Differential Geometry

Mathematical Tripos Part II

June 8, 2023

Note. Knowledge of what a diffeomorphism, homotopy, and isotopy are is assumed.

1 Differential Topology

1.1 Manifolds

Definition 1.1 (Manifold)**.** We say that $X \subseteq \mathbb{R}^n$ is a **k-dimensional manifold** if each $x \in X$ has a neighborhood $V \subseteq X$ diffeomorphic to an open set of $\mathbb{R}^k$.

Definition 1.2 (Parameterisation and Chart)**.** A diffeomorphism $\phi : U \to V$, where U is an open set of $\mathbb{R}^k$ is a **parameterisation** of the neighborhood V. The inverse diffeomorphism $\phi^{-1} : V \to U$ is called a **chart** on V.

If we have manifolds X and Z with $Z \subseteq X$ we say that Z is a **submanifold** of X. In this case, the **codimension** of Z in X is $\dim X - \dim Z$.

Definition 1.3 (Tangent Space of a Manifold)**.** Let $X \subseteq \mathbb{R}^n$ be a manifold, $x \in X$. Let $\phi : U \to X$ be a parameterisation with $\phi(0) = x$. The **tangent space** $T_x X$ is[1] $d\phi_0(\mathbb{R}^k)$.

Let $f : X \to Y$ be a smooth map between manifolds. We say that f is a **local diffeomorphism** at x if f maps a neighbourhood of x diffeomorphically onto a neighbourhood of $f(x)$.

Theorem 1.4 (Inverse Function Theorem)**.** *Suppose that $f : X \to Y$ is a smooth map whose derivative df_x at the point x is an isomorphism. Then f is a local diffeomorphism at x.*

1.2 Regular values and Sard's theorem

Let $f : X \to Y$ be a smooth map between manifolds. Let C be the set of all points $x \in X$ such that $df_x : T_x X \to T_{f(x)} Y$ is not surjective.

Definition 1.5. A point in C will be called a **critical point**. A point in $f(C)$ will be called a **critical value**. A point in the complement of $f(C)$ will be called a **regular value**.

Theorem 1.6 (Preimage Theorem)**.** *Let y be a regular value of $f : X \to Y$ with $\dim X \geq \dim Y$. Then the set $f^{-1}(y)$ is a submanifold of X with $\dim f^{-1}(y) = \dim X - \dim Y$.*

[1] Viewing ϕ as a function onto $\mathbb{R}^n$.

Proof. Let $x \in f^{-1}(y)$. Since y is a regular value, the derivative df_x maps T_xX onto T_yY. The kernel of df_x is a subspace K of T_xX of dimension $p := \dim X - \dim Y$. Suppose $X \subset \mathbb{R}^N$ and let $T : \mathbb{R}^N \to \mathbb{R}^p$ be any linear map such that $\mathrm{Ker}(T) \cap K = \{0\}$. Consider the map $F : X \to Y \times \mathbb{R}^p$ given by

$$F(z) = (f(z), T(z)).$$

The derivative of F is given by

$$dF_x(v) = (df_x(v), T(v))$$

which is clearly nonsingular by our choice of T. By the inverse function theorem, F is a local diffeomorphism at x, i.e. F maps some neighbourhood U of x diffeomorphically onto a neighbourhood V of $(y, T(x))$. Hence F maps $f^{-1}(y) \cap U$ diffeomorphically onto $(\{y\} \times \mathbb{R}^p) \cap V$ which proves that $f^{-1}(y)$ is a manifold with $\dim f^{-1}(y) = p$. $\square$

Theorem 1.7 (Stack of Records Theorem)**.** *Let $f : X \to Y$ be a smooth map between manifolds of the same dimension with X compact. Let y be a regular value of f and write $f^{-1}(y) = \{x_1, \ldots, x_k\}$. Then there exists a neighbourhood U of y in Y such that $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_k$, where V_i is an open neighbourhood of x_i and f maps each V_i diffeomorphically onto U.*

Proof. By the inverse function theorem we can pick disjoint neighbourhoods W_i of x_i such that f maps W_i diffeomorphically onto a neighbourhood of y. Observe that $f(X - \cup_i W_i)$ is a compact set which does not contain y. Now take $U = \bigcap_i f(W_i) - f(X - \cup_i W_i)$. $\square$

If we let $\# f^{-1}(y)$ be the cardinality of $f^{-1}(y)$, the theorem implies that the function $y \mapsto \# f^{-1}(y)$ is locally constant as y ranges over regular values of f.

Theorem 1.8 (Sard's Theorem)**.** *The set of critical values of a smooth map $f : X \to Y$ has measure zero.*

1.3 Transversality

Definition 1.9 (Transversal)**.** A smooth map $f : X \to Y$ is said to be **transversal** to a submanifold $Z \subset Y$ if for every $x \in f^{-1}(Z)$ we have

$$\mathrm{Image}\,(df_x) + T_{f(x)}Z = T_{f(x)}Y.$$

We write $f \pitchfork Z$.

Theorem 1.10 (Transversal Preimage Theorem)**.** *If the smooth map $f : X \to Y$ is transversal to a submanifold $Z \subset Y$, then $f^{-1}(Z)$ is submanifold of X. Moreover, $f^{-1}(Z)$ and Z have the same codimension.*

An important special case occurs when f is the inclusion of a submanifold X of Y and Z is another submanifold of Y. In this case the condition of transversality reduces to

$$T_x X + T_x Z = T_x Y$$

for every $x \in X \cap Z$. If this is the case, then $X \cap Z$ is a submanifold of codimension given by

$$\operatorname{codim}(X \cap Z) = \operatorname{codim} X + \operatorname{codim} Z.$$

1.4 Degree Modulo 2

Lemma 1.11 (Homotopy Lemma)**.** *Let $f, g : X \to Y$ be smooth maps which are smoothly homotopic. Suppose X is compact, has the same dimension as Y and $\partial X = \emptyset$. If y is a regular value for both f and g, then*

$$\# f^{-1}(y) = \# g^{-1}(y) \pmod 2.$$

Lemma 1.12 (Homogeneity Lemma)**.** *Let X be a smooth connected manifold, possibly with boundary. Let y and z be points in* $\operatorname{Int}(X)$*. Then there exists a diffeomorphism $h : X \to X$ smoothly isotopic to the identity such that $h(y) = z$.*

In what follows suppose that X is compact and without boundary and Y is connected and with the same dimension as X. Let $f : X \to Y$ be a smooth map.

