Differential Equations – A Review of Calculus

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To study differential equations, one needs to have a working knowledge of calculus. The goal of this handout is to review some of the ideas that you (hopefully) already know, possibly through a slightly different lense (the ideas of orders of magnitude). To emphasize: *this handout will in no way teach you how to do calculus.* It is merely a review before we begin studying the subject of *differential equations.*

Derivatives

Definition 1 (Derivative)**.** Let $f : I \to \mathbb{R}$ be a function defined on some interval $I \subseteq \mathbb{R}$. The **derivative** of f is the limit

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided it exists. If this limit exists for all $x \in I$, then we say f is **differentiable** over I.

If a function is sufficiently smooth, we can differentiate multiple times. The nth derivative of f is written

$$f^{(n)}(x) = \frac{\mathrm{d}^n f}{\mathrm{d}x^n}.$$

Theorem 2 (Elementary Properties of the Derivative)**.** *For sufficiently differentiable functions, the following all hold.*

(i) The Chain Rule. *Suppose $f(x) = F(g(x))$ for functions $F(x)$ and $g(x)$. Then*

$$f'(x) = F'(g(x)) \cdot g'(x).$$

(ii) The Leibniz Rule. *Suppose $f(x) = u(x)v(x)$ for functions $u(x)$ and $v(x)$. Then*

$$\begin{aligned} f^{(n)}(x) &= \sum_{k=0}^{n} \binom{n}{k} u^{(k)}(x)v^{(n-k)}(x) \\ &= u^{(n)}(x)v(x) + nu^{(n)}(x)v'(x) + \cdots + u(x)v^{(n)}(x). \end{aligned}$$

In particular,

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$