List of Exercises in Problem Sets

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Preliminary Exercises

Exercise 1. First Steps. Set your working directory (equivalent to the cd in Stata):

```
import os
os.chdir("your folder path")
```

Read the Zen of Python by running the following line

```
import this
```

What does it say about programming? How does it relate to some of the good coding practices we saw in class?

Exercise 2. Copy-paste the following paragraph into a string variable called *str_x*. Count the number of times each of the following words appears in the string: *wealth*, *distribution*, *households*, and *assets*. **Hint:** Google how to create a string in Python. Use the count method of a str class.

"Facts about the distribution of wealth have been highlighted in a large number of studies, including Wolff (1992, 1998), Cagetti and De Nardi (2008), and Moritz and Rios-Rull (2015). A striking aspect of the wealth distribution in the US is its degree of concentration. Over the past 30 years or so, for instance, households in the top % of the wealth distribution have held about one-third of the total wealth in the economy, and those in the top 5% have held more than half. At the other extreme, more than 10% of households have little or no assets. While there is agreement that the share held by the richest few is very high, the extent to which the shares of the richest have changed over time (and why) is still the subject of some debate (Piketty 2014, Kopczuk 2014, Saez and Zucman 2014, and Bricker et al. 2015)."

This paragraph is from the column "Quantitative macro models of wealth inequality: A survey" by Mariacristina de Nardi at VOXEU-CEPR, 2015.

Exercise 3. Compute and plot 1-D functions.

a. Create a grid—or linear space—x from 0 to 10, with 100 points. Create a variable y that is equal to the log of x, and a variable z that is equal to the sinus of x. Explore your outcomes on the variable explorer. What type is y? what size is y? and z? **Hint:** Check the NumPy documentation and familiarize yourself with the NumPy *linspace*, *log*, and *sin* routines. To import the NumPy library in your file run

```
import numpy as np
```

and then call the NumPy functions via np.linspace, np.log, and np.sin.

b. Go to Matplotlib documentation. Use some of the routines in the library to plot (y, x) and (z, x).

Exercise 4. Compute the values of z, $z = f(x, y) = log(x) + sin(y)$, where both x and y are grids from 0 to 10 with 100 points. Make a 3-D plot of $f(x, y)$ using Matplotlib 3D surface.

Exercise 5. Let's solve the following consumer problem. The consumer maximizes utility over goods x_1, x_2 subject to the budget constraint.

$$Max_{x_1,x_2} x_1^{\alpha} x_2^{\alpha-1}$$
$$\text{St: } p_1 x_1 + p_2 x_2 \leq y$$

Where y is the consumer's income and x_i, p_i, and y are strictly positive numbers. First, set up the problem in Python using the following code

```
import numpy as np
# Parameters values
alpha = 0.5
y = 10
p1 = 1
p2 = 2

def utility_function(x1,alpha,y,p1,p2):
    x2 = (y-p1*x1)/p2  # Income not spent first good is spent on the second
    utility = x1**alpha * x2**(1-alpha)
    return utility
```

Go to SciPy.optimize documentation and use *minimize_scalar* to find the x_1 and x_2 that solve the consumer problem. **Hint:** Note that maximizing a function f is the same a minimizing a function $g = -f$. Note that to apply minimize scalar you need to write a new function based on the *utility_function* with x1 as the only argument. One way to this is the following

```
obj_func = lambda x1: -utility_function(x1,alpha,y,p1,p2)
```

Exercises

Exercise 6. Arrays and matrix operations (20 points). Given the following list

```
l1 = [2, 5, 6, 4, 5 ,9, 3, 2, 2]
```

a. Convert the list into a 3×3 matrix (2-d array). The outcome should be the matrix

$$A = \begin{bmatrix} 2 & 5 & 6 \\ 4 & 5 & 9 \\ 3 & 2 & 2 \end{bmatrix}$$

b. Find the maximum of the matrix A. Find the index of the maximum.

c. Transpose the matrix$A$.

d. Squared the matrix (i.e. AA'). Also, raise to the power of 2 all the elements in matrix A.

e. Compute the eigenvalues of matrix A.

f. Multiply matrix A by a matrix of zeros (f1), by the identity matrix (f2),and by a matrix of ones (f3).

g. Create an even grid from 1 to 9 with 9 elements. Convert the grid into a 3x3 matrix called B. Multiply matrix A by matrix B.

