Baire Category Theorem

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The Baire category theorem is a lovely result that is missing from the Part IB 'Analysis and Topology' course. This article is a presentation of this result, along with a number of applications of it.

Theorem (Baire Category Theorem)

Let $U_1, U_2, \dots$ be a sequence of dense open sets in a complete metric space X. Then $U = \cap_{n=1}^{\infty} U_n$ is dense in X.

Proof. Since U_n is a dense open subset of X for all $x \in X$ and $r > 0$ the open ball $D_r(x)$ intersects U_n in a non-empty open set, so there is $y \in X$ and $s > 0$ such that

$$B_s(y) \subset D_{2s}(y) \subset Y_n \cap D_r(x).$$

□