Lecture Notes For: Functional Analysis

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0. Contents

1. Point-set Topology

We will review some basic notions of the topology, and then we will present solved solutions for the related problems.

Definition 1.1 Let $(X, \mathcal{T})$ be a topological space an let $A \subseteq X$ be a subset. Then

- The *interior* of A denoted by A° is defined as

$$A^\circ = \bigcup_{\substack{V \subset A,\\ V \text{open}}} V.$$

  In words, the interior of a set is the union of all open sets contained in the set.

- The *closure* of A denoted by $\overline{A}$ is defined as

$$\overline{A} = \bigcap_{\substack{F \supset A,\\ F \text{closed}}} F.$$

  In words, the closure of a set is the intersection of all closed sets containing A.

- The *boundary* or A is defined as

$$\partial A = \overline{A} \backslash A^\circ.$$

- A is *dense* in X if

$$\overline{A} = X.$$

- A is *nowhere dense* if

$$(\overline{A})^\circ = \varnothing.$$

■ **Remark** Consider the following remarks for the definition above.

- By the definition above, if $x \in A^\circ$, then there exists $V \in \mathcal{T}$ such that $x \in V \subset A$. Also, we can interpret the interior of A as the largest open set contained in A.
- We can interpret $\overline{A}$ as the smallest closed set containing A. There is a very interesting parallel between this definition and the notion of smallest σ-algebra containing a collection. The smallest σ-algebra containing a collection is the intersection of all σ-algebra that contains

that collection.

Proposition 1.1 — Basic Properties. Let $(X, \mathcal{T})$ be a topological space, and $A, F \subseteq X$ a subset. Then we have

(a) $A^\circ \subseteq A \subseteq \overline{A}$.

(b) A° is open and $\overline{F}$ is closed.

(c) A is open iff $A = A^\circ$.

(d) F is closed iff $F = \overline{F}$.

(e) $(\overline{A})^c = (A^c)^\circ$.

(f) $(A^\circ)^c = \overline{(A^c)}$.

(g) A is open iff it is a neighborhood of all of its points.

(h) If $A_1 \subseteq A_2$ then $A_1^\circ \subseteq A_2^\circ$ as well as $\overline{A_1} \subseteq \overline{A_2}$.

(i) $(A^\circ)^\circ = A^\circ$, and $\overline{(\overline{A})} = \overline{A}$.

(j) $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$.

(k) $(A_1 \cap A_2)^\circ = A_1^\circ \cap A_2^\circ$.

(l) $\overline{A} = A \cup A'$, where A' is the derived set of A.

(m) A is closed iff $A' \subset A$. In words, A is closed iff it contains all of its accumulation points.

Proof. (a) Let $x \in A^\circ$. Then $\exists V \in \mathcal{T}$ such that $x \in V \subseteq A$. Thus $x \in A$, so $A^\circ \subseteq A$. For the second part, Let $x \in A$. Then $x \in F$ for every F that contains A. Consider the intersection of all such Fs that are also closed. x also belongs to their intersection, which is by definition $\overline{A}$. So $A \subseteq \overline{A}$.

(b) A° is open since it is the union of open sets. $\overline{F}$ is closed since it is the intersection of closed sets.

(c) First, we assume A is open. Since $A^\circ = \bigcup V$ for all $V \subseteq A$ and V open, we can take the collections of open sets on the RHS to be only A, and it proves that $A^\circ = A$. For the other direction, we assume $A = A^\circ$. We know that A° is open. Thus A is also open.

(d) First, we assume that F is closed. Then since $\overline{F} = \bigcap A$ where $F \subseteq A$ and A is closed, we can take the union on the RHS to be F and this proves that $F = \overline{F}$. For the converse, we assume $F = \overline{F}$. Since $\overline{F}$ is open this implies that F is closed.