Exercise 7. Conditional statements and operations (20 points). Given the list from exercise 1

```
l1 = [2, 5, 6, 4, 5 ,9, 3, 2, 2]
```

a. Create a new list that contains only the elements in list l1 that are smaller than 5.

b. Create a new list that contains only the elements in list l1 bigger or equal than 3 and smaller than 7.

c. Given matrix A from exercise 2, write a code that checks whether 5 belongs to A.

d. Create a new matrix B that is equal to matrix A but where numbers below 4 are replaced by zeroz.

e. Write a code that counts the number of zeros in matrix B.

Exercise 8. CES production function and comparative statistics (30 points). The Constant Elasticity of Substitution production function is a commonly used production function that takes the following form

$$Y = F(K, L) = A\left(\alpha K^{\frac{\sigma-1}{\sigma}} + (1-\alpha)L^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

Where K and L are the capital and labor factors. A represents the total factor productivity, α is the capital's share, and σ is the constant elasticity of substitution between the two production factors. A, α, σ are strictly positive. It can be shown (using L'Hôpital's rule) that when $\sigma \to 1$ the CES production function boils down to a Cobb-Douglas production function

$$Y = F(K, L) = AK^{\alpha}L^{1-\alpha}$$

a. Create a function that given the arguments K, L, A, α, σ, returns output Y. You can have inside the function an if statement for when $\sigma = 1$ the output Y comes from Cobb-Douglass production function, else from the CES function.

   From now on work with the following parameterization: $A = 1.5, \alpha = 0.33$.

b. **Cobb-Douglass production function**. First consider the Cobb-Douglass case with $\sigma = 1$. Compute output Y for an even-spaced grid of K, $G_k = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and a fixed $L = 3$. Plot the resuls—make an x-y plot with the grid of K in the x axis and output Y in the axis y.

c. From b, recompute output Y for the 3 cases $\alpha = 0.25, \alpha = 0.5, \alpha = 0.75$. Make an $x - y$ plot with the 3 production functions in the same graph.

d. **CES production function.** Redo exercise b but for $\sigma = 0.33$.

e. Keeping $\alpha = 0.33$, plot output Y vs the grid of capital for the cases of $\sigma = 0.25, \sigma = 0.5, \sigma = 1, \sigma = 2, \sigma = 4$.

f. How does output Y changes along K for the different σ specifications? Can you provide the economic interpretation? **Hint:** σ captures the relative degree of substitutability/complementarity between the two inputs K, L.

Exercise 9. Transitions in the Solow model of growth (30 points) Consider a basic Solow model with a Cobb-Douglass production function $Y_t(K_t, L_t) = AK_t^{\alpha}L_t^{1-\alpha}$, no population growth, and depreciation rate δ. The capital per worker law of motion is

$$k_{t+1} = sAk_t^{\alpha} + (1-\delta)k_t \tag{1}$$

The parameter values are $\alpha = 0.3$, $\delta = 0.07$, $A = 4$, $s = 0.4$.

a. What is the steady state capital per worker and output per worker in the economy?

b. Suppose the economy in the first three periods, $t = 0$ to $t = 2$, is in steady state. Then, at period $t = 3$ the economy experiences a permanent negative shock such that A decreases by 25 percent. Compute and plot the levels of capital per worker and output per worker from period $t = 0$ till the economy reaches the new steady state. Use a tolerance level of $\varepsilon = 0.5$ to find convergence to the new steady state.

c. Consider the economy is in the original steady state with $A = 4$ from period $t = 0$ to period $t = 2$. Then, from period $t = 3$ to $t = 10$ the economy experiences a temporary shock that leads the saving rates up to $s = 0.6$ and after period $t = 10$ the saving rate goes back to $s = 0.4$. Compute and plot the output per worker level from period $t = 0$ to period $t = 100$. Explain how a temporary change in the savings rate affects output.

Use the 2011-2012 wave of the Ugandan National Panel Survey. The survey is under the umbrella of the ISA-LSMS surveys and offers nationally representative household data on consumption, income, wealth and other key variables in Uganda. The file *UNPS_1112_PS2.xls* contains the data while the file *variables_description_UNPS_1112_PS2.xls* contains a description of the variables. Import the data and answer the following exercises.

