Lecture Notes For: The Complex Analysis and Applications

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The content of this lecture note will be mostly based on the course MATH 305 (Applied Complex Analysis) at UBC during Winter2, 2023 term. However, I have expanded the content and examples using the following text books as well:

- Fundamentals of Complex Analysis for Mathematics, Science and Engineering,(Third Edition) by E. Saff, A. Snider.
- Visual Complex Functions: An Introduction with Phase Portraits by Elias Wegert

1 Fundamentals

Complex numbers can be thought as an extension to the real number system in which we have a solution for the $x^2+1=0$ equation. There are some mathematically rigorous ways to construct the complex numbers from real numbers. However, those mathematically rigorous ways came out only recently (20th century) and there were not around when the first ideas of complex numbers were forming around 18th century. For that reason I have not discussed the detailed mathematical construction of the complex numbers here.

As discussed earlier, complex numbers system is a system in which we have a solution for the $x^2+1=0$ equation which is represented as $i=\sqrt{-1}$. A complex number is written like $z=a+bi$ in which $a,b\in\mathbb{R}$ and the set of all complex numbers is denoted as $\mathbb{C}$. It is easy to check that complex numbers still satisfy the *commutative*, *associative*, and *distributive* properties similar to the real numbers.

Definition: Set of Complex Numbers and Basic Definitions $\mathbb{C}$

- The set of complex numbers: The set of complex numbers $\mathbb{C}$ is:

$$\mathbb{C}=\{z=a+bi : a,b\in\mathbb{R}, i=\sqrt{-1}\}$$

In which we call a as the *real part* and b as the *complex part* of the complex number z and we show that as:

$$\mathrm{Re}(z)=a$$
$$\mathrm{Im}(z)=b$$

- Two complex numbers $z_1=a_1+b_1i$ and $z_2=a_2+b_2i$ are equal if:

$$\mathrm{Im}(z_1)=\mathrm{Im}(z_2)$$
$$\mathrm{Re}(z_1)=\mathrm{Re}(z_2)$$

- Polar representation of the complex numbers: A complex number $z=a+bi$ can be written as:

$$z=re^{i\varphi}$$

in which r is the modulus of z and φ is the argument of z which is defined as: $\varphi=\arctan(\frac{a}{b})$. Also we define the set Arg as:

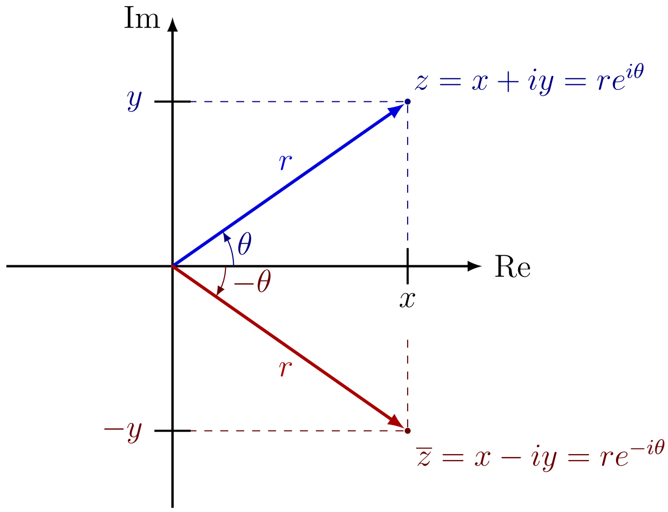


Figure 1.1: A summary of the polar and Cartesian representation of the complex numbers.

$$Arg(z) = \{arg(z) + 2\pi n : n \in \mathbb{Z}\}$$

- Complex conjugate: The complex conjugate of a complex number $z = a + bi$ is defined as:
$$\overline{z} = a - bi$$

- Modulus of a complex number: The modulus of a complex number $z = a + bi$ is defined as:
$$|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$$

The following figure summarizes the basic properties of the complex numbers discussed above.

Proposition: Fundamental Properties of Complex Numbers

Using the definition of the complex numbers, we can show that they satisfy the following properties.

