Differential Equations – A Review of Calculus

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To study differential equations, one needs to have a working knowledge of calculus. The goal of this handout is to review some of the ideas that you (hopefully) already know, possibly through a slightly different lense (big-O and little-o notation). To emphasise: *this handout will in no way teach you how to do calculus*. It is merely a review before we begin studying the subject of *differential equations*. We will not be rigorous – the joy of that will be left to other courses.

Derivatives

Definition 1 (Derivative)

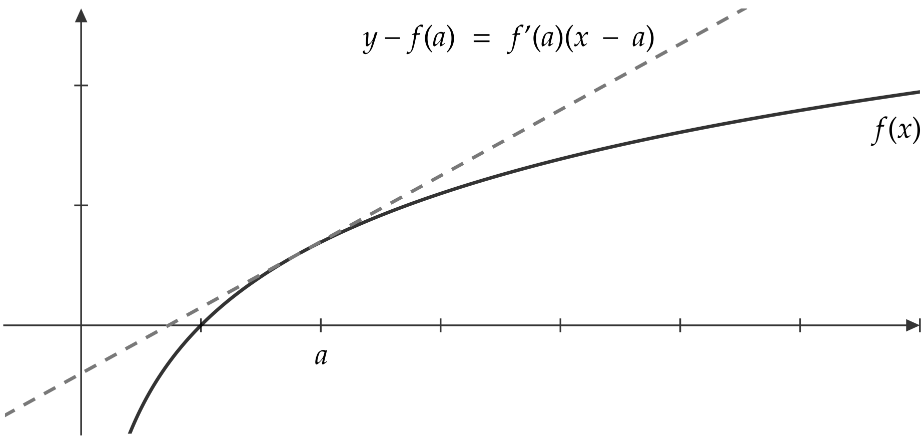
Let $f : I \to \mathbb{R}$ be a function defined on some interval $I \subseteq \mathbb{R}$. The **derivative** of f is the limit

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided it exists.[a] If this limit exists for all $x \in I$, then we say f is **differentiable** over I.

[a] We will use the notation $f^{(n)}(x) = \frac{\mathrm{d}^n f}{\mathrm{d}x^n}$ to mean taking the derivative n times.

Informally, a derivative is a way of studying the *local* behaviour of a function, and it can be used as a linear approximation to a function at a point. One can think of the derivative as being the slope of its graph at a particular point.



Theorem 2 (Elementary properties of the derivative)

For sufficiently differentiable functions, the following all hold.

(i) *Linearity.* Differentiation is a linear operator, with

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)+g(x)] = f'(x)+g'(x), \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}[cf(x)] = cf'(x)$$

(ii) *The Chain Rule.* Suppose $f(x) = F(g(x))$ for functions $F(x)$ and $g(x)$. Then

$$f'(x) = F'(g(x)) \cdot g'(x).$$

(iii) *The Leibniz Rule.* Suppose $f(x) = u(x)v(x)$ for functions $u(x)$ and $v(x)$. Then

$$\begin{aligned} f^{(n)}(x) &= \sum_{k=0}^{n} \binom{n}{k} u^{(k)}(x)v^{(n-k)}(x) \\ &= u^{(n)}(x)v(x) + nu^{(n)}(x)v'(x) + \cdots + u(x)v^{(n)}(x). \end{aligned}$$

In particular,

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$