Vector Calculus

Adam Kelly

January 30, 2021

This set of notes is a work-in-progress account of the course 'Vector Calculus', originally lectured by Dr Anthony Ashton in Lent 2020 at Cambridge. These notes are not a transcription of the lectures, but they do roughly follow what was lectured (in content and in structure).

These notes are my own view of what was taught, and should be somewhat of a superset of what was actually taught. I frequently provide different explanations, proofs, examples, and so on in areas where I feel they are helpful. Because of this, this work is likely to contain errors, which you may assume are my own. If you spot any or have any other feedback, I can be contacted at ak2316@cam.ac.uk.

Contents

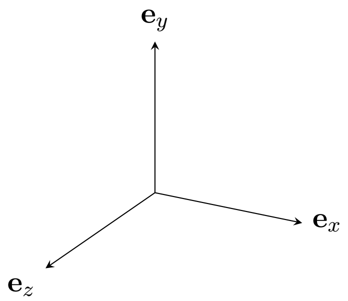
0 Introduction

§0.1 Notation

Throughout this course, a column vector

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

is to be interpreted as the vector $\mathbf{x} = a\mathbf{e}_x + b\mathbf{e}_y + c\mathbf{e}_z$, where $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ are basis vectors aligned with the fixed Cartesian x, y and z axes in $\mathbb{R}^3$.



We may also use the notation $\mathbf{e}_1 = \mathbf{e}_x$, $\mathbf{e}_2 = \mathbf{e}_y$ and $\mathbf{e}_3 = \mathbf{e}_z$, and then we can write $\mathbf{x} = x_i\mathbf{e}_i$.

We will also be using the summation convention frequently, which you can read more about in the 'Vectors and Matrices' course notes.

1 Differential Geometry of Curves

We will begin by studying the differential geometry of curves in $\mathbb{R}^3$, which sounds exciting but isn't.

§1.1 Parameterized Curves & Arc Length

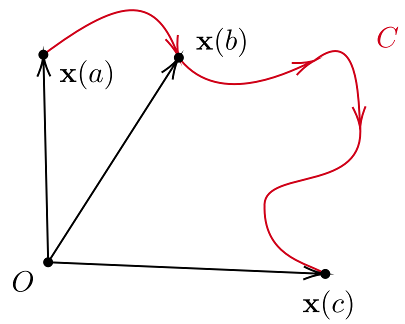
The first object we shall study is *parameterized curves.*

Definition 1.1.1 (Parameterized Curve)
A **parameterized curve** C in $\mathbb{R}^3$ is the image of a continuous map $\mathbf{x} : [a, b] \to \mathbb{R}^3$ in which $t \mapsto \mathbf{x}(t)$.

In Cartesian coordinates, we can write

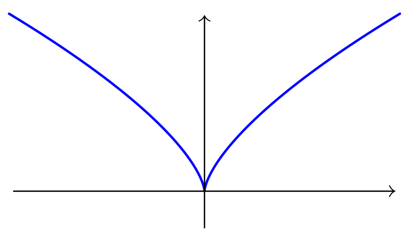
$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.$$

We also give the curve an orientation, based on the map going from the image of A to the image of B.



Definition 1.1.2 (Differentiability and Smoothness)
We say that a curve C is **differentiable** if each of the components $\{x_1(t), x_2(t), x_3(t)\}$ are differentiable, and say C is **regular** if $|\mathbf{x}'(t)| \neq 0$. If C is differentiable and regular then we say C is **smooth**.

Remark. The 'regular' condition exists to avoid 'spikes' in the curve. For example, consider the curve $\mathbf{x}(t) = (t^3, t^2)$. Clearly this is differentiable, but $\mathbf{x}(t)$ has a cusp at $t = 0$. At $t = 0$, we have $|\mathbf{x}(0)| = 0$. If this was not the case, there would be no cusp.

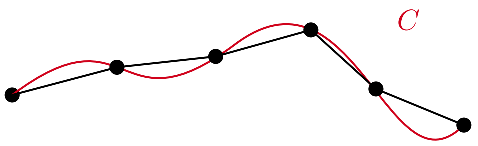


Recall that the function $x_i(t)$ is differentiable at t if $x_i(t+h) = x_i(t) + x_i'(t)h + o(h)$, where $\frac{o(h)}{h} \to 0$ as $h \to 0$. We can write this in terms of vectors, where

$$\mathbf{x}(t+h) = \mathbf{x}(t) + \mathbf{x}'(t)h + o(h),$$

where $o(h)$ is a vector for which $\frac{|o(h)|}{h} \to 0$ as $h \to 0$.

