PQM Alimo – Hurdles to Greatness

Cambridge Mathematical Tripos Part II – Adam Kelly (`ak2316`)
June 10, 2023

Note. Likelihood ratings are between 1 and 5, and are conditioning on interaction picture and angular momentum questions being the most likely to come up.

2010 Paper 1 *Likelihood Rating: 2 (Probably too straightforward and doesn't require much more than Paper 3 question).*

$|j_1 + j_2\ j_1 + j_2\rangle = |j_1\ j_1\rangle\,|j_2\ j_2\rangle$. Other combined angular momentum states can be found by applying the lowering operator $J_- = J_-^{(1)} + J_-^{(2)}$ to find lowered M states, and by finding orthogonal vectors to get lowered J states.

If $j_1 = j_2 = j$, then states with $M = 0$ are the sum of terms of the form $|j\ m\rangle\,|j\ {-m}\rangle$ for $m \in \{-j, \dots, j\}$. Noting that $J_+\,|0\ 0\rangle = 0$, applying on the right and noting the coefficients of basis states must sum to zero, we get $a_m = -a_{m+1}$. So by normalisation, we have $2j+1$ states and hence $\alpha_m = \frac{1}{\sqrt{2j+1}}(-1)^m$.

The probability of measuring combined total angular momentum zero is $|\langle 0\ 0|\,(|j\ j\rangle\,|j\ {-j}\rangle)|^2 = \|\alpha_j\|^2 = \frac{1}{2j+1}$ by the Born rule.

2010 Paper 2 *Likelihood Rating: 5 (Disjoint enough and hard enough).*

$U(n, \theta) = e^{-i\theta n\cdot J/\hbar}$. A state with zero orbital angular momentum would transform the only the spin component of the state by the same rotation (about the origin with respect to spin).

When $j = 1/2$, we ge that $J = \frac{1}{2}\hbar\sigma$. We can explicitly calculate

$$J \cdot a = \frac{h}{2}\begin{pmatrix} a_z & a_x - ia_y \\ a_x + ia_y & -az \end{pmatrix},$$

and so $(J\cdot a)^2 = \frac{h^2}{4}\|a\|^2 I$. The eigenvalues of $J \cdot a$ are $\pm\frac{h}{2}\|a\|$. For $j = 1/2$, we can write

$$U(n, \theta) = e^{-i\theta n\cdot J/\hbar} = e^{-i\theta n\cdot\sigma/2} = \cos(-\theta n\cdot\sigma/2) + i\sin(-\theta n\cdot\sigma/2),$$

which, by linearity and considering the action on the eigenstates, gives $U(n, \theta) = \cos\theta/2 - in\cdot\sigma\sin\theta/2$.

We have $n'\cdot J\left|\frac{1}{2}\ m\right\rangle_\theta = \hbar m'$. So the probability of measuring $m'\hbar$ along the direction n' is

$$\left|\left\langle \frac{1}{2}\ m\middle|\frac{1}{2}\ m'\right\rangle_\theta\right|^2 = \left|\left\langle \frac{1}{2}\ m\middle|U(y,\theta)\middle|\frac{1}{2}\ m'\right\rangle\right|^2,$$

using $\left|\frac{1}{2}\ m'\right\rangle_\theta = U(y,\theta)\left|\frac{1}{2}\ m'\right\rangle$, with y the y-axis unit vector. So if $m = \frac{1}{2}$, $m' = -\frac{1}{2}$, then substituting in gives our probability as $\left|\left\langle\frac{1}{2}\ \frac{1}{2}\right|\left(\sin\frac{\theta}{2}\left|\frac{1}{2}\ \frac{1}{2}\right\rangle + \cos\frac{\theta}{2}\left|\frac{1}{2}\ \frac{1}{2}\right\rangle\right)\right|^2 = \sin^2\frac{\theta}{2} = 1 - \cos^2\frac{\theta}{2}$.

2010 Paper 4 *Likelihood Rating: 4 (Exact probability calculation and exact condition for approximation to be valid similar to Paper 2 qn).*

Our time evolution equation is $i\hbar\frac{d}{dt}\left|\psi_I(t)\right\rangle = V_I(t)\left|\psi_I(t)\right\rangle$. We want to compute $|\langle b|\psi(t)\rangle|^2$ where $|\psi(0)\rangle = |a\rangle$. We can integrate our time evolution equation to get

$$\begin{aligned}
\left|\psi_I(t)\right\rangle &= -\frac{i}{\hbar}\left|a\right\rangle + \frac{i}{\hbar}\int_0^t V_I(t')\left|\psi_I(t')\right\rangle \mathrm{d}t' \\
\implies |\langle b|\psi_I(t)\rangle|^2 &= \frac{1}{\hbar^2}\left|\int_0^t \left\langle b\right| V_I(t')\left|\psi_I(t')\right\rangle \mathrm{d}t\right|^2 \\
&= \frac{1}{\hbar^2}\left|\int_0^t \left\langle b\right| V_I(t')\left|a\right\rangle \mathrm{d}t\right|^2 \\
&= \frac{1}{\hbar^2}\left|\int_0^t dt' \,\langle b|V(t')|a\rangle\, e^{i(E_b-E_a)t'/\hbar}\right|^2,
\end{aligned}$$

to order $V(t)^2$ as required.

