Numbers and Sets

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This document is an account of the Cambridge Mathematical Tripos course 'Numbers and Sets', lectured by Prof. Imre Leader in Michaelmas 2020. This is a work in progress, and is likely to to contain errors, which you may assume to be my own.

Contents

§0 Introduction

Numbers and sets is one of the first course in pure mathematics that you will take as an undergraduate at Cambridge. In a sense, it is the 'starting course', in that it will introduce you to the 'pure maths' way of thinking about things. This introduction will happen through the lense of thinking about objects, beginning with the natural and real numbers. You will be introduced to the 'thoughtful way' of thinking about such objects, that you can carry through to almost every other course in pure mathematics.

§0.1 Structure of the Course

This course is divided into four chapters.

1. *Elementary Number Theory*

   This is a chapter that almost everybody enjoys. We deal with number theory first, which is elementary not in the sense that it is easy but in the sense that it is our 'first steps' in the subject. The main aim of this chapter is to get used to the additive and multiplicative structure of the natural numbers.

   It is like that some of you will be familiar with this material already, but nothing in this chapter will be assumed, and everything will be built from the ground up.

2. *The Reals*

   This chapter has a different perspective, centering on the questions of *what is a real number* and *what can we assume about them?* This is one of the harder parts of this course, and many of the definitions contain a subtlety that is not present in other chapters.

3. *Sets and Functions*

   This is a 'terminology' chapter. There is no exciting theorems, mostly notation, definitions, and so on. It is a short chapter, but it is somewhat boring in that sense.

4. *Countability*

   This chapter is best described as 'fun with infinite sets'. It is to do with the concepts introduced in Chapter 3 (in the sense that we are thinking about sets and functions), but it has a very different flavour. You will find results in this chapter that are both interesting and surprising. Almost everyone likes this chapter.

Everything in the chapters above makes up the 'course'. If you are wondering what is examinable, it will be everything in these lecture notes (unless otherwise stated). For a more formal answer to that question, have a look at the schedules.

§0.2 Books

As with most mathematics courses in Cambridge, you will not need a textbook to follow this course. What is covered in lectures is enough to do both the example sheets and the examinations for this course. Still, you might find that a textbook can provide a different perspective, additional worked examples, and additional material that you may find informative, helpful or fun.

In particular, the following books are quite relevant/good, but there is no expectation that you will look at these.

- R. T. Allenby, *Numbers and Proofs*.

  This book is readable, easy to understand and clear.

- A. G. Hamilton, *Numbers, Sets and Axioms*.

  Another readable and clear book, but with a different flavour to the previous book.

- H. Davenport, *The Higher Arithmetic*.

  This book can be thought of as showing 'where things go next'. It is very interesting, and goes quite a bit beyond this course. It is worth noting however that this book contains no exercises.

You should be able to find all of these books in either your college library or the university library.

§0.3 Example Sheets

As is normal for a 24 lecture course, there will be 4 example sheets. You should be able to have a good go at the first one after lecture 3 or 4.

§0.4 A Brief Note About These Notes

In the original lecture course, there was two lectures that (informally) introduced the idea of a proof, along with examples and non-examples of what a proof is. This material has been purposefully excluded, and familiarity with proofs (and common logical notation such as $\forall$, $\exists$, and $\implies$) is assumed.

If you are interested in reading a brief introduction to proofs, I will direct you to this quite readable introduction.

§1 Elementary Number Theory

This chapter is looking at the properties of the natural numbers. We will begin by defining exactly what they are, in a way that hopefully matches your own intuition.

§1.1 The Peano Axioms

Intuitively, the natural numbers $\mathbb{N}$ consist of the list of numbers

$$1, 1+1, 1+1+1, 1+1+1+1, \dots$$

In this list, every 'number' is distinct from the previous, and then it goes on forever with some vague notion of '$\dots$'. Let's try and make this precise.

Instead of trying to say what a natural number *is*, we will instead define *how they work*, that is, *what we can assume about them.* We do this by specifying the axioms that the natural numbers satisfy, that (hopefully) define that structure in a way that matches our intuitive idea of the natural numbers.

Definition 1.1 (Peano Axioms)

The natural numbers, written $\mathbb{N}$, is a set containing an element '1', and an operation '+1' that satisfies the following axioms.

(i) For all $n \in N$, $n + 1 \neq 1$.

(ii) If $n \neq m$, then $n + 1 \neq m + 1$.

(iii) For any property $p(n)$, if it is the case that $p(1)$ is true, and for every n we have $p(n) \implies p(n+1)$, then $p(n)$ is true for all $n \in \mathbb{N}$. This is the **induction axiom**.

Remark. We will write 2 for $1 + 1$ and so on.

With the operation $+1$ defined, we can define the operation $+k$ for any $k \in \mathbb{N}$.

Definition 1.2 (Addition)

We define the operation of *addition* so that

$$n + (k + 1) = (n + k) + 1,$$

for every natural number $k \in \mathbb{N}$.

This is defined for all natural numbers by induction. In a similar way, we can define multiplication, powers, order, etc, and we can prove the basic properties that they satisfy. Some of these are listed below.

Proposition 1.3

For all $a, b \in \mathbb{N}$:

(i) $a + b = b + a$.

(ii) $ab = ba$.

(iii) $a + (b + c) = (a + b) + c$.

(iv) $a(bc) = (ab)c$.

(v) $a(b + c) = ab + ac$.

(vi) If $a < b$ then

- $a + c < b + c$.
- $ac < bc$.
- If $b < c$ then $a < c$.

Remark. This is the last time that we'll dump a bunch of statements – that's not what this class is about.