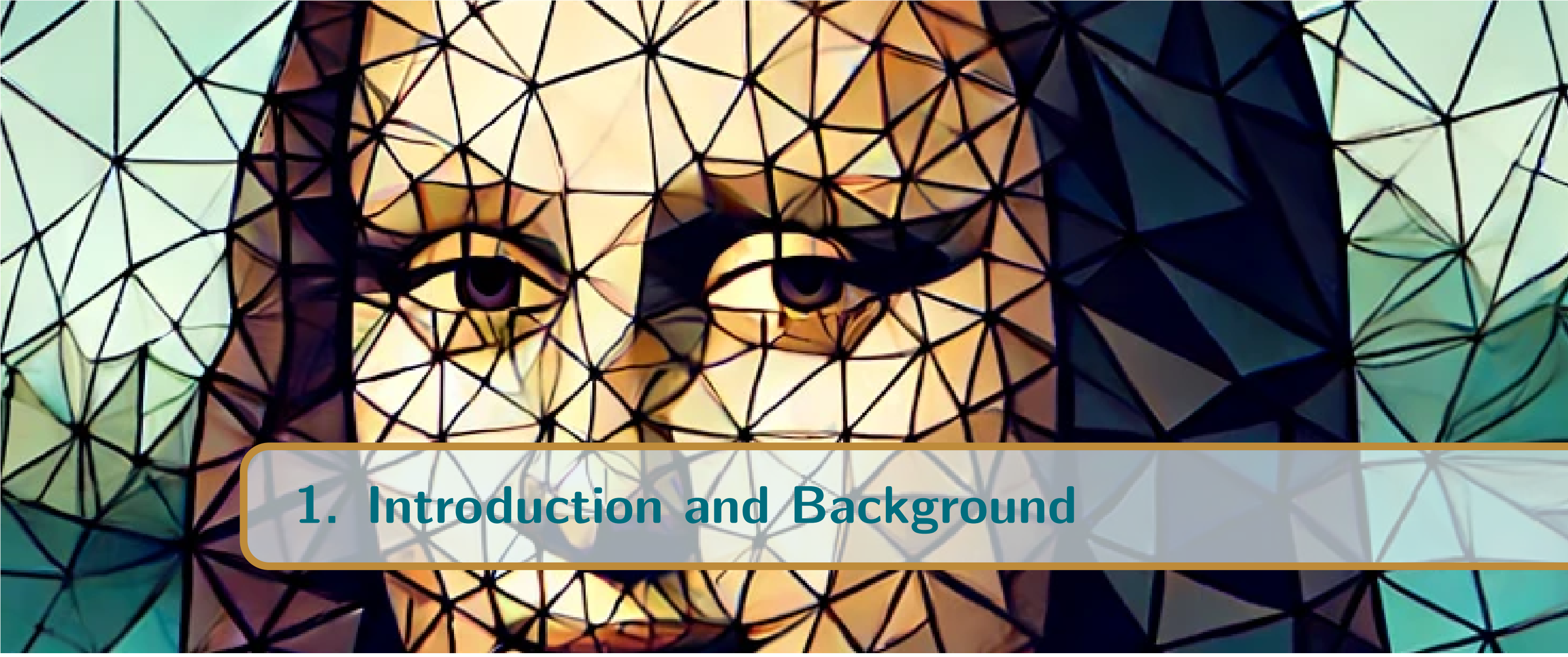
Numerical Analysis

Ali Fele Paranj
alifele@student.ubc.ca

June 16, 2024

0. Contents



1. Introduction and Background

1.1 Summary

Summary 1.1 — Continuous functions on Ω and $\overline{\Omega}$. Let $\Omega \subset \mathbb{R}^n$ be an open set. $C(\Omega)$ denotes the set of all continuous functions defined on Ω. Similarly, $C(\overline{\Omega})$ denotes the set of all continuous functions defined on the closure of Ω.

For any $f \in C(\Omega)$ we have $f \in C(\overline{\Omega})$. However, for the converse, if $g \in C(\overline{\Omega})$, then if g is uniformally continuous and Ω is bounded, then g can continuously be extended to $\partial\Omega$. Note that $C(\Omega)$ functions can behave badly near $\partial\Omega$. For instance, consider the function $f : (0, 1) \to \mathbb{R}$ given by $f(x) = \sin\big(1/x\big)$.