In the case of H_0 and V given, we can compute that $\langle 2|vt\sigma_1|1\rangle = vt$, and hence our transition probability is given by

$$\frac{1}{\hbar^2}\left|\int_0^t vt'\mathrm{d}t'\right|^2 = \frac{v^2t^4}{4\hbar^2}.$$

We can clearly see that the given state is a solution to the TDSE, so letting $|\psi(0)\rangle = |1\rangle$, we can substitute in and take inner products with $|2\rangle$. Then (after rewriting the exponential to get only the σ_1 terms and rearranging the sum) we get that the exact transition probability is $\sin^2(\frac{vt^2}{2\hbar})$. So our approximation is valid if $vt^2/2\hbar$ is small.

2011 Paper 1 *Likelihood Rating: 1* (Requires same angular momentum calculation skills).

2011 Paper 2 *Likelihood Rating: 3* (Not on a specific topic but he seems to like the 'from the start' questions).

2011 Paper 3 *Likelihood Rating: 5* *(Disjoint enough and covers end part of course).*

(i) $|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$. $s = \frac{1}{2}\hbar\sigma$. We can then manually check using the given matrices that $[s_i, s_j] = \varepsilon_{ijk}s_k$ and $[s^2, s_3] = 0$. Lastly we check that $(n\cdot s)^2 = I$ which gives us that the eigenvalues are as desired. We also have

$$s_x\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix},\quad s_x\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix},\quad s_y\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0\\i\end{pmatrix},\quad s_y\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}-i\\0\end{pmatrix}.$$

These can be rewritten in terms of $|\uparrow\rangle$ and $|\downarrow\rangle$ as desired.

(ii) It's easy to check that the eigenvalues are $-\frac{\hbar^3}{8}, -\frac{\hbar^3}{8}, -\frac{\hbar^3}{8}, \frac{\hbar^3}{8}$.

(iii) Let $\tilde{s}_x^{(i)}$ take on value $\frac{\hbar}{2}$ with probability $p_x^{(i)}$, and $-\frac{\hbar}{2}$ otherwise. Define $\tilde{s}_y^{(i)}$ similarly. Then from the previously calculated eigenvalues we have that $\mathbb{P}(\tilde{s}_x^{(1)})\tilde{s}_y^{(2)}\tilde{s}_y^{(2)} = -\frac{\hbar^3}{8}) = 1$ and so on. Multiplying gives that the probability they are all -1 is 1, and hence $\mathbb{P}(\tilde{s}_x^{(1)})\tilde{s}_x^{(2)})\tilde{s}_x^{(2)}) = -\frac{\hbar^3}{8}) = 1$ (since the $\tilde{s}_y^{(i)}$ terms all square to 1). Hence there is a classical unique possibility for this of $-\frac{\hbar^3}{8}$ with probability 1.

This lets us test quantum mechanics experimentally as we can produce this state and measure to get -1 which contradicts this 'hidden classical variables' theory.

2012 Paper 2 *Likelihood Rating: 3* (Heisenberg picture plus dirac formalism not really examined yet).

2012 Paper 4 *Likelihood Rating: 3.5* (Disjoint enough but still quite easy, covers the angular momentum content not tested in Paper 3).

2013 Paper 2 *Likelihood Rating: 1* (Easy spin question, same year as a previous repeat, calculates commutators like in Paper 1 and simple addition of angular momentum which was examined in Paper 3)

2014 Paper 1 *Likelihood Rating: 2* (Dirac notation question, basically just position and momentum space with not much else going on)

2014 Paper 3 *Likelihood Rating: 3.5* (Angular momentum question disjoint to the previously asked stuff)

For states $|\psi_S\rangle$ and operators A_S in the Schrödinger picture, we define the interaction picture states and operators by

$$|\psi_I(t)\rangle = e^{iH_0t/\hbar}|\psi_S(t)\rangle, \quad A_I(t) = e^{iH_0t/\hbar}A_S e^{-iH_0t/\hbar}.$$

We can then check that matrix elements are preserved as then they give the same physical predictions, and indeed if $|\psi\rangle$, $|\phi\rangle$ and A are operators then

$$\langle\phi_S|A_S|\psi_S\rangle = \left\langle\phi_S\left|e^{-iH_0t/\hbar}e^{iH_0t/\hbar}A_S e^{-iH_0t/\hbar}e^{iH_0t/\hbar}\right|\psi_S\right\rangle = \langle\phi_I|A_I(t)|\psi_I\rangle,$$

so our theory is the same. We can then just differentiate to obtain the equation of motion

$$i\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle = V_I(t)|\psi_I(t)\rangle.$$

The rest isn't PQM content it's just DEs so you just follow the instructions and you get what you're told.