Now given some curve C, we may want to find the length of the curve. We can try to do this by approximating the curve using straight lines.



For $C : [a, b] \to \mathbb{R}^3$ with $t \mapsto \mathbf{x}(t)$, we introduce a partition P of $[a, b]$ with t_0, $t_N = b$ and $t_0 < t_1 < \cdots < t_N$, and we set $\Delta t_i = t_{i+1} - t_i$ and $\Delta t = \max_i \Delta t_i$.

Definition 1.1.3 (Length Relative to a Partition)

For some curve C and partition P as above, we define the **length** of C relative to P by

$$\ell(C, P) = \sum_{i=0}^{N-1} |\mathbf{x}(t_{i+1}) - \mathbf{x}(t_i)| .$$

We would expect that this length would get closer to the true length of C as $\Delta t \to 0$. Indeed, we will define the length of C in this way.

Definition 1.1.4 (Length of a Curve)

We define the length of a curve C by

$$\ell(C) = \lim_{\Delta t \to 0} \sum_{i=0}^{N-1} |\mathbf{x}(t_{i+1}) - \mathbf{x}(t_i)| = \lim_{\Delta t \to 0} \ell(C, P).$$

if this limit does not exist, we say the curve is **non-rectifiable**.

In this course, we will not worry too much about non-rectifiable curves, so we will assume that this notion is always well defined.

Now suppose C is a differentiable curve. Then we have that

$$\begin{aligned}
\mathbf{x}(t_{i+1}) &= \mathbf{x}(t_i + t_{i+1} - t_i) \\
&= \mathbf{x}(t_i + \Delta t_i) \\
&= \mathbf{x}(t_i) + \mathbf{x}'(t_i)\Delta t_i + o(\Delta t_i).
\end{aligned}$$

It follows that

$$|\mathbf{x}(t_{i+1}) - \mathbf{x}(t_i)| = |\mathbf{x}'(t_i)|\Delta t_i + o(\Delta t_i).$$

So if C is differentiable, we get the expression

$$\ell(C, P) = \sum_{i=0}^{N-1} \left(\left| \mathbf{x}'(t_i) \right| \Delta t_i + o(\Delta t_i) \right).$$

Recall that $o(\Delta t_i)$ represents a function for which $\frac{o(\Delta t_i)}{\Delta t_i} \to 0$ as $\delta t_i \to 0$. So for any $\epsilon > 0$, if $\Delta t = \max_i \Delta t_i$ is sufficiently small, then we have

$$|o(\Delta t_i)| < \left(\frac{\epsilon}{b-a} \right) \Delta t_i,$$

for $i = \{0, \dots, N-1\}$. So

$$\begin{aligned} \left| \ell(C, P) - \sum_{i=0}^{N-1} |\mathbf{x}'(t_i) \Delta t_i \right| &= \left| \sum_{i=0}^{N-1} o(\Delta t_i) \right| \\ &< \frac{\epsilon}{b-a} \sum_{i=0}^{N-1} \Delta t_i = \epsilon. \end{aligned}$$

Thus the LHS tends to zero as $\Delta t \to 0$. So we get that

$$\begin{aligned} \ell(C) = \lim_{\Delta t \to 0} \ell(C, P) &= \lim_{\Delta t \to 0} \sum_{i=0}^{N-1} |\mathbf{x}'(t_i)| \Delta t_i \\ &= \int_a^b \left| \mathbf{x}'(t) \right| \, \mathrm{d}t, \end{aligned}$$

by the definition of the Riemann integral. So, we can restate our definition in this other form.

