Dynamics and Relativity

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This set of notes is a work-in-progress account of the course 'Dynamics and Relativity', originally lectured by Prof Peter Haynes in Lent 2020 at Cambridge. These notes are not a transcription of the lectures, but they do roughly follow what was lectured (in content and in structure).

These notes are my own view of what was taught, and should be somewhat of a superset of what was actually taught. I frequently provide different explanations, proofs, examples, and so on in areas where I feel they are helpful. Because of this, this work is likely to contain errors, which you may assume are my own. If you spot any or have any other feedback, I can be contacted at ak2316@cam.ac.uk.

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§1 Newtonian Dynamics – Basic Concepts

A central aspect of this course is Newtonian dynamics. In this chapter we will develop some of the ideas and definitions needed to discuss this in detail.

§1.1 Particles

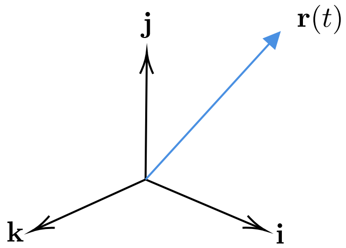
When dealing with Newtonian dynamics, we will often use and refer to *particles*, as a way of describing phenomina.

Definition 1.1 (Particle)
A **particle** is an object of negligible size. It has some mass $m > 0$, and can also have other properties such as (perhaps) an electric charge q.

A particle is completely described by a **position vector**, usually denoted $\mathbf{r}(t)$ or $\mathbf{x}(t)$, with respect to some origin O. The cartesian coordinates of $\mathbf{r}$ are (x, y, z), where

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ being orthonormal basis vectors.



The choice of coordinate axes defines a **frame of reference** S.

Of course, we will be considering particles that are moving, so we will define the velocity, momentum and acceleration of the particle.

Definition 1.2 (Velocity)
The **velocity** of a particle is $\mathbf{u}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = \dot{\mathbf{r}}$, and is tangent to the path (or trajectory) of the particle.

Definition 1.3 (Momentum)
The **momentum** of a particle is $\mathbf{p} = m\mathbf{u} = m\dot{\mathbf{r}}$, where m is the mass of the particle.

Definition 1.4 (Acceleration)
The **acceleration** of the particle is $\dot{\mathbf{u}} = \ddot{\mathbf{r}} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}(t)$.

§1.2 Newton's Laws of Motion

We can now write down Newton's three laws of motion, which govern the motion of particles. All of these statements about particles can be extended to finite bodies (which are composed of many particles).

Law (Newton's First Law/Galileo's Law of Inertia)
There exist inertial frames of reference in which a particle remains at rest or moves

at constant velocity unless it is acted on by a force.

Law (Newton's Second Law)

In an inertial frame the rate of change of momentum of a particle is equal to the force acting on it.

Law (Newton's Third Law)

To every action there is an equal and opposite reaction. That is, forces excreted between two particles are equal in magnitude and opposite in direction.

We will see how these are used in the coming sections.

§1.3 Inertial Frames & Galilean Transformations

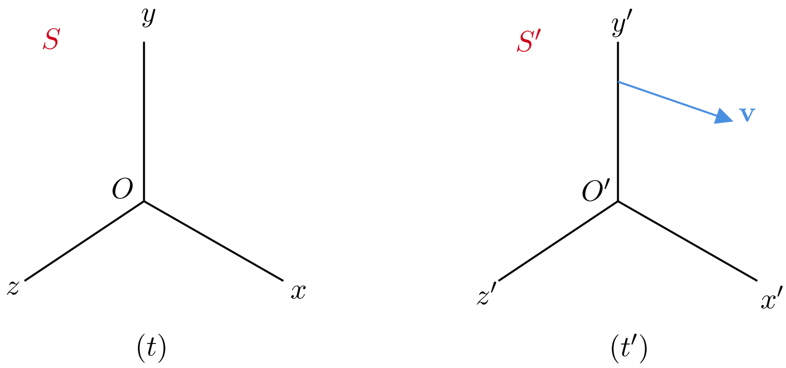
We met inertial frames in the previous section. In this section we will look at what inertial frames are and how they relate to each other.

Definition 1.5 (Inertial Frame)

In an **inertial frame**, the acceleration of a particle is zero if the force is zero. That is

$$\ddot{\mathbf{r}} = \mathbf{0} \iff \mathbf{F} = \mathbf{0}.$$

Inertial frames are not unique. For example, if S is an inertial frame then any other frame S' moving with constant velocity relative to S is also an inertial frame.



