

Linear Algebra

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This article constitutes my notes for the ‘Linear Algebra’ course, held in Michaelmas 2021 at Cambridge. These notes are *not a transcription of the lectures*, and differ significantly in quite a few areas. Still, all lectured material should be covered.

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§1 Vector Spaces

§1.1 Vector Spaces and Subspaces

Linear algebra is, somewhat obviously, primarily about studying objects that are *linear* in nature. The objects we really care about are *vector spaces*, settings in which we can add elements and multiply by scalars. We are also going to consider *linear maps*, functions on vector spaces which preserve that linear structure – but more on that later.

Throughout the following discussion (and this course), \mathbb{F} is going to denote an arbitrary field¹

Definition 1.1 (\mathbb{F} -Vector Space)

An **\mathbb{F} -vector space** is an abelian group $(V, +)$ together with a function $\mathbb{F} \times V \rightarrow V$, written $(\lambda, v) \mapsto \lambda v$ such that the following axioms hold:

- (i) *Distributivity in V* . $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$,

¹A field \mathbb{F} is a set \mathbb{F} equipped with two operations $+$ (‘addition’) and \cdot (‘multiplication’). We require \mathbb{F} with addition to form an abelian group, and multiplication must be associative and have an identity element 1. We also require every element except 0 to have an inverse with respect to multiplication, and multiplication must be distributive over addition.

Informally, you can think of a field as something you can do arithmetic in.

- (ii) *Distributivity in \mathbb{F} .* $(\lambda_1 + \lambda_2)v = \lambda_1v + \lambda_2v$,
- (iii) *Associativity.* $\lambda(\mu v) = (\lambda\mu)v$,
- (iv) *Identity.* $1v = v$.

We usually call elements of V **vectors** and elements of \mathbb{F} **scalars**. The identity element in V is usually called the zero vector, and is written 0_V (or just 0 if the context is clear).

If \mathbb{F} is \mathbb{R} or \mathbb{C} , we use the terms ‘real vector space’ and ‘complex vector space’, since they’re so common.

Example 1.2 (Examples of Vector Spaces)

- (i) The set of triples

$$\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

forms a real vector space called \mathbb{R}^3 , because you can add any two triples component wise.

- (ii) The set

$$\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$$

is a \mathbb{Q} -vector space, where we add elements and scale by rational numbers in the obvious way.

- (iii) The set $\mathcal{C}[0, 1]$ of all continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ forms a real vector space.

As with many new objects, it’s helpful to be able to discuss its substructure. In the case of a vector space V , there’s a pretty natural notion for what it means for a subset $U \subseteq V$ to still act like a vector space.

Definition 1.3 (Subspace)

Let V be a \mathbb{F} -vector space. A subset $U \subseteq V$ is a **subspace** of V if U is also an \mathbb{F} -vector space. If U is a subspace of V , we will write $U \leq V$.

Example 1.4 (Examples of Subspaces)

- (i) The set of vectors $\{(x, y, z) \mid x, y, z \in \mathbb{R}, x + y + z = 0\}$ is a subspace of \mathbb{R}^3 .
- (ii) The set of polynomials with terms of even degree $\{a_0 + a_2x^2 + a_4x^4 + \cdots + a_{2k}x^{2k} \mid a_{2i} \in \mathbb{R}, k \in \mathbb{N}\}$ is a subspace of $\mathbb{R}[X]$, the vector space of polynomials with coefficients in \mathbb{R} .

As you would expect, checking that something is a subspace is usually easier than checking all of the axioms for a vector space. In particular, to check that U is a subspace of an \mathbb{F} -vector space V , you can just check that the following hold:

- *Zero vector*². $0_V \in U$,
- *Closure under addition*. $u_1, u_2 \in U$ to imply $u_1 + u_2 \in U$,

²You may wonder why we need to check this when we already check that we are closed under scaling. To see why, notice that we still have to ensure U is non-empty!

- *Closure under scaling.* $\lambda \in \mathbb{F}$ and $u \in U$ to imply $\lambda u \in U$.

There are various ways in which we can manipulate subspaces, for example we can take the intersection of two subspaces, and we will get back another subspace.

