

# Differential Equations

ADAM KELLY

Michaelmas 2020, Updated October 15, 2020

None of the notes here have been reviewed at all, and are just exactly what was taken down live in the lectures. I would turn around now and come back in a few days, when I have gone back, cleaned things up, fixed explanations and added some structure.

This set of notes is a work-in-progress account of the course ‘Differential Equations’, originally lectured by Dr. John Taylor in Michaelmas 2020 at Cambridge. These notes are not a transcription of the lectures, but they do roughly follow what was lectured (in content and in structure).

These notes are my own view of what was taught, and should be somewhat of a superset of what was actually taught. I frequently provide different explanations, proofs, examples, and so on in areas where I feel they are helpful. Because of this, this work is likely to contain errors, which you may assume are my own. If you spot any or have any other feedback, I can be contacted at [ak2316@cam.ac.uk](mailto:ak2316@cam.ac.uk).

## Contents

<b>1 Calculus</b>	<b>2</b>
1.1 Derivatives and Limits . . . . .	2
1.2 Rules for Differentiation . . . . .	3
1.3 Order of Magnitude . . . . .	4

## §1 Calculus

### §1.1 Derivatives and Limits

We will consider limits in an informal capacity.

#### Definition 1.1 (Limits)

We say that the **limit**  $\lim_{x \rightarrow x_0} f(x) = A$  if  $f(x)$  can be made arbitrarily close  $A$  by making  $x$  sufficiently close to  $x_0$ .

We can define limits from two side:

- *Left Limits.*  $\lim_{x \rightarrow x_0^-} f(x) = A$ , requiring  $x < x_0$ .
- *Right Limits.*  $\lim_{x \rightarrow x_0^+} f(x) = A$ , requiring  $x > x_0$ .

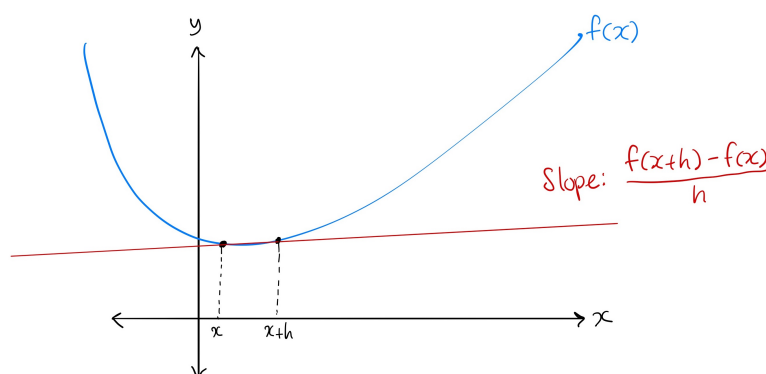
Using the definition of a limit, we can define what a *derivative* is.

#### Definition 1.2 (Derivatives)

We define the **derivative** of a function  $f(x)$  with respect to it's argument  $x$  to be the function

$$\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

*Pictorially*



For the derivative to exist at a point  $x$ , we require

$$\lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h}.$$

#### Example 1.3

The function  $f(x) = |x|$  is not differentiable at  $x = 0$ , but it is everywhere else.

There are a number of ways of writing derivatives.

- *Leibniz Notation.*  $\frac{df}{dx}$
- *Lagrange Notation.*  $f'(x)$

- *Newton Notation.*  $\dot{f}(x)$  – This is typically only used for the derivative with respect to time.

For sufficiently smooth functions (the derivative exists at each step), we can define ‘higher order derivatives’ recursively.

#### Definition 1.4 (Higher-Order Derivatives)

We define the notion of higher order derivatives recursively with

$$\frac{d^0 f}{dx^0} = f \quad \text{and} \quad \frac{d}{dx} \left( \frac{d^n f}{dx^n} \right) = \frac{d^{n+1} f}{dx^{n+1}}.$$

In Lagrange notation, we write  $\frac{d^n f}{dx^n} = f^{(n)}(x)$ .

## §1.2 Rules for Differentiation

In this section, we review some rules for computing derivatives that should be familiar to you. These can be proven directly from the definitions.

#### Theorem 1.5 (The Chain Rule)

Consider  $f(x) = F(g(x))$ . Then

$$f'(x) = F'(g(x)) \cdot g'(x) = \frac{dF}{dg} \cdot \frac{dg}{dx}.$$

#### Theorem 1.6 (The Product Rule)

Consider  $f(x) = u(x)v(x)$ . Then

$$\frac{df}{dx} = u'(x)v(x) + u(x)v'(x).$$

We can generalise the product rule to higher order derivatives. The generalisation bears a large resemblance to the binomial theorem.

#### Theorem 1.7 (Leibniz Rule)

Consider  $f(x) = u(x)v(x)$ . Then

$$f^{(n)}(x) = \sum_{k=0}^n \binom{n}{k} u^{(k)}(x) v^{(n-k)}(x).$$

*Proof Sketch.* Induction on  $n$ . □

For example, for  $n = 3$  we have

$$f'''(x) = u'''v + 3u''v' + 3u'v'' + uv'''.$$

### §1.3 Order of Magnitude

Suppose you wished to compare the ‘size’ of two functions, in the vicinity of some specific points (that is, you want to study its behaviour locally, not globally). We can do this by utilizing two concepts: Little- $o$  and Big- $O$ .

#### Definition 1.8 (Little- $o$ )

We say that  $f(x) = o(g(x))$  as  $x \rightarrow x_0$  if

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0.$$

Informally, we can think ‘ $f(x)$  is much smaller than  $g(x)$  around  $x_0$ ’.

#### Example 1.9

$x^2 = o(x)$  as  $x \rightarrow 0$ , because

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = 0.$$

#### Definition 1.10 (Big- $O$ )

We say that  $f(x) = O(g(x))$  as  $x \rightarrow x_0$  if:

- (i)  $x_0$  is finite, and  $\exists m, \delta \in \mathbb{R}^+$  such that  $\forall x$  where  $|x - x_0| < \delta$ , we have  $|f(x)| \leq M|g(x)|$ .
- (ii)  $x_0$  is infinite, and  $\exists$  constants  $m, x_1 \in \mathbb{R}^+$  such that  $\forall x > x_1$ ,  $|f(x)| \leq M|g(x)|$ .

#### Example 1.11

$x^2 = O(x)$  as  $x \rightarrow 0$ . To see this, take  $\delta = 1$  and  $M = 1$ . We also have  $x \neq O(x^2)$  as  $x \rightarrow 0$ .

By convention, we choose the most restrictive bound.

Consider the equation of the tangent to  $f(x)$  at  $x = x_0$ . The slope of this tangent is given by

$$\left. \frac{df}{dx} \right|_{x=x_0} = \frac{f(x_0 + h) - f(x_0)}{h} + \frac{o(h)}{h}, \quad \text{as } h \rightarrow 0.$$

This implies that

$$f(x_0 + h) = f(x_0) + f'(x_0)h + o(h), \quad \text{as } h \rightarrow 0.$$