Q01.14

Question

For given n, row vector $[B_{0,n}(u), B_{1,n}(u), \cdots, B_{n,n}(u)]$ can be written as $[1, u, u^2, \cdots, u^n]M$, where M is an $(n+1) \times (n+1)$ matrix. Thus, a Bézier curve can be written in matrix form, $\mathbf{C}(u) = [u^i]^T M[\mathbf{P}_i]$. Compute the matrix M for $n = 1, 2, 3$. Notice that setting $[\mathbf{a}_i] = M[\mathbf{P}_i]$ yields the conversion of a Bézier curve to power basis form. Assuming $0 \le u \le 1$, $[\mathbf{P}_i] = M^{-1}[\mathbf{a}_i]$ gives the conversion from power basis to Bézier form.

Solution

According to the definition of Bernstein basis function in Eq.(1.8), we have

$B_{i,n}(u) = \binom{n}{i} u^i (1-u)^{n-i}$.

The coefficient on u^k in $B_{i,n}(u)$ is denoted as $\mathcal{C}^k_{i,n}$. It has the following form:

$$\mathcal{C}^k_{i,n} = \begin{cases} \binom{n}{i}\binom{n-i}{k-i}(-1)^{k-i} & k \ge i \\ 0 & k < i \end{cases}$$

Therefore we have the following form of matrix element m_{ij} for M:

$m_{ki} = \mathcal{C}^k_{i,n}$.

The full matrix is

$$M_n = \begin{bmatrix} \mathcal{C}^0_{0,n} & \mathcal{C}^0_{1,n} & \mathcal{C}^0_{2,n} & \cdots & \mathcal{C}^0_{n,n} \\ \mathcal{C}^1_{0,n} & \mathcal{C}^1_{1,n} & \mathcal{C}^1_{2,n} & \cdots & \mathcal{C}^1_{n,n} \\ \mathcal{C}^2_{0,n} & \mathcal{C}^2_{1,n} & \mathcal{C}^2_{2,n} & \cdots & \mathcal{C}^2_{n,n} \\ \vdots & & & \ddots & \vdots \\ \mathcal{C}^n_{0,n} & \mathcal{C}^n_{1,n} & \mathcal{C}^n_{2,n} & \cdots & \mathcal{C}^n_{n,n} \end{bmatrix}$$

Since we have $m_{ki} = 0$ for $k < i$, the above matrix is a lower triangular matrix.

Let us write down the first few matrices. For $n = 1$ we have

$$M_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Code

The matrix element is computed by the following Julia code.

```julia
C(k,i,n) = (-1)^(k-i) * binomial(n,i) * binomial(n-i,k-i)
M(n) = [C(k,i,n) for k=0:n, i=0:n]
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