Lecture Notes For: Solved Problems in $\mathbb{R}$eal Analysis

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October 19, 2023

Introduction:
In this document, I have presented interesting questions in $\mathbb{R}$eal analysis. The questions are gathered from different books and websites (all of which are referenced). However, the solutions are written by myself, thus there is always a chance of mistake. Please let me know via email if you spot any.

Contents

1 Sets and Functions

1.1 Basic Set Theory

Question 1. If A, B and C are sets. prove the followings:

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(b) $C \backslash (A \cup B) = (C \backslash A) \cap (C \backslash B)$.

(c) $C \backslash (A \cap B) = (C \backslash A) \cup (C \backslash B)$.

Answer.

(a) *Proof.* Let P, Q, and R be logical statements. We can show (using the truth table) that the following biconditional implication is a tautology.

$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R).$$

Now let $x \in A \cap (B \cup C)$. Then $x \in A \wedge (x \in B \vee x \in C)$. Using the tautology above we can write $(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$, thus $x \in (A \cap B) \cup (A \cap C)$, which means $A \cap (B \cap C) \subset (A \cap B) \cup (A \cap C)$. Conversely, let $x \in (A \cap B) \cup (A \cap C)$. By definition $(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$. With the similar logic as above we can infer $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. Thus $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

(b) *Proof.* Let $x \in C \backslash (A \cup B)$. By definition $x \in C \wedge x \notin (A \cup B) \Leftrightarrow x \in C \wedge x \in \overline{A \cup B} \Leftrightarrow x \in C \wedge x \in \bar{A} \cap \bar{B} \Leftrightarrow x \in C \wedge (x \notin A \wedge x \notin B)$. Finally, using the following tautology

$$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge (P \wedge R),$$

we can write $(x \in C \wedge x \notin A) \wedge (x \in C \wedge x \notin B) \Leftrightarrow x \in (C \backslash A) \cap (C \backslash B)$, thus $C \backslash (A \cup B) subset (C \backslash A) \cap (C \backslash B)$. The converse can be shown is true following the similar logic as below, thus inferring $C \backslash (A \cup B) = (C \backslash A) \cap (C \backslash B)$. □

(c) *Proof.* Let $x \in C \backslash (A \cap B)$. By definition $x \in C \wedge x \notin (A \cap B) \Leftrightarrow x \in C \wedge x \in \overline{A \cap B} \Leftrightarrow x \in C \wedge x \in \bar{A} \cup \bar{B} \Leftrightarrow x \in C \wedge (x \notin A \vee x \notin B)$. Using the tautology in section (a), we can write $(x \in C \wedge x \notin A) \vee (x \in C \wedge x \notin B) \Leftrightarrow x \in (C \backslash A) \cup (C \backslash B)$, thus $C \backslash (A \cap B) \subset (C \backslash A) \cup (C \backslash B)$. The converse can be shown is true following the similar logic as below, thus inferring $C \backslash (A \cap B) = (C \backslash A) \cup (C \backslash B)$. □

1.2 Functions

Question 2. Let $f : A \to B$. Let C, C_1, and C_2 be subsets of A, and let D be a subset of B. Prove the followings.

(a) If f is one-to-one, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$.

(b) If f is one-to-one, then $f^{-1}(f(C)) = C$.

(c) If f is onto, then $f(f^{-1}(D)) = D$.

Answer.

(a) *Proof.* Let $C_1, C_2 \in A$. Then $f(C_1 \cap C_2) = \{y \in B : y = f(x), x \in C_1 \cap C_2\} = \{y \in B : y = f(x), x \in C_1 \wedge x \in C_2\}$. Let $\tilde{y} \in f(C_1 \cap C_2)$. Then from the definition above, $\tilde{y} = f(\tilde{x})$ s.t. $\tilde{x} \in C_1 \wedge \tilde{x} \in C_2$. Using the definition of function, we can write $\tilde{y} \in f(C_1) \wedge \tilde{y} \in f(C_2) \Leftrightarrow \tilde{y} \in f(C_1) \cap f(C_2)$. Thus $f(C_1 \cap C_2) \subset f(C_1) \cap f(C_2)$. Conversely, let $y \in (f(C_1) \cap f(C_2))$. Then by definition

$$y \in f(C_1) \Rightarrow \exists x_1 \in C_1 \text{ s.t. } y = f(x_1),$$
$$y \in f(C_2) \Rightarrow \exists x_2 \in C_2 \text{ s.t. } y = f(x_2).$$