Exercise 10. Exploring the data (25 points).

a. Are there duplicate households in the data? That is check if there are repeated observations in the unique household identifier variable. How many observations are there in the data?

b. Present some basic summary statistics for the following variables: *head_gender, head _age, familysize, consumption, income, wealth*. Comment your results in 2 lines. In particular, you might mention if there are missing observations or potential outliers for some of the variables.

c. Using the *head_gender* variable, create a dummy variable for household head being female (1=female, 0=male). What is the proportion of households where the head is female?

d. Using the groupby method, compute the average consumption, average household size, and average household head age for households where the head is male vs where the head is female. Do we observe noticeable differences across the two groups?

Exercise 11. Inequality in Uganda (50 points).

a. Create the variables log_c, log_inc, log_w that are the log of consumption, income, and wealth, respectively. Plot in the same graph the distribution of the log of consumption and the log of income. Do the distributions resemble some known distribution? Is inequality higher in consumption or in income?

b. A commonly used statistic to measure inequality is the variance of the logs. Compute the variance of the log of consumption, of the log of income, and of the log of wealth. How do these measures of inequality in Uganda compare to the same measures of inequality in the United States? Use table 3, column 5–PSID in De Magalhães, L., & Santaeulàlia-Llopis, R. (2018) for the comparison.

c. Measuring between rural and urban inequality in Uganda. Compute the average consumption, income, and wealth for rural and urban areas separately (groupby). Are the differences between the two areas large?

d. Measuring within rural and urban inequality in Uganda. Compute the variance of the log of consumption, income, and wealth for rural and urban areas separately.

e. Compute the Gini coefficient in consumption, in income, and in wealth in Uganda. Compare these values with the Gini coefficients in the United States—table 3, column 5–PSID in De Magalhães, L., & Santaeulàlia-Llopis, R. (2018)

f. Compute the share of the wealth that the bottom 50 percent hold. Compute the share of the wealth that the top 10, 5, and 1 percent hold.

g. Although in the last years, there has been a big debate on inequality, the debate has mostly focused on rich countries. From your results of this exercise, discuss whether inequality is relatively large in Uganda with respect to rich countries.

h. The few previous studies on income inequality in Africa had to rely on consumption measures to estimate income inequality. See, for example, Alvaredo & Gasparini (2005). Debate on the advantages and disadvantages of using consumption measures to study income inequality.

Exercise 12. The lifecycle of male vs female head households in Uganda (25 points).

Before going to the plots, you might want to drop ages for where there are few households—as above 80 years old and below 18 years old. You might also want to group the ages in bins so that that the plots are more smooth.

Then, using seaborn lineplot with the argument *hue='female'*, or any other variable to distinguish the gender of the head,

a. Plot the lifecycle of the log of consumption for households where the head is male and for households where the head is female.

b. Redo the same plot but for the log of income (i) and for the log of wealth (ii).

c. What are the differences in the lifecycle of consumption, income, and wealth of households across the gender of the household heads? Comment your results.

Exercise 13. Monte Carlo simulation: OLS asymptotic behavior. (20 points) Prove that the OLS estimator is distributed asymptotically normal using Monte Carlo simulation.

$$\sqrt{n}\left(\hat{\beta}-\beta\right) \overset{as}{\sim} N\left(\beta, \sigma^2\left(\mathbb{E}(x_i x_i')\right)^{-1}\right)$$

To prove that, consider the model or data generating process seen in class

$$y_i = 2 + 0.5x_{2,i} + e_i$$
$$e_i \sim N(0, 25)$$

and, as a preliminary step, using the following code create a sample with size N=100, a fictitious $x_2 \sim N(0,1)$, and estimate the model by OLS.

```python
# import statsmodels for OLS regression
import statsmodels.api as sm
# Let's create a sample (a data), and estimate B* by OLS
N= 100 # Sample size
sigma_e = 5
#  fictisiously create our regressor x2.
x2 =  np.random.normal(0,1,N);  #x2 follows normal (0,1)
# e is an error term (what we don't observe in the data)
e = np.random.normal(0,sigma_e,N)
# And our dependent variable is:
y1 = 2 + 0.5*x2 +e
# Let's estimate the model by OLS
X = sm.add_constant(x2)  # We need to add constant, B0
ols1 = sm.OLS(y1,X).fit()
print(ols1.summary())
```

a. Now, following the steps in lecture 4, slide 12, run a Monte Carlo simulation to show the asymptotic properties of $\hat{\beta}_{OLS}$ and **replicate figure 1 and figure 2 from slide 13**. Set the number of simulations equal to T=10000. For each repetition of the experiment (step 3) use a sample size of $N = 100$.

b. Redo exercise a) with a sample size $N = 1000$. How do the $\hat{\beta}_{OLS}$ asymptotic distributions change when the sample size increases? **Hint:** For a visual demonstration, keep the same x-axis in the plots in a) and b) using *plt.xlim()*.