Theorem 1.13 (Degree Mod 2)**.** *If y and z are regular values of f then*

$$\# f^{-1}(y) = \# f^{-1}(z) \pmod 2.$$

This common residue class is called the ***degree mod 2*** *of f,* $\deg_2 f$*, and only depends on the homotopy class of f.*

Corollary 1.14 (Smooth Brouwer Fixed Point Theorem)**.** *Any smooth map $f : B^k \to B^k$ has a fixed point.*

Proof. Suppose f has no fixed point. Define $g : B^k \to S^{k-1}$ so that $g(x)$ is the point where the line segment starting at $f(x)$ passing through x hits the boundary. This is obviously smooth, and restricts to the identity on S^{k-1}.

Now the identity map on a compact boundaryless manifold has $\deg_2 = 1$, and the constant map has $\deg_2 = 0$. So they are never homotopic. This implies that there is no smooth map $f : B^k \to S^{k-1}$ which restricts to the identity on S^{k-1}, as otherwise we could construct a homotopy $H : S^k \times [0,1] \to S^k$ between the constant map and the identity given by $H(x,t) = f(tx)$. So f must have a fixed point. $\square$

2 Length, Area and Curvature

2.1 Curves

Definition 2.1 (Curve)**.** Let $I \subset \mathbb{R}$ be an interval and let X be a manifold. A **curve** in X is a smooth map $\alpha : I \to X$. The curve is said to be **regular** if α is an immersion, i.e., if the velocity vector $\dot{\alpha}(t) = d\alpha_t(1) \in T_{\alpha(t)}X$ is never zero.

By definition, given $t \in I$, the arc-length of $\alpha : I \to \mathbb{R}^3$ from the point t_0 is given by

$$s(t) := \int_{t_0}^{t} |\dot{\alpha}(\tau)| d\tau.$$

If the interval I has endpoints a and $b, a < b$, the length of α is

$$\ell(\alpha) := \int_a^b |\dot{\alpha}(t)| dt.$$

The curve is said to be parametrized by arc-length if $|\dot{\alpha}(t)| = 1$ for all $t \in$ I. From now on we will assume that curves are parametrized by arc-length.

Definition 2.2. The **tangent** at $s \in I$ is $t(s) = \dot{\alpha}(s)$. The **curvature** of α at s is the number $k(s) = |\ddot{\alpha}(s)|$. The **normal vector** at s is $n(s)$, where $\ddot{\alpha}(s) = k(s)n(s)$. The **binormal vector** at s is $b(s) = t(s) \wedge n(s)$. We have $\dot{b}(s) = \tau(s)n(s)$, where $\tau(s)$ is the **torsion** at s.

Proposition 2.3 (Frenet Formulas)**.** *We have*

$$\dot{t} = kn, \quad \dot{n} = -kt - \tau b, \quad \textit{and} \quad \dot{b} = \tau n$$

Theorem 2.4 (Fundamental Theorem of the Local Theory of Curves)**.** *Given smooth functions $k(s) > 0$ and $\tau(s), s \in I$, there exists a regular curve $\alpha : I \to \mathbb{R}^3$ such that s is arc-length, $k(s)$ is the curvature, and $\tau(s)$ is the torsion of α. Moreover any other curve $\bar{\alpha}$, satisfying the same conditions, differs from α by an isometry.*

2.2 Isoperimetric Inequality

Lemma 2.5 (Wirtinger's Inequality)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function which is periodic with period L. Suppose $\int_0^L f(t)\,\mathrm{dt} = 0$. Then*

$$\int_0^L |f'(t)|^2 \,\mathrm{dt} \geq \frac{4\pi^2}{L^2} \int_0^L |f(t)|^2 \,\mathrm{dt},$$

with equality if and only if there exist constants $a_{\pm 1}$ such that $f(t) = a_{-1}e^{-2\pi it/L} + a_1 e^{2\pi it/L}$.

Theorem 2.6 (Isoperimetric Inequality in the Plane)**.** *Let Ω be a domain, that is, a connected open set. We assume that Ω has compact closure and that its* *boundary $\partial\Omega$ is a connected 1-manifold of class C^1. Let $A(\Omega)$ be the area of Ω. Then*

$$\ell^2(\partial\Omega) \geq 4\pi A(\Omega)$$

with equality if and only if Ω is a disk.

