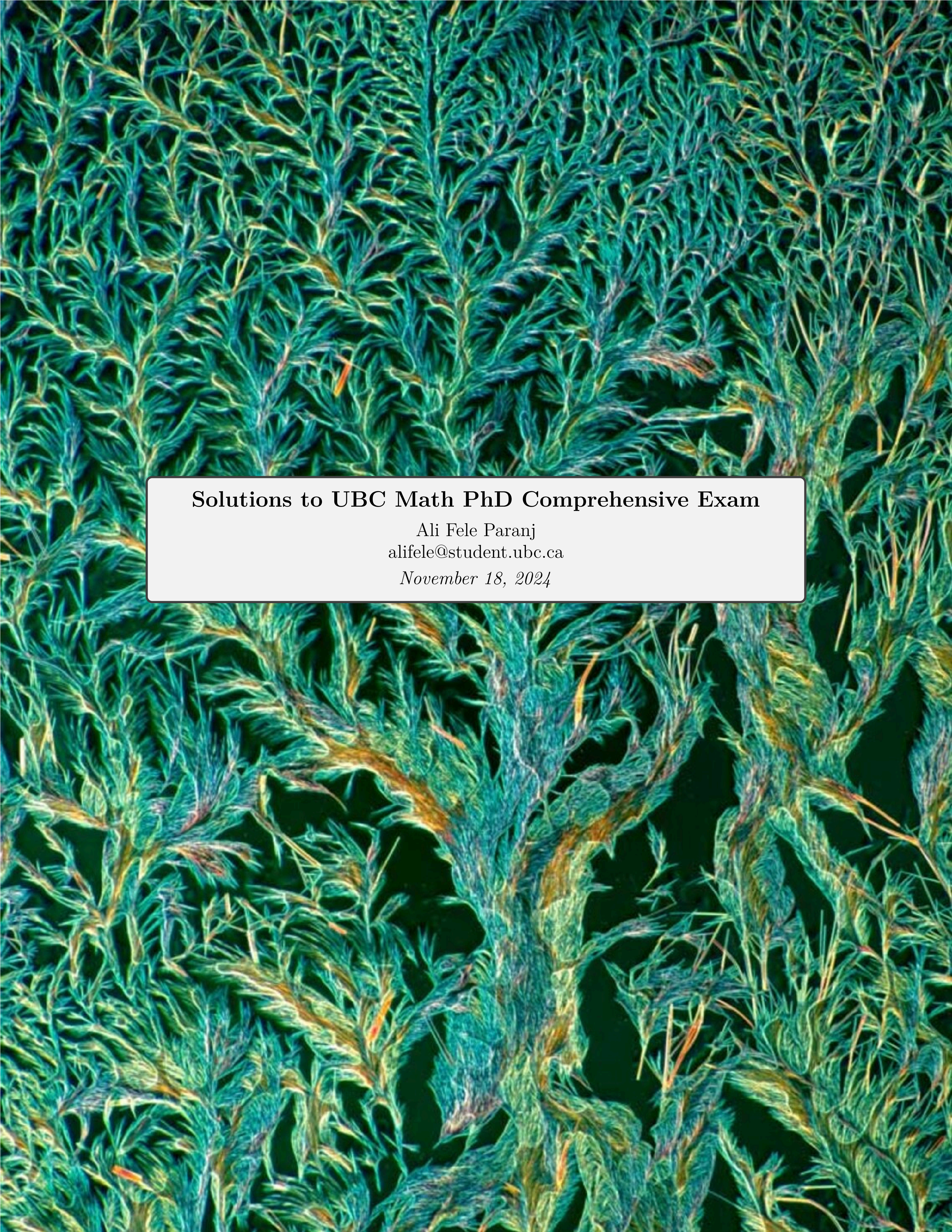

Solutions to UBC Math PhD Comprehensive Exam

