Graph Theory

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This set of notes is a work-in-progress account of the course 'Graph Theory', originally lectured by Dr Julian Sahasrabudhe in Lent 2020 at Cambridge. These notes are not a transcription of the lectures, but they do roughly follow what was lectured (in content and in structure).

These notes are my own view of what was taught, and should be somewhat of a superset of what was actually taught. I frequently provide different explanations, proofs, examples, and so on in areas where I feel they are helpful. Because of this, this work is likely to contain errors, which you may assume are my own. If you spot any or have any other feedback, let me know.

Contents

1 Introduction

For many people, 'Graph Theory' is a first course in combinatorics. It's an area with a big focus on problem solving, and it can give a perspective on many other areas of mathematics.

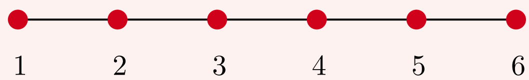
§1.1 Definitions

We will begin our course in graph theory naturally by defining what a graph is.

Definition 1.1.1 (Graph)
A **graph** is an ordered pair $G = (V, E)$ where V is the set of **vertices**, and $E \subseteq \{\{x, y\} \mid x, y \in V, x \neq y\}$ is a set of unordered pairs of vertices called **edges**.

We have a natural way of drawing a graph. For each vertex we have a point in the plane, and for each edge we draw a line between the corresponding pair of vertices.

Example 1.1.2 (Example of a Graph)
The ordered pair (V, E) where $V = \{1, 2, \ldots, 6\}$ and $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{5, 6\}\}$ is a graph.

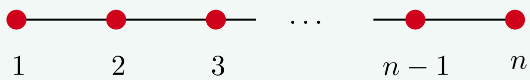


This graph is known as P_6, a path on 6 vertices.

§1.1.1 Common Graphs

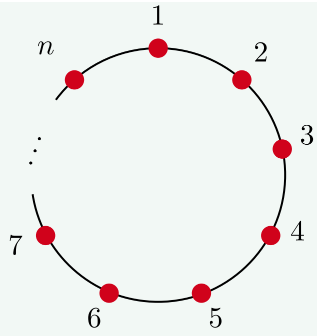
There are some graphs that will appear repeatedly throughout the course, and we will define them now.

Definition 1.1.3 (Path)
We define P_n to be the graph $V = \{1, \ldots, n\}$, $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}\}$ as shown.



We call this a **path** on n vertices, and say it has **length** $n - 1$.

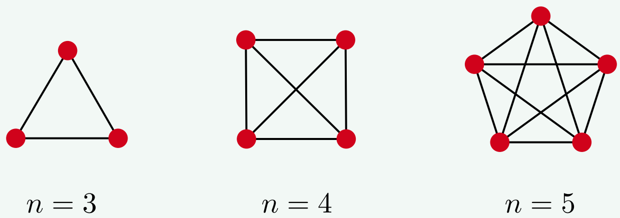
Definition 1.1.4 (Cycle)
We define C_n (for $n \geq 3$) to be the graph $V = \{1, \ldots, n\}$, $E = \{\{1, 2\}, \ldots, \{n - 1, n\}, \{n, 1\}\}$ as shown.



We call this the **cycle** on n vertices.

Definition 1.1.5 (Complete Graph)

The **complete graph** on n vertices K_n is the graph $\{1, \ldots, n\}$ and $E = \{\{i, j\} \mid i \neq j \in V\}$.

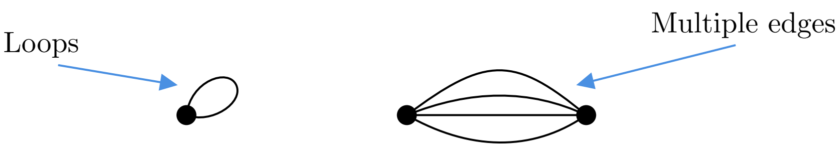


Note that there is an edge between every pair of vertices.

Definition 1.1.6 (Empty Graph)

We define the **empty graph** on n vertices $\overline{K_n}$ to have $V = \{1, \ldots, n\}$ but $E = \emptyset$.

Remark. In our definition of a graph, we *don't allow* loops, and there *cannot* be multiple edges between the same set of vertices.



These limitations are inherent in our definition, where we use sets rather than multisets. You can define graphs where such things are allowed, but for now we will outlaw them. We also note that edges are *unordered pairs*, so for now edges have no direction.

To be slightly more succinct, we will use some shorthand notation.

Notation. If $G = (V, E)$ is a graph, and we have some edge $\{x, y\} \in E$, we will denote it by xy. We will also define $|G| = |V|$, and $e(G) = |E|$.

Example 1.1.7 (Vertices and Edges of K_n)

Consider the graph K_n. We have $|K_n| = n$, and $e(K_n) = \binom{k}{2}$, as there is an edge between any pair of vertices.

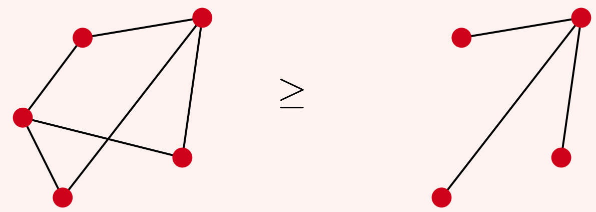
§1.1.2 Subgraphs

Now we will define the notion of a *subgraph*, in the natural way.

Definition 1.1.8 (Subgraph)
We say that $H = (V', E')$ is a **subgraph** of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

Informally, H is a subgraph of G if we can remove vertices and edges from G to get H. Let's look at some examples.

Example 1.1.9 (Example of a Subgraph)
The graph on the right is a subgraph of the graph on the left.



We are also going to use some notation for removing an edge or a vertex from a graph. Of course, when removing a vertex you also have to remove the edges connecting to it.

Notation (Adding/Removing Vertices & Edges)**.** For an edge xy or a vertex x, we define $G - xy$ to be the graph G with the edge xy removed, and $G - x$ to be G with vertex x removed, along with all edges incident to x. We will also define $G + xy$ to be G with the edge xy, and $G + x$ to be G with the vertex x.

An easy way to get a subgraph is by taking a subset of the vertices and seeing what edges you get from the original graph.

Definition 1.1.10 (Inducted Subgraph)
If $G = (V, E)$ is a graph and $X \subseteq V$, the **subgraph inducted by X** is defined to be $G[X] = (X, \{xy \in E \mid x, y \in X\})$.

§1.1.3 Graph Isomorphism

Now that we have defined graphs, it's natural to define some notion of isomorphism.

Definition 1.1.11 (Graph Isomorphism)
Let $G = (V, E)$ and $H = (V', E')$ be graphs. We say that $f : V \to V'$ is a **graph isomorphism** if $f(u)f(v) \in E' \iff uv \in E$.

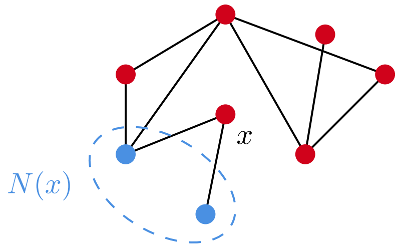
If there is a graph isomorphism between G and H then we say they are **isomorphic**.

Now for the following discussion, fix some graph $G = (V, E)$, and let $x \in V$.

Definition 1.1.12 (Neighbourhood)
If $xy \in E$, then we say that x and y are **adjacent**. We define the **neighborhood**

of x to be the set $N(x) = \{y \in V \mid xy \in E\}$ of all vertices adjacent to x.

Note that as in the diagram below, x is not in its own neighborhood.



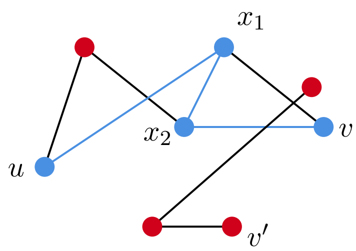
Definition 1.1.13 (Degree)
We define the **degree** of a vertex x to be $d(x) = |N(x)|$. This is equal to the number of edges that are incident to x.

Definition 1.1.14 (Regularity)
A graph G is said to be **regular** if all of the degrees are the same. We say G is k-regular if $d(x) = k$ for all $x \in V$.

Example 1.1.15 (Regular and Non-Regular Graphs)
The graphs K_n is $n-1$ regular, and C_n is 2-regular. The graph P_n is not regular.

§1.1.4 Connectivity

We now want to define some notion of *connectivity*, where a vertex u is connected to vertex v if you can follow some path in the graph to get from u to v.



For example, in the graph above we want to say somehow that u and v are connected, but u and v' are not. To do this, we will introduce some more definitions.

Definition 1.1.16 (uv Path)
A uv **path** is a sequence $x_1, x_2, \ldots, x_l$ where $x_1, \ldots, x_l$ are distinct, $x_1 = u$, $x_l = v$ and $x_i x_{i+1} \in E$.

In the example above, ux_1x_2v is a uv path.

The slight subtlety in this condition is the *distinctness* condition. For example, if $x_1 \ldots x_l$ is a uv path and $y_1 \ldots y_{l'}$ is a vw path, then $x_1 \ldots x_l y_1 \ldots y_{l'}$ may *not* be a uw path since we may have reused an edge. Of course, we can just not reuse edges by avoiding cycles.

Proposition 1.1.17 (Joining Paths)

If $x_1 \dots x_l$ is a uv path and $y_1 \dots y_{l'}$ is a vw path, then $x_1 \dots x_l y_1 \dots y_{l'}$ contains a uw path.

Proof. Choose a minimal subsequence $w_1 \dots w_r$ of $x_1 \dots x_l y_1 \dots y_{l'}$ such that

1. $w_i w_{i+1} \in E$.
2. $w_1 = u$, $w_r = w$.

We now claim that $w_1 \dots w_r$ is a uw path. If this was not the case, then it must fail on distinctness, so there would exist some z such that the sequence is

$$w_1 \dots w_a z w_{a+2} \dots w_b z w_{b+2} w_r,$$

but now note that

$$w_1 \dots w_a z w_{b+2} \dots w_r$$

also satisfies the conditions for the subsequence, but is strictly shorter length. This contradicts the minimality condition. □

Now given $G = (V, E)$, let's define an equivalence relation $\sim$ on V, where

$$x \sim y \iff \text{there exists an } xy \text{ path in } G.$$

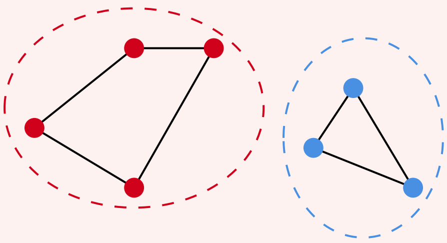
Proposition 1.1.18

$\sim$ is an equivalence relation.

Proof. Note that $\sim$ is reflexive and symmetric, and we get transitivity from our previous proposition. □

Example 1.1.19

In the graph below, the vertices that are the same colour are in the same equivalence class under $\sim$.



Definition 1.1.20 (Connected Graph)

If there is a path between any two vertices in G then we say that G is **connected**.

Definition 1.1.21 (Connected Components)

We call the equivalence classes of $\sim$ on G the **components** or **connected components** of G.

§1.1.5 Edges and Distance

We can also introduce some useful definitions relating to the edges of a graph.

Definition 1.1.22 (Minimum/Maximum Degree)
Let G be a graph. The **maximum degree** of G, $\triangle(G)$ is defined to be $\triangle(G) = \max_{x\in V} d(x)$. Similarly, we define the **minimum degree** of G, $\delta(G)$ to be $\delta(G) = \min_{x\in V} d(x)$.

