ALGEBRAIC TOPOLOGY

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1. Covering Spaces

1.1. **Definitions and Lifting.** We now start to develop some machinery which will allow us to compute fundamental groups.

Definition 1.1 (Evenly Covered Set)

Suppose $p : \hat{X} \to X$ is continuous, we say that $U \subseteq X$ is evenly covered if $p^{-1}(U) = \sqcup_\alpha V_\alpha$, where $p|_{v_\alpha} : V_\alpha \to U$ is a homeomorphism.

Definition 1.2 (Covering Map)

A map $p : \hat{X} \to X$ is a covering map if for all $x \in X$, there exists an open neighbourhood U_x which is evenly covered. In this case, we call $\hat{X}$ a covering space for X.

Definition 1.3 (Lift)

Suppose $p : \hat{X} \to X$ is a covering map, $f : Z \to X$ continuous. Then we say that $\hat{f} : Z \to \hat{X}$ is a lift of f if $p \circ \hat{f} = f$, that is, DIAGRAM commutes.

Lemma 1.4 (Lebesgue Covering)

Suppose X is a compact metric space, $\{U_\alpha\}_\alpha$ is an open cover of X. Then there exists $\delta > 0$ such that for all $x \in X, B_\delta(x) \subseteq U_\alpha$ for some α.

Proof. Given $x \in X$, let $\alpha(x)$ and $\delta(x) > 0$ be such that $B_{2\delta(x)}(x) \subseteq U_{\alpha(x)}$. Then $\left\{B_{\delta(x)}\right\}_{x \in X}$ is an open cover of X. Therefore, by compactness there exists a finite subcover $\left\{B_{\delta(x_i)}(x_i)\right\}_{i=1}^n$. Let $\delta = \min\{\delta(x_1), \ldots, \delta(x_n)\}$. Then for all $y \in X, y \in B_{\delta(x_i)}(x_i)$ for all i. Then

$$B_\delta(y) \subseteq B_{2\delta(x_i)}(x_i) \subseteq U_{\alpha(x_i)}$$

□

Notation. We say a path γ with $\gamma(0) = x_0$ has the (unique) lifting property if for all $\hat{x}_0 \in p^{-1}(x_0)$, there exists a (unique) lift $\hat{\gamma}$ of γ with $\hat{\gamma}(0) = \hat{x}_0$.

Lemma 1.5

If $f : Z \to U, Z$ connected, $\operatorname{im}(f) \subseteq U$, where U is evenly covered, then γ has the unique lifting property.

Proof. Since U is evenly covered, $p^{-1}(U) = \sqcup_\alpha V_\alpha$. Then $\hat{x} \in V_{\alpha_0}$ for some α_0. Then $p' = (p \mid v_{\alpha_0})^{-1} : U \to \hat{X}$ is continuous, with $p'(x_0) = \hat{x}_0$, so $\hat{f} = p' \circ f$ is a lift of f.

For uniqueness, notice that $p^{-1}(U) = U_{\alpha_0} \sqcup (\sqcup_{\alpha \neq \alpha_0} V_\alpha)$, which disconnects $p^{-1}(U)$, and as $\operatorname{im}(f)$ is connected, $\operatorname{im}(\hat{f}) \subseteq V_{\alpha_0}$. But p' above is a homeomorphism, so $\hat{\gamma}$ is unique. □

Lemma 1.6

Suppose $\gamma : [a, b] \to X$ with $a' \in [a, b]$, if $\gamma|_{[a,a']}$ has the ULP at a and $\gamma|_{[a',b]}$ has the ULP at a', then γ has the ULP at a.

Proof. We have a lift $\hat{\gamma}_1 : [a, a'] \to \hat{X}$ of $\gamma|_{[a,a']}$ at a, and a lift $\hat{\gamma}_2 : [a', b] \to \hat{X}$ of $\gamma|_{[a',b]}$ at a', such that $\hat{\gamma}_1(a') = \hat{\gamma}_2(a')$. So $\hat{\gamma}_1\hat{\gamma}_2$ is a lift of γ at a.

For uniqueness, suppose $\hat{\eta}$ is any other lift. Then $\hat{\eta}|_{[a,a']}$ is a lift of $\gamma|_{[a',a]}$, so $\left.\hat{n}\right|_{[a,a']} = \hat{\gamma}_1$. This means that □

Theorem 1.7 (Path Lifting)

Any $\gamma : I \to X$ has the ULP.