Definition 1.1.5 (Length of a Curve)
If $C : [a, b] \to \mathbb{R}^3$ is a differentiable curve with $t \mapsto \mathbf{x}(t)$, then its **length** is

$$\ell(C) = \int_a^b |\mathbf{x}'(t)| \, \mathrm{d}t = \int_C \mathrm{d}s,$$

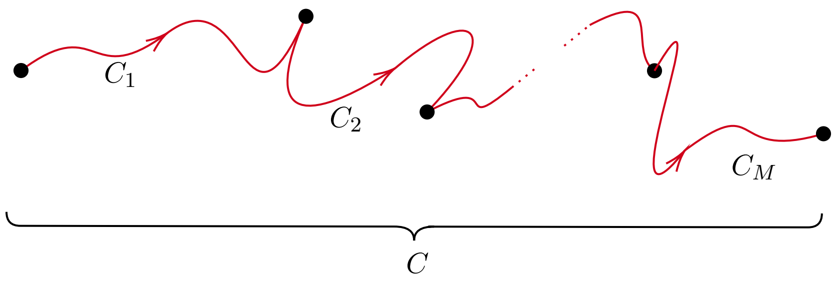
where $\mathrm{d}s$ is the **arc-length element**, $\mathrm{d}s = |\mathbf{x}'(t)| \, \mathrm{d}t$.

With this defined, we can define the integral of a function along a curve.

Definition 1.1.6 (Integral on a Curve)
For a function $f : \mathbb{R}^3 \to \mathbb{R}$ and a differentiable curve $C : [a, b] \to \mathbb{R}^3$ with $t \mapsto \mathbf{x}(t)$, we define the **integral of f along C** to be

$$\int_C f(\mathbf{x}) \, \mathrm{d}s = \int_a^b f(\mathbf{x}(t)) |\mathbf{x}'(t)| \, \mathrm{d}t.$$

Now consider a curve C made up of M smooth curves $C_1, C_2, \dots, C_M$.



Then writing $C = C_1 + C_2 + \cdots + C_M$, we can define

$$\int_C f(\mathbf{x})\ \mathrm{d}s = \sum_{i=1}^{M} \int_{C_i} f(\mathbf{x})\ \mathrm{d}s,$$

so we can integrate over piecewise smooth curves.

Note that informally, we have

$$\mathrm{d}s = |\mathbf{x}'(t)|\ \mathrm{d}t = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\ \mathrm{d}t.$$

So being somewhat sacrilegious, we have

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

Example 1.1.7 (Circumference of a Circle)

Let C be a circle of radius $r > 0$ in $\mathbb{R}^3$,

$$\mathbf{x}(t) = \begin{pmatrix} r\cos t \\ r\sin t \\ 0 \end{pmatrix}, \qquad t \in [0, 2\pi].$$

We can differentiate this to get

$$\mathbf{x}'(t) = \begin{pmatrix} -r\sin t \\ r\cos t \\ 0 \end{pmatrix}, \qquad t \in [0, 2\pi].$$

Then integrating over C we have

$$\begin{aligned} \int_C \mathrm{d}s &= \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\ \mathrm{d}t \\ &= \int_0^{2\pi} r\ \mathrm{d}t \\ &= 2\pi r, \end{aligned}$$

as we would expect.

Example 1.1.8 (Integrating over a Circle)

With C being a circle as before, we can integrate over the curve. For example

$$\begin{aligned}\int_C x^2 y \, \mathrm{d}s &= \int_0^{2\pi} (r\cos t)^2 (r \sin t) \underbrace{r \, \mathrm{d}t}_{|\mathbf{x}'(t)| \, \mathrm{d}t} \\ &= 0\end{aligned}$$

There is one subtlety that we have looked over - does $\ell(C)$ depend on the parameterization used?

For example, if we had $\mathbf{x}(t) = (r\cos t, r\sin t, 0)$ with $t \in [0, 2\pi]$ and $\tilde{\mathbf{x}}(t) = (r\cos(2t), r\sin(2t), 0)$ with $t \in [0, \pi]$, these represent the same circle and they should have the same length. We will clear up this possible ambiguity now.

Proposition 1.1.9

If $f : \mathbb{R}^3 \to \mathbb{R}$ is a function and C is a differentiable curve, then $\int_C f(\mathbf{x}) \, \mathrm{d}s$ is independent of the parameterization of C used.