Proof. Define the vector field $X(x, y) = (x, y)$, and let n be the outward pointing normal vector field along $\partial\Omega$. The divergence theorem gives us that

$$\int_\Omega \operatorname{div} X \, \mathrm{d}A = \int_{\partial\Omega} \langle X, n \rangle \, \mathrm{d}s.$$

But $\operatorname{div}(X) = 2$, so by Cauchy-Schwarz we have

$$2A(\Omega) \leq \int_{\partial\Omega} |X| \, \mathrm{d}s.$$

By Cauchy-Schwarz again we have

$$\begin{aligned} 2A(\Omega) &\leq \left(\int_{\partial\Omega} |X|^2 \, \mathrm{d}s \right)^{1/2} \left(\int_{\partial\Omega} \mathrm{d}s \right)^{1/2} \\ &= \ell(\partial\Omega)^{1/2} \left(\int_{\partial\Omega} |X|^2 \, \mathrm{d}s \right)^{1/2}. \end{aligned}$$

Since we parameterise $\partial\Omega$ by arc length, $X(s) = (x(s), y(s))$ along $\partial\Omega$ are C^1 and periodic with period $L = \ell(\partial\Omega)$. Hence by Wirtinger's inequality we have

$$\begin{aligned} \left(\int_{\partial\Omega} |X|^2 \, \mathrm{d}s \right)^{1/2} &\leq \left(\frac{\ell(\partial\Omega)^2}{4\pi^2} \int_{\partial\Omega} |X'|^2 \, \mathrm{d}s \right)^{1/2} \\ &= \frac{\ell(\partial\Omega)^{3/2}}{2\pi} \end{aligned}$$

which gives the desired result. Equality occurs if and only if we have equality in all of the above, in particular in the second $s \mapsto |X(s)|$ is constant, so Ω is a disk. □

2.3 First Fundamental Form

Definition 2.7 (First Fundamental Form)**.** Let $S \subset \mathbb{R}^3$ be a surface. The quadratic form I_p on T_pS given by

$$I_p(w) := \langle w, w \rangle = |w|^2$$

is called the **first fundamental form** of the surface at p.

Definition 2.8. Two surfaces S_1 and S_2 are said to be **isometric** if there exists a diffeomorphism $f : S_1 \to S_2$ such that for all $p \in S_1, df_p$ is a linear isometry between T_pS_1 and $T_{f(p)}S_2$.

Let $\phi : U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ be a parametrization of a neighbourhood of a point $p \in S$. We will denote by (u, v) points in U and let

$$\begin{aligned} \phi_u(u,v) &= \frac{\partial \phi}{\partial u}(u,v) \in T_{\phi(u,v)}S, \\ \phi_v(u,v) &= \frac{\partial \phi}{\partial v}(u,v) \in T_{\phi(u,v)}S. \end{aligned}$$

Note these are linearly independent. Set

$$\begin{aligned} E &= \langle \phi_u, \phi_u \rangle_{\phi(u,v)}, \\ F &= \langle \phi_u, \phi_v \rangle_{\phi(u,v)}, \\ G &= \langle \phi_v, \phi_v \rangle_{\phi(u,v)}. \end{aligned}$$

Since a tangent vector $w \in T_pS$ is the tangent vector of a curve $\alpha(t) = \phi(u(t), v(t))$, $t \in (-\varepsilon, \varepsilon)$, with $p = \alpha(0) = \phi(u_0, v_0)$ we have

$$\begin{aligned} I_p(\dot{\alpha}(0)) &= \langle \dot{\alpha}(0), \dot{\alpha}(0) \rangle_p \\ &= E(\dot{u})^2 + 2F\dot{u}\dot{v} + G(\dot{v})^2. \end{aligned}$$

We can compute the length of a curve in S then by integrating $\sqrt{E(\dot{u})^2 + 2F\dot{u}\dot{v} + G(\dot{v})^2}$. Note also that $|\phi_u \wedge \phi_v| = \sqrt{EG - F^2}$.

Definition 2.9 (Area)**.** Let $\Omega \subset S$ be a bounded domain contained in the image of a parametrization $\phi : U \to S$. The positive number

$$A(\Omega) = \int_{\phi^{-1}(\Omega)} |\phi_u \wedge \phi_v| \, \mathrm{d}u \, \mathrm{d}v$$

is called the area of Ω.

2.4 The Gauss Map

Given a parametrization $\phi : U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ around a point $p \in S$, we can choose a unit normal vector at each point of $\phi(U)$ by setting

$$N(q) = \frac{\phi_u \wedge \phi_v}{|\phi_u \wedge \phi_v|}(q).$$

Definition 2.10 (Orientable)**.** A surface $S \subset \mathbb{R}^3$ is **orientable** if it admits a smooth field of unit normal vectors. The choice of such a field is called an **orientation**.

Definition 2.11 (Gauss Map)**.** Let S be an oriented surface and $N : S \to S^2$ the smooth field of unit normal vectors defining the orientation. The map N is called the **Gauss map** of S.

Since T_pS and $T_{N(p)}S^2$ are parallel planes, we will regard dN_p as a linear map $dN_p : T_pS \to T_pS$.

Proposition 2.12. *The linear map $dN_p : T_pS \to T_pS$ is self-adjoint.*

Proof. Let $\phi : U \to S$ be a parametrization around p. If $\alpha(t) = \phi(u(t), v(t))$ is a curve in $\phi(U)$ with $\alpha(0) = p$ we have

$$\begin{aligned} dN_p(\dot\alpha(0)) &= dN_p\left(\dot u(0)\phi_u + \dot v(0)\phi_v\right) \\ &= \left.\frac{d}{dt}\right|_{t=0} N(u(t), v(t)) \\ &= \dot u(0)N_u + \dot v(0)N_v \end{aligned}$$

In particular $dN_p\left(\phi_u\right) = N_u$ and $dN_p\left(\phi_v\right) = N_v$ and since $\{\phi_u, \phi_v\}$ is a basis of the tangent plane, we only have to prove that

$$\langle N_u, \phi_v\rangle = \langle N_v, \phi_u\rangle$$

To prove the last equality, observe that $\langle N, \phi_u\rangle = \langle N, \phi_v\rangle = 0$. Take derivatives with respect to v and u to obtain:

$$\begin{aligned} \langle N_v, \phi_u\rangle + \langle N, \phi_{uv}\rangle &= 0, \\ \langle N_u, \phi_v\rangle + \langle N, \phi_{vu}\rangle &= 0 \end{aligned}$$

which gives the desired equality. □

Definition 2.13 (Second Fundamental Form)**.** The quadratic form defined on T_pS by $II_p(w) = -\langle dN_p(w), w\rangle$ is called the **second fundamental form** of S at p.

