Introduction To Quantum Mechanics

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1 The Wave Function

What we are looking for is the **wave function $\mathbf{\Psi}$**.

Law 1.1 (Schrodinger Equation).

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi.$$

For simplicity, we always rewrite it as:

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}\partial_x^2\Psi + V\Psi.$$

Born's statistical interpretation:

$$\int_a^b |\Psi(x,t)|^2 \,\mathrm{d}x = \text{probability of finding the particle between } a \text{ and } b \text{ at time } t.$$

Law 1.2 (Normalization).

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \,\mathrm{d}x = 1.$$

Proposition 1.1. The wave function will always stay NORMALIZED.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \,\mathrm{d}x = 0.$$

Proof. By Schrodinger EQ.,

$$\mathrm{LHS} = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{+\infty}.$$

□

Definition 1.1.

$$\langle x\rangle \xlongequal{def} \int_{-\infty}^{\infty} x|\Psi|^2 \,\mathrm{d}x$$

and

$$\langle p\rangle \xlongequal{def} m\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t}.$$

Theorem 1.1.

$$\langle x\rangle = \int \Psi^*(x)\Psi \,\mathrm{d}x$$

and

$$\langle p\rangle = \int \Psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi \,\mathrm{d}x.$$

Remark 1.1 (Operator)**.** We say that the operator x represents position, and the operator $-i\hbar\,\partial/\partial x$ represents momentum. Also,

$$\langle Q(x,p)\rangle = \int_{-\infty}^{\infty} \Psi^*\left[Q(x,-i\hbar\frac{\partial}{\partial x})\right]\Psi \,\mathrm{d}x.$$

2 Time-independent Schrodinger Equation

2.1 Stationary states

We look for solutions that are simple products,

$$\Psi(x,t) = \psi(x)\varphi(t).$$

Theorem 2.1. By the method of separation of variables,

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + V\psi = E\psi$$

and

$$\varphi(t) = e^{-iEt/\hbar}.$$

The first is called the **time-independent Schrodinger equation**.

Definition 2.1 (Hamiltonian)**.** In classical mechanics, the total energy (kinetic plus potential) is called Hamiltonian:

$$H(x,p) = \frac{p^2}{2m} + V(x).$$

Now we introduce **Hamiltonian operator**:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x).$$

Thus the time-independent Schrodinger EQ. can be written

$$\hat{H}\psi = E\psi.$$

Remark 2.1. Intriguingly and intuitively,

$$\langle H\rangle = E.$$

Also, if the equation yields an infinite collection of solutions $(\psi_1(x), \psi_2(x), \cdots)$, each with its associated value of the separation constant $(E1, E2, \cdots)$; thus the wave function is:

$$\Psi(x,t) = \sum_{n=1}^{+\infty} c_n\psi_n(x)e^{-iE_nt/\hbar}.$$

2.2 The infinite square wall

Suppose

$$V(x) = \begin{cases} 0 & \text{if } 0 \le x \le a \\ \infty & \text{otherwise} \end{cases}.$$