Combinatorics

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This article constitutes my notes for the 'Combinatorics' course, held in Michaelmas 2021 at Cambridge. These notes are *not a transcription of the lectures*, and differ significantly in quite a few areas. Still, all lectured material should be covered.

Contents

§1 Set Systems

We will begin our study of combinatorics by considering *set systems* – collections of subsets of a set (which will typically be $X = [n] = \{1, 2, \ldots, n\}$).

Definition 1.1 (Set Systems)

Let X be a set. A **set system** on X or a **family of subsets** of X is a family $\mathcal{A} \subset \mathcal{P}(X)$.

It's often useful to think about the power set of a set X, $\mathcal{P}(X)$, as a graph. We can do this by joining two elements A and B if $|A \triangle B| = 1$, where $\triangle$ is the symmetric difference. This graph is the **discrete cube** Q_n[1].

§1.1 Chains and Antichains

We are first going to look at what happens when sets are contained or not contained in each-other. If you know anything about posets, this will likely be familiar.

Definition 1.2 (Chain and Antichain)

We say that $\mathcal{A} \subset \mathcal{P}(X)$ is a vocabchain if for all $A, B \in \mathcal{A}$ we have $A \subset B$ or $B \subset A$.

We say that $\mathcal{A}$ is a **antichain** if for all $A, B \in \mathcal{A}$ with $A \neq B$ we have $A \not\subset B$ and $B \not\subset A$.

[1] This is the same graph as the boolean hypercube, in the obvious way.

Example 1.3

] $\{\{1,4\},\{1,4,7,8\},\{1,2,4,7,8\}\}$ is a chain, and $\{\{1,4\},\{1,7,8\},\{32,3,8\}\}$ is an antichain.

A natural first question is how big can a chain be? We can easily get $|\mathcal{A}| = n-1$ by taking

$$\mathcal{A} = \{\emptyset, \{1\}, \{1,2,\}, \dots, [n]\}.$$

Can we beat this? No, since $\mathcal{A}$ must meet $X^{(r)}$ (the set of r element subsets of X) at at most one point.

How about antichains? We can achieve $|\mathcal{A}| = n$ by taking all singleton sets, but can we do any better? Well with the same idea we can take each $\lfloor n/2 \rfloor$-element subset of $[n]$, giving $|\mathcal{A}| = \binom{n}{\lfloor n/2 \rfloor}$. Can we do better than *this*? It's not quite obvious (and it's this type of question that we will come across frequently in this course...).