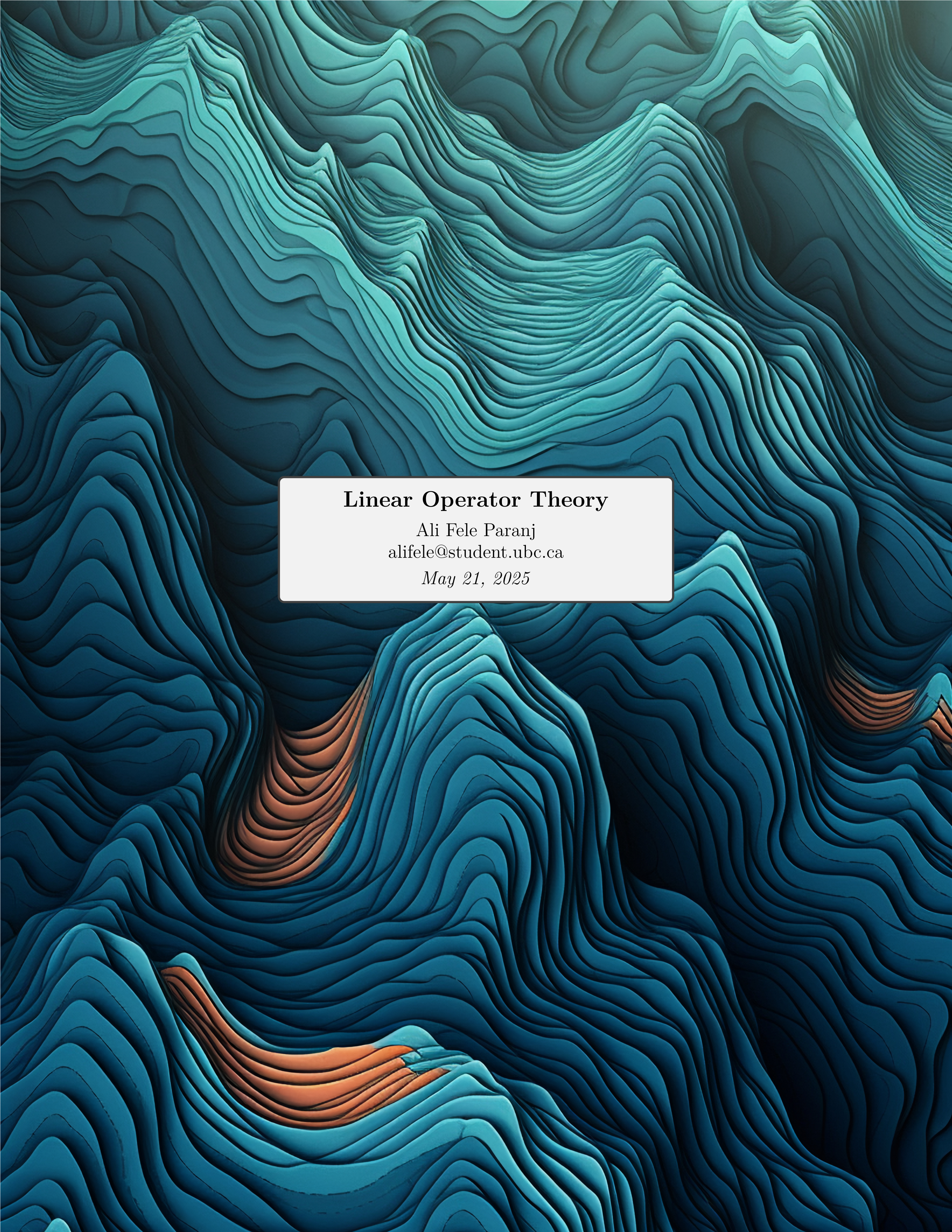

Linear Operator Theory

Ali Fele Paranj
alifele@student.ubc.ca

May 21, 2025

0. Contents

1. Topological Spaces

1.1 Solved Problems

■ **Problem 1.1** Let d be a metric on X and let $d_\alpha = \alpha d$, where $0 < \alpha < 1$. Show that $d_\alpha \not\equiv d$ if X has more than two points.

Solution Let X have more than one point. Then $\exists x, y \in X$ such that $d(x,y) \neq 0$. Then $d(x,y) \neq \alpha d(x,y) = d_\alpha(x,y)$. So $d \not\equiv d_\alpha$.

■ **Problem 1.2** Show that $d(x,y) = |x-y|$ is a metric on the real number $\mathbb{R}$. Show that it is also a metric on the set of complex numbers $\mathbb{C}$.

Solution First, we show $(\mathbb{R}, d)$ is a metric space. We need to show:

(i) First, we show that d is always positive. Since $|x| = \max\{x, -x\}$, we have $\pm x \leq |x|$. Adding to two inequalities we will get $0 \leq |x|$. Now we want to show that $|x-y| = 0$ then $x = y$. $|x-y| = 0$ implies $x - y = 0$, and then we will have $x = y$.