Proposition 1.5 (Intersecting Subspaces)

Let $U, W \leq V$. Then $U \cap W \leq V$.

Proof. Since U and W are both subspaces of V , we have $0_V \in U \cap W$, and also since they are both closed under addition and scaling, $u_1, u_2 \in U \cap W$ implies that $u_1 + u_2 \in U \cap W$, and $\lambda \in \mathbb{F}$ implies $\lambda u \in U \cap W$. Thus $U \cap W$ is a subspace of V . \square

However we can't manipulate subspaces however we want and expect magic. For example, the union of two subspaces is generally *not* a subspace, as it is typically not closed under addition. In fact, the union is only ever a subspace if one of the subspaces is contained in the other.³

We can however try to 'complete' the union so that it becomes a subspace.

Definition 1.6 (Sum of Subspaces)

Let V be a vector space over \mathbb{F} , and let $U, W \leq V$. We define the **sum** of U and W to be the set

$$U + W = \{u + w \mid u \in U, w \in W\}.$$

This definition immediately forces $U + W \leq V$, and indeed it is the minimal such space (in that any subspace of V containing both U and W must also contain $U + W$).

§1.2 Quotient Spaces

Since a vector space V forms an abelian group $(V, +)$, we are able to take the quotient by any subspace $U \leq V$.

Definition 1.7 (Quotient Space)

Let V an \mathbb{F} -vector space, and let $U \leq V$. The **quotient space** V/U is the abelian group V/U equipped with the scalar multiplication $F \times V/U \rightarrow V/U$ written $(\lambda, v + U) \mapsto \lambda v + U$.

With this definition, we need to check that this scalar multiplication operation is well defined. Indeed, if $v_1 + U = v_2 + U$ then

$$\begin{aligned} v_1 - v_2 &\in U \\ \implies \lambda(v_1 - v_2) &\in U \\ \implies \lambda v_1 + U &= \lambda v_2 + U \in V/U, \end{aligned}$$

so our operation is indeed well defined.

As you would expect, taking a quotient gives you back a vector space.

³There are some more exercises of this flavour on the example sheet.

Proposition 1.8 (Quotient Spaces are Vector Spaces)

V/U is an \mathbb{F} -vector space.

Proof Sketch. Check definitions (most properties are inherited from V being a vector space). \square

§1.3 Basis and Dimension

You are likely informally familiar with the idea of *dimension*, a measure how much freedom exists in a system. Dimensionality is a rather natural concept with respect to vector spaces, but we will need to move through some technicalities to establish the results we want.

To discuss the amount of freedom, we first need a way to quantify what it means for a set of vectors to be independent from one another. This is the idea of *linear independence*.

Definition 1.9 (Linear Independence)

We say that $\{v_1, \dots, v_n\} \in V$ are **linearly independent** if

$$\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = 0$$

implies that $\lambda_1 = \dots = \lambda_n = 0$.

Remark. For an infinite subset $S \subseteq V$, we say it's linearly independent if every finite subset is linearly independent.

If a set of vectors is *not* linearly independent, then there's some vector in that set that can be written as a linear combination of the others – so it's not independent of them!

The next idea we need to pin down is being able to see if our set of vectors can 'generate' the rest of our vector space.

Definition 1.10 (Span)

Let V be a vector space over \mathbb{F} , and let $S \subset V$. We define the **span** of S , $\langle S \rangle$ to be the set of finite combinations of elements of S .

If $\langle S \rangle = V$, then we say S is **spans** or **generates** V .

Remark. By convention, we also take $\langle \emptyset \rangle = \{0\}$. An equivalent definition is that $\langle S \rangle$ is the smallest subspace of V that contains S .

Example 1.11 (Quadratic Polynomials)

Let V be the vector space of quadratic polynomials over \mathbb{R} ,

$$V = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}.$$

Then the subset $S \subseteq V$ with $S = \{1, x, x^2\}$ spans V .

Putting these two concepts together gives us the idea of *bases*, which are sets of linearly

independent vectors that span a vector space.