Exercise 14. Solving the earnings distribution in an Aiyagari economy (40 points). In the Aiyagari model—one of the fundamental models in quantitative macroeconomics—"there is a very large number of households" whose log of labor earnings (y_t) follows the next AR(1) process

$$y_t = \rho y_{t-1} + \varepsilon_t \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2) \tag{2}$$

For this exercise let's use the following parameterization of the AR(1) process:
$\rho = 0.95$, $\sigma_\varepsilon = 0.25$ and an initial value $y_0 = 0$.

a. Simulate and plot the AR(1) process given by equation (1) for T=50 periods.

b. Create a function that simulates N AR(1) processes for T periods.

- The function should have the following inputs $(N, T, rho, y0, sigma_e)$
- The output of the function should be a 2-D array with size (N,T)

c. Simulate and plot 5 AR(1) processes given by equation (1) for t=50 periods.

d. **The stationary earnings distribution.** Simulate for a large T (like T=1000) the AR(1) process of 10000 individuals. Use the result of the last period to plot the stationary distribution. From the stationary distribution, compute the variance of the log of earnings (y_t), and the Gini coefficient of earnings (e^{y_t}).

e. For computational reasons, Aiyagari discretizes the AR(1) process into a Markov process of 7 states. Using the Rouwenhorst method, discretize the AR(1) process of this exercise into a 3 states Markov process. What is the resulting transition matrix P? What is the resulting stationary distribution ψ^*?

Exercise 15. Simulating distributions and computing expectations (20 points).

a. Simulate a binomial distribution with $n = 4$ and $p = 0.5$. Plot the resulting distribution.

b. Compute the expected value of a function $g(x) = x^2$ where x follows a Poisson distribution with $\lambda = 2$. Use Monte Carlo integration.
Is your result equal to $g(\mathbb{E}(x))$? where $\mathbb{E}(x) = \lambda = 2$ if $x \sim Poisson(\lambda)$. Why not?

Exercise 16. Simulating and computing expectations in an economy (20 points) Consider an economy where individual's income (y) follows a log-normal distribution. That is $log(y) \sim N(\mu, \sigma^2)$ where $\mu = 7.5$, $\sigma = 0.8$.

a. Simulate y for $N = 100000$ and compute the average, the variance, and the Gini of y.

b. Now consider that individuals follow a consumption rule that takes the following functional form:

$$c_i = (y_i)^{0.8} + 0.5y_i + 500$$

Compute the average, the variance, and the Gini coefficient of c.

c. Plot the distribution of y and c in the same graph.

Exercise 17. Testing algorithms with the Rosenbrock function (30 points). Let's use again the Rosenbrock function to test the different algorithms seen in class. For each algoritm report: the value $\mathbf{x}^*$ that minimizes the function and how much time it took to compute the solution. If the computation takes too much time (let's say 5 minutes) stop the computation and report that the algorithm took too much time. [1] The Rosenbrock function is

$$f(\mathbf{x}) = \sum_{i}^{N-1} \left((1 - x_i)^2 + (x_{i+1} - x_i^2)^2 \right)$$

Where N is the number of variables.

a. define the Rosenbrock function for a general number of variables N. [2]

b. **Testing the Brute-force algorithm**. Using a range of $(-2, 2)$ for all x_i, find the minimum of the Rosenbrock function using the brute-force method for the three following cases: $N = 3$, $N = 4$, and $N = 5$. How does computational time increases as the number of variables N increases? linearly or exponentially on N?

