Graph Theory – Main Results

Mathematical Tripos Part II

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1 Planar Graphs

Definition 1.1 (Face)**.** Let G be a planar graph (with an associated planar diagram). Then one of the finitely many connected components of $\mathbb{R}^2 \backslash G$ is a *face*.

Theorem 1.2 (Euler)**.** *Let G be a connected planar graph where the planar drawing has F faces. Then $V - E + F = 2$.*

Proof. Induct on E. If $E = 0$, we have $V = F = 1$. Now if G is acyclic, then G is a tree and $V = E+1$, and $F = 1$ so we are done. Now if G has a cycle, remove an edge in the cycle then we still have a connected graph but have reduced the number of faces by 1, and then we are done by induction. □

Definition 1.3 (Subdivision)**.** A *subdivision* of a graph G is a new graph where some of the edges are replaced by (disjoint) paths.

Theorem 1.4 (Kuratowski)**.** *G is planar if and only if it contains no subdivision of $K_{3,3}$ or K_5.*

2 Hall's Theorem

Definition 2.1 (Matching)**.** Let $G = (X \sqcup Y, E)$ be a bipartite graph. A *matching* from X to Y is a set of edges $E' = \{xy_x \mid x \in X, y_x \in Y\} \subseteq E$ such that the map $x \mapsto y_x$ is injective.

Theorem 2.2 (Hall)**.** *Let $G = (X \sqcup Y, E)$ be a bipartite graph. There exists a matching from X to Y if and only if* Hall's criterion *holds: $|A| \leq |N(A)|$ for all $A \subseteq X$.*

Proof. The condition is clearly needed, so for sufficient, we induct on $|X|$. If there's $A \subsetneq X$ such that $|A| = |N(A)|$, then Hall's condition holds on $A \sqcup N(A)$ and $X \backslash \sqcup Y \backslash N(A)$ so we can take matchings on both of these by induction. If no such A exists, we can just pick any edge xy and add it to our matching, and Hall still holds on the remaining graph so we can complete our matching by induction. □

3 Connectivity

Definition 3.1 (Vertex Connectivity)**.** The *vertex connectivity parameter* κ of G is the least number of vertices needed to be removed from G to make it either disconnected or a single vertex.

We say G is *k-connected* if $k \leq \kappa$.

Definition 3.2. Let G be a graph with $a \neq b \in V(G)$, where $a \not\sim b$. We say $S \subseteq V(G) \backslash \{a, b\}$ is a *a-b separator* if $G - S$ disconnects a and b.

Theorem 3.3 (Menger, Form 1)**.** *Let G be a connected graph with $a \neq b \in V(G)$, where $a \not\sim b$. The minimum size of an a-b separator is maximum number of disjoint paths from a to b. Equivalently, if all a-b separators have size at least k, then there exist k disjoint a-b paths.*

Proof. We will prove the equivalent condition. For $k = 1$, this is trivial so assume $k \geq 2$.

Pick a minimal counterexample in both k and $e(G)$, and let S be our a-b separator.

Suppose $S \not\subset N(a)$ or $S \not\subset N(b)$. Then we can construct two graphs G_a and G_b where we replace the a and b component of $G \backslash S$ with a single point a' and b' respectively, connected to every point in S. Then minimality of $e(G)$ implies we then have k disjoint paths from b (resp. a) to the vertices in S. We can then take these paths and join them to get k disjoint paths from a to b in our original graph, which contradicts us having a counterexample.

So we must have S contained in one of these neighborhoods. We first note that it can't be both, as $N(a) \cap N(b) = \emptyset$ (as otherwise we could take a point in the neighborhood and remove it to get a smaller counterexample for $k-1$).