(ii) Since $|x-y| = \max x-y, y-x$, we will have $|y-x| = \max y-x, x-y = |x-y|$.

(iii) Triangle inequality: From the definition of absolute value we have $|x| = \max\{x, -x\}$. So $x \leq |x|$ and $-|x| \leq x$. Let $a, b \in \mathbb{R}$. Then

$$a \leq |a|,\ b \leq |b| \quad \Longrightarrow \quad a + b \leq |a| + |b|.$$

Furthermore,

$$-|a| \leq a,\ -|b| \leq b \quad \Longrightarrow \quad -|a| - |b| \leq a + b.$$

Combining these two inequalities, we will get

$$-|a| - |b| \leq a + b \leq |a| + |b|.$$

So

$$|a + b| \leq |a| + |b|.$$

Now we show that $(\mathbb{C}, d)$ is a metric space. For this purpose we can utilize the properties of the complex conjugation and work with the definition $|z_1 - z_2| = \sqrt{(z_1 - z_2)\overline{(z_1 - z_2)}}$. But instead, we want to show that there is a bijection from $(\mathbb{C}, d)$ (we don't know yet if it is a metric space, otherwise we could call this map an isometric bijection) to $(\mathbb{R}^2, d_{\mathbb{R}^2})$. Recall

$$\mathbb{C} = \{a + ib : \ a, b \in \mathbb{R}\}.$$

Let $\phi : \mathbb{C} \to \mathbb{R}^2$ given by $a + ib \mapsto (a, b)$. This is a bijection, since for any $a + ib \in \mathbb{C}$ we have a unique $(a, b) \in \mathbb{R}^2$ (uniquely determined by a, b), and for any $(a, b) \in \mathbb{R}^2$ we have a uniquely determined $a + ib \in \mathbb{C}$. Also we claim that

$$d(z, w) = d_{\mathbb{R}^2}(\phi(z), \phi(w)).$$

This is true because

$$d(z, w) = d(z_1+iz_2, w_1+iw_2) = \sqrt{(z_1 - w_1)^2 + (z_2 - w_2)^2} = d_{\mathbb{R}^2}((z_1, z_2), (w_1, w_2)) = d_{\mathbb{R}^2}(\phi(z), \phi(w)).$$

We can use this map to transfer all of the properties of $d_{\mathbb{R}^2}$ in $\mathbb{R}^2$ to d in $\mathbb{C}$. So $(\mathbb{C}, d)$ is indeed a metric space. So now we can call ϕ an isometric bijection.

■ **Remark 1.1** In the question above, for $(\mathbb{R}, d)$ we used the definition $|x| = \max\{x, -x\}$, and for $(\mathbb{C}, d)$ we used the definition $|x| = \sqrt{x\overline{x}}$.

■ **Problem 1.3** Let $d(x, y)$ be a metric on X. Show that

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \qquad d_2(x, y) = \min\{1, d(x, y)\},$$

are also metrics on X. Show that every set in the metric space (X, d_1) and (X, d_2) is bounded.

Solution (i) Showing that $d_1(x, y)$ is a metric: Positive definiteness: since the enumerator and denumenator are both positive, then $d_1(x, y)$ is also positive for all $x, y \in X$. Let $x, y \in X$ such that $d_1(x, y) = 0$. This implies $d(x, y) = 0$, hence $x = y$. Symmetry follows from d being symmetric. For the triangle inequality we use the fact that $f(x) = \frac{x}{1+x}$ is strictly increasing for $x \geq 0$ (because it has positive derivative), and also it is subadditive. To see the subadditivity, observe that

$$f(x + y) \leq \frac{x + y}{1 + x + y} = \frac{x}{1 + x + y} + \frac{y}{1 + x + y} \leq \frac{x}{1 + x} + \frac{y}{1 + y} = f(x) + f(y).$$

Using the fact that for all $x, y, z \in X$ we have

$$d(x, z) \leq d(x, y) + d(y, z),$$

using the monotinicity of f and then its sub additivity, we will get

$$d_2(x, z) \leq d_2(x, y) + d_2(y, z).$$

(See the summary box below for a detailed argument of a more general setting).

