Geometry

Adam Kelly (ak2316@cam.ac.uk)

January 26, 2022

This article constitutes my notes for the 'Geometry' course, held in Lent 2022 at Cambridge. These notes are *not a transcription of the lectures*, and differ significantly in quite a few areas. Still, all lectured material should be covered.

Contents

§1 Topological and Smooth Surfaces

§1.1 Topological Surfaces

We will begin immediately with a definition that will occupy us for some time.

Definition 1.1 (Topological Surface)

A topological surface is a topological space Σ such that

(i) Each $p \in \Sigma$ has an open neighbourhood U with $p \in U$ such that U is homeomorphic to $\mathbb{R}^2$, with its usual Euclidean topology.

(ii) Σ is Hausdorff and second countable.

Recall that a space X is Hausdorff if for $p \neq q$ in X there exists disjoint open sets $p \in U$ and $q \in V$ in X, and that a space is second countable it's topology has a countable basis. In some ways, the real nature of topological spaces comes from the condition (a), and the condition (b) is really there for technical honesty.

§1.2 Examples of Topological Surfaces

Let's now take some to consider some examples of topological surfaces.

Example 1.2 ($\mathbb{R}^2$)

The plane $\mathbb{R}^2$ is a topological surface.

Example 1.3 (Subsets of $\mathbb{R}^2$)
Any open subset of $\mathbb{R}^2$ is a topological surface. For example

(i) $\mathbb{R}^2 \backslash \{0\}$ is a topological surface;

(ii) Let $Z = \{(0,0)\} \cup \{(1, 1/n) \mid n \in \mathbb{N}\}$, then $\mathbb{R}^2 \backslash Z$ is a topological surface.

Example 1.4 (Graphs)
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then the **graph**

$$\Gamma_f = \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\} \subseteq \mathbb{R}^3 \text{ (subspace topology).}$$

Recall that if X and Y are topological spaces, the product topology on $X \times Y$ has basis open sets $U \times V$ with $U \subseteq X$ and $V \subseteq Y$ both open sets.

It has the feature that $f : Z \to X \times Y$ is continuous if and only if $\pi_x \circ f : Z \to X$ and $\pi_y \circ f : Z \to Y$ are continuous.

So if $\Gamma_f \subseteq X \times Y$ and $f : X \to Y$ is continuous then Γ_f is homeomorphic to X, with the map $s : x \mapsto (x, f(x))$, so that $\pi|_{\Gamma_f}$ and s are inverse homeomorphisms.

So $\Gamma_f \cong \mathbb{R}^2$ for *any* continuous $f : \mathbb{R}^2 \to \mathbb{R}$, and Γ_f is a topological surface.

As a note, the topological surface Γ_f is independent of f. Later on as we develop more tools in geometry we will be able to better reflect the structure of the function f.

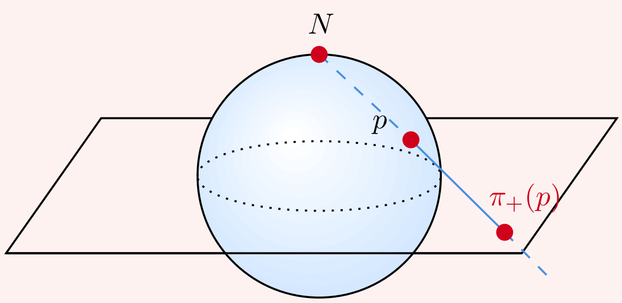
Example 1.5 (Stereographic Projection)
Consider the sphere

$$S^2 = \{(x, y, t) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

We can consider the stereographic projection

$$\begin{aligned} \pi_+ : S^2 \backslash \{(0,0,1)\} &\to \mathbb{R}^2 (z = 0) \subseteq \mathbb{R}^3 \\ (x, y, t) &\mapsto \left(\frac{x}{1-t}, \frac{y}{1-t} \right). \end{aligned}$$

Such a projection is shown below.



Note that π_+ is continuous and has an inverse

$$(u,v) \mapsto \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

So π_+ is a continuous bijection with continuous inverse and hence a homeomorphism.

Of course we could also have projected from the south pole, to get a homeomorphism π_- from $S^2 \backslash \{0,0,-1\}$ to $\mathbb{R}^2$, so indeed every point lies in an open set which is homeomorphic through either π_+ or π_- to $\mathbb{R}^2$. So S^2 is a topological surface.

Remark. S^2 is compact as a topological space, since it is a closed bounded set in $\mathbb{R}^3$.