Definition 2.14 (Normal Curvature)**.** Let $\alpha : (-\varepsilon, \varepsilon) \to S$ be a curve, $\alpha(0) = p$. Then the **normal curvature** of α at p is defined by $k_n(p) = \langle N, kn\rangle$ where N is the Gauss map, k is the curvature of α and n is the unit normal to α at p (i.e. $kn = \ddot\alpha$).

Proposition 2.15. $k_n(p) = II_p(\dot\alpha(0))$.

Definition 2.16 (Principal Curvatures and Directions)**.** As $dN_p : T_pS \to T_pS$ is self adjoint, it can be diagonalised. Let $e_1, e_2 \in T_pS$ be such that, with respect to this basis, we have

$$dN_p = \begin{pmatrix} -k_1 & 0 \\ 0 & -k_2 \end{pmatrix}$$

where $k_1 \geq k_2$. We call k_1, k_2 the **princial curvatures**, and e_1, e_2 the **principal directions**.

From standard linear algebra we get that k_1 (respectively k_2) is the maximum (minimum) value of II_p on the set of unit vectors in T_pS. That is, they are the extreme values of the normal curvature at p.

Definition 2.17 (Gaussian and Mean Curvature)**.** The determinant of dN_p is the **Gaussian curvature** $K(p)$ of S at p. Minus half of the trace of dN_p is the **mean curvature** $H(p)$ of S at p.

Clearly $K = k_1 k_2$ and $H = \frac{k_1+k_2}{2}$.

A point $p \in S$ of a surface is called **elliptic** if $K(p) > 0$, **hyperbolic** if $K(p) < 0$, **parabolic** if $K(p) = 0$ and $dN_p \neq 0$, and **planar** if $dN_p = 0$. A point $p \in S$ is called **umbilical** if $k_1 = k_2$.

2.5 Local Coordinates

Let $\phi : U \to S$ be a parametrization around a point $p \in S$. Let us express the second fundamental form in the basis $\{\phi_u, \phi_v\}$. Since $\langle N, \phi_u \rangle = \langle N, \phi_v \rangle = 0$ we have

$$\begin{aligned} e &= -\langle N_u, \phi_u \rangle = \langle N, \phi_{uu} \rangle, \\ f &= -\langle N_v, \phi_u \rangle = \langle N, \phi_{uv} \rangle = -\langle N_u, \phi_v \rangle, \\ g &= -\langle N_v, \phi_v \rangle = \langle N, \phi_{vv} \rangle. \end{aligned}$$

If α is a curve passing at $t = 0$ through p we can write:

$$\begin{aligned} II_p(\dot\alpha(0)) &= -\langle dN_p(\dot\alpha(0)), \dot\alpha(0) \rangle \\ &= e(\dot u)^2 + 2f\dot u\dot v + g(\dot v)^2. \end{aligned}$$

With respect to the basis ϕ_u, ϕ_v, we can express $\mathrm{d}N_p$ as a matrix, namely

$$\begin{aligned} \mathrm{d}N_p(\phi_u) &= N_u = a_{11}\phi_u + a_{21}\phi_v \\ \mathrm{d}N_p(\phi_v) &= N_v = a_{12}\phi_u + a_{22}\phi_v \end{aligned}$$

Taking inner products of the above equations with ϕ_u, ϕ_v we get

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

But with respect to the basis ϕ_u, ϕ_v, $\mathrm{d}N_p$ has matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Linear algebra then gives

Corollary 2.18. *We can write*

$$K = \frac{eg - f^2}{EG - F^2}, \text{ and } H = \frac{eG - 2fF + gE}{2(EG - F^2)}.$$

2.6 Theorema Egregium

Theorem 2.19 (Theorema Egregium)**.** *The Gaussian curvature K of a surface is invariant under isometries.*

Proof. It suffices to write K in terms only of the coefficients E, F, G of the first fundamental form and their derivatives. Let $\phi : U \to S$ be a parameterisation. Then at each point in the image we have a basis of $\mathbb{R}^3$ given by $\{\phi_u, \phi_v, N\}$. We can then express the derivatives of ϕ_u and ϕ_v in this basis:

$$\begin{aligned}
\phi_{uu} &= \Gamma_{11}^1 \phi_u + \Gamma_{11}^2 \phi_v + eN, \\
\phi_{uv} &= \Gamma_{12}^1 \phi_u + \Gamma_{12}^2 \phi_v + fN, \\
\phi_{vu} &= \Gamma_{21}^1 \phi_u + \Gamma_{21}^2 \phi_v + fN, \\
\phi_{vv} &= \Gamma_{22}^1 \phi_u + \Gamma_{22}^2 \phi_v + gN,
\end{aligned}$$

where the Γ_{ij}^k are the **Christoffel symbols**. Taking inner products with ϕ_u and ϕ_v, we can see that we can solve for the Christoffel symbols in terms of E, F, G and their derivatives. So Christoffel symbols are invariant under isometries.

Consider $\phi_{uuv} = \phi_{uvu}$, and differentiating our previous expressions and substituting in gives (after some manipulation)

$$\begin{aligned}
&\left(\Gamma_{12}^2\right)_u - \left(\Gamma_{11}^2\right)_v + \Gamma_{12}^1 \Gamma_{11}^2 + \\
&\Gamma_{12}^2 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{11}^1 \Gamma_{12}^2 \\
= -f a_{21} + e a_{22} = -E \frac{eg - f^2}{EG - F^2} &= -EK.
\end{aligned}$$

Thus K can be expressed solely in terms of the coefficients of the first fundamental form and their derivatives as required. $\square$

Definition 2.20 (Isothermal Parameterisation)**.** A parameterization is **isothermal** if $E = G = \lambda^2(u, v)$ and $F = 0$.