(ii) Showing that $d_2(x, y) = d(x, y) \wedge 1$ is a metric: Positive definiteness: Since $d(x, y) \geq 0$ then $d(x, y) \wedge 1 \geq 0$. Let $x, y \in X$ such that $d(x, y) \wedge 1 = 0$. So $d(x, y) = 0$. This implies $x = y$. So d_2 is positive definite. Symmetric: Let $x, y \in X$. Then $d_2(x, y) = d(x, y) \wedge 1 = d(y, x) \wedge 1 = d_2(y, x)$, where we have used the fact that $d(x, y) = d(y, x)$. So $d_2(x, y) = d_2(y, x)$. For the triangle inequality we use the following lemma:

Lemma 1.1 Let $a, b \geq 0$. Then

$$a < b \qquad \Longrightarrow \qquad a \wedge 1 \leq b \wedge 1.$$

Also

$$(a+b) \wedge 1 \leq a \wedge 1 + b \wedge 1.$$

Proof. For the first implication, when $a < b$, then there are three cases: $0 \leq a < b \leq 1$, $0 \leq a \leq 1 < b, 1 < a < b$. In the first case $a \wedge 1 = a, b \wedge 1 = b$ So $a \wedge 1 \leq b \wedge 1$ holds. This holds for the other cases as well.

For the second implication, again we will show in cases: When $a+b < 1$, $a+b = 1$, $a+b > 1$. For the first case we can only have $a, b < 1$. So $(a+b) \wedge 1 = a+b$, $a \wedge 1 = a$, $b \wedge 1 = b$. So the desired inequality holds. For the second case, again $a = a \wedge 1$, $b = b \wedge 1$, and $(a+b) \wedge 1 = a+b = 1$. So the desired inequality holds. For the third case, WLOG we can assume that $a \leq b$. Since $a + b \geq 1$, then at least one of a or b should be larger than 1. WLOG we can assume $a \geq 1$. So $(a+b) \wedge 1 = 1$ and $a \wedge 1 = 1$. So the desired inequality holds. □

So by the triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$, and using the lemma above we can conclude that

$$d(x,z) \wedge 1 \leq d(x,y) \wedge 1 + d(y,z) \wedge 1.$$

Now to show that all of the sets in X, with the metrics above are bounded, it is enough to observe that for all $x, y \in X$ we have $d_1(x,y) \leq 1$ and $d_2(x,y) \leq 1$.

Summary 1.1 — Transformation of the metric. Using the proof ideas of the examples above, we can generalize it in the following proposition.

Proposition 1.1 Let (X,d) be a metric space and $\phi : [0, +\infty) \to \mathbb{R}$ be a real valued functions such that

(i) $\phi(x) = 0$ if and only if $x = 0$.

(ii) monotone increasing,

(iii) subadditive: $\forall x, y \geq 0$ we have $\phi(x+y) \leq \phi(x) + \phi(y)$. Then $\tilde{d} = \phi \circ d$ is also a metric, and $(X, \tilde{d})$ is a metric space.

Proof. Since $\phi(x) = 0$ if and only if $x = 0$, then $\tilde{d} = 0$ if and only if $d = 0$, if and only if $x = y$. So $\tilde{d}$ is positive definite. $\tilde{d}$ is also symmetric, because $\forall x, y \in X$, we have $d(x,y) = d(y,x)$ that implies $\tilde{d}(x,y) = \phi(d(x,y)) = \phi(d(y,x)) = \tilde{d}(y,x)$. For the triangle inequality, we have

$$d(x,z) \leq d(x,y) + d(y,z).$$

We apply (ii) above and we will get

$$\tilde{d}(x,z) \leq \phi(d(x,y) + d(y,z)),$$

and we now apply (iii) above to get

$$\tilde{d}(x,z) \leq \tilde{d}(x,y) + \tilde{d}(y,z).$$

□

Summary 1.2 — Metric transformation examples. Using the example above, and the remark below, the followings are example functions that can transform a metric d to a new metric $\tilde{d} = f(d)$.

(i) $f_1(x) = x \wedge 1$.

(ii) $f_2(x) = x/(1+x)$.

(iii) $f_3(x) = x^\alpha/(1+x^\alpha)$ for $0 < \alpha \leq 1$.

■ **Remark 1.2** The reason that the example (iii) above works is that f_3 satisfies all the properties in the summary box above. To see the sub-additivity, it is enough to show the sub-additivity of x^α when $0 < \alpha \leq 1$. Let $\alpha = 1 - \beta$. Then

$$(x+y)^\alpha = \frac{x+y}{(x+y)^\beta} < \frac{x}{(x+y)^\beta} + \frac{y}{(x+y)^\beta} \leq \frac{x}{x^\beta} + \frac{y}{y^\beta} = x^\alpha + y^\alpha.z$$

■ **Problem 1.4** A real-valued function $\rho(x, y)$ is said to be a pseudometric on X if it satisfies conditions (M1), (M3), and (M4).

(a) Show that $\rho(x, y) \equiv 0$ is a pseudometric on any set X.

(b) Show that $\rho((x_1, x_2), (y_1, y_2)) = |x_1 - y_1|$ is a pseudometric in the plane $\mathbb{R}^2$.

