Special Relativity

Adam Kelly (ak2316)
June 10, 2021

Axioms of Special Relativity

1. *Galilean relativity.* The laws of physics are the same in all inertial reference frames.
2. *Speed of light.* The speed of light in a vacuum is the same in all inertial reference frames.

Lorentz Transformations

In special relativity we think in terms of events: instantaneous point-like occurrences. These are specified by four coordinates, one of time and three of position, like (t, x, y, z). These coordinates will be measured differently in different inertial frames, and to make our axioms hold we need to use a new set of transformation laws.

If we have two inertial frames $\mathcal{F}$ and $\mathcal{F}'$, and $\mathcal{F}'$ is moving at speed v relative to $\mathcal{F}$ in the x direction, then we have

$$\begin{aligned} x' &= \gamma \left(x - \frac{v}{c} ct \right) \\ y' &= y \\ z' &= z \\ ct' &= \gamma \left(ct - \frac{v}{c} x \right), \end{aligned}$$

where

$$\gamma = \sqrt{\frac{1}{1 - v^2/c^2}}.$$

Relativistic Physics

1. *Relativity of simultaneity.* Events with the same t no longer correspond to events with equal t', so what is simultaneous in one frame is not necessarily simultaneous in another.
2. *Causality.* While observers can disagree about the temporal ordering of events, if an event is within the 'light cone' of an event P (within the region of a space-time diagram traced out by light passing through P) then all observers will agree on a causal ordering.
3. *Time Dilation.* Consider a clock sitting stationary at the origin of the frame $\mathcal{F}'$, ticking at intervals of T'. The tick events in frame $\mathcal{F}'$ will occur at $(t'_1, 0), (t'_1 + T', 0), \dots$.

   In the frame $\mathcal{F}$, using the Lorentz transformations, we see that the time interval between ticks is $T = \gamma T'$. So the ticks are longer in the stationary frame.
4. *Length Contraction.* Consider a rod of length L', stationary in the frame $\mathcal{F}'$. The endpoints of the rod are given by $x' = 0$ and $x' = L'$, which are then mapped into $x = vt$ and $x = vt + L'/\gamma$. So in $\mathcal{F}$, the length of the rod is L'/γ, and thus lengths of moving objects are contracted in the direction of motion. To deal with this, we define *proper length* to be the length measured in an objects rest frame.

5. *Composition of Velocities.* Suppose a particle moves with constant velocity u' in frame $\mathcal{F}'$, which moves with velocity v relative to $\mathcal{F}$. We want to find its velocity u in the frame $\mathcal{F}$.

   In $\mathcal{F}'$, for the particle we have $x' = u't'$. Substituting this into the Lorentz transformation laws, we have
$$u = \frac{x}{t} = \frac{\gamma(x' + vt')}{\gamma(t' + vx'/c^2)} = \frac{u' + v}{1 + u'v/c}.$$

6. *Newtonian Limit.* When v/c is very small, the Lorentz transformations approximate the Galilean transformations that we use in Newtonian mechanics.

Geometry of Minkowski Space

Consider two events P_1 and P_2 have coordinates (t_1, x_1) and (t_2, x_2) in the frame $\mathcal{F}$. These events are separated by $\Delta t = t_1 - t_2$ in time and $\Delta x = x_1 - x_2$.

We define the *invariant interval* between P_1 and P_2 to be
$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2.$$

We say it is invariant because it is the same in all inertial reference frames, that is, it is invariant under Lorentz transformations[1].

It is possible for Δs^2 to be either positive or negative. If it is positive, we say the events are *timelike* separated, and if it is negative we say they are *spacelike* separated, and if it is zero, we say they are *lightlike* separated. Events that are in each other's light-cones are timelike, and can influence one another.

4-Vectors

We can view Minkowski space as a vector space equipped with the *Minkowski inner product*[2]. The coordinates of some event P in the frame $\mathcal{F}$ can be written as a *4-vector*
$$X = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},$$

and then the Minkowski inner product is given by
$$X \cdot Y = X^T \eta Y, \qquad \text{where} \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Taking $X \cdot X$ gives the invariant interval between the origin and P, and is known as the *Minkowski metric*[3].

[1] Feel free to check this.

[2] This is not an inner product in the normal sense since it's not positive definite

[3] Again, this isn't actually a metric in the normal sense since it's not positive definite

The Lorentz Group

Using 4-vectors, we can view Lorentz transformations as linear transformations from the coordinates of one inertial frame $\mathcal{F}$ to another $\mathcal{F}'$. This would be represented by a 4×4 matrix Λ, with

$$X' = \Lambda X.$$

Since such transformations must preserve the invariant interval, they must preserve the Minkowski inner product[4] and hence satisfy

$$\Lambda^T \eta \Lambda = \eta.$$

The group of all such matrices is the *Lorentz group*, and is generated by

$$\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & \mathrm{R} & \\ 0 & & & \end{array}\right) \qquad \text{and} \qquad \left(\begin{array}{cc|cc} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right),$$

where the first corresponds to a rotation of space, leaving time intact (so we need R to be orthogonal), and the second corresponds to a Lorentz boost along the x direction.