In the example above, it is easy to relate the coordinate systems in each system.

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t,$$

where $\mathbf{v}$ is the velocity of S' relative to S. This transformation is called a **boost**.

For a particle with position vector $\mathbf{r}(t)$ in S and $\mathbf{r}'(t')$ in S', we can relate the velocity and acceleration as measured in S and S'. Let the velocity be $\mathbf{u} = \dot{\mathbf{r}}(t)$ in S and the acceleration be $\mathbf{a} = \ddot{\mathbf{r}}(t)$ in S. Then these relate to the corresponding quantities in S' by

$$\mathbf{u}' = \mathbf{u} - \mathbf{v}, \qquad \mathbf{a}' = \mathbf{a}.$$

Boosts aren't the only transformations of frames that preserve inertial frames.

Definition 1.6 (Galilean Frames)

A **Galilean transformation** preserves inertial frames. The set of all Galilean transformations forms the **Galilean group**.

Galilean frames combine boosts with some combination of the following:

- Translations of space, $\mathbf{r}' = \mathbf{r} - \mathbf{r}_0$ where $\mathbf{r}_0$ is constant.
- Translations of time, $t' = t - t_0$ where t_0 is constant.
- Rotations and reflections in space, $\mathbf{r}' = R\mathbf{r}$ where R is an orthogonal matrix.

This set generates the Galilean group.

For any Galilean transformation we have

$$\ddot{\mathbf{r}} = \mathbf{0} \iff \ddot{\mathbf{r}}' = \mathbf{0},$$

that is, S inertial if and only if S' is inertial.

Law (Galilean Relativity)

The principle of Galilean relativity is that the laws of Newtonian physics are the same in all inertial frames.

This principle tells us that the the laws of physics are the same at any point in space, at any point in time, in whatever direction I face, and at whatever constant velocity I move with. Thus any set of equations which describe Newtonian physics must have these properties, known as **Galilean invariance**.

Note that the measurement of velocity is then not absolute, but measurement of acceleration is.

§1.4 Newton's Second Law and Equations of Motion

We previously stated Newton's second law, it says that for a particle subject to a force $\mathbf{F}$, the momentum $\mathbf{p}$ satisfies

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F},$$

where the momentum $\mathbf{p} = m\mathbf{u} = m\dot{\mathbf{r}}$.

For now, we will assume that m, the mass of the particle, is constant (we will return to the variable mass case later in the course). That gives us

$$m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = m\ddot{\mathbf{r}} = \mathbf{F}.$$

In this way, we can describe mass as a measure of the 'reluctance to accelerate', the inertia of the particle.

If $\mathbf{F}$ is specified as a function of $\mathbf{r}, \dot{\mathbf{r}}$ and t, $\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$, then we have a second order differential equation for $\mathbf{r}$. To solve this completely, we would need to prove the initial position $\mathbf{r}(t_0)$ and the initial velocity $\dot{\mathbf{r}}(T_0)$, then we get a unique solution. The path/trajectory of the particle is then determined (at all future and past times).

To get further, we will need to consider some possible forms of vvF.

§1.5 Examples of Forces

§1.5.1 Gravitational Force

Consider two particles with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ and masses m_1 and m_2. Then we can describe the gravitational forces between them as follows.

Law (Newton's Gravitation Law)

Newton's law of gravitation states that the force of gravity between the particles is

$$\mathbf{F}_1 = \frac{Gm_1m_2(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} = -\mathbf{F}_2,$$

where G is the gravitational constant.

Note that $\mathbf{F}_1 = \mathbf{F}_2$ is proportional to $|\mathbf{r}_1 - \mathbf{r}_2|^{-1}$, so we call this law an **inverse square law**.

We will explore the details of gravitational forces later on in the course.

§1.5.2 Electromagnetic Forces

Consider a particle with electric charge q, in the presence of an electric field $\mathbf{E}(\mathbf{r}, t)$ and a magnetic field $\mathbf{B}(\mathbf{r}, t)$. Then we can describe the electromagnetic force as follows.

Law (Lorentz Force Law)

The electromagnetic force on the particle is

$$\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) = q(\mathbf{E}(\mathbf{r}, t) + \dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}, t)).$$

Example 1.7

Consider the case of $\mathbf{E} = \mathbf{0}$ and $\mathbf{B}$ being a constant vector.

