Rotating Frames

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Motion in Rotating Frames

Suppose that $\mathcal{F}$ is an inertial frame, and $\mathcal{F}'$ is rotating about the z axis with angular velocity $\boldsymbol{\omega} = \omega \mathbf{e}_z$ with respect to $\mathcal{F}$.

Suppose we have basis vectors $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_i'\}$ in $\mathcal{F}$ and $\mathcal{F}'$ respectively. If a particle is at rest in $\mathcal{F}'$, then in $\mathcal{F}$ its velocity is given by

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\mathcal{F}} = \boldsymbol{\omega} \times \mathbf{r}.$$

Of course, this also applies to the basis vectors in $\mathcal{F}'$, with

$$\left(\frac{d\mathbf{e}_i'}{dt}\right)_{\mathcal{F}} = \boldsymbol{\omega} \times \mathbf{e}_i'.$$

Now for some vector $\mathbf{a}$, we can write it in the $\{\mathbf{e}_i'\}$ basis as

$$\mathbf{a} = \sum_i a_i'(t)\mathbf{e}_i'.$$

In the frame $\mathcal{F}'$, the basis vectors $\mathbf{e}_i'$ are constant, and thus the derivative of $\mathbf{a}$ is given by

$$\left(\frac{d\mathbf{a}}{dt}\right)_{\mathcal{F}'} = \sum_i \frac{da_i'(t)}{dt}\mathbf{e}_i'.$$

In the frame $\mathcal{F}'$ however, the basis vectors $\{\mathbf{e}_i'\}$ are not constant, and we have

$$\left(\frac{d\mathbf{a}}{dt}\right)_{\mathcal{F}} = \sum_i \frac{da_i'(t)}{dt}\mathbf{e}_i' + \sum_i a_i'(t)\boldsymbol{\omega} \times \mathbf{e}_i' = \left(\frac{d\mathbf{a}}{dt}\right)_{\mathcal{F}'} + \boldsymbol{\omega} \times \mathbf{a}$$

Change of Frame Operator

Let $\mathcal{F}$ be an inertial frame, and $\mathcal{F}'$ be rotating relative to $\mathcal{F}$ with angular velocity $\boldsymbol{\omega}$. Then we have

$$\left(\frac{d}{dt}\right)_{\mathcal{F}} = \left(\frac{d}{dt}\right)_{\mathcal{F}'} + \boldsymbol{\omega}\times$$

Velocity and Acceleration

Using the change of frame operator, we can see that

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\mathcal{F}} = \left(\frac{d\mathbf{r}}{dt}\right)_{\mathcal{F}'} + \boldsymbol{\omega} \times \mathbf{r},$$

and applying the operator again (and noting that $\dot{\boldsymbol{\omega}}$ is the same in both frames), we have

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\mathcal{F}} = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\mathcal{F}'} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{\mathcal{F}'} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

Force in the Rotating Frame

Since $\mathcal{F}$ is an inertial frame, we have $m\left(d^2\mathbf{r}/dt^2\right)_{\mathcal{F}} = F$ by Newton's laws. This allows us to write down what the force appears to be in the frame $\mathcal{F}'$ (as if the observer in the frame was trying to apply Newton's laws).

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\mathcal{F}'} = \mathbf{F} - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{\mathcal{F}'} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

The additional terms on the right are known as *fictitious forces*, each with a different name.

1. *Coriolis force.* $-2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{\mathcal{F}'}$.
2. *Euler force.* $-m\dot{\boldsymbol{\omega}} \times \mathbf{r}$.
3. *Centrifugal force.* $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$.