1. Commutative property for the sum and product:

$$a + b = b + a \tag{1.1}$$
$$ab = ba \tag{1.2}$$

2. Associative property for the sum and product

$$a + (b + c) = (a + b) + c \tag{1.3}$$
$$a(bc) = (ab)c \tag{1.4}$$

3. Distributive property:

$$a(b + c) = ab + ac \tag{1.5}$$

Proof. All of the statements can be proved by writing complex numbers as $z = x + yi$ and substituting in the equations. □

Utilizing the basic definitions along with the mathematical logic, we can derive different properties for complex numbers. The following properties come in handy when solving problems:

Proposition: Basic Properties of Complex Numbers

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \tag{1.6}$$
$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2} \tag{1.7}$$
$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}} \tag{1.8}$$
$$\overline{(\overline{z})} = z \tag{1.9}$$
$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2} \tag{1.10}$$
$$\mathrm{Im}(z) = \frac{z - \overline{z}}{2i} \tag{1.11}$$
$$(\overline{z})^k = \overline{(z^k)} \tag{1.12}$$
$$|z_1 z_2| = |z_1||z_2| \tag{1.13}$$

Proof. The proof for some of the properties:

- $|z_1 z_2| = |z_1||z_2|$: To show this we first need to raise the both sides to the power 2:

  $$|z_1 z_2|^2 = (z_1 z_2)\overline{(z_1 z_2)} = (z_1 \overline{z_1})(z_2 \overline{z_2}) = |z_1|^2 |z_2|^2$$

  . So by taking the square root of the both sides we will arrive at: $|z_1 z_2| = |z_1||z_2|$

- $(\overline{z})^k = \overline{(z^k)}$: Let's start from the right hand side:

$$\overline{(z^k)} = \overline{(\underbrace{z * z * z * \ldots * z}_{\text{k times}})} = \underbrace{\overline{z} * \overline{z} * \overline{z} * \ldots * \overline{z}}_{\text{k times}} = (\overline{z})^k$$

□

Example: Ordering Property in Complex Numbers

Question. Show that we can not define any ordering property in the complex numbers.

Proof. Suppose that we *can* define ordering property in the complex numbers. So two cases might arise for the i. It will be $i > 0$ or $i < 0$.

- Let's assume that $i > 0$. Since $f(x) = x^2$ is a strictly increasing function, then by arising two sides of the inequality to the power of two we will have: $i^2 = -1 > 0$ which is a contradiction.
- Let's assume that $i < 0$. Since i is smaller than zero, we can multiply the two sides of the inequality by i but we should change the direction of the inequality sign. So we will have $i * i = -1 > 0$ which again arises a mathematical contradiction.

So we can conclude that we can not have any ordering property in the set of complex numbers. □

We can also have some important inequalities in the complex numbers that often come very handy in solving problems. The following box summarizes some of them.

Theorem: Important Inequalities in Complex Numbers

1. Basic Inequalities

$$\begin{aligned} \mathrm{Re}(z) &\leq |z| \\ \mathrm{Im}(z) &\leq |z| \end{aligned}$$

2. Triangle Inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof.

- Triangle Inequality (proof number 1): Let's start by raising the right hand side to the power 2 and simplify the equation:

$$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + \overline{z_1}z_2$$

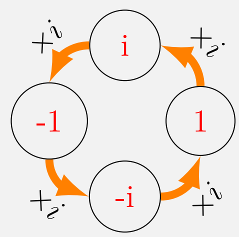
□

Example: Cyclic Property of i

By using the basic property of the complex number $i = \sqrt{-1}$, for every $k \in \mathbb{Z}$ we can show that:

$$
\begin{aligned}
i^{4k} &= 1 \\
i^{4k+1} &= i \\
i^{4k+2} &= -1 \\
i^{4k+3} &= -i
\end{aligned}
$$

This also represent that fact that multiplying a complex number by i means a $\pi/2$ counterclockwise rotation. This becomes very clear if we consider the polar representation of complex numbers.



Example: Karatsuba Algorithm

Suppose that we want to multiply two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ in the computer and return $z = z_1 + z_2$. To do that we should get the imaginary and real part of two numbers and then return:

$$
\begin{aligned}
\mathrm{Re}(z) &= ac - bd \\
\mathrm{Im}(z) &= bc + ad
\end{aligned}
$$

So to multiply two complex numbers in the computer, we need to do 4 multiplications and 2 additions. So performing multiplication takes considerably more clock cycles in CPU. However we can do a trick to reduce the number of multiplications to 3 with a cost of some extra additions. To do so we need to calculate three intermediate variables each of which require one multiplication.

$$
\begin{aligned}
t_1 &= ac \\
t_2 &= bd \\
t_3 &= (a + b)(c + d)
\end{aligned}
$$

So we will have:

$$\begin{aligned} \mathrm{Re}(z) &= t_1 - t_2 \\ \mathrm{Im}(z) &= t_3 - (t_1 + t_2) \end{aligned}$$

So using the Karatsuba algorithm we will have 3 multiplications, and 5 addictions. There is a very interesting story behind this discovery by the Karatsuba in 1960 (when he was a 23-year-old graduate student) that he somehow proved the Kolmogorov's statement is wrong! You can read more on this in the Wikipedia page of Karatsuba algorithm.