Definition 1.12 (Basis)

A subset S of a vector space V is a **basis** if S is a set of linearly independent vectors that span V .

Example 1.13 (Basis for \mathbb{R}^n)

The **canonical basis** of \mathbb{R}^n is the set of vectors

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}.$$

Importantly, this is not the *only* basis of \mathbb{R}^n , just one that is quite convenient most of the time.

Remark. Note that in the definition of a basis there is no requirement for the set of basis vectors $S \subseteq V$ to be finite – only that any element in V must be representable using finitely many elements of S .

We'd intuitively want to say that the *dimension* of a vector space is the number of elements in its basis. However, we first need to check that this is a well defined notion. We can at this point distinguish between finite and infinite dimensional vector spaces at this point though⁴.

Definition 1.14 (Finite & Infinite Dimension)

We say a vector space V is **finite dimensional** if it has a finite basis, and we say it is **infinite dimensional** otherwise.

The next result about bases we will prove is that they induce *unique* representations of elements in the vector space.

Lemma 1.15 (Unique Representations with a Basis)

Let V be a vector space over \mathbb{F} . Then $S \subseteq V$ is a basis of V if and only if any vector $v \in V$ can be written uniquely as a linear combination of elements $v_1, \dots, v_n \in S$.

Proof. Suppose that S was a basis for V . Then if $v \in V$ can't be written as such a linear combination, then S wouldn't span V , contradicting it being a basis. Also, if v can be written as such a linear combination non-uniquely, then taking

$$v = \lambda_1 v_1 + \dots + \lambda_n v_n = \mu_1 v_1 + \dots + \mu_n v_n.$$

where $\lambda_i \neq \mu_i$ for at least one value of i , we'd have $0 = v - v = (\lambda_1 - \mu_1)v_1 + \dots + (\lambda_n - \mu_n)v_n$, and at least one of these coefficients must be non-zero, contradicting S being linearly independent.

⁴Can you see why this is well defined already?

Alternatively, if any element in V can be written uniquely, then if $v_1, \dots, v_n \in S$ with $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$ implies that $\lambda_1 = \dots = \lambda_n = 0$, giving that S must be linearly independent. Since S is also spanning by definition, we see that it therefore must be a basis of V . \square

With that out of the way, we can prove some results about finite dimensional vector spaces which will help us get towards our definition of dimension.

Lemma 1.16 (Spanning Sets Contain a Basis)

Let V be a finite dimensional vector space, and let $S = \{v_1, \dots, v_n\}$ be a set of vectors that spans V . Then there is some subset of S that is a basis of V .

Proof. If $\{v_1, \dots, v_n\}$ is linearly independent, then we are done. If it's not, then (up to reordering) we have $v_n \in \langle \{v_1, \dots, v_{n-1}\} \rangle$. But then $\langle \{v_1, \dots, v_n\} \rangle = \langle \{v_1, \dots, v_{n-1}\} \rangle$, so we can not include v_n in our subset. Not including elements in this way repeatedly, since there is finitely many elements in S , we must eventually get a linearly independent set that still spans V . \square

Theorem 1.17 (Steinitz Exchange Lemma)

Let V be a finite dimensional vector space. Then if $\{v_1, \dots, v_m\}$ is a set of linearly independent vectors, and $\{w_1, \dots, w_n\}$ spans V , then

- (i) $m \leq n$
- (ii) up to reordering, $\{v_1, \dots, v_m, w_{m+1}, \dots, w_n\}$ spans V .

Proof. We will prove this by induction. Suppose we have replaced $\ell \geq 0$ of the w_i , and that

$$\langle \{v_1, \dots, v_\ell, w_{\ell+1}, \dots, w_n\} \rangle = V.$$

If $m = \ell$, we are done, so assume that $\ell < m$. Then since this set is spanning, we can write $v_{\ell+1} \in V$ as

$$v_{\ell+1} = \alpha_1 v_1 + \dots + \alpha_\ell v_\ell + \beta_{\ell+1} w_{\ell+1} + \dots + \beta_n w_n.$$

Since having $\beta_i = 0$ for all $\ell + 1 \leq i \leq n$ would violate linear independence, we can suppose without loss of generality that $\beta_{\ell+1} \neq 0$. We also note that this implies that $\ell + 1 \leq n$, as otherwise this would not be possible.

