Linear Algebra

Adam Kelly (ak2316@cam.ac.uk)

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This article constitutes my notes for the 'Linear Algebra' course, held in Michaelmas 2021 at Cambridge. These notes are *not a transcription of the lectures*, and differ significantly in quite a few areas. Still, all lectured material should be covered.

Contents

§1 Vector Spaces

§1.1 Vector Spaces and Subspaces

Let $\mathbb{F}$ be an arbitrary field.

Definition 1.1 (Vector Space Over $\mathbb{F}$)

A **vector space over $\mathbb{F}$** is an abelian group $(V, +)$ equipped with a function $\mathbb{F} \times V \to V$, $(\lambda, v) \mapsto \lambda v$ such that

(i) $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$,

(ii) $(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$,

(iii) $\lambda(\mu v) = (\lambda\mu)v$,

(iv) $1v = v$.

Example 1.2 (Examples of Vector Spaces)

(i) $\mathbb{F}^n$ with $n \in \mathbb{N}$, the set of column vectors of size n with entries in $\mathbb{F}$.

(ii) Take any set X, and define $\mathbb{R}^X = \{f : X \to \mathbb{R}\}$, the set of real valued functions on X. This is a vector space over $\mathbb{R}$.

(iii) $\mathcal{M}_{n,m}$, the set of $n \times m$ matrices with entries in $\mathbb{F}$.

Remark. The axioms of scalar multiplication imply that $0v = 0$, for any $v \in V$.

Definition 1.3 (Subspace)

Let V be a vector space over $\mathbb{F}$. The subset U of V is a **vector subspace** of V, denoted $U \leq V$, if:

(i) $0 \in U$,

(ii) $u_1, u_2 \in U$ implies that $u_1 + u_2 \in U$,

(iii) $\lambda \in \mathbb{F}$, $u \in U$ implies that $\lambda u \in U$.

Clearly if V is an $\mathbb{F}$ vector space and $U \leq V$, then U is an $\mathbb{F}$ vector space.

Example 1.4 (Examples of Subspaces)

(i) If V is the set of functions $\mathbb{R} \to \mathbb{R}$, then the set of continuous functions $\mathcal{C}(\mathbb{R}) \leq V$ is a subspace.

(ii) The set of vectors

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x_1, x_2, x_3 \in \mathbb{R}, x_1 + x_2 + x_3 = t \right\}$$

is a subspace of $\mathbb{R}^3$ for $t = 0$ only.

Proposition 1.5 (Intersecting Subspaces)

Let $U, W \leq V$. Then $U \cap W \leq V$.

Proof. Since $0 \in U$ and $0 \in W$, we have $0 \in U \cap W$. Now if $\lambda_1, \lambda_2 \in \mathbb{F}$ and $v_1, v_2 \in U \cap W$, then $\lambda_1 v_1 + \lambda_2 v_2 \in U$ and V, and thus is in $U \cap V$. Thus $U \cap W \leq V$. $\square$

The union of two subspaces is generally *not* a subspace, as it is typically not closed by addition. In fact, the union is only ever a subspace if one of the subspaces is contained in the other.

We can however try to 'complete' the union so that it becomes a subspace.

Definition 1.6 (Sum of Subspaces)

Let V be a vector space over $\mathbb{F}$, and let $U, W \leq V$. We define the **sum** of U and W to be the set

$$U + W = \{u + w \mid u \in U, w \in W\}.$$

This definition immediately forces $U + W \leq V$, and indeed it is the minimal such space (in that any subspace of V containing both U and W must also contain $U + W$).