Proposition 2.21. *For isothermal parameterization, $K = -\frac{1}{\lambda^2}\Delta(\log \lambda)$, where Δ is the Laplacian in (u, v)-coordinates.*

3 Geodesics & Minimal Surfaces

3.1 Geodesics

Let $S \subseteq \mathbb{R}^3$ be a surface with $p, q \in S$. Let $\Omega(p, q)$ be the set of all curves $\alpha : [0, 1] \to S$ with $\alpha(0) = p$ and $\alpha(1) = q$.

Definition 3.1 (Energy Functional)**.** The **energy** $E : \Omega(p, q) \to \mathbb{R}$ is given by

$$E(\alpha) = \frac{1}{2} \int_0^1 |\dot{\alpha}|^2 \, \mathrm{d}t.$$

Let $\alpha_s \in \Omega(p, q)$ be a smooth one parameter family of curves, with $s \in (-\varepsilon, \varepsilon)$. Let $E(s) = E(\alpha_s)$. Then we have that

$$\frac{\mathrm{d}E}{\mathrm{d}s} = \int_0^1 \left\langle \frac{\partial}{\partial s} \frac{\partial \alpha_s}{\partial t}, \frac{\partial \alpha_s}{\partial t} \right\rangle \mathrm{d}t$$

Integrating by parts we get

$$\left.\frac{\mathrm{d}E}{\mathrm{d}s}\right|_{s=0} = \langle J(1), \dot{\alpha}(1)\rangle - \langle J(0), \dot{\alpha}(0)\rangle$$
$$- \int_0^1 \langle J(t), \ddot{\alpha}(t)\rangle \mathrm{d}t$$

where

$$J(t) = \left.\frac{\partial \alpha_s(t)}{\partial s}\right|_{s=0}$$

Since $\alpha_s \in \Omega(p, q)$, $J(0) = J(1) = 0$. So we get that

$$\left.\frac{\mathrm{d}E}{\mathrm{ds}}\right|_{s=0} = -\int_0^1 \langle J(t), \ddot{\alpha}(t)\rangle \mathrm{d}t$$

Now notice that for each $t \in [0, 1]$, $J(t) \in T_{\alpha(t)}S$, since $s \mapsto \alpha_s(t)$ is a curve in s. So if α is such that $\ddot{\alpha} \perp T_{\alpha(t)}S$ for all t, then α extremises E.

Definition 3.2 (Geodesic)**.** A curve $\alpha : I \to S$ is a **geodesic** if for all $t \in I$, $\ddot{\alpha}(t)$ is orthogonal to $T_{\alpha(t)}S$.

3.2 Covariant Derivative

Definition 3.3 (Vector Field)**.** Let $\alpha : I \to S$ be a curve. A **vector field** along α is a smooth map $V : I \to \mathbb{R}^3$ such that for all t, $V(t) \in T_{\alpha(t)}S$.

Definition 3.4 (Covariant Derivative)**.** The **covariant derivative** of a vector field V along α is

$$\frac{\mathrm{D}V}{\mathrm{d}t}(t) = \mathrm{proj}_{T_{a|t\rangle}S}\left(\frac{\mathrm{d}V}{\mathrm{d}t}\right)$$

where $\mathrm{proj}_{T_{\alpha(t)}}$ is the orthogonal projection onto $T_{\alpha(t)}S$.

Proposition 3.5. *A curve α is a geodesic if and only if $\frac{\mathrm{D}\dot{\alpha}}{\mathrm{d}t} = 0$ for all t.*

Definition 3.6 (Parallel)**.** A vector field V along α is **parallel** if $\frac{\mathrm{D}V}{\mathrm{d}t} = 0$.

Proposition 3.7. *Let V, W be parallel vector fields along α. Then $\langle V(t), W(t)\rangle$ is constant.*

Corollary 3.8. *If α is a geodesic, then $|\dot{\alpha}|$ is constant. So geodesics are parametrised proportional to arc length.*

3.3 Local Coordinates

Let $\phi : U \to S$ be a parametrisation, $\alpha : I \to S$ a curve, with $\alpha(/) \subseteq \phi(U)$. Write $\alpha(t) = \phi(u(t), v(t))$. Let V be a vector field along α. Then there are functions $a(t), b(t)$ such that

$$V(t) = a(t)\phi_u + b(t)\phi_v$$

Differentiating this, we get that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = a\left(\phi_{uv}\dot{u} + \phi_{uv}\dot{v}\right) + b\left(\phi_{vu}\dot{u} + \phi_{vv}\dot{v}\right) + a\phi_u + b\phi_v$$

The covariant derivative is just the ϕ_j and ϕ_v components of this, since N is orthogonal to $T_{a(t)}S$. Therefore, in terms of Christoffel symbols, we have that

$$\begin{aligned}\frac{\mathrm{DV}}{\mathrm{d}t} &= \left(\dot{a} + a\dot{u}\Gamma_{11}^1 + a\dot{v}\Gamma_{12}^1 + b\dot{u}\Gamma_{12}^1 + b\dot{v}\Gamma_{22}^1\right)\phi_u \\ &+ \left(b + a\dot{u}\Gamma_{11}^2 + a\dot{v}\Gamma_{12}^2 + b\dot{u}\Gamma_{12}^2 + b\dot{v}\Gamma_{22}^2\right)\phi_v\end{aligned}$$

From this expression, we see that the covariant derivative only depends on the first fundamental form.