Solution (a) $\rho(x, y) \equiv 0$ satisfies $(M1), (M3)$, and $(M4)$ vacuously. However, if X has only one element, then ρ is indeed a metric. But if X has more than two elements, then $\exists x, y \in X$ such that $x \neq y$ but $\rho(x, y) = 0$.

(b) Being symmetric (M3) and the triangle inequality (M4) follows from the properties of $|\cdot|$. However, for all $x, y \in \mathbb{R}^2$ that have the same first component (i.e. points on the vertical lines) have $d(x, y) = 0$. However, this will be a metric on the quotient space $\mathbb{R}^2/W$ where $W = \text{Span}\{(0, 1)\}$.

■ **Problem 1.5** Show that if A is nonempty, in a metric space (X, d), then $\operatorname{diam} A = 0$ if and only if A consist of a single point. Is this true in a pseudometric space?

Solution For the forward direction assume for $A \subset X$ we have $\operatorname{diam} A = 0$. So

$$\sup_{x,y \in A} \{d(x, y)\} = 0.$$

This implies that $\forall x, y \in A$ we have $d(x, y) = 0$. Since d is a metric, then $A = \{x\}$ is a singleton. For the converse, let $A = \{x\}$. Then it follows immediately that $\operatorname{diam} A = 0$.

No this is not true in the case of the pseudometric spaces. In our proof above, in the forward direction, the logic breaks if d is a pseudometric. I.e. it is possible to have $\sup_{x,y \in A}\{d(x, y)\} = 0$ but A is not singleton.

■ **Problem 1.6** Let $X = \mathbb{R}^2$ and let

$$d(x, y) = (|x_1 - y_1|^{1/2} + |x_2 - y_2|^{1/2})^2.$$

Is (X, d) a metric space?

Solution No. Because the triangle inequality breaks. Let

$$x = (0,0), \qquad y = (0,1/4), \qquad z = (1/4,1/4).$$

It is easy to check that the triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$ does not hold. Because

$$1 = d(x,z) \leq d(x,y) + d(y,z) = 1/4 + 1/4 = 1/2,$$

is a contradiction.

■ **Remark 1.3** With a similar idea of proof, it is easy to show that d_p fails to satisfy the triangle inequality if $0 < p < 1$.

■ **Problem 1.7** In $\mathbb{R}^2$ let $A\{x = (x_1, x_2) : (|x_1|^2 + |x_2|^2)^{1/2} < 1\}$, and

$$d(x,y) = \{|x_1 - y_1|^2 + |x_2 - y_2|^2\}^{1/2}.$$

Compute $d(x,A)$. Show that $d(x,A) = 0$ if and only if

$$|x_1|^2 + |x_2|^2 \leq 1.$$

Solution Let $x = (x_1, x_2) \in \mathbb{R}^2$ (WLOG we assume that $x_1 > 0$) that does not belong to A. The ray that passes through x is given by $y = \frac{x_2}{x_1}x$, and this ray intersects the unit circle at

$$\tilde{x} = (\frac{1}{\sqrt{1+(x_2/x_1)^2}}, \frac{x_2/x_1}{\sqrt{1+(x_2/x_1)^2}}).$$

So the distance between x and the point above is

$$d(x,\tilde{x}) = \left(|\frac{1}{\sqrt{1+(x_2/x_1)^2}} - x_1|^2 + |\frac{x_2/x_1}{\sqrt{1+(x_2/x_1)^2}} - x_2|^2 \right)^{1/2}.$$

To show that $d(x,A) = 0$ if and only if $x_1^2 + x_2^2 \leq 1$, we first do the forward direction. Let $d(x,A) = 0$. From our explicit formula above, it is easy to see that we should have

$$x_1 = \frac{1}{\sqrt{1+(x_2/x_1)^2}}, \qquad x_2 = \frac{x_2/x_1}{\sqrt{1+(x_2/x_1)^2}}.$$

It is easy to check that $x_1^2 + x_2^2 = 1$ which implies $x_1^2 + x_2^2 \leq 1$. For the converse direction, we assume $x_1^2 + x_2^2 \leq 1$. When $x_1^2 + x_2^2 < 1$, then $x \in A$, and by the definition of $d(x,A)$ we see $d(x,A) = 0$. When $x_1^2 + x_2^2 = 1$, then we can write $1 + (x_1/x_2)^2 = 1/x_1^2$. Using the fact that $x_1 > 0$ this implies that

$$d(x,A) = 0,$$

where we have used the explicit formula above.

■ **Problem 1.8** In example 5, the metric d_∞ was defined with a "sup" instead of "max". In order to see the necessity of this let $x = (x_1, x_2, \cdots), y = (y_1, y_2, \cdots)$ be given by

$$x_n = \frac{1}{n+1}, \qquad y_n = \frac{n}{n+1}.$$

And argue why "sup" should be used instead of "max"? What about Example 8 and Example 11?