Methods

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This article constitutes my notes for the 'Methods' course, held in Michaelmas 2021 at Cambridge. These notes are *not a transcription of the lectures*, and differ significantly in quite a few areas. Still, all lectured material should be covered.

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§1 Fourier Series

§1.1 Periodic Functions

We will begin our study of method and in particular Fourier series by considering some periodic functions.

Definition 1.1 (Perioidic)
A function $f(x)$ is **periodic** if $f(x+T) = f(x)$ for all x, where T is the **period**.

Example 1.2 (Simple Harmonic Motion)
Many physical objects are described by *simple harmonic motion*, with the position given by

$$y = A \sin \omega t.$$

We call A the **amplitude**, and the period is $T = 2\pi/\omega$. The **frequency** is $1/T$.

Fourier series is all about trying to write periodic functions as particular sums of sines and cosines. Consider the set of functions

$$g_n(x) = \cos \frac{n\pi x}{L}, \quad \text{and} \quad h_n(x) = \sin \frac{n\pi x}{L},$$

where we take $n \in \mathbb{R}^+$. These functions are periodic on the interval $[0, 2L]$.

You may recall the following set of identities:

$$\cos A \cos B = \frac{1}{2} (\cos(A-B) + \cos(A+B))$$

$$\begin{aligned}\sin A \sin B &= \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right) \\ \sin A \cos B &= \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right).\end{aligned}$$

We are going to try and define an inner product on this domain $[0, 2L]$, and using that we will by able to multiply these functions together and talk about their relative orthogonality.

Definition 1.3

We define the inner product $\langle f, g\rangle = \int_0^{2L} f(x)g(x)\,\mathrm{d}x$.

We can then obtain some orthogonality conditions for h_n and g_n with respect to this inner product. We can compute for $n \neq m$

$$\begin{aligned}\langle h_n, h_m\rangle &= \int_0^{2L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^{2L}\left(\cos\frac{(n-m)\pi}{L}x - \cos\frac{(n+m)\pi}{L}x\right)\mathrm{d}x \\ &= \frac{1}{2}\frac{L}{\pi}\left[\frac{\sin(n-m)\pi x/L}{n-m} - \frac{\sin(n+m)\pi x/L}{n-m}\right]_0^{2L} \\ &= 0,\end{aligned}$$

and for $n = m$

$$\begin{aligned}\langle h_n, h_n\rangle &= \int_0^{2L}\sin^2\frac{n\pi x}{L}\,\mathrm{d}x \\ &= \int_0^{2L}\frac{1}{2}\left(1-\cos\frac{2\pi n x}{L}\right)\mathrm{d}x \\ &= L.\end{aligned}$$

Hence we obtain the orthogonality condition

$$\langle h_n, h_m\rangle = \begin{cases} L\delta_{mn} & \text{if } n, m \neq 0, \\ 0 & \text{if } m = 0.\end{cases}$$

Similarly, it's straightforward to check that

$$\langle g_n, g_m\rangle = \begin{cases} L\delta_{mn} & \text{if } n, m \neq 0, \\ 2L\delta_{0n} & \text{if } m = 0.\end{cases}$$

and

$$\langle h_n, g_m\rangle = 0.$$

These orthogonality conditions are important because we are going to use these functions as a complete orthogonal set which spans the space of 'well-behaved periodic functions'.

§1.2 Definition of a Fourier Series

We can express any 'well-behaved' periodic function $f(x)$ with period $2L$ as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$$

where a_n, b_n are constants such that the RHS is convergent for all x where f is continuous. At a discontinuity, the Fourier series approaches the midpoint of the upper and lower limits at that point.

Consider taking the inner product $\langle h_n, f\rangle$ and substitute the expression for f above, to get

$$\int_0^{2L} \sin\frac{m\pi x}{L} f(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} Lb_n\delta_{nm} = Lb_m.$$

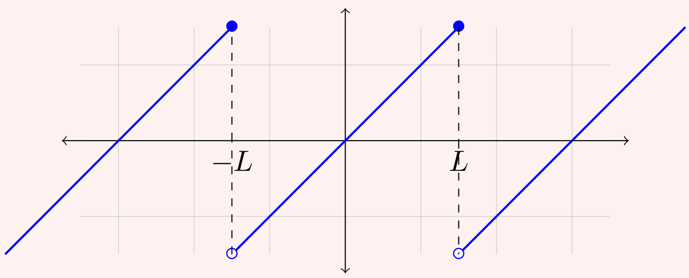
Hence we find that (doing something similar with g_n)

$$\begin{aligned} b_n &= \frac{1}{L}\int_0^{2L} g(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x,\\ \text{and}\quad a_n &= \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x. \end{aligned}$$

Now, this expression for a_n includes the case $n = 0$, and says that it is the average value of the function. Also, the range of integration is one period, and we can equivalently integrate over $[-L, L]$ instead of $[0, 2L]$.

Example 1.4 (The Sawtooth Wave)

Consider the function $f(x) = x$ for $-L \leq x \leq L$, with the function being periodic elsewhere.



Here we have

$$a_n = \frac{1}{L}\int_{-L}^{L} x\cos\frac{n\pi x}{2}\,\mathrm{d}x = 0, \qquad \text{(integrating an odd function)}$$

for all n, and

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}\,\mathrm{d}x\\ &= \frac{-2}{n\pi}\left[x\cos\frac{n\pi x}{L}\right]_0^L + \frac{2}{n\pi}\int_0^h \cos\frac{n\pi x}{L}\,\mathrm{d}x\\ &= -\frac{2L}{n\pi}\cos n\pi + \frac{2L}{(n\pi)^2}\cancel{\sin n\pi}\\ &= \frac{2L}{n\pi}(-1)^{n+1}. \end{aligned}$$

So the sawtooth Fourier series is

$$2L\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi x}{L}\right)=\frac{2L}{\pi}\left[\sin\left(\frac{\pi x}{L}\right)-\frac{1}{2}\sin\left(\frac{2\pi x}{L}\right)+\frac{1}{3}\sin\left(\frac{3\pi x}{L}\right)+\cdots\right].$$

which is slowly convergent.