Then $w_{\ell+1} \in \langle \{v_1, \dots, v_{\ell+1}, w_{\ell+2}, \dots, w_n\} \rangle$, and this set spans V .

Repeating this process, we will be done after m steps, and we have also shown (at the final step) that $m \leq n$. \square

Corollary 1.18 (Dimension)

Let V be a finite dimensional vector space over \mathbb{F} . Then any two bases of V have the same number of elements, called the **dimension** of V , denoted $\dim V$ or $\dim_{\mathbb{F}} V$.

Proof. Immediate by Steinitz exchange lemma. \square

So using Steinitz exchange lemma we have finally been able to pin down exactly what is meant by the dimension of a vector space – it's the size of its basis. This should match up to the intuitive idea of 'freedom' that you had at the start of this section. Freedom in a vector space comes from varying coefficients, and in a basis we can both freely vary coefficients and also reach any element in a vector space uniquely, so the number of independent parameters really is the size of the basis.

Steinitz also gives us a few useful results for free.

Corollary 1.19

Let V be a vector space over \mathbb{F} with finite dimension $n = \dim V$. Then

- (i) Any independent set of vectors has at most n elements, with equality if and only if it's a basis.
- (ii) Any spanning set has at least n elements, with equality if and only if it's a basis.

Proof. Immediate by Steinitz exchange lemma. \square

Working with basis and dimension can make the study of vector spaces much easier. For example, Steinitz allows us to take a subspace and nicely extend a basis for that subspace to a basis for the entire space. Working with ideas like this can make many results easier to prove, as we will see in the following propositions.

Proposition 1.20 (Dimension of the Sum of Subspaces)

Let U, W be subspaces of a vector space V . if U and W are finite dimensional, then so is $U + W$, and $\dim U + W = \dim U + \dim W - \dim U \cap W$.

Proof. Pick a basis $\{v_1, \dots, v_a\}$ of $U \cap W$, and extend by Steinitz exchange lemma to a basis $\{v_1, \dots, v_a, u_1, \dots, u_b\}$ of U , and to a basis $\{v_1, \dots, v_a, w_1, \dots, w_c\}$ of W .

It suffices to prove that $\{v_1, \dots, v_a, u_1, \dots, u_b, w_1, \dots, w_c\}$ is a basis of $U + W$.

Clearly this set of vectors spans $U + W$, so we just need to check that they are linearly independent. Suppose that

$$\sum_{i=1}^a \alpha_i v_i + \sum_{i=1}^b \beta_i u_i + \sum_{i=1}^c \gamma_i w_i = 0.$$

Rewriting,

$$\sum_{i=1}^a \alpha_i v_i + \sum_{i=1}^b \beta_i u_i = - \sum_{i=1}^c \gamma_i w_i, \quad (\dagger)$$

where the LHS is in U and the RHS is in W . This implies that $\sum_{i=1}^c \gamma_i w_i \in U \cap W$, and can be written as $\sum_{i=1}^c \gamma_i w_i = \sum_{i=1}^a \mu_i v_i$, for some μ_i , and then substituting

this back into (\dagger) ,

$$\sum_{i=1}^a (\alpha_i + \mu_i) v_i + \sum_{i=1}^b \beta_i u_i = 0$$

which forces $\beta_i = 0$. A similar argument also gives $\gamma_i = 0$, which then finally forces $\alpha_i = 0$, since $\{v_1, \dots, v_a\}$ is a basis. \square

Proposition 1.21 (Dimension of the Subspace its Quotient)

If V is a finite dimensional vector space over \mathbb{F} and $U \subseteq V$, then U and V/U are also finite dimensional, and $\dim V = \dim U + \dim V/U$.

Proof. Let $\{u_1, \dots, u_\ell\}$ be a basis of U , and extend it via Steinitz exchange lemma to a basis $\{v_1, \dots, v_\ell, w_{\ell+1}, \dots, w_n\}$ of V .