Proof. Suppose C has two different parameterization,

$$\begin{aligned}\mathbf{x} &= \mathbf{x}_1(t), \qquad a \leq t \leq b \\ \mathbf{x} &= \mathbf{x}_2(t), \qquad \alpha \leq t \leq \beta\end{aligned}$$

We must then have $\mathbf{x}_2(\tau) = \mathbf{x}_1(t(\tau))$ for some function $t(\tau)$. We can assume that $\frac{\mathrm{d}t}{\mathrm{d}\tau} \neq 0$ so the map between t and τ is invertible and differentiable (this is the inverse function theorem, covered in Analysis and Topology in Part IB). Note then that

$$\begin{aligned}\mathbf{x}_2'(\tau) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{x}_2(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{x}_1(t(\tau)) \\ &= \frac{\mathrm{d}t}{\mathrm{d}\tau}\mathbf{x}_1'(t(\tau)).\end{aligned}$$

Then from our definition,

$$\int_C f(\mathbf{x}) \, \mathrm{d}s = \int_a^b f(\mathbf{x}_1(t)) |\mathbf{x}_1'(t)| \, \mathrm{d}t.$$

We can make the substitution $t = t(\tau)$, and assuming $\mathrm{d}t/\mathrm{d}\tau > 0$, the later integral becomes

$$\int_\alpha^\beta f(\mathbf{x}_2(\tau)) \underbrace{|\mathbf{x}_1'(t(\tau))| \frac{\mathrm{d}t}{\mathrm{d}\tau} \, \mathrm{d}\tau}_{|\mathbf{x}_2'(\tau)| \, \mathrm{d}\tau},$$

which is precisely the same as $\int_C f(\mathbf{x}) \, \mathrm{d}s$ using the $\mathbf{x}_2(t)$ parameterization.

We assumed here that $\frac{\mathrm{d}t}{\mathrm{d}\tau} > 0$, but the same holds if it is negative. Thus our definition of integrating over a curve C does not depend on the choice of parameterization of C. □

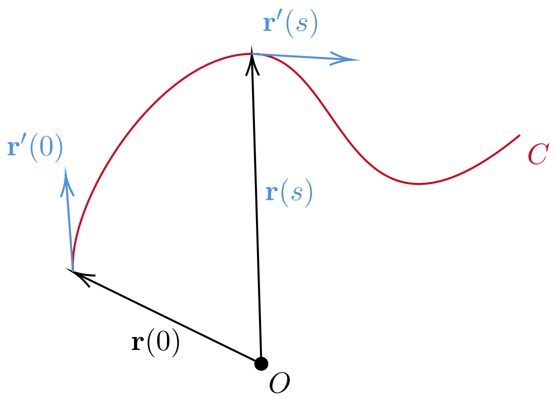
We can parameterize curves in many different ways. One useful way to parameterise regular cuves is with respect to arc-length. If we write $\mathbf{r}(s) = \mathbf{x}(t(s))$ where $0 \leq s \leq \ell(C)$, then by the chain rule

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{1}{\mathrm{d}s/\mathrm{d}t} = \frac{1}{|\mathbf{x}'(t(s))|},$$

so

$$\begin{aligned} \mathbf{r}'(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{x}(t(s)) \\ &= \frac{\mathrm{d}t}{\mathrm{d}s}\mathbf{x}'(t(s)) \\ &= \frac{\mathbf{x}'(t(s))}{|\mathbf{x}'(t(s))|}, \end{aligned}$$

that is, $|\mathbf{r}'(s)| = 1$. This (consistently) gives $\ell(C) = \int_0^{\ell(C)} |\mathbf{r}'(s)|\ \mathrm{d}s) = \int_0^{\ell(C)}\ \mathrm{d}s$. So the way to picture $\mathbf{r}'(s)$ is as the unit tangent vector to the curve.



§1.2 Curvature and Torsion

Throughout this section, we will talk about a generic regular curve C, parameterized by arc-length, and we will write $s \mapsto \mathbf{r}(s)$.

Definition 1.2.1 (Tangent Vector)

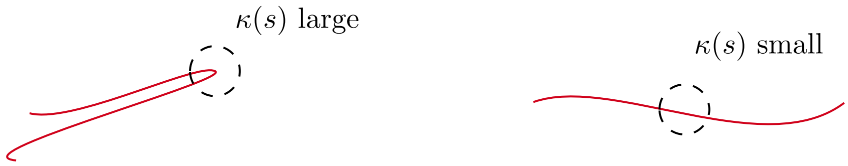
For a regular curve C parameterized by arc length, we define the **tangent vector** to be $\mathbf{t}(s) = \mathbf{r}'(s)$.

