PRINCIPLES OF QUANTUM MECHANICS

ADAM KELLY – MATHEMATICAL TRIPOS PART II

1. Uncertainty Principle

If two operators do not commute, we cannot expect to measure both exactly. This is quantified in an *uncertainty principle*: Let A, B be Hermitian. Then taking $C = A + i\lambda B, \lambda \in \mathbb{R}$,

$$C^\dagger C = A^2 + \lambda^2 B^2 + \lambda i[A, B].$$

The first three are Hermitian, so $i[A, B]$ is also Hermitian. We have

$$\begin{aligned} 0 \leqslant \langle C\psi, C\psi\rangle &= \left\langle \psi | C^\dagger C | \psi \right\rangle \\ &= \left\langle A^2 \right\rangle_\psi + \lambda^2 \left\langle B^2 \right\rangle_\psi + \lambda \langle i[A, B]\rangle_\psi. \end{aligned}$$

For this to always be nonnegative, can have at most one real root, so discriminant gives

$$\left\langle A^2 \right\rangle_\psi \left\langle B^2 \right\rangle_\psi \geqslant \frac{1}{4} \left(\langle i[A, B]\rangle_\psi \right)^2.$$

This works for any A, B, so if we apply it to $\tilde{A} = A - \langle A\rangle_\psi$ and $\tilde{B} = B - \langle B\rangle_\psi$, we find $[\tilde{A}, \tilde{B}] = [A, B]$ and hence

$$(\Delta A)_\psi (\Delta B)_\psi \geqslant \frac{1}{2} \left| \langle [A, B]\rangle_\psi \right|.$$

Most famous is Heisenberg's uncertainty principle from applying this to (38),

$$(\Delta x)(\Delta p) \geqslant \frac{1}{2}\hbar$$

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