It's easy to see that $\{w_{\ell+1} + U, \dots, w_n + U\}$ is a basis of V/U , as it clearly spans and linear independence is inherited from it being a basis of V . The result then follows. \square

§1.4 Direct Sums

Previously, we were able to look at the substructure of a vector space by looking at subspaces. Given two subspaces, we were then able to construct their sum, which is the set of all linear combinations of elements in each subspace.

When studying a vector space using its subspaces in this way (considering their sum), it can be useful to impose an *additional* constraint about the way in which the subspaces interact. In particular, it can be useful to impose a uniqueness constraint on the linear combinations that are created.

Definition 1.22 (Direct Sum)

Let V be a vector space over \mathbb{F} , and let $U, W \leq V$. We say that V is the **direct sum** of U and W , written $V = U \oplus W$ if and only if every element $v \in V$ can be decomposed as

$$v = u + w$$

with $u \in U$ and $w \in W$, with this decomposition being unique.

Of course, we can generalize the notion of a direct sum naturally to the case of multiple subspaces, in the way that you would expect.

Remark (Warning). We say that W is a direct complement of U in V . There is *no uniqueness* of such a complement!

Lemma 1.23

Let V be a vector space and let $U, W \leq V$. Then the following are equivalent.

- (i) $V = U \oplus W$
- (ii) $V = U + W$ and $U \cap W = \{0\}$

(iii) For any basis B_1 of U and B_2 of W , the union $B = B_1 \cup B_2$ is a basis of V .

Proof. (ii) implies (i). Let $V = U + W$ with $U \cap W = \{0\}$. Then for all $v \in V$, we can write $v = u + w$ with $u \in U$ and $w \in W$. To see that this is unique, suppose that

$$v = u + w = u' + w'.$$

Then $(u - u') = -(w - w')$, and thus they are both in $U \cap W$, but the only element of this is 0, and thus $u = u'$ and $w = w'$, giving us uniqueness.

(i) implies (iii). Let B_1 be a basis of U and B_2 be a basis of W . Then $B = B_1 \cup B_2$ clearly spans V , so we need to check that it's linearly independent. Indeed, if

$$\sum_{u \in B_1} \lambda_u u + \sum_{w \in B_2} \lambda_w w = 0,$$

then $V = U \oplus W$ implies that since this is the sum of an element of U and an element of W , then each is zero, and linear independence follows from B_1, B_2 being sets of linearly independent vectors.

(iii) implies (ii). Clearly we have $V = U + W$, so we just need to check that $U \cap W = \{0\}$. Let $v \in U \cap W$. Then we can write

$$v = \sum_{u \in B_1} \lambda_u u = \sum_{w \in B_2} \lambda_w w \implies \sum_{u \in B_1} \lambda_u u - \sum_{w \in B_2} \lambda_w w = 0,$$

and since $B_1 \cup B_2$ is a basis for V , we must have $\lambda_u, \lambda_w = 0$ for all $u, w \in B_1, B_2$, implying that $v = 0$. \square

This result extends as you would expect for the case of direct products using multiple subspaces.

§2 Linear Maps

§2.1 Linear Maps and Isomorphisms

We are now going to turn our attention to *linear maps*, which are maps that preserve the underlying vector space structure that they act on.

Definition 2.1 (Linear Map)

Let V, W be vector spaces over \mathbb{F} . A map $\alpha : V \rightarrow W$ is a **linear map** if for all $\lambda_1, \lambda_2 \in \mathbb{F}$ and $v_1, v_2 \in V$ we have

$$\alpha(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \alpha(v_1) + \lambda_2 \alpha(v_2).$$

Example 2.2 (Examples of Linear Maps)

- (i) Consider the vector space $\mathcal{C}[0, 1]$ of continuous functions $f : [0, 1] \rightarrow \mathbb{R}$. Then for any fixed point $x \in [0, 1]$, the map $\alpha : \mathcal{C}[0, 1] \rightarrow \mathbb{R}$ with $\alpha(f) = f(x)$ is a linear map.