We already established that $|\mathbf{t}(s)| = 1$, and thus as the second derivative $\mathbf{r}''(s) = \mathbf{t}'(s)$ only measures the change in *direction*.

Intuitively, if $|\mathbf{r}''(s)|$ is large, then the curve rapidly changes in direction, whereas if $|\mathbf{r}''(s)|$ is small, we would expect the curve to be approximately flat.

Definition 1.2.2 (Curvature)

We define the **curvature** of C to be $\kappa(s) = |\mathbf{r}''(s)| = |\mathbf{t}'(s)|$.



Now since $\mathbf{t} = \mathbf{r}'(s)$ is a unit vector, if we differentiate $\mathbf{t} \cdot \mathbf{t} = 1$ we get $\mathbf{t} \cdot \mathbf{t}' = 0$. Because of this we can introduce the following definition.

Definition 1.2.3 (Principle Normal)
The **principle normal** $\mathbf{n}$ is the vector given by $\mathbf{t}' = \kappa \mathbf{n}$.

The principle normal $\mathbf{n}$ is everywhere normal to the curve C, since it's always perpendicular to the tangent vector ($\mathbf{t} \cdot \mathbf{n} = 0$). Then we have two perpendicular vectors, so we can extend $\{\mathbf{t}, \mathbf{n}\}$ to an orthonormal basis by defining the following.

Definition 1.2.4 (Binormal)
The **binormal** $\mathbf{b}$ is the unit vector perpendicular to $\mathbf{t}$ and $\mathbf{n}$, $\mathbf{b} = \mathbf{t} \times \mathbf{n}$.

Since $|\mathbf{b}| = 1$, we have $\mathbf{b}' \cdot \mathbf{b} = 0$. Also since $\mathbf{t} \cdot \mathbf{b} = 0$ and $\mathbf{n} \cdot \mathbf{b} = 0$,

$$\begin{aligned} 0 = (\mathbf{t} \cdot \mathbf{b})' &= \mathbf{t}' \cdot \mathbf{b} + \mathbf{t} \cdot \mathbf{b}' \\ &= \cancel{\kappa \mathbf{n} \cdot \mathbf{b}} + \mathbf{t} \cdot \mathbf{b}', \end{aligned}$$

thus $\mathbf{b}'$ is orthogonal to $\mathbf{t}$ and $\mathbf{b}$, and thus it is parallel to $\mathbf{n}$. We define the following.

Definition 1.2.5 (Torsion)
We define the **torsion** τ by $\mathbf{b}' = -\tau \mathbf{n}$.

Informally, torsion measures a twisting motion along a curve. So we have two equations.

$$\mathbf{t}' = \kappa \mathbf{n}, \qquad \mathbf{b}' = -\tau \mathbf{n}.$$

Proposition 1.2.6 (Fundamental Theorem of Curves)
The curvature $\kappa(s)$ and torsion $\tau(s)$ define a curve up to translation and orientation.

Proof Sketch. Since $\mathbf{n} = \mathbf{b} \times \mathbf{t}$, we have

$$\mathbf{t}' = \kappa(\mathbf{b} \times \mathbf{t}), \quad \mathbf{b}' = -\tau(\mathbf{b} \times \mathbf{t}).$$

This gives six equations for six unknowns. Given $\kappa(s)$, $\tau(s)$, $\mathbf{t}(0)$ and $\mathbf{b}(0)$, we can construct the functions $\mathbf{t}(s)$, $\mathbf{b}(s)$, and hence $\mathbf{n} = \mathbf{b} \times \mathbf{t}$. Hence the result. $\square$