Example: Complex Roots

Find all of the roots of the following equations

1. $z^4 - 16 = 0$

   Solution. Writing the complex number in the form of its polar representation will ease finding the roots of this equation significantly. So let's assume

   $$z = re^{i\varphi}$$

   Then by substituting in the equation we will have:

   $$r^4 e^{4i\varphi} = 16 = 16e^{2n\pi i}$$

   So we will have: $r = 2$ and $\varphi = \{n\pi/2 : n \in \mathbb{R}\} = \{0, \pi/2, \pi, 3\pi/2\}$. □

2. $\frac{z}{1-z} = 1 - 5i$

   Solution.

   $$\begin{aligned} z &= (1-z)(1-5i) \\ &= 1 - 5i - z - 5iz \\ &= \frac{1-5i}{2-5i} = \frac{(1-5i)(2+5i)}{29} \end{aligned}$$

   □

Proposition: Roots of Polynomial

if z_0 is the root of the following polynomial equation (in which $a_i \in \mathbb{R}$):

$$z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n = 0$$

then $\overline{z_0}$ is also the root of the equation.

Proof. This can easily be shown by taking the complex conjugate from both sides of the equation, and then using the fact that $(\overline{z})^k = \overline{(z^k)}$.

□

Example: Matrices with Complex Entries

TO BE COMPLETED

2 Planar Sets and Geometry

The interesting fact about the complex plane that I like the most is that we can construct different good looking (!) sets in the complex plain $\mathbb{C}$ with quite simple statements. For example the inequality $|z| \leq 9$ represents a circle in the complex plane centred at the origin and has a radius of 3.

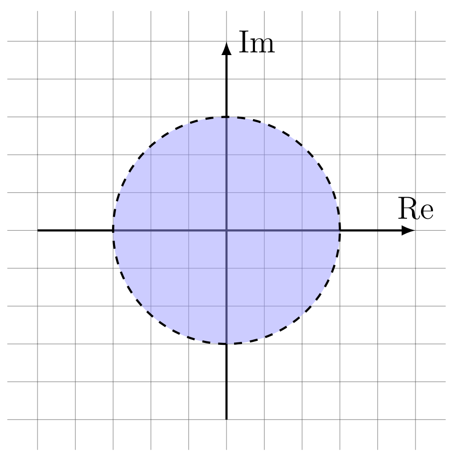


Figure 2.1: The planar set representing $|z| \leq 3$

Example:

Question. Derive the points in $\mathbb{C}$ that satisfy following conditions.

1. $|z+2| = |z-1|$

Solution. There is not a single method that works the best for all of the problems. For this question, it is better to raise both sides of the eqution to the power 2.

$$|z+2|^2 = |z-1|^2$$
$$(z+2)\overline{(z+2)} = (z-1)\overline{(z-1)}$$
$$(z+2)(\overline{z}+2) = (z-1)(\overline{z}-1)$$
$$z\overline{z} + 2z + 2\overline{z} + 4 = z\overline{z} - z - \overline{z} + 1$$
$$z + 1 = -\overline{z}$$

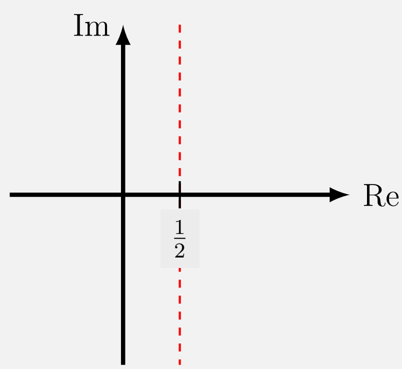
So let's assume z is in the form $z = x + iy$. By inserting this in equation above we will get:

$$(x+1) + iy = -x + iy$$

So:

$$x = \frac{1}{2}$$
$$y = \text{any real number}$$

So the set of points satisfying this equation will be:



