Optimisation

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At its heart, analysis is the study of ideas that depend on the notion of *limits*. The main concepts of analysis (such as convergence, continuity, differentiation and integration) will all depend quite fundamentally on a limiting process.

This article constitutes my notes for the 'Analysis I' course, held in Lent 2021 at Cambridge. These notes are *not a transcription of the lectures*, and differ significantly in quite a few areas. Still, all lectured material should be covered.

Currently up to lecture 1.

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This course is roughly split into:

- Convex Optimisation (1-3)
- General optimisation with Lagrange Multipliers (4-5)
- Linear Programming (minimizing linear functions), and applications (6-12)

§1 Optimisation Problems

In optimization, we are interested in solving problems that look like: minimize $f(x)$ for $x \in X$, or minimize $f(x)$ for $x \in X$, $h(x) = b$ where $h : \mathbb{R}^n \to \mathbb{R}^m$.

We will always be looking at minimizing functions.

Terminology:

- f is called the objective function.
- The components of x are the decision variables.
- In the above, '$h(x) = b$' is a functional constraint.
- '$x \in X$' is the regional constraint.

- The set $\{x \in X : h(x) = b\}$ is called the feasible set $X(b)$. A problem is feasible if this set is not empty, and otherwise it is in-feasible. We say the problem is bounded if the minimum is bounded.
- A point $x^* \in X(b)$ is optimal if it minimises f over $X(b)$. The value of $f(x^*)$ is called the optimal constraint.

So why do we only care about constraints $h(x) = b$? Well consider the constraint $h(x) \leq b$, then $h(x) + s = b$, $s \geq 0$. So we can always convert an inequality constraint into a functional and regional constraint.

§2 Convex Optimisation

Definition 2.1 (Convex Set)
A set $S \subseteq \mathbb{R}^n$ is **convex** if for all $x, y \in S$, and all $\lambda \in [0, 1]$, $x(1 - \lambda) + y(\lambda) \in S$. That is, the line segment joining x and y lies in S.

insert diagram 1 - convex here.

We want to study the notion of *convex functions*.

Definition 2.2 (Convex Functions)
A function $f : S \to \mathbb{R}$ is **convex** if S is convex, and for all $x, y \in S$, and $\lambda \in [0, 1]$ we have

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y).$$

The function is **strictly convex** if the inequality is strict, and is **concave** if $-f$ is convex.

In one dimension, such functions look 'u-shaped', and points between x and y always lies below the chord.

§2.1 Unconstrained Optimisation

We are going to look at the problem of minimizing $f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function.

Convex functions have certain properties that allow us to minimize them somewhat easily. The most important property is that *local* information gives us *global* information. Informally, if you know what a convex function looks like in a small neighbourhood, you can always say something about what it looks like outside of that.

Let's have a think about how to tell if a function is convex.

A first order condition for convexity is that a tangent line always lies below the curve. That is,

$$f(y) \geq f(x) + (y - x)f'(x).$$

In higher dimensions, we will have to replace $f'(x)$ with $\nabla f(x)$, and take a dot product (treating it as a vector). Note that if $\nabla f(x) = 0$, then $f(y) \geq f(x)$, thus x minimizes f.

Let's write this all down properly.

Theorem 2.3 (First Order Convexity Conditions)
A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if for $x, y \in \mathbb{R}^n$ we have

$$f(y) \geq f(x) + (y - x) \cdot \nabla f(x).$$

Proof. $\square$