Numerical Analysis

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This article constitutes my notes for the 'Numerical Analysis' course, held in Lent 2022 at Cambridge. These notes are *not a transcription of the lectures*, and differ significantly in quite a few areas. Still, all lectured material should be covered.

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§1 Polynomial Interpolation

§1.1 Lagrange and Newton Polynomials

Let $f : [a, b] \to \mathbb{R}$ be a real-valued continuous function defined on some interval $[a, b]$ and let $(x_i)_{i=0}^{n}$ be $n+1$ distinct points in $[a, b]$. We wish to construct a polynomial p of degree n which interpolates f at these points, i.e., satisfies

$$p\left(x_i\right) = f\left(x_i\right), \quad i = \overline{0, n}, \quad p \in \mathcal{P}_n$$

Theorem 1.1 (Existence and Uniqueness)
Given $f \in C[a, b]$ and $n+1$ distinct points $(x_i)_{i=0}^{n} \in [a, b]$, there is exactly one polynomial $p \in \mathcal{P}_n$ such that $p\left(x_i\right) = f\left(x_i\right)$ for all i.

Proof. Existence. There is at least one polynomial interpolant $p \in \mathcal{P}_n$, the one in the Lagrange form,

$$p(x) = \sum_{i=0}^{n} f\left(x_i\right) \ell_i(x) \quad \text{with} \quad \ell_i(x) := \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}, i = 0, \ldots, n, \tag{†}$$

the ℓ_i s are called the **fundamental Lagrange polynomials**. Each ℓ_i is the product of n linear factors, hence $\ell_i \in \mathcal{P}_n$. It also equals 1 at x_i and vanishes at $x_j \neq x_i$, i.e., $\ell_i\left(x_j\right) = \delta_{ij}$. Therefore $p \in \mathcal{P}_n$ and

$$p\left(x_j\right) = \sum_{i=0}^{n} f\left(x_i\right) \ell_i\left(x_j\right) = f\left(x_j\right).$$

> *Uniqueness.* There is at most one polynomial interpolant $p \in \mathcal{P}_n$ to f on $(x_i)_{i=0}^n$. For if there are two, $p, q \in \mathcal{P}_n$, then the polynomial $r := p - q$ is of degree n and vanishes at $n+1$ points, whence $r \equiv 0$. $\square$

Let us introduce the so-called **nodal polynomial**

$$\omega(x) = \prod_{i=0}^{n} (x - x_i) .$$

Then, in the expression (†) for ℓ_i, the numerator is simply $\omega(x) / (x - x_i)$ while the denominator is equal to $w'(x_i)$. With that we arrive to a compact Lagrange form

$$\boxed{p(x) = \sum_{i=0}^{n} f(x_i) \ell_i(x) = \sum_{i=0}^{n} \frac{f(x_i)}{\omega'(x_i)} \frac{\omega(x)}{x - x_i}}.$$

The lagrange forms above and in (†) for interpolating polynomials are easy to manipulate, but they are unsuitable for numerical evaluation. An alternative is the *Newton form* which has an *adaptive* nature.

Method 1.2 (Newton Form)

For $k = 0, 1, \ldots, n$, let $p_k \in \mathcal{P}_k$ be the polynomial interpolant to f on $x_0, \ldots, x_k$. Then two subsequent p_{k-1} and p_k interpolate the same values $f(x_i)$ for $i \le k - 1$, hence their difference is a polynomial of degree k that vanishes at k points $x_0, \ldots, x_{k-1}$. Thus

$$p_k(x) - p_{k-1}(x) = A_k \prod_{i=0}^{k-1} (x - x_i) ,$$

with some constant A_k which is seen to be equal to the leading coefficient of p_k.

It follows that $p = p_n$ can be built step by step as one constructs the sequence $(p_0, p_1, \ldots)$, with p_k obtained from p_{k-1} by addition the term from the right-hand side of (1.3), so that finally

$$p(x) = p_n(x) = p_0(x) + \sum_{k=1}^{n} [p_k(x) - p_{k-1}(x)] = \sum_{k=0}^{n} A_k \prod_{i=0}^{k-1} (x - x_i) .$$

Definition 1.3 (Divided Difference)

Given $f \in C[a, b]$ and $k+1$ distinct points $(x_i)_{i=0}^k \in [a, b]$, the **divided difference** $f[x_0, \ldots, x_k]$ of order k is the leading coefficient of the polynomial $p_k \in \mathcal{P}_k$ which interpolates f at these points.

The divided difference is a symmetric function of the variables $[x_0, \ldots, x_k]$, and if $f(x) = x^m, m \le k$, then $f[x_0, \ldots, x_k] = \delta_{km}$.

With this definition we arrive at the **Newton formula** for the interpolating polynomial.