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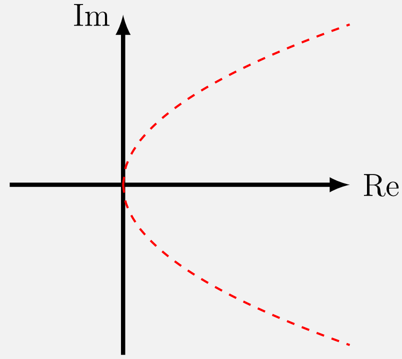
2. $|z-1| = \operatorname{Re} z + 1$

Solution. For this question it is better to start with a general form of $z = x+iy$ and then substitute in the equation.

$$|(x-1) + yi| = x + 1$$
$$(x-1)^2 + y^2 = (x+1)^2$$
$$z^2 - 2x + 1 + y^2 = x^2 + 2x + 1$$
$$y^2 = 4x$$

So the points of complex plain that satisfy the equation will be like the curve:



2.1 * Discussion on the Argument of Complex Numbers

As discussed earlier, we can have Cartesian or polar representation of the complex number $z = x + iy$. The polar presentaion will be in the form $z = re^{i\varphi}$ in which r is the modulus of z and φ is so called the argument of the complex number. The modulus is defined as $r = \sqrt{x^2 + y^2} = |z|$. However, the argument definition is quite tricky.

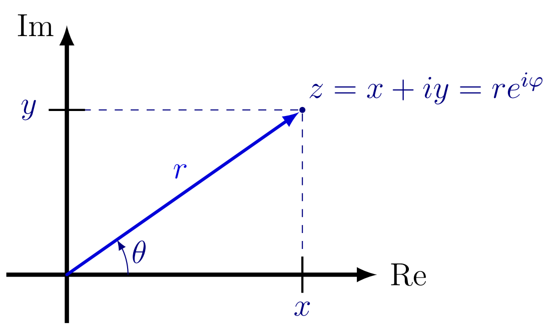


Figure 2.2: Polar and Cartesian representaion of complex number $z = x + iy$.

Based on the geometrical interpretation of the complex numbers, it feels natural that we define the argument using the inverse tangent function. $\varphi = \arctan(\frac{x}{y})$. But according to the figure 2.1, the range of this function is defined in $(-\pi/2, \pi/2)$ interval. So for the complex points in the first and fourth quadrant we get a meanungful φ value but for the points in the second and the third quadrant, the interpretation of the φ does not work well. This probelm can be solved by considering the signs of the Re and Im parts of the complex number or considering the $\cos(\theta) = \frac{x}{r}$ and $sin(\theta) = \frac{y}{r}$ seperatly. The latter works because sin and cos uniquely determine the position (and hence the argument) of a complex number. For the former, the following definition determines a unique argument even for the points in the second and the fourth quadrant.

$$arg(z) = \begin{cases} \arctan(y/x) + \pi/2(1 - \operatorname{sgn}(x)) & \text{if } x \neq 0 \\ \pi/2 \operatorname{sgn}(y) & \text{if } x = 0, y \neq 0 \\ \text{undefined} & \text{if } x = y = 0 \end{cases}$$

in which $\operatorname{sgn}(x)$ is the sign function that returns the sign of its argument.

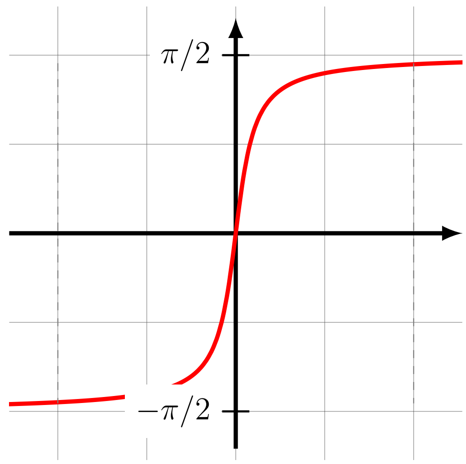


Figure 2.3: Plot of $y = \arctan(\theta)$ function.

3 Complex Maps

3.1 Linear Map

3.2 Inverse Map

3.3 Mobius Map

3.4 Quadratic Map

3.5 Exponential Map

4 Calculus for Complex variables

4.1 Limit

4.2 Continuity

4.3 Differentiability

5 Quick Facts

Fact 5.1 (Harmonic Function)**.** Let $f : D \to \mathbb{C}$ be a holomorphic function in the connected open set $D \subseteq \mathbb{C}$, where $f(x, y) = u(x, y) + iv(x, y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$. Then the function u, v are harmonic functions.