We can then use Newton's second law to get the equations of motion,

$$m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}.$$

We will choose axes such that $\mathbf{B} = Bv\hat{v}z$. Hence

$$m\ddot{z} = 0 \implies z = z_0 + ut,$$

for constants z_0 and u. We also get

$$\begin{aligned} m\ddot{x} &= qB\dot{y} \\ m\ddot{y} &= -qB\dot{x}. \end{aligned}$$

Then defining $\omega = qB/m$, we get the solution

$$\begin{aligned} x &= x_0 - \alpha\cos(\omega(t - t_0)), \\ y &= y_0 + \alpha\sin(\omega(t - t_0)), \end{aligned}$$

where x_0, x_0, t_0 and α are constants determined by the initial conditions.

Geometrically, the motion of the particle is made up of circular motion in x, y and constant velocity in z, which is helical motion in the direction of the magnetic field. The motion is also clockwise from the direction of $\mathbf{B}$.

§2 Dimensional Analysis

§2.1 Units and Dimensional Quantities

Many of the problems we think about in dynamics involve three basic dimensional quantities:

- L – length,
- M – mass,
- T – time.

The dimensions of some quantity $[x]$[1] can be expressed in terms of L, M, T. For example, $[\text{density}] = M \cdot L^{-3}$ and $[\text{force}] = M \cdot L \cdot T^{-2}$.

There are some rules that apply to the dimensions of a quantity. Specifically, we only allow power law functions of L, M, T. For example, if X was some dimensional quantity, then $e^X = 1 + X + X^2/2 + \cdots$ wouldn't make any sense because we would be using different dimensions and units.

We usually use the standard SI set of units for L, M and T. We have m (meters) for length L, kg (kilograms) for mass M and s (seconds) for time T. Many other physical quantities can be formed out of these basic units.

Example 2.1 (Newton's Gravitational Constant)

Consider the gravitational constant G appearing in Newton's Law of Gravitation, described previously.

We had the force $F = \frac{Gm_1m_2}{r^2}$ for masses m_1, m_2 and length r, Thus we have

$$G = \frac{F \cdot r^2}{m_1 m_2} \implies [G] = \frac{M \cdot L \cdot L^2}{T^2 \cdot M \cdot M} = \frac{L^3}{M \cdot T^2}.$$

So the natural units for G will be $\mathrm{m^3\,kg^{-1}\,s^{-2}}$. Indeed, if we choose those units we would find that $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$.

The general principle is that dynamical/physical equations must work for *any* consistent choice of units.

§2.2 Scaling

Suppose that we had some dimensional quantity Y, depending on some other set of dimensional quantities $X_1, X_2, \dots, X_n$. Let the dimensions be $[Y] = L^a M^b T^c$, and $[X_i] = L^{a_i} M^{b_i} T^{c_i}$ for $i \in \{1, \dots, n\}$.

[1] It will be convenient throughout this section to use the notation $[x]$ to denote the dimensions of x, using the shorthand of L, M and T as described above.

If $n \leq 3$, then $Y = C \cdot X_1^{p_1} X_2^{p_2} X_3^{p_3}$, and p_1, p_2, p_3 can be determined by balancing dimensions:

$$L^a M^b T^c = (L^{a_1} M^{b_1} T^{c_1})^{p_1} \cdots$$

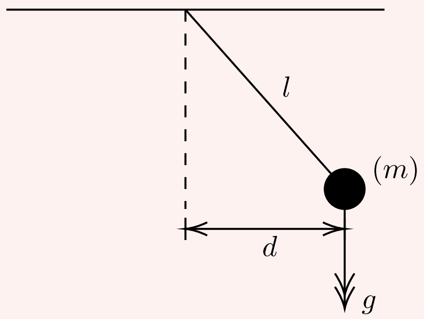
hence we have $a = a_1 p_1 + a_2 p_2 + a_3 p_3$, and similarly for b and c. Then we have three unknowns for p_1, p_2, p_3 if the dimensions X_1, X_2, X_3 are independent.

If $n > 3$, then the property of dimensional independence cannot extend for $X_1, X_2, \ldots,$ but we can choose X_1, X_2 and X_3 that are (without loss of generality) dimensionally independent and $n-3$ dimensionless quantities $\lambda_1 = \frac{X_4}{X_1^{q_{11}} X_2^{q_{12}} X_3^{q_{13}}}$ and so on, where q_{mn} are chosen to balance the dimensions.

