, \mathbf{n} \in \mathbb{R}^2$ 2) Write $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$	No
General form	$ax + by = c$	1) Find the normal form 2) Let $\mathbf{x} = [x, y], \mathbf{n} = [a, b]$, and write $c = \mathbf{n} \cdot \mathbf{p}$ 3) Simplify as follows $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ $\implies [a, b] \cdot ([x, y] - \mathbf{p}) = 0$ $\implies [a, b] \cdot [x, y] - c = 0$ $\implies ax + by = c$	No
Vector form	$\mathbf{x} = \mathbf{p} + t\mathbf{d}, t \in \mathbb{R}$	1) Find $\mathbf{p}, \mathbf{d} \in \mathbb{R}^2$ 2) Write $\mathbf{x} = \mathbf{p} + t\mathbf{d}, t \in \mathbb{R}$	Yes
Parametric equations	$x = p_1 + td_1$ $y = p_2 + td_2$ for $t \in \mathbb{R}$	1) Find the vector form 2) Write equations using each corresponding components of $\mathbf{x}, \mathbf{p}$ and $\mathbf{d}$	Yes

2.19 Skewness of lines

Skewness of lines
Two lines are "skew" if they are NOT parallel and do NOT intersect. (applicable only to $\mathbb{R}^3$ and above since all non-parallel lines in $\mathbb{R}^2$ has to intersect)

2.20 Forms of planes in $\mathbb{R}^3$

<table>
<tr><td colspan="4">Let
$\mathbf{p}$: position vector pointing to the line
$\mathbf{n}$: normal vector (vector orthogonal to the line)
$\mathbf{u}, \mathbf{v}$: direction vector (vector parallel to the line)</td></tr>
<tr><th>Name</th><th>Form</th><th>How to obtain</th><th>Survives $\mathbb{R}^n$</th></tr>
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2) Write $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$</td><td>No</td></tr>
<tr><td>General form</td><td>$ax + by + cz = d$</td><td>1) Find the normal form
2) Let $\mathbf{x} = [x, y, z], \mathbf{n} = [a, b, c]$, and write $d = \mathbf{n} \cdot \mathbf{p}$
3) Simplify as follows
$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p} = 0)$
$\implies [a, b, c] \cdot ([x, y, z] - \mathbf{p}) = 0$
$\implies ax + by + cz = d$</td><td>No</td></tr>
<tr><td>Vector form</td><td>$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$
for $s, t \in \mathbb{R}$</td><td>1) Find $\mathbf{p}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$
2) Write $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$, for $s, t \in \mathbb{R}$</td><td>Yes</td></tr>
<tr><td>Parametric equations</td><td>$x = p_1 + su_1 + tv_1$
$y = p_2 + su_2 + tv_2$
$z = p_3 + su_3 + tv_3$
for $s, t \in \mathbb{R}$</td><td>1) Find the vector form
2) Write equations using each corresponding components of $\mathbf{x}, \mathbf{p}, \mathbf{u}$ and $\mathbf{v}$</td><td>Yes</td></tr>
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