Now take the shortest a-b path, say $ax_1x_2 \dots x_r b$, where $r \geq 2$. Consider $G - x_1x_2$. Then we have separators of size $k-1$ in $G - x_1x_2$, by minimality, say S'. Then $S' \cup \{x_1\}$ and $S' \cup \{x_2\}$ are separators in G. Now $x_1 \notin N(b)$ implies $S' \cup \{x_1\} \subset N(a)$, and $x_2 \notin N(a)$ implies $S' \cup \{x_2\} \subset N(b)$. But then $S' \subset N(a) \cap N(b) = \emptyset$, which is a contradiction. So no such counterexample exists. □

Corollary 3.4 (Menger, Form 2)**.** *Let G be a connected graph with $|G| \geq 2$. Then G is k-connected if and only if all pairs of distinct vertices a, b admit k disjoint a-b paths.*

Proof. k-connected from having k disjoint a-b paths is trivial. For the other direction, we just need to check the case of $a \sim b$ (and possibly remove this edge) and apply the previous form of Menger's theorem. □

Definition 3.5 (Edge Connectivity)**.** The *edge connectivity parameter* λ of G is the least number of edges that need to be removed to disconnect G.

We say G is *k-edge connected* if $k \leq \lambda(G)$.

Theorem 3.6 (Menger – Edge Version, Form 1)**.** *Let G be a connected graph, and $a \neq b$ be vertices. Then, if every $W \subseteq E(G)$ that separates a from b has size at least k, then there exist k edge-disjoint $a - b$ paths.*

Proof. Let G' be the *line graph*[1] of G, along with the distinguished vertices a' and b' which are connected to the edges incident to a and b respectively. Then there's an a-b path in G if and only if there's an a'-b' path in G'. So we can then apply the vertex form of Menger to G' to get our edge disjoint a-b paths in G. □

Theorem 3.7 (Menger – Edge Version, Form 2)**.** *Let G be a connected graph. Then $\lambda(G) \geq k$ if and only if all all pairs of vertices $a \neq b$ admit k edge-disjoint $a - b$ paths.*

Proof. Analogous to the second form of vertex Menger. □

4 Vertex Colouring

Proposition 4.1 (Six Colour Theorem)**.** *Let G be planar. Then G admits a 6-colouring.*

Proof. We note that G planar implies $\delta(G) \leq 5$. Then we just apply induction to G. So we induct on $|G|$. If $|G| \leq 6$ we're done. Otherwise there's a vertex v of degree at most 5. Take a 6-colouring of $G - x$, then x has at most 5 neighbours so there's a free colour left for x. □

Proposition 4.2 (Five Colour Theorem)**.** *Let G be planar. Then G admits a 5-colouring.*

Proof. As before, take $|G| > 5$, and let $x \in G$ with $d(x) \leq 5$. Take a 5-colouring of $G - x$. We are then done unless $d(x) = 5$ and all 5 colours appear in $N(x)$.

So lets say $N(x)$ is $x_1, \dots, x_5$ (clockwise) with x_i having colour i.

If there is no 1-3 path from x_1 to x_3 (going through vertices which are coloured 1 and 3 alternately), then we can take the 1-3 component of x_1, H and swap the colours 1 and 3. We then still have a valid colouring of $G - x$ and we can colour x 1.

If there is such a path, there must be no $2 - 4$ path from x_2 to x_4 (as it would have to meet the $1 - 3$ path by planarity), so we just finish then as before. □

Now we consider colourings in general.

Theorem 4.3. *If G is not a regular graph, then $\chi(G) \leq \Delta(G)$.*

Proof. Start with a vertex x with $\deg x \leq \Delta(G)$. Then perform a breath first search of the graph starting at this vertex. Label the vertices v_i by the order in which they are encountered (with $v_1 = v$). Now colour the vertices in backwards order.

For each vertex v_i, at most $\Delta(G) - 1$ neighbors are among the $v_{i+1}, \dots, v_n$ that receive a colour before v_i, so we can just pick a (say minimal) missing colour. □

Theorem 4.4 (Brooks)**.** *Let G be a connected graph. If G is not an odd cycle or a complete graph, $\chi(G) \leq \Delta(G)$.*

Proof. By the previous theorem we can assume that G is regular, and also that $\Delta(G) \geq 3$ (as the other cases are trivial). WLOG, G is 2-connected (as otherwise we could take a cut vertex x and colour the components of $G - x$ with x, colouring x some fixed colour in all of them).