Example 1.6 (Real Projective Plane)

The group $\mathbb{Z}/2\mathbb{Z}$ acts on S^2 by homeomorphisms via the **antipodal map** $a : S^2 \to S^2$ with

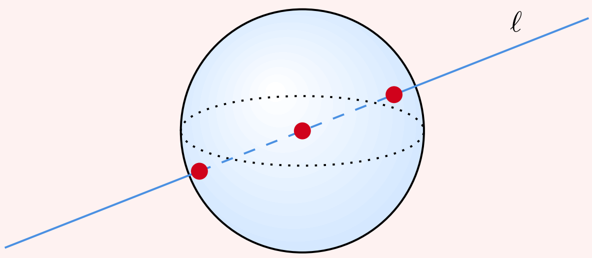
$$a(x,y,t) = (-x,-y,-t).$$

That is, there exists a homeomorphism $\mathbb{Z}/2\mathbb{Z} \to \text{Homeo}(S^2)$ sending the non-identity element to a.

The **real projective plane** is the quotient space of S^2 given by identifying every point with its antipodal image: $\mathbb{RP}^2 = S^2/(\mathbb{Z}/2\mathbb{Z}) = S^2/\sim$ with $x \sim a(x)$.

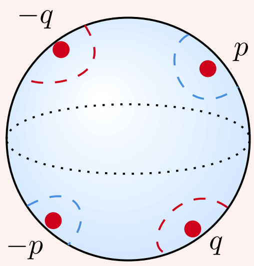
Note that $\sim$ is the equivalence relation of belonging to the same orbit under the given action.

As a set, $\mathbb{RP}^2$ is naturally in bijection with the set of straight lines in $\mathbb{R}^3$ through the origin, with the bijection given by mapping lines with the identified points on the sphere that they pass through.



We can also check that $\mathbb{RP}^2$ is a topological surface.

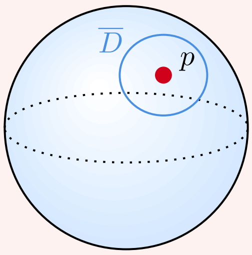
We must first check that it is Hausdorff. Recall that if X is a space and $q : X \to Y$ is a quotient map, then $V \in Y$ is open if and only if $q^{-1}(V) \in X$ is open.



If $[p] \neq [q] \in \mathbb{RP}^2$ then $\pm p$ and $\pm q \in S^2$ are distinct antipodal points. We can then take small open discs[a] about these in S^2 as in the picture, which give us disjoint open neighborhoods of $[p]$ or $[q]$ in $\mathbb{RP}^2$. That is, it's Hausdorff.

We can also check that $\mathbb{RP}^2$ is second countable. We know that U be a countable basis for the topology on S^2, and without loss of generality for all $u \in U$, the antipodal image is in U. Let $\overline{U}$ be the family of opens ets in $\mathbb{RP}^2$ of the form $q(u) \cup q(-u), u \in U$. Now if $V \subseteq \mathbb{RP}^2$ is open, by definition $q^{-1}(V)$ is open in S^2, and so $q^{-1}(V)$ contains some $u \in U$ and hence contains $u \cup (-u)$. So $\overline{U}$ is a countable base for the quotient topology on $\mathbb{RP}^2$.

Finally, let $p \in S^2$ and let $[p] \in \mathbb{RP}^2$ be it's image. Let $\overline{D}$ be a small[b] closed disc neighborhood of $p \in S^2$.



Then the quotient map $q|_{\overline{D}} : \overline{D} \to q(\overline{D}) \subseteq \mathbb{RP}^2$ is a continuous map from a compact space to a Hausdorff space. Also, on $\overline{D}$ the map q is injective. So by the topological inverse function theorem, this map $q|_{\overline{D}}$ is a homeomorphism. This induces (by taking interiors) a homeomorphism $q|_D : D \to q(D) \in \mathbb{RP}^2$. So $[p] \in q(D)$ has an open neighborhood in $\mathbb{RP}^2$ homeomorphic to an open disk, and we are done.

[a]We could take small balls $B_{\pm p}(\delta)$ and $B_{\pm q}(\delta)$ ($\delta \ll 1$ small), which meet S^2 in open sets. If $q : S^2 \to \mathbb{RP}^2$ is the quotient map, then $q(B_p(\delta))$ *is* open since $q^{-1}(q(B_p(\delta))) = B_p(\delta) \cup (-B_p(\delta))$, the union with the antipodal image.

[b]Contained in an open hemisphere

Example 1.7 (Torus)

Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, and define the **torus** $S_1 \times S_1$ with the subspace topology from $\mathbb{C}^2$ (which is the product topology). We will show that the torus is a topological surface.

