Quantum Mechanics

Adam Kelly (ak2316@cam.ac.uk)

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Quantum mechanics is the mathematical framework used to describe nature at the scale of subatomic particles. This course will primarily focus on building up this mathematical framework, and not so much on the actual applications or physical background to it.

This article constitutes my notes for the 'Quantum Mechanics' course, held in Michaelmas 2021 at Cambridge. These notes are *not a transcription of the lectures*, and differ significantly in quite a few areas (more so than some of my other notes). Still, all lectured material should be covered. In particular, these notes are heavily influenced by the 'Principles of Quantum Mechanics' course, along with numerous books.

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§1 Foundations of Quantum Mechanics

§1.1 Classical Mechanics

In classical mechanics, particle motion is described by Newton's laws. These are a set of second order differential equations, which when coupled with two initial conditions then specify the future motion of the particle for all time. These initial conditions are typically taken to be position and momentum $(x(t_0), p(t_0))$. So Newton's laws really say that the entire state of a particle is encoded by a vector $(x(t), p(t))$ in *phase space* M of position and momentum. As time passes, the particles state traces out a path in phase space, but the whole state is indeed determined by initial vector (along with the differential equation given by Newton's laws).

But of course a particle is really corresponding to something physical, it's not really an element of an abstract vector space M. We need some way to go from this Newtonian framework of phase space to physical space. This is done by introducing *Observables*, which are functions $O : M \to \mathbb{R}$.

Observables can be all sorts of things – position, kinetic energy, angular momentum, and everything else. These observables are defined for vectors in M, but we typically want to evaluate them at the vector corresponding to our particles state at some time

t. Still, we don't factor time into our definition of observables, it's just baked into the point at which we evaluate them at.

Classical mechanics is a game of juggling initial conditions, inputs to Newton's laws (such as forces and potentials), and observables. We will see that in the abstract, quantum and classical mechanics are similar in this way.

But generally, classical mechanics and quantum mechanics are fundamentally different - you can't derive quantum mechanics from classical mechanics, and attempts to introduce it like that will fail. In reality, you should look at the mathematical framework of quantum mechanics and use that to build intuition. To help with this, looking at the *mathematical* framework of classical mechanics may also help, but it shouldn't be where you derive your intuition.

§1.2 Postulates of Quantum Mechanics

We begin by clearly setting out the mathematical framework that quantum mechanics uses.

Instead of phase space, everything we want to know about a quantum mechanical system is contained in a vector[1] $|\psi\rangle$ in 'Hilbert space'.

Axiom 1.1 (State)

The state of a quantum mechanical system is given by a non-zero **state vector** $|\psi\rangle$ in a **Hilbert space** $\mathcal{H}$, a complex vector space with a complete inner product $\langle\cdot|\cdot\rangle$.

The exact choice of Hilbert space $\mathcal{H}$ will depend on the system being studied, and there is non-trivial physical systems where both finite and infinite dimensional spaces are used.

Axiom 1.2 (Observables)

The observables of a quantum mechanical system O are given by self-adjoint linear operators $O : \mathcal{H} \to \mathcal{H}$.

Axiom 1.3 (Time evolution)

There is a distinguished quantum observable, the Hamiltonian H. Time evolution of states $|\psi(t)\rangle \in \mathcal{H}$ is given by the Schrödinger equation

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H|\psi(t)\rangle.$$

The operator H has eigenvalues that are bounded below.

§1.3 Postulates of Quantum Mechanics

Okay now we are just going to dump a bunch of stuff and say that it's intuitive and makes sense. The truth is that you have to start somewhere and this is where it's going

[1] This notation is Dirac notation, and it has a lot of nice features which are useful when doing quantum mechanics. This notation wasn't used in the IB course, but since it's so ubiquitous I just decided to use it from the beginning.

to be.

I'm also going to just use dirac notation because literally everyone actually uses it, and since we are doing things properly there's no reason not to – this is a mathematics course after all.

Axiom 1.4 (States)
The state of a quantum mechanical system is given by a non-zero vector $|\psi\rangle$ in a complex vector space $\mathcal{H}$ with Hermitian inner product $\langle \cdot, \cdot \rangle$.

We won't specify more about what $\mathcal{H}$ is here, as it depends on the system you're working with. There's real world systems where it can make sense to have $\mathcal{H}$ being finite or infinite dimensional.

Also of note: $\mathcal{H}$ is a *vector space*, and the linear combination of states is also a state. It's also a *complex* vector space, where linear combinations allow complex numbers. This is necessary – I won't elaborate here.

Axiom 1.5 (Observables)
The observables of a quantum mechanical system are given by self-adjoint linear operators on $\mathcal{H}$.

Axiom 1.6 (Dynamics)
There is a distinguished quantum observable, the Hamiltonian H. Time evolution of states $|\psi(t)\rangle \in \mathcal{H}$ is given by the Schrödinger equation

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H|\psi(t)\rangle.$$

The operator H has eigenvalues that are bounded below.

The Hamiltonian observable has the physical interpretation in terms of the energy, and boundedness implies there's a stable lowest energy state. $\hbar$ is a dimensionless constant called Planck's constant.

It's very much a logical leap but I'm just going to claim that for a particle moving in a potential U, we can take the Hamiltonian to be

$$H = -\frac{\hbar^2}{2m}\nabla^2 + U.$$

Good stuff.

We now need a way in our framework to go from what's happening on a quantum scale to what's happening in the real world. You'll notice that in classical mechanics, an observable is a function that takes a state and produces some numerical value – something that can be thought of as reading out a value from some measurement device. For quantum mechanics, since we work with states in a vector space, if we had two different states with two different values in an observable, since the sum of states is also a state – what would that observable be? This is a core quantum effect, and the way we deal with it is as follows.

Axiom 1.7 (Measurement)

States for which the value of an observable can be characterized by a well-defined number are the states that are eigenvectors for the corresponding self-adjoint operator. The value of the observable in such a state will be a real number, the eigenvalue of the operator.

Axiom 1.8 (The Born Rule)

Given an observable O with two unit-norm states $|\psi_1\rangle$ and $|\psi_2\rangle$ that are eigenvectors of O with distinct eigenvalues λ_1 and λ_2,

$$O|\psi_1\rangle = \lambda_1 |\psi_1\rangle, \quad O|\psi_2\rangle = \lambda_2 |\psi_2\rangle,$$

the complex linear combination state

$$\alpha_1 |\psi_1\rangle + \alpha_2 |\psi_2\rangle$$

will not have a well-defined value for the observable O. If one attempts to measure this observable, one would get either λ_1 or λ_2, with probabilities

$$\frac{|\alpha_1^2|}{|\alpha_1^2| + |\alpha_2^2|}$$

and

$$\frac{|\alpha_2^2|}{|\alpha_1^2| + |\alpha_2^2|}$$

respectively.

Note that this axiom implies there is a global phase on states that makes no difference to these probabilities. This is ignored, but one must keep in mind that *relative* differences in phase *can* make a difference.

§1.4 Conservation of Probability

Because the schrodinger equation

Because the norm of a state is preserved by the Schrodinger equation, we really get probability conservation.

This can be written something like

$$\frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \nabla \cdot J = 0,$$

where

$$J(\mathbf{x}, t) =$$