Vectors and Matrices

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December 17, 2020

This set of notes is a work-in-progress account of the course 'Vectors and Matrices', originally lectured by Dr. Jonathan Evans in Michaelmas 2020 at Cambridge. These notes are not a transcription of the lectures, but they do roughly follow what was lectured (in content and in structure).

These notes are my own view of what was taught, and should be somewhat of a superset of what was actually taught. I frequently provide different explanations, proofs, examples, and so on in areas where I feel they are helpful. Because of this, this work is likely to contain errors, which you may assume are my own. If you spot any or have any other feedback, I can be contacted at ak2316@cam.ac.uk.

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0 Introduction

Vectors and Matrices covers topics in both algebra and geometry, and the way in which they relate to one another. The course uses approaches that are quite varied in their nature (can be abstract or more concrete, conceptual or more computational, and so on). You will need to be able to fluently switch between these approaches.

The course assumes that you are vaguely familiar with Euclidean and coordinate geometry, along with the idea of geometric transformations.

§0.1 Course Structure

This course is divided into a number of chapters.

1. *Complex Numbers*

   This chapter takes the point of view of thinking of points in the plane as pairs of real numbers, and defining 'multiplication' on it to turn it into the complex numbers.

2. *Vectors in Three Dimensions*

   Here, we will recap on the relationship between three dimensional vectors and some of their geometrical applications, and we will discuss things like the dot and cross product. Towards the end of that discussion, we will introduce the 'index notation', a powerful and helpful notation for dealing with vectors. We will also introduce the 'summation convention', which is also incredibly useful.

3. *Vectors in a General Setting*

   This chapter will discuss what vectors are in general, and different ways of looking at them. We will be particularly concerned with vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$, that is, vectors who's entries are in $\mathbb{R}$ and $\mathbb{C}$ respectively.

4. *Matrices and Linear Maps*

   Picking up on the idea of generalizing vectors, this chapter will consider the idea of a 'linear map', an abstraction of matrices.

5. *Determinants and Inverses*

   This chapter will detail how to define and compute determinants of general $n \times n$ matrices. This will take two points of view, in that we need to be able to compute them but we also must understand what they mean. The relation between determinants and finding inverses of matrices will also be considered.

6. *Eigenvalues and Eigenvectors*

   This chapter also involves both geometry and algebra. The core question of this chapter is: given a linear map or matrix, what does it act on in a very straightforward way?

7. *Changing Basis, Canonical Forms and Symmetries*

In this final chapter, we will consider a set of far reaching results by trying to describing an arbitrary linear map. These ideas are far reaching, and immensely useful.

§0.2 Differences to the Lecture Course

This set of notes may diverge slightly from the lectures. If this occurs, I will attempt to describe the differences in this section. For now, while these notes are incomplete, I will leave it up to the reader to check themselves what is included or missing.

1 Complex Numbers

The complex numbers arose through the study of polynomials, but there is hardly an area of modern mathematics where they are not of use, with applications stretching from number theory to geometry to quantum mechanics.

§1.1 Defining the Complex Numbers

We are going to start right at the beginning, though some familiarity with complex numbers is assumed. We will construct the set of complex numbers, denoted by $\mathbb{C}$, from the real numbers $\mathbb{R}$ by adjoining an element i with the property that $i^2 = -1$.

Definition 1.1.1 (Complex Numbers)

A **complex number** is a number $z \in \mathbb{C}$ of the form $z = x + yi$ with $x, y \in \mathbb{R}$ such that $i^2 = -1$.

We call $x = \mathrm{Re}(z)$ the **real part**, and $y = \mathrm{Im}(z)$ the **imaginary part** of z.

The reals are a subset of the complex numbers, as for $x \in \mathbb{R}$, we have $x + 0i \in \mathbb{C}$.

§1.1.1 Basic Operations

With the complex numbers defined, we have to come up with some things to do with them. We can add, subtract, multiply and divide complex numbers in a sensible way, and indeed they form a *field* (we will elaborate on this later).