Proof. $p : \hat{X} \to X$ is a covering map, so every $x \in X$ has an evenly covered neighbourhood U_x. Then $\{U_x \mid x \in X\}$, so $\left\{\gamma^{-1}(U_x) \mid x \in X\right\}$ is an open cover of I. Thus, by the Lebesgue covering lemma, there exists $\delta > 0$ such that $B_\delta(t) \subseteq \gamma^{-1}\left(U_{x(t)}\right)$ for any t

Choose n such that $1/n < \delta, a_i = i/n \in I$. Then $[a_i, a_{i+1}] \subseteq B_\delta(a_i)$, so $\gamma([a_i, a_{i+1}]) \subseteq U_{x_i}$, where $a_i = \gamma(a_i)$. As U_{x_i} is evenly covered, $\gamma|_{[a_i,a_{i+1}]}$ has the ULP at a_i. By induction and the previous lemma, γ has the ULP. □

Theorem 1.8 (Homotopy Lifting)

Suppose $p : \hat{X} \to X$ is a covering map, $H : I \times I \to X$ is a homotopy, then H has the lifting property at $(0, 0)$.

Proof. Suppose $\{U_x \mid x \in X\}$ is an open cover of X by evenly covered neighbourhoods. Since l^2 is compact, by the Lebesgue covering lemma, there exists $\delta > 0$ such that $B_\delta(v) \subseteq H^{-1}\left(U_{H(v)}\right)$ for each $v \in R^2$.

Choose n such that $\sqrt{2}/n < \delta$. Then divide R^2 into squares with side lengths $1/n$. Enumerate them $A_1, A_2, \ldots, A_{n^2}$, starting from the bottom left and going right then up. Label the bottom left corner of A_i as v_i. Now note that $H(A_i) \subseteq H(B_\delta(v_i)) \subseteq U_{x_i}$ is evenly covered. Thus, H_{A_i} has the ULP at v_i, as l^2 is connected. Let $B_k = \bigcup_{i=1}^k A_i$

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We will prove by induction that $H|_{B_k}$ has LP at $(0,0)$. For $k = 1, B_1 = A_1$, so we are done. Now suppose $H|_{B_k}$ has a lift $\hat{H} : B_k \to X$ with $\hat{H}_k(0,0) = \hat{X}$. Now as $H|_{A_k}$ has the lifting property at v_{k+1}. So choose a lift $\hat{h}_k : A_{k+1} \to \hat{X}$ with $\hat{h}_k(v_{k+1}) = \hat{H}^k(v_{k+1})$.

Now note that $B_k \cap A_{k+1}$ is either one or two edges of A_{k+1}, both coming from v_{k+1}. By uniqueness of path lifting, $\hat{H}_k\Big|_{A_{k+1}\cap B_k} = \hat{h}_k \mid A_{k+1} \cap B_k$, so by the gluing lemma we have a well defined lift $\hat{H}_{k+1}$ of H on B_{k+1}. ☐

Proposition 1.9

Suppose $\gamma_0, \gamma_1 \in \Omega(X, x_0, x_1), \gamma_0 \sim_e \gamma_1$. Suppose $\hat{\gamma}_i$ is a lift of $\hat{X}$ with $\hat{\gamma}_i(0) = \hat{x}_0$. Then $\hat{\gamma}_0 \sim_e \hat{\gamma}_1$. In particular, $\hat{\gamma}_0(1) = \hat{\gamma}_1(1)$.

Proof. Suppose $H : R^2 \to X$ is a homotopy between γ_0 and γ_1.

DIAGRAM

By homotopy lifting, we have a lift $\hat{H} : I^2 \to \hat{X}$ with $\hat{H}(0,0) = \hat{x}_0$. Let $\alpha_i(t) = \hat{H}(t,i)$ and $\beta_i(t) = \hat{H}(i,t)$. By uniqueness of path lifting.

DIAGRAM

That is, $\hat{\gamma}_0 \sim e\hat{\gamma}_1$ via $\hat{H}$. ☐

Corollary 1.10

$p_* : \pi_1(\hat{X}, \hat{x}_0) \to \pi_1(X, x_0)$ is injective.

Proof.

$$\begin{aligned} p_*[\gamma_0] = p_*[\gamma_1] &\implies p \circ \gamma_0 \sim_e p \circ \gamma_1 \\ &\implies p \circ \gamma_0 \sim_e \widehat{\rho \circ \gamma_1} \\ &\implies \gamma_0 \sim_e \gamma_1 \end{aligned}$$

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