Proposition 3.9 (Geodesic Equations)**.** *$\alpha(t) = \phi(u(t), v(t))$ is a geodesic if and only if*

$$\begin{aligned}\ddot{u} + \Gamma_{11}^1\dot{u}^2 + 2\Gamma_{12}^1\dot{u}\dot{v} + \Gamma_{22}^1\dot{v}^2 = 0 \\ \dot{v} + \Gamma_{11}^2\dot{u}^2 + 2\Gamma_{12}^2\dot{u}\dot{v} + \Gamma_{22}^2\dot{v}^2 = 0\end{aligned}$$

Proposition 3.10 (Parallel Transport)**.** *Given $v_0 \in T_{\alpha(t_0)}S$, there exists a unique parallel vector field $V(t)$ along $\alpha(t)$, with $V(t_0) = v_0$. We call $V(t_1)$ the **parallel transport** of v_0 along α at t_1.*

Corollary 3.11 (Geodesic Existence)**.** *Given $p \in S, v \in T_pS$, there exists $\varepsilon > 0$, and a unique geodesic $\gamma : (-\varepsilon, \varepsilon) \to S$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$*

Definition 3.12. Let $\alpha \in \Omega(p,q)$. Define $P : T_pS \to T_qS$ the map sending $v \in T_pS$ to the **parallel transport** of v along α at q.

3.4 Exponential Map

Proposition 3.13. *Given $p \in S, v \in T_PS$, let $\gamma_v : (-\varepsilon, \varepsilon) \to S$ by the unique geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Then $\gamma_{\lambda v}$ is defined on $(-\varepsilon/\lambda, \varepsilon/\lambda)$. Furthermore, $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$.*

Definition 3.14 (Exponential Map)**.** Suppose $v \in T_pS$ nonzero is such that $\gamma_v(1)$ is defined, we define the **exponential map** $\exp_p(v) = \gamma_v(1)$.

We note exists $\varepsilon > 0$ such that $\exp_p : B_\varepsilon(0) \to S$ is well defined and smooth.

Proposition 3.15. *If S is closed, then $\exp_p$ is defined on all of T_pS.*

Proposition 3.16. *$\exp_p : B_\varepsilon(0) \to S$ is a diffeomorphism onto its image in a neighbourhood $U \subseteq B_\varepsilon(0)$ of $0 \in T_pS$*

Proof. By the inverse function theorem, suffices to show $d\left(\exp_p\right)_0$ is nonsingular. Let $\alpha(t) = tv$ for some fixed $v \in T_pS$. Then $\exp_p(tv) = \gamma_v(t)$ at $t = 0$ has tangent vector v. So $d\left(\exp_p\right)_0(v) = v$. $\square$

Definition 3.17 (Normal Neighbourhood)**.** Let U be as in the previous proposition. Then $V = \exp_p(U)$ is called a **normal neighbourhood** of p.

Corollary 3.18. *$\exp_p : U \to V$ is a parametrisation.*

Proposition 3.19. *If we choose cartesian coordinates on T_pS, then with the $\exp_p$ parametrisation, we have the first fundamental form*

$$E(p) = G(p) = 1 \text{ and } F(p) = 0$$

Definition 3.20 (Geodesic Polars)**.** If we choose polar coordinates (r, θ) for T_pS, then we have the **geodesic polar coordinates**. That is,

$$\begin{aligned}\phi(r,\theta) &= \exp_p\left(r\left(\cos(\theta)e_1 + \sin(\theta)e_2\right)\right)\\ &= \exp_p(rv(\theta)) = \gamma_{v(\theta)}(t)\end{aligned}$$

where $v(\theta) = \cos(\theta)e_1 + \sin(\theta)e_2$.

Definition 3.21 (Geodesic Circles, Radial Geodesics)**.** The images of circles centred in the origin under the map ϕ are called **geodesic circles** (i.e. $r =$ const). Similarly, the images of lines through the origin (i.e. $\theta =$ const) are called **radial geodesics**.

Proposition 3.22. *For geodesic polars we have*

$$\begin{aligned}E = 1, \quad F = 0, \quad G(0,\theta) = 0\\ \textit{and} \quad (\sqrt{G})_r(0,\theta) = 1\end{aligned}$$

Moreover, the Gaussian curvature can be written as

$$K = -\frac{(\sqrt{G})_{rr}}{\sqrt{G}}$$

Proof. By definition of ϕ, we have that $\phi_r = \dot{\gamma}_{v(\theta)}(r)$, so $E = 1$ as $v(\theta)$ is a unit vector and geodesics travel at constant speed. Now let $w = \frac{dv}{d\theta}$. Then by chain rule, we have that

$$\phi_\theta = \mathrm{d}\left(\exp_p\right)_{rv}(rw) = r\ \mathrm{d}\left(\exp_p\right)_{rv}(w)$$

So we have that

$$\begin{aligned}F &= r\left\langle \dot{\gamma}_v(r), \mathrm{d}\left(\exp_p\right)_{rv}(w)\right\rangle\\ G &= r^2\left|\ \mathrm{d}\left(\exp_p\right)_{rv}(w)\right|^2\end{aligned}$$

Clearly $F(0,\theta) = 0$, and from the last equality, we find that

$$(\sqrt{G})_r(0,\theta) = \left|\mathrm{d}\left(\exp_p\right)_0(w)\right| = |w| = 1$$

Finally, we can compute

$$\begin{aligned} F_r &= \langle \phi_{rr}, \phi_\theta \rangle + \langle \phi_r, \phi_{\theta r} \rangle \\ &= \langle \phi_r, \phi_{\theta r} \rangle \\ &= \frac{1}{2} \frac{\partial}{\partial \theta} \langle \phi_r, \phi_r \rangle \\ &= \frac{1}{2} E_\theta \\ &= 0 \end{aligned}$$

where we used the fact that $\phi(\cdot, \theta) = \gamma_v$ is a geodesic, so $\phi_{rr} = \ddot{\gamma}_v$ is normal to T_pS. So $F = 0$ identically. We omit the computation for K, and note that it can be computed using Christoffel symbols. □

