Dynamics and Relativity

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This set of notes is a work-in-progress account of the course 'Dynamics and Relativity', originally lectured by Prof Peter Haynes in Lent 2020 at Cambridge. These notes are not a transcription of the lectures, but they do roughly follow what was lectured (in content and in structure).

These notes are my own view of what was taught, and should be somewhat of a superset of what was actually taught. I frequently provide different explanations, proofs, examples, and so on in areas where I feel they are helpful. Because of this, this work is likely to contain errors, which you may assume are my own. If you spot any or have any other feedback, I can be contacted at ak2316@cam.ac.uk.

Contents

1 Newtonian Dynamics – Basic Concepts

A central aspect of this course is Newtonian dynamics. In this chapter we will develop some of the ideas and definitions needed to discuss this in detail.

§1.1 Particles

When dealing with Newtonian dynamics, we will often use and refer to *particles*, as a way of describing phenomina.

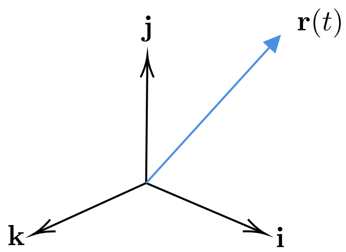
Definition 1.1.1 (Particle)

A **particle** is an object of negligible size. It has some mass $m > 0$, and can also have other properties such as (perhaps) an electric charge q.

A particle is completely described by a **position vector**, usually denoted $\mathbf{r}(t)$ or $\mathbf{x}(t)$, with respect to some origin O. The cartesian coordinates of $\mathbf{r}$ are (x, y, z), where

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ being orthonormal basis vectors.



The choice of coordinate axes defines a **frame of reference** S.

Of course, we will be considering particles that are moving, so we will define the velocity, momentum and acceleration of the particle.

Definition 1.1.2 (Velocity)

The **velocity** of a particle is $\mathbf{u}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = \dot{\mathbf{r}}$, and is tangent to the path (or trajectory) of the particle.

Definition 1.1.3 (Momentum)

The **momentum** of a particle is $\mathbf{p} = m\mathbf{u} = m\dot{\mathbf{r}}$, where m is the mass of the particle.

Definition 1.1.4 (Acceleration)

The **acceleration** of the particle is $\dot{\mathbf{u}} = \ddot{\mathbf{r}} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}(t)$.

§1.2 Newton's Laws of Motion

We can now write down Newton's three laws of motion, which govern the motion of particles. All of these statements about particles can be extended to finite bodies (which are composed of many particles).

Law (Newton's First Law/Galileo's Law of Inertia)
There exist inertial frames of reference in which a particle remains at rest or moves at constant velocity unless it is acted on by a force.

Law (Newton's Second Law)
In an inertial frame the rate of change of momentum of a particle is equal to the force acting on it.

Law (Newton's Third Law)
To every action there is an equal and opposite reaction. That is, forces excreted between two particles are equal in magnitude and opposite in direction.

We will see how these are used in the coming sections.

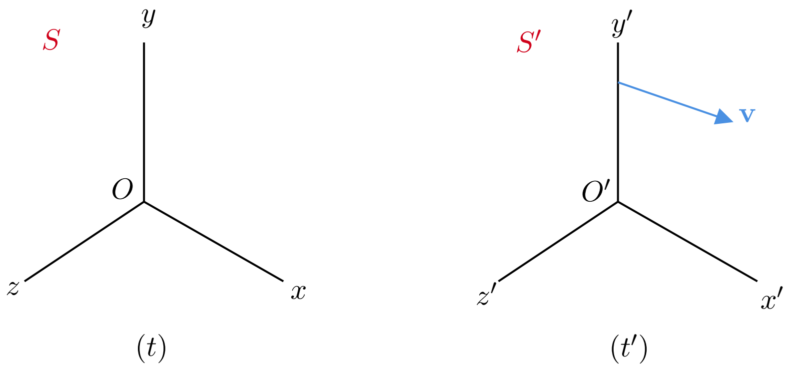
§1.3 Inertial Frames & Galilean Transformations

We met inertial frames in the previous section. In this section we will look at what inertial frames are and how they relate to each other.

Definition 1.3.1 (Inertial Frame)
In an **inertial frame**, the acceleration of a particle is zero if the force is zero. That is

$$\ddot{\mathbf{r}} = \mathbf{0} \iff \mathbf{F} = \mathbf{0}.$$

Inertial frames are not unique. For example, if S is an inertial frame then any other frame S' moving with constant velocity relative to S is also an inertial frame.



In the example above, it is easy to relate the coordinate systems in each system.

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t,$$

where $\mathbf{v}$ is the velocity of S' relative to S. This transformation is called a **boost**.

