Lecture Notes For: Functional Analysis

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0. Contents

1. Point-set Topology

We will review some basic notions of the topology, and then we will present solved solutions for the related problems.

Definition 1.1 Let $(X, \mathcal{T})$ be a topological space an let $A \subseteq X$ be a subset. Then

- The *interior* of A denoted by A° is defined as
$$A^\circ = \bigcup_{\substack{V \subset A, \\ V \text{open}}} V.$$
In words, the interior of a set is the union of all open sets contained in the set.
- The *closure* of A denoted by $\overline{A}$ is defined as
$$\overline{A} = \bigcap_{\substack{F \supset A, \\ F \text{closed}}} F.$$
In words, the closure of a set is the intersection of all closed sets containing A.
- The *boundary* or A is defined as
$$\partial A = \overline{A} \backslash A^\circ.$$
- A is *dense* in X if
$$\overline{A} = X.$$
- A is *nowhere dense* if
$$(\overline{A})^\circ = \varnothing.$$

■ **Remark** Consider the following remarks for the definition above.

- By the definition above, if $x \in A^\circ$, then there exists $V \in \mathcal{T}$ such that $x \in V \subset A$. Also, we can interpret the interior of A as the largest open set contained in A.
- We can interpret $\overline{A}$ as the smallest closed set containing A. There is a very interesting parallel between this definition and the notion of smallest σ-algebra containing a collection. The smallest σ-algebra containing a collection is the intersection of all σ-algebra that contains

that collection.

Proposition 1.1 — Basic Properties. Let $(X, \mathcal{T})$ be a topological space, and $A, F \subseteq X$ a subset. Then we have

(a) $A^\circ \subseteq A \subseteq \overline{A}$.

(b) A° is open and $\overline{F}$ is closed.

(c) A is open iff $A = A^\circ$.

(d) F is closed iff $F = \overline{F}$.

(e) $(\overline{A})^c = (A^c)^\circ$.

(f) $(A^\circ)^c = \overline{(A^c)}$.

(g) A is open iff it is a neighborhood of all of its points.

(h) If $A_1 \subseteq A_2$ then $A_1^\circ \subseteq A_2^\circ$ as well as $\overline{A_1} \subseteq \overline{A_2}$.

(i) $(A^\circ)^\circ = A^\circ$, and $\overline{(\overline{A})} = \overline{A}$.

(j) $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$.

(k) $(A_1 \cap A_2)^\circ = A_1^\circ \cap A_2^\circ$.

(l) $\overline{A} = A \cup A'$, where A' is the derived set of A.

(m) A is closed iff $A' \subset A$. In words, A is closed iff it contains all of its accumulation points.

Proof. (a) Let $x \in A^\circ$. Then $\exists V \in \mathcal{T}$ such that $x \in V \subseteq A$. Thus $x \in A$, so $A^\circ \subseteq A$. For the second part, Let $x \in A$. Then $x \in F$ for every F that contains A. Consider the intersection of all such Fs that are also closed. x also belongs to their intersection, which is by definition $\overline{A}$. So $A \subseteq \overline{A}$.

(b) A° is open since it is the union of open sets. $\overline{F}$ is closed since it is the intersection of closed sets.

(c) First, we assume A is open. Since $A^\circ = \bigcup V$ for all $V \subseteq A$ and V open, we can take the collections of open sets on the RHS to be only A, and it proves that $A^\circ = A$. For the other direction, we assume $A = A^\circ$. We know that A° is open. Thus A is also open.

(d) First, we assume that F is closed. Then since $\overline{F} = \bigcap A$ where $F \subseteq A$ and A is closed, we can take the union on the RHS to be F and this proves that $F = \overline{F}$. For the converse, we assume $F = \overline{F}$. Since $\overline{F}$ is open this implies that F is closed.

