Daily Notes

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1. October

1.1 Manifolds

For a quick review on the manifolds (on 8 October), I am reading through the Lee's text book.

Proposition 1.1 — Some properties of the Hausdorff spaces. Let $(X, \mathcal{T})$ be a Hausdorff topological spaces. I.e. for any distinct points $x, y \in X$ there exists $U, V \in T$ such that $x \in U, y \in V$ and $U \cap V = \varnothing$. Then we have:

(a) Any convergent sequence in X has a unique limit point.

(b) $(X, \mathcal{T})$ is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed under the product topology.

(c) Every compact set in X is closed.

Proof. (a) Assume otherwise. I.e. the sequence $\{x_n\}$ converges to two different distinct points $p, q \in X$. Since X is Hausdorff, then there exists disjoint $U, V \in \mathcal{T}$ such that $U \cap V = \varnothing$ and $x \in U, y \in V$. Since $x_n \to p$ and $x_n \to q$ as $n \to \infty$ then there exists $N_1, N_2 \in \mathbb{N}$ such that $\forall n > N_1$ we have $x_n \in U$ and $\forall n > N_2$ we have $x_n \in V$. This is a contradiction as we assume that $U \cap V = \varnothing$.

(b) Forward direction $\boxed{\implies}$: We assume that X is Hausdorff. Let $(x, y) \in (X \times X) \backslash \Delta$. Since X is Hausdorff, then there exists $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$ and $U \cap V = \varnothing$. Thus the set $U \times V$ is also open in $X \times X$ with the subspace topology and does not contain (x, x) nor (y, y). Thus for any point that is not on the diagonal we can find a open set in $X \times X$ that is disjoint with the diagonal. This proves that Δ^c is open thus Δ is closed.

Backward direction $\boxed{\impliedby}$: We assume that Δ is closed in $X \times X$. Then for any $(x, y) \in (X \times X) \backslash \Delta$ we can find and open set M such that $M \cap \Delta = \varnothing$. We can find an open set $U \times V \in M$ such that $x \in U$ and $y \in V$. Since $M \cap \Delta = \varnothing$ then $U \cap V = \varnothing$. Since the point (x, y) was arbitrary, then X is Hausdorff.

(c) Let K be a compact set in X and let $x \in K^c$. Since the space is Hausdorff then for any $q \in K$ we can find U_q, V_q such that $x \in U_q$ and $q \in V_q$. The collection $\{V_q\}_{q \in K}$ covers K. Since K is compact, then we have a finite sub-cover, call $\{V_{q_1}, \cdots, V_{q_n}\}$. Let $U = U_{q_1} \cap \cdots \cap U_{q_n}$. U is open (finite intersection of open sets) and is disjoint with all of the sets in the finite sub-cover (i.e. $U \cap V_{p_i} = \varnothing$ for $i = 1, \cdots, n$). Thus $U \cap K^c = \varnothing$. This implies that K is closed.

□

The history behind the following proposition is funny! When I first had analysis course, then had this view that the uniqueness of the limit of a convergent sequence comes from the triangle inequality. Then today, in proposition above, I saw that it comes form the space being Hausdorff. The following proposition and its proof makes my intuition about the limits of the convergent sequences and triangle inequality more clear.

Proposition 1.2 Every metric space (X, d) is a Hausdorff space.

Proof. Let $x, y \in X$ distinct. Then $d(x, y) > 0$. let $d = d(x, y)/2$ and define the open balls

$$U = B_{d/2}(x), \qquad V = B_{d/2}(y).$$

We claim $U \cap V = \varnothing$. To see the proof, assume otherwise. Then there exists $z \in U \cap V$. By the triangle inequality we have

$$d = d(x, y) \leq d(x, z) + d(z, y) < d/2 + d/2 = d,$$

which is a contradiction. □

Proposition 1.3 — Properties of first countable space. Let $(X, \mathcal{T})$ be first countable. I.e. for all $x \in X$ there is a neighborhood basis at x. Let $A \subset X$ be any subset, and $x \in X$.