Summary 1.2 — The space of continuous 2π periodic functions. Consider the space of continouse functions defined on $\mathbb{R}$, i.e. $C(\mathbb{R})$. An important subset of this set is $C_p(2\pi)$ which is the set of all continuous 2π periodic functions where for $f \in C_p(2\pi)$ we have

$$f(x + 2\pi) = f(x), \qquad x \in \mathbb{R}.$$

This set, is in one-to-one correspondence with the set of all continuous function defined from the manifold S^1, or equivalently $\mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{R}$.

Summary 1.3 — Basis for the set of polynomials. Let $\mathbb{P}_n$ denote the set of all polynomials defined on $\mathbb{R}$ with degree less than or equal to n. Then a basis for this linear space is

$$\mathbb{B} = \{1, x, \cdots, x^n\}.$$

Thus the dimension of this space is $n + 1$.

Now, consider a *linear subspace* of this space, the set of all polynomials that vanishes at 0 and 1 denoted by

$$\mathbb{P}_{n,0} = \{p \in \mathbb{P}_n \mid p(0) = p(1) = 0\}.$$

A basis for this linear subspace can be given as

$$\mathbb{B}_{n,0} = \{x(1-x), x^2(1-x), \cdots, x^{n-1}(1-x)\}.$$

Thus the dimension of this linear subspace is $\dim(\mathbb{P}_n) - 2$. The difference two in the dimension comes from the fact that polynomials in $\mathbb{P}_{n,0}$ vanished at two points of their domain. Thus the set of all polynomials of degree n that vanish at n points of their domain is a 1 dimensional linear subspace of $\mathbb{P}_n$.

Summary ✎ 1.4 — Normed space $\mathbb{R}^d$. Consider the linear space $\mathbb{R}^d$. Then the followings are the common norms for this space.

$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p}, \qquad \text{for } 1 \leq p < \infty.$$
$$\|x\|_\infty = \max_{1\leq i\leq d} |x_i|.$$

Proposition 1.1 In $\mathbb{R}^d$ we have for all $x \in \mathbb{R}^d$

$$\|x\|_\infty \leq \|x\|_p \leq d^{1/p}\|x\|_\infty.$$

■ **Remark 1.1** As a simple corollary of the proposition above we can see

$$\|x\|_\infty = \lim_{p\to\infty} \|x\|_p.$$

Proposition 1.2 — Equivalence of norms. On a finite dimensional space all norms are equivalent.

■ **Remark 1.2** The proposition above dose not hold true on infinite dimensional spaces. In those space, some norms has more stronger sense of convergence than others.

Summary ✎ 1.5 — Normed space $C(\Omega)$. Let $V = C[0,1]$ denote the linear space of all continuous function defined on $[0,1]$. Define the following norms

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p}, \qquad 1 \leq p < \infty$$
$$\|f\|_\infty = \sup_{0\leq x\leq 1} |f(x)|$$

The norm $\|x\|_\infty$ is a natural norm for this space that is also called *uniform norm*.

Proposition 1.3 For the norms given above we have

$$\|v\|_p \le \|v\|_\infty \qquad \forall v \in V.$$

This implies that the convergence under the uniform norm $\|\cdot\|_\infty$ implies the convergence under the norm $\|\cdot\|_p$. Note that the converse is not true.

■ **Remark 1.3** A very good example to see the proposition above is $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 - nx, & 0 \le x \le 1/n, \\ 0, & 1/n < x \le 1. \end{cases}$$

Proposition 1.4 Let $\Omega \subset \mathbb{R}^d$ and let $\overline{\Omega}$ denote its closure. Then the space $C(\overline{\Omega})$ with the norm $\|\cdot\|_\infty$ is a Banach space. However, this space is not a Banach space with $\|\cdot\|_p$ for $1 \le p < \infty$.

■ **Remark 1.4** The proposition above is true since the continuoity of a sequence of functions persists under the uniform continuity. A good example for the second statement is the function $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 0, & 0 \le x \le 1/2 - 1/(2n), \\ n(x - 1/2 + 1/(2n)), & 1/2 - 1/(2n) \le x \le 1/2 + 1/(2n), \\ 1, & 1/2 + 1/(2n) \le x \le 1. \end{cases}$$

Summary 1.6 — Normed space $C^k(\Omega)$. Let $\Omega \subset \mathbb{R}$. The space $C^k(\overline{\Omega})$ is the set of all k times continuously differentiable functions. Define the following metric on this space

$$\|f\|_{k,p} = \left(\sum_{i=1}^{k} \left\| f^{(i)} \right\|_p^p \right)^{1/p}, \qquad 1 \le p < \infty.$$

$$\|f\|_{k,\infty} = \max_{1 \le i \le k} \left\| f^{(i)} \right\|_p.$$

The natural basis for this space is $\|f\|_{k,\infty}$.