Case 1. G is 3-connected.

We want an ordering of the vertices such that each vertex is adjacent to an edge later in the list (other than the last) but so that the last vertex has two neighbours of the same colour.

Pick any x_n, and then there must be $x_1, x_2 \in N(x_n)$ not adjacent (otherwise we get a $K_{\Delta+1}$). Now $G - \{x_1, x_2\}$ is connected, so order its vertices as before using breath first search starting with x_3. Then run our previous algorithm on this ordering and we use at most Δ colours.

Case 2. G is not 3-connected.

Choose a separator $\{x, y\}$, and let G_i be the components of $G - \{x, y\}$, together with x and y. Then each G_i has a Δ-colouring (since $d_{G_i}(x) \leq \Delta - 1$).

If $x \sim y$, then $x$and y have different colours in the colouring of G_i for each i, so we can recolour and combine to form a Δ-colouring of G.

So take $x \not\sim y$. Then if each G_i has at least one of $d_{G_i}(x), d_{G_i}(y) \leq \Delta - 2$, we can recolor to ensure x and y have different colours in G_i and then combine to form a full colouring.

So we are done unless some G_i has $d_{G_i}(x) = d_{G_i}(y) = \Delta - 1$, say $i = 1$. Then in G_2 they have degree 1, say $N_{G_2}(X) = \{u\}$ and $N_{G_2}(y) = \{v\}$. Then $\{x, v\}$ is a separator not of this form so we can use our previous argument. □

5 Edge Colouring

Theorem 5.1 (Vizing)**.** *Let G be a graph. Then $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$.*

Proof. We first observe that if we have $\Delta + 1$ colours, then every vertex in the graph has at least one free colour. Also, if two adjacent vertices share a free colour then we can switch the current edge colour with the free colour and won't break the colouring.

Now take an edge xv_1, and colour $G - xv_1$ using $\Delta + 1$ colours (which we can assume is possible by induction).

We define a sequence of vertices $v_1, \dots, v_k \in N(x)$ and corresponding colours $c_1, \dots, c_k$ such c_i is free at v_i and the colour of edge xy_{i+1} is c_i. This chain continues until we find a colour free at x, or repeat a colour.

If we find a colour free at x, we can re-colour xv_k with c_k for all k (starting at the end of the chain) and we are done.

Otherwise, suppose that $c_k = c_i$ for some $i < k$. Note that we can take $i = 1$ by uncolouring xv_{i-1} and performing the process above.

Now let c_0 be free at x. If v_1 is not in the same $\{c_0, c_1\}$ component as x, then we can swap the colours on the $\{c_0, c_1\}$ component containing v_1 so that c_0 is also missing at v_1. We can then colour xv_1 c_0 and we are done.

Similarly, if v_k is not in the same $\{c_0, c_1\}$ component as x we can swap the colours on the $\{c_0, c_1\}$ component containing v_k and colour xv_k by c_1, and then xy_i by c_i as before.

If this all fails, then x, v_1 and v_k are in the same $\{c_0, c_1\}$ component, but each is missing one of $\{c_0, c_1\}$. Also the $\{c_0, c_1\}$ component is made up of disjoint paths, even cycles and isolated vertices. So we'd need each of these vertices to be the endpoint of a path which is a contradiction since we have only 2 endpoints. □

6 Colouring Graphs on Surfaces

Fact 6.1 (Euler Characteristic)**.** *For G on a surface of genus g (a sphere with g handles attached) we have $n - m + f \geq E$, where $E = 2 - 2g$ is the Euler characteristic.*

Theorem 6.2 (Heawood's Theorem)**.** *Let G be a graph drawn on a surface of Euler characteristic $E \leq 0$. Then*

$$\chi(G) \leq H(E) = \left\lfloor \frac{7 + \sqrt{49 - 24E}}{2} \right\rfloor.$$

Proof. Let G be an edge minimal graph drawn on the surface, with $\chi(G) = k$. Then $\delta(G) \geq k - 1$ by minimality and $n \geq k$.