In a k-regular graph as mentioned above, we have $\triangle(G) = \delta(G) = k$.

Definition 1.1.23 (Graph Distance)
Let $G = (V, E)$ be a graph. The associated **graph distance** $d : V \times V \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ is defined so that $d(x, y)$ is the minimum path length from x to y if it exists, and ∞ otherwise.

Proposition 1.1.24 (Graph Distance is a Metric)
Let $G = (V, E)$ be a connected graph and let d be the associated graph distance. Then (V, d) defines a metric space.

Proof Sketch. We have $d(x, y) = 0$, and $d(x, y) = d(y, x)$ (taking the shortest path in the opposite direction), and $d(x, z) \leq d(x, y) + d(y, z)$ as we can find path from x to z by taking paths from x to y and y to z and adjoining them, and this puts an upper bound on $d(x, z)$. $\square$

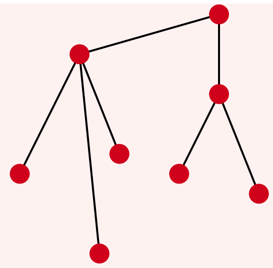
§1.2 Trees

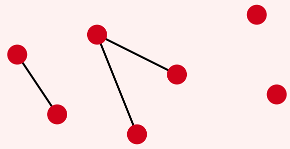
We will now discuss a special class of graph called *trees*. This class is quite restrictive (yet is quite useful), and they have some nice properties.

To define what a tree is, we first need a notion of when a graph is acyclic.

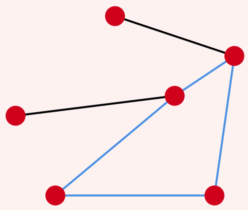
Definition 1.2.1 (Acyclic)
A graph G is said to be **acyclic** if it does not contain any subgraph isomorphic to a cycle, C_n.

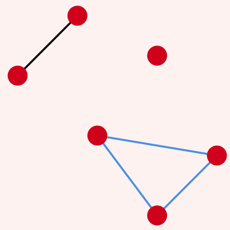
Example 1.2.2 (Example of Acyclic/Non-Acyclic Graphs)
In the example below, the two graphs are both *acyclic*.





Two *non-acyclic* graphs are shown below. The subgraphs isomorphic to C_4 and C_3 are highlighted.



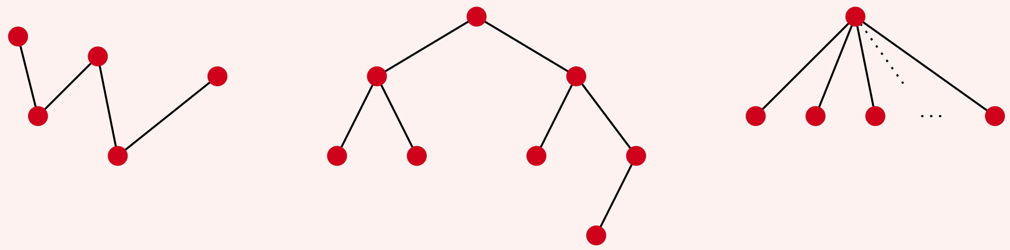


Definition 1.2.3 (Tree)

A **tree** is a connected, acyclic graph.

Example 1.2.4 (Examples of Trees)

The following three graphs are trees.



Proposition 1.2.5 (Characterising Trees)

The following are equivalent.

(a) G is a tree.

(b) G is a maximal acyclic graph (adding any edge creates a cycle).

(c) G is a minimal connected graph (removing any edge disconnects the graph).

Proof. *(a) $\implies$ (b).* By definition G is acyclic. Let $x, y \in V$ such that $xy \notin E$. As G is connected, there is an xy path P. So xPy then defines a cycle.

(b) $\implies$ (a). By definition G is acyclic. So for a contradiction assume G is not connected and let x, y be vertices from different components. Now note $G + xy$ is acyclic, but this contradicts the claim that G is maximally acyclic.

(a) $\implies$ *(c).* By definition G is connected. Suppose, for a contradiction, that there exists some vertices $x, y \in E$ with $x \neq y$ and $G - xy$ is connected. But then there is some xy path P that does not use the edge xy, so xPy is then a cycle, contradicting that G is acyclic.

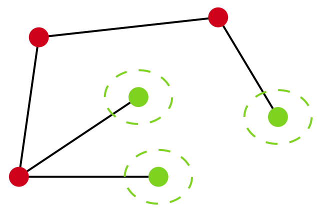
(c) $\implies$ *(a).* By definition G is connected. Again for a contradiction, assume that G contains a cycle C. Then let xy be an edge on C. We claim $G - xy$ is still connected. If $u, v \in V(G - xy)$ then let P be a path in G from u to v. If xy does not appear as consecutive vertices on this path, then u is connected to v. Otherwise, we can consider a new path where we replace x, y with the other vertices in $C - xy$ in order. Thus u and v are still connected. This contradicts the minimal connectedness of G.

□

Definition 1.2.6 (Leaf)

Let G be a graph. A vertex $v \in V(G)$ is a **leaf** if $d(v) = 1$.

For example, the tree below has three leaves.



In general, trees has a leaf.

Proposition 1.2.7 (Trees Have Leaves)

Every tree T with $|T| \geq 2$ has a leaf.

Proof. Let T be a tree with $|T| \geq 2$, and let P be a path of maximum length in T, with $P = x_1 \dots x_k$. We claim that $d(x_k) = 1$. Observe that $\deg(x_k) \geq 1$, since $x_k x_{k-1} \in E$. If x_k is adjacent to another vertex $y \neq x_{k-1}$, then either $y \in \{x_1, \dots, x_{k-2}\}$, which would imply thar T contains a cycle, or $y \notin \{x_1, \dots, x_{k-2}\}$, then $x_1 \dots x_k y$ is a path longer than P, which violates its maximality. □

Remark. This proof gives us two leaves in T, which is the best we can hope for considering P_n is a tree with exactly two leaves.

Proposition 1.2.8 (Edges of a Tree)

Let T be a tree. Then $e(T) = |T| - 1$.

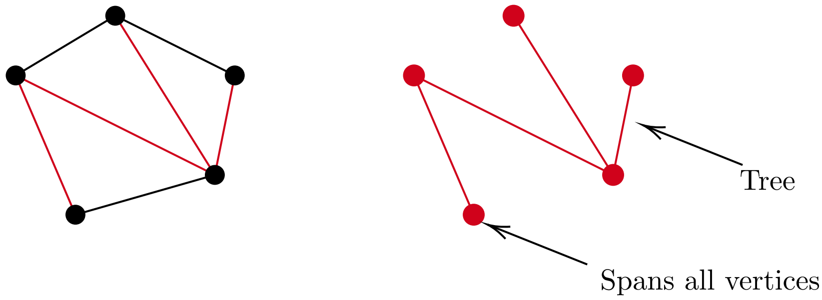
Proof. We will do induction on $n = |T|$. If $n = 1$, this is trivial as there is only one edge. Now given T with at least 2 vertices, let x be a leaf in T, and define $T' = T - x$.

T' must be acyclic, since we have only removed vertices. T' must also be connected since for all $u, v \in V(T')$ there exists a path from u to v in T that does not use x, so it is also a path from u to v in T'. Thus T' is a tree. Thus by induction, T' has $n-2$ edges, and $e(T) = e(T') + 1 = |T| - 1$. □

Now lets think about trees as subgraphs of other graphs.

Definition 1.2.9 (Spanning Tree)

Let G be a graph. We say T is a **spanning tree** of G if T is a tree on $V(G)$ and is a subgraph of G.



Spanning trees are useful in a number of contexts, one of which is giving a sensible ordering to the vertices of a graph. They are particularly useful because of the following result.

Proposition 1.2.10 (Connected Graphs have Spanning Trees)

Every connected graph contains a spanning tree.

Proof. A tree is a minimal connected graph. So take the connected graph and remove edges until it becomes a minimal connected graph. Then this will be a subgraph of the original graph, and will thus be a spanning tree. □

§1.3 Bipartite Graphs

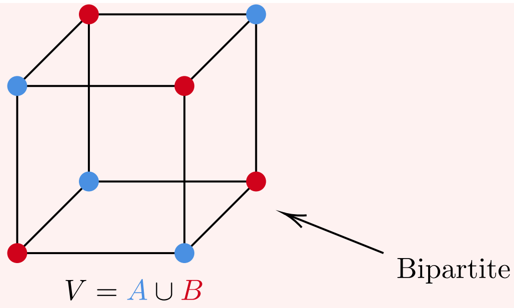
The next type of graph we will look at is *bipartite* graphs.

Definition 1.3.1 (Bipartite Graphs)

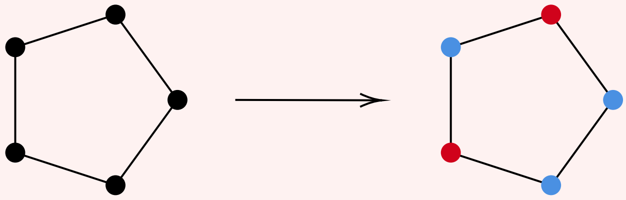
A graph $G = (V, E)$ is **bipartite** if $V = A \cup B$ where $A \cap B = \emptyset$ and all edges $xy \in E$ have either $x \in A, y \in B$ or $x \in B, y \in A$.

Example 1.3.2 (Example of Bipartite Graphs)

The graph below is bipartite, with vertices in the set A being coloured red and vertices in the set B being coloured blue.



An example of a non-bipartite graph is C_5. To see this, we can start by choosing a vertex to be in A (without loss of generality), then the adjacent vertices must be in B, but then their adjacent vertices must be in A, but then there is an edge between two vertices in A. This is shown below.



C_5 can't be bipartite

The argument given for C_5 works in general.

Proposition 1.3.3 (Bipartite Cyclic Graphs)
The cycle C_{2k+1} is *not* bipartite, and the cycle C_{2k} *is* bipartite.

Proof. Assume that C_{2k+1} is bipartite. Then there must be disjoint sets A and B, and as $2k+1$ is odd, we must have (without loss of generality), $|A| > |B|$. Now let's count the edges between A and B. This must be $2|A|$ and also $2|B|$, as every vertex has degree 2. But then $|A| = |B|$, which is a contradiction.

For C_{2n}, we can let $v_i \in A$ if i is even and $v_i \in B$ if i is odd. Then a vertex i only has edges to vertices $i-1$ and $i+1 \pmod 2$, which have opposite parity. Thus all edges are between A and B, as required. □

There is then a natural question: given some arbitrary graph G, how do we determine if a given graph is bipartite? It turns out that there is a nice check for 'bipartness'. We will state the result and then do some setup before we prove it.

Proposition 1.3.4 (Bipartite Criterion)
A graph G is bipartite if and only if G contains no odd cycles.

We need to first develop some theory regarding *circuits*. Informally, a circuit is like a cycle where we can revisit vertices.

Definition 1.3.5 (Circuit)
A circuit is a sequence $x_1 \dots x_l$ where $x_1 = x_l$ and $x_i x_{i+1} \in E$. The **length** of the circuit is $l-1$, the number of edges traversed in the circuit.

Definition 1.3.6 (Odd Circuits)
If the length of a circuit is odd, then we say it is an **odd circuit**.

Proposition 1.3.7
An odd circuit contains an odd cycle.

Proof. We will prove this by induction on the length of the circuit. For a circuit of length 3, the circuit must be a cycle. In general, let $C = x_1 \dots x_l$ be our circuit. If $x_1, \dots, x_{l-1}$ are distinct, then C is a cycle and we are done.

