Derivations of a working example based on Di Tella (2017)

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This document presents a second working example for the Matlab toolbox from d'Avernas et al. (2020)[1]. We derive the model and relate the different expressions to the equations needed to solve the model numerically using the toolbox.
In this example, we present the model derived in Di Tella (2017) where two agents have Epstein and Zin (1989) utility functions and agents face a time-varying, idiosyncratic volatility.

Technology Experts can use capital to produce consumption goods with this production technology:

$$y_t^i = \left[a - \iota(g_t^i)\right] k_t^i$$

where g^i is the growth rate of the capital stock of agent i. In order to reach such as growth rate, agent i must invest $\iota(g_t^i)$ consumption goods. Agent i's capital stock follows an Ito process:

$$\frac{dk_t^i}{k_t^i} = g_t^i dt + \sigma dZ_t + v_t dW_t^i$$

where Z is an aggregate Brownian motion, and W is an idiosyncratic Brownian motion of expert i. The exposure to the aggregate shock σ is assumed to be constant, while the load on the idiosyncratic shock is time-varying and follows an exogenous stochastic process of the form:

$$dv_t = \lambda(\bar{v} - v_t)dt + \sigma^v \sqrt{v_t} dZ_t$$

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[1]Available for download under https://github.com/adavernas/toolbox

The idiosyncratic shocks cancel out in aggregate. The law of motion of aggregate capital $K_t = \int_{[0,1]} k_t^i di$ is given by:

$$dK_t = \left(\int_{[0,1]} g_t^i k_t^i di \right) dt + \sigma K_t dZ_t$$

Preferences There are two types of agents: households $h \in H$ and experts $i \in I$. Both agents have stochastic differential utility, as developed by Duffie and Epstein (1992). The utility of agent j over his consumption process c_t^j is defined as

$$U_t^j = \mathbb{E}_t \left(\int_t^\infty f\left(c_s^j, U_s^j\right) ds \right).$$

The function $f_j(c, u)$ is a normalized aggregator of consumption and continuation value in each period defined as

$$f(c, U) = \frac{1-\gamma}{1-1/\zeta} U \left[\left(\frac{c}{((1-\gamma)U)^{1/(1-\gamma)}} \right)^{1-1/\zeta} - \rho \right]$$

where ρ is the rate of time preference, γ is the coefficient of relative risk aversion, and ζ determines the elasticity of intertemporal substitution.

Markets Experts can trade capital at a competitive market price q_t, which has the following law of motion:

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t$$

Financial markets are complete with a stochastic discount factor η_t that follows the process:

$$\frac{d\eta_t}{\eta_t} = -r_t dt - \pi_t dZ_t$$

where r_t is the risk-free rate and π_t is the price of **aggregate** risk.

Households Households cannot hold capital of a single firm, but invest in the complete financial market. They choose their optimal consumption c_t^j and exposure to aggregate risk σ^w_t in order to maximize discounted infinite life time expected utility U^j_t. Taking the stochastic discount factor η_t as given, at any time, they solve the following maximization problem:

$$\max_{c^h \geq 0, \sigma^w} U(c^h)$$
$$\text{subject to } \frac{dn^h_t}{n^h_t} = (r_t + \sigma^w_t \pi_t - \mathfrak{c}^h_t)dt + \sigma^w_t dZ_t$$

where n^h_t is the wealth of a household, $\mathfrak{c}^h_t = c^h_t / n^h_t$ her consumption rate, and Z_t is a standard Brownian motion that reflects aggregate risk.

