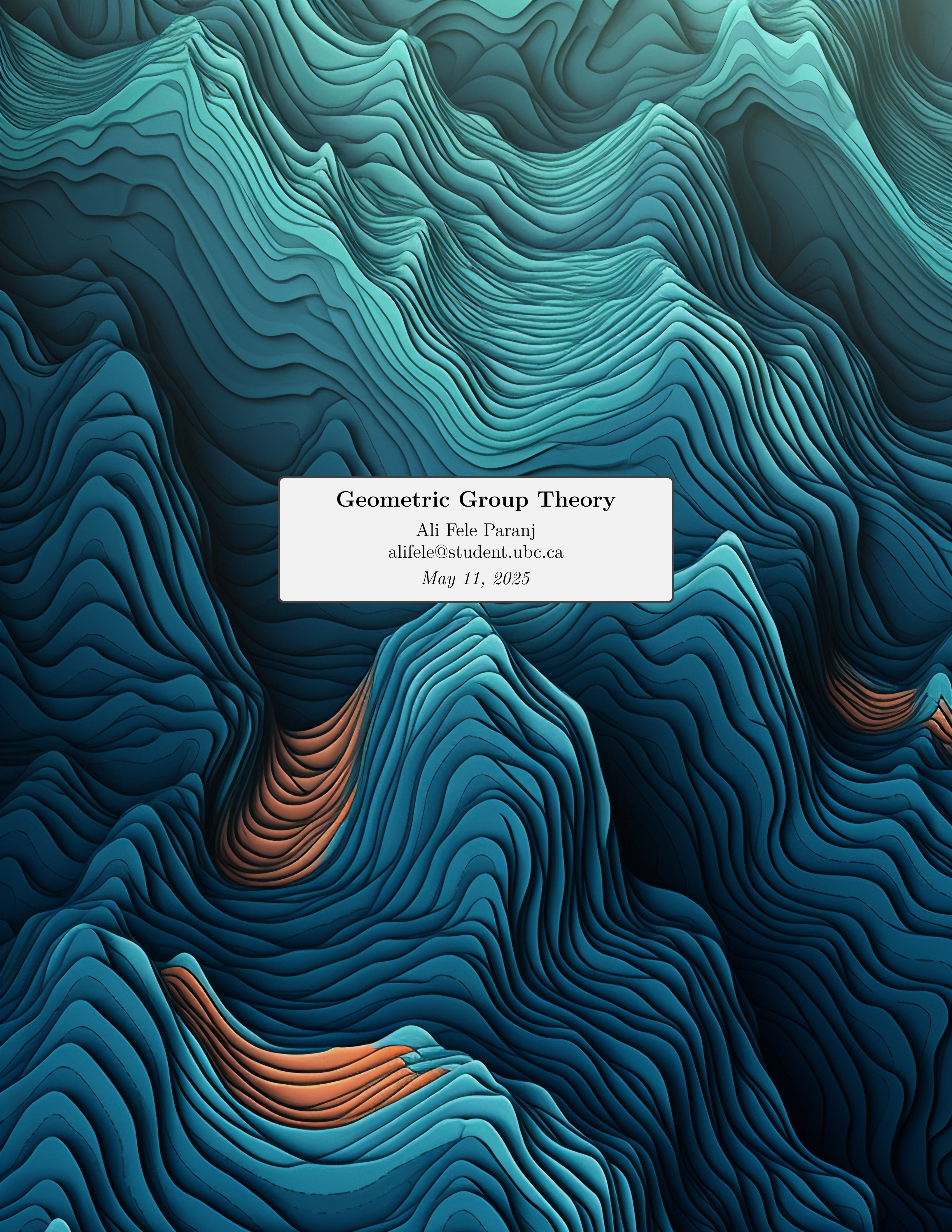

# Geometric Group Theory


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# 0. Contents

# 1. Topological Spaces and Continuous Function

**Definition 1.1 — Topology on a set.** Let $X$ be a set. A topology on $X$ is a collection of subsets of $X$, called *open sets* and denoted by $\mathcal{T}$, that satisfies