Otherwise, there exists some $z \in C$ that is repeated. We write

$$C = x_1 \dots x_a z x_{a+2} \dots x_b z x_{b+2} \dots x_l.$$

We define $C' = x_1 \dots x_a z x_{b+2} \dots x_l$ and $C'' = z x_{a+2} \dots x_b z$. The length of C' and C'' is strictly less than the length of C. One of these circuits must have odd length, and by induction that odd circuit contains an odd cycle. □

We can now prove our original bipartness criterion, that a graph is bipartite if and only if it contains no odd cycles.

Proof (Bipartite Criterion). If G was bipartite and contained an odd cycle, then there exists an odd cycle that is bipartite. But this is a contradiction.

Now if G is not bipartite, we can induct on the number of vertices. For $|G| = 1$, this holds. Now if G is not connected, let $C_1, \dots, C_k$ be the components of G. We may now apply our induction to each component of G to obtain a bipartition $V(C_i) = A_i \cup B_i$ for each $i \in 1, \dots, k$. Then $A = A_1 \cup \dots A_k$ and $B = B_1 \cup B_k$ is a bipartition for the whole graph.

We may now assume without loss of generality that G is connected. Fix some vertex $v \in V$, and define

$$A = \{u \in V \mid d(u, v) \text{ is odd}\}$$
$$B = \{u \in V \mid d(u, v) \text{ is even}\}$$

We claim that $A \cup B$ is a bipartition.

Assume (for a contradiction) that u_1 is adjacent to u_2 and $d(u_1, v) \cong d(u_2, v) \pmod 2$. Then there exists paths P_1 from v to u_1 and P_2 from u_2 to v with $|P_1| \equiv |P_2|$. But this implies that $vP_1u_1u_2P_2$ defines a odd circuit in G. Therefore, by our previous proposition, G contains an odd cycle, which is a contradiction. □

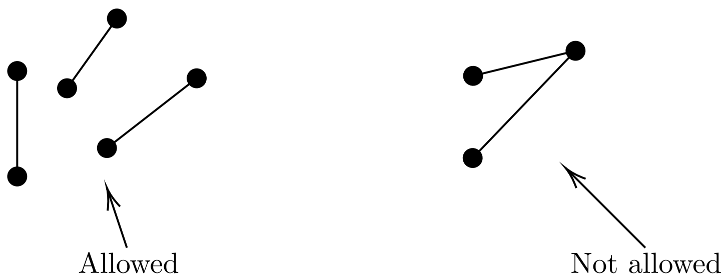
2 Hall's Theorem

In this chapter, we will build up our knowledge of *matchings* so that we can prove the first theorem of the course – Hall's theorem.

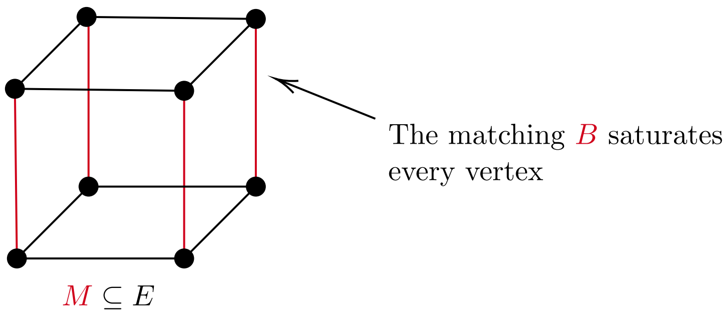
§2.1 Matchings

An appropriate place to start is probably by defining what a matching is.

Definition 2.1.1 (Matching)
A **matching** of G is a collection of edges $M \subseteq E$ so that $\forall e_1, e_2 \in M$ with $e_1 \neq e_2$ has $e_1 \cap e_2 = \emptyset$.



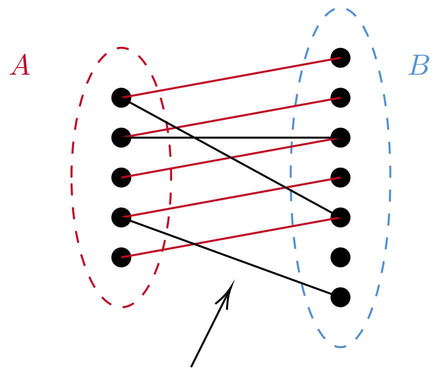
Definition 2.1.2 (Saturated)
Given a graph $G = (V, E)$ and a matching M in G, we say that a vertex $v \in V$ is **saturated by M** if there exists an edge in M containing v.



A matching where every vertex is saturated (such as the above) is known as a **perfect matching**. However, there is no restriction in general on how many vertices are saturated by a matching.

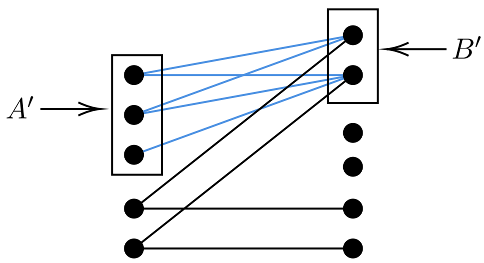
§2.2 Matching in Bipartite Graphs

Matchings are particularly interesting in bipartite graphs. We will be interested in the following question: given a bipartite graph $G = (A \cup B, E)$, when can I find a matching saturating A?



This question is the same as asking when is there a function $f : A \to B$ where $xf(x) \in E$ that is an injection.

In trying to answer this question, we might try and think about why it may not be possible. The simplest reason is when B isn't large enough to have an injection, when $|B| \leq |A|$. In a similar way, we might have a graph is big enough, but that isn't true for a small part of the graph, like below.



What Hall's theorem says is that this issue is the only obstruction to creating such a matching.

Definition 2.2.1 (Neighbourhood of a Vertex Set)

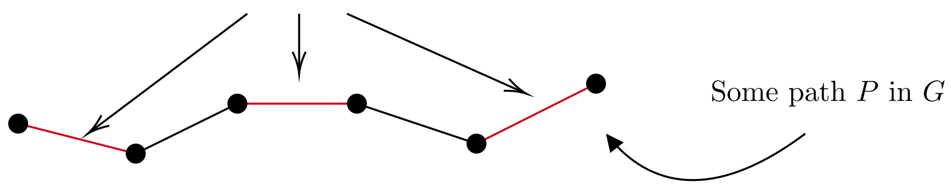
If $X \subseteq V$, then we define $N(X)$, the **neighbourhood of X** to be $N(X) = \bigcup_{x \in X} N(x)$.

With the notion of the neighborhood of a set of vertices, we can rephrase the issue mentioned about as $|N(A')| < |A'|$ for some subset $A' \subseteq A$. We will use this notation in our statement for Hall's theorem.

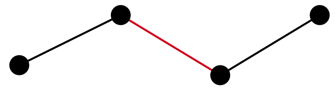
Before we prove the theorem, we will need to prepare a little bit.

Definition 2.2.2 (M-Alternating Path)

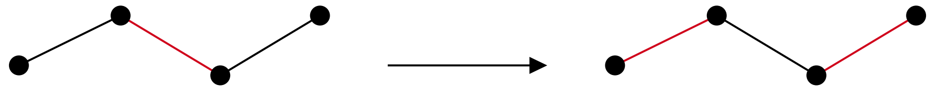
Let $G = (V, E)$ be a graph with a matching M in G. We say that a path $P = x_1 \dots x_l$ is **M-alternating** if $x_i x_{i+1}$ is alternately in M and not in M.



Another example of an M-alternating path is the one below.



If we saw the path above in the graph, and we knew that the end vertices was not saturated, then we could change the edges that are in M, so we would still have a matching. This move will be key in our proof of Hall's theorem.



We are going to call this configuration an *augmented path*.

Definition 2.2.3 (M-Augmenting Path)

Given a graph $G = (V, E)$ and a matching M in G, an M-alternating path $P = x_1 \dots x_l$ is said to be **M-augmenting** if x_1 and x_l are not saturated.

Proposition 2.2.4

If M is a matching in G of maximum size, then there are no M-augmenting paths.

Proof. If there is an M-augmenting path in G, then we can flip the edges of M along P to find a strictly larger matching. □

An important observation is that an M-alternating path in a bipartite graph $G = (A \cup B, E)$ with $P = x_1 \dots x_l$, with $x_1 \in A$ and $x_1 x_2 \in M$, then $x_{2k+1} x_{2k+2} \in M$, and vice versa.

§2.3 Hall's Theorem

We are now ready to state and proof Hall's matching theorem, which formalises the ideas mentioned in the previous section.

Theorem 2.3.1 (Hall's Theorem)

Let $G = (A \cup B, E)$ be a bipartite graph. Then there exists a matching saturating A if and only if every subset $A' \subseteq A$ satisfies $|N(A')| \geq |A'|$.

Proof. First we prove the forward (and easy) direction. Let $A' = \{x_1, \dots, x_t\} \subseteq A$. We have matching edges $x_1 y_1, \dots, x_t y_t \in M \subseteq E$, and thus $\{y_1, \dots, y_t\} \subseteq N(A')$, and thus $|N(A')| \geq |A'|$.

Now for the other (harder) direction, which is that this condition implies the existence of such a matching. Choose a matching M in G with $|M|$ maximized. For a contradiction, assume there is some vertex $a_0 \in A$ that is not saturated by M.

We inductively define sets $A_i \subseteq A$, $B_t \subseteq B$ by setting $A_0 = \{a_0\}$, $B_0 = \emptyset$. We will maintain, for all t, that

1. $|A_t| = t + 1$, $|B_t| = t$.
2. Every vertex in $A_t \cup B_t$ is the endpoint of an alternating path that started at a_0.
3. $A_t \backslash \{a_0\}$ is matched to B_t.

So given A_t, B_t, we need to define A_{t+1}, B_{t+1}.

First consider $N(A_t)$. We have $|N(A_t)| \geq |A_t| = t + 1 > |B_t|$. So $N(A_t) \backslash B_t$ is non-empty. So let $b_{t+1} \in N(A_t) \backslash B_t$. Observe that b_{t+1} can be reached along an alternating path which started at a_0. Call $y \in A_t$ such that $yb_{t+1} \in E$. Then y can be reached along an alternating path starting at a_0. Let $P = a_0 x_1 \ldots x_l y$ be such an alternating path. Since $a_0 x_1 \notin M$, $x_l y \in M$. Hence Pb_{t+1} is a alternating path.

Now, b_{t+1} is saturated by M, as otherwise Pb_{t+1} would be an M-augmenting path, which is a contradiction. So let $a_{t+1} b_{t+1} \in M$. We claim $a_{t+1} \notin A_t$, since A_t is matched to B_t, and $b_{t+1} \notin B_t$. So we may define $A_{t+1} = A_t \cup \{a_{t+1}\}$ and $B_{t+1} = B_t \cup \{b_{t+1}\}$. We can check that what we claimed before holds.

Since $a_{t+1} \notin A_t$, and $b_{t+1} \notin B_t$, we have $|A_{t+1}| = |A_t| + 1$ and $|B_{t+1}| = |B_t| + 1$. Also every vertex is the endpoint of an alternating path starting at a_0 by construction. Also $A_{t+1} \backslash \{a_0\}$ is matched to B_{t+1} since $A_t \backslash \{a_0\}$ is matched to B_t, and by construction.