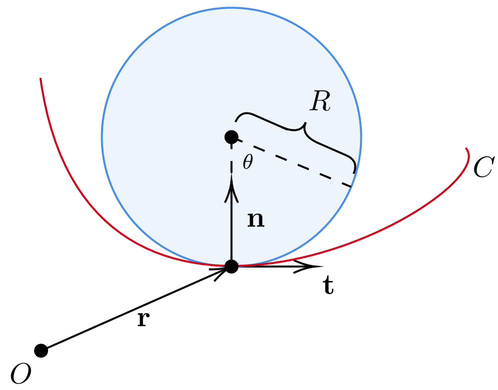
§1.3 Radius of Curvature

Consider some generic curve $s \mapsto \mathbf{r}(s)$, and consider the Taylor expansion about $s = 0$. Write $\mathbf{t} = \mathbf{t}(0)$, $\mathbf{n} = \mathbf{n}(0)$, etc. Then we get

$$\begin{aligned}\mathbf{r}(s) &= \mathbf{r}(0) + s\mathbf{r}'(0) + \frac{1}{2}s^2\mathbf{r}''(0) + o(s^2)\\ &= \mathbf{r} + s\mathbf{t} + \frac{1}{2}s^2\kappa\mathbf{n} + o(S^2).\end{aligned}$$

So this gives us a good approximation of the curve around $s = 0$, and now we want to try and draw a circle through the point $\mathbf{r}(0)$ such that the circle is tangent to the curve at that point. Specifically, what would the radius of that circle be?

Suppose, without loss of generality, that $\mathbf{t}$ is horizontal.



A point on this circle can be described by

$$\mathbf{x}(\theta) = \mathbf{r} + R(1 - \cos\theta)\mathbf{n} + R\sin\theta\mathbf{t}.$$

Then expanding for $|\theta|$ small, we get

$$\mathbf{x}(\theta) = \mathbf{r} + R\theta\mathbf{t} + \frac{1}{2}R\theta^2\mathbf{n} + o(\theta^2).$$

On this circle, the arc-length is $s = R\theta$. Thus if we express the above in terms of arc-length, we get

$$\mathbf{x}(\theta) = \mathbf{r} + S\mathbf{t} + \frac{1}{2R}s^2\mathbf{n} + o(s^2).$$

As we want this to match the curve, we compare it to the equation for the curve near $\mathbf{r}$, and to match quadratic terms we need $R = 1/\kappa$.

Definition 1.3.1 (Radius of Curvature)
We define $R(s) = 1/\kappa(s)$ to be the **radius of curvature** of the curve $s \mapsto \mathbf{r}(s)$.

2 Coordinates, Differentials and Gradients

§2.1 Differentials and First Order Changes

Recall that for $f = f(u_1, \ldots, u_n)$, we define the **differential** of f, written $\mathrm{d}f$, by

$$\mathrm{d}f = \frac{\partial f}{\partial u_i}\mathrm{d}u_i,$$

using the summation convention. We call $\{\mathrm{d}u_i\}$ **differential forms**. We will treat these differential forms as formal objects that are the elements of some abstract vector space, and treat them as linear independent if $\{u_1, \ldots, u_n\}$ are independent, that is, if $\alpha_i \mathrm{d}u_i = 0$ implies $\alpha_i = 0$ for $i = 1, \ldots, n$.

Similarity, if we had $\mathbf{x} = \mathbf{x}(u_1, \ldots, u_n)$, we define

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial u_i}\mathrm{d}u_i.$$

Example 2.1.1 (Using Differentials)
If $f(u, v, w) = u^2 + w\sin(v)$, then

$$\mathrm{d}f = 2u\mathrm{d}u + w\cos(v)\mathrm{d}v + \sin(v)\mathrm{d}w.$$

Similarity, if we had $\mathbf{x}(u, v, w) = \begin{pmatrix} u^2 - v^2 \\ w \\ e^v \end{pmatrix}$, then

$$\mathrm{d}\mathbf{x} = \begin{pmatrix} 2u \\ 0 \\ 0 \end{pmatrix}\mathrm{d}u + \begin{pmatrix} -2v \\ 0 \\ e^v \end{pmatrix}\mathrm{d}v + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\mathrm{d}w.$$

Differentials encode information about how a function/field changes when we 'wobble' our coordinates. Indeed, by calculus we know

$$f(u_1 + \delta u_1, \ldots, u_n + \delta u_n) - f(u_1, \ldots, u_n) = \frac{\partial f}{\partial u_i}\delta u_i + o(\delta \mathbf{u}),$$

where $\delta \mathbf{u} = (\delta u_1, \ldots, \delta u_n)$. So differentials allow to discuss first order changes in f.

