Geometry

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This article constitutes my notes for the 'Geometry' course, held in Lent 2022 at Cambridge. These notes are *not a transcription of the lectures*, and differ significantly in quite a few areas. Still, all lectured material should be covered.

Contents

§1 Topological and Smooth Surfaces

§1.1 Topological Surfaces

We will begin immediately with a definition that will occupy us for some time.

Definition 1.1 (Topological Surface)

A topological surface is a topological space Σ such that

(i) Each $p \in \Sigma$ has an open neighbourhood U with $p \in U$ such that U is homeomorphic to $\mathbb{R}^2$, with its usual Euclidean topology.

(ii) Σ is Hausdorff and second countable.

Recall that a space X is Hausdorff if for $p \neq q$ in X there exists disjoint open sets $p \in U$ and $q \in V$ in X, and that a space is second countable it's topology has a countable basis. In some ways, the real nature of topological spaces comes from the condition (a), and the condition (b) is really there for technical honesty.

§1.2 Examples of Topological Surfaces

Let's now take some to consider some examples of topological surfaces.

Example 1.2 ($\mathbb{R}^2$)

The plane $\mathbb{R}^2$ is a topological surface.

Example 1.3 (Subsets of $\mathbb{R}^2$)
Any open subset of $\mathbb{R}^2$ is a topological surface. For example

(i) $\mathbb{R}^2 \backslash \{0\}$ is a topological surface;

(ii) Let $Z = \{(0,0)\} \cup \{(1, 1/n) \mid n \in \mathbb{N}\}$, then $\mathbb{R}^2 \backslash Z$ is a topological surface.

Example 1.4 (Graphs)
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then the **graph**

$$\Gamma_f = \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\} \subseteq \mathbb{R}^3 \text{ (subspace topology)}.$$

Recall that if X and Y are topological spaces, the product topology on $X \times Y$ has basis open sets $U \times V$ with $U \subseteq X$ and $V \subseteq Y$ both open sets.

It has the feature that $f : Z \to X \times Y$ is continuous if and only if $\pi_x \circ f : Z \to X$ and $\pi_y \circ f : Z \to Y$ are continuous.

So if $\Gamma_f \subseteq X \times Y$ and $f : X \to Y$ is continuous then Γ_f is homeomorphic to X, with the map $s : x \mapsto (x, f(x))$, so that $\pi|_{\Gamma_f}$ and s are inverse homeomorphisms.

So $\Gamma_f \cong \mathbb{R}^2$ for *any* continuous $f : \mathbb{R}^2 \to \mathbb{R}$, and Γ_f is a topological surface.

As a note, the topological surface Γ_f is independent of f. Later on as we develop more tools in geometry we will be able to better reflect the structure of the function f.

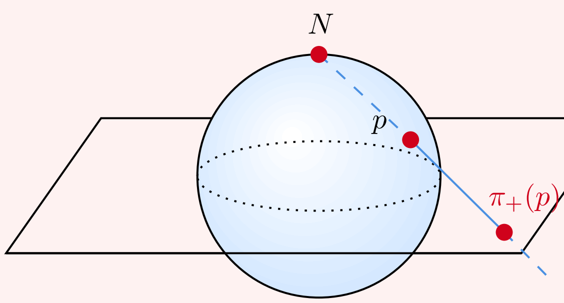
Example 1.5 (Stereographic Projection)
Consider the sphere

$$S^2 = \{(x, y, t) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

We can consider the stereographic projection

$$\begin{aligned} \pi_+ : S^2 \backslash \{(0,0,1)\} &\to \mathbb{R}^2(z = 0) \subseteq \mathbb{R}^3 \\ (x, y, t) &\mapsto \left(\frac{x}{1-t}, \frac{y}{1-t} \right). \end{aligned}$$

Such a projection is shown below.



Note that π_+ is continuous and has an inverse

$$(u,v) \mapsto \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right).$$

So π_+ is a continuous bijection with continuous inverse and hence a homeomorphism.

Of course we could also have projected from the south pole, to get a homeomorphism π_- from $S^2 \backslash \{0,0,-1\}$ to $\mathbb{R}^2$, so indeed every point lies in an open set which is homeomorphic through either π_+ or π_- to $\mathbb{R}^2$. So S^2 is a topological surface.

Remark. S^2 is compact as a topological space, since it is a closed bounded set in $\mathbb{R}^3$.

Example 1.6 (Real Projective Plane)

The group $\mathbb{Z}/2\mathbb{Z}$ acts on S^2 by homeomorphisms via the **antipodal map** $a : S^2 \to S^2$ with

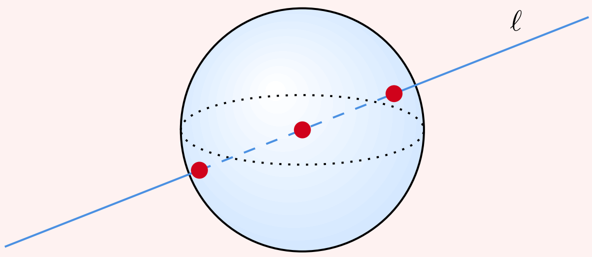
$$a(x,y,t) = (-x,-y,-t).$$

That is, there exists a homeomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Homeo}(S^2)$ sending the non-identity element to a.

The **real projective plane** is the quotient space of S^2 given by identifying every point with its antipodal image: $\mathbb{RP}^2 = S^2/(\mathbb{Z}/2\mathbb{Z}) = S^2/\sim$ with $x \sim a(x)$.

Note that $\sim$ is the equivalence relation of belonging to the same orbit under the given action.

As a set, $\mathbb{RP}^2$ is naturally in bijection with the set of straight lines in $\mathbb{R}^3$ through the origin, with the bijection given by mapping lines with the identified points on the sphere that they pass through.



We can also check that $\mathbb{RP}^2$ is a topological surface.

We must first check that it is Hausdorff. Recall that if X is a space and $q : X \to Y$ is a quotient map, then $V \in Y$ is open if and only if $q^{-1}(V) \in X$ is open.

If $[p] \neq [q] \in \mathbb{RP}^2$ then $\pm p$ and $\pm q \in S^2$ are distinct antipodal points. We can then take small open discs about these in S^2