From the Euler characteristic (and say counting (e, f) pairs in two ways) we get that $m \leq 3(n - E)$. So the sum of all degrees is $2m \leq 6(n - E)$. Thus $\delta(G) \leq 6 - 6E/n$.

Thus $k - 1 \leq \delta(G) \leq 6 - 6E/n \leq 6 - 6E/k$ (as $n \geq k$ and $E \leq 0$). So $k^2 - k \leq 6k - 6E$, that is, $k^2 - 7k + 6E \leq 0$. The result follows. □

[1] The graph on the edges of G with two adjacent if they share an endpoint

7 Hamiltonian Cycles

Theorem 7.1 (Dirac's Theorem)**.** *Let G be a graph on $n \geq 3$ vertices. Then if $\delta(G) \geq n/2$, G is Hamiltonian.*

Proof. We first note that G must be connected as otherwise we couldn't have the minimal degree condition. Now consider the longest path $x_0x_1 \cdots x_k$ in G. We note that $N(x_0)$ and $N(x_k)$ are contained in this path as otherwise we could extend it.

We find a cycle as follows. Define

$$A = \{i \in [k] \mid x_0 \sim x_i\}$$
$$B = \{i \in [k] \mid x_k \sim x_{i-1}\}.$$

Then if $A \cap B \neq \emptyset$, we get a cycle. To see that this must be the case, note $|A \cup B| \leq k < n$, but $|A| + |B| \geq \frac{n}{2} + \frac{n}{2} = n$.

Now we have a cycle $v_0v_1 \cdots v_iv_0$ of length $k+1$. So to get a Hamiltonian cycle, we need only to have $k = n-1$. If this was not the case, there would be v_i in the cycle with $w \sim v_i$ and w not in the cycle. But then $wu_iu_{i+1} \cdots u_{i-1}$ is a path of length $k+1$ which contradicts maximality. □

8 Forcing Triangles

Theorem 8.1 (Mantel's Theorem)**.** *Let G be a graph on n vertices, and $n^2/4 < e(G)$. Then G contains a triangle.*

Proof. Suppose the graph contains no triangles, and assume $n \geq 3$. Let $x, y \in V(G)$ such that $x \sim y$. Then $\deg x + \deg y \leq n - 2 + 2 = n$. Then since $x \mapsto x^2$ is convex, we have

$$\begin{aligned} e(G) &= \frac{1}{2}\sum_{x\sim y} 1 \\ &\geq \frac{1}{2n}\sum_{x\sim y} \deg x + \deg y = \frac{1}{n}\sum_x \deg^2 x \\ &\geq \frac{1}{n}\left(\frac{1}{n}\sum_x \deg x\right)^2 = \frac{1}{n}\left(\frac{2e(G)}{n}\right)^2, \end{aligned}$$

and $n^2/4 \geq e(G)$. □

9 Forcing Cliques

Theorem 9.1 (Turan, Form 1)**.** *Let G be a graph on n vertices, and $(1 - \frac{1}{r})\frac{n^2}{2} < e(G)$ for $r \geq 1$. Then G contains a subgraph of the form K_{r+1}.*

Proof. For a given r, we will induct on n (assuming that it holds for all lower values of r) and will then induct on r.

Let G be a graph that contains no $(r+1)$-clique. Suppose $r \geq 2$ (otherwise the result is trivial). Then we can find an r-clique by induction on r. Let K be such a clique. Then each vertex in $V(G)\backslash K$ has at most $r-1$ neighbours in K, otherwise we could expand K to an $(r+1)$-clique. So

$$\begin{aligned} e(G) &\leq \binom{r}{2} + (r-1)(n-r) + e(G\backslash K) \\ &\leq \binom{r}{2} + (r-1)(n-r) + \left(1 - \frac{1}{r}\right)\frac{(n-1)^2}{2} \\ &= \left(1 - \frac{1}{r}\right)\frac{n^2}{2}. \end{aligned}$$

□

Definition 9.2. The *Turán graph $T_{r,n}$* is the complete r-partite graph $K_{n_1,\ldots,n_r}$ where $\sum n_i = n$ and $n_1, \ldots, n_r$ differ by at most 1.