For a particle with position vector $\mathbf{r}(t)$ in S and $\mathbf{r}'(t')$ in S', we can relate the velocity and acceleration as measured in S and S'. Let the velocity be $\mathbf{u} = \dot{\mathbf{r}}(t)$ in S and the acceleration be $\mathbf{a} = \ddot{\mathbf{r}}(t)$ in S. Then these relate to the corresponding quantities in S' by

$$\mathbf{u}' = \mathbf{u} - \mathbf{v}, \qquad \mathbf{a}' = \mathbf{a}.$$

Boosts aren't the only transformations of frames that preserve inertial frames.

Definition 1.3.2 (Galilean Frames)

A **Galilean transformation** preserves inertial frames. The set of all Galilean transformations forms the **Galilean group**.

Galilean frames combine boosts with some combination of the following:

- Translations of space, $\mathbf{r}' = \mathbf{r} - \mathbf{r}_0$ where $\mathbf{r}_0$ is constant.
- Translations of time, $t' = t - t_0$ where t_0 is constant.
- Rotations and reflections in space, $\mathbf{r}' = R\mathbf{r}$ where R is an orthogonal matrix.

This set generates the Galilean group.

For any Galilean transformation we have

$$\ddot{\mathbf{r}} = \mathbf{0} \iff \ddot{\mathbf{r}}' = \mathbf{0},$$

that is, S inertial if and only if S' is inertial.

Law (Galilean Relativity)

The principle of Galilean relativity is that the laws of Newtonian physics are the same in all inertial frames.

This principle tells us that the the laws of physics are the same at any point in space, at any point in time, in whatever direction I face, and at whatever constant velocity I move with. Thus any set of equations which describe Newtonian physics must have these properties, known as **Galilean invariance**.

Note that the measurement of velocity is then not absolute, but measurement of acceleration is.

§1.4 Newton's Second Law and Equations of Motion

We previously stated Newton's second law, it says that for a particle subject to a force $\mathbf{F}$, the momentum $\mathbf{p}$ satisfies

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F},$$

where the momentum $\mathbf{p} = m\mathbf{u} = m\dot{\mathbf{r}}$.

For now, we will assume that m, the mass of the particle, is constant (we will return to the variable mass case later in the course). That gives us

$$m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = m\ddot{\mathbf{r}} = \mathbf{F}.$$

In this way, we can describe mass as a measure of the 'reluctance to accelerate', the inertia of the particle.

If $\mathbf{F}$ is specified as a function of $\mathbf{r}, \dot{\mathbf{r}}$ and t, $\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$, then we have a second order differential equation for $\mathbf{r}$. To solve this completely, we would need to prove the initial position $\mathbf{r}(t_0)$ and the initial velocity $\dot{\mathbf{r}}(T_0)$, then we get a unique solution. The path/trajectory of the particle is then determined (at all future and past times).

To get further, we will need to consider some possible forms of vvF.

§1.5 Examples of Forces

§1.5.1 Gravitational Force

Consider two particles with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ and masses m_1 and m_2. Then we can describe the gravitational forces between them as follows.

Law (Newton's Gravitation Law)
Newton's law of gravitation states that the force of gravity between the particles is

$$\mathbf{F}_1 = \frac{Gm_1 m_2(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} = -\mathbf{F}_2,$$

where G is the gravitational constant.

Note that $\mathbf{F}_1 = \mathbf{F}_2$ is proportional to $|\mathbf{r}_1 - \mathbf{r}_2|^{-1}$, so we call this law an **inverse square law**.

We will explore the details of gravitational forces later on in the course.

§1.5.2 Electromagnetic Forces

Consider a particle with electric charge q, in the presence of an electric field $\mathbf{E}(\mathbf{r}, t)$ and a magnetic field $\mathbf{B}(\mathbf{r}, t)$. Then we can describe the electromagnetic force as follows.

Law (Lorentz Force Law)
The electromagnetic force on the particle is

$$\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) = q(\mathbf{E}(\mathbf{r}, t) + \dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}, t)).$$

Example 1.5.1
Consider the case of $\mathbf{E} = \mathbf{0}$ and $\mathbf{B}$ being a constant vector.

We can then use Newton's second law to get the equations of motion,

$$m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}.$$

We will choose axes such that $\mathbf{B} = Bv\hat{v}z$. Hencce

$$m\ddot{z} = 0 \implies z = z_0 + ut,$$

for constants z_0 and u. We also get

$$
\begin{aligned}
m\ddot{x} &= qB\dot{y} \\
m\ddot{y} &= -qB\dot{x}.
\end{aligned}
$$

Then defining $\omega = qB/m$, we get the solution

$$
\begin{aligned}
x &= x_0 - \alpha\cos(\omega(t - t_0)), \\
y &= y_0 + \alpha\sin(\omega(t - t_0)),
\end{aligned}
$$

where x_0, x_0, t_0 and α are constants determined by the initial conditions.

Geometrically, the motion of the particle is made up of circular motion in x, y and constant velocity in z, which is helical motion in the direction of the magnetic field. The motion is also clockwise from the direction of $\mathbf{B}$.