THE SIMPLICITY OF A_n

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Here we establish the simplicity of A_n.

Lemma. *A_n is generated by 3-cycles.*

Proof. All elements of A_n are, by definition, generated by an even number of transposition. It thus suffices to show that a product of two transpositions can be written as a product of 3-cycles. Explicitly,

$$(a\ b)(c\ d) = (a\ c\ b)(a\ c\ d); \quad (a\ b)(b\ c) = (a\ b\ c).$$

□

Lemma. *If $n \geq 5$, all 3-cycles in A_n are conjugate (in A_n).*

Proof. We claim that every 3-cycle is conjugate to $(1\ 2\ 3)$. If $(a\ b\ c)$ is a 3-cycle, we have $(a\ b\ c) = \sigma(1\ 2\ 3)\sigma^{-1}$ for some $\sigma \in S_n$. If $\sigma \in A_n$, then the proof is finished. Otherwise $\sigma \mapsto \sigma(4\ 5) \in A_n$ suffices, since $(4\ 5)$ commutes with $(1\ 2\ 3)$. □

Theorem 0.1. *A_n is simple for $n \geq 5$.*

Proof. Suppose $1 \neq N \lhd A_n$. To disprove normality, it suffices to show that N contains a 3-cycle by the lemmas above, since the normality of N would imply N contains all 3-cycles and hence all elements of A_n.

Let $1 \neq \sigma \in N$, writing σ as the product of disjoint cycles.

(1) Suppose σ contains a cycle of length $r \geq 4$. Without loss of generality, let $\sigma = (1\ 2\ 3\ \dots\ r)\tau$, where τ fixes $1, \dots, r$. Now let $\delta = (1\ 2\ 3)$. We have

$$\underbrace{\sigma^{-1}}_{\in N}\underbrace{\delta^{-1}\sigma\delta}_{\in N} = (r\ \dots\ 2\ 1)(1\ 3\ 2)(1\ 2\ \dots\ r) = (2\ 3\ r)$$

So N contains a 3-cycle.

(2) Suppose σ contains two 3-cycles, which can be written without loss of generality as $(1\ 2\ 3)(4\ 5\ 6)\tau$. Then let $\delta = (1\ 2\ 4)$, and then

$$\sigma^{-1}\delta^{-1}\sigma\delta = (1\ 3\ 2)(4\ 6\ 5)(1\ 4\ 2)(1\ 2\ 3)(4\ 5\ 6)(1\ 2\ 3) = (1\ 2\ 4\ 3\ 6).$$

Therefore, there existst an element of N which contains a cycle of length $5 \geq 4$, which reduces our problem to the previous case.

(3) Finally, suppose that σ contains two 2-cycles, which will be written $(1\ 2)(3\ 4)\tau$. Then let $\delta = (1\ 2\ 3)$ and

$$\sigma^{-1}\delta^{-1}\sigma\delta = (1\ 2)(3\ 4)(1\ 3\ 2)(1\ 2)(3\ 4)(1\ 2\ 3) = (1\ 4)(2\ 3) = \pi.$$

Let $\varepsilon = (2\ 3\ 5)$. Then

$$\underbrace{\pi^{-1}}_{\in N}\underbrace{\varepsilon^{-1}\pi\varepsilon}_{\in N} = (1\ 4)(2\ 3)(2\ 5\ 3)(1\ 4)(2\ 3)(2\ 3\ 5) = (2\ 3\ 5),$$

So N contains a 3-cycle.

There are now three remaining cases, where σ is a transposition, a 3-cycle, or a transposition composed with a 3-cycle. Note that the remaining cases containing transpositions cannot be elements of A_n. If σ is a 3-cycle, we already know A_n contains a 3-cycle, namely σ itself. □

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