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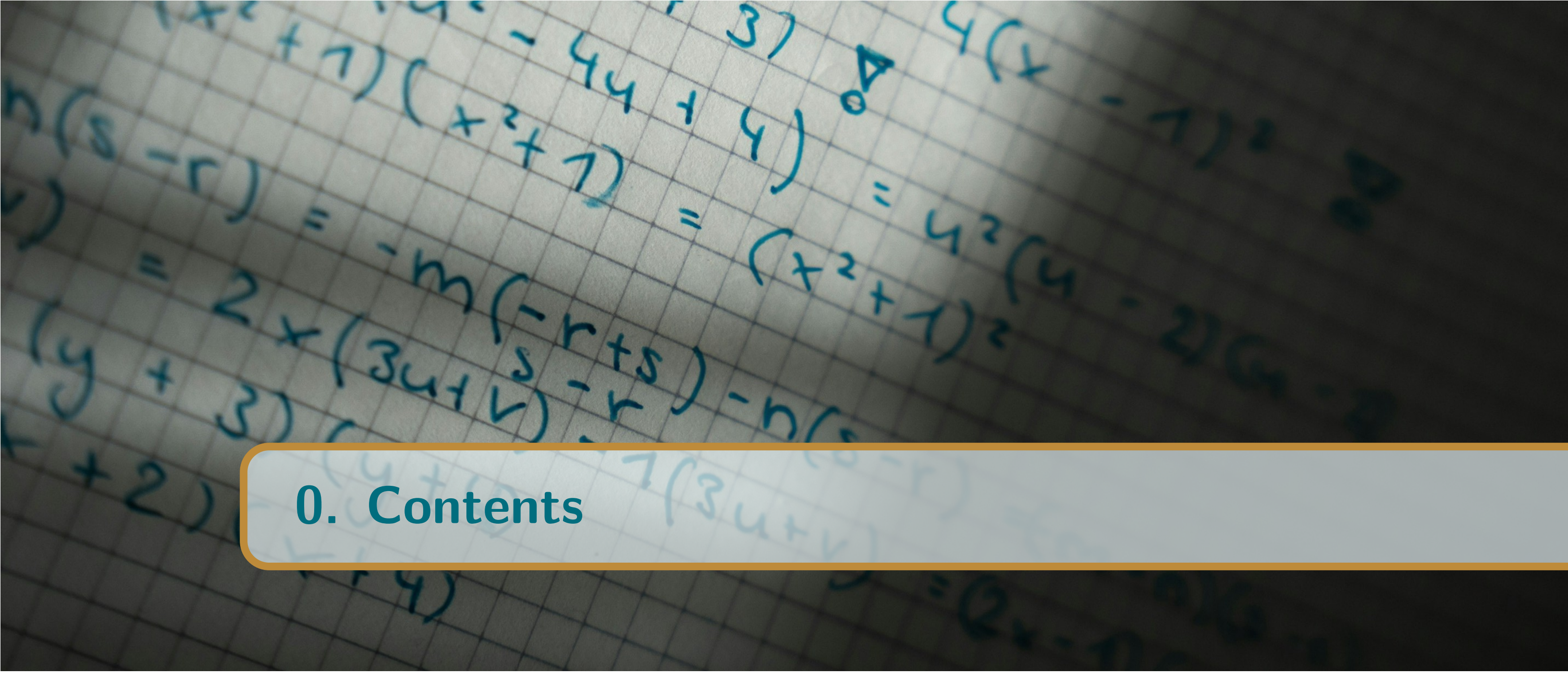
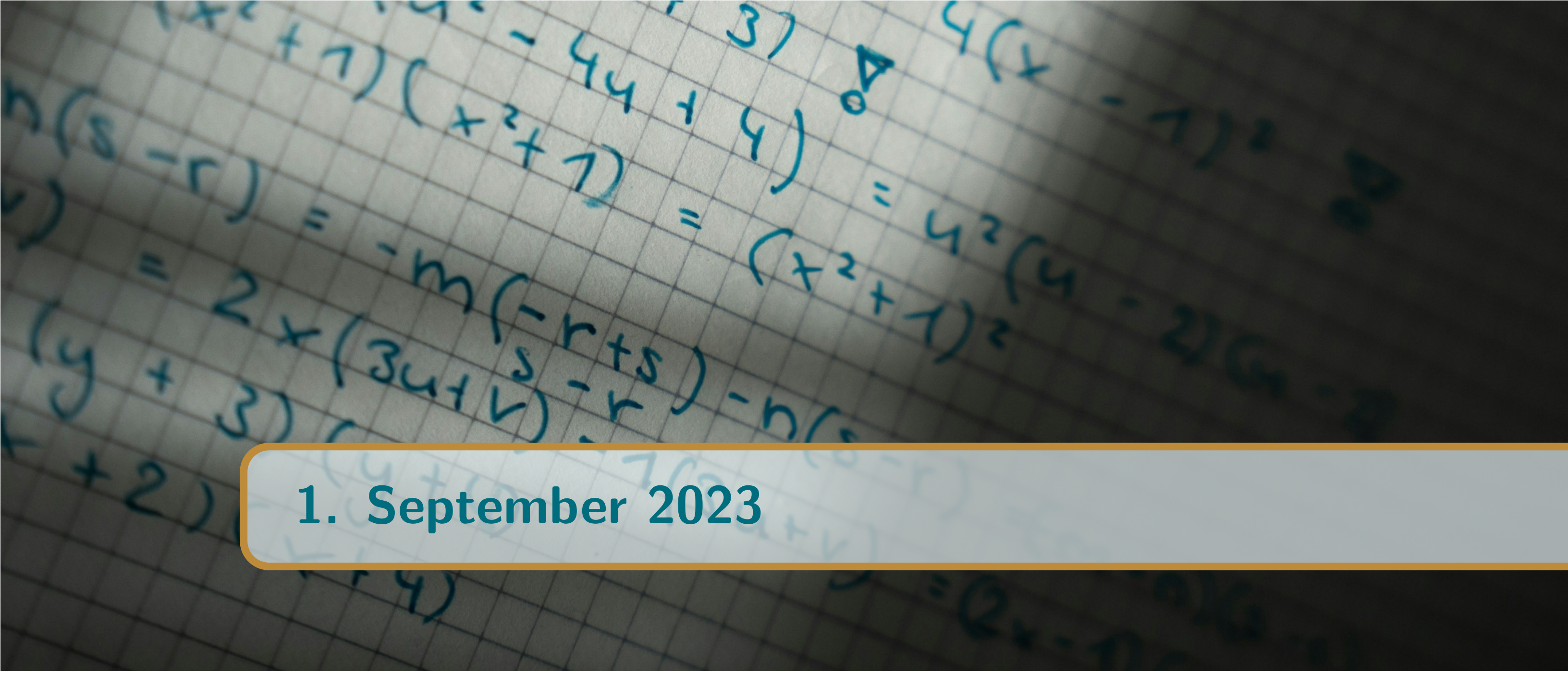


