Lab 6 – Warm-Up Activity Worksheet – Propagating uncertainties in quadrature
Reference: Taylor, Section 3.5, 3.6, 3.7

Group: ___________________________

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1. The rules for combining uncertainties presented earlier in the course can be summarized as follows: When measured quantities area added or subtracted, the uncertainties add; when measured quantities are multiplied or divided, the fractional uncertainties add. We called these the provisional rules. However, in certain conditions, these calculations can yield uncertainties that are unnecessarily large. In this worksheet we will explore one common situation, the case of independent and random uncertainties.

   Case 1. Adding and Subtracting

   Can we just add the uncertainties when adding two (or more) quantities?

   Consider a measurement of the length *l* and width *w* of a desk. What impact does the uncertainty have on the sum of these two quantities? Consider the following measurements and fill in the table below.

Measurement	Length Measurement	Width Measurement	Length + Width
1	too big	too big	much too big
2	too big	too small	about right
3	too small	too big	
4	too small	too small	

   What fraction of the time are you likely to get the right answer?

   This effect of adding quantities with random uncertainties is accounted for quantitatively by combining the uncertainties as follows: square each of them, add the squares and then take the square root. This called "adding in quadrature".

   To see how this works in a practical example, consider a measurement of length = 1.21 ± 0.04 m and width = 0.85 ± 0.03 m. Complete the following table:

half of maximum range of (l + w)	
$\sqrt{(\Delta l)^2 + (\Delta w)^2}$	

   Adding uncertainties in quadrature, our best estimate of the result is that length + width = 2.06 +/- 0.05 m. Using the provisional rule, we would have ascribed an error of ±0.07 m to the sum the length and the width. If the two measurements are independent of each other, then the result that we get by adding in quadrature, 0.05 m, accounts for the fact that one measurement could be above the mean and the other below the mean of the distributions of the two measurements. The uncertainty estimate obtained by adding in quadrature will always be less that the result obtained by the provisional rule, but is a more reasonable estimate of the uncertainty.

If we need to subtract quantities, the uncertainties still add in quadrature. After all, we don't know if the measurement would be above or below the mean so the result is still uncertain by the same amount as if we were adding. In this example, (length – width) = 0.36 +/- 0.05 m. As we have seen previously, subtracting two quantities with random errors often results in a small number with a large relative uncertainty.

We can summarize as follows:

$$\text{If } Y = A \pm B \pm C,$$

$$\Delta Y = \sqrt{(\Delta A)^2 + (\Delta B)^2 + (\Delta C)^2}$$

Case 2. Multiplying and Dividing

The rule for multiplying is similar to that of addition but instead of using the absolute uncertainty, *Δx*, one adds the relative uncertainty, *Δx/x*, in quadrature. The relative uncertainty of a product of two numbers is calculated by adding the relative uncertainties of the numbers in quadrature.

Consider a measurement of length = 1.91 ± 0.04 m and width = 0.45 ± 0.03 m. Complete the following table:

half of maximum range of (l · w)	
$\frac{\Delta l}{l}$	
$\frac{\Delta w}{w}$	
$\sqrt{\left(\frac{\Delta l}{l}\right)^2 + \left(\frac{\Delta w}{w}\right)^2}$	
Δ (l · w)	

Our best estimate of the result is that (length x width) = 0.86 +/- 0.06 m^2.

Notice that when one of the uncertainties is more than double that of the other, the result of adding in quadrature is not much different than just the largest error alone — there's really no need to calculate in this case.

Similarly, when one divides two quantities, the relative uncertainty of the result is the sum of the relative errors, added in quadrature.

We can summarize as follows:

If $Y = ABC, Y = ABC^{-1}, Y = AB^{-1}C^{-1},$ etc.,

$$\frac{\Delta Y}{Y} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2 + \left(\frac{\Delta C}{C}\right)^2}$$

Case 3. Constants

If x is measured with uncertainty δx, and is used to calculate the quantity $y=ax$ where a is a constant, can we treat this as a product and use the rule above? Explain your reasoning.

Case 4. Powers

If x is measured with uncertainty δx, and is used to calculate the power $y=x^n$, can we treat this as a product and use the rule above? Explain your reasoning.