Proposition 9.3. *Let G be an r-partite graph on n vertices. Then $e(G) \leq e(T_{r,n})$.*

Proof. To have maximal edges we must be complete, and then we can check that making the parts as equal as possible maximizes the number of edges (by say smoothening). □

We will skim over the details in the next theorem.

Theorem 9.4 (Turán, Form 2)**.** *Let G be a graph on n vertices and $r \geq 2$. Then if G does not contain an $(r+1)$-clique, $e(G) \leq e(T_{r,n})$.*

Proof. Induct on n. Suppose $e(G) \geq e(T_{r,n})$, and $n > r+1$. Then we can delete edges to form a subgraph H with $|H| = n$ and $e(G) = e(T_{r,n})$, with $K_{r+1} \not\subset H$. Let $v \in H$ have minimal degree, and let $K = H - v$. We know that $|H| = |T_{r,n}|$ and $e(H) = e(T_{r,n})$, so H and $T_{r,n}$ have the same average degree. But the degrees in $T_{r,n}$ are as equal as possible by construction. So $\delta(H) < \delta(T_{r,n})$.

Thus $|K| = n-1$, $K_{r-1} \not\subset K$ and

$$\begin{aligned} e(K) &= e(H) - \delta(H) \geq e(H) - \delta(T_{r,n}) \\ &= |e(T_{r,n})| - \delta(T_{r,n}) = e(T_{r,n-1}). \end{aligned}$$

So by induction, we have $K \cong T_{r,n-1}$, and so to recover H we must add a vertex and $e(T_{r,n}) - e(T_{r,n-1})$ edges to K without creating a K_{r+1}, which forces $H \cong T_{r,n}$. But then we can't add any edges to H without creating a K_{r+1} by our previous proposition, so $G \cong T_{r,n}$. □

10 Zarankiewicz Problem

Now Turán for bipartite graphs.

Definition 10.1. The *Zarankiewicz number* $Z(n,t)$ is the maximum number of edges in a bipartite graph $G = (X \sqcup Y, E)$ with $|X| = |Y| = n$ such that G does not contain a $K_{t,t}$.

Theorem 10.2 (Zarankiewicz Upper Bound)**.** *For $t \geq 2$, $Z(n,t) \leq n^{2-1/t}t^{1/t} + nt$. In particular, $Z(n,t) \leq 2n^{2-1/t}$ for n sufficiently large.*

Proof. Note we can assume that $\deg y \geq t-1$ for all $y \in Y$, as otherwise we could add a vertex and preserve that G has no $K_{t,t}$. We now note that if $x_1, \ldots, x_t \in X$ are distinct vertices, then $|N(x_1) \cap \cdots N(x_t)| \leq t-1$, otherwise we would have a $K_{t,t}$. Now we apply Jensens.

$$\begin{aligned} (t-1)\binom{n}{t} &\geq \sum_{x_1,\ldots,x_t \text{ distinct}} |N(x_1) \cap \cdots \cap N(x_t)| \\ &= \sum_y \binom{\deg y}{t} \\ &\geq n\binom{\overline{d}}{t}, \end{aligned}$$

where $\overline{d} = e(G)/n$, using that $\deg y \geq t-1$ implies $x \mapsto \binom{x}{t}$ is convex. So

$$\begin{aligned} (t-1)\binom{n}{t} &\geq n\binom{\overline{d}}{t} \\ \implies \frac{tn^t}{t!} &\geq \frac{n(\overline{d}-t)^t}{t!} \\ \implies t^{\frac{1}{t}}n^{2-\frac{1}{t}} + tn &\geq e(G), \end{aligned}$$

as required. □

We can use the probabilistic method for the lower bound.