- (ii) The map $\mathbb{R}^3 \rightarrow \mathbb{R}$ given by $(a, b, c) \mapsto 4a + 2b + c$ is a linear map.
- (iii) The map $\alpha : Q[\sqrt{2}] \rightarrow Q[\sqrt{2}]$ of ‘multiplying by $\sqrt{2}$ ’ is a linear map, with $a + b\sqrt{2} \mapsto 2b + a\sqrt{2}$.

It’s easy to see that the identity map is a linear map, and also that linearity is preserved by composing linear maps.

One nice thing about linear maps is that they interact well with basis, because of linearity. Indeed, specifying a linear map on the basis is sufficient to define that map on all of a vector space.

Lemma 2.3 (Extension by Linearity)

Let V and W be vector spaces, and let B be a basis for V . Then if $\alpha_0 : B \rightarrow W$ is *any* map, then there exists a *unique* linear map $\alpha : V \rightarrow W$ extending α_0 , so that for all $v \in B$ we have $\alpha(v) = \alpha_0(v)$.

Proof. For $v \in V$, let $v = \sum_{i=1}^n \lambda_i v_i$. Then necessarily by linearity, we define

$$\alpha(v) = \alpha\left(\sum_{i=1}^n \lambda_i v_i\right) = \sum_{i=1}^n \lambda_i \alpha_0(v_i).$$

By construction this is a linear map and it extends α_0 . □

As with many other areas of pure mathematics, we have a special case for the structure preserving bijections on a given object.

Definition 2.4 (Isomorphism)

If V and W are vector spaces over \mathbb{F} , the linear map $\alpha : V \rightarrow W$ is an **isomorphism** if it also bijective. We say two vector spaces V and W are **isomorphic**, written $V \cong W$, if there is an isomorphism between them.

Remark. if $\alpha : V \rightarrow W$ is an isomorphism, then the inverse map α^{-1} exists and importantly it’s *also* linear.

Lemma 2.5 (Isomorphism is an Equivalence Relation)

\cong is an equivalence relation on the class of all vector spaces over \mathbb{F} .

Proof. For reflexivity, we note that the identity map $id : V \rightarrow V$ is an isomorphism. For symmetry, we note that if $\alpha : V \rightarrow W$ is an isomorphism, then $\alpha^{-1} : W \rightarrow V$ is also an isomorphism. Lastly, we get transitivity because if $\alpha : U \rightarrow V$ and $\beta : V \rightarrow W$ are isomorphisms, then $\beta \circ \alpha : U \rightarrow W$ is also an isomorphism. □

From one perspective, isomorphism allows us to know when two objects are really just different representations of the same thing. With this in mind, the following result tells us something interesting (yet unsurprising) about the nature of finite dimensional vector spaces.

Theorem 2.6 (Isomorphism for Finite Dimensional Vector Spaces)

If V is an n dimensional vector space over \mathbb{F} , then $V \cong \mathbb{F}^n$, the vector space

$$\mathbb{F}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mid x_1, \dots, x_n \in \mathbb{F} \right\},$$

where addition is component-wise.

Proof. Let $B = \{v_1, \dots, v_n\}$ be a basis for V . Then we claim that the map $\alpha : V \rightarrow \mathbb{F}^n$ defined by

$$v = \sum_{i=1}^n \lambda_i v_i \mapsto \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

is an isomorphism. Indeed, it's easy to see that we have linearity inherited from \mathbb{F} , and also the map is a bijection, so indeed we do have $V \cong \mathbb{F}^n$. \square

In particular, this result tells us that when we choose a basis for a finite dimensional vector space V , it's really the same as picking some isomorphism from V to \mathbb{F}^n .

This theorem also directly implies (with isomorphism being an equivalence relation) a criterion for two finite dimensional vector spaces to be isomorphic.

Theorem 2.7 (Isomorphism Criterion)

Let V and W be vector spaces over \mathbb{F} . Then they are isomorphic if and only if they have the same dimension.

Stepping back from isomorphisms, we need to define some frequently used concepts relating to linear maps.