2014 Paper 4 *Likelihood Rating: 4* (Interaction Picture question, pretty disjoint from previous stuff)

2015 Paper 2 *Likelihood Rating: 2* (Simple spin half stuff combined with already done calculations for addition of angular momentum)

2016 Paper 4 *Likelihood Rating: 5* (Interaction picture with the part of spin we didn't discuss).

(a) Interaction picture bookwork as above, except we get out the operator derivative $i\hbar\frac{dA_I}{dt} = [A_I(t), H_0]$.

(b) Define $S = \frac{1}{2}\hbar\sigma$. Then we can check $(n\cdot S)^2 = \hbar^2/4$ for any unit vector n, and also $[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k$ so we have the correct spin algebra.

(c) In general, to check that U acts correctly as the transformation T with infinitesimal parameter, we need to check

$$U(\delta\theta)QU(\delta\theta)^{-1} = T(\delta\theta)Q$$

The operators x, p, σ all transform as vectors, so we can just check for a 'vector transforming object' V that this holds. We work to first order in $\delta\theta$ to get

$$\left(I + \frac{i\delta\theta}{\hbar}n\cdot J\right)V\left(I - \frac{i\delta\theta}{\hbar}n\cdot J\right) + O(\delta\theta^2) = V + \frac{i\delta\theta}{\hbar}[n\cdot J, V] + O(\delta\theta^2)$$

then we use that $[J_i, V_j] = i\hbar\varepsilon_{ijk}V_k$ to get the above is

$$V + \delta\theta n \times V + O(\delta\theta^2),$$

which is the correct form of our transformation for infinitesimal rotations.

(d) Define $J_i^I(t)$ and J_i in the obvious way. Then (a) gives

$$i\hbar\frac{dJ_i^I(t)}{dt} = \left[J_i^I(t), H_0\right] = e^{iH_0t/\hbar}\left[J_i, H_0\right]e^{-iH_0t/\hbar}$$

We can evaluate $[J_i, H_0]$ with (using index notation):

$$\begin{aligned}[J_i, H_0] &= \frac{1}{2m}\left[J_i, p_kp_k\right] + \frac{\alpha}{m\hbar}\left[J_i, L_kS_k\right] \\ &= \frac{1}{2m}\left(\left[J_i, p_k\right]p_k + p_k\left[J_i, p_k\right]\right) + \frac{\alpha}{m\hbar}\left(\left[J_i, L_k\right]S_k + L_k\left[J_i, S_k\right]\right)\end{aligned}$$

Using the commutation relation for J_i with a vector operator, we have

$$[J_i, H_0] = \frac{1}{2m}\left(\epsilon_{ikj}p_jp_k + \epsilon_{ikj}p_kp_j\right) + \frac{\alpha}{m\hbar}\left(\epsilon_{ikj}L_jS_k + \epsilon_{ikj}L_kS_j\right) = 0$$

since both terms are the product of something antisymmetric on k, j and something symmetric on k, j. We deduce that $\frac{dJ_i^I(t)}{dt} = 0$ as required.

In the Heisenberg picture, we instead must use the Heisenberg equation of motion:

$$i\hbar\frac{dJ_i^H(t)}{dt} = \left[J_i^H(t), H\right] = e^{iHt/\hbar}\left[J_i, H\right]e^{-iHt/\hbar}.$$

We have already seen that $[J_i, H_0] = 0$. So we are left to evaluate the commutator:

$$[J_i, B\sigma_3] = \frac{2B}{\hbar}\left[J_i, S_3\right] = \frac{2B}{\hbar}i\hbar\epsilon_{i3k}S_k = i\hbar B\epsilon_{i3k}\sigma_k$$

Hence we're left with:

$$\frac{dJ_i^H(t)}{dt} = B\epsilon_{i3k}e^{iHt/\hbar}\sigma_ke^{-iHt/\hbar} \neq 0,$$

unless $i = 3$. So the vector $J^H(t)$ in the Heisenberg picture is not independent of time.