Theorem 1.4 (Newton Formula)

Given $n+1$ distinct points $(x_i)_{i=0}^n$, let $p_n \in \mathcal{P}_n$ be the polynomial that interpolates

f at these points. Then it may be written in the Newton form

$$\begin{aligned} p_n(x) = & f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots \\ & \cdots + f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1}), \end{aligned}$$

or, more compactly,

$$p_n(x) = \sum_{k=0}^{n} f[x_0, \ldots, x_k] \prod_{i=0}^{k-1} (x - x_i)$$

To make this formula of any use, we need an expression for $f[x_0, \ldots, x_k]$. One such can be derived from the Lagrange formula by identifying the leading coefficient of p. This turns out to be

$$f[x_0, \ldots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\omega'(x_i)}, \quad \omega(x) := \prod_{i=0}^{n} (x - x_i).$$

However, this expression has computational disadvantages as the Lagrange form itself. A useful way to calculate divided difference is again an adaptive (or recurrence) approach.

Theorem 1.5 (Recurrence Relation)
For distinct $x_0, x_1, \ldots, x_k$ with $k > 1$, we have

$$f[x_0, \ldots, x_k] = \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}.$$

Proof. Let $q_0, q_1 \in \mathcal{P}_{k-1}$ be the polynomials such that q_0 interpolates f on $(x_0, x_1, \ldots, x_{k-1})$ and q_1 interpolates on $(x_1, \ldots, x_k)$. Consider the polynomial

$$p(x) = \frac{x - x_0}{x_k - x_0} q_1(x) + \frac{x_k - x}{x_k - x_0} q_0(x), \quad p \in \mathcal{P}_k.$$

One readily sees that $p(x_i) = f(x_i)$ for all i, hence, p is the k-th degree interpolating polynomial by f. Moreover, the leading coefficient of p is equal to the difference of those of q_1 and q_0 divided by $x_k - x_0$, and that is exactly what the recurrence (1.5) says. □

The recursive relation allows for the fast evaluation of the **divided difference table**

x_i	f_i	$f[*,*]$	$f[*,*,*]$	$\cdots$	$f[*,\cdots,*]$
x_0	$f[x_0]$				
		$f[x_0, x_1]$			
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$		
		$f[x_1, x_2]$		$\ddots$	
x_2	$f[x_2]$		$f[x_2, x_3, x_4]$	$\cdots$	$f[x_0, x_1, \cdots, x_n]$
		$f[x_2, x_3]$		$\iddots$	
x_3	$f[x_3]$		$\iddots$		
$\vdots$	$\vdots$	$\iddots$			
x_n	$f[x_n]$				

This can be done in $\mathcal{O}\left(n^2\right)$ operations and the outcome is the numbers $\{f\left[x_0,\ldots,x_k\right]\}_{k=0}^{n}$ at the head of the columns which can be used in the Newton form.

Finally evaluation of p at a given point x using Newton formula (provided that divided differences $A_k = f[x_0,\ldots,x_k]$ are known) requires just n multiplications, as long as we do it by the **Horner scheme**

$$\begin{aligned} p_n(x) =&\{\cdots\{\{A_n \times (x-x_{n-1}) + A_{n-1}\} \times (x - x_{n-2}) + A_{n-2}\} \\ &\times (x - x_{n-3}) + \cdots + A_1\} \times (x - x_0) + A_0. \end{aligned}$$

§1.2 Examples

We will now look at some examples of these methods in use.

Example 1.6 (Interpolating Polynomial)

Given the data

x_i	0	1	2	3
$f(x_i)$	-3	-3	-1	9

we want to find the interpolating polynomial $p \in \mathcal{P}_3$ in both Lagrange and Newton forms.

Fundamental Lagrange polynomials. We have

$$\begin{aligned} \ell_0(x) &= \frac{(x-1)(x-2)(x-3)}{-6} = -\frac{1}{6}\left(x^3 - 6x^2 + 11x - 6\right), \\ \ell_1(x) &= \frac{x(x-2)(x-3)}{2} = \frac{1}{2}\left(x^3 - 5x^2 + 6x\right), \\ \ell_2(x) &= \frac{x(x-1)(x-3)}{-2} = -\frac{1}{2}\left(x^3 - 4x^2 + 3x\right), \\ \ell_3(x) &= \frac{x(x-1)(x-2)}{6} = \frac{1}{6}\left(x^3 - 3x^2 + 2x\right). \end{aligned}$$

Lagrange form. Using the polynomials above, we can write

$$\begin{aligned} p(x) &= (-3)\cdot\ell_0(x) + (-3)\cdot\ell_1(x) + (-1)\cdot\ell_2(x) + 9\cdot\ell_3(x) \\ &= \left(\frac{1}{2} - \frac{3}{2} + \frac{1}{2} + \frac{3}{2}\right)x^3 + \left(-3 + \frac{15}{2} - 2 - \frac{9}{2}\right)x^2 + \left(\frac{11}{2} - 9 + \frac{3}{2} + 3\right)x - 3 \\ &= x^3 - 2x^2 + x - 3. \end{aligned}$$