(a) $x \in A^\circ$ if and only if every sequence in X converging to x is eventually in A.

(b) $x \in \overline{A}$ if and only if x is a limit of a sequence of points in A.

(c) Every metric space is first countable.

Proof. (a) First, we prove the forward direction $\boxed{\Longrightarrow}$. We prove this by contrapositive. Assume that there exists a sequence $\{a_n\}$ in X converging to x such that for all $N \in \mathbb{N}$ we can find $n > N$ such that $a_n \notin A$. Since $a_n \to x$ as $n \to \infty$ then for all $U \in \mathcal{N}(x)$ we have $a_n \in U$ for all n large enough. However, we just observed that for n large enough we can find some $a_n \notin A$. Thus for all neighborhood U of x there exists some a_n that $a_n \notin A$, i.e. $U \cap A^c \neq \varnothing$ for all $U \in \mathcal{N}(x)$. Thus $x \notin A^\circ$.

For the converse direction $\boxed{\Longleftarrow}$, again we use the prove by contrapositive. Let $x \notin A^\circ$. Then for all neighborhoods $U \in \mathcal{N}(x)$ we have $A^c \cap U \neq \varnothing$. Since the space is first-countable, in particular for all $\{V_i\}_{i\in\mathbb{N}} \subset \mathcal{N}(x)$ we have $A^c \cap V_i \neq \varnothing$. For each such V_i we choose $a_i \in A^c \cap V_i$. Then the sequence $\{a_n\}$ is converging to x that is not eventually in A.

(b) First, we prove the forward direction $\boxed{\Longrightarrow}$. Let $x \in \overline{A}$. Then for all $U \in \mathcal{N}(x)$ we have $U \cap A \neq \varnothing$. In particular for the local base $\{V_i\}_{i\in\mathbb{N}} = B \subset \mathcal{N}(x)$ we have $V_i \cap A \neq \varnothing$ for all $i \in \mathbb{N}$. Let $a_n \in V_n \cap A$. Then by definition a_n converges to x and this completes the proof.

Then we prove the backward direction $\boxed{\Longleftarrow}$. Let $\{a_n\}$ be a sequence of points in A such that $a_n \to x$ as $n \to \infty$. Then by definition $\forall U \in \mathcal{N}(x)$ we have $a_n \in U$ for all n large enough. This implies that $U \cap A \neq \varnothing$ for all $U \in \mathcal{N}(x)$. This implies that $x \in \overline{A}$.

(c) We provide a constructive prove. Let (X, d) be a metric space with $x \in X$ be some point. Then a local countable basis for x is given by the collection of all open balls with rational radius and centered at x.

□

Proposition 1.4 — Properties of second-countable spaces. (a) Every second-countable topological space is separable (i.e. admits a countable dense subset).

(b) Every second-countable space is first-countable.

Proof. (a) We demonstrate a constructive proof. Since the space X is second-countable, then there exists a countable basis for the topology, call it $\{U_n\}_{n\in\mathbb{N}}$. From each U_n get x_n (assuming that U_n is non-empty, otherwise discard it) and call the collection of all such points D. Let V be any open set. Then there exists $U_n \subseteq V$ and due to construction $x_n \in V$ thus $D \cap V \neq 0$. Since V was arbitrary, then D is dense.

(b) For a point $x \in X$ let $B(x)$ (the local base) be the set of all open sets in the countable topology base such that contains the point x.

□

■ **Remark 1.1** Note that all metric spaces are first-countable, however, only separable metric spaces are second-countable.

Reviewing the nice properties of the second-countable spaces and the Hausdorff spaces reveals the rationale behind our choice that why a topological manifold should be second-countable and Hausdorff. In fact, we want the space be well-behaved when with respect to the sequences (i.e. unique limit of converging sequences, some useful sequencial characterizations, etc).

■ **Problem 1.1 — Exercise in Lee.** Show that equivalent definitions of topological manifolds are obtained if instead of allowing U to be homeomorphic to any open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$ or to $\mathbb{R}^n$ itself.

Proof. $\boxed{\Longrightarrow}$. □