Definition 2.8

Let V, W be vector spaces over F , and let $\alpha : V \rightarrow W$ be a linear map. We define the **kernel** of α as

$$\ker \alpha = \{v \in V \mid \alpha(v) = 0\},$$

and the **image** of α as

$$\text{img } \alpha = \{w \in W \mid w = \alpha(v), v \in V\}.$$

Lemma 2.9 (Image and Kernel are Subspaces)

$\ker \alpha$ and $\text{img } \alpha$ are subspaces of V and W respectively.

Proof Sketch. Check definitions. \square

With one eye back on group theory, we can use the kernel and image of a linear map to establish an isomorphism.

Theorem 2.10 (First Isomorphism Theorem For Vector Spaces)

Let V and W be \mathbb{F} vector spaces, and let $\alpha : V \rightarrow W$ be a linear map. Then $V/\ker \alpha \cong \text{img } \alpha$.

Proof Sketch. Consider the isomorphism $\bar{\alpha} : V/\ker \alpha \rightarrow \text{img } \alpha$ defined by $\bar{\alpha}(v + \ker \alpha) = \alpha(v)$, and check definitions. \square

Now while the kernel and image being subspaces are rather straightforward, slightly more interesting is the *dimension* of these subspaces. This is described by the *rank-nullity theorem*.

Definition 2.11 (Rank and Nullity)

Let α be a linear map. Then the **rank** of α , $r(\alpha)$ is the dimension of the image, $r(\alpha) = \dim \text{img } \alpha$. The **nullity** of α , $n(\alpha)$, is the dimension of the kernel, $n(\alpha) = \dim \ker \alpha$.

Theorem 2.12 (Rank-Nullity Theorem)

Let U and V be vector spaces over \mathbb{F} such that U is finite dimensional. Let $\alpha : U \rightarrow V$ be a linear map. Then

$$\dim U = r(\alpha) + n(\alpha).$$

Proof. We know that $U/\ker \alpha \cong \text{img } \alpha$. Thus $\dim(U/\ker \alpha) = \dim(\text{img } \alpha)$, that is, $\dim U - \dim \ker \alpha = \dim \text{img } \alpha$, or $\dim U = r(\alpha) + n(\alpha)$. \square

This result has an important consequence which is incredibly useful for quickly checking if a map is an isomorphism.

Lemma 2.13 (Characterisation of Isomorphism)

Let V and W be vector spaces over \mathbb{F} with *equal finite dimension*. Then the following are equivalent:

- (i) α is injective,
- (ii) α is surjective,
- (iii) α is an isomorphism.

Proof Sketch. Follows directly from the rank-nullity theorem. \square

§2.2 Linear Maps $V \rightarrow W$ and Matrices

We are now going to consider linear maps between two vector spaces V and W .

Definition 2.14 (Space of Linear Maps)

Let V and W be vector spaces over \mathbb{F} . We define

$$\mathcal{L}(V, W) = \{\alpha : V \rightarrow W \mid \alpha \text{ is a linear map}\}.$$

Proposition 2.15

$\mathcal{L}(V, W)$ is a vector space over \mathbb{F} .

Moreover, if V and W are finite dimensional then so is $\mathcal{L}(V, W)$, and $\dim \mathcal{L}(V, W) = \dim V \cdot \dim W$.

Proof Sketch. Check definitions. □

It is likely that you are familiar with representing linear maps using *matrices*.

Definition 2.16 (Matrix)

An $m \times n$ **matrix** over \mathbb{F} is an array with m rows and n columns, with entries in \mathbb{F} :

$$A = (a_{ij}),$$

with $1 \leq i \leq m$ and $1 \leq j \leq n$.

The space of matrices is the set $\mathcal{M}_{m,n}(\mathbb{F})$ over $m \times n$ matrices over \mathbb{F} .

Proposition 2.17 (Matrices form a Vector Space)

$\mathcal{M}_{m,n}(\mathbb{F})$ is an \mathbb{F} -vector space.

Proof Sketch. Check definitions. □

Proposition 2.18 (Dimension of the Space of Matrices)

$\dim \mathcal{M}_{m,n}(\mathbb{F}) = m \times n$.