(e) Let $x \in (\overline{A})^c$. This implies $x \in (\overline{A})^c = (\bigcap_{\substack{A \subseteq F, \\ F \text{closed}}} F)^c = \bigcup_{\substack{A \subseteq F, \\ F \text{closed}}} F^c$. Let $F^c = V$. Then we can write

$$x \in \bigcup_{\substack{V \subseteq A^c \\ V \text{open}}} V = (A^c)^\circ.$$

So $(\overline{A})^c \subseteq (A^c)^\circ$. For the converse, let $x \in (A^c)^\circ$. This implies $x \in \bigcup_{\substack{V \subseteq A^c, \\ V \text{open}}} V$. Or equivalently $x \notin \bigcap_{\substack{V\subseteq A^c,\\ V\text{open}}} V^c$. Let $F = V^c$. Then we can write

$$x \notin \bigcap_{\substack{A\subseteq F,\\ F\text{closed}}} F = \overline{A}.$$

So $x \in (\overline{A})^c$. Thus we conclude that $(\overline{A})^c = (A^c)^\circ$.

(f) Let $x \in (A^\circ)^c$. Then

$$x \in (\bigcup_{\substack{V\subseteq A,\\ V\text{open}}} V)^c = \bigcap_{\substack{V\subseteq A,\\ V\text{open}}} V^c = \bigcap_{\substack{A^c\subseteq F,\\ F\text{closed}}} F = \overline{A^c}.$$

This implies $(A^\circ)^c \subseteq \overline{A^c}$. For the converse let $x \in \overline{A^c}$. Then $x \in \bigcap_{\substack{A^c\subseteq F,\\ F\text{closed}}} F$. This implies

$$x \notin \bigcup_{\substack{A^c\subseteq F,\\ F\text{closed}}} F^c = \bigcup_{\substack{V\subseteq A,\\ V\text{open}}} V = A^\circ.$$

This implies that $x \in (A^\circ)^c$. Thus $\overline{A^c} \subseteq (A^\circ)^c$.

(g) We assume that A is open. Then for any $x \in A$ we have $x \in A \subseteq A$. Thus A is a neighborhood of x. For the converse, we assume that A is a neighborhood of all of its points. So for any $x \in A$ there exits $V_x \in \mathcal{T}$ such that $x \in V \subseteq A$. A can be written as $A = \bigcup_x V_x$ where V_x is as above. This A is open.

(h) Let $x \in A_1^\circ$. Then $\exists V \in \mathcal{T}$ such that $x \in V \subseteq A_1$. From assumption we also have $x \in V \subseteq A_2$. This implies that $x \in A_2^\circ$. For the second statement, let $x \in \overline{A_1}$.

(i) to be added.

(j) to be added.

(k) to be added.

(l) $\boxed{\Rightarrow}$. We want to show $\overline{A} \subseteq A \cup A'$. We will prove by contrapositive. I.e. we equivalently prove $A^c \cap (A')^c \subseteq \overline{A}^c$. Let $x \in A^c \cap (A')^c$. This implies that $x \notin A$ as well as $x \notin A'$. So $\exists U \in \mathcal{T}$ such that $A \cap U = \varnothing$ (note that we both used $x \notin A$ and $x \notin A'$). Thus $x \in U \subseteq A^c$. This implies $x \in (A^c)^\circ = \overline{A}^c$.

$\boxed{\Leftarrow}$. We know that $A \subseteq \overline{A}$. So it suffices to show $A' \subseteq \overline{A}$. It is easier to prove the contrapositive, i.e. $(\overline{A})^c \subseteq (A')^c$ or equivalently $(A^c)^\circ \subseteq (A')^c$. Let $x \in (A^c)^\circ$. This implies $\exists U \in \mathcal{T}$ such that $x \in U \subset A^c$. So $A \cap (U \backslash \{x\}) = \varnothing$, thus $x \notin A'$, or equivalently $x \in (A')^c$.

(m) $\boxed{\Rightarrow}$. Assume A is closed. Then $A = \overline{A}$. Using above we will have $\overline{A} = A \cup A'$ it implies that $A' \subseteq A$.

$\boxed{\Leftarrow}$. Assume $A' \subseteq A$. Then from above $\overline{A} = A \cup A'$ it implies that $\overline{A} = A$, hence A is closed. $\square$

■ **Remark** In item (e), by taking the complement from both sides we will have

$$\overline{A} = ((A^c)^\circ)^c$$

1.1 Sporadic Notes

In this section I will include the nodes that do not fit with the current layout of the document and will be added later when I start writing the corresponding sections.

observation 1.1 The subspace topology is the weakest topology that makes the inclusion map continuous. Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. The topological space $(A, \mathcal{T}_A)$ is a topological space and $\mathcal{T}_A$ is called the subspace topology. Define the inclusion map

$$\iota : A \to X.$$

The subspace topology is the weakest topology for which ι is continuous. Let $U \in \mathcal{T}$. Then $\iota^{-1}(U) = A \cap U \in \mathcal{T}_A$. TODO: I above I just showed that under the subspace topology, the inclusion map is continuous. However, I also need to show that $\mathcal{T}_A$ is the smallest such topology.