Proposition 1.5 — C^k is a Banach space. The space C^k is complete under the norm $\|\cdot\|_{k,\infty}$.

Summary 1.7 — Completion of $C(\overline{\Omega})$. The space $C(\overline{\Omega})$ is not complete under the norm $\|\cdot\|_p$ for $1 \le p < \infty$. Its completion is the space of *Lebesgue integrable functions* L^p.

Summary 1.8 — Completion of $C^k(\overline{\Omega})$. The space $C^k(\overline{\Omega})$ is not complete under the norm $\|\cdot\|_{k,p}$ for $1 \le p < \infty$. Its completion is the *Sobolev spaces.*

Summary 1.9 — Norm changing the topology in action!. Consider the spaces $V = C^1[0,1] \subset$

$C[0,1]$, and $W = C[0,1]$, and the linear operator

$$T = \frac{d}{dx} : V \to W.$$

Consider the same infinity norm $\|\cdot\|_\infty$ for both V and W. Let $\{f_n\}$ be a sequence of functions in V defined as

$$f_n(x) = \frac{1}{n}\sin(2^n \pi x).$$

It is evident that $\|f_n\|_\infty \to 0$ as $n \to 0$. However, $\|f_n'\|_\infty \to \infty$ as $n \to \infty$. Geometrically, we can feel that the sequence $\{f_n\}$ sort of moves towards the origin of the space $(V, \|\cdot\|_\infty)$ while $\{f_n'\}$ shoots to infinity in the space $(W, \|\cdot\|_\infty)$.

However, if we change the norm of space V to the standard norm of $C^1[0,1]$, i.e.

$$\|f\|_\infty^1 = \max\{\|f\|_\infty, \|f'\|_\infty\},$$

then we can see that the sequence $\{f\}$ is not moving towards the origin, but shoots of to the infinity of the space $(V, \|\cdot\|_\infty^1)$. From this example it is clear that how norm induces topology. A sequence that originally was moving towards the origin in one topology, shoots off to the infinity in another topology.

Summary 1.10 — Continuity of differentiation operator. According to the summary box above, the differentiation operator

$$T_1 = \frac{d}{dx} : C^1[0,1] \subset (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$$

is not continuous, but the operator

$$T_2 = \frac{d}{dx} : (C^1[0,1], \|\cdot\|_\infty^1) \to (C[0,1], \|\cdot\|_\infty)$$

is continuous.

1.2 Solved Problems

■ **Problem 1.1 — The space of solutions of an ODE (from Atkinson).** Show that the set of all continuous solutions of the differential equation $u''(x) + u(x) = 0$ is a finite-dimensional linear space. Is the set od all continuous solutions of $u''(x) + u(x) = 1$ is a linear space?

Solution Denote the set of all solutions for the ODE $u'' + u' = 0$ as

$$S = \{f \in C(\mathbb{R}) \mid f'' + f = 0\}.$$

We claim that S is a linear space. Because

- Closed under addition: Let $f, g \in S$. Then $f'' + f = 0$ and $g'' + g = 0$. From the linearity of derivative it follows that $(f+g)'' + (f+g) = 0$, hence $f + g \in S$.
- Existence of zero element: The function $g \equiv 0$ is in S.
- Existence of inverse element: Let $f \in S$. Then $f'' + f = 0$. Multiplying both sides by -1 we will get $(-f)'' + (-f) = 0$. Thus $-f \in S$.
- Closed under scalar multiplication: Let $f \in S$. Then $f'' + f = 0$. Multiplying both sides by $a \in \mathbb{R}$ we will get $(af)'' + (af) = 0$. Thus $af \in S$.

- Commutativity, associativity, distributivity, and follows from the same properties for the addition of functions.

To show that the dimension of this linear space is finite, consider two solutions $u_1, u_2 \in S$ such that their Wronskian is non-zero. I.e.

$$W(t) = \det \begin{pmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{pmatrix} \neq 0.$$

For the particular ODE given in this question, we can consider $u_1(t) = \cos(t)$ and $u_2(t) = \sin(t)$. Take any solution $v \in S$. Assume $v(0) = a$ and $v'(0) = b$. Consider $w(t) = pu_1(t) + qu_2(t)$ where $p, q \in \mathbb{R}$ chosen such that $v(0) = w(0)$ and $v'(0) = w'(0)$. Since both w, v are solutions of the ODE, then from the existence-uniqueness theorem, it follows that $v(t) = w(t)$. This shows that we can write every solution of the ODE in terms of u_1 and u_2. Thus S is a linear space of dimension 2.