So if δf denotes a change in $f(u_1, \ldots, u_n)$ under perturbation of coordinates $(u_1, \ldots, u_n) \mapsto (u_1 + \delta u_1, \ldots, u_n + \delta u_n)$, we have, to first order,

$$\delta f \approx \frac{\partial f}{\partial u_i}\delta u_i.$$

Similarly for vector fields,

$$\delta \mathbf{x} \approx \frac{\partial \mathbf{x}}{\partial u_i}\delta u_i.$$

§2.2 Coordinates and Line Elements

You have already seen at least two sets of coordinates before taking this course for $\mathbb{R}^2$: Cartesian coordinates (x, y), and polar coordinates (r, θ). They have an invertible (except at the origin) relationship,

$$x = r\cos\theta, y = r\sin\theta.$$

A general set of coordinates (u, v) on $\mathbb{R}^2$ can be specified by their relationship to (x, y), the Cartesian coordinates. That is, we can specify smooth functions $x = x(u, v)$, $y = y(u, v)$ that are a bijection. Of course we have the same thing for $\mathbb{R}^3$.

Let's look at some examples.

Example 2.2.1 (Cartesian Coordinates)

In the standard Cartesian coordinate system we have $\mathbf{x}(x, y) = \begin{pmatrix} x \\ y \end{pmatrix} = x\mathbf{e}_x + y\mathbf{e}_y$, where $\{\mathbf{e}_x, \mathbf{e}_y\}$ are orthogonal vectors. We have $\mathbf{e}_x$ is in the direction of changing x with y fixed.

Said differently, we have

$$\mathbf{e}_x = \frac{\frac{\partial}{\partial x}\mathbf{x}(x, y)}{\left|\frac{\partial}{\partial x}\mathbf{x}(x, y)\right|}, \quad \text{and} \quad \mathbf{e}_y = \frac{\frac{\partial}{\partial y}\mathbf{x}(x, y)}{\left|\frac{\partial}{\partial y}\mathbf{x}(x, y)\right|}.$$

A feature of Cartesian coordinates is that

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial x}\mathrm{d}x + \frac{\partial \mathbf{x}}{\partial y}\mathrm{d}y = \mathrm{d}x\mathbf{e}_x + \mathrm{d}y\mathbf{e}_y,$$

that is, a change in coordinate $x \mapsto x + \delta x$ cause a change to the vector (to the first order) by $\mathbf{x} \mapsto \mathbf{x} + \delta x\mathbf{e}_x$. We call $\mathrm{d}\mathbf{x}$ the **line element**, which tells us how small changes in coordinates produce changes in position vectors.

That was slightly tame, we're going to look at something slightly more interesting.

Example 2.2.2 (Polar Coordinates)

For polar coordiantes (r, θ) we have

$$\mathbf{x}(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = r\mathbf{e}_r.$$

We have used the basis vectors $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ where

$$\mathbf{e}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \mathbf{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

Note that $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ are orthonormal at each (r, θ), but not the same for each (r, θ).

Note that as before,

$$\mathbf{e}_r = \frac{\frac{\partial}{\partial r}\mathbf{x}(r,\theta)}{\left|\frac{\partial}{\partial r}\mathbf{x}(r,\theta)\right|}, \quad \text{and} \quad \mathbf{e}_\theta = \frac{\frac{\partial}{\partial \theta}\mathbf{x}(r,\theta)}{\left|\frac{\partial}{\partial \theta}\mathbf{x}(r,\theta)\right|}.$$

Since $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ are orthogonal, it makes sense to call (r, θ) **orthogonal**, **curvilinear coordinates**.

We also have the line element

$$\mathrm{d}x = \frac{\partial \mathbf{x}}{\partial r}\mathrm{d}r + \frac{\partial \mathbf{x}}{\partial \theta}\mathrm{d}\theta = \mathbf{e}_r \mathrm{d}r + r\mathrm{d}\theta \mathbf{e}_\theta,$$

and so we can see that a change $\theta \mapsto \theta + \delta\theta$ produces a (first order) change

$$\mathbf{x} \longmapsto \mathbf{x} + r\delta\theta\mathbf{e}_\theta,$$

which is notably *not* $\mathbf{x} \mapsto \mathbf{x} + \delta\theta\mathbf{e}_\theta$.

§2.2.1 Orthogonal Curvilinear Coordinates

We can state properly what it means for a coordinate system to be orthogonal curvilinear.