This completes the construction of A_t, B_t for all t. Then if $t > |A|$, then $|A_t| > |A|$, but $A_t \subseteq A$, which is a contradiction. $\square$

§2.4 Corollaries of Hall's Theorem

So Hall's theorem gives us a nice way to guarantee the existence of matchings that saturate a part of a bipartite graph. We are going to use Hall's theorem to prove some interesting results.

Corollary 2.4.1

A k-regular bipartite graph contains a perfect matching.

Proof. Let $G = (A \cup B, E)$, and let $A' \subseteq A$. We want to show that $|N(A')| \geq |A'|$ so that we may apply Hall's theorem.

We will count the number of edges between A' and $N(A')$ in two different ways. We know that each vertex in A' has degree k, so there is $k|A|$ edges. But on the other hand, the number of edges in $N(A')$ is $|N(A')k|$. Thus $|N(A')k| \geq |A'|k$, so $|N(A')| \geq |A'|$, and by Hall's theorem there exists a matching saturating A.

We claim that $|B| = |A|$. Indeed, the number of edges between A and B is $k|A|$ and is also $k|B|$, and thus $|A| = |B|$. So a matching saturating A also saturates B. So there exists a perfect matching. $\square$

We can also consider an extension of Hall's theorem. Hall's theorem tells us when, in a bipartite graph $G = (A \cup B, E)$, there is a matching saturating A, but what if we only wanted a matching of size k?

Let's say that a matching in G has **deficiency** d if it saturates $|A| - d$ vertices.

Corollary 2.4.2

Let G be a bipartite graph. Then G contains a matching saturating $|A| - d$ vertices in A if and only if for all $A' \subseteq A$, we have $|N(A')| \geq |A'| - d$.

Proof. If we have a matching with deficiency d, then this clearly holds.

Now if this condition holds for some graph $G = (A \cup B, E)$, we define a new graph $\tilde{G}$ by by setting $\tilde{B} = B \cup \{z_1, \ldots, z_d\}$ where $z_1, \ldots, z_d$ are distinct from $x \in A \cup B$, and we define $\tilde{E} = E \cup \{e_i a \mid i \in \{1, \ldots, d\}, a \in A\}$. So $\tilde{G} = (A \cup \tilde{B}, \tilde{E})$.

This is a bipartite graph, and we observe that is satisfies Hall's condition. Thus $\tilde{G}$ has a matching M saturating A. So if we define M' by removing all of the edges with an endpoint in $\{z_1, \ldots, z_d\}$, then M' saturates all but at most d vertices in G. □

Another way of stating Hall's theorem is with set systems.

Definition 2.4.3 (System of Distinct Representatives)

Gives sets $S_1, \ldots, S_n \subset X$ where S_i is finite for all i. Then we say that $x_1, \ldots, x_n \in X$ is a **system of distinct representatives** (or SDR) if they are distinct and $x_i \in S_i$.

The question is, for what set systems does there exist a system of distinct representatives? The answer is if they satisfy some Hall-like condition.

Corollary 2.4.4 (Existence of SDRs)

Given $S_1, \ldots, S_n \subseteq X$, with S_i finite, then $S_1, \cdots S_n$ has a system of distinct representatives if and only if

$$\left| \bigcup_{i \in I} S_i \right| \geq |I|,$$

for all $I \in \{1, \ldots, n\}$.

Proof. If such an SDR exists, then this condition clearly holds.

In the other direction, given that this condition holds, we define a graph G by setting $A = \{S_1, \ldots, S_n\}$ and $B = \bigcup_{i=1}^n S_i$, with $E = \{\{S_i, x\} \mid x \in A,\ i \in \{1, \ldots, n\},\ x \in S_i\}$. Then $G = (A \cup B, E)$. Now observe that a SDR is exactly a matching in G that saturates A.

We check Hall's condition. Given $A' \subseteq A$, then $N(A') = \bigcup_{S_i \in A'} S_i$. Thus

$$|N(A')| = \left| \bigcup_{S_i \in A'} S_i \right| \geq |A'|,$$

by our condition. So such a matching exists. □

Remark. The existence of SDRs is equivalent to Hall's theorem.

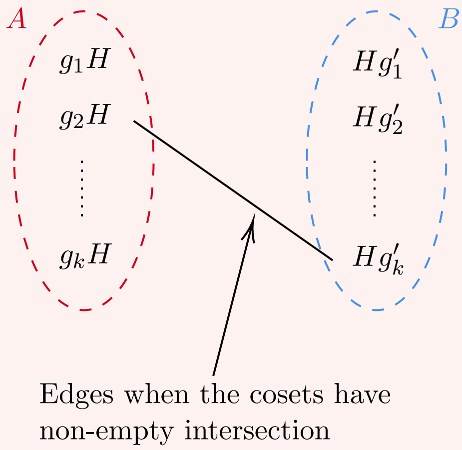
The last example of using Hall's theorem will be an application to group theory.

Example 2.4.5 (Left and Right Coset Representatives)

Let G be a group and let $H < G$, with G finite. We have the left cosets $g_1H, \dots, g_kH$ and right cosets $Hg'_1, \dots, Hg'_k$.

We want to know does there exists $h_1, \dots, h_k$ such that $h_1H, \dots, h_kH$ are all of the left cosets and $Hh_1, \dots, Hh_k$ are all of the right cosets. It turns out that this question is entirely combinatorial, and we can use Hall's theorem.

Let's form the bipartite graph $(A \cup B, E)$ with $A = \{g_1H, \dots, g_kH\}$ and $B = \{Hg'_1, \dots, Hg'_k\}$ so that there is an edge between g_iH and Hg'_j if the cosets have non empty intersection.



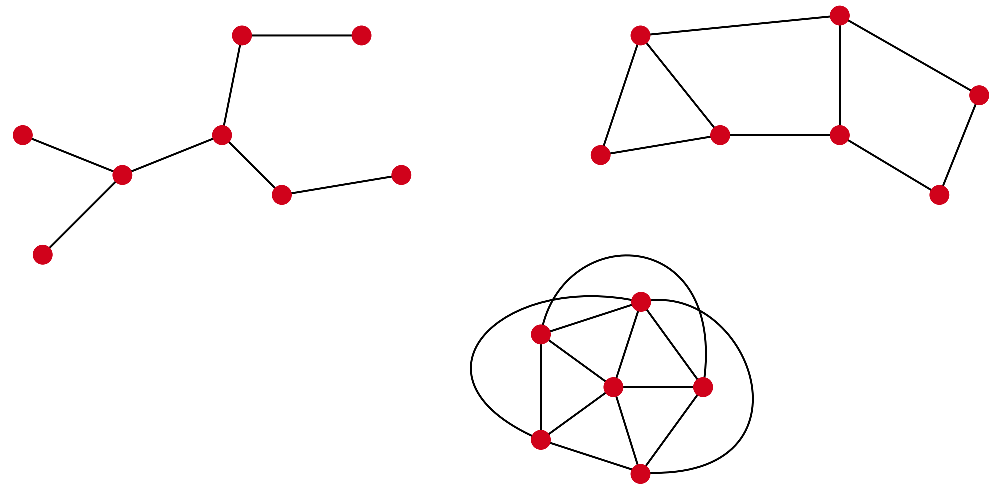
If we had a perfect matching in this graph, then we could pick an element h_i in the (non-empty) intersection of the matched left and right cosets, and then $h_1, \dots, h_k$ would be a set of coset representatives for all of the k left and right cosets.

So we want to show that Hall's theorem is satisfied. Given $I \subseteq \{1, \dots, k\}$, we want to show that the number of right cosets intersecting $\bigcup_{i \in I} g_iH$ is at least $|I|$.

Observe that $\left|\bigcup_{i \in I} g_iH\right| = |H||I|$. Since the right cosets partition G, and each coset has size $|H|$, so the number of right cosets intersecting $\bigcup g_iH$ is at least $|I|$. Thus $|N(\{g_iH \mid i \in I\})| \geq |I|$, and thus Hall's theorem is satisfied, and we have a perfect matching in the graph.

3 Connectivity

We have already defined what it means for a graph to be connected, but consider the following connected graphs:



Clearly these are connected, but they are all 'connected to different extents'. For example, in the first graph, removing any vertex disconnects the graph. In the second graph, any vertex could be removed and the graph would stay connected. We can also see that in the first graph, there's only one path from one vertex to another, whereas in the third graph there is many. This also seems to correlate with 'how connected' a graph is.

§3.1 Measuring Connectivity

So looking at the graphs above, we two natural notions for 'how connected a graph is' can be informally described as

1. A 'deletion notion' of connectivity, where we consider how connected the graph is after some vertices are removed.
2. A 'paths notion' of connectivity, where we consider how many independent paths there is between vertices.

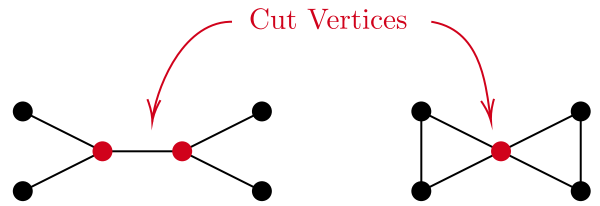
One of the main goals of this chapter will be turning this vague notion into a concrete concept, and proving an interesting result about how the two notions relate to each other.

Notation (Removing Vertex Sets)**.** In this chapter (and for the rest of this course), if $G = (V, E)$ is a graph and $S \subseteq V$, we define $G - S = G - x_1 - x_2 - \cdots - x_l$ where $S = \{x_1, \ldots, x_l\}$.

We will begin with a definition.

Definition 3.1.1 (Cut Vertex)
Let $G = (V, E)$ be a connected graph. We say that $v \in V$ is a **cut vertex** if $G - v$

is disconnected.



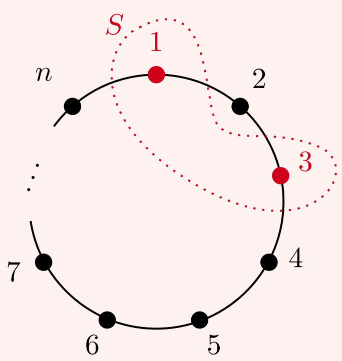
Definition 3.1.2 (Seperator)
If $G = (G, V)$ is a connected graph, we say that a subset $S \subseteq V$ is a **separator** (or separating set) if $G - S$ is disconnected.

With these concepts defined, we can define our 'deletion' notion of connectivity.

Definition 3.1.3 (Connectivity)
Let $G = (V, E)$ be a graph. The **connectivity** of G, denoted $\kappa(G)$, is the size of the smallest set $S \subseteq V$ such that $G - S$ is disconnected, or just a single vertex[a].

[a] This is needed for the case of a complete graph.

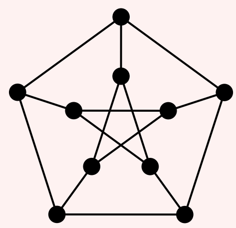
Example 3.1.4 (Connectivity of C_n)
Consider the graph C_n for $n \geq 3$.



Removing one vertex will never disconnect C_n.
However, removing two vertices can disconnect it.

$$\implies \kappa(C_n) = 2$$

Example 3.1.5 (Connectivity of the Petersen Graph)
We define the **Petersen graph** with 10 vertices and 15 edges as shown below.



We can see that the connectivity is at most 3, since that is the degree of each vertex, and also removing two vertices won't disconnect the graph. Thus the connectivity of the Petersen graph is exactly 3.

Definition 3.1.6 (k-Connected)
We say that a graph G is **k-connected** if $\kappa(G) \geq k$. In particular, any set $S \leq V(G)$ with $|S| < k$ will have $G - S$ connected.

