

The background of the entire page is an abstract, fluid, and swirling pattern in shades of blue and white, set against a dark, almost black, background. The patterns resemble smoke, liquid, or perhaps mathematical curves, creating a sense of movement and depth. The colors transition from deep navy blue to lighter, ethereal whites and light blues, with some areas appearing more translucent than others.

## Lecture Notes For: Functional Analysis

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# 1. Point-set Topology

We will review some basic notions of the topology, and then we will present solved solutions for the related problems.

**Definition 1.1** Let  $(X, \mathcal{T})$  be a topological space and let  $A \subseteq X$  be a subset. Then

- The *interior* of  $A$  denoted by  $A^\circ$  is defined as

$$A^\circ = \bigcup_{\substack{V \subset A, \\ V \text{ open}}} V.$$

In words, the interior of a set is the union of all open sets contained in the set.

- The *closure* of  $A$  denoted by  $\overline{A}$  is defined as

$$\overline{A} = \bigcap_{\substack{F \supset A, \\ F \text{ closed}}} F.$$

In words, the closure of a set is the intersection of all closed sets containing  $A$ .

- The *boundary* of  $A$  is defined as

$$\partial A = \overline{A} \setminus A^\circ.$$

- $A$  is *dense* in  $X$  if

$$\overline{A} = X.$$

- $A$  is *nowhere dense* if

$$(\overline{A})^\circ = \emptyset.$$

■ **Remark** Consider the following remarks for the definition above.

- By the definition above, if  $x \in A^\circ$ , then there exists  $V \in \mathcal{T}$  such that  $x \in V \subset A$ . Also, we can interpret the interior of  $A$  as the largest open set contained in  $A$ .
- We can interpret  $\overline{A}$  as the smallest closed set containing  $A$ . There is a very interesting parallel between this definition and the notion of smallest  $\sigma$ -algebra containing a collection. The smallest  $\sigma$ -algebra containing a collection is the intersection of all  $\sigma$ -algebra that contains

that collection.

**Proposition 1.1 — Basic Properties.** Let  $(X, \mathcal{T})$  be a topological space, and  $A, F \subseteq X$  a subset. Then we have

- (a)  $A^\circ \subseteq A \subseteq \overline{A}$ .
- (b)  $A^\circ$  is open and  $\overline{F}$  is closed.
- (c)  $A$  is open iff  $A = A^\circ$ .
- (d)  $F$  is closed iff  $F = \overline{F}$ .
- (e)  $(\overline{A})^c = (A^c)^\circ$ .
- (f)  $(A^\circ)^c = \overline{(A^c)}$ .
- (g)  $A$  is open iff it is a neighborhood of all of its points.
- (h) If  $A_1 \subseteq A_2$  then  $A_1^\circ \subseteq A_2^\circ$  as well as  $\overline{A_1} \subseteq \overline{A_2}$ .
- (i)  $(A^\circ)^\circ = A^\circ$ , and  $\overline{(\overline{A})} = \overline{A}$ .
- (j)  $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$ .
- (k)  $(A_1 \cap A_2)^\circ = A_1^\circ \cap A_2^\circ$ .
- (l)  $\overline{A} = A \cup A'$ , where  $A'$  is the derived set of  $A$ .
- (m)  $A$  is closed iff  $A' \subset A$ . In words,  $A$  is closed iff it contains all of its accumulation points.

*Proof.* (a) Let  $x \in A^\circ$ . Then  $\exists V \in \mathcal{T}$  such that  $x \in V \subseteq A$ . Thus  $x \in A$ , so  $A^\circ \subseteq A$ . For the second part, Let  $x \in A$ . Then  $x \in F$  for every  $F$  that contains  $A$ . Consider the intersection of all such  $F$ s that are also closed.  $x$  also belongs to their intersection, which is by definition  $\overline{A}$ . So  $A \subseteq \overline{A}$ .

(b)  $A^\circ$  is open since it is the union of open sets.  $\overline{F}$  is closed since it is the intersection of closed sets.

(c) First, we assume  $A$  is open. Since  $A^\circ = \bigcup V$  for all  $V \subseteq A$  and  $V$  open, we can take the collections of open sets on the RHS to be only  $A$ , and it proves that  $A^\circ = A$ . For the other direction, we assume  $A = A^\circ$ . We know that  $A^\circ$  is open. Thus  $A$  is also open.

(d) First, we assume that  $F$  is closed. Then since  $\overline{F} = \bigcap A$  where  $F \subseteq A$  and  $A$  is closed, we can take the union on the RHS to be  $F$  and this proves that  $F = \overline{F}$ . For the converse, we assume  $F = \overline{F}$ . Since  $\overline{F}$  is open this implies that  $F$  is closed.

(e) Let  $x \in (\overline{A})^c$ . This implies  $x \in (\overline{A})^c = (\bigcap_{\substack{A \subseteq F, \\ F \text{ closed}}} F)^c = \bigcup_{\substack{A \subseteq F, \\ F \text{ closed}}} F^c$ . Let  $F^c = V$ . Then we can write

$$x \in \bigcup_{\substack{V \subseteq A^c \\ V \text{ open}}} V = (A^c)^\circ.$$

So  $(\overline{A})^c \subseteq (A^c)^\circ$ . For the converse, let  $x \in (A^c)^\circ$ . This implies  $x \in \bigcup_{\substack{V \subseteq A^c \\ V \text{ open}}} V$ . Or equiva-

lently  $x \notin \bigcap_{\substack{V \subseteq A^c, \\ V \text{ open}}} V^c$ . Let  $F = V^c$ . Then we can write

$$x \notin \bigcap_{\substack{A \subseteq F, \\ F \text{ closed}}} F = \overline{A}.$$

So  $x \in (\overline{A})^c$ . Thus we conclude that  $(\overline{A})^c = (A^c)^\circ$ .

(f) Let  $x \in (A^\circ)^c$ . Then

$$x \in \left( \bigcup_{\substack{V \subseteq A, \\ V \text{ open}}} V \right)^c = \bigcap_{\substack{V \subseteq A, \\ V \text{ open}}} V^c = \bigcap_{\substack{A^c \subseteq F, \\ F \text{ closed}}} F = \overline{A^c}.$$

This implies  $(A^\circ)^c \subseteq \overline{A^c}$ . For the converse let  $x \in \overline{A^c}$ . Then  $x \in \bigcap_{\substack{A^c \subseteq F, \\ F \text{ closed}}} F$ . This implies

$$x \notin \bigcup_{\substack{A^c \subseteq F, \\ F \text{ closed}}} F^c = \bigcup_{\substack{V \subseteq A, \\ V \text{ open}}} V = A^\circ.$$

This implies that  $x \in (A^\circ)^c$ . Thus