Haniyeh Ehsani Oskouie / 306300374

No.

Date

Problem 1:

$G=(V,E) \rightarrow A_{ij}=1$ if $(i,j)\in E$ $\quad L=D-A$

$ncut(S)=\frac{cut(S)}{vol(S)}+\frac{cut(\bar{S})}{vol(\bar{S})}$ $\qquad x_i=\begin{cases}\sqrt{\frac{vol(\bar{S})}{vol(S)}}, & i\in S\\ -\sqrt{\frac{vol(S)}{vol(\bar{S})}}, & i\in \bar{S}\end{cases}$

1. $L=\sum_{(i,j)\in E}(\vec{1}_i-\vec{1}_j)(\vec{1}_i-\vec{1}_j)^T$

$=\sum_{(i,j)\in E}\vec{1}_i\vec{1}_i^T-\vec{1}_j\vec{1}_i^T-\vec{1}_i\vec{1}_j^T+\vec{1}_j\vec{1}_j^T$

$=\sum_{(i,j)\in E}\vec{1}_i\vec{1}_i^T+\vec{1}_j\vec{1}_j^T-\sum_{(i,j)\in E}\vec{1}_i\vec{1}_j^T+\vec{1}_j\vec{1}_i^T=D-A=L$

$1_i1_i^T=\begin{bmatrix}0\\ \vdots\\ 1\\ \vdots \\ 0\end{bmatrix}\rightarrow i\ [0\ \dots\ 1\ \dots\ 0]=1$ $\qquad D_{ii}=\sum_j A_{ij}=d_i$

2. $x^TLx \overset{?}{=} \sum_{(i,j)\in E}(x_i-x_j)^2$

$x^TLx=x^T(D-A)x=x^TDx-x^TAx=[x_1\dots x_n]\begin{bmatrix}d_1 & & 0\\ & \ddots & \\ 0 & & d_n\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$

$-[x_1\dots x_n]\begin{bmatrix}A_{11}&\dots&A_{1n}\\ \vdots&\ddots&\vdots\\ A_{n1}&\dots&A_{nn}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\sum_i x_i^2d_i-\sum_i\sum_j x_ix_jA_{ij}$

$=\sum_i\sum_j x_i^2A_{ij}-\sum_i\sum_j x_ix_jA_{ij}$ ①

$\sum_{(i,j)\in E}(x_i-x_j)^2=\frac{1}{2}\sum_i\sum_j A_{ij}(x_i-x_j)^2=\frac{1}{2}\sum_i\sum_j A_{ij}x_i^2+\frac{1}{2}\sum_i\sum_j A_{ij}x_j^2$

$-\frac{1}{2}\sum_i\sum_j 2A_{ij}x_ix_j=\sum_i\underbrace{\left(\sum_j A_{ij}\right)}_{d_i}x_i^2-\sum_i\sum_j x_ix_jA_{ij}=\sum_i x_i^2d_i-\sum_i\sum_j x_ix_jA_{ij}$ ②

① = ② $\square$

KOKUYO LOOSE-LEAF ノ-836B 6 mm ruled×36 lines

No.

Date

3. $x^T L x \overset{?}{=} c \cdot ncut(S)$

$$x^T L x = \frac{1}{2} \sum_i \sum_j A_{ij} (x_i - x_j)^2 \rightarrow \text{from } ②$$

$$= \sum_{\substack{i \in S \\ j \in \bar{S}}} A_{ij} (x_i - x_j)^2$$ → because we count x_i and x_j only once in this case

$$= \sum_{\substack{i \in S \\ j \in \bar{S}}} A_{ij} \left(\sqrt{\frac{vol(\bar{S})}{vol(S)}} + \sqrt{\frac{vol(S)}{vol(\bar{S})}} \right)^2 = \sum_{\substack{i \in S \\ j \in \bar{S}}} A_{ij} \left(\frac{vol(\bar{S})}{vol(S)} + \frac{vol(S)}{vol(\bar{S})} + 2 \right)$$

$$= \sum_{\substack{i \in S \\ j \in \bar{S}}} A_{ij} \left(\frac{vol^2(\bar{S}) + vol^2(S) + 2 vol(S) vol(\bar{S})}{vol(S) vol(\bar{S})} \right) = \sum_{\substack{i \in S \\ j \in \bar{S}}} A_{ij} \frac{(vol(\bar{S}) + vol(S))^2}{vol(S) vol(\bar{S})}$$

$$= \underbrace{(vol(\bar{S}) + vol(S))}_{c} \underbrace{\underbrace{\left(\sum_{\substack{i \in S \\ j \in \bar{S}}} A_{ij} \right)}_{cut(S)} \left(\frac{1}{vol(S)} + \frac{1}{vol(\bar{S})} \right)}_{ncut(S)} = c \, ncut(S)$$