0. Contents



1. September 2023

1.1 Algebra

■ **Problem 1.1** (a) Prove that any module over $\mathbb{Z}[i]$ is a direct sum of a free module and a torsion module. Is the same true for modules over $\mathbb{Z}[\sqrt{-5}]$?

(b) Let $I = 3 + 2i$ be the ideal in $\mathbb{Z}[i]$ generated by the element $3 + 2i$. Describe the quotient $\mathbb{Z}[i]/I$.

1.2 Linear Algebra

■ **Problem 1.2** (a) Find a lower triangular matrix L such that $LL^T = A$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 5 & 2 & 2 \\ 0 & 2 & 2 & 1 \\ 2 & 2 & 1 & 6 \end{pmatrix}$$

(b) Compute $\det(A)$.

(c) Find the volume in $\mathbb{R}^4$ of the set $S_A = \{x \in \mathbb{R}^4 : x^T A x \leq 1\}$.

Solution (a) Let L be

$$L = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix}$$

So $A = LL^T$ we will have

$$A = \begin{pmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} & l_{11}l_{41} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} & l_{21}l_{41} + l_{22}l_{42} \\ l_{11}l_{31} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 & l_{31}l_{41} + l_{32}l_{42} + l_{33}l_{43} \\ l_{41}l_{11} & l_{41}l_{21} + l_{42}l_{22} & l_{41}l_{31} + l_{42}l_{32} + l_{43}l_{33} & l_{41}^2 + l_{42}^2 + l_{43}^2 + l_{44}^2 \end{pmatrix}$$

By solving the equations formed by the first row we have

$$l_{11} = 1,\ l_{21} = 1,\ l_{31} = 0,\ l_{41} = 2.$$

Furthermore

$$l_{22} = 2,\ l_{32} = 1,\ l_{33} = 1,$$

and lastly

$$l_{42} = 0, l_{43} = 1, l_{44} = 1.$$

So the matrix L will be

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

(b) We will expand relative to the last column as it has more zeros.

