Lecture Notes For: Advanced Linear Algebra

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This lecture note is mostly based on the course MATH 257 Partial Differential Equations at UBC (2023). However I have also expanded the contents and examples from the following books.

1 Basics

Partial differential equations relate the partial derivatives of a function to each other. For example f can be a function of spacial coordinates (like x, y, z in the case of Cartesian coordinates), dynamical variable (like time), or any other kind of variables (like the space of genotypes g). For example suppose that $\Phi(x, y)$ represents the electric potential of a point charge. Such function should satisfy the Laplace equation:

$$\partial_{xx}\Phi + \partial_{yy}\Phi = 0$$

.

Note that the symbols ∂_{xx} and ∂_{yy} are short symbols for $\frac{\partial}{\partial x^2}$ and $\frac{\partial}{\partial y^2}$ respectively.

Definition: Order of PDE

The order of a PDE is the highest derivative that occurs in the equation.

Based on the definition above, the Laplace equation is a second order partial differential equation.

1.1 Classification of The Second Order PDEs

There are three categories of the second order PDEs that every other type of a second order PDE can be converted to one of these kinds. The most general type of a second order PDE can be written as:

$$A\partial_{xx}u + B\partial_{xy}u + C\partial_{yy}u + D\partial_x u + E\partial_y u + Fu = k \tag{1.1}$$

In which the coefficients are all a function of x, y (but not u in which case the PDE will be nonlinearx). Equation 1.1 can be summarized in a more compact form using the derivative operator L:

$$\mathrm{L}\, u = 0$$

,

in which:

$$\mathrm{L} = A\partial_{xx} + B\partial_{xy} + C\partial_{yy} + D\partial_x + E\partial_y + F$$

.

Because of the similarities of the equation 1.1 with the generic quadratic equation describing the conic sections, we call each class of second order PDEs with its corresponding conic section. The generic equation describing the conic sections is:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + K = 0. \tag{1.2}$$

PDE	Analogous Quad Surf	Δ	Class	Application
$u_t = u_{xx}$	$T = x^2$	0	parabolioc	Diffusion - Heat Equation
$u_{tt} = u_{xx}$	$T^2 = x^2$	$\Delta > 0$	Hyperbolic	Wave Equation
$u_{xx} + u_{yy} = 0$	$x^2 + y^2 = 0$	$\Delta < 0$	Elliptic	Laplace
$u_{xx} + u_{yy} = c$	$x^2 + y^2 = k$			Poisson

Table 1: A summary of the three class of second order linear PDE.

All of the conic sections (ellipse, parabola, hyperbola) can be described with the equation 1.2 which is determined with the discriminant $\Delta = B^2 - 4AC$. for $\Delta = 0$, $\Delta > 0$, and $\Delta < 0$ the conic section will be parabolic, hyperbolic, and elliptic respectively. Table 1 summarizes special categories of the linear second order PDEs that frequently occur in physical applications.

1.2 Intuitive Derivation of the Second Order PDEs