The continuous solutions of the ODE $u'' + u = 1$ is not a linear space as it does not contain the zero element. However, we can show that this is an affine space.

■ **Problem 1.2 — Linear space (from Atkinson).** When is the set $\{v \in C[0,1] \mid v(0) = a\}$ a linear space?

Solution This set is a linear space only when $a = 0$. Otherwise, this set can not contain the zero function (to be served as the zero element of the vector space). Also, if $a \neq 0$, then this set will not be closed under addition and scalar multiplication.

■ **Problem 1.3 — Zero vector and linear independence (from Atkinson).** Show that in any linear space V, a set of vectors is always linearly dependent if one of the vectors is zero.

Solution Let $\{u_1, u_n, f\}$ be a collection of vectors where f is the zero vector. Let $\alpha_1 = \cdots = \alpha_n = 0$ and $\beta \neq 0$ and consider the sum

$$\alpha_1 u_1 + \cdots + \alpha_n u_n + \beta f = 0.$$

There is one non-zero coefficient β, thus the collection of vectors are linearly dependent.

■ **Problem 1.4 — Unique expansion in terms of basis vectors (from Atkinson).** Let $\{v_1, \cdots, v_n\}$ be a basis of an n-dimensional space V. Show that for any $v \in V$, there are scalars $\alpha_1, \cdots, \alpha_n$ such that

$$v = \sum_{i=1}^{n} \alpha_i v_i,$$

and the scalars $\alpha_1, \cdots, \alpha_n$ are uniquely determined by v.

Solution Let $\mathbb{B} = \{v_1, \cdots, v_n\}$ be a basis and let $v \in V$ be any vectors. Since $\mathbb{B}$ is a basis, then by definition the vectors $v_1, \cdots, v_n$ are

(I) linearly independent, and
(II) spans the whole space.

(II) implies the existence of the scalars $\alpha_1 \cdots \alpha_n$ such that

$$v = \sum_{i}^{n} \alpha_i v_i.$$

Furthermore, (I) implies the uniqueness of these scalars. To see this we will use the proof by contradiction. Consider the $\beta_1, \cdots, \beta_n$ where we have $\beta_i \neq \alpha_i$ at least for one $1 \leq i \leq n$. Then

$$v = \sum_{i=1}^{n} \alpha_i v_i, \qquad v = \sum_{i=1}^{n} \beta_i v_i.$$

Subtracting these two expressions we will get

$$0 = \sum_{i=1}^{n} (\alpha_i - \beta_i) v_i.$$

Since $\alpha_i \neq \beta_i$ for at least one index i. From the definition of linear dependence, this implies that the collection of vectors in $\mathbb{B}$ is linearly dependent that contradicts (I) which is a contradiction.

■ **Problem 1.5 — Cartesian product of linear spaces (from Atkinson).** Assume U and V are finite dimensional linear spaces, and let $\{u_1, \cdots, u_n\}$ and $\{v_1, \cdots, v_m\}$ be bases for them, respectively. Using these bases, create a basis for $W = U \times V$. Determine $\dim W$.

Solution Consider the following basis for U and V

$$\mathbb{B}_U = \{u_1, \cdots, u_n\}, \qquad \mathbb{B}_V = \{v_1, \cdots, v_m\}.$$

Construct the sets

$$\mathcal{B}_U = \{(u_i, 0_V) \mid u_i \in \mathbb{B}_U,\ 0_V \in V\}, \qquad \mathcal{B}_V = \{(0_U, v_i) \mid v_i \in \mathbb{B}_V, u_0 \in U\}.$$

Then the following collection will be a basis for $U \times V$.

$$\mathbb{B}_{U\times V} = \mathcal{B}_U \cup \mathcal{B}_V.$$

The linearindependentness of the vectors in $\mathbb{B}_{U\times V}$ follows immediately from the linearindependentness of $\mathbb{B}_1$ and $\mathbb{B}_2$. Similarly, it follows immediately from the spanning property of $\mathbb{B}_U$ and $\mathbb{V}_B$ that $\mathbb{B}_{U\times V}$ spans the whole space $U \times V$. This construction reveals that the space $U \times V$ has dimension $n + m$.