Definition 2.2.3 (Orthogonal Curvilinear)

We say that (u, v, w) are a set of **orthogonal, curvilienar coordinates** if the vectors

$$\mathbf{e}_u \frac{\partial \mathbf{x}/\partial u}{|\partial \mathbf{x}/\partial u|}, \quad \mathbf{e}_v \frac{\partial \mathbf{x}/\partial v}{|\partial \mathbf{x}/\partial v|}, \quad \mathbf{e}_w \frac{\partial \mathbf{x}/\partial w}{|\partial \mathbf{x}/\partial w|}$$

form a right-handed[a], orthonormal basis for each (u, v, w).

[a] $\mathbf{e}_u \times \mathbf{e}_v = \mathbf{e}_w$.

Just as with polar coordinates, we have these vectors $\{\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w\}$ form an orthonormal basis for $\mathbb{R}^3$ with each choice of (u, v, w), but *not* necessarily the same basis at each point.

It is standard to write

$$h_u = \left|\frac{\partial x}{\partial u}\right|, \quad h_v = \left|\frac{\partial x}{\partial v}\right|, \quad h_w = \left|\frac{\partial x}{\partial w}\right|,$$

which we call **scale factors**. Note that the line element is

$$\begin{aligned} \mathrm{d}\mathbf{x} &= \frac{\partial \mathbf{x}}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{x}}{\partial v}\mathrm{d}v + \frac{\partial \mathbf{x}}{\partial w}\mathrm{d}w \\ &= h_u\mathbf{e}_u\mathrm{d}u + h_v\mathbf{e}_v\mathrm{d}v + h_w\mathbf{e}_w\mathrm{d}w. \end{aligned}$$

Scale factors tell us how small changes in coordinates 'scale-up' to changes in position $\mathbf{x}$.

§2.2.2 Cyclindrical Polar Coordinates

We define the coordinate system (ρ, ϕ, z) by

$$\mathbf{x}(\rho, \phi, z) = \begin{pmatrix} \rho \cos \phi \\ \rho \sin \phi \\ z \end{pmatrix},$$

with $0 \leq \rho < \infty$, $0 \leq \phi < 2\pi$ and $-\infty < z < \infty$.

We find that

$$\mathbf{e}_\rho = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}, \qquad \mathbf{e}_\phi = \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix}, \qquad \mathbf{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and $h_\rho = 1$, $h_\phi = \rho, h_z = 1$. Thus the line element is

$$\mathrm{d}\mathbf{x} = \mathrm{d}\rho \mathbf{e}_\rho + \rho \mathrm{d}\phi \mathbf{e}_\phi + \mathrm{d}z \mathbf{e}_z.$$

Note also that we can write

$$\mathbf{x} = \begin{pmatrix} \rho \cos \theta \\ \rho \sin \theta \\ z \end{pmatrix} = \rho \begin{pmatrix} \cos \phi \\ \sin \phi \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \rho \mathbf{e}_\rho + z \mathbf{e}_z.$$

§2.2.3 Spherical Polar Coordinates

We can define the spherical polar coordinates (r, θ, ϕ) by

$$\mathbf{x}(r, \theta, \phi) = \begin{pmatrix} r \cos \phi \sin \theta \\ r \sin \phi \sin \theta \\ r \cos \theta \end{pmatrix},$$

with $0 \leq r < \infty$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.

We have the corresponding basis vectors

$$\mathbf{e}_r = \begin{pmatrix} \cos \phi \sin \theta \\ \sin \phi \sin \theta \\ \cos \theta \end{pmatrix}, \quad \mathbf{e}_\theta = \begin{pmatrix} \cos \phi \cos \theta \\ \sin \phi \cos \theta \\ -\sin \theta \end{pmatrix}, \quad \mathbf{e}_\phi = \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix}.$$

We also have $h_r = 1$, $h_\theta = r$ and $h_\theta = r \sin \theta$, so

$$\mathrm{d}\mathbf{x} = \mathrm{d}r \mathbf{e}_r + r \mathrm{d}\theta \mathbf{e}_\theta + r \sin \theta \mathrm{d}\phi \mathbf{e}_\phi,$$

and we can write

$$\mathbf{x} = r \mathbf{e}_r.$$