Variational Principles

Adam Kelly (ak2316@cam.ac.uk)

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From finding geodesics to expressing the fundamental laws of nature, variational principles and the calculus of variations arise naturally in pure & applied mathematics, and theoretical physics. Here, we will study such ideas, with the underlying goal of considering problems involving minimizing (or maximizing) quantities that depend on an entire function.

This article constitutes my notes for the 'Variational Principles' course, held in Easter 2021 at Cambridge. These notes are *not a transcription of the lectures*, and differ significantly in quite a few areas. Still, all lectured material should be covered.

Contents

§1 Introduction

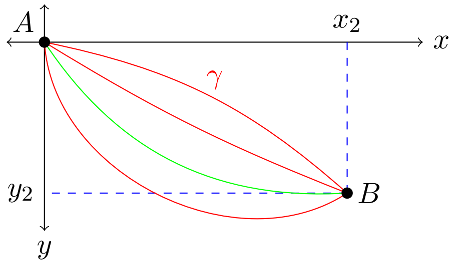
We will begin our study of variational principles by getting a feel for the type of problems that we care about. As a field, the study of variational principles can be traced back to a single problem, posed by Johann Bernoulli in 1696. It is there that we will begin.

§1.1 A Motivating Problem – The Brachistochrone

One of the earliest problems in the calculus of variations is the **Brachistochrone** problem.

Example 1.1 (Brachistochrone)
Consider a particle sliding on a wire between two fixed points A and B, under the influence of gravity. Which shape of the wire will give the shortest travel time, when the particle starts from rest?

It is not immediately obvious what the answer to this problem is. We won't solve it right now, but we can still can take some steps in the right direction. We first setup a coordinate system, setting A to be the origin, and B to be some point (x_2, y_2).



In this coordinate system, for a given curve γ, we can write down the travel time

$$T = \int_\gamma \mathrm{d}t = \int_\gamma \frac{1}{v}\,\mathrm{d}s,$$

where v is the velocity of the particle. Since the particle starts at $y = 0$, conservation of energy implies that

$$\frac{1}{2}mv^2 + mgy = 0$$
$$\implies v = \sqrt{-2gy}.$$

Thus to solve our problem, we really want to minimize

$$T[y] = \frac{1}{\sqrt{2g}} \int_0^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{-y}}\,\mathrm{d}x,$$

subject to y being a smooth function with $y(0) = 0$, and $y(x_2) = y_2$.

§1.2 Problems of Interest

Let's jump to another problem from pure mathematics, finding geodesics.

Example 1.2 (Geodesic Problem)
Find the shortest path γ between two points on a surface Σ (if one exists).

If Σ is $\mathbb{R}^2$, we can again adopt a coordinate system, and try to find the shortest curve γ between the two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$. To solve this problem, we really want to minimize

$$D[y] = \int_{x_1}^{x_2} \sqrt{1+(y')^2}\,\mathrm{d}x,$$

subject to y being a smooth function with $y(x_1) = y_1$ and $y(x_2) = y_2$.

Both of these problems have a similar flavour: we are trying the minimize or maximize something like

$$F[y] = \int_{x_1}^{x_2} f(x, y(x), y'(x))\,\mathrm{d}x, \qquad (*)$$

among all functions such that $y(x_1) = y_1$ and $y(x_2) = y_2$. This quantity $(*)$ is called a **functional**, that is, a function on the space of functions[1].

[1] In standard calculus, we usually consider functions which map numbers → numbers. In the calculus of variations, we consider *functional* which map functions → numbers. An intuitive example is that of arc-length, which takes some curve and gives back a number corresponding to its length.

In this article, we are going to study the *calculus of variations*, in which we will be finding extrema (minima, maxima, and saddle points) of functionals. We will start by looking back at and extending our knowledge of extrema in finite dimensions, before using this knowledge to approach problems involving functionals.

Notation. Throughout this article, we will frequently refer to different spaces of functions. We will let $C(\mathbb{R})$ be the space of continuous functions on $\mathbb{R}$, and $C^k(\mathbb{R})$ be the space of functions with continuous kth derivative. We will also write $C_{(}\alpha, \beta)^k(\mathbb{R})$ for the subset of functions f with $f(\alpha) = f(\beta)$.

§2 Calculus for Functions on $\mathbb{R}^n$

For this section, we are going to consider a function $f \in C^2(\mathbb{R}^n)$, that is, $f : \mathbb{R}^n \to \mathbb{R}$, with continuous second derivative.

§2.1 Stationary Points

Recall the definition of a stationary point for a function $f \in C^2(\mathbb{R}^n)$.

Definition 2.1 (Stationary Point)
We say that $a \in \mathbb{R}^n$ is a **stationary point** of f if $\nabla f(a) = 0$.

If a is a stationary point of f, then we can look at the Taylor expansion of f around a to determine the nature of the stationary point. For x near a, usiing the summation convention we have

$$f(x) = f(a) + \cancel{(x-a)\cdot\nabla f(a)} + \frac{1}{2}(x_i - a_i)(x_j - a_j)\,\partial^2_{ij} f\big|_a + O(|x-a|^2),$$

where $\partial_i f = \partial f/\partial x_i$.