We note this immediately implies that G is 1-connected if and only if it's connected, and it is 2-connected if and only if it has no cut vertex.

We can note some basic properties of connnectivity.

Lemma 3.1.7 (Increasing/Reducing Connectivity)
If $G = (V, E)$ is a k-connected graph and $v \in V$ then $G - v$ is $(k-1)$-connected[a]. Also, if we have some $e \in E$, then $G - E$ is $(k-1)$-connected.

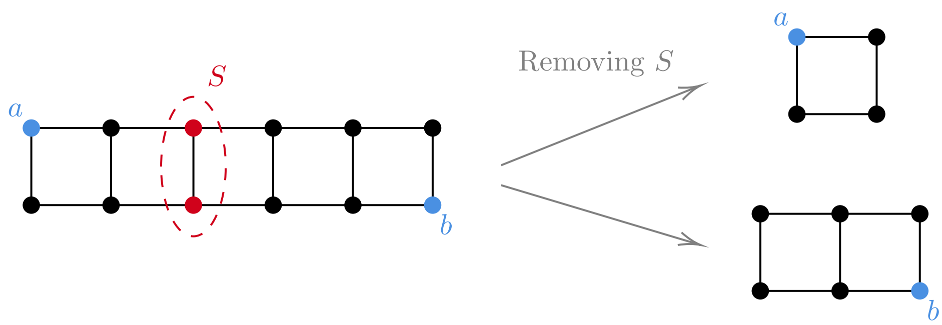
[a] Note that it is also possible that $G - v$ is $(k+1)$-connected, so connectivity can also increase by deleting a vertex.

Proof Sketch. Check definitions. □

§3.2 Preparing for Menger's Theorem

We are going to prove Menger's theorem about connectivity measures, but we have to do some setting up beforehand.

Definition 3.2.1 (ab-Seperator)
If $G = (V, E)$ is a graph and $a, b \in V$ are distinct vertices, we say that S is an **ab seperator** if a and b lie in different components of $G - S$ (so $a, b \notin S$).



Definition 3.2.2 (Seperator)
If $G = (V, E)$ is a graph and $F \subseteq E$, we say that F is a **separator** of G if $G - F$ is disconnected.

We then have a number of useful lemmas, that we will employ later on.

Notation. We let $\kappa_{a,b}(G)$ be the minimum size of an ab separator.

Lemma 3.2.3
Let $G = (V, E)$ be a graph. Then $\kappa_{a,b}(G) \geq \kappa_{a,b}(G - v)$, where $v \in V$ and $v \neq a, b$.

Proof. If $G - v$ has an ab separator S then $S \cup \{v\}$ is an ab separator in G. □

Lemma 3.2.4

For a graph $G = (V, E)$, $\kappa_{a,b}(G - e) \geq \kappa_{a,b}(G) = 1$, for $e \in E$ and $a \sim b$.

Proof. If $G - e$ has an ab separator S then $S \cup \{x\}$ and $S \cup \{y\}$ are ab separators in G, where $e = xy$. □

Lemma 3.2.5

Let $G = (V, E)$ be a graph with distinct non-adjacent vertices $a, b \in V$. Also let $\kappa_{a,b}(G) \geq k$. Let S be an ab separator in G, and say $G - S = A \cup C$, where A is the connected component containing a. Then define $\tilde{G}$ as the induced graph $G[A \cup S]$ with a vertex x joined to all of S. Then $\kappa_{a,x}(\tilde{G}) \geq k$.

Proof Sketch. Check definitions. □

We can now formalize our 'independent paths' notion of connectivity.

Definition 3.2.6 (Independent Paths)

We say that $P_1, \dots, P_k$ are **independent ab paths** if each of $P_1, \dots, P_k$ are ab-paths and all of the vertices in these paths are distinct, apart from a and b.

§3.3 Actually Menger's Theorem

Finally, we can state and prove Menger's theorem.

Theorem 3.3.1 (Menger's Theorem, First Form)

Let $G = (V, E)$ be a graph, with distinct and non-adjacent $a, b \in V$. If every ab separator in G has size at least k then we can find k independent ab paths.

Proof. Suppose for a contradiction, assume this is false and let G be the counterexample that

(1) Minimizes $\kappa_{a,b}(a)$

(2) Subject to (1), minimizes the number of edges in the graph.

Then we observe that $\kappa_{a,b}(G - e) = \kappa_{a,b}(G) - 1$ for any edge $e \in E$. We will let $k = \kappa_{a,b}(G)$.

Claim. There exists an ab separator S with $|S| = k$ so that $S \not\subseteq N(a)$ and $S \not\subseteq N(b)$.

We first observe that $N(a) \cap N(b)$ is empty. To see this, let $x \in N(a) \cap N(b)$ and $G' = G - x$. Then $\kappa_{a,b}(G') \geq \kappa_{a,b}(G) - 1$, thus there are $k - 1$ independent paths $P_1, \dots, P_{k-1}$ in G', then $P_1, \dots, P_{k-1}, axb$ are k independent paths in G, and our graph would then not be a counterexample.

Now choose a shortest path $P = ax_1 \dots x_l b$. Let $G' = G - x_1x_2$. We know that $\kappa_{a,b}(G') = k - 1$. Therefore there exists an ab separator S' of size $k - 1$. Let A be the component of a in $G' - S'$, and B be the component of b in $G' - S'$. Note that x_1x_2 must have $x_1 \in A$ and $x_2 \in B$.

Note $x_1 \sim a$, and $x_2 \not\sim a$ since P is the shortest path. Also note that $x_2 \neq b$, since $N(a) \cap N(b) = \emptyset$.

If $S' \subseteq N(a)$, then $S' \cup \{x_2\}$ is an ab separator of size k, and $S \not\subseteq N(a), N(b)$. If $S' \subseteq N(b)$, then $S = S' \cup \{x_1\}$ is an ab separator of size k, and $S \not\subseteq N(b), N(a)$. Lastly, if $S' \not\subseteq N(a)$ and $S' \not\subseteq N(b)$, then $S = S' \cup \{x_1\}$ is an ab separator of size k that is not in $N(a)$ or $N(b)$. So our claim holds.

So let S be an ab separator with $|S| = k$ and $S \not\subseteq N(a), N(b)$. Define A, B to be the components containing a and b in $G - S$. We also define $\tilde{G}_a$ as $G[A \cup S]$ with a vertex x that joins to all of S. define $\tilde{G}_b$ likewise. We now have

$$e(\tilde{G}_a) < e(G), \quad e(\tilde{G}_b) < e(G).$$

We also have $\kappa_{a,b}(\tilde{G}_a) = k = \kappa_{a,b}(\tilde{G}_b)$. Thus $\tilde{G}_a$ and $\tilde{G}_b$ satisfy the theorem, by minimality.

So we can find independent ax paths $P_1, \dots, P_k \in \tilde{G}_a$ and yb paths $Q_1, \dots, Q_k \in \tilde{G}_b$. Thus we can find k independent ab paths by concatenation and reordering in G, as desired. □

Remark. It should be noted that we need the non-adjacent condition, otherwise there is no ab separator. Before we write down the proof, we will isolate some notable facts about connectivity. Also, this result implies Hall's theorem.

Another form of Menger's theorem is more common.

Theorem 3.3.2 (Menger's Theorem, Second Form)

Let $G = (V, E)$ be a graph. Then G is k-connected if and only for all $u, v \in V$ with $u \neq v$, there exists k independent uv-paths.

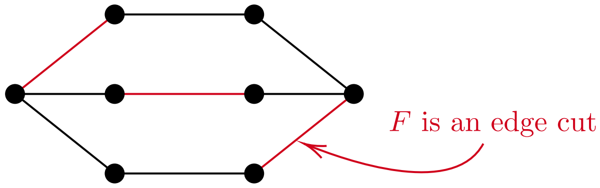
Proof. If u is not adjacent fo v, then apply Menger's theorem (first form) to find k-independent uv paths. If they are adjacent, then $G' = G - uv$ is $k - 1$ connected, thus Menger's theorem (first form) tells us that there are uv independent paths $P_1, \dots, P_{k-1}$ in $G - uv$. Thus $P_1, \dots, P_{k-1}, uv$ are k independent paths. The other direction is straightforward. □

§3.4 Edge Connectivity

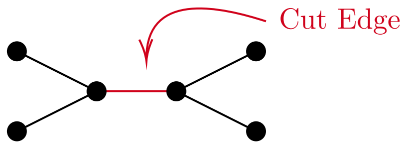
We have so far been looking at connectivity related to removing vertices and considering independent paths. We will now look at connectivity related to removing edges, and we will see that Menger's theorem is still useful.

Notation. If $G = (V, E)$ is a graph and $F \subseteq E$, define $G - F = (V, E \backslash F)$.

Definition 3.4.1 (Edge Cut)
An **edge cut** in a graph $G = (V, E)$ is a set $F \subseteq E$ so that $G - F$ is disconnected.



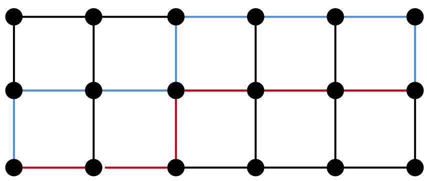
Definition 3.4.2 (Cut Edge)
A **cut edge** $e \in E$ is an edge so that $G - e$ is disconnected.



Now similarly to how we had independent paths before (that didn't share vertices), we can define a notion of edge independent paths.

Definition 3.4.3 (Edge Disjoint Paths)
We say that the uv paths $P_1, \ldots, P_k$ are **edge disjoint** if $E(P_i) \cap E(P_j) = \emptyset$ for all $i \neq j$.

The blue and red paths are edge disjoint



We can then define edge connectivity (our deletion notion).

Definition 3.4.4 (Edge Connectivity)
Define the **edge connectivity** of G, $\lambda(G)$ to be the smallest $|S|$ with $S \subseteq E$ such that $G - S$ is disconnected.

Definition 3.4.5 (k-Edge-Connected)
We say a graph G is **k-edge-connected** if $\lambda(G) \geq k$. In other words, $G - F$ is connected for all $|F| \leq k - 1$, with $F \subseteq E$.

We note that G is 1-edge-connected if and only if it is connected, and it is 2-edge-connected if and only if there is no cut edge.

We have a Menger's theorem for this notion of connectivity also.

Theorem 3.4.6 (Menger's Theorem, Edge Version)

Let $G = (V, E)$ be a graph, and u, v be distinct vertices of G. If every set of edges $F \subseteq E$ that separates u from v has size greater than or equal to k, then there exists k edge disjoint paths from u to v.

We also have a similar second version.

Theorem 3.4.7 (Menger's Theorem, Edge Version Two)

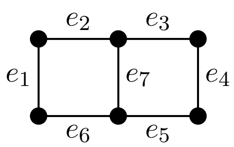
Let $G = (V, E)$ be a graph. Then G is k-edge-connected if and only if for every $u, v \in V$ with $u \neq v$ there exists k edge disjoint uv-paths $P_1, \dots, P_k$.

We are going to prove this by constructing a graph that we can then get the required result from my applying vertex Menger. The construction will be based on the idea of a *line graph*.

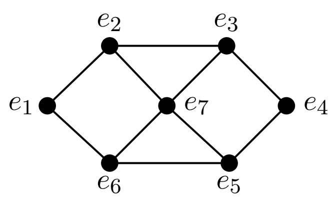
Definition 3.4.8 (Line Graph)

Given a graph $G = (V, E)$, we define $L(G)$ to be the **line graph** as follows. $V(L(G)) = E$, and for $e, f \in E$, we have $ef \in E(L(G))$ if $e \cap f \neq \emptyset$.

