Vector Calculus

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This set of notes is a work-in-progress account of the course 'Vector Calculus', originally lectured by Dr Anthony Ashton in Lent 2020 at Cambridge. These notes are not a transcription of the lectures, but they do roughly follow what was lectured (in content and in structure).

These notes are my own view of what was taught, and should be somewhat of a superset of what was actually taught. I frequently provide different explanations, proofs, examples, and so on in areas where I feel they are helpful. Because of this, this work is likely to contain errors, which you may assume are my own. If you spot any or have any other feedback, I can be contacted at ak2316@cam.ac.uk.

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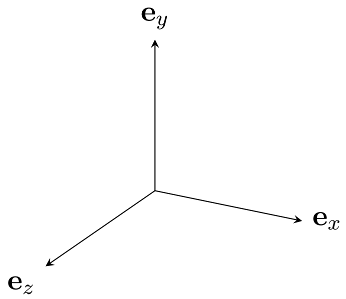
0 Introduction

§0.1 Notation

Throughout this course, a column vector

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

is to be interpreted as the vector $\mathbf{x} = a\mathbf{e}_x + b\mathbf{e}_y + c\mathbf{e}_z$, where $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ are basis vectors aligned with the fixed Cartesian x, y and z axes in $\mathbb{R}^3$.



We may also use the notation $\mathbf{e}_1 = \mathbf{e}_x$, $\mathbf{e}_2 = \mathbf{e}_y$ and $\mathbf{e}_3 = \mathbf{e}_z$, and then we can write $\mathbf{x} = x_i\mathbf{e}_i$.

We will also be using the summation convention frequently, which you can read more about in the 'Vectors and Matrices' course notes.

1 Differential Geometry of Curves

We will begin by studying the differential geometry of curves in $\mathbb{R}^3$, which sounds exciting but isn't.

§1.1 Parameterized Curves & Arc Length

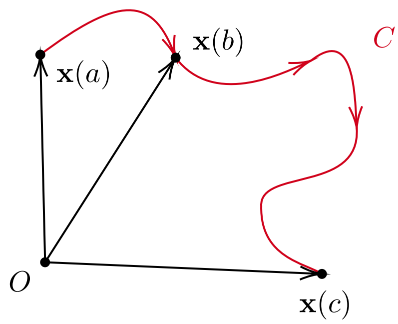
The first object we shall study is *parameterized curves*.

Definition 1.1.1 (Parameterized Curve)
A **parameterized curve** C in $\mathbb{R}^3$ is the image of a continuous map $\mathbf{x} : [a, b] \to \mathbb{R}^3$ in which $t \mapsto \mathbf{x}(t)$.

In Cartesian coordinates, we can write

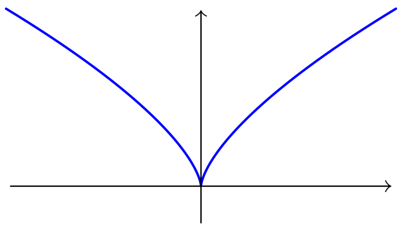
$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.$$

We also give the curve an orientation, based on the map going from the image of A to the image of B.



Definition 1.1.2 (Differentiability and Smoothness)
We say that a curve C is **differentiable** if each of the components $\{x_1(t), x_2(t), x_3(t)\}$ are differentiable, and say C is **regular** if $|\mathbf{x}'(t)| \neq 0$. If C is differentiable and regular then we say C is **smooth**.

Remark. The 'regular' condition exists to avoid 'spikes' in the curve. For example, consider the curve $\mathbf{x}(t) = (t^3, t^2)$. Clearly this is differentiable, but $\mathbf{x}(t)$ has a cusp at $t = 0$. At $t = 0$, we have $|\mathbf{x}(0)| = 0$. If this was not the case, there would be no cusp.

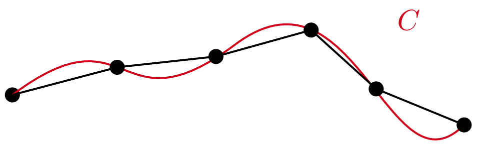


Recall that the function $x_i(t)$ is differentiable at t if $x_i(t+h) = x_i(t) + x_i'(t)h + o(h)$, where $\frac{o(h)}{h} \to 0$ as $h \to 0$. We can write this in terms of vectors, where

$$\mathbf{x}(t+h) = \mathbf{x}(t) + \mathbf{x}'(t)h + o(h),$$

where $o(h)$ is a vector for which $\frac{|o(h)|}{h} \to 0$ as $h \to 0$.

Now given some curve C, we may want to find the length of the curve. We can try to do this by approximating the curve using straight lines.