- $X, \varnothing \in \mathcal{T}$,
- For an *arbitrary* collection of open sets $\{U_\alpha\}_{\alpha \in J}$ we have

$$\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}.$$

- For a *finite* collection of open sets $\{U_1, \cdots, U_n\}$ for some $n \in \mathbb{N}$ we have

$$\bigcap_{i=1}^{n} U_i \in \mathcal{T}$$

**Definition 1.2 — Finer and Coarse Topologies.** Let $X$ be a set and $\mathcal{T}_1$ and $\mathcal{T}_2$ two topologies. The we say the topology $\mathcal{T}_1$ is finer if we have $\mathcal{T}_\in \subset \mathcal{T}_\infty$. Or alternatively, we can call the topology $\mathcal{T}_2$ to be coarser.

**Definition 1.3 — Bases for a topology.** Let $X$ be a set. A collection of subsets of $X$, denoted by $\mathbb{B}$, is called a basis for the topology on $X$ if we have

- $\forall x \in X$ there exists $B \in \mathbb{B}$ such that $x \in B$.
- If for sets $B_1, B_2 \in \mathbb{B}$ we have $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathbb{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

**Definition 1.4 — Topology Generated by a Basis.** Let $X$ be a set and $\mathbb{B}$ a basis for a topology. We *define the topology $\mathcal{T}$ generated by* $\mathbb{B}$ as follow: For any $U \in \mathcal{T}$ and $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

■ **Remark 1.1** We can now check that if $\mathcal{T}$ generated by $\mathbb{B}$ is really a topology. We need to show that $\mathcal{T}$ has the topology properties.

- The statement $\varnothing \in \mathcal{T}$ is vacuously true. To see $X \in \mathcal{T}$, Consider $x \in X$. Then from the definition of a basis $\exists B \in \mathbb{B}$ such that $x \in B$ and since $X$ is the whole space, thus $x \in B \subset X$.

- Let $\{U_\alpha\}_{\alpha\in J}$ be an arbitrary collection of sets in $\mathcal{T}$. Consider $x \in \bigcup_\alpha U_\alpha$. Then there exists an index $\alpha$ such that $x \in U_\alpha$. Since $U_\alpha \in \mathcal{T}$ thus there exists $B \in \mathbb{B}$ such that $x \in B \subset U_\alpha$ which implies $x \in B \subset \bigcup_\alpha U_\alpha$, hence $\bigcup_\alpha U_\alpha \in \mathcal{T}$.
- To see that the intersection of a finite collection of sets in $\mathcal{T}$ is also in $\mathcal{T}$, first if $U_1 \in \mathcal{T}$ and $U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$. That is because for $x \in U_1 \cap U_2$ we know that $\exists B_1, B_2 \in \mathbb{B}$ such that $x \in B_1 \cap B_2$. Thus from the definition of a basis $\exists B_3 \in \mathbb{B}$ such that $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$, hence $U_1 \cap U_2 \in \mathcal{T}$. Then by induction we can conclude that for any finite collection of sets in $\mathcal{T}$ their intersection is also in $\mathcal{T}$.

**Proposition 1.1 — Characterization of Open Sets in Terms of Basis.** Let $X$ be a set and $\mathbb{B}$ a basis for a topology $\mathcal{T}$ on $X$. Then $\mathcal{T}$ is the collection of all unions of $\mathbb{B}$.