An example of a graph and its corresponding line graph is shown below.



G



$L(G)$

We can now prove the edge version of Menger's theorem.

Proof (Menger's Theorem, Edge Version). We are given a graph $G = (V, E)$ and distinct vertices $u, v \in V$ such that every u, v separator has size $\geq k$.

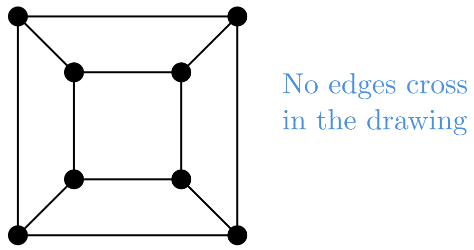
Construct the graph $\tilde{G}$ which is $L(G)$ along with vertices u and v with u joined to all $e \in V(L(G))$ such that $u \in e$, and likewise for v.

Then applying Menger's theorem (the vertex version, form one) to $\tilde{G}$ with u, v as the distinguished vertices. □

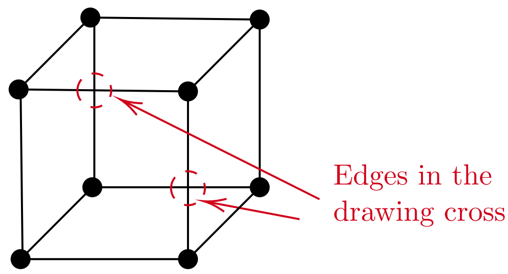
4 Planar Graphs

Informally, a graph is **planar** if it can be drawn in the plane without any pair of edges crossing.

For example the cube graph we mentioned earlier is planar, as we can draw it as below.



Of course, a graph is planar only if *there is some drawing* where the edges don't intersect. For example, we could draw the cube graph as below (where edges intersect), and the graph would still be planar.



§4.1 Defining Planar Graphs

Let's formalize the notion of 'planar graphs' a bit. First we will define (a somewhat obvious notion) what we mean for a graph to be in a plane.

Definition 4.1.1 (Plane Graph)
A **plane graph** is a finite set of points $V \subseteq \mathbb{R}^2$ and a collection of disjoint polygonal curves (representing the edges) with start and endpoints in V.

Then we can define what it means for a graph to be *planar*.

Definition 4.1.2 (Planar Graph)
A graph G is **planar** if there exists a graph isomorphism from G to some plane graph.

And again, informally this says that a graph is planar if there is some way to draw it in the plane so that edges don't intersect.

We can also define the notion of 'faces' of a plane graph, by looking at the components.

Definition 4.1.3 (Faces)

The **faces** of a plane graph G are the connected components of $\mathbb{R}^2 - G$.

With this we get a nice relation between the vertices, edges and faces of a plane graph.

Theorem 4.1.4 (Euler's Formula)

Let G be a connected plane graph with V vertices, E edges and F faces. Then

$$V - E + F = 2.$$

Proof. We apply induction on the number of edges of G. The base case is $E = 0$, and then since G is connected, $V = 1$, and $F = 1$, and the formula holds. Now if G contains a cycle C, then let e be on C. Then $G' = G - e$ is still connected, and the number of faces increases by 1. The face enclosed is lost. So considering G' and applying Euler's formula, $V-(E-1)+(F-1) = 2$, and $V-E+F = 2$, as required.

If G does not contain a cycle, then G is acyclic and connected and is then a tree. Then $E = V - 1$, $F = 1$, and $V - E + F = 2$, as required. □

Corollary 4.1.5 (Planar Graphs are Sparse)

Let $G = (V, E)$ be a planar graph, with $|V| \geq 3$. Then $|E| \leq 3|V| - 6$. This bound is also sharp[a].

[a]By considering plane graphs where every face is a triangle.

Proof. We may assume without loss of generality that G is connected (if not, then we can add edges until it is connected). Also draw G in the plane so that there is no edge crossings. Then by Euler's formula we have $V - E + F = 2$.

Now since ever face has at least three edges on its boundary and every edge on the boundary is incident to at most two faces, we obtain

$$3F \leq \left|\{(e', f') \mid e' \in E, f' \text{ is a face, and } e' \text{ is on the boundary of } f'\}\right| \leq 2E.$$

Thus $3F \leq 2E$, so $3(2 - V + E) \leq 2E$, and $E \leq 3V - 6$, as required. □

§4.2 What Graphs are Planar?

In this chapter we really care about the following question:

What graphs are planar?

The theorems proved at the end of the last section already give us a small amount of control over planar graphs. For example, we can show that K_5 is not planar.

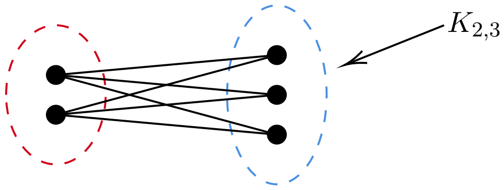
Example 4.2.1 (K_5 is Non-Planar)

We will show that K_5 is not planar. Since the number of edges is 10, and the number of vertices is 5, by the previous result we would need $10 \leq 3 - 5 + 6$, which is not the case.

Let's consider another type of graph.

Definition 4.2.2 (Bipartite Complete Graph)

Define the **bipartite complete graph** $K_{n,m}$ to be the graph with vertex set $V = A \cup B$ where $A \cap B = \emptyset$ so that $|A| = n$, $|B| = m$, and $E(K_{n,m}) = \{ab \mid a \in A, b \in B\}$.



If you play around a bit, you can see that $K_{3,2}$ is planar, but we get a problem if we try use $K_{3,3}$.

Example 4.2.3 ($K_{3,3}$ is Non-Planar)

We will show that $K_{3,3}$ is not planar. Note that there is 9 edges, and 6 vertices. Then by the result in the previous section, we would need $9 \leq 3 \cdot 6 - 6$, which holds.

However, recall in the proof that of our result that we had each face having at least three edges on its boundary. But that for bipartite graphs, we can get something stronger, as each face has at least four edges on its boundary (since there is no cycles of length 3).

So if we repeat the proof of the previous theorem with the stronger bound, we obtain $E \leq 2V - 4$, which does not hold for this graph. Thus $K_{3,3}$ is not planar.

Now these two examples were interesting, but of course we care about whether *any* graph is planar. It turns out though that these are (in some sense) the *only* fundamentally non-planar graphs, in that any non planar graph will have one of these graphs 'behind it'.

To look at this formally, we need to look at the idea of a subdivision.

Definition 4.2.4 (Subdivision)

A **subdivision** of a graph G is a graph $\tilde{G}$, obtained by replacing the edges of G with disjoint paths.

Lemma 4.2.5

If G is non-planar, then a subdivision of G is non-planar also.

Proof. Given a plane drawing of the subdivided graph, then by disregarding the vertices on the subdivided paths, we obtain a plane drawing of G (which is a contradiction). □

Corollary 4.2.6

Subdivisions of $K_{3,3}$ and K_5 are non-planar.

Proof. Follows directly from the previous lemma. □

What ties all of this together is Kuratowski's theorem, which gives us a nice necessary and sufficient condition for a graph to be planar.

Theorem 4.2.7 (Kuratowski's Theorem)

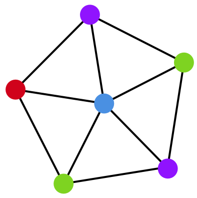
A graph G is planar if and only if G contains no subdivisions of $K_{3,3}$ or K_5.

Proof. Omitted. □

We will leave the topic of planar graphs here (for now), but we note that there are some other interesting notions that are related to planarity. For example, for what graphs is it possible to draw on a torus with no edge crossings?

5 Graph Colouring

Informally, a graph colouring is just a way of colouring in different vertices of a graph, so that adjacent vertices are different colours. An example of a graph colouring is shown below.



§5.1 Basic Concepts

Of course, we need to define what this means in a slightly more mathematical sense, so we will define a colouring as follows.

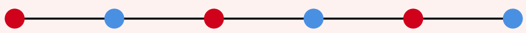
Notation. We will write $[n] = \{1, 2, \ldots, n\}$.

Definition 5.1.1 (r-Colouring)
Let $G = (V, E)$ be a graph. An ***r*-colouring** of G is a function $c : V \to [r]$ that satisfies $xy \in E \implies c(x) \neq c(y)$.

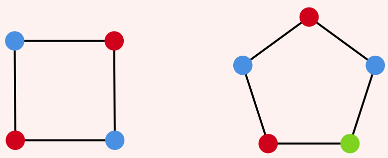
An r-colouring divides up a graph into r different **colour classes**, where there is only edges between the different colour classes.

Definition 5.1.2 (Chromatic Number)
The **chromatic number** of a graph G, denoted $\chi(G)$ is the smallest r for which there exists an r-colouring.

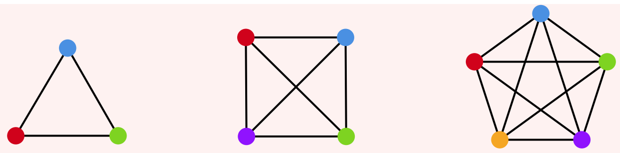
Example 5.1.3 (Examples of Chromatic Numbers)
In the graph P_n, we have $\chi(P_n) = 2$ for $n \geq 2$, and $\chi(P_1) = 1$.



The graph C_n has $\chi(C_n) = 2$ if n is even, and $\chi(C_n) = 3$ if n is odd.



The complete graph K_n has $\chi(K_n) = n$.



The case for C_{2n} follows from a more general fact about bipartite graphs.

Proposition 5.1.4 (Chromatic Number of Bipartite Graphs)
If G is a bipartite graph, then $\chi(G) \leq 2$, and indeed $\chi(G) = 2$ unless $E = \emptyset$.

Proof Sketch. Colour the vertices in each part separately. □

Indeed, one way to think about chromatic number is as a generalization of bipartite graphs to multiple parts.

A straightforward observation to make is that the maximum degree of a graph (Δ) puts a bound on the chromatic number.

Proposition 5.1.5 (Degree Bound for Chromatic Number)
For a graph G, we have $\chi(G) \leq \Delta(G) + 1$. This bound is also sharp[a].

[a]Take the complete graph or odd cycles. These are the only such examples though

To prove this, we are going to use a type of greedy algorithm.

Definition 5.1.6 (Greedy Colouring)
Given a graph $G = (V, E)$ with vertices $V = \{v_1, v_2, \ldots, v_n\}$, the **greedy colouring** of G is a function $c_g : V \to \mathbb{N}$ defined inductively by

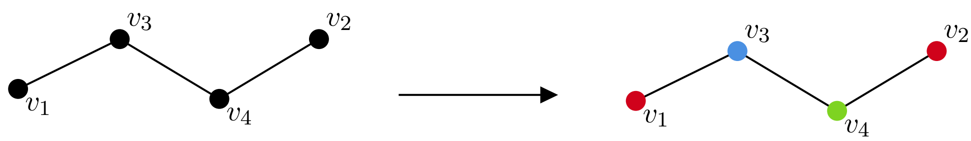
- $c_g(v_1) = 1$,
- Given coloured $v_1, \ldots, v_t$, we have $c_g(v_{t+1}) = \min(\mathbb{N} \backslash \{c(v_i) \mid v_i \sim v_{t+1}, i \leq t\})$.

