Markov Chains

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This article constitutes my notes for the 'Markov Chains' course, held in Michaelmas 2021 at Cambridge. These notes are *not a transcription of the lectures*, and differ significantly in quite a few areas. Still, all lectured material should be covered.

Contents

§1 Introduction

For this whole course, I will be a finite or countable set. All of our random variables will also be defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1.1 (Markov Chain)

A **stochastic process** $(X_n)_{n\geq 0}$ is called a **Markov chain** if for all $n \geq 0$ and all $x_0, \ldots, x_{n+1} \in I$, we have[a]

$$\mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n, \ldots, X_0 = x_0) = \mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n).$$

[a] We assume here that we are not conditioning on a zero probability event.

Remark. This definition gives a *discrete time* Markov chain. It is possible to define a continuous time Markov chain, but we won't worry about that for now.

If $\mathbb{P}(X_{n+1} = y \mid X_n = x)$ for all $x, y \in I$ is independent of n, then X is called **time-homogeneous**. Otherwise, it is **time-inhomogeneous**. In this course, we will only study time-homogeneous Markov chains.

We will write $P(x, y) = \mathbb{P}(X_1 = y \mid X_0 = x)$, where $x, y \in I$. We call P a **stochastic matrix**, because

$$\sum_{y\in I} P(x, y) = 1,$$

that is, the sum of each row is 1.

Remark. The index set does not have to be $\mathbb{N}$, it could be say $\{0, 1, \ldots, N\}$ for $N \in \mathbb{N}$.

So to characterize a Markov chain, we need this matrix P, giving the probability of passing from a state x to a state y. We call this matrix the **transition matrix** of X.

Definition 1.2 (Markov)

We say that X is Markov(λ, P) if X_0 has distribution λ and P is the transition matrix. That is,

(i) $\mathbb{P}(X_0 = x_0) = \lambda_{x_0}$, $x_0 \in I$,

(ii) $\mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n, \ldots, X_0 = x_0) = P(x_n, x_{n+1}) = P_{x_n x_{n+1}}$.

We usually represent a Markov chain by its diagram corresponding to the allowed transitions.

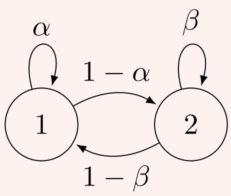
Example 1.3 (Diagram of a Markov Chain)

Let $\alpha, \beta \in (0, 1)$. We consider the matrix

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}.$$

This is a transition matrix on two states which we can call 1 and 2. Here α is the probability of staying at 1, and $1-\alpha$ is the probability of moving from state 2 when at state 1.

A diagram of this is given below. This is a directed graph with the relevant probabilities labelling each edge.

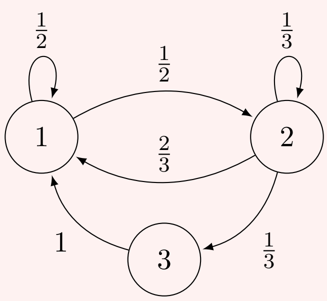


Example 1.4

Suppose that we have the transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 \end{pmatrix}.$$

This is a transition matrix on three states and corresponds with the diagram below.



Theorem 1.5

The process X is Markov(λ, P) if and only if for all $n \geq 0$ and all $x_0, \dots, x_n \in I$ we have

$$\mathbb{P}(X_0 = x_0, \dots, X_n = x_n) = \lambda_{x_0} P(x_0, x_1) \cdots P(x_{n-1}, x_n).$$

Proof. First suppose that X is Markov(λ, P). Then

$$\begin{aligned}
\mathbb{P}(X_n = x_n, \dots, X_0 = x_0) &= \mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1}, \dots, X_0 = x_0) \\
&\quad \cdot \mathbb{P}(X_{n-1} = x_{n-1}, \dots, X_0 = x_0) \\
&= P(x_{n-1}, x_n) \cdot \mathbb{P}(X_{n-1} = x_{n-1}, \dots, X_0 = x_0) \\
&= P(x_0, x_1) \dots P(x_{n-1}, x_n) \mathbb{P}(X_0 = x_0) \\
&= \lambda_{x_0} P(x_0, x_1) \cdots P(x_{n-1}, x_n),
\end{aligned}$$

as required.

Now suppose that the property holds. Then $n = 0$ gives $\mathbb{P}(X_0 = x_0) = \lambda_{x_0}$, so our base case holds. Then

$$\begin{aligned}
\mathbb{P}(X_n = x_) n \mid X_{n-1} = x_{n-1}, \dots, X_0 = x_0) &= \frac{\lambda_{x_0} P(x_0, x_1) \cdots P(x_{n-1}, x_n)}{\lambda_{x_0} P(x_0, x_1) \cdots P(x_{n-2}, x_{n-1})} \\
&= P(x_{n-1}, x_n)
\end{aligned}$$

□

Now we are going to define some useful notation.

Definition 1.6 (δ_i-mass)

For $i \in I$, the **δ_i-mass** of i is defined as $\delta_{ij} = \mathbb{1}(i = j)$.

Recall the notion of independence for random variables. Let $X_1, \dots, X_n$ be discrete random variables. They are *independent* if for all $x_1, \dots, x_n \in I$, we have

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = \prod_{i=1}^{n} \mathbb{P}(X_i = x_i).$$

We have a similar notion for sequences of random variables. We say a sequence $(X_n)_{n \geq 0}$ is *independent* if for all $i_1 < i_2 < \cdots < i_k$ and all $x_1, \dots, x_k$,

$$\mathbb{P}(X_{i_1} = x_1, \dots, X_{i_k} = x_k) = \prod_{j=1}^{k} \mathbb{P}(X_{i_j} = x_j).$$

If $X = (X_n)_{n \geq 0}$ and $Y = (Y_n)_{n \geq 0}$ are two sequences of random variables, they are independent if for all k, m and $i_1 < \cdots < i_k$, $j_1 < \cdots j_m$ we have

$$\begin{aligned}
&\mathbb{P}(X_{i_1} = x_1, \dots, X_{i_k} = x_k, Y_{j_1} = y_1, \dots, Y_{j_m} = y_m) \\
&= \mathbb{P}(X_{i_1} = x_1, \dots, X_{i_k} = x_k) \cdot \mathbb{P}(Y_{j_1} = y_1, \dots, Y_{j_m} = y_m)
\end{aligned}$$

Theorem 1.7 (Simple Markov Property)

Suppose that X is Markov(λ, P). Fix $m \in \mathbb{N}$ and $i \in I$. Conditional on $X_m = i$, the process $(X_{m+n})_{n \geq 0}$ is Markov(δ_i, P) and it is independent of $X_0, \ldots, X_m$.

Proof. We need to show that

$$\mathbb{P}(X_m = x_0, \ldots, X_{m+n} = x_n \mid X_m = i) = \delta_{ix_0} P(x_0, x_1) \ldots P(x_{n-1}, x_n).$$

(To be completed next lecture) □

Remark. Informally, this theorem says 'past and future are independent given the present'.