Since f is one-to-one, then $f(x_1) = f(x_2) \Rightarrow x = x_1 = x_2$. Thus $x \in C_1 \cap C_2, y = f(x) \in f(C_1 \cap C_2)$. So $f(C_1) \cap f(C_2) \subset f(C_1) \cap f(C_2)$. Finally we can conclude $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$. □

(b) *Proof.* Let $C \in A$, and $a \in f^{-1}(f(C))$. Since $f^{-1}(f(C)) = \{x \in A : f(x) \in f(C)\}$, then $f(a) \in f(C)$. Thus $\exists x_c \in C$ s.t. $f(a) = f(x_c)$. Since f is one-to-one, then $a = x_c \Leftrightarrow a \in C$. So we will have $f^{-1}(f(C)) \subset C$ .Conversely, let $a \in C$. Then by definition $f(a) \in f(C) \Leftrightarrow f(a) \in f^{-1}(f(C))$. So $C \subset f^{-1}(f(C))$. So we can conclude $f^{-1}(f(C)) = C$. □

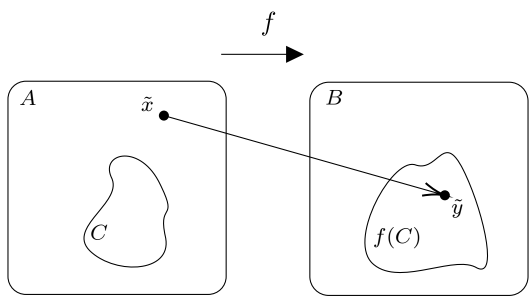
(c) *Proof.* Let $D \subset B$, $b \in D$. Since f is onto, then $\exists a \in A$ s.t. $b = f(a)$, thus $a \in f^{-1}(D)$. And by definition $b \in f(f^{-1}(D))$, thus $D \subset f(f^{-1}(D))$. Conversely, Let $b \in f(f^{-1}(D))$. Then $\exists a \in f^{-1}(D)$ s.t. $f(a) = b$, hence from definition it follows that $b \in D$, implying $f(f^{-1}(D)) \subset D$. So we conclude that $f(f^{-1}(D)) = D$. □

Question 3. For all sub-sections of Question 2 construct a specific example in which the indicated equation fails. (Of course the given hypothesis will have to be false too.)

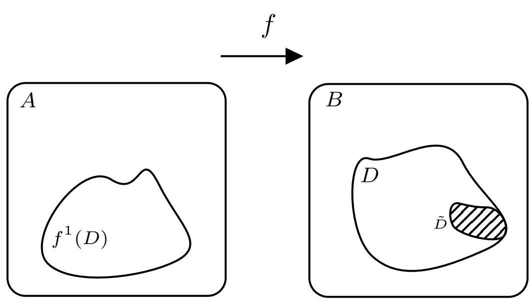
Answer.

(a) includeQ4a.tex

(b) From the figure below, it is clear that $\tilde{x} \in f^{-1}(f(C))$ but $\tilde{x} \notin C$, thus $f^{-1}(f(C)) \not\subset C$.



(c) In the figure below, the set $\tilde{D} \subset D$, shown as a shaded region, has not been mapped (since we assume that f is not onto). Then for any $b \in \tilde{D}$ we have $b \notin f(f^{-1}(D))$, thus $D \not\subset f(f^{-1}(D))$.



Question 4. Question 8. Let $f : A \to B$ and let $C \subseteq A$. Answer the followings.

(a) Proof or counterexample: $f(A \backslash C) \subseteq f(A) \backslash f(C)$.

(b) Proof or counterexample: $f(A \backslash C) \supseteq f(A) \backslash f(C)$.

(c) What condition on f will guarantee $f(A \backslash C) = f(A) \backslash f(C)$.

Answer.

(a) *Proof.* Let $y \in B$ s.t. $y \in f(A \backslash C)$. From definition $y = f(\tilde{x})$ for some $\tilde{x} \in A \backslash C \Leftrightarrow \tilde{x} \in A \wedge \tilde{x} \notin C$. Now construct the function in a way that $\exists x_c \in C$ s.t. $f(x_c) = f(\tilde{x})$. Thus f is **not** one-to-one. So $y = f(\tilde{x}) = f(x_c) \in f(C)$. We can write:

$$\begin{aligned} &y \in f(C), \\ &y \in f(C) \cup \overline{f(A)}, \\ &y \in \overline{(\overline{f(C)} \cap f(A))}, \\ &y \notin \overline{f(C)} \cap f(A), \\ &y \notin f(A) \backslash f(C). \end{aligned}$$

Thus $f(A \backslash C) \not\subset f(A) \backslash f(C)$. □

(b) *Proof.* Let $y \in B$ s.t. $y \in f(A) \backslash f(C)$. So $y \in f(A) \wedge y \notin f(C)$. From definition $y = f(x_1)$ for some $x_1 \in A$ and $y = f(x_2)$ for some $x_2 \notin C$. Then if f is one-to-one, $y = f(x_1) = f(x_2) \Rightarrow x = x_1 = x_2$, in which $x \in A \wedge x \notin C$, hence $x \in A \backslash C$. So from definition $y \in f(A \backslash C)$ which implies $f(A \backslash C) \supseteq f(A) \backslash f(C)$. □

(c) f should be one-to-one to guarantee $f(A \backslash C) = f(A) \backslash f(C)$. Proof is provided in (a) and (b).