3.5 Geodesic Curvature

Definition 3.23 (Algebraic Value of the Covariant Derivative)**.** Let W be a differentiable field fo unit vectors along a curve $\alpha : I \to S$ along an oriented surface S. Then

$$\left[\frac{\mathrm{D}W}{\mathrm{d}t} \right] = \left\langle \frac{\mathrm{d}W}{\mathrm{d}t}, N \wedge W \right\rangle$$

Proposition 3.24. *Let W be a field of unit vectors along α. Then $\frac{\mathrm{D}W}{\mathrm{d}t}$ is parallel to $N \wedge W$, and we have that*

$$\frac{\mathrm{D}W}{\mathrm{d}t} = \left[\frac{\mathrm{D}W}{\mathrm{d}t} \right] (N \wedge W)$$

Definition 3.25 (Geodesic Curvature)**.** Let $\alpha : I \to S$ be a regular curve parametrised by arc length. The algebraic value of the covariant derivative

$$\kappa_g(s) = \left[\frac{\mathrm{D}\dot{\alpha}}{\mathrm{d}t} \right] = \langle \ddot{\alpha}, N \wedge \dot{\alpha} \rangle$$

is called the **geodesic curvature** of α at α(s).

Proposition 3.26. *α is a geodesic if and only if its geodesic curvature is identically zero.*

Proposition 3.27. *Let k and n be the curvature and unit normal for α. Then we have that*

$$\ddot{\alpha} = k_n N + k_g (N \wedge \dot{\alpha})$$

where κ_n, κ_g are the normal and geodesic curvatures respectively.

Proof. Since W has norm 1, we have that $\langle W, W \rangle = 0$, so $\left\langle \frac{\mathrm{d}W}{\mathrm{d}t}, W \right\rangle = 0$. Hence $\frac{\mathrm{d}W}{\mathrm{d}t}$ is perpendicular to W. Thus, $\frac{\mathrm{D}W}{\mathrm{d}t}$ must be perpendicular to both W and N, so it is parallel to $N \wedge W$. □

Definition 3.28 (Perpendicular Vector Field)**.** Let V be a unit vector field along $\alpha : I \to S$. Let $iV(t)$ be the unique vector field along α such that for every $t \in I, V(t), iV(t), N(t)$ forms a positively oriented orthonormal basis of $\mathbb{R}^3$. That is,

$$V(t) \wedge iV(t) = N(t)$$

Proposition 3.29. *Let V, W be unit vector fields along $\alpha : I \to S$. Then there exists smooth functions a, b, such that*

$$W(t) = a(t)V(t) + b(t)iV(t)$$

with $a^2 + b^2 = 1$. Furthermore, if we fix $t_0 \in I$, and let φ_0 be such that

$$a(t_0) = \cos(\varphi_0) \quad \text{and} \quad b(t_0) = \sin(\varphi_0)$$

then there exists a smooth function $\varphi : l \to S$ such that

$$a(t) = \cos(\varphi(t)), \quad b(t) = \sin(\varphi(t)) \quad \text{and} \quad \varphi(t_0) = \varphi_0$$

Proof. $V(t), iV(t)$ is an orthonormal basis of $T_{\alpha(t)}S$. The construction of φ is as in the construction of the winding number in Complex Analysis. □

Definition 3.30 (Smooth Determinantion of Angle)**.** φ from the previous proposition is called a **smooth determination of the angle** from V to W.

Proposition 3.31. *Let V, W be unit vector fields along $\alpha : I \to S$ and φ by a smooth determination of angle from V to W. Then*

$$\left[\frac{\mathrm{D}W}{\mathrm{d}t}\right] - \left[\frac{\mathrm{D}V}{\mathrm{d}t}\right] = \frac{\mathrm{d}\varphi}{\mathrm{d}t}$$

Proposition 3.32. *Let $\alpha : I \to S$ be a curve parametrised by arc length, $V(s)$ a parallel unit vector field along α, φ a smooth determination of angle from V to $\dot{\alpha}$. Then*

$$\kappa_g(s) = \frac{\mathrm{d}\varphi}{\mathrm{d}s}$$

Proof. $\left[\frac{\mathrm{D}V}{dt}\right] = 0$ as V is parallel. □

4 Gauss-Bonnet

Theorem 4.1 (Gauss's Theorem for Geodesic Triangles)**.** *Let T be a geodesic triangle on a surface S. Suppose T is small enough so that it is contained in a normal neighbourhood of one of its vertices, then*

$$\int_T K \, \mathrm{d}A = \alpha_1 + \alpha_2 + \alpha_3 - \pi$$

where K is the Gaussian curvature of S, and $0 < \alpha_i < \pi$ are the internal angles of T.

Proof. We can assume without loss of generality that we have geodesic polar coordinates centred at one of the vertices of T, one of the edges corresponds to $\theta = 0$ and another corresponds to $\theta = \theta_0$. The remaining edge is a geodesic segment γ.

First notice that γ can be written in the form $r = h(\theta)$. Suppose not, then there exists such that $\dot{\gamma}(s)$ is parallel to ϕ_r. But radial segments are geodesics, so this means that γ is radial. Contradiction. Hence we can write γ as $r = h(\theta)$. Then

$$\begin{aligned}\int_T K \, \mathrm{d}A &= \int_T K\sqrt{G} \, \mathrm{d}r \, \mathrm{d}\theta \\ &= \int_0^{\theta_0} \left(\lim_{\varepsilon \to 0} \int_\varepsilon^{h(\theta)} K\sqrt{G} \, \mathrm{d}r \right) \mathrm{d}\theta\end{aligned}$$

But in geodesic polar coordinates, we have $K\sqrt{G} = -(\sqrt{G})_{rr}$, and $\lim_{r\to 0}(\sqrt{G})_r = 1$, so

$$\lim_{\varepsilon \to 0} \int_\varepsilon^{h(\theta)} K\sqrt{G} \, \mathrm{d}r = 1 - (\sqrt{G})_r(h(\theta), \theta)$$