4. $x^T D \mathbb{1} \overset{?}{=} 0$

$$x^T D \mathbb{1} = [x_1 \cdots x_n] \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = x_1 d_1 + \cdots + x_n d_n$$

$$= \sum_{i \in S} x_i d_i + \sum_{i \in \bar{S}} x_i d_i$$

$$= \sum_{i \in S} \sqrt{\frac{vol(\bar{S})}{vol(S)}} d_i - \sum_{i \in \bar{S}} \sqrt{\frac{vol(S)}{vol(\bar{S})}} d_i = \sqrt{\frac{vol(\bar{S})}{vol(S)}} \underbrace{\left(\sum_{i \in S} d_i \right)}_{vol(S)} - \sqrt{\frac{vol(S)}{vol(\bar{S})}} \underbrace{\left(\sum_{i \in \bar{S}} d_i \right)}_{vol(\bar{S})}$$

$$= \sqrt{vol(\bar{S})} \sqrt{vol(S)} - \sqrt{vol(S)} \sqrt{vol(\bar{S})} = 0 \quad \square$$

No.

Date

5. $x^T D x \overset{?}{=} 2m$

$$x^T D x = \sum_i x_i^2 d_i = \frac{vol(\bar{S})}{vol(S)} \sum_{i \in S} d_i + \frac{vol(S)}{vol(\bar{S})} \sum_{i \in \bar{S}} d_i$$

$$= \frac{vol(\bar{S})}{vol(S)} vol(S) + \frac{vol(S)}{vol(\bar{S})} vol(\bar{S}) = vol(\bar{S}) + vol(S) = \sum_{i \in S} d_i + \sum_{i \in \bar{S}} d_i = \sum_i d_i = 2m$$

Problem 2:

$$\min_{x \in R^n} \frac{x^T L x}{x^T D x} \qquad z = D^{\frac{1}{2}} x \Rightarrow x = D^{-\frac{1}{2}} z$$

s.t. $x^T D 1 = 0$

$x^T D x = 2m$

$$\frac{x^T L x}{x^T D x} = \frac{D^{-\frac{1}{2}} z L D^{-\frac{1}{2}} z}{z^T D^{-\frac{1}{2}} D D^{-\frac{1}{2}} z} = \frac{z^T D^{-\frac{1}{2}} L D^{-\frac{1}{2}} z}{z^T z} = J(z)$$

$$\text{Lagrangian} \Rightarrow \frac{z^T D^{-\frac{1}{2}} L D^{-\frac{1}{2}} z}{z^T z} + \lambda_1 (x^T D 1) + \lambda_2 (x^T D x - 2m) = L(z)$$

$$L(z) = \frac{z^T D^{-\frac{1}{2}} L D^{-\frac{1}{2}} z}{2m} + \lambda_1 (z^T D^{-\frac{1}{2}} D 1) + \lambda_2 (z^T z - 2m)$$

$$\frac{\partial L(z)}{\partial z} = \frac{2 D^{-\frac{1}{2}} L D^{-\frac{1}{2}} z}{2m} + \lambda_1 D^{\frac{1}{2}} 1 + \lambda_2 (2z) = 0$$

$\times \tilde{L}$

$$\Rightarrow \frac{\tilde{L}(D^{-\frac{1}{2}} L D^{-\frac{1}{2}} z)}{m} + \underbrace{\lambda_1 \tilde{L} D^{\frac{1}{2}} 1}_{0 \text{ because } \tilde{L} \text{ is symmetric}} + 2\lambda_2 \tilde{L} z = 0$$

$\times \tilde{L}^{-1}$

$$\Rightarrow \tilde{L}^{-1} \tilde{L} (D^{-\frac{1}{2}} L D^{-\frac{1}{2}} z) + 2m\lambda_2 \tilde{L}^{-1} \tilde{L} z = 0$$

$$\Rightarrow D^{-\frac{1}{2}} L D^{-\frac{1}{2}} z + 2m\lambda_2 z = 0$$

$$\Rightarrow \underbrace{D^{-\frac{1}{2}} L D^{-\frac{1}{2}}}_{\tilde{L}} z = \underbrace{-2m\lambda_2}_{\text{eigen value}} z \qquad \Rightarrow x = D^{-\frac{1}{2}} z \text{ is our answ}$$

KOKUYO LOOSE-LEAF ノ-836BT 6 mm ruled × 36 lines