Expert's problem Experts can trade capital and use capital for production, as well as participate in the aggregate financial market. The cumulative return from investing in capital for an expert i is given by $R^{k,i}_t$ that follows:

$$dR^{k,i}_t = \underbrace{\left[\frac{a - \iota(g^i_t)}{q_t} + g^i_t + \mu^q_t + \sigma\sigma^q_t\right]}_{\mathbb{E}_t[dR^{k,i}_t]} dt + (\sigma + \sigma^q_t)dZ_t + v_t dW^i_t$$

Thus, an expert chooses her consumption and trading strategies to maximize her lifetime expected utility

$$\max_{c^i \geq 0, g, k \geq 0, \theta} U(c^i)$$
$$\text{subject to } \frac{dn^i_t}{n^i_t} = (\mu^{n,i}_t - \mathfrak{c}^i_t)dt + \sigma^{n,i}_t dZ_t + \tilde{\sigma}^{n,i}_t dW^i_t$$

where

$$\mu^{n,i}_t = r_t + q_t \hat{k}^i_t (\mathbb{E}_t[dR^{k,i}_t] - r_t) - (1-\phi) q_t \hat{k}^i_t (\sigma + \sigma^q_t)\pi_t + \theta^i_t \pi_t$$
$$\sigma^{n,i}_t = \phi q_t \hat{k}^i_t (\sigma + \sigma^q_t) + \theta^i_t$$
$$\tilde{\sigma}^{n,i}_t = \phi q_t \hat{k}^i_t v_t$$

and $\hat{k}_t^i$ is k_t^i normalized by wealth n_t^i.

Solving the HJB We will guess and verify that the homotheticity of preferences allows us to write the value function for agents of type j as:

$$U\left(n_t^j, \xi_t^j\right) = \frac{\left(n_t^j \xi_t^j\right)^{1-\gamma}}{1-\gamma},$$

where variable ξ_t^j follows

$$\frac{d\xi_t^j}{\xi_t^j} = \mu_t^{\xi,j} dt + \sigma_t^{\xi,j} dZ_t$$

Using the guess of the value function, the Hamilton-Jacobi-Bellman (HJB) equation for experts i is equal to

$$\begin{aligned} 0 = \max_{\mathfrak{c}_t^i, g_t^i, \hat{k}_t^i, \theta_t^i} \Bigg\{ & \left(\frac{\rho}{1-1/\zeta}\right)\left[\left(\frac{\mathfrak{c}_t^i}{\xi_t^i}\right)^{1-1/\zeta} - 1\right] \\ & + \mu_t^{\xi,i} + \mu_t^{n,i} - \mathfrak{c}_t^i - \frac{\gamma}{2}(\sigma_t^{n,i})^2 - \frac{\gamma}{2}(\sigma_t^{\xi,i})^2 \\ & + (1-\gamma)\sigma_t^{\xi,i}\sigma_t^{n,i} - \frac{\gamma}{2}(\tilde{\sigma}_t^i)^2 \Bigg\} \end{aligned}$$

while households face the following HJB:

$$\begin{aligned} 0 = \max_{\mathfrak{c}_t^h, \sigma_t^w} \Bigg\{ & \left(\frac{\rho}{1-1/\zeta}\right)\left[\left(\frac{\mathfrak{c}_t^h}{\xi_t^h}\right)^{1-1/\zeta} - 1\right] \\ & + \mu_t^{\xi,h} + \mu_t^{n,h} - \mathfrak{c}_t^h - \frac{\gamma}{2}(\sigma_t^{n,h})^2 - \frac{\gamma}{2}(\sigma_t^{\xi,h})^2 \\ & + (1-\gamma)\sigma_t^{\xi,h}\sigma_t^{n,h} \Bigg\} \end{aligned}$$

Optimality Conditions Defining $\psi = 1/\zeta$, the first order conditions with respect to consumption give:

$$\mathfrak{c}_t^i = \rho^{1/\psi}(\xi_t^i)^{(\psi-1)/\psi}$$
$$\mathfrak{c}_t^h = \rho^{1/\psi}(\xi_t^h)^{(\psi-1)/\psi}$$

The first order condition for g is equal to

$$\iota'(g_t^i) = q_t$$

From the first order condition for θ, we get

$$\sigma_t^{n,i} = \frac{\pi_t}{\gamma} - \frac{\gamma-1}{\gamma}\sigma_t^{\xi,i}$$

while the first order condition for volatility from the households' maximization problem yields

$$\sigma_t^{n,h} = \frac{\pi_t}{\gamma} - \frac{\gamma-1}{\gamma}\sigma_t^{\xi,h}$$