$$\det(L) = \det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Again, expanding relative to the last column we will have

$$\det(L) = \det \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = 2.$$

(c) First, we find a suitable linear transformation that can map the volume to another volume for which we know how to calculate its volume. Consider the linear transformation $x = L^{-T}y$. Since $x^T Ax = (y^T L^{-1})LL^T(L^{-T}x) = y^T y$, thus the set $x^T Ax \leq 1$ will be mapped to the 4-ball (4 dimensional ball). So its volume will be

$$|x^T Ax \leq 1| = \det(L)|y^T y \leq 1| = 2|S^4|.$$

(Note: The volume of the 4-ball is $\pi^2/2$.)

■ **Problem 1.3** Let P_n be the n+1-dimensional space of polynomials of degree n with real coefficients, and let $< \cdot, \cdot >$ be the inner product defined as

$$< p, q >= \int_{-1}^{1} p(x)q(x)\ dx.$$

(a) Find an orthogonal basis $\{u_0, u_1, u_2\}$ for P_2 such that $u_j \in P_j$, and $u_j(1) > 0$.

(b) Using the basis in part (a), express the operator $F[p] := \int_{-1}^{1} p(x)dx$ acting on P_2 as a 1×3 matrix.

(c) Using the basis in part (a), express the derivative operator $D[p] := \frac{d}{dx}p(x)$ as a 3×3 matrix.

Solution (a) We start with the Gram-Schmidt orthogonalization. Let $\{c, x, x^2\}$ be the basis vectors that we want to orthogonalize. Let $u_0 = c$. Then the first vector will be $\hat{u}_0 = u_0/\|u_0\| = 1/\sqrt{2}$. To find the second vector we have

$$u_1 = x - < x, \hat{u}_0 > \hat{u}_0 = x - \frac{1}{2}\int_{-1}^{1} x\ dx = x.$$

So the second normal vector will be $\hat{u}_1 = u_1/\|u_1\| = \sqrt{\frac{3}{2}}x$. To find the third vector we have

$$\begin{aligned} u_2 &= x^2 - (< x^2, \hat{u}_0 > \hat{u}_0 + < x^2, \hat{u}_1 > \hat{u}_1) \\ &= x^2 - (\frac{1}{2}\int_{-1}^{1} x^2 dx + \frac{3}{2}\int_{-1}^{1} x^3 dx) \\ &= x^2 - (1/3 + 0) = x^2 - 1/3. \end{aligned}$$

So the third normal vector will be $\hat{u}_2 = u_2/\|u_2\| = 3/2\sqrt{\frac{5}{2}}(x^2 - 1/3)$. However, since the question has not asked for normal basis vectors, for the following sections of the question we will use the following orthogonal (not orthonormal) basis vectors

$$u_1 = 1, \quad u_1 = x, \quad u_2 = x^2 - \frac{1}{3}.$$

(b) It is enough to see what is the effect of this operator on the basis vectors

$$F[u_0] = \int_{-1}^{1} 1dx = 2, \quad F[u_1] = \int_{-1}^{1} xdx = 0, \quad F[u_2] = \int_{-1}^{1} (x^2 - 1/3)dx = 2/3 - 2/3 = 0.$$

So the matrix representation of this operator will be

$$F = \begin{pmatrix} 2 & 0 & 0 \end{pmatrix}.$$

(c) Similarly to the solution above,

$$D[u_0] = 0, \quad D[u_1] = 1, D[u_2] = 2x,$$

so the matrix representation will be

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

■ **Problem 1.4** Recall that an orthogonal projection matrix is a matrix P that satisfies

$$P^2 = P, \quad P = P^T.$$

Suppose P is an $n \times n$ projection matrix with rank$(P) = k$. In the following, I_m denotes the $m \times m$ identity matrix.

(a) List all the eigenvalues of P, including multiplicity. Be sure to justify your reasoning.

(b) Show that $P = AA^T$ for some $n \times k$ matrix A such that $A^T A = I_k$.

(c) Suppose P_1, P_2 are two $n \times n$ projection matrices with rank k. Show that there exists an $n \times n$ orthonormal matrix U (i.e. such that $U^T U = I_n, UU^T = I_n$) such that $P_2 = UP_1U^T$.