(e) Let $x \in (\overline{A})^c$. This implies $x \in (\overline{A})^c = (\bigcap_{\substack{A \subseteq F, \\ F \text{closed}}} F)^c = \bigcup_{\substack{A \subseteq F, \\ F \text{closed}}} F^c$. Let $F^c = V$. Then we can write

$$x \in \bigcup_{\substack{V \subseteq A^c \\ V \text{open}}} V = (A^c)^\circ.$$

So $(\overline{A})^c \subseteq (A^c)^\circ$. For the converse, let $x \in (A^c)^\circ$. This implies $x \in \bigcup_{\substack{V \subseteq A^c, \\ V \text{open}}} V$. Or equivalently $x \notin \bigcap_{\substack{V\subseteq A^c,\\ V\text{open}}} V^c$. Let $F = V^c$. Then we can write

$$x \notin \bigcap_{\substack{A\subseteq F,\\ F\text{closed}}} F = \overline{A}.$$

So $x \in (\overline{A})^c$. Thus we conclude that $(\overline{A})^c = (A^c)^\circ$.

(f) Let $x \in (A^\circ)^c$. Then

$$x \in (\bigcup_{\substack{V\subseteq A,\\ V\text{open}}} V)^c = \bigcap_{\substack{V\subseteq A,\\ V\text{open}}} V^c = \bigcap_{\substack{A^c\subseteq F,\\ F\text{closed}}} F = \overline{A^c}.$$

This implies $(A^\circ)^c \subseteq \overline{A^c}$. For the converse let $x \in \overline{A^c}$. Then $x \in \bigcap_{\substack{A^c\subseteq F,\\ F\text{closed}}} F$. This implies

$$x \notin \bigcup_{\substack{A^c\subseteq F,\\ F\text{closed}}} F^c = \bigcup_{\substack{V\subseteq A,\\ V\text{open}}} V = A^\circ.$$

This implies that $x \in (A^\circ)^c$. Thus $\overline{A^c} \subseteq (A^\circ)^c$.

(g) We assume that A is open. Then for any $x \in A$ we have $x \in A \subseteq A$. Thus A is a neighborhood of x. For the converse, we assume that A is a neighborhood of all of its points. So for any $x \in A$ there exits $V_x \in \mathcal{T}$ such that $x \in V \subseteq A$. A can be written as $A = \bigcup_x V_x$ where V_x is as above. This A is open.

(h) Let $x \in A_1^\circ$. Then $\exists V \in \mathcal{T}$ such that $x \in V \subseteq A_1$. From assumption we also have $x \in V \subseteq A_2$. This implies that $x \in A_2^\circ$. For the second statement, let $x \in \overline{A_1}$.

(i) to be added.

(j) to be added.

(k) to be added.

(l) $\boxed{\Rightarrow}$. We want to show $\overline{A} \subseteq A \cup A'$. We will prove by contrapositive. I.e. we equivalently prove $A^c \cap (A')^c \subseteq \overline{A}^c$. Let $x \in A^c \cap (A')^c$. This implies that $x \notin A$ as well as $x \notin A'$. So $\exists U \in \mathcal{T}$ such that $A \cap U = \varnothing$ (note that we both used $x \notin A$ and $x \notin A'$). Thus $x \in U \subseteq A^c$. This implies $x \in (A^c)^\circ = \overline{A}^c$.

$\boxed{\Leftarrow}$. We know that $A \subseteq \overline{A}$. So it suffices to show $A' \subseteq \overline{A}$. It is easier to prove the contrapositive, i.e. $(\overline{A})^c \subseteq (A')^c$ or equivalently $(A^c)^\circ \subseteq (A')^c$. Let $x \in (A^c)^\circ$. This implies $\exists U \in \mathcal{T}$ such that $x \in U \subset A^c$. So $A \cap (U \backslash \{x\}) = \varnothing$, thus $x \notin A'$, or equivalently $x \in (A')^c$.

(m) $\boxed{\Rightarrow}$. Assume A is closed. Then $A = \overline{A}$. Using above we will have $\overline{A} = A \cup A'$ it implies that $A' \subseteq A$.