*Proof.* Let $\mathcal{A}$ be the collection of all unions of $\mathbb{B}$. We need to show the equality of the sets

$$\mathcal{A} = \mathcal{T}.$$

Let $A \in \mathcal{A}$. Thus $A = \bigcup_\alpha B_\alpha$ for $B_\alpha \in \mathbb{B}$. Let $x \in A$. Then we can choose any $B_\alpha$ and we will have $x \in B_\alpha \subset A$. Thus $A \in \mathcal{T}$. To show the converse let $A \in \mathcal{T}$. Thus for any $x \in A$ we have $B_x \in \mathbb{B}$ such that $x \in B_x \subset A$. We can write

$$A = \bigcup_{x \in A} B_x.$$

Thus $A \in \mathcal{A}$. This completes the proof. □

**Be Careful Here! 1.0.1 — Be Aware of the Terminology.** From the proposition above, we can say that a subset $U \subset X$ is open (i.e. $U \in \mathcal{T}$) if it can be written as union of sets in $\mathbb{B}$. This union, however, need not be unique. This is a very important difference with the notion of basis in linear algebra, where given a basis for a space, we can then write any vector in the space uniquely by the basis vectors.

**Proposition 1.2 — Basis of a Given Topology.** Let $(X, \mathcal{T})$ be a given topological spaces. A collection of open sets $\mathcal{C}$ is a basis for the topology if $\forall U \in \mathcal{T}$ and $x \in U$ there exists $C \in \mathcal{C}$ such that $x \in C \subset U$.

*Proof.* This proof has two steps. First, we need to show that the collection $\mathcal{C}$ is indeed a basis, and then we need to show that the topology that this basis generates is the same as the topology of the set itself.

**Showing that $\mathcal{C}$ is a basis**. To show the first property of being a basis, we need to show that $\forall x \in X$ we can find an element of $\mathcal{C}$ that contains $x$. Since $X \in \mathcal{T}$ and by hypothesis we can find $C \in \mathcal{C}$ such that $x \in C \subset X$, thus $\mathcal{C}$ has the first property of being a basis. For the second property, we need to show that if $x \in C_1 \cap C_2$ then there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$. Let $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C}$. Since $C_1, C_2$ are open sets, then $C_1 \cap C_2$ is also open. Thus by hypothesis there exists $C \in \mathcal{C}$ such that $x \in C \subset C_1 \cap C_2$. This completes the proof that the collection $\mathcal{C}$ is indeed a basis.

**Showing that the basis $\mathcal{C}$ generate the topology $\mathcal{T}$.** To show this, we can directly use Definition 1.4 or use the characterization in Proposition 1.1, where we will use the latter. I.e. it suffices to show that for any given $U \in \mathcal{T}$ we can write it as a union of basis sets. Let $U \in \mathcal{T}$ and $x \in U$. Then by hypothesis we know that $\exists C_x \in \mathcal{C}$ such that $x \in C_x \subset U$. Thus we can write $U$ as

$$U = \bigcup_{x \in U} C_x,$$

which completes the proof. $\square$

**Proposition 1.3 — Comparing Topologies in Terms of Their Bases.** Let $X$ be a set and $\mathcal{B}$, $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ on $X$ respectively. Then the followings are equivalent.

(i) $\mathcal{T}'$ is finer than $\mathcal{T}$.

(ii) For any $x \in X$ and any basis element $B \in \mathcal{B}$ containing $x$, there exists a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

*Proof.* We will have two parts

$\boxed{(i) \implies (ii)}$ By Definition 1.2 hypothesis implies $\mathcal{T} \subset \mathcal{T}'$. Let $x \in X$ and $B \in \mathcal{B}$ by any basis element that contains it. Since the topology $\mathcal{T}$ is generated by $\mathcal{B}$, thus $B \in \mathcal{T}$. By hypothesis we have $B \in \mathcal{T}'$. Since $\mathcal{B}'$ generates the topology $\mathcal{T}'$, thus there exists some $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

$\boxed{(ii) \implies (i)}$ Let $U \in \mathcal{T}$ and consider $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$ then $\exists B \in \mathcal{B}$ such that $x \in B \subset U$. By hypothesis there exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B$, hence $x \in B' \subset U$. This implies that $U \in \mathcal{T}'$.

$\square$

**Proposition 1.4 — Three Topologies on Real Line $\mathbb{R}$.** Consider the real line $\mathbb{R}$. Then the followings are three topologies on this set.