For $C : [a, b] \to \mathbb{R}^3$ with $t \mapsto \mathbf{x}(t)$, we introduce a partition P of $[a, b]$ with t_0, $t_N = b$ and $t_0 < t_1 < \cdots < t_N$, and we set $\Delta t_i = t_{i+1} - t_i$ and $\Delta t = \max_i \Delta t_i$.

Definition 1.1.3 (Length Relative to a Partition)

For some curve C and partition P as above, we define the **length** of C relative to P by

$$\ell(C, P) = \sum_{i=0}^{N-1} |\mathbf{x}(t_{i+1}) - \mathbf{x}(t_i)|\,.$$

We would expect that this length would get closer to the true length of C as $\Delta t \to 0$. Indeed, we will define the length of C in this way.

Definition 1.1.4 (Length of a Curve)

We define the length of a curve C by

$$\ell(C) = \lim_{\Delta t \to 0} \sum_{i=0}^{N-1} |\mathbf{x}(t_{i+1}) - \mathbf{x}(t_i)| = \lim_{\Delta t \to 0} \ell(C, P).$$

if this limit does not exist, we say the curve is **non-rectifiable**.

In this course, we will not worry too much about non-rectifiable curves, so we will assume that this notion is always well defined.

Now suppose C is a differentiable curve. Then we have that

$$\begin{aligned}\mathbf{x}(t_{i+1}) &= \mathbf{x}(t_i + t_{i+1} - t_i)\\ &= \mathbf{x}(t_i + \Delta t_i)\\ &= \mathbf{x}(t_i) + \mathbf{x}'(t_i)\Delta t_i + o(\Delta t_i).\end{aligned}$$

It follows that

$$|\mathbf{x}(t_{i+1}) - \mathbf{x}(t_i)| = |\mathbf{x}'(t_i)|\Delta t_i + o(\Delta t_i).$$

So if C is differentiable, we get the expression

$$\ell(C, P) = \sum_{i=0}^{N-1} \left(\left| \mathbf{x}'(t_i) \right| \Delta t_i + o\left(\Delta t_i\right) \right).$$

Recall that $o(\Delta t_i)$ represents a function for which $\frac{o(\Delta t_i)}{\Delta t_i} \to 0$ as $\delta t_i \to 0$. So for any $\epsilon > 0$, if $\Delta t = \max_i \Delta t_i$ is sufficiently small, then we have

$$|o(\Delta t_i)| < \left(\frac{\epsilon}{b-a} \right) \Delta t_i,$$

for $i = \{0, \ldots, N-1\}$. So

$$\begin{aligned} \left| \ell(C, P) - \sum_{i=0}^{N-1} |\mathbf{x}'(t_i) \Delta t_i \right| &= \left| \sum_{i=0}^{N-1} o(\Delta t_i) \right| \\ &< \frac{\epsilon}{b-a} \sum_{i=0}^{N-1} \Delta t_i = \epsilon. \end{aligned}$$

Thus the LHS tends to zero as $\Delta t \to 0$. So we get that

$$\begin{aligned} \ell(C) = \lim_{\Delta t \to 0} \ell(C, P) &= \lim_{\Delta t \to 0} \sum_{i=0}^{N-1} |\mathbf{x}'(t_i)| \Delta t_i \\ &= \int_a^b \left| \mathbf{x}'(t) \right| \, \mathrm{d}t, \end{aligned}$$

by the definition of the Riemann integral. So, we can restate our definition in this other form.

Definition 1.1.5 (Length of a Curve)

If $C : [a, b] \to \mathbb{R}^3$ is a differentiable curve with $t \mapsto \mathbf{x}(t)$, then its **length** is

$$\ell(C) = \int_a^b |\mathbf{x}'(t)| \, \mathrm{d}t = \int_C \mathrm{d}s,$$

where $\mathrm{d}s$ is the **arc-length element**, $\mathrm{d}s = |\mathbf{x}'(t)| \, \mathrm{d}t$.

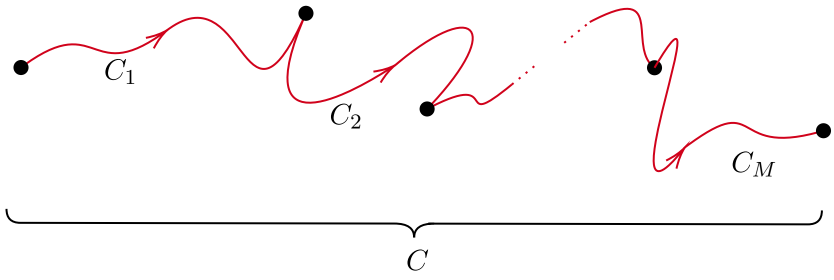
With this defined, we can define the integral of a function along a curve.