$\boxed{\Leftarrow}$. Assume $A' \subseteq A$. Then from above $\overline{A} = A \cup A'$ it implies that $\overline{A} = A$, hence A is closed. □

■ **Remark** In item (e), by taking the complement from both sides we will have

$$\overline{A} = ((A^c)^\circ)^c$$

1.1 Sporadic Notes

In this section I will include the nodes that do not fit with the current layout of the document and will be added later when I start writing the corresponding sections.

observation 1.1 The subspace topology is the weakest topology that makes the inclusion map continuous. Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. The topological space $(A, \mathcal{T}_A)$ is a topological space and $\mathcal{T}_A$ is called the subspace topology. Define the inclusion map

$$\iota : A \to X.$$

The subspace topology is the weakest topology for which ι is continuous. Let $U \in \mathcal{T}$. Then $\iota^{-1}(U) = A \cap U \in \mathcal{T}_A$. TODO: I above I just showed that under the subspace topology, the inclusion map is continuous. However, I also need to show that $\mathcal{T}_A$ is the smallest such topology.

2. Solution Manual Folland

Here is a list of theorems that are used in the problem sets.

Proposition 2.1 — Some properties. 1. Let $T \in L(X, X)$. Then $\|T^n\| \leq \|T\|^n$.

Proof. (a) We demonstrate the statement for the case where $n = 2$, and the general result follows by induction. Observe that

$$\left\|T^2 x\right\| = \|T(Tx)\| \leq \|T\|\|Tx\| \leq \|T\|^2\|x\|.$$

Since $\left\|T^2\right\|$ is smallest constant C such that $\left\|T^2 x\right\| \leq C\|x\|$ for all $x \in X$, the it follows that $\left\|T^2\right\| \leq \|T\|^2$.

□

2.1 Elements of Functional Analysis

■ **Problem 2.1 — Folland: Ch5,P7.**

Solution (a) First, we want to show that the series $\sum_{n=0}^{\infty}(I-T)^n$ converges. First, observe that this series converges absolutely. Because

$$\sum_{n=0}^{\infty}\left\|(I-T)^n\right\| \leq \sum_{n=0}^{\infty}\|I-T\|^n \leq \sum_{n=0}^{\infty}\gamma^n = \frac{1}{1-\gamma} < \infty.$$

Using the fact that X is a Banach space (hence complete), it follows that $L(X, X)$ is also complete, thus by Theorem 5.1 Folland the absolutely convergent series converges in $L(X, X)$. Let

$$L(X, X) \ni X = \sum_n (I-T)^n.$$

Now we want to prove that X is left and right inverse of T. To see this we can write

$$(I-T)X = \sum_{n=0}^{\infty}(I-T)^{n+1} = \sum_{n=1}^{\infty}(I-T)^n = \sum_{n=0}^{\infty}(I-T)^n - I = X - I.$$

This implies

$$TX = X.$$

With a similar argument we can get $X(I-T) = X - I$, thus $XT = I$. So we conclude that X is the right and the left inverse of T, thus T is a bijection and $X = T^{-1}$.

■ **Remark** I think in above, when we proved that $X \in L(\mathcal{X}, \mathcal{X})$, we automatically proved that T^{-1} is bounded. However, in the solution manual that I got the idea of proof, the author separately proves that T^{-1} is bounded. For the sake of completeness I will do the same here as well.

To show that T^{-1} is bounded, consider the sequence of partial sums of T^{-1}

$$S_n = \sum_{i=1}^{n} (I-T)^n.$$

Using the continuity of $\|\cdot\|$ we can write

$$\left\|T^{-1}x\right\| = \left\|\lim_n S_i x\right\| = \lim_n \|S_n x\| \leq \lim_n \sum_{i=0}^{n} \left\|(I-T)^n x\right\| \leq \lim_n \sum_{i=0}^{n} \|I-T\|^n \|x\| \leq \frac{\|x\|}{1-\gamma}.$$

(b) Observe that

$$\left\|(ST^{-1} - I)\right\| = \left\|(ST^{-1} - I)TT^{-1}\right\| = \left\|ST^{-1} - TT^{-1}\right\| \leq \|(S-T)\| \left\|T^{-1}\right\| < 1.$$

So ST^{-1} has an inverse $A \in L(\mathcal{X}, \mathcal{X})$ and we have $A = TS^{-1}$. So $S^{-1} = T^{-1}A$. It is also easy to see that S^{-1} is bounded. Because

$$\left\|S^{-1}x\right\| \leq \left\|T^{-1}\right\| \|A\| \|x\|.$$