Theorem 10.3 (Zarankiewicz Lower Bound)**.** *Let $t \geq 2$. Then $Z(n,t) \geq \frac{1}{2}n^{2-\frac{2}{t+1}}$.*

Proof. Consider a random bipartite graph $G = (X \sqcup Y, E)$ with $|X| = |Y| = n$, and each edge included with probability $p = n^{-\frac{2}{t-1}}$. We construct from this a $K_{t,t}$ free graph by removing an edge from each $K_{t,t}$ until there's no such subgraphs. Call this modified graph $\tilde{G}$. It then suffices to show that $\mathbb{E}[e(\tilde{G})]$ is at least our desired lower bound.

We have

$$\begin{aligned} \mathbb{E}[e(\tilde{G})] &\geq \mathbb{E}[e(G)] - \mathbb{E}[\#K_{t,t} \subset G] \\ &= pn^2 - \binom{n}{t}^2 p^{t^2} \geq pn^2 - \frac{n^{2t}}{2}p^{t^2} \\ &= \frac{1}{2}n^{2-\frac{2}{t+1}}, \end{aligned}$$

as required. □

11 Erdös-Stone

Definition 11.1. Let H be a fixed graph, and $n \in \mathbb{N}$. Then we define the *extremal number* $\mathrm{ex}(n,H) = \max\{e(G) \mid |G| = n, H \not\subset G\}$.

Theorem 11.2 (Erdös-Stone)**.** *Let H be a fixed nonempty graph. Then*

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = 1 - \frac{1}{\chi(H)-1}.$$

12 Ramsey Theory

Definition 12.1. We define $R(s,t)$ to be the minimal n such that every 2-edge colouring of K_n contains either a red K_s or a blue K_t. We define $R(s) = R(s,s)$ to be the *sth Ramsey number.*

Theorem 12.2 (Ramsey)**.** *For all s,t, the Ramsey number $R(s,t)$ exists and $R(s,t) \leq R(s-1,t) + R(s,t-1)$.*

Proof. Induct on $s+t$. For $s,t \leq 2$ this holds. Now for $s+t > 2$, consider K_n where $n = a+b$ with $a = R(s-1,t)$ and $b = R(s,t-1)$. Pick a vertex v. Then v has $a+b-1$ edges, and so has either a red edges or b blue edges. WLOG take a red edges, and consider the subgraph of vertices v is adjacent to via red edges. Then $a = R(s-1,t)$, so either we get a monochromatic K_{s-1} red subgraph (to which we adjoin v and are done), or a monochromatic K_t blue subgraph, and we are then also done. □

Corollary 12.3 (Ramsey Upper Bound)**.** *For all $s,t \geq 2$, $R(s,t) \leq 2^{s+t}$, so $R(s) \leq 4^s$.*

Proof. Induct on $s+t$ and use the previous result. □

We can use the probabilistic method to get a lower bound (though we can construct $R(s) > (s-1)^2$ easily).

Theorem 12.4 (Erdös Lower Bound for Ramsey Numbers)**.** *Let $s \geq 3$. Then $R(s) \geq 2^{s/2}$.*

Proof. Take $G = K_n$ for $n \leq 2^{s/2}$, and colour each edge red or blue with 1/2 probability. Then we compute the probability of having a monochromatic K_s is upper bounded by

$$\begin{aligned} &\mathbb{P}\left(\bigcup_{K\in[n]^{(s)}} \{K \text{ monochromatic}\}\right) \\ &\leq \binom{n}{2} 2 \cdot 2^{-\binom{s}{2}} \\ &< 2\left(\frac{n}{(s!)^{\frac{1}{s}}}2^{-\frac{s-1}{2}}\right) \\ &\leq 2\left(\frac{2^{1/2}}{(s!)^{1/s}}\right)^s \leq 1, \end{aligned}$$

and since this probability is less than 1, there must exist a colouring that has no monochromatic K_s. □

Theorem 12.5 (Infinite Ramsey)**.** *Whenever the edges of K_∞ are k-coloured, we have a monochromatic K_∞ subgraph.*