2. Solution Manual Folland

Here is a list of theorems that are used in the problem sets.

Proposition 2.1 — Some properties. 1. Let $T \in L(X, X)$. Then $\|T^n\| \leq \|T\|^n$.

Proof. (a) We demonstrate the statement for the case where $n = 2$, and the general result follows by induction. Observe that

$$\left\|T^2 x\right\| = \left\|T(Tx)\right\| \leq \|T\|\|Tx\| \leq \|T\|^2\|x\|.$$

Since $\left\|T^2\right\|$ is smallest constant C such that $\left\|T^2 x\right\| \leq C\|x\|$ for all $x \in X$, the it follows that $\left\|T^2\right\| \leq \|T\|^2$.

□

2.1 Elements of Functional Analysis

■ **Problem 2.1 — Folland: Ch5, P1.** The essence of the proof is to show that if u is close to u' and v is close to v' the $u + v$ is close to $u' + v'$. This is true because

$$\left\|(u + v) - (u' + v')\right\| \leq \left\|u - u'\right\| + \left\|v - v'\right\| < 2\epsilon,$$

where we choose $\left\|u - u'\right\| < \epsilon$ and $\left\|v - v'\right\| < \epsilon$.

To prove the continuity of the multiplication, we also use a similar idea. Want to show that if u is close to u' then αu is also close to $\alpha u'$. To see this we can write

$$\left\|\alpha u - \alpha u'\right\| = \alpha\left\|u - u'\right\| < \alpha\epsilon$$

where we choose $\left\|u - u'\right\| < \epsilon$.

To prove the continuity of the norm we need to show that if u is close to v then $\|u\|$ is close to $\|v\|$ (as real numbers). So we need to prove

$$|\|u\| - \|v\|| \leq \|u - v\|.$$

This is reverse triangle inequality. To prove this we can write

$$\|y\| = \|y \pm x\| \leq \|y - x\| + \|x\|.$$

This implies

$$\|y\| - \|x\| \leq \|y - x\|.$$

By a similar argument we can also write

$$\|x\| - \|y\| \leq \|y - x\|.$$

These two inequalities implies that

$$|\|x\| - \|y\|| \leq \|x - y\|.$$

■ **Problem 2.2 — Folland: Ch5, P4.** Briefly, we want to prove that given any $\epsilon > 0$ we can make T is close enough to S and x close enough to y such that Tx is closer to Sy that ϵ. Fix $(T, y) \in L(X, Y) \times X$. We want to show continuity at this point. For the given $\epsilon > 0$ as above if we have $\|T - S\| < \epsilon/\|y\|$ and $\|x - y\| < \epsilon/\|T\|$, then

$$\|Tx - Sy\| \leq \|Tx - Ty\| + \|Sy - Ty\| \leq \|T\|\|x - y\| + \|S - T\|\|y\| < 2\epsilon.$$

■ **Problem 2.3 — Folland: Ch5, P7.**

Solution (a) First, we want to show that the series $\sum_{n=0}^{\infty}(I - T)^n$ converges. First, observe that this series converges absolutely. Because

$$\sum_{n=0}^{\infty} \left\|(I - T)^n\right\| \leq \sum_{n=0}^{\infty} \|I - T\|^n \leq \sum_{n=0}^{\infty} \gamma^n = \frac{1}{1 - \gamma} < \infty.$$

Using the fact that X is a Banach space (hence complete), it follows that $L(X, X)$ is also complete, thus by Theorem 5.1 Folland the absolutely convergent series converges in $L(X, X)$. Let

$$L(X, X) \ni X = \sum_n (I - T)^n.$$

Now we want to prove that X is left and right inverse of T. To see this we can write

$$(I - T)X = \sum_{n=0}^{\infty}(I - T)^{n+1} = \sum_{n=1}^{\infty}(I - T)^n = \sum_{n=0}^{\infty}(I - T)^n - I = X - I.$$

This implies

$$TX = X.$$

With a similar argument we can get $X(I - T) = X - I$, thus $XT = I$. So we conclude that X is the right and the left inverse of T, thus T is a bijection and $X = T^{-1}$.