   For the rest of the exercises use an initial value of zero for all the variables. That is $\mathbf{x}_0 = [x_{01}, x_{02}, .., x_{0N}] = [0, 0, ..., 0]$.

c. Find the minimum of the Rosenbrock function with $N = 30$ for the following three cases: using the BFGS method, using the Nelder-Mead method and using the Powell method. Which algorithm is faster?

d. **Let's test the algorithms further**. Minimize the Rosenbrock function for 100 variables, N=100. Use the BFGS method, the Nelder-Mead method and the Powell method. If N=100 with N=100 you can use another N like N=50.

e. **Algorithms comparison.** For this exercise we see that there is one algorithm that tends to to do better than the rest (especially when N is large). Which algorithm is and why? you might want to comment on the properties of the function we are minimizing and the characteristics of the algorithm.

Exercise 18. Solving Cournot markets using root-finding routines (30 points).
Duopoly à la Cournot. Consider a market that is controlled by two firms that compete with each other on quantities. For this duopoly, the inverse of the demand function is given by

$$P(q) = q^{-\alpha}$$

and both firms face quadratic costs

$$C_1 = \frac{1}{2} c_1 q_1^2$$

$$C_2 = \frac{1}{2} c_2 q_2^2$$

. Thus, firm's profits are

$$\pi_1(q_1, q_2) = P(q_1 + q_2)q_1 - C_1(q_1)$$

$$\pi_2(q_1, q_2) = P(q_1 + q_2)q_2 - C_2(q_2)$$

a. Given that each firm maximizes its profits taking as given the other firm's output, find the first order conditions of each firm.

[1] In Spyder you can stop the computation by clicking on the squared button in the console (or just close the console). If some of the fans in your laptop do not work and your laptop gets too hot, also stop the computation.

[2] You need to create the function, if you directly import it from somewhere else you will get a 0 for the whole exercise.

Considering the following parameterization: $\alpha = 0.625$, $c_1 = 0.6$ and $c_2 = 0.8$.

[resume]find the Cournot equilibrium (q_1^*, q_2^*, p^*). Solve the system equations given by the previous first order conditions and report the equilibrium quantities of q_1^* and q_2^* and equilbrium price p^*. Provide an interpretation of the results.

Oligopolies à la Cournot. Assume the market is controlled by N firms that compete with each other. Firms face the same inverse demand function as before, the same quadratic costs, and each firm maximizes its profits taking as given the other firms' output.

[resume]Create a function that for N>1 computes the set of N first order conditions that characterize the Cournot equilibrium quantities. Compute the Cournot equilibrium (q_i^*, p^*) under 3 firms with cost parameters $\mathbf{c} = [0.6, 0.8, 0.5]$ and under 6 firms with $\mathbf{c} = [0.6, 0.8, 0.5, 0.5, 0.4, 0.2]$. Compute the Cournot equilibrium (q_i^*, p^*) for $N = 10$ and $N = 15$. Firms are identical with cost parameter $c_i = 0.6$ for all $i \in N$. Given your previous results, plot the equilibrium prices, p^*, of Cournot oligopolies with number of firms $N = 2, 3, 6, 10, 15$. Provide an interpretation of the results.

Exercise 19. Steady states and transitions in a Solow economy (30 points). Consider a basic Solow model with a Cobb-Douglass production function $Y_t(K_t, L_t) = AK_t^{\alpha}L_t^{1-\alpha}$, no population growth and depreciation rate δ. The capital per worker law of motion is

$$k_{t+1} = sAk_t^{\alpha} + (1-\delta)k_t \tag{3}$$

And the capital per worker in steady state is

$$k^* = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}} \tag{4}$$

The parameter values are $\alpha = 0.3$, $\delta = 0.1$.

a. **Calibration exercise**. What are the values of the saving rate, s, and the TFP, A, such that in steady state the output per worker is equal to 10, and the captial-output ratio is equal to 4?

$$y^* = 10$$
$$\frac{k^*}{y^*} = 4$$

b. **A new steady state.** Now, suppose there is a positive TFP shock and A doubles with respect to the value you found on a). Keeping everything else unchanghed, that is with the s you found on a) and $\alpha = 0.3$, $\delta = 0.1$, what is the new steady state capital per worker, k^*, and output per worker, y^*?

c. **Computing the transition to the new steady state.** Plot the capital per worker and output per worker dynamics from the steady state in a) to the new steady state in b). To do so,

   - Notice that the capital per worker dynamics under Solow are given by the law of motion given by equation (1). Where now $A = 2A_{old}$.
   - You can use a while loop that starts with the capital value in the steady state at a), and in each iteration computes the next capital value k_{t+1} given by the law of motion (1) until it reaches the point where the next capital value capital k_{t+1} is approximately equal to the steady state in b). [3]
   - To find the approximately equal value we can use a small number $\varepsilon = 0.5$ so that your while loop looks something like this

[3] This is a suggestion. It is perfectly fine to solve the exercise using a different code like a for loop with some stopping criteria.

```python
eps=0.5
t=0
k = k_s #the steady state k in a)
while np.abs(k-k_s2)>= eps:  #k_s2 is the steady state k in b)
```

- For each iteration store the iteration number (t), the capital k_t, and the output y_t.
- Plot the capital per worker and output per worker paths you found with the while loop.