Proof of Proposition 5.1.5. Apply the greedy colouring to G with an arbitrary vertex ordering $v_1, \ldots, v_n$. Then we note that

$$|\{c(v_i) \mid v_i \sim v_{t+1}, i \leq t\}| \leq \Delta(G),$$

and thus $c_g(v_{t+1}) \in [\Delta + 1]$. □

The 'greedy' approach need not give you any colouring that's in any way optimal. For example, if we have the graph P_4 with vertices labelled as below, we get the following colouring from our greedy approach.



This is clearly not optimal.

§5.2 Brooks' Theorem

One step up from the greedy approach to obtaining a colouring comes in the form of Brooks' Theorem. Before we look at that, we will make an observation.

Proposition 5.2.1

Let G be a connected graph for which $\delta(G) < \Delta(G)$. Then $\chi(G) \leq \Delta(G)$.

Proof. We find a better ordering to apply the greedy colouring to. First define $v_n = v$, where $d(v) \leq \Delta(G) - 1$. Now choose an ordering of $v_1, \dots, v_{n-1}$ so that

$$d(v_1, v) \geq d(v_2, v) \geq \cdots \geq d(v_{n-1}, v).$$

We claim that each vertex v_i with $i \in [n-1]$ has at most $\Delta(G) - 1$ neighbors in $v_1, \dots, v_{i-1}$.

This is true for v_n by definition. For v_i with $i \neq n$, we observe that a shortest path P from v_i to v_n contains a neighbor v_j of v_i with $d(v_j, v) < d(v_i, v)$. Thus v_j comes later in the ordering (that is, $j > i$).

So the greedy colouring gives each vertex one of $\Delta(G)$ colours. □

We can now extend this idea to prove Brook's theorem.

Theorem 5.2.2 (Brooks' Theorem)

Let G be a connected graph that is not complete or an odd cycle. Then $\chi(G) \leq \Delta(G)$.

Proof. Let G be a counterexample with a minimal number of edges. We may assume G is a regular graph (Δ-regular), and also that $\Delta \geq 3$.

Since G is not complete, there exists a vertex $v \in V(G)$ so that $G[N(V)]$ is not complete, and let $x, y \in N(v)$ where $x \sim y$ and $x \neq y$.

If G is 3-connected then $G' = G - x - y$ is connected. In this case, we order the vertices of G' by $v_1 = x$, $v_2 = y$, and $v_n = v$. Then we define $v_3, \dots, v_{n-1}$ by ordering the vertices of G' so that

$$d_{G'}(v_3, v) \geq d_{G'}(v_4, v) \geq \cdots \geq d_{G'}(v_{n-1}, v).$$

As we saw before, the greedy colouring with the ordering $v_1, \dots, v_n$ will give us a Δ-colouring of G.

If G has a cut vertex $w \in V$, then let $G - w = C_1 \cup \cdots \cup C_k$ be the components, and $G_i = G[C_i \cup \{w\}]$. The maximum degree in G_i is bounded by Δ, and therefore G_i has a Δ-colouring for each i, by the minimality of the counter example. Note that G_i cannot be a complete graph on $\Delta + 1$ vertices since $d_{G_i}(w) \leq \Delta - 1$. By permuting the colours, we can assume each colouring gives the vertex w colour 1. Thus we have a colouring of G with Δ colours, which is a contradiction.

If $G - \{w_1, w_2\}$ is disconnected for $w_1 \neq w_2$ with $w_1, w_2 \in V$, then let $C_1, \dots, C_k$ be the components of $G - \{w_1, w_2\}$. We may assume that G has no cut vertex by the previous case. Now let $G_i = G[C_i \cup \{w_1, w_2\}] + w_1 w_2$. Note that $e(G_i) < e(G)$ since w_1, w_2 both send at least one edge to each component $C_1, \dots, C_k$. Also $d_{G_i}(w_1) \le \Delta$ and likewise $d_{G_i}(w_2) \le \Delta$, thus the maximum degree $G_i \le \Delta$ and thus we can find a Δ-colouring of G_i for each i. That gives w_1, w_2 different colours. After permuting these colours, I can assume w_1 is coloured 1 in each, and w_2 is coloured 2 in each. Thus we can put all of the colourings together to obtain a colouring of G. □

§5.3 Colouring Planar Graphs

We will now consider colourings on graphs that are planar. It is in this section that the theorem ever-present in the popular maths psyche is kept (without proof).

Theorem 5.3.1 (Four-Colour Theorem)
Let G be a planar graph. Then $\chi(G) \le 4$.

Or informally: any map can be coloured with only four colours, where neighboring regions have different colours. Note that this is the duel form of the theorem above. This theorem was proved in 1976 by Appel & Haken and centers around reducing the theorem to a large number of cases, that were checked by computer.

We are going to prove two slightly weaker versions, that are still quite interesting.

Theorem 5.3.2 (Six Colour Theorem – Warmup)
Let G be a planar graph. Then $\chi(G) \le G$.

Proof. We will use induction on $n = |G|$. For $n = 1$, this is trivial. Now inductively, we claim that there is a vertex v with $\deg(v) \le 5$. We note

$$\frac{1}{n}\left[\sum_{x \in V} d(x)\right] = \frac{2E}{n} \le 6 - \frac{12}{n} < 6.$$

Thus there is a vertex v with $\deg(v) \le 5$. By induction, $G - v$ is 6-colourable, and since v has at most 5 neighbors, there is a colour in $[6]$ that does not appear in $N(v)$. Colouring v with this colour, we get that the graph is 6-colourable. □

Now we are going to kick it up a notch shortly, by introducing one more ingredient.

Definition 5.3.3
Given a graph G and an r-colouring of G, let $v \in V(G)$, and define the **$\{i, j\}$-component of v** to be all of the vertices that can be reached starting at v along a path using only colours i and j.

We make the following observation, which gives us an extra 'move' to use in the stronger proof

Proposition 5.3.4

Given a graph G with an r-colouring c, and for $i, j \in [v]$ with i, j, we can *swap* the colour on an $\{i, j\}$-component to obtain a new colouring.

Proof Sketch. This works because of the 'being reached' condition in the $\{i, j\}$-component definition. □

Now we can prove the five colour theorem, using a lot of the ideas from the proof of the six colour theorem.

Theorem 5.3.5 (Five Colour Theorem)

Let G be a planar graph. Then $\chi(G) \leq G$.

Proof. By induction on $n = |G|$, we note $n = 1$ is trivial. Now we proceed with the induction step. Let v be a vertex with $d(v) \leq 5$. Then apply induction to get a 5-colouring of $G - v$. Let $N(V) = \{x_1, \dots, x_5\}$, where $x_1, \dots, x_5$ are arranged in a clockwise manner. We may assume that $c(x_i) = i$ (otherwise we can colour v with the missing colour).

Now consider the $\{1, 3\}$-component containing x_1. If x_3 is not in this component, we can swap colours on this $\{1, 3\}$ component so that x_1 is colored with 3, and then colour v with 1.

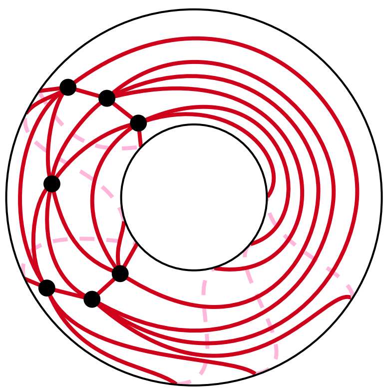
So we may assume there exists a path $x_1 \to x_3$ using colours 1 and 3 only. By the same argument there exists a path $x_2 \to x_4$ using colours 2 and 4 only. But then these paths must share a vertex, which is a contradiction. □

§5.4 Colouring Graphs on (Other) Surfaces

Following on from the last section, we will consider a related guiding question:

If G is a graph, drawn on the torus, what can we say about $\chi(G)$?

More generally, if G is a graph drawn on a surface of genus g, what can we say about $\chi(G)$? For example, the graph K_7 can be drawn on a torus without edge crossings.



We may wonder if there is an Euler's formula for surfaces, and indeed there is, but instead of equality, we get a bound.

Theorem 5.4.1 (Euler's Formula for Surfaces)

If G is drawn in a surface of genus g, then $V - E + F \geq 2 - 2g$, where F is the number of connected components of (Surface $- G$).

Proof Sketch. A similar inductive proof to the planar case. □

Remark. By a surface of genus g, we mean a compact orientable surface of genus g. Informally this is the surface formed from taking a sphere and adding g 'handles' to it. Also note that $2 - 2g$ is the **Euler characteristic** of a surface of genus g.

We can then use this to get a bound on the edges of a graph drawn on a surface with no edge crossings.

Proposition 5.4.2

If $G = (V, E)$ is a graph drawn on a surface of genus g, then $|E| \leq 3(|G| - (2 - 2g))$.

Proof Sketch. We have $3F \leq 2E$, then apply Euler's formula for surfaces. □

We can now get a bound (similarly to the planar case) on the chromatic number of a graph drawn on a surface.

Theorem 5.4.3 (Heawood's Theorem)

If G is a graph drawn on a surface of Euler characteristic E, with $E \leq 0$[a], then

$$\chi(G) \leq \left\lfloor \frac{7 + \sqrt{49 - 24E}}{2} \right\rfloor .$$

[a]This condition really is needed, as otherwise for the planar case we get $\chi(G) \leq 4$, which is correct but this is not a proof for it!

Proof. Let $G = (V, E)$ be a given graph with $\chi(G) = k$. We may assume that G has the minimum number of edges, subject to $\chi(G) = k$. Observe that $\delta(G) \geq k - 1$, as otherwise there would be a vertex v with $d(v) = k - 1$, and thus $\chi(G - v) = k - 1$, and thus $\chi(G) = k - 1$. Also $k \leq n$, where $n = |V|$. Now the average degree of each vertex is

$$\frac{1}{n}\left[\sum_{v \in V} d(v)\right] = \frac{2e}{n} \leq 6\left(1 - \frac{E}{n}\right),$$

where e is the number of edges and E is the Euler characteristic.

Thus

$$k - 1 \leq \delta(G) \leq \text{avg degree} \leq 6\left(1 - \frac{E}{n}\right) \leq 6\left(1 - \frac{E}{k}\right),$$

and so $k^2 - k \leq 6(k - E)$, then solving gives the required result. □

Remark. It turns out that this estimate is sharp. Calling $H(E) = \left\lfloor \frac{7+\sqrt{49-24E}}{2} \right\rfloor$, we find that $K_{H(E)}$ can be drawn on a surface of Euler characteristic E.

§5.5 Edge Colouring

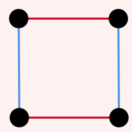
Now instead of assigning vertices colours, we will assign them to edges.

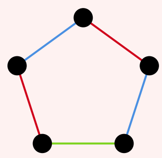
Definition 5.5.1 (Edge Colouring)
Let $G = (V, E)$ be a graph. A **k-edge colouring** is a function $c : E \to [k]$ so that $c(e) \neq c(f)$ for all $e, f \in E$ such that $e \cap f \neq \emptyset$.

We have a similar notion to chromatic number for edge colourings.

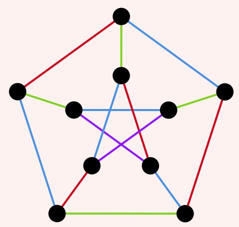
Definition 5.5.2 (Edge Chromatic Number/Chromatic Index)
The **edge chromatic number** or **chromatic index** of a graph G is $\chi'(G)$, the minimum k for which a k-edge colouring exists.