Then $Y = X_1^{p_1} X_2^{p_2} X_3^{p_3} \cdot C(\lambda_1, \ldots, \lambda_{n-3})$. This is 'Bridgeman's theorem'.

Example 2.2 (Period of a Pendulum)

Consider a simple pendulum as shown below.



We want to know how the period of oscillation P depends on the quantities m, l, d and g. Here, we have Y being the period and X_1, X_2, X_3 and X_4 being the four quantities m, l, d and g. So let's think about the dimensions.

$$\begin{aligned}
[P] &= T \\
[m] &= M \\
[g] &= L \cdot T^{-2} \qquad \text{(gravity)} \\
[l] &= L \\
[d] &= L
\end{aligned}$$

Now we form the one dimensionless group by taking the ratio d/l, writing

$$P = f\left(\frac{d}{l}\right) m^{p_1} l^{p_2} g^{p_3},$$

which is the dimensional statement

$$T = M^{p_1} L^{p_2} \left(\frac{L}{T^2}\right)^{p_3}.$$

Balancing dimensions, we have that $M : p_1 = 0$, $L : p_2 + p_3 = 0$, and $T : 1 = -2p_3$. Solving these we get $p_1 = 0, p_2 = 1/2$ and $p_3 = -1/2$. Thus we get

$$P = f(d/l) l^{1/2} g^{-1/2}.$$

There is of course a lot of freedom in this formula (as we haven't really considered the physical situation), but it does contain useful information. For example, if $d \mapsto 2d$ and $l \mapsto 2l$, then $p \mapsto \sqrt{2}p$. But if $d \mapsto 2d$ and $l \mapsto l$, then we couldn't say much because it depends on the precise form of f.

§3 Forces

§3.1 Force and Potential Energy in 1 Spacial Dimension

Consider a mass m moving in a straight line with position $x(t)$. Let's suppose we had some force that depends only on position x, and not velocity $\dot{x}$ or time t. We will call the force $F(x)$, and define potential energy as follows.

Definition 3.1 (Potential Energy)
The **potential energy** $V(X)$ is given by $F(x) = -\frac{\mathrm{d}V}{\mathrm{d}x}$, so

$$V(x) = -\int^{x} F(x')\,\mathrm{d}x'.$$

We know that the force satisfies newton's second law, $m\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}$. We also define kinetic energy.

Definition 3.2 (Kinetic Energy)
The **kinetic energy** $T(\dot{x}) = \frac{1}{2}m\dot{x}^2$. In higher dimensions, this is generalized to $\frac{1}{2}, |\dot{\mathbf{x}}|^2$.

Now, if we take the kinetic and potential energy we get an important invariant.

Proposition 3.3 (Conservation of Energy)
The total energy $E = T + V$ is conserved. That is, $\frac{\mathrm{d}E}{\mathrm{d}t} = 0$.

Proof. Differentiating we have

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}v\dot{x}^2 + V(x)\right) \\ &= m\dot{x}\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\dot{x} \\ &= \dot{x}\left(m\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\right) = 0,\end{aligned}$$

as required. □

Note that for conservation of $\frac{1}{2}m\dot{x}^2 + \Phi$, we require $\dot{F} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$. In principle, Φ may depend on $x, \dot{x}$ and t. But usually it is the case that there is no such Φ if F depends on $\dot{x}$ and/or t.

Example 3.4 (Harmonic Oscillator)

Consider a system where $F(x) = -kx$ (physical examples of such system is Hooke's law for an elastic string). Then

$$V(x) = -\int^x (-kx')\,\mathrm{d}x' = \frac{1}{2}kx^2,$$

with an appropriate choice of arbitrary constant. We can then seek an explicit expression for $x(t)$. We know that $m\ddot{x} = -kx$, hence

$$x(t) = A\cos\omega t + B\sin\omega t,$$

and

$$\dot{x} = -\omega A\sin\omega t + \omega B\cos\omega t,$$

for suitable constants A and B and $\omega = (k/m)^{1/2}$.

We can then verify that energy is conserved. The total energy is $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}(-\omega A\sin\omega t + \omega B\cos\omega t)^2 m + \frac{1}{2}k(A\omega\sin\omega t + B\sin\omega t)^2 = \frac{1}{2}k(A^2 + B^2)$, which is a constant.

Let's now think about more general potentials. In one dimension, conservation of energy gives useful information about the motion.