Market Clearing Conditions We start by providing the definition of such an equilibrium in the state variables $\{x_t, v_t\}$, where x_t is defined as the share of wealth in the hands of the intermediaries:

$$x_t = \frac{n_t^i}{n_t^h + n_t^i} = \frac{n_t^i}{q_t k_t}.$$

Then, we can use the market clearing condition for consumption to find q_t. Market clearing for consumption dictates that consumption from both types of agents equals the surplus from the production technology. So we have:

$$\left(\mathfrak{c}_t^i x_t + \mathfrak{c}^h(1-x_t)\right) q_t = a - \iota(g_t^i)$$

The market for capital clears:

$$q_t k_t^i x_t = 1$$

And financial markets clear:

$$\sigma_t^{n,i} x_t + \sigma_t^{n,h}(1 - x_t) = \sigma + \sigma_t^q$$

Further, using our definition of x_t and Ito's lemma, we can derive the law of motion of x_t as:

$$\begin{aligned}\frac{dx_t}{x_t} =& \Big(\mu_t^{ni} - \mathfrak{c}_t^i - \mu_t^q - g_t^i - \sigma_t^q \sigma + (\sigma + \sigma_t^q)^2 - \sigma_t^{ni}(\sigma + \sigma_t^q)\Big) dt \\ &+ \big(\sigma_t^{ni} + \sigma + \sigma_t^q\big) dZ_t\end{aligned}$$

Linking the model to the code In this paragraph, we explain how we derive the expressions for the different endogenous and secondary variables used in the code. Expressions are referenced by their number in the `mod_DiTella.m` file. We start by collecting parameters and variables. The model parameters are reported in table 1. The parameters have to be specified (as well as given a value) in the section *Parameters* in the file `par_DiTella.m` of the toolbox. Model specific variables are shown in Table 2. Endogenous variables are specified in the array `vars` in section *Variables* in the file `mod_DiTella.m` of the toolbox, while secondary variables are listed in `vars_` of the same script. The two state variables are

$$x_t = \texttt{e} \tag{1}$$

$$v_t = \texttt{z} \tag{2}$$

In the code, the wealth multipliers are

$$\xi_t^i = \texttt{vi} \tag{3}$$

$$\xi_t^h = \texttt{vh} \tag{4}$$

Table 1: Model parameters

Parameter	Definition
γ	relative risk aversion
ζ	intertemporal elasticity of substitution (IES)
ψ	inverse IES
ρ	discount rate
δ	depreciation
a	productivity
κ_p	investment costs
$\bar{v}$	average volatility
σ	exposure of capital to aggregate risk
ϕ	expert exposure to own return
λ	mean reversion parameter
A	calibration parameter
B	calibration parameter

Table 2: Variables

Variables	Definition
Endogenous	q_t, μ_t^x, σ_t^{nh}, σ_t^x, θ_t
Secondary	$\mathfrak{c}_t^i$, $\mathfrak{c}_t^h$, x, v, $\tilde{\sigma}_t^z$, μ_t^q, r_t, μ_t^{ni},μ_t^{nh}, σ_t^{ni}, $\sigma_t^{\xi i}$, $\sigma_t^{\xi h}$, g_t^i, μ_t^v, π_t, ι^i, σ_t^q

Consumption-to-wealth ratio is given by the first order condition:

$$c_t^i = \rho^{1/\psi}(\xi_t^i)^{\frac{\psi-1}{\psi}} \tag{5}$$
$$c_t^h = \rho^{1/\psi}(\xi_t^h)^{\frac{\psi-1}{\psi}} \tag{6}$$