Proof. Since f is holomorphic in the open connected subset $D \subseteq \mathbb{C}$, then it means that the function u, v satisfy the Cauchy-Riemann equations, the first derivatives exists, and are continuous. As we will show later, in fact u, v are smooth function, so any higher derivative exists and is continuous. From Cauchy-Riemann we have

$$u_x = v_y, \quad u_y = -v_x.$$

Calculating the second derivatives will yield

$$u_{xx} = v_{yx}, \quad u_{yy} = -v_{xy}.$$

Note that since u, v are smooth, then $v_{xy} = v_{yx}$. Then we can conclude that

$$u_x x + u_y y = 0,$$

which indicates that u is a harmonic function (it satisfies the Laplace equation). The same reasoning works for v. $\square$

Fact 5.2 (Harmonic Conjugate Function)**.** Let $u : \Omega \to R$ be a harmonic function define on the connected open set $\Omega \subseteq \mathbb{R}^2$. Then $v : \Omega \to \mathbb{R}$ is a *harmonic conjugate* of u if and only if the function f of the complex variable $z := x + iy \in \Omega$ is holomorphic.

Example 5.1. Construct an analytic function whose real part is $u(x, y) = x^3 - 3xy^2 + y$.

Answer. Let $f(z) = u(x, y) + iv(x, y)$ holomorphic as required. Then it should satisfy the Cauchy-Riemann equations.

$$v_y = 3x^2 - 3y^2, \quad v_x = 6xy - 1.$$

Integrating the first expression yields in $v(x, y) = 3x^2y - y^3 + f(x)$ and differentiating it and comparing it with the second expression above yields $f(x) = -x + C$ for some $C \in \mathbb{R}$. Thus a complex conjugate of $u(x, y)$ will be

$$v(x, y) = 3x^2y + y^3 - x + C.$$

Fact 5.3. Let $f : \Omega \to \mathbb{C}$ where $\Omega \subseteq \mathbb{C}$ is an open connected set. Assume $f(x + iy) = u(x, y) + iv(x, y)$. If f is holomorphic at Ω, then v is a harmonic conjugate function of u. Further more, the level sets of these function are perpendicular to each other everywhere in the domain Ω.

Proof. let $(x_0, y_0) \in \Omega$. The gradient of u and v are given as

$$\nabla u = (u_x, u_y), \quad \nabla v = (v_x, v_y).$$

When evaluated at (x_0, y_0), they will be the vectors perpendicular to the corresponding level curves. We calculate the inner product of these vectors.

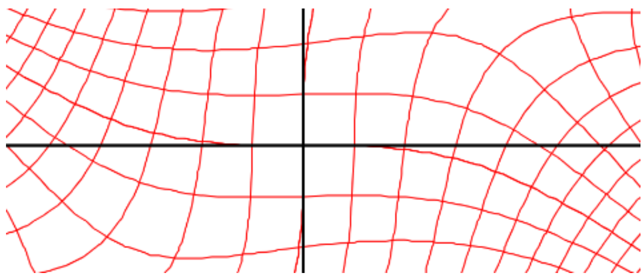
$$\nabla u(x_0, y_0) \cdot \nabla v(x_0, v_0) = u_x(x_0, y_0) y_x(x_0, y_0) + u_y(x_0, y_0) v_y(x_0, y_0)$$

Since f is holomorphic at Ω, then it satisfies the Cauchy-Riemann equations. Thus we get

$$\nabla u(x_0, y_0) \cdot \nabla v(x_0, v_0) = 0.$$

This implies that the level curves are perpendicular. □

The following figure shows this fact for the harmonic conjugate functions we calculated in Example 5.1.



Fact 5.4 (Harmonic Conjugacy is Not Symmetric)**.** If v is a harmonic conjugate of u, then it means that $f(x + iy) = u(x, y) + iv(x, y)$ is holomorphic at the open, connected disk of definition. Then this implies that $if(x + iy) = -v + iu$ is holomorphic as well. Thus we conclude that u is a harmonic conjugate of $-v$.

Fact 5.5 (The parallel between Picard iteration and Newton's method)**.** As we know from Galois theory, there are no closed form formula for the roots of polynomials of order 5 and higher. On the other hand, from the fundamental theorem of algebra we know that such roots exists and the number of roots is in fact the same as the degree of polynomials (multiplicity counted). However, in general, to find the roots of the degree 5 or higher polynomials we use the methods like the Newton's method. There is a beautiful parallel between this and the Picard iteration in finding the solution of an ODE. In both cases, we use an iterative approach to find the solution of a particular equation.