We can think of conservation as the first integral of Newton's second law. If we have

$$E = \frac{1}{2}m\dot{x}^2 + V(x) \qquad \text{constant},$$

then

$$\dot{x} = \pm\sqrt{\frac{2}{3}(E - V(X))},$$

where E is set by the initial conditions. Hence

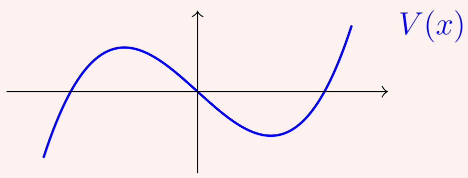
$$\pm\int_{x_0}^{x} \frac{\mathrm{d}x'}{\sqrt{\frac{2}{3}(E - V(X))}} = t - t_0$$

with $x = x_0$ when $t = t_0$. We get an implicit solution for $x(t)$. Of course, there is a limit to how much can be taken analytically from this integral, but in principle (possibly numerically) this integral could be evaluated to find $x(t)$.

We can gain some qualitative insights from the conservation of energy.

Example 3.5

Take $V(x) = \lambda(x^3 - 3\beta^2 x)$ where λ, β are positive constants.



We can wonder what happens if we release the particle from rest at $x = x_0$ for various choices of x_0.

The assumption of stating from rest implies that the total energy is $V(x_0)$. Then conservation of energy implies that $V(x) \leq V(x_0)$.

If the left maxima is at $x = -\beta$ and the right minima is at $x = \beta$, then we can consider cases of the value of x.

If $x_0 < -\beta$, then the particle will keep moving to the left, going to $-\infty$. If $-\beta < x_0 < 2\beta$, then the particle must stay confined to this region. If $x_0 > 2\beta$, then the particle will move to the left, will reach $x = -\beta$ and will then continue to $-\infty$.

Some special cases are $x_0 = -\beta$, where the particle will stay at $-\beta$, and $x_0 = \beta$ where the particle will stay at β. Lastly, if $x_0 = 2\beta$, then the particle will move to the left, and will come to rest at $x = -\beta$.

§3.2 Equilibrium Points

Equilibrium points are points at which a particle can stay at rest for all time. The general condition for an equal point is

$$V'(x) = 0,$$

that is, the derivative of the potential vanishes.

It is often useful to analyse the motion near the equilibrium at $x = x_0$, so $V'(x_0) = 0$. If we assume that $x - x_0$ is small, we can expand $V(x)$ as a taylor series.

$$V(x) \approx V(x_0) + \cancel{(x - x_0)V'(x_0)} + \frac{1}{2}(x - x_0)^2 V''(x) + \cdots .$$

So the equation of motion is

$$m\ddot{x} = -V'(x) \approx -(x - x_0)V''(x_0).$$

We then have two cases.

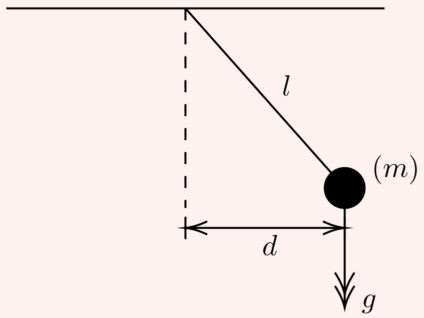
- If $V''(x_0) > 0$, then we have a local minimum of $V(x)$, and we have the harmonic oscillator equation. We can also see that the frequency of oscillation will be $\sqrt{V''(x_0)/m}$. We call this scenario a **stable equilibrium**.
- If $V''(x_0) < 0$, then we have a local maximum of $V(x)$. This will have exponentially increasing and exponentially decaying solutions, and will almost always excite the exponentially increasing part. We have the growth rate $\gamma = \sqrt{-V''(x_0)/m}$. We call this an **unstable equilibrium**.
- If $V''(x_0) = 0$, then we need to go to higher terms in the Taylor series.

Example 3.6 (Pendulum)

Consider again a the same pendulum as before. If θ is the angle between the pendulum and the vertical, we get from Newton's second law that

$$ml\ddot{\theta} = -mg\sin\theta,$$

which comes from considering acceleration perpendicular to the string.



We can write this as

$$ml\ddot{\theta} = -\frac{\mathrm{d}}{\mathrm{d}\theta}(-mg\cos\theta),$$

hence $E = T + V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$. We can then note that $\frac{\mathrm{d}E}{\mathrm{d}t} = 0$.

We can see that there is a stable equilibrium at $\theta = 0$ and an unstable equilibrium at $\theta = \pi$ (this may be an equilibrium point if for example a rigid rod).