Question 5.

(a) Suppose $f : X \to X$ is a function, and define $g = f \circ f$. Prove: if $g(x) = x$ for all $x \in X$, then f is one-to-one and onto.

(b) Extend the results in (a) to the function $g = f \circ f \circ \ldots \circ f$ defined by compositing f by itself n times. Show that the results is valid for each $n \in \mathbb{N}$.

Answer.

(a) *Proof.* Lets first prove that f is one-to-one. Let $x_1 \neq x_2 \in X$ s.t. $f(x_1) = f(x_2)$. By definition $f(f(x_1)) = f(f(x_2)) \Leftrightarrow g(x_1) = g(x_2) \Leftrightarrow x_1 = x_2$, thus f is one-to-one.

Now we prove that f is onto. $\forall y \in X$, setting $x = f(y) \in X$ will yield $y = f(x) = f(f(y)) = g(y) = y$. Thus f is onto. □

(b) *Proof.* First we prove that f is one-to-one. Let $x_1 \neq x_2 \in X$ s.t. $f(x_1) = f(x_2)$. By definition $f(f(x_1)) = f(f(x_2)) \Leftrightarrow f(f(f(x_1))) = f(f(f(x_2))) \Leftrightarrow f(f(\ldots f(x_2)\ldots)) = f(f(\ldots f(x_1)\ldots)) \Leftrightarrow g(x_1) = g(x_2) \Leftrightarrow x_1 = x_2$, thus f is one-to-one.

Now we prove that f is onto. $\forall y \in X$, setting $x = (\overbrace{f \circ f \circ \ldots \circ f}^{n-1})(y) \in X$ will yield $y = f(x) = (\overbrace{f \circ f \circ \ldots \circ f}^{n})(y) = g(y) = y$. Thus f is onto.

Now by induction we can infer that this is true $\forall n \in \mathbb{N}$. □

Question 6. Let $f : A \to B$ and $g : B \to C$ be given functions. Use the symbol $g \circ f$ to denote the function from A to C defined by $(g \circ f)(x) = g(f(x))$ for all $x \in A$. Prove the followings.

(a) If f and g are one-to-one, then $g \circ f$ is one-to-one.

(b) If $g \circ f$ is one-to-one, then f is one-to-one.

(c) If f is onto and $g \circ f$ is one-to-one, then g is one-to-one.

(d) It can happen that $g \circ f$ is one-to-one, but g is not.

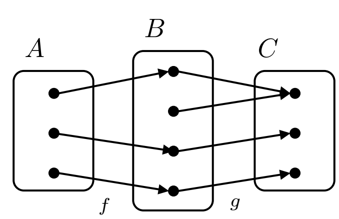
Answer.

(a) *Proof.* Let $x_1 \neq x_2 \in A$ s.t. $(g \circ f)(x_1) = (g \circ f)(x_2)$. Since g is one-to-one then $f(x_2) = f(x_1)$. Similarly, since f is one-to-one, then $x_1 = x_2$, hence $g \circ f$ is one-to-one. □

(b) *Proof.* Let $x_1 \neq x_2 \in A$ s.t. $f(x_1) = f(x_2)$. By definition of a function we can write $g(f(x_1)) = g(f(x_2)) \Leftrightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$. Since $g \circ f$ is one-to-one, then $x_1 = x_2$. This then implies that f is one-to-one. □

(c) *Proof.* Let $b_1 \neq b_2 \in B$ s.t. $g(b_1) = g(b_2)$. Since f is onto, then for $b_1, b_2 \in B, \exists a_1, a_2 \in A$, s.t. $b_1 = f(a_1), b_2 = f(a_2)$. So we can write $g(f(a_1)) = g(f(a_2))$. Since $g \circ f$ is one-to-one, $a_1 = a_2$. From definition of a function it follows $b_1 = f(a_1) = f(a_2) = b_2$, hence implies g is one-to-one. □

(d) *Proof.* The following figure shows a specific example in which $g \circ f$ is one-to-one, however g is not.



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