Definition 1.1.6 (Integral on a Curve)

For a function $f : \mathbb{R}^3 \to \mathbb{R}$ and a differentiable curve $C : [a, b] \to \mathbb{R}^3$ with $t \mapsto \mathbf{x}(t)$, we define the **integral of f along C** to be

$$\int_C f(\mathbf{x}) \, \mathrm{d}s = \int_a^b f(\mathbf{x}(t)) |\mathbf{x}'(t)| \, \mathrm{d}t.$$

Now consider a curve C made up of M smooth curves $C_1, C_2, \ldots, C_M$.



Then writing $C = C_1 + C_2 + \cdots + C_M$, we can define

$$\int_C f(\mathbf{x})\ \mathrm{d}s = \sum_{i=1}^{M} \int_{C_i} f(\mathbf{x})\ \mathrm{d}s,$$

so we can integrate over piecewise smooth curves.

Note that informally, we have

$$\mathrm{d}s = |\mathbf{x}'(t)|\ \mathrm{d}t = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\ \mathrm{d}t.$$

So being somewhat sacrilegious, we have

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

Example 1.1.7 (Circumference of a Circle)
Let C be a circle of radius $r > 0$ in $\mathbb{R}^3$,

$$\mathbf{x}(t) = \begin{pmatrix} r\cos t \\ r\sin t \\ 0 \end{pmatrix}, \qquad t \in [0, 2\pi].$$

We can differentiate this to get

$$\mathbf{x}'(t) = \begin{pmatrix} -r\sin t \\ r\cos t \\ 0 \end{pmatrix}, \qquad t \in [0, 2\pi].$$

Then integrating over C we have

$$\begin{aligned} \int_C \mathrm{d}s &= \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\ \mathrm{d}t \\ &= \int_0^{2\pi} r\ \mathrm{d}t \\ &= 2\pi r, \end{aligned}$$

as we would expect.

Example 1.1.8 (Integrating over a Circle)

With C being a circle as before, we can integrate over the curve. For example

$$\begin{aligned}\int_C x^2 y \, \mathrm{d}s &= \int_0^{2\pi} (r\cos t)^2 (r \sin t) \underbrace{r \, \mathrm{d}t}_{|\mathbf{x}'(t)| \, \mathrm{d}t} \\ &= 0\end{aligned}$$

There is one subtlety that we have looked over - does $\ell(C)$ depend on the parameterization used?

For example, if we had $\mathbf{x}(t) = (r\cos t, r\sin t, 0)$ with $t \in [0, 2\pi]$ and $\tilde{\mathbf{x}}(t) = (r\cos(2t), r\sin(2t), 0)$ with $t \in [0, \pi]$, these represent the same circle and they should have the same length. We will clear up this possible ambiguity now.

Suppose C has two different parameterization,

$$\begin{aligned}\mathbf{x} &= \mathbf{x}_1(t), \qquad a \leq t \leq b \\ \mathbf{x} &= \mathbf{x}_2(t), \qquad \alpha \leq t \leq \beta\end{aligned}$$

We must then have $\mathbf{x}_2(\tau) = \mathbf{x}_1(t(\tau))$ for some function $t(\tau)$. We can assume that $\frac{\mathrm{d}t}{\mathrm{d}\tau} \neq 0$ so the map between t and τ is invertible and differentiable (this is the inverse function theorem, covered in Analysis and Topology in Part IB). Note then that

$$\begin{aligned}\mathbf{x}_2'(\tau) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{x}_2(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{x}_1(t(\tau)) \\ &= \frac{\mathrm{d}t}{\mathrm{d}\tau}\mathbf{x}_1'(t(\tau)).\end{aligned}$$

Then from our definition,

$$\int_C f(\mathbf{x}) \, \mathrm{d}s = \int_a^b f(\mathbf{x}_1(t)) |\mathbf{x}_1'(t)| \, \mathrm{d}t.$$

We can make the substitution $t = t(\tau)$, and assuming $\mathrm{d}t/\mathrm{d}\tau > 0$, the later integral becomes

$$\int_\alpha^\beta f(\mathbf{x}_2(\tau)) \underbrace{|\mathbf{x}_1'(t(\tau))| \frac{\mathrm{d}t}{\mathrm{d}\tau} \, \mathrm{d}\tau}_{|\mathbf{x}_2'(\tau)| \, \mathrm{d}\tau},$$

which is precisely the same as $\int_C f(\mathbf{x}) \, \mathrm{d}s$ using the $\mathbf{x}_2(t)$ parameterization.

We assumed here that $\frac{\mathrm{d}t}{\mathrm{d}\tau} > 0$, but the same holds if it is negative. Thus our definition of integrating over a curve C does not depend on the choice of parameterization of C.