VARIATIONAL PRINCIPLES

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To-do: we should probably write actual proofs for the examinable bookwork...

1. The Euler Lagrange Equations

Method 1.1 (Lagrange Multipliers). *To extremize $f : \mathbb{R}^n \to \mathbb{R}$ subject to $g : \mathbb{R}^n \to \mathbb{R}$ with $g(x) = 0$, define $h(x, \lambda) = f(x) - \lambda g(x)$, then extremize h without constraints by solving $\nabla h = 0$.*

Lemma 1.1 (Fundamental Lemma of the Calculus of Variations). *If $g : [\alpha, \beta] \to \mathbb{R}$ is continuous and for all ∇ continuous with $\nabla(\alpha) = \nabla(\beta) = 0$ we have $\int_\alpha^\beta g(x)\nabla(x)\,\mathrm{d}x = 0$, then $g(x) = 0$ for all x.*

Proof (Sketch). If there's a c with $g(c) \neq 0$, then take an interval $[a, b]$ on which g is non-zero (exists by continuity) and consider $\nabla(x) = (x - a)(x - b)$ on $[a, b]$ with $\nabla(x) = 0$ elsewhere. Then the integral is non-zero, which is a contradiction. □

Theorem 1.2 (Euler-Lagrange). *Suppose $y \in C^2_{[\alpha,\beta]}(\mathbb{R})$ is a function with fixed endpoints that extremizes the functional*

$$F[y] = \int_\alpha^\beta f(x, y(x), y'(x))\,\mathrm{d}x.$$

Then y satisfies the Euler-Lagrange equations*,*

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

Proof (Sketch). Let y be an extreme point and consider a pertubation $y+\varepsilon\nabla$, where ∇ is zero at the endpoints. Then Taylor expanding $F[y+\varepsilon\nabla]$ in ε gives a first order term that we want to be zero. We can then integrate by parts to get an equation that we can apply our Lemma to. □

First Integrals. If f does not depend explicitly on y, then the Euler Lagrange equations simplify to $\partial f/\partial y' = c$, for some constant c, which we can solve.

If f does not depend explicitly on x, then noting by computation that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f - y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x},$$

we get a different first integral condition, $f - y'\frac{\partial f}{\partial y'} = c$, for some constant c, which we can solve.

Multiple Dependent Variables. If we had a function $y : \mathbb{R} \to \mathbb{R}^m$ that we wanted to extremize as before, following the same derivation as the single variable Euler Lagrange yields that the Euler-Lagrange equations hold in each component. We can also obtain the first integrals as before.

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Multiple Independent Variables. Repeating the derivation of the Euler Lagrange equations in the case that we have multiple independent variables, we can obtain (using the divergence theorem) the Euler-Lagrange equation for multiple independent variables,

$$\frac{\mathrm{d}f}{\mathrm{d}\phi} - \nabla \cdot \left(\frac{\partial f}{\partial \phi_x}, \frac{\partial f}{\partial \phi_y}, \frac{\partial f}{\partial \phi_z} \right) = 0.$$

Higher Derivatives. If we want to work with higher derivatives of y, then we obtain (with the standard derivation of the Euler-Lagrange equations along with more integration by parts) a variant of the Euler-Lagrange equations of the form

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial f}{\partial y'} + \cdots + (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{\partial f}{\partial y^(n)} = 0.$$

2. Variational Principles

We will now look at some examples of laws of nature that arise in the form of variational principles.

Fermat's Principle. As light travels between two points, it takes the path of least time.

Principle of Least Action. Consider a particle moving in $\mathbb{R}^3$ with kinetic energy T and potential energy V. We define the *Lagrangian* to be $L(x, \dot{x}, t) = T - V$. We then define the *action* to be $S[x] = \int_{t_1}^{t_2} L \ \mathrm{d}t$. The principle of least (or stationary) action then says that on the path of motion of a particle this functional is extremized.

Noether's Theorem. Consider a functional $F[y] = \int_\alpha^\beta f(y_i, y_i', x) \ \mathrm{d}x$ with $i = 1, \dots, n$.

Suppose there was a one-parameter family of transformations $y_i(x) \mapsto \mathcal{Y}_i(x,s)$ with $\mathcal{Y}_i(x,0) = y_i(x)$. This family is a *continuous symmetry* of the Lagrangian f if

$$\frac{\mathrm{d}}{\mathrm{d}x} f(\mathcal{Y}_i(x,s), \mathcal{Y}_i'(x,s), x) = 0.$$

Theorem 2.1 (Noether's Theorem). *Given a continuous symmetry $\mathcal{Y}_i(x,s)$ of f,*

$$\sum_{i=1}^n \frac{\partial f}{\partial y_i'} \frac{\partial \mathcal{Y}_i}{\partial s}\bigg|_{s=0}$$

is a first integral of the Euler-Lagrange equation.

Proof (Sketch). Start with $\left.\mathrm{d}f/\,\mathrm{d}s\right|_{s=0} = 0$ and expand. □

3. The Legendre Transform

Definition 3.1 (Legendre Transform). The Legendre transform of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a function f^* given by

$$f^*(p) = \sup_x (p \cdot x - f(x)).$$

We take the domain of f^* to be such that the supremum above is finite.

In one dimension, $f^*(p)$ can be thought of as the maximum vertical distance between $y = f(x)$ and $y = px$.

Proposition 3.2. *If the domain of f^* is non-empty, it is a convex-set, and f^* is convex.*

4. Second Variations

Theorem 4.1 (Condition on Local Minima)**.** *Let y be a solution to the Euler-Lagrange equation and define*

$$P = \frac{\partial^2 f}{\partial (y')^2} \quad \text{and} \quad Q = \frac{\partial^2 f}{\partial y^2} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial^2 f}{\partial y \partial y'}.$$

If $Q\eta^2 + P(\eta')^2 > 0$ for all η which vanish at α and β, then y is a local minimizer of $F[y]$.

Proof (Sketch). Expand to the second order in ε and integrate the last term by parts to get $\delta^2 F[y] > 0$. □

Theorem 4.2 (Legendre Condition)**.** *If y_0 is a local minimum then $P = \left.\frac{\partial^2 f}{\partial (y')^2}\right|_{y_0} \geq 0$.*

Theorem 4.3. *If $-(Pu')' + Qu = 0$ has a solution with $u \neq 0$ on $[\alpha, \beta]$, then $F[y]$ is a local minima.*