(i) **Standard Topology.** Standard topology is the topology $\mathcal{T}_s$ generated by the bases elements given as the collection of open intervals which are of the form

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}.$$

(ii) **Lower Limit Topology.** Is the topology $\mathcal{T}_l$ generated by the bases elements of the form

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}.$$

(iii) **$K$-Topology**. Let $K = \{1/n \mid n \in \mathbb{N}\}$. Then the topology $\mathcal{T}_K$ is the topology generated by the bases elements of the form

$$(a, b), \quad \text{and} \quad (a, b) - K.$$

Then $\mathcal{T}_l$ and $\mathcal{T}_K$ are strictly finer than $\mathcal{T}_s$ but are not comparable.

■ **Remark 1.2** When we talk about the real number line $\mathbb{R}$ the default topology is the standard topology.

*Proof.* First, we need to show that the sets specified above are bases. This is true as for any $x \in \mathbb{R}$ we can find a basis element of each type that contains $x$. Also, since the intersection of any two basis element in each of the bases sets is again in the set, thus the second property of being a basis set is also satisfies, hence these sets are bases sets.
**Showing that $\mathcal{T}_s \subset \mathcal{T}_l$.** To show this let $(a, b) \in \mathcal{T}_s$. We show that we can write this as a union of bases elements in $\mathcal{B}_l$. The following union does the job

$$(a, b) = \bigcup_{n \in \mathbb{N}} [a + \frac{a+b}{2n}, b).$$

To show that $\mathcal{T}_l$ is strictly finder than $\mathcal{T}_s$, let $[a,b) \in \mathcal{T}_l$. Assume that there is a collection of basis elements in $\mathcal{B}_s$ whose union in $[a,b)$. Then $a \in [a,b)$ is in one of the basis elements, say $a \in B \in \mathcal{B}_s$. Then $\exists x$ less than $a$ such that $x \in B$. Then $x < a$ should also be in the union, which is a contradiction as $x \notin [a,b)$.
**Showing that $\mathcal{T}_s \subset \mathcal{T}_K$.** Let $(a,b) \in \mathcal{T}_s$. Since $(a,b) \in \mathcal{B}_K$ as well, then $(a,b) \in \mathcal{T}_K$. To show the strict relation, let $(-1,1) - K \in \mathcal{T}_K$. We claim that there is no collection of basis elements in $\mathcal{B}_s$ whose union is $(-1,1) - K$. Assume otherwise. Then $0 \in (-1,1) - K$ should be on one of the open sets in the collection, say $0 \in B \in \mathcal{B}_s$. Then we can find $N \in \mathbb{N}$ large enough such that $1/n \in B$ for all $n > N$. This is a contradiction as $(-1,1) - K$ does not have any such elements.
**Showing that $\mathcal{T}_l$ and $\mathcal{T}_K$ are not comparable**. Let $[-2,-1) \in \mathcal{T}_l$. Following a similar argument as above, there is no collection of bases elements in $\mathcal{T}_K$ whose union is $[-2,-1)$. Furthermore, consider $(-1,1) - K \in \mathcal{B}_K$. We claim that there is no collection of bases elements in $\mathcal{B}_l$ whose union os $(-1,1) - K$. To see this consider $0 \in (-1,1) - K$. Then 0 should be in one basis element of the form $[0,a)$ or $[b,a)$ where $b < 0 < a$. In either of cases we can find $N \in \mathbb{N}$ large enough such that $\forall n > N$ we have the elements of $1/n$ in these sets. This is a contraction. This completes the proof that these two topologies are not comparable. □

**Definition 1.5 — Subbasis of a Topology.** A collection $\mathcal{S}$ of subsets of $X$ is a subbasis for the topology $\mathcal{T}$ on $X$ if

(i) The union of the elements of $\mathcal{S}$ is $X$.
(ii) $\mathcal{T}$ is all the union of finite intersection of the elements in $\mathcal{S}$