Example 5.5.3 (Examples of Edge Chromatic Number)
The graph C_n has $\chi'(C_n) = 2$ if n is even, and $\chi'(C_n) = 3$ for n odd.





The Petersen graph G has $\chi'(G) = 4$.



In contrast to the chromatic number, it is much easier to get a handle on the edge chromatic number, as we will see in the following theorem.

Theorem 5.5.4 (Vizing's Theorem)
Let G be a graph. Then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Proof. We proceed by induction on $e(G)$. The basis step is trivial, then for the inductive step we are given a graph G, and let's assume for a contradiction that $\chi'(G) > \Delta + 1$.

By induction, $G - e$ has a $\Delta + 1$ colouring, so let's write $e = xy$ with $x \neq y \in G$. We define vertices $y_1, \ldots, y_k$ inductively by setting $y_1 = y$. Now assume $y_1, \ldots, y_t$ are defined, and the colours missing from $y_1, \ldots, y_t$ are $c_1, \ldots, c_t$ respectively. Then if $c_t \notin \{c_1, \ldots, c_{t-1}\}$, then let y_{t+1} be so that xy_{y+1} receives colour c_t. Note that such a vertex exists as otherwise we could recolour xy_1 with c_1, xy_2 with c_2, and so on until xy_t is with c_t, to obtain a $\Delta + 1$ colouring, which is a contradiction.

Now if $c_t \in \{c_1, \dots, c_{t-1}\}$, then stop. Say we stop after k steps, so I have defined $y_1, \dots, y_k$ with missing colours $c_1, \dots, c_k$ and $c_k = c_i$ for some $i < k$. We may assume that $i = 1$ (otherwise uncolour the edge xy_i and recolour xy_1 with c_1, and so on until xy_{i-1} with c_{i-1}).

Let's call the colour missing at x, c_0. Consider the $\{c_0, c_1\}$ component C, containing y_1. If $x \notin C$, then we can flip colours on C so that c_0 is missing at y_1, then colour xy_1 with c_0. Likewise, the $\{c_0, c_1\}$-component containing y_k must contain x, as otherwise we flip colours on this component so that the colour c_0 is missing at y_k. Then recolour xy_k to c_0 and xy_i to c_i for $i < k$.

Thus $x, y_1, y_k \in C$, the $\{c_0, c_1\}$-component. But this is impossible since x, y_1, y_k all have one of the colours $\{c_0, c_1\}$ missing, thus $d_C(x), d_C(y_1), d_C(y_k) \leq 1$. But this is is impossible for a path or cycle. $\square$

Remark. This theorem is *not* true if we generalize to multigraphs, which are graphs that have multiple edges between vertices.

6 Extremal Graph Theory

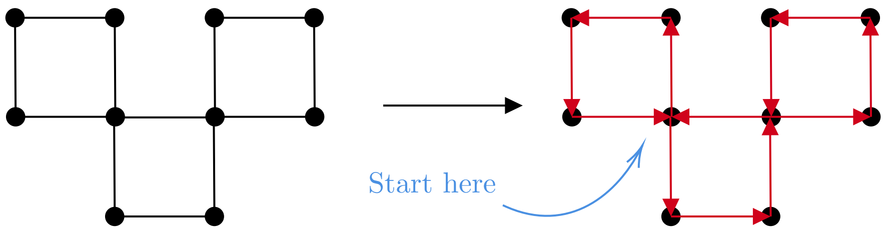
Welcome to the chapter on extremal graph theory, where we think about problems involving things with an 'extreme' flavour.

§6.1 Eulerian Circuits and Hamiltonian Cycles

We are going to begin our chapter on extremal graph theory with something that's not *really* extremal graph theory. Still, it will lead us nicely into the more extreme parts of this chapter and the course. Let's state a definition.

Definition 6.1.1 (Eulerian Circuit)

An **Eulerian circuit** is a circuit in a graph G that crosses each edge exactly once. If a G has an Eulerian circuit, we say that the graph is **Eulerian**.



What's nice is that there is a straightforward characterisation of Eulerian graphs.

Theorem 6.1.2 (Euler's Theorem)

A connected graph has an Eulerian circuit if and only if every vertex has even degree.

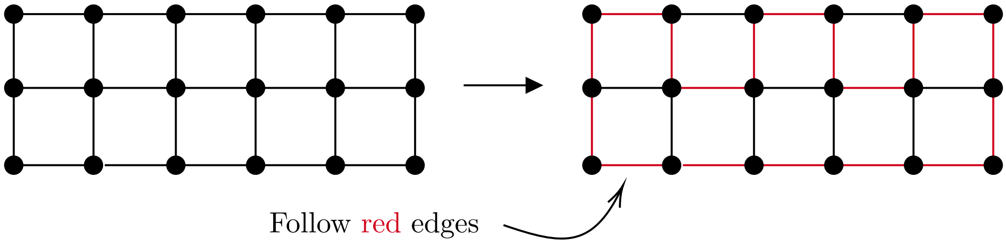
Proof. If a graph G has an Eulerian circuit, then the degree of each vertex must be even. This is because a fixed vertex x is entered and exited a fixed number of times.

Now if each vertex of a graph G has even degree, we will apply induction on $e(G)$. If $e(G) = 0$, then we are done. Now if $d(x) \geq 1$ for all vertices x, then $d(x) \geq 2$, and G contains a cycle C. Define $G' = G - E(C)$. Let $G_1, \dots, G_k$ be the components of G'. The degree of each G_i has all degrees even. Thus by induction, there is an Eulerian circuit $W_1, \dots, W_k$ for each $G_1, \dots, G_k$ respectively. Thus we can combine C with $W_1, \dots, W_k$ to obtain an Eulerian circuit for all of G. $\square$

Now we are going to define a similar looking notion, but it will not end up being so well behaved as Eulerian circuits.

Definition 6.1.3 (Hamiltonian Cycle)

Let G be a graph, then a **Hamiltonian cycle** in G is a cycle that visits each vertex exactly once. We say G is **Hamiltonian** if it contains a Hamiltonian cycle.



Our first question might be 'is there an iff type condition for Hamiltonian cycles', and so far there is no such property known. However there are some results that give us information about whether a graph is Hamiltonian.

Theorem 6.1.4 (Dirac's Theorem)

Let G be a graph with $n \geq 3$ such that $\delta(G) \geq n/2$. Then G contains a Hamiltonian cycle.

Proof. For a contradiction, assume that G is a counterexample to the theorem. Note that G is connected, since $x \not\sim y$ then $N(x), N(y) \subset V \setminus \{x, y\}$ and $|N(x)|, |N(y)| \geq \frac{n}{2}$, and thus $N(x) \cap N(y) \neq \emptyset$, and thus $d(x, y) \leq 2$.

Let $x_1, \dots, x_\ell$ be the longest path in G. Note that $x_1, \dots, x_\ell$ does not form a cycle, as otherwise if $\ell = n$, then we have a contradiction, and if $\ell < n$ then there exists $y \notin \{x_1, \dots, x_\ell\}$ that is $y \sim x_i$. Thus we can find a longer path (which is also a contradiction).

Now if there exists $i \in \{1, \dots, \ell - 1\}$ so that $x_i \sim x_\ell$ and $x_{i+1} \sim x_1$, then the vertices $x_1, \dots, x_\ell$ form a cycle. This contradicts the above.

Define
$$N^+(x_\ell) = \{x_i \mid x_{i-1} \in N(X_l), 2 \leq i \leq \ell\}.$$
We have $N^+(x_\ell) \cap N(x_1) =$, but this is impossible since $|N^+(x_\ell)| \geq n/2$, $|N(x_1)| \geq n/2$, and $N^+(x_\ell), N(x_1) \subseteq \{x_2, \dots, x_\ell\}$. Thus we have a contradiction. □

Remark. We never *really* used this $\delta(G) \geq n/2$ condition fully. It suffices to have $d(x) + d(y) \geq n$ for $x \not\sim y$.

We can use this same argument to prove a more general result about paths.

Proposition 6.1.5

Let G be a connected graph. Let $k < n$ and assume $\delta(G) \geq k/2$. Then $G \geq P_{k+1}$.

Proof Sketch. Similar to Dirac's theorem. Choose a longest path $x_1, \dots, x_\ell$ in G. They don't form a cycle, and then we can force a configuration like $x_1 \sum x_{i+1}$, $x_\ell \sum x_i$ by using the minimum degree condition. □

We may wonder in the above if it's possible to replace 'path' with 'cycle' in the above. The answer is, sadly, no.

A natural question is to wonder how many edges are needed to 'force' a (for example) triangle. For a Hamilton cycle, this sort of question would have been a bit strange, and a more natural question would be what is the minimum value of $\delta(G)$ to ensure that

G contains a Hamilton cycle. In the next theorem, we will give a result that answers a simpler sort of question.

Theorem 6.1.6

Let G be a graph. If $e(G) > \frac{n}{2}(k-1)$, then G contains a path of length k.

Proof. We will prove the contrapositive: If G is P_{k+1} free, then $e(G) \leq \frac{n}{2}(k-1)$.

We will apply induction on n. This is true for $n = 2$. Then given a graph G on $|G| = n \geq 3$ vertices, if G is disconnected let $G_1, \dots, G_k$ be the components of G. Then each of these has $G_i \not\supseteq P_{k+1}$. Then by induction we have $e(G_i) \leq \frac{n(G_i)(k-1)}{2}$. Thus

$$e(G) = \sum_{i=1}^{k} e(G_i) \leq \sum \frac{n(G_i)(k-1)}{2} = \left(\frac{k-1}{2}\right) n,$$

and we are done.

Now if there is a vertex v with $d(v) \leq \frac{k-1}{2}$, then consider $G - v$. Then $e(G - v) = e(G) - d(v)$, and also $G - v$ does not contain P_{k+1}. Thus $e(G - v) \leq \frac{n-1}{2}(k-1)$. So

$$e(G) = e(G - v) + d(v) \leq \frac{n-1}{2}(k-1) + \frac{k-1}{2} \leq \frac{n}{2}(k-1).$$

Thus we can assume G is connected and $d(v) \geq \frac{k}{2}$. We we can apply Proposition 6.1.5 to find a path of length k, that is, $P_{k+1} \subseteq G$, which is a contradiction.

[We may also assume $k < n$ for if $k = n$ then $e(G) \leq \frac{k}{2}(n-1)$, which reduces to showing $e(G) \leq \binom{n}{2}$, which is trivial]. □

§6.2 Complete Subgraphs and Turan's Theorem

Let's return to the question of how many edges are needed in a graph to guarantee the existence of a triangle. Later on, we will look more generally at the question of the number of edges needed to guarantee a K_n subgraph.

Theorem 6.2.1 (Mantel's Theorem)

If $e(G) > \frac{n^2}{4}$, then $G \supset K_3$, and this is sharp.

Proof. Given a K_3-free graph G and $x, y \in V$ such that $x \sim y$ and $d(x) + d(y) \leq n$, and let $m = e(G)$. Then summing we get

$$\sum_{xy \in E} d(x) + d(y) \leq mn.$$

We have that this is $\sum_x \sum_y d(x) \mathbb{1}(xy \in E) = \sum_{x \in V} (d(x))^2$. Now $\sum_{x \in V} d(x) = 2m$, and by Cauchy-Schwarz we have $(\sum d(x))^2 \leq n \sum d(x)^2$, and thus

$$mn^2 \geq (2m)^2 \implies \frac{n^2}{4} = e(G).$$

To see that this is sharp, consider the complete bipartite graph with n vertices. Then this has $n^2/4$ edges, but no triangle. Thus this is sharp. $\square$