Relativistic Kinematics

1. *Proper Time.* To do kinematics, it helps to have a consistant notion of time. We do this by defining *proper time* τ, which is the time experienced by a particle in its own reference frame. It is given by

$$\Delta\tau = \frac{\Delta s}{c},$$

in all reference frames.

2. *Relating Proper Time to Measured Time.* By considering small changes, if $\mathbf{u}$ is the velocity of the particle in the frame $\mathcal{F}$, we get that

$$d\tau = \frac{1}{c} ds = \frac{1}{c}\sqrt{c^2 dt^2 - |d\mathbf{x}|^2} = \sqrt{1 - u^2/c^2}\, dt, \quad \text{so} \quad \frac{dt}{d\tau} = \gamma_u.$$

We can then get the total time experienced by a particle as

$$T = \int d\tau = \int \frac{1}{\gamma_u} dt.$$

3. 4-*Position.* Using proper time, we can parametrise the trajectory of a particle using a 4-vector

$$X(\tau) = \begin{pmatrix} ct(\tau) \\ \mathbf{x}(\tau) \end{pmatrix}.$$

4. 4-*Velocity.* Then we can define the 4-*velocity* as

$$U = \frac{dX}{d\tau} = \begin{pmatrix} c\,dt/d\tau \\ d\mathbf{x}/d\tau \end{pmatrix} \qquad \text{that is,} \qquad U = \frac{dt}{d\tau}\begin{pmatrix} c \\ \mathbf{u} \end{pmatrix} = \gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix},$$

where $\mathbf{u} = d\mathbf{x}/dt$. Since this is a 4-vector, it transforms like $U' = \Lambda U$.

[4] These are the analogue of orthogonal matrices for Minkowski space

5. 4-*Momentum.* For a particle of mass m, the 4-*momentum* is defined to be

$$P = mU = \begin{pmatrix} mc\gamma_u \\ m\gamma_u \mathbf{u} \end{pmatrix}.$$

The spacial components of P give us the *relativistic* 3-*momentum*, $\mathbf{p} = m\gamma_u \mathbf{u}$. 4-momentum is conserved in the absence of external forces, and for a system of particles the total 4-momentum is the sum of the 4-momenta of the particles.

6. *Relativistic Energy.* We define the *relativistic energy* of a particle to be $E = P^0 c$, so that

$$E = m\gamma c^2 = mc^2 + \frac{1}{2}m|\mathbf{u}|^2 + \cdots,$$

and for a stationary particle we have $E = mc^2$.

Since we can calculate the Lorentz invariant quantity $P \cdot P$ in the particles rest frame, we have

$$P \cdot P = \frac{E^2}{c^2} - |\mathbf{p}|^2 = m^2c^2,$$

so generally we have

$$E^2 = |\mathbf{p}|^2c^2 + m^2c^4.$$

Relativistic energy is a conserved quantity, which includes mass (just another form of energy, it's not conserved separately).

7. *Massless Particles.* For particles that have zero mass (like photons), they can still have momentum and energy. However, $P \cdot P = 0$, and we can't define proper time for the particles. We still have $E^2 = |\mathbf{p}|^2c^2$, and thus

$$P = \frac{E}{c}\begin{pmatrix} 1 \\ \mathbf{n} \end{pmatrix},$$

where $\mathbf{n}$ is a unit vector in the direction of propagation. We also have $E = hc/\lambda = hf$, where h is Planck's constant, λ is the wavelength, and f is the frequency of the photon/particle.

Rapidities

Consider a 2×2 matrix corresponding to a Lorentz boost by v in the x direction. Then if $\beta = v/c$, we have

$$\Lambda[\beta] = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}, \quad \text{and the composition law} \quad \Lambda[\beta_1]\Lambda[\beta_2] = \Lambda\left[\frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}\right],$$

exactly our composition law from before.

A slightly nicer way to deal with this is with *rapidities.* We define the *rapidity* of a Lorentz boost ϕ such that

$$\beta = \tanh\phi, \quad \gamma = \cosh\phi, \quad \gamma\beta = \sinh\phi.$$

Then

$$\Lambda[\beta] = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} = \Lambda(\phi), \quad \text{and also} \quad \Lambda(\phi_1)\Lambda(\phi_2) = \Lambda(\phi_1 + \phi_2),$$

which is much nicer. This also shows that Lorentz boosts correspond to hyperbolic rotations in spacetime.