Consider the map

$$\begin{aligned} \mathbb{R}^2 &\xrightarrow{e} S^1 \times S^1 \subseteq \mathbb{C} \times \mathbb{C} \\ (s,t) &\longmapsto (e^{2\pi i s}, e^{2\pi i t}). \end{aligned}$$

Note that this induces a map (in a set theoretic sense)

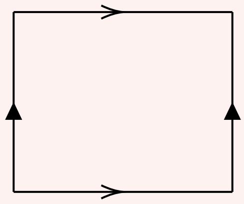
$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "e"] \arrow[d] & S^1 \times S^1 \\ \mathbb{R}^2/\mathbb{Z}^2 \arrow[ur, "\hat{e}"] & \end{tikzcd}$$

that is, on the equivalence relation on $\mathbb{R}^2$ given by translating by π^2, e is constant on equivalence classes and so induces a map of sets $\mathbb{R}^2/\mathbb{Z}^2 \to S^1 \times S^1$. Note we viewing $\mathbb{R}^2/\mathbb{Z}^2$ as the quotient space for $q : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$.

So the map $[0,1]^2 \hookrightarrow \mathbb{R}^2 \xrightarrow{q} \mathbb{R}^2/\mathbb{Z}^2$ is onto, *so* $\mathbb{R}^2/\mathbb{Z}^2$ is compact. So $\hat{e}$ is a continuous map from a compact space to a Hausdorff space and is a bijection, and is thus a homeomorphism by the topological inverse function theorem.

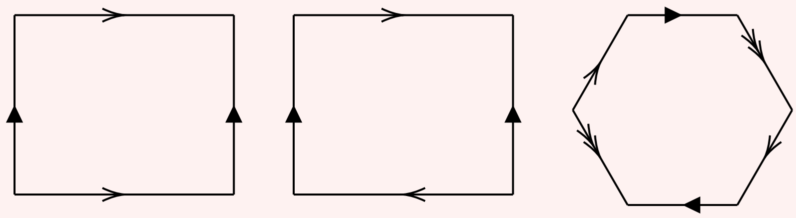
Note we already know that $S^1 \times S^1$ is compact and Hausdorff (closed and bounded in $\mathbb{R}^4$). As for $S^2 \to \mathbb{RP}^2$, pick $[p] = q(p)$ for $p \in \mathbb{R}^2$ and some small closed disc $\overline{D}(p) \subseteq \mathbb{R}^2$ such that for all non-zero integers n, m we have $\overline{D}(p) \cap (\overline{D}(p) + (n, m)) = \emptyset$. Then $e|_{\overline{D}(p)}$ is injective and $q|_{\overline{D}(p)}$ is injective. Now restricting to the open disc as before, we get an open disc neighborhood of $[p] \in S^1 \times S^1$. Since $[p]$ is arbitrary, $S^1 \times S^1$ is a topological surface.

Another viewpoint: $\mathbb{R}^2/\mathbb{Z}^2$ is also given by imposing on $[0,1]^2$ the equivalence relation generated by $(x, 0) \sim (x, 1)$, $(0, y) \sim (1, y)$.



Example 1.8 (Gluing Edges)

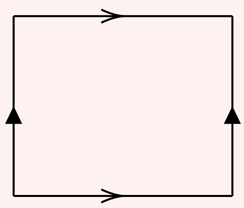
Let P be a planar Euclidean polygon. We will assume the edges are *oriented* and paired. For simplicity, we can suppose the Euclidean length of e and e' if $\{e, e'\}$ are paired.



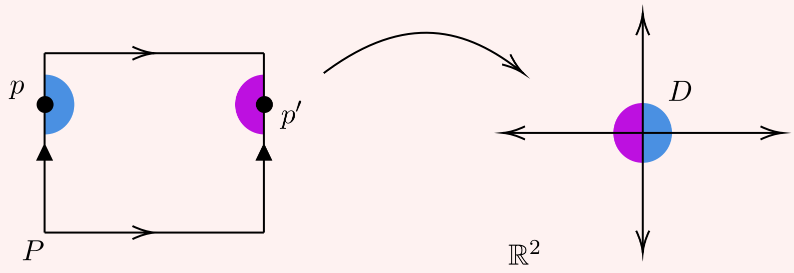
If $\{e, e'\}$ are paired edges, there is a unique isometry from e to e' respecting their orientations, say $f_{ee'} : e \to e'$. These maps generate an equivalence relation on P, where I identify $x \in P$ with $f_{ee'}(x)$ whenever $x \in e$.

Lemma. $P/\sim$ (with the quotient topology) is a topological surface.

Before we prove this, we will consider the specific example of the torus as $[0,1]^2/\sim$.



If $P = [0,1]$ and p is in the interior of P, then picking $\delta > 0$ sufficiently small so that $B_p(\delta)$ and $\overline{B_p(\delta)}$ in $\mathbb{R}^2$ lie in the interior of P. Now we argue as before: the quotient map is injective on $\overline{B_p(\delta)}$ and a homeomorphism on its interior. If p is on an edge of P, then we take two half disks of sufficiently small radius so that they don't meet a vertex of P.



Then we can define a map from the union of these half disks to the disk of the same radius at the origin of $\mathbb{R}^2$ as above.