If $-mgl < E < mgl$, then the pendulum will oscillate around the position of the stable equilibrium. If $E > mgl$, then $\dot{\theta} > 0$ or $\dot{\theta} < 0$ for all time.

We can work out the period of motion. The period is four times as long as it takes for the pendulum to go from 0 to θ_0, which is

$$\begin{aligned} P &= 4\int_0^{\theta_0} \frac{1}{\sqrt{2gl(\cos\theta - \cos\theta_0)/l^2}}\,\mathrm{d}\theta \\ &= 4\sqrt{\frac{l}{g}}\int_0^{\theta_0} \frac{1}{\sqrt{2\cos\theta - 2\cos\theta_0}}\,\mathrm{d}\theta \\ &= 4\sqrt{\frac{l}{g}}F(\theta_0), \end{aligned}$$

which matches what we got from the dimensional analysis.

If θ_0 is small, we can approximate

$$F(\theta_0) \approx \int_0^{\theta_0} \frac{1}{\sqrt{\theta_0^2 - \theta^2}}\,\mathrm{d}\theta = \frac{\pi}{2}.$$

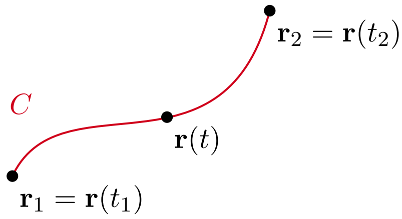
Hence the period is approximately $P = 2\pi\sqrt{\frac{l}{g}}$, for small angles.

§3.3 Force and Potential for Motion in 3D

Consider a particle moving in three dimensions under a force $\mathbf{F}$. Newton's second law then gives $m\dot{\mathbf{r}} = \mathbf{F}$. The kinetic energy of the particle is $T = \frac{1}{2}m|\dot{\mathbf{r}}|^2 = \frac{1}{2}m|\mathbf{u}|^2$. Then

$$\frac{\mathrm{d}T}{\mathrm{d}t} = m\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = \mathbf{F} \cdot \dot{\mathbf{r}} = \mathbf{F} \cdot \mathbf{u}.$$

Now considering the total word done by the force on the particle as it moves along a finite path:



The total work is

$$\int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{u} \, \mathrm{d}t = \int_{t_1}^{t_2} \mathbf{F} \cdot \dot{\mathbf{r}} \, \mathrm{d}t = \int_{\mathbf{r}(t_1)}^{\mathbf{r}(t_2)} \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where we integrate over the curve C. We can write this in terms of components, with

$$\int_{t_1}^{t_2} F_x \mathrm{d}x + F_y \mathrm{d}y + F_z \mathrm{d}z,$$

where $\mathbf{F} = (F_x, F_y, F_z)$.

Now suppose that $\mathbf{F}$ is a function of $\mathbf{r}$ only (and not of velocity). We say that $\mathbf{F}(\mathbf{r})$ defines a 'force field'. Specifically, this is a *conservative force field.*

Definition 3.7 (Conservative Force Field)
A **conservative force field** is one such that $\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})$ for some function $V(\mathbf{r})$.

That is, we have $F_i = \frac{\partial V}{\partial x_i}$. The condition required for $vvF(\mathbf{r})$ to be conservative is $\nabla \times \mathbf{F}(\mathbf{r}) = \mathbf{0}$.

Proposition 3.8 (Conservation of Energy)
If $\mathbf{F}$ is conservative, then the energy $E = T + V(\mathbf{r})$ is conserved.

Proof. We have

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t} V(\mathbf{r}) \\
&= m\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} + \nabla V \cdot \dot{\mathbf{r}} \\
&= \dot{\mathbf{r}} \cdot (m\ddot{\mathbf{r}} + \nabla V) \\
&= \dot{\mathbf{r}} \cdot (m\ddot{\mathbf{r}} - \mathbf{F}) = 0,
\end{aligned}$$

so energy is conserved. □

We can consider the total work done by a conservative force.

$$W = \int_C \mathbf{F} \cdot \, \mathrm{d}\mathbf{r} = -\int_C \nabla V \cdot \mathrm{d}\mathbf{r} = V(\mathbf{r}_1) - V(\mathbf{r}_2),$$

which follows from the properties of ∇. Note that this is independent of the path taken between $\mathbf{r}_1$ and $\mathbf{r}_2$. Then, if C is closed, no work is done by the force.