Proof. We will write down an explicit basis. We define the elementary matrix $E_{i,j}$ such that every entry is 0 except for the i, j th entry which is 1. The set $\{E_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is clearly spanning and is also clearly linearly independent, so it is a basis. Counting elements, we see that this set has size $m \times n$, as required. □

What's important about matrices is the way they relate to linear maps. Let V, W be vector spaces over \mathbb{F} , and let $\alpha : V \rightarrow W$ be a linear map.

Let $B = \{v_1, \dots, v_n\}$ be a basis for V , and $C = \{w_1, \dots, w_m\}$ be a basis for W . Then if $v \in V$, we can write

$$v = \sum_{j=1}^n \lambda_j v_j = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \in \mathbb{F}^n,$$

which are the ‘coordinates of v ’ with the respect to the basis B , which we will call $[v]_B$.

Similarly, for $w \in W$, we can let $[w]_C$ be the ‘coordinates’ of w in the basis C .

Definition 2.19 (Matrix of α in the B, C Basis)

We define $[\alpha]_{B,C}$, the **matrix of α** with respect to B and C by

$$[\alpha]_{B,C} = ([\alpha(v_1)]_C, [\alpha(v_2)]_C, \dots, [\alpha(v_n)]_C) \in \mathcal{M}_{m,n}(\mathbb{F}).$$

Lemma 2.20

For all $v \in V$, we have $[\alpha(v)]_C = [\alpha_{BC}] \cdot [v]_B$.

Proof. Let $v \in V$, and let $v = \sum_{j=1}^n \lambda_j v_j$. Then

$$\alpha(v) = \sum_{j=1}^n \lambda_j \alpha(v_j) = \sum_{j=1}^n \lambda_j \sum_{i=1}^m a_{ij} w_i = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} \lambda_j \right) w_i,$$

as required. \square

Lemma 2.21

Let $\alpha : V \rightarrow W$ and $\beta : U \rightarrow W$ be linear maps. Then if A is a basis of U , B is a basis of V and C is a basis of W , then

$$[\alpha \circ \beta]_{A,C} = [\alpha]_{B,C} \cdot [\beta]_{A,B}.$$

Proof Sketch. Write out the definition of matrix multiplication and check that it corresponds to the correct thing. \square

Proposition 2.22

If V and W are vector spaces over \mathbb{F} such that $\dim V = n$ and $\dim W = m$, then $\mathcal{L}(V, W) \cong \mathcal{M}_{m,n}(\mathbb{F})$.

Proof. Fix a basis B, C of V and W respectively. Then consider the linear map $\theta : \mathcal{L}(V, W) \rightarrow \mathcal{M}_{m,n}(\mathbb{F})$ given by $\alpha \mapsto [\alpha]_{B,C}$.

This map is clearly linear, and it’s also clearly a bijection thus it’s an isomorphism and our result follows. \square

Remark. This implies that $\dim \mathcal{L}(L, W) = m \times n$.

Example 2.23

Let V and W be finite dimensional vector spaces. Consider the linear map $\alpha : V \rightarrow W$, and suppose $Y \leq V$. Consider the image of Y under α , $\alpha(Y) = \{\omega \in W \mid \omega = \alpha(y), y \in Y\}$.

Define the sets $B' = \{v_1, \dots, v_k\}$ and $B'' = \{v_{k+1}, \dots, v_n\}$ such that the union of

disjoint sets $B = B' \cup B''$ is a basis for V , and B' is a basis of Y (that is, we extend using Steinitz exchange lemma).

Similarly we define $C' = \{w_1, \dots, w_\ell\}$ and $C'' = \{w_{\ell+1}, \dots, w_m\}$ such that the union of disjoint sets $C = C' \cup C''$ is a basis for W and C' is a basis for $\alpha(Y)$.

Then we have the matrix

$$[\alpha]_{B,C} = \begin{pmatrix} \alpha(v_1) & \cdots & \alpha(v_k) & \alpha(v_{k+1}) & \cdots & \alpha(v_n) \end{pmatrix},$$

and this will look like the block matrix

$$\left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array} \right],$$

where A is $k \times \ell$.