Definition 1.1.2 (Basic Operations in $\mathbb{C}$)

Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ be two complex numbers. Then we define **addition** and **multiplication** as

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + (y_1 + y_2)i, \\ z_1 \times z_2 &= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i. \end{aligned}$$

We can see from this definition that the operation of multiplication respects our original requirement that $i^2 = -1$. From these definitions we can also immediately define **subtraction** and **division** as the inverse operations of these, with

$$\begin{aligned} z_1 - z_2 &= (x_1 - x_2) + (y_1 - y_2)i \\ \frac{z_1}{z_2} &= \frac{x_1 + y_1 i}{x_2 + y_2 i} \cdot \frac{x_2 - y_2 i}{x_2 - y_2 i} = \left(\frac{x_1 y_1 + x_2 y_2}{x_2^2 + y_2^2}\right) + \left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right) i. \end{aligned}$$

From the definitions of addition and multiplication, it is clear that both operations are commutative and associative. Indeed, as we also have inverse operations, the complex numbers form groups.

Proposition 1.1.3 ($\mathbb{C}$ is a Group)

$\mathbb{C}$ with the operation $+$ is an abelian group with identity 0, and $\mathbb{C}\setminus\{0\}$ with the operation $\times$ is an abelian group with identity 1.

Proof Sketch. Check definitions and verify that $+$ and $\times$ are indeed associative. ☐

Aside: The History of Complex Numbers

The complex numbers arose through the study of polynomials, and their serious study really began in the study of the *cubic* polynomials of the form[1]

$$x^3 - 3px - 2q = 0.$$

The sixteenth century mathematicians del Ferro, Tartaglia and Cardano discovered that this equation could be solved explicitly, with the result being published in Cardano's 1545 work *Ars Magna* (where he mentioned del Ferro as a first author, and noted that Tartaglia had independently found the same result). The solution they had obtained was

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}.$$

For example, for the cubic $x^3 - 12x - 20$ we obtain the solution $x = \sqrt[3]{16} + \sqrt[3]{2}$.

Some 30 years later, Rafael Bombelli discussed one particular cubic in his work *l'Algebra*. He looked at $x^3 - 15x - 4 = 0$. One might observe (through mere trial and error) that $x = 4$ is a solution, but using Cardano's formula, we obtain the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

Indeed, Bombelli noticed this and gave an explanation for how these two solutions were essentially the same, even with the presence of the rather puzzling term $\sqrt{-121}$. His explanation essentially began by noting that if we wrote $\sqrt{-121} = 11i$, then we could find that

$$\begin{aligned} \sqrt[3]{2 + \sqrt{-121}} &= 2 + ni \\ \sqrt[3]{2 - \sqrt{-121}} &= 2 - ni, \end{aligned}$$

and adding would then result in $2 + ni + 2 - ni = 4$, our other solution.

Inherent in this explanation was an expectation that this i would obey many of the same rules of arithmetic that we would expect of, say, a real number. After this work, contributions from mathematicians such as Descartes, Wallis, de Moivre, Euler, Wessel, Argand, Hamilton, Gauss, Cauchy and many others began to build up a rich theory of the complex numbers, later developing the field of complex analysis, all centered on this construction of some i with the property that $i^2 = -1$.

Looking again at the solutions to polynomials it turns out that the complex numbers are sufficient to solve *every* polynomial equation. This result is known as the 'Fundamental Theorem of Algebra', and was proved in the 18th century. We will use it frequently in this course, but you will have to wait until a later course to see a proof.

[1] This form of a cubic equation is a 'depressed', and any general cubic can be expressed in this form.

Theorem 1.1.4 (Fundamental Theorem of Algebra)

Let $p(z)$ be a polynomial of degree $n \geq 1$ with complex coefficients. Then $p(z) = 0$ has precisely n (not necessarily distinct) complex roots, counted with multiplicity.