Now let $H_{ij} = \partial^2_{ij} f$ be the Hessian matrix, which we note is symmetric. If we shift the origin such that $a = 0$ and write $H = H(0)$, we can diagonalize this matrix to get

$$H' = R^T H(0) R = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}.$$

Then we have

$$f(x') - f(0) = \frac{1}{2}\sum_{i=1}^{n} \lambda_i (x_i')^2 + O(|x'|^2).$$

The behavior then depends on if the eigenvalues of $H(0)$ are positive or negative:

(i) If all $\lambda_i > 0$, then $f(x') > f(0)$ in all directions, and we have a local minimum.

(ii) If all $\lambda_i < 0$, then $f(x') < f(0)$ in all directions, and we have a local maximum.

(iii) If some of the eigenvalues are positive and some are negative, then $f(x')$ is increasing in some directions and is decreasing in others, and we have a saddle point.

(iv) If some (or all) of the eigenvalues are 0, then we need to consider higher order derivatives in the Taylor expansion.

In the special case where $n = 2$, then $\det H = \lambda_1 \lambda_2$ and $\operatorname{tr} H = \lambda_1 + \lambda_2$. From this we obtain the conditions:

- If $\det > 0, \operatorname{tr} > 0$ we have a local minimum.
- If $\det > 0, \operatorname{tr} < 0$ we have a local maximum.
- If $\det < 0$ we have a saddle point.
- If $\det = 0$, then we need to look at higher derivatives.

Remark. If a function $f : D \to \mathbb{R}$ is defined on some domain D, then the minimum or maximum could be obtained on the boundary of D, where we may not have $\nabla f = 0$.

Example 2.2 (Finding and Classifying Stationary Points)

Consider the function $f(x, y) = x^3 + y^3 - 3xy$. Then at a stationary point we have

$$\nabla f = \begin{pmatrix} 3x^2 - 3y \\ 3y^2 - 3x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

These imply that at a stationary point, $y^4 = y$. Thus we have the points $(0, 0)$ and $(1, 1)$. We can write down the Hessian as

$$H = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}.$$

At $(0, 0)$, we have $\det H = -9 < 0$, so f has a saddle point there. At $(1, 1)$, we have $\det H = 27 > 0$, and $\operatorname{tr} H = 12 > 0$, so f has a local minimum there.

§2.2 Constraints and Lagrange Multipliers

Consider the following problem, and two natural ways to solve it.

Example 2.3

Find the circle centered at $(0, 0)$ with smallest radius such that the circle intersects the parabola $y = x^2 - 1$.

Direct Solution. One way to solve this problem is to substitute the constraint $y = x^2 - 1$ into the expression we are trying to minimize.

$$r^2 = x^2 + y^2 = x^2 + (x^2 - 1)^2 = x^4 - x^2 + 1.$$

To have a stationary point, the derivative of the RHS must be zero, so $4x^3 - 2x = 0$. This gives us two solutions: $x = \pm 1/\sqrt{2}, y = -1/2$ and $x = 0, y = -1$. These give the two possible radii of $\sqrt{3}/2$ and 1, of which the first is the smallest. □

While this solution works, it's not hard to imagine that we may not be able to say solve for y. An alternate solution is given below.

Lagrange Multipliers Solution. Define the function $h(x, y, \lambda) = f(x, y) - \lambda g(x, y)$, where $f(x, y) = x^2 + y^2 = r^2$, $g(x, y) = 0$ is the constraint, and λ is the Lagrange multiplier. So $x^2 + y^2 - \lambda(y - x^2 + 1)$. We are going to extremize this over 3 variables

> with no constraints.
>
> $$\frac{\partial h}{\partial x} = 2x + 2\lambda x = 0,$$
> $$\frac{\partial h}{\partial y} = 2y - \lambda = 0,$$
> $$\frac{\partial h}{\partial \lambda} = y - x^2 + 1 = 0,$$
>
> where the third condition is our constraint. From these we get either $x = 0, y = -1$ and $f = 1, \lambda = -2$ or $y = -1/2, x = \pm 1/\sqrt{2}$ and $f = 3/4, \lambda = -1$, giving the radius $\sqrt{3}/2$. □

So why does the second solution work? Well suppose that we wanted to minimize a function f subject to $g = 0$. Then ∇g is perpendicular to $g = 0$. Also, ∇f is perpendicular to $f(x) = c$, for any constant c. So at the extremum, we must have that ∇f and ∇g are parallel, that is,

$$\nabla f - \lambda \nabla g = 0,$$

for some λ. Thus it suffices to consider the extrema of $h = f - \lambda g$.

This method generalizes to multiple variables and multiple constraints. For example, if we wanted to extremize $f : \mathbb{R}^n \to \mathbb{R}$, subject to $g_\alpha(x) = 0$, where $g_\alpha : \mathbb{R}^n \to \mathbb{R}$ and $\alpha = 1, \dots, k$, we would define

$$h(x_1, \dots, x_n, \lambda_1, \dots, \lambda_k) = f - \sum_{\alpha=1}^{k} \lambda_\alpha g_\alpha,$$

and would extremize this function with respect to the $n + k$ variables.