■ **Remark** I think in above, when we proved that $X \in L(\mathcal{X}, \mathcal{X})$, we automatically proved that T^{-1} is bounded. However, in the solution manual that I got the idea of proof, the author separately proves that T^{-1} is bounded. For the sake of completeness I will do the same here as well.

To show that T^{-1} is bounded, consider the sequence of partial sums of T^{-1}

$$S_n = \sum_{i=1}^{n}(I - T)^n.$$

Using the continuity of $\|\cdot\|$ we can write

$$\left\|T^{-1}x\right\| = \left\|\lim_n S_i x\right\| = \lim_n \|S_n x\| \leq \lim_n \sum_{i=0}^{n} \left\|(I - T)^n x\right\| \leq \lim_n \sum_{i=0}^{n} \|I - T\|^n \|x\| \leq \frac{\|x\|}{1 - \gamma}.$$

(b) Observe that

$$\left\|(ST^{-1} - I)\right\| = \left\|(ST^{-1} - I)TT^{-1}\right\| = \left\|ST^{-1} - TT^{-1}\right\| \leq \|(S - T)\|\left\|T^{-1}\right\| < 1.$$

So ST^{-1} has an inverse $A \in L(\mathcal{X}, \mathcal{X})$ and we have $A = TS^{-1}$. So $S^{-1} = T^{-1}A$. It is also easy to see that S^{-1} is bounded. Because

$$\left\|S^{-1}x\right\| \leq \left\|T^{-1}\right\|\|A\|\|x\|.$$

3. Notes For the Illustrative Analysis Book

3.1 Chapter 11: Linear Operators On Normed Spaces

observation 3.1 — Elaboration on Example 11.6. Let

$$P := \{p : [0,1] \to \mathbb{R} \mid p \in \mathbb{R}[x]\}.$$

In Example 11.6 we show that the differentiation operator defined as $Tp = p'$ is not a continuous operator in P. In this box I will proved a second way to see this. First, note that the relation between P as a linear subspace of $C([0,1],\mathbb{R})$ is similar to the relation between $\mathbb{Q}$ and $\mathbb{R}$ when they are defined over the field $\mathcal{F}_2$. So similar to the dyadic expansion of a real number and representing it as a sequence of zeros and ones, we can also express the elements of $C([0,1],\mathbb{R})$ as a Taylor expansion and record the coefficients as a sequence of real numbers (don't take this analogy to serious, since we have smooth functions that are not analytic, hence do not have a Taylor's expansion!). So the elements of P will be sequences of real numbers with finite support. We summarize our note so far as below.

Summary 3.1 Elements of P can be seen as the sequence of real numbers with finite support. Then the $\|\cdot\|_\infty$ of $p \in P$ with coordinates $(\xi_0, \xi_1, \cdots,)$ will be

$$\|p\|_\infty = \sum_{i\in\mathbb{N}} \xi_i,$$

that is a finite sum because the sequence if finitely supported.

Now consider the polynomials $p_n(t) = t^n/n$. The coordinates of this polynomial is

$$p_n = (0, \cdots, 0, \underbrace{1/n}_{n^{\text{th}}\text{ position}}, 0, \cdots),$$

So

$$\|p_n\|_\infty = \frac{1}{n}.$$

It is easy to see

$$p_n' = (0, \cdots, 0, \underbrace{1}_{(n-1)^{\text{th}} \text{ position}}, 0, \cdots),$$

whose norm is

$$\left\| p_n' \right\|_\infty = 1.$$

So with this norm p_n is converging to zero, but p_n' is not. So T is not continuous.

Proposition 3.1 See proposition 11.7 in textbook.

Proof. (i) $\implies$ (ii): Let $B \subset X$ by any bounded set. Then $\exists r \in \mathbb{R}$ such that $B \subset \mathbb{B}(0, r)$. We claim that $T\mathbb{B}(0, r) \subset \mathbb{B}_Y(0, Mr)$. Let $y \in TB(0, r)$. So $\exists x \in X$ such that $\|x\| \le r$ and $Tx = y$. By hypothesis $\|Tx\| \le M\|x\| \le Mr$. So $Tx \in \mathbb{B}_Y(0, Mr)$.