Proof. Take $v_1 \in K_\infty$. The vertex v_1 is in infinitely many edges, so infinitely many edges from v_1 are the same colour. Let A_1 be an infinite set of vertices of K_∞ such that for all $u \in A_1, v_1u$ has colour c_1. Now pick $v_2 \in A_1$. Similarly, we can find an infinite $A_2 \subset A_1$ such that all edges v_2u for $(u \in A_2)$ have colour c_2. Keep going. We get infinite sequences $v_1, v_2, v_3, \ldots$ of vertices, $c_1, c_2, c_3, \ldots$ of colours and $A_1 \supset A_2 \supset A_3 \supset \ldots$ such that

1. for $i \geq 2, v_i \in A_{i-1}$
2. for $i \geq 1$, for all $u \in A_i, v_iu$ is an edge of colour c_i.

In particular, if $i < j$ then v_iv_j has colour c_i. Now, infinitely many of the c_i are the same. Say $i_1 < i_2 < i_3 < \ldots$ such that $c_{i_1} = c_{i_2} = c_{i_3} = \ldots$. Consider $v_{i_1}, v_{i_2}, v_{i_3}, \ldots$ Any edge between two of these vertices has colour c_{i_1}. So we have a monochromatic K_∞ □

13 Graphs of Large Girth and Chromatic Number

Definition 13.1. The *girth* of a graph is the length of the shortest cycle.

Proposition 13.2. *Let G be a graph. Then $\chi(G) \geq |G|/\alpha(G)$, where $\alpha(G)$ is the size of the largest independent set (non-adjacent vertices) in G.*

Proof. Let c be a colouring of G with $k = \chi(G)$ vertices. Let C_i be the set of vertices coloured i, each of which is independent. Then $|G| = |C_1| + \cdots + |C_k| \leq k\alpha(G) = \chi(G)\alpha(G)$. □

Theorem 13.3. *For all k, $g \geq 3$, there exists a graph G with $\chi(G) \geq k$ and girth at least g.*

Proof. Let $G \sim \mathcal{G}(n,p)$ with $p = n^{-1+1/g}$. Let X_i be the number of cycles of length i. Let $X = X_3 + \cdots + X_{g-1}$. Note that $\mathbb{P}(X \geq \frac{n}{2}) \leq \frac{2}{n}\mathbb{E}[X]$. Now we compute

$$\begin{aligned} \mathbb{E}[X] &= \sum_{i=3}^{g-1}\mathbb{E}[X_i] = \sum_{i=3}^{g-1}\binom{n}{i}\frac{i!}{i}p^i \\ &\leq \sum_{i=3}^{g-1}(np)^i = \sum_{i=3}^{g-1} n^{\frac{i}{g}} \\ &\leq cn^{-\frac{1}{g}} < \frac{1}{2}, \end{aligned}$$

for some constant c. Now let Y be the number of independent sets of $s = n/2k$ vertices (up to rounding). Then

$$\begin{aligned} \mathbb{P}(Y \geq 1) \leq \mathbb{E}[Y] &= \binom{n}{s}(1-p)^{\binom{s}{2}} \\ &\leq n^s e^{-p\binom{s}{2}} = \left(n^2 e^{-p(s-1)}\right)^{\frac{s}{2}} \\ &\leq \left(2n^2 e^{-\frac{n^{\frac{1}{g}}}{2k}}\right)^{\frac{s}{2}} < 1/2 \end{aligned}$$

for n sufficiently large. Thus G has at most $n/2$ cycles of length at most $g-1$ with probability at least 1/2, and G has $\alpha(G) \leq n/2k$ with probability at least 1/2. Hence there is a graph G with both probabilities. Let $\tilde{G}$ be G with a vertex deleted from each cycle of length less than g. Then $\tilde{G}$ has girth at least G, and $\chi(\tilde{G}) \geq |\tilde{G}|/\alpha(\tilde{G}) \geq k$, as needed. □