Exercise 20. Optimal life-cycle consumption paths: (40 points). Consider the problem of an individual that maximizes his/her lifetime utility by choosing consumption levels across periods. The per period utility funciton takes a CRRA form

$$u(c_t) = \frac{c_t^{1-\theta}}{1-\theta}$$

where we assume a constant relative risk aversion parameter of $\theta = 1.5$. The individual discounts future consumption at discount rate $\beta = 0.96$. The individual can save and borrow in a risk-free asset a_t at an interest rate r. For simplicity, let's assume $r = \frac{1}{\beta} - 1$.

a. **A two-period model**: consider the problem of an individual that lives for two periods ($t = 0, t = 1$) and maximizes his/her lifetime utility, subject that the budget constraint at each period hold. The maximization problem is

$$\max_{c_0, c_1, a_1} \{u(c_0) + \beta u(c_1)\}$$
$$\text{st:}$$
$$c_0 + a_1 = a_0 + y_0$$
$$c_1 = (1+r)a_1 + y_1$$

Where a_0 is an initial assets endowment, y_0, y_1 are the income flows at period 0 and period 1 respectively. Using the following parameter values $a_0 = 0, y_0 = 10, y_1 = 5$, find the optimal consumption c_0^*, c_1^* that solves the previous problem. plot the life-cycle of income and optimal consumption. That is a line plot where the x-axis are the time periods: $t = 0, 1$, line-1 is y_0, y_1, and line-2 is c_0^*, c_1^*.

b. **A life-cycle model with 4 periods**. Solve the following maximization problem

$$\max_{\mathbf{c}, \mathbf{a}} \{u(c_0) + \beta u(c_1) + \beta^2 u(c_3) + \beta^3 u(c_2)\}$$
$$\text{st:}$$
$$c_0 + a_1 = a_0 + y_0$$
$$c_1 + a_2 = (1+r)a_1 + y_1$$
$$c_2 + a_3 = (1+r)a_2 + y_2$$
$$c_3 = (1+r)a_3 + y_3$$

and plot the life-cycle of income and optimal consumption. Use the following parameter values: $a_0 = 0$, and $\mathbf{y} = [y_0, y_1, y_2, y_3] = [5, 10, 15, 0]$

c. **Comparative statics.** Solve the problem in b) for r=0.01 (case 1), for r=0.04 (case 2) and for r=0.08 (case 3). Plot the optimal consumption path along the 4 periods for each of the three cases. How does the optimal life-cycle consumption changes as r changes?

d. **Optimal consumption paths under income risk**. Now let's solve the original 2-period model with uncertainty. Suppose that period 1 income, y_1, is unknown: with probability

$p^l = 0.5$ it takes a low realization $y_1^l = 2.5$, with probability $p^h = 0.5$ it takes a high realization $y_1^h = 7.5$. The rest of parameters and assumptions are exactly as in exercise a). solve the optimal consumption and saving decisions that maximize the expected lifetime utility. That is c_0, c_1, s_1 that solves the following problem

$$\begin{aligned} \max_{c_0,c_1,a_1} \{u(c_0) + \beta \mathbb{E}[u(c_1)]\} = \\ = \max_{c_0,c_1,a_1} \{u(c_0) + \beta \left[p^l u(c_1(y_1^l)) + p^h u(c_1(y_1^h))\right]\} \\ \text{st:} \\ c_0 + a_1 = a_0 + y_0 \\ c_1 = (1+r)a_1 + y_1 \end{aligned}$$

Plot the income and optimal consumption paths when y_t takes a low realization and when y_t takes a high realization.

e. **Precautionary Savings.** Compare the savings in a world with certainty (case a), with respect to the world with uncertainty (d). Why do savings increase as there is income risk? Provide an economic or "real life" intuition.