Solution (a) The projection matrix P has only two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0$. The eigenvectors corresponding to λ_2 eigenvalue are the basis vectors of the null-space of P (that has dimension $n - k$), and the eigenvectors corresponding to λ_1 are the basis vectors of the orthogonal sub-space to the null space of P. Thus λ_2 has multiplicity $n - k$ and λ_1 has multiplicity k.

(b) Since P is symmetric, then we can choose the eigenvectors to form an orthogonal basis. Let $U = [U_1\ U_2]$ be the matrix that its columns are the eigenvectors and U_1 are the one with eigenvalue 1 while U_2 are the one with eigenvalue 0. Note that since the columns in U are orthogonal, then $U^{-1} = U^T$. Let D be the diagonal matrix of P, then we can write

$$P = UDU^T = [U_1\ U_2]\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}\begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} = U_1U_1^T.$$

Note that $U_1^TU_1 = I_k$ since U is orthogonal matrix.

(c) From part (b) we know that there are matrices A_1, A_2 such that

$$P_1 = A_1A_1^T, \qquad P_2 = A_2A_2^T.$$

Define $U = A_1A_2^T$. Then

$$P_1 = A_1A_1^T = A_1(A_2^TA_2A_2^TA_2)A_1^T = (A_1A_2^T)A_2A_2^T(A_2A_1^T) = UP_2U^T.$$

1.3 Real Analysis

■ **Problem 1.5** Let S be the part of the paraboloid $z = 2 - x^2 - y^2$ above the cone $z = \sqrt{x^2 + y^2}$, with upward orientation. Let

$$F = (\tan\sqrt{z}) + \sin\left(y^3\right)\hat{i} + e^{-x^2}\hat{j} + z\hat{k}.$$

Evaluate the flux integral $\iint_S F \cdot dS$.

■ **Problem 1.6** Let $f(x) = \sum_{n=1}^{\infty} \sin(nx)x^n$ for those x for which the series converges. Note that this is NOT a power series.

(a) Show that f is defined and continuous on $(-1, 1)$.

(b) Shoat the f is differentiable and that f' is continuous on $(-1, 1)$.

Solution (a) We use the Weierstrass M-test to show that the series converges uniformly on $(-1, 1)$. To see this let $r \in (0, 1)$. Then on $[-r, r]$ we have $|\sin(nx)x^n| \le r^n$. Since $\sum r^n < \infty$ (by the geometric series) then by Weierstrass M-test the series $\sum_n \sin(nx)x^n$ converges on $[-r, r]$ uniformly and absolutely for all $r \in (0, 1)$. This implies uniform and absolute converges on $(-1, 1)$. To show the continuity of the series, observe that $\sin(nx)x^n$ is continuous for all n. Thus this continuity carries over through the uniform converges.

(b) First, observe that each term in the sum is continuously differentiable. Denote the n-th term by $f_n(x)$, then we will have

$$f_n(x) = n\cos(nx)x^n + nx^{n-1}\sin(nx) = nx^{n-1}(x\cos(nx) + \sin(nx)).$$

For $x \in (-1, 1)$, the term inside the parenthesis will be

$$|x\cos nx + \sin nx| \le |x||\cos nx| + |\sin nx| \le 2.$$

Let $r \in (0, 1)$ and $x \in [-r, r]$, then we will have

$$nx^{n-1} \le nr^{n-1} \quad \forall x \in [-r, r].$$

Observe that the series $\sum_n nr^{n-1}$ converges for $r \in (-1,1)$ (you can see this easily by the ratio test). So for $r \in (0,1)$

$$f_k(x) \leq M_k \qquad \forall x \in (-r,r).$$

for some $M_k > 0$ (to be precise $M_k = 2kr^{k-1}$) where $\sum_k M_k < \infty$. So by the Weierstrass M-test $\sum_k f_k'(x)$ converges uniformly on any compact subset $[-r,r]$ for $r \in (0,1)$, and because each term is continuous, the f' is also continuous on $(-1,1)$. So far have observed that $\sum_k f_k(x)$ converges on compact subsets of $(-1,1)$, $\sum_k f_k'(x)$ converges on compact subsets of $(-1,1)$, and f_k is continuously differentiable. So we can do a term by term differentiation.