(ii) $\implies$ (i): Image of any bounded set is bounded. In particular $T\mathbb{B}(0, 1) \subset \mathbb{B}_Y(0, M)$. So

$$\|Tx\| = \|x\| \left\| T \frac{x}{\|x\|} \right\| \le M\|x\|,$$

where we have used the fact that $\frac{x}{\|x\|} \in B(0, 1)$. $\square$

4. Random Question

■ Problem 4.1 Let $(X, \|\cdot\|)$ be a finite dimensional vector space over K, where $K = \mathbb{R}$ or $\mathbb{C}$. Then there is a finite basis $(e_i)_{i=1}^n$ for the vector space X. Therefore, the linear map $\phi : K^n \to X$ defined by

$$\phi(a_1, \cdots, a_n) = \sum_{i=1}^{n} a_i e_i \qquad \text{for all } (a_1, \cdots, a_n) \in K^n,$$

is a bijection. Let K^n be equipped with the norm $\left\|(a_1, \cdots, a_n)\right\|_1 = \sum_{i=1}^n |a_i|$ for all $i = 1, \cdots, n$.

(a) Show that ϕ is a bounded linear map from $(K^n, \|\cdot\|_1)$ to $(X, \|\cdot\|)$.

(b) Show that the unit sphere $S = \{(a_1, \cdots, a_n) \in K^n \ : \ \sum_{i=1}^n |a_i| = 1\}$ in $(K^n, \|\cdot\|_1)$ is a compact subset of K^n.

(c) Show that $\phi^{-1} : (X, \|\cdot\|) \to (K^n, \|\cdot\|_1)$ is also bounded.

(d) Conclude that any two norms on X are equivalent.

(e) Show that any finite dimensional normed vector space is a Banach space.

Solution (a) First, we will show that ϕ is a linear map. Let $a, b \in K^n$ and $\alpha \in K$. Then

$$\phi(\alpha a + b) = \sum_{i=1}^{n} (\alpha a + b)_i e_i = \sum_{i=1}^{n} (\alpha a_i + b_i) e_i = \alpha \sum_{i=1}^{n} a_i e_i + \sum_{i=1}^{n} b_i e_i = \alpha\phi(a) + \phi(b).$$

Further, we need to show that this operator is bounded. Let $a \in K^n$. Then we can write

$$\begin{aligned} \|\phi a\| &= \left\| \sum_{i=1}^{n} a_i e_i \right\| \\ &\le \sum_{i=1}^{n} |a_i| \|e_i\| \\ &\le M \sum_{i=1}^{n} |a_i| \\ &= M \|a\|_1, \end{aligned}$$

where $M = \max_i \{\|e_i\|\}$.

(b) Recall that the norm function is a continuous function (in the topology that it is generating). So the pre-image of closed sets is closed. This implies that the pre-image of $\{1\}$ is closed. So the unit balls is closed. Unit ball is also trivially bounded (contained in unit ball which as a finite radius). On the other hand, in finite dimensional vector spaces, the closed and bounded sets are compact. So unit ball is a finite dimensional vector space is compact.

(c) Since X and K^n are finite dimensional vector spaces, they are Banach spaces. Since ϕ is a bijective linear map (in particular it is surjective), then by open mapping theorem it is an open mapping. Let $V \subset F^n$ be open. Then $(\phi^{-1})^{-1}(V) = T(V)$ is open. So ϕ^{-1} is continuous. Hence bounded. (The proof idea is coming from Theorem 11.30 Illustrative analysis book).

(d) Let $\|\cdots\|$ and $\|\cdot\|_*$ be two norms on X. Then for $v \in X$ we can write

$$\begin{aligned}\|v\| = \|\phi x\| &\leq \|\phi\| \|x\|_1 \\ &\leq \|\phi\| \left\|\phi^{-1} v\right\|_1 \\ &\leq \|\phi\| \left\|\phi^{-1}\right\| \|v\|_*.\end{aligned}$$

Similarly, we can write

$$\|v\|_* \leq \|\phi\| \left\|\phi^{-1}\right\| \|v\|.$$

So we will have

$$\frac{\|v\|_*}{\|\phi\| \|\phi^{-1}\|} \leq \|v\| \leq \|\phi\| \left\|\phi^{-1}\right\| \|v\|_*.$$

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