Examples & Counterexamples in Group Theory

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This is a collection of groups that have specific interesting properties, such as being a counterexample to some non-obvious statements about groups. Most of these are well known.

A group isomorphic to every non-trivial subgroup. $\mathbb{Z}$.

A group of order n which has no subgroup of k, where $k \mid n$. A_5, since it is simple and thus has no subgroup of index 2. Another example is A_4 (which is the smallest such group), but it's a little harder to show that it works.

Two non-isomorphic groups with the same order type. $C_4 \times C_4$ and $C_2 \times Q_8$. Both have 1 element of order 1, 3 elements of order 2, and 12 elements of order 4. If this property holds for two abelian groups, then they are isomorphic.

A non-abelian group with all non-identity elements of order p. The group of upper triangular 3×3 matrices over $\mathbb{Z}/p\mathbb{Z}$ with 1s on the diagonal.

A non-abelian group of order p^3. Same as above.

An infinite group whose proper subgroups are all finite. The group $\{k/2^n : k, n \in \mathbb{N}, k < 2^n\}$ with addition modulo 1. This is the Prüfer 2-group.

A group G with $N \trianglelefteq G$ and $H \trianglelefteq N$ such that $H \ntrianglelefteq G$. In D_8, we have $\langle s \rangle \trianglelefteq \langle r^2, s \rangle \trianglelefteq \langle r, s \rangle = D_4$, but $\langle s \rangle \ntrianglelefteq D_4$.

A group in which every group generated by n elements has a surjective homomorphism to it. The free group with a basis of n elements.

An infinite group with every non-identity element of order 2. $C_2 \times C_2 \times C_2 \times \cdots$.

An infinite non-abelian group with every element of finite order. $S_3 \times C_2 \times C_2 \times \cdots$.

A group G with $G \cong G \times G$. $G = C_2 \times C_2 \times C_2 \times \cdots$. We can also get $G \cong G \times G \times G \times \cdots$ in the natural way.

Two non-zero elements of $\mathbb{R}$ that generate a subgroup not isomorphic to $\mathbb{Z}$. 1 and $\sqrt{2}$.

A group with two subgroups whose product is not a subgroup. Consider D_3, with the distinct reflections $s, s' \in D_3$. Then $\{e, s\} \times \{e, s'\} \not\leq D_3$.

A quotient group of a finite group that is not isomorphic to a subgroup. $Q_8/\{-1, 1\}$.

A group isomorphic to its automorphism group. $S_3 \cong \mathrm{Aut}(S_3)$.

A non-abelian infinite group where the set of elements of finite order is a subgroup. $\mathrm{GL}_2(\mathbb{Q})$. We note that this also holds for all abelian groups, and indeed any group where the elements of finite order commute with each other.

A group that is not a semi-direct product. Q_8.

A group G with two isomorphic subgroups H, K where G/H and G/K are not isomorphic. Take $G = C_4 \times C_2$, with $H = \langle(2, 0)\rangle$ and $K = \langle(0, 1)\rangle$. Then $G/H \cong C_2 \times C_2$, and $G/K \cong C_4$.