Groups

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Michaelmas 2020

This document is an account of the Cambridge Mathematical Tripos course 'Groups', lectured by Dr. Ana Khukhro. in Michaelmas 2020. This is a work in progress, and is likely to to contain errors, which you may assume to be my own.

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§1 Groups

§1.1 Definition

In this section we will formally introduce the notion of a group, and we will consider some examples of groups along with their basic properties.

Definition 1.1

A **group** is a set G with a binary operation $*$ on G such that:

- *Identity.* G has an **identity element** e such that $e * g = g * e = g$ for all $g \in G$.
- *Inverses.* Each element $g \in G$ has an **inverse**, that is, an element $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.
- *Associativity.* The operation $*$ is associative, that is $(g * h) * k = g * (h * k)$ for any $g, h, k \in G$.

Remark (A pedantic point)**.** In some cases, people will add an additional 'closure' axiom, stating that if $g, h \in G$ then $g * h \in G$. However, this is redundant as it is implied by stating that $*$ is a binary operation on G. You must keep it in mind however when checking if something is a group.

Remark (Bracketing)**.** The 'associativity' axiom means that we can write $g * h * k$ without specifying what order it should be done first.

Notation. It's proper to state that '$(G, *)$ is a group', but this is regularly abbreviated to saying 'G is a group', whenever the operation being used is clear.

So that's what a group is, let's dive straight into some examples.

Example 1.2 (Examples of Groups)

The following are all examples of groups.

1. $G = \{e\}$, along with the binary operation $*$ satisfying $e * e = e$ (the 'trivial group').
2. G being the set of symmetries of a shape, along with $g * h$ defined to be 'performing h followed by g' where $g, h \in G$ is a group.
3. $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$ and $(\mathbb{C}, +)$ are all groups.
4. The nonzero[a] real numbers $\mathbb{R} \backslash \{0\}$ with multiplication is a group.
5. $(\mathbb{R}, *)$ where $r * s = r + s + 5$ for any $r, s \in \mathbb{R}$ is a group.
6. $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with addition modulo n is a group.
7. A vector space with vector addition is a group.
8. The set of invertible 2×2 matrices with real coefficients, $GL_2(\mathbb{R})$ is a group with respect to matrix multiplication.

[a] You should consider why we need to exclude zero for $\mathbb{R}$ to be a group.

Proof Sketch. Check that each construction satisfies all of the axioms stated in the definition of a group. □

Let's also look at some structures that are *not* groups.

Example 1.3 (Non-Examples of Groups)

The following are all *not* groups.

1. $G = \{0, 1, 2, \dots, n-1\}$ with addition.
2. $(\mathbb{Z}, \times)$.
3. $(\mathbb{R}, *)$ where $r * s = r^2 s$ for $r, s \in \mathbb{R}$.
4. $G = \{0, 1, 2, \dots\}$ and the operation $*$ such that $m * n = |n - m|$ for $m, n \in G$.