Methods

Adam Kelly (ak2316@cam.ac.uk)

October 9, 2021

This article constitutes my notes for the 'Methods' course, held in Michaelmas 2021 at Cambridge. These notes are *not a transcription of the lectures*, and differ significantly in quite a few areas. Still, all lectured material should be covered.

Contents

§1 Fourier Series

§1.1 Periodic Functions

We will begin our study of method and in particular Fourier series by considering some periodic functions.

Definition 1.1 (Perioidic)
A function $f(x)$ is **periodic** if $f(x+T) = f(x)$ for all x, where T is the **period**.

Example 1.2 (Simple Harmonic Motion)
Many physical objects are described by *simple harmonic motion*, with the position given by

$$y = A \sin \omega t.$$

We call A the **amplitude**, and the period is $T = 2\pi/\omega$. The **frequency** is $1/T$.

Fourier series is all about trying to write periodic functions as particular sums of sines and cosines. Consider the set of functions

$$g_n(x) = \cos \frac{n\pi x}{L}, \quad \text{and} \quad h_n(x) = \sin \frac{n\pi x}{L},$$

where we take $n \in \mathbb{R}^+$. These functions are periodic on the interval $[0, 2L]$.

You may recall the following set of identities:

$$\cos A \cos B = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$$

$$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$
$$\sin A \cos B = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right).$$

We are going to try and define an inner product on this domain $[0, 2L]$, and using that we will by able to multiply these functions together and talk about their relative orthogonality.

Definition 1.3

We define the inner product $\langle f, g \rangle = \int_0^{2L} f(x)g(x)\,\mathrm{d}x$.

We can then obtain some orthogonality conditions for h_n and g_n with respect to this inner product. We can compute for $n \neq m$

$$\begin{aligned}
\langle h_n, h_m \rangle &= \int_0^{2L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{2L} \left(\cos\frac{(n-m)\pi}{L}x - \cos\frac{(n+m)\pi}{L}x\right)\mathrm{d}x \\
&= \frac{1}{2}\frac{L}{\pi}\left[\frac{\sin(n-m)\pi x/L}{n-m} - \frac{\sin(n+m)\pi x/L}{n-m}\right]_0^{2L} \\
&= 0,
\end{aligned}$$

and for $n = m$

$$\begin{aligned}
\langle h_n, h_n \rangle &= \int_0^{2L} \sin^2\frac{n\pi x}{L}\,\mathrm{d}x \\
&= \int_0^{2L} \frac{1}{2}\left(1 - \cos\frac{2\pi n x}{L}\right)\mathrm{d}x \\
&= L.
\end{aligned}$$

Hence we obtain the orthogonality condition

$$\langle h_n, h_m \rangle = \begin{cases} L\delta_{mn} & \text{if } n, m \neq 0, \\ 0 & \text{if } m = 0. \end{cases}$$

Similarly, it's straightforward to check that

$$\langle g_n, g_m \rangle = \begin{cases} L\delta_{mn} & \text{if } n, m \neq 0, \\ 2L\delta_{0n} & \text{if } m = 0. \end{cases}$$

and

$$\langle h_n, g_m \rangle = 0.$$

These orthogonality conditions are important because we are going to use these functions as a complete orthogonal set which spans the space of 'well-behaved periodic functions'.

§1.2 Definition of a Fourier Series

We can express any 'well-behaved' periodic function $f(x)$ with period $2L$ as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$$

where a_n, b_n are constants such that the RHS is convergent for all x where f is continuous. At a discontinuity, the Fourier series approaches the midpoint of the upper and lower limits at that point.

Consider taking the inner product $\langle h_n, f\rangle$ and substitute the expression for f above, to get

$$\int_0^{2L} \sin\frac{m\pi x}{L} f(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} Lb_n\delta_{nm} = Lb_m.$$

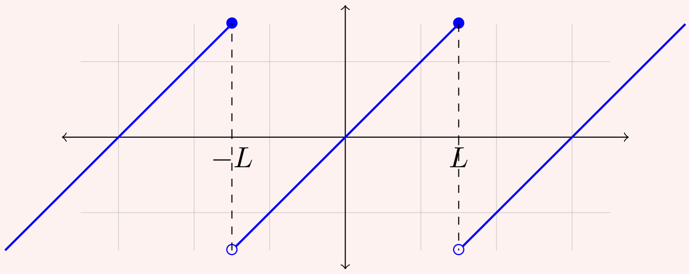
Hence we find that (doing something similar with g_n)

$$\begin{aligned} b_n &= \frac{1}{L}\int_0^{2L} g(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x, \\ \text{and}\quad a_n &= \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x. \end{aligned}$$

Now, this expression for a_n includes the case $n = 0$, and says that it is the average value of the function. Also, the range of integration is one period, and we can equivalently integrate over $[-L, L]$ instead of $[0, 2L]$.

Example 1.4 (The Sawtooth Wave)

Consider the function $f(x) = x$ for $-L \le x \le L$, with the function being periodic elsewhere.



Here we have

$$a_n = \frac{1}{L}\int_{-L}^{L} x\cos\frac{n\pi x}{2}\,\mathrm{d}x = 0,$$

for all n, and

$$b_n = \frac{2}{L}\int_0^{L} x\sin\frac{n\pi x}{L}\,\mathrm{d}x$$