Now suppose $\gamma(s) = \phi(r(s), \theta(s))$ makes an angle $\varphi(s)$ with ϕ_r, that is, the curves $\theta = \text{const}$. Then the previous corollary $(u = r, v = \theta)$ gives that

$$(\sqrt{G})_r \frac{\mathrm{d}\theta}{\mathrm{d}s} + \frac{\mathrm{d}\varphi}{\mathrm{d}s} = 0$$

as γ is a geodesic. Therefore, we have that

$$\begin{aligned}\int_T K \, \mathrm{d}A &= \int_0^{\theta_0} \Big(1 - (\sqrt{G})_r(h(\theta), \theta)\Big) \, \mathrm{d}\theta \\ &= \int_0^{\theta_0} \, \mathrm{d}\theta - \int_0^{s_0} (\sqrt{G})_r \frac{\mathrm{d}\theta}{\mathrm{d}s} \mathrm{d}s \\ &= \theta_0 + \int_0^{s_0} \frac{\mathrm{d}\varphi}{\mathrm{d}s} \, \mathrm{d}s \\ &= \theta_0 + \int_{\varphi(0)}^{\varphi(s_0)} \mathrm{d}\varphi \\ &= \theta_0 + \varphi(s_0) - \varphi(0)\end{aligned}$$

Finally, by the orientations, we have the result. □

Definition 4.2 (Triangulation)**.** Let S be a compact surface. A **triangulation** of S is a finite number of closed subsets $T_1, \ldots, T_n$ which cover S, each T_i is homeomorphic to a Euclidean triangle in the plane. Moreover, any two distinct triangles are either disjoint, share a vertex, or share an edge.

Theorem 4.3. *Triangulations always exist. Furthermore, we can choose it so that each T_i is diffeomorphic to a Euclidean triangle, and each edge is a geodesic segment.*

Definition 4.4 (Euler Characteristic)**.** Given a triangulation of S, let F be the number of faces, E the number of edges, V the number of vertices. Then

$$x(S) = V - E + F$$

is the **Euler characteristic** of S.

This is independent of the choice of triangulation.

Proposition 4.5 (Classification of Compact Orientable Surfaces)**.** *All compact orientable surfaces are diffeomorphic to some Σ_g where g is a g-holed torus. g is called the **genus** of Σ_g. Furthermore,*

$$\chi\left(\Sigma_g\right) = 2 - 2g$$

Theorem 4.6 (Global Gauss-Bonnet)**.** *Let S be a compact surface without boundary. Then*

$$\int_S K dA = 2\pi\chi(S)$$

Proof. Consider a triangulation by geodesic triangles $T_1, \ldots, T_F$. We can assume wlog that each T_i is contained in a normal neighbourhood of one of its vertices. Let $\alpha_i, \beta_i, \gamma_i$ be the interior angles of T_i. Then by Gauss's theorem for triangles, we have that

$$\int_{T_i} K \, \mathrm{d}A = \alpha_i + \beta_i + \gamma_i - \pi$$

Summing over all i, we have that

$$\int_S K \, \mathrm{d}A = \sum_{i=1}^{F} (\alpha_i + \beta_i + \gamma_i) - \pi F$$

Now notice that the sum of the angles at every vertex is 2π, so

$$\sum_{i=1}^{F} (\alpha_i + \beta_i + \gamma_i) = 2\pi V$$

Finally, for a triangulation, every edge belongs to two triangles, so $2E = 3F$. Putting this all together we get that

$$\int_S K \, \mathrm{d}A = \pi(2V - F) = 2\pi\chi(S).$$

□

Theorem 4.7 (Local Gauss-Bonnet)**.** *Let $\phi : U \to S$ be an orthogonal parametrisation of an oriented surface S, U is a disc in $\mathbb{R}^2$, and ϕ is compatible with the orientation of S. Let $\alpha : I \to \phi(U)$ be a smooth simple closed curve enclosing* *a domain R. Suppose α is positively oriented and parametrised by arc length. Then*

$$\int_l k_g(s)ds + \int_R KdA = 2\pi$$

where k_g is the geodesic curvature of α.

Theorem 4.8 (Gauss-Bonnet with Boundary)**.** *Let $R \subseteq S$ be a connected open relatively compact*[2] *subset.* [2] *Suppose ∂R contains of n piecewise smooth simple closed curves $\alpha_i : I_i \to S$, where the images do not intersect. Suppose the α_i are parametrised by arc length, and are positively oriented. Let θ_i be the external angles of the vertices of these curves. Then*

$$\sum_{i=1}^{n} \int_{l_i} k_g(s)\mathrm{d}s + \int_R K \ \mathrm{d}A + \sum_i \theta_i = 2\pi\chi(R)$$

Theorem 4.9. *Suppose S is a compact orientable surface with $K > 0$. Then S is diffeomorphic to S^2. Moreover, if α, β are simple closed geodesics on S, then they must intersect.*

Proof. Gauss-Bonnet gives us that $\chi(S) > 0$, so S is diffeomorphic to S^2. Now suppose α, β do not intersect. Then they bound a domain R with $\chi(R) = 0$. But then Gauss-Bonnet means that R must in fact have measure zero. Contradiction. □

4.1 Minimal Surfaces

Definition 4.10 (Minimal Surface)**.** A surface S is **minimal** if its mean curvature vanishes everywhere.

Definition 4.11 (Normal Variation)**.** Let $\phi : U \to S$ be a parametrisation, $D \subseteq U$ bounded open connected, with $\bar{D} \subseteq U$. Let $h : \bar{D} \to \mathbb{R}$ be smooth. Then the **normal variation** of $\phi(\bar{D})$ determined by h is the map $\rho : \bar{D} \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ given by

$$\rho(u, v, t) = \phi(u, v) + th(u, v)N(u, v)$$

[2]That is, the closure is compact.