From the market clearing conditions, we have:

$$k = \frac{1}{q_t x_t} \tag{7}$$

The investment maximization problem yields

$$g_t^i = \frac{1}{2A}(q_t - B) - \delta \tag{8}$$

and ι has the functional form:

$$\iota_t^i = A(g_t^i + \delta)^2 + B(g_t^i + \delta) \tag{9}$$

where A and B are calibrated to achieve GDP growth of 2% per year. By assumption, we have:

$$\mu_t^v = \kappa_z(\bar{v} - v) \tag{10}$$

The drift of the state variable σ_t was assumed to be

$$\tilde{\sigma}_t^v = \sigma_t^v \sqrt{v_t} \tag{11}$$

Using Ito's lemma, we derive

$$\sigma_t^{\xi,i} = \frac{\xi_x^i}{\xi^i}\sigma^x x + \frac{\xi_v^i}{\xi^i}\tilde{\sigma}_t^v \tag{12}$$
$$\sigma_t^{\xi,h} = \frac{\xi_x^h}{\xi^h}\sigma^x x + \frac{\xi_v^h}{\xi^h}\tilde{\sigma}_t^v \tag{13}$$

Similarly, we have

$$\sigma_t^q = \frac{q_x}{q_t}\sigma_t^x x_t + \frac{q_v}{q_t}\tilde{\sigma}_t^v \tag{14}$$

$$\begin{aligned} \mu_t^q =& \frac{q_x}{q_t}\mu_t^x + \frac{q_v}{q_t}\mu_t^v v_t \\ &+ \frac{1}{2}\frac{q_{xx}}{q_t}(\sigma_t^x)^2 \\ &+ \frac{1}{2}\frac{q_{vv}}{q_t}(\tilde{\sigma}_t^v)^2 v_t \\ &+ \frac{q_{xv}}{q_t}\sigma_t^x\tilde{\sigma}_t^v\sqrt{v_t} \end{aligned} \tag{15}$$

Next, from budget constraint of the experts we define

$$\sigma_t^{n,i} = \phi q_t k_t(\sigma + \sigma_t^q); \tag{16}$$

And using Ito's lemma and law of motion of the Stochastic Discount Factor, we find

$$\pi_t = \gamma\sigma_t^{n,h} + (\gamma - 1)\sigma_t^{\xi,h} \tag{17}$$

After some algebra using the experts' FOCs, we get an expression for the risk-free rate:

$$\begin{aligned} r_t =& (a - \iota_t^i)/q_t + g_t^i + \mu_t^q + \sigma\sigma_t^q - (1-\phi)(\sigma + \sigma_t^q)\pi_t \\ &- \gamma(\sigma + \sigma_t^q)(\phi q_t k_t(\sigma + \sigma_t^q) + \theta_t) \\ &+ (1-\gamma)\phi(\sigma + \sigma_t^q)\sigma_t^{\xi,i} \\ &- \gamma q_t k_t(\phi v_t)^2 \end{aligned} \tag{18}$$

And from the two budget constraints, we have:

$$\mu_t^{n,i} = r_t + \gamma(\phi v)^2/x^2 + \pi_t\sigma_t^{n,i} \tag{19}$$

$$\mu_t^{n,h} = r_t + \pi_t\sigma_t^{n,h} \tag{20}$$

The model is closed be defining the endogenous variables:

$$\texttt{eqmux} = (\mu_t^{n,i} - c_t^i - \mu_t^q - g_t^i - \sigma_t^q \sigma + (\sigma + \sigma_t^q)^2 - \sigma_t^{n,i}(\sigma + \sigma_t^q)x_t - \mu_t^v \tag{21}$$

$$\texttt{eqq} = (c_t^i x_t + c_t^h(1 - x_t))q_t - (a - g_t^i) \tag{22}$$

$$\texttt{eqsignh} = \sigma_t^{n,i} x_t + \sigma_t^{n,h}(1 - x_t) - (\sigma + \sigma_t^q) \tag{23}$$

$$\texttt{eqsigx} = (\sigma_t^{n,i} + \sigma + \sigma_t^q)x_t - \sigma_t^x \tag{24}$$

$$\texttt{eqtheta} = \pi_t + (1 - \gamma)\sigma_t^{\xi,i} - \gamma\phi q_t k_t(\sigma + \sigma_t^q) - \gamma\theta_t \tag{25}$$