■ **Remark 1.3** We can check to see if $\mathcal{T}$ has the properties of a topology. Note that $X \in \mathcal{T}$. To see this consider the union of the intersection of each element of $\mathcal{S}$ with itself. By hypothesis (i) this union will be the whose set $X$. To see $\varnothing \in \mathcal{T}$ consider the union of the intersection of each element of $\mathcal{S}$ with the empty set $\varnothing$ which will be the empty set, hence $\varnothing \in \mathcal{T}$. To show that arbitrary union of the elements of $\mathcal{T}$ is in $\mathcal{T}$ note that this since each element of $\mathcal{T}$ is already a union of finite intersections of $\mathcal{S}$, thus any arbitrary union of the element of $\mathcal{T}$ will still be a union of finite intersections of elements of $\mathcal{S}$, thus by hypothesis it will be in $\mathcal{T}$.

## 1.1 Solved Problems

■ **Problem 1.1 — Finite Complement Topology.** Let $X$ be a set and let $\mathcal{T}_f$ be the collection of all subsets $U$ of $X$ such that $X - U$ is either finite ir is all of $X$. Show that $\mathcal{T}_f$ is a topology on $X$.

**Solution** First, observe that since $X$ is the whole space, for any subset $U \subset X$ we have $X - U = U^c$. So we can write $\mathcal{T}_f$ as

$$\mathcal{T}_f = \{U \subset X \mid U^c = X \text{ or } U^c \text{ is finite}\}.$$

We need to check if $\mathcal{T}_f$ has the properties of topology.

- It is immediate that $\varnothing \in \mathcal{T}_f$ and $X \in \mathcal{T}_f$.
- Let $\{A_\alpha\}_{\alpha \in I}$ be an arbitrary collection where $A_\alpha \in \mathcal{T}_f$ for all $\alpha \in I$. Let $C = \bigcup_\alpha A_\alpha$. For $C$ to be in $\mathcal{T}_f$, $C^c$ needs to be $X$ or finite.

$$C^c = \bigcap_\alpha A_\alpha^c.$$

Since all $A_\alpha$ are in $\mathcal{T}_f$, then $A_\alpha^c$ are $X$ or finite. Thus $C^c$ is also $X$ or finite, hence $C \in \mathcal{T}_f$

- Let $\{A_1, \cdots, A_n\}$ be finite collection of open sets. Consider $D = \bigcap_i A_i$. Since

$$D^c = \bigcup_i A_i^c,$$

and $A_i^c$ are finite or $X$, then $D^c$ is also finite or $X$, hence $D \in \mathcal{T}_f$.

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■ **Problem 1.2** Construct a set and give it a finite complement topology.

**Solution** Consider the set

$$X = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}.$$

The collection of sets $\mathcal{T} = \{A_n\}_{n \in \mathbb{N}}$, where

$$A_n = X - \{\frac{1}{1}, \frac{1}{2}, \cdots, \frac{1}{n}\},$$

is a finite complement topology.

# 2. Review of the Category of Groups

**Summary 2.1 — Random Notes.**

(a) In example 2.1.11 Löh, we discuss the notion of the isometry groups. One good example for such group is the group of unitary matrices. The reason is that these matrices preserve the inner product, and hence the norm induced by the inner product. Indeed $U^{-1} = U\dagger$, so

$$\|Ua\| = \langle Ua, Ua\rangle^{1/2} = \langle UU^{\dagger}a, a\rangle^{1/2} = \langle a, a\rangle = \|a\|.$$

(b) Dihedral group is the symmetry group (isometry group) of regular polygon. A bit of interpretation is needed to clarify we mean here. When we say, the symmetry group of a regular polygon, implicitly, we consider the regular polygon as a metric space. And one natural question is, what is the metric? We can let the metric be the one inherited by restricting the Euclidean metric to the polygon (assuming it is embedded in $\mathbb{R}^n$), or we can use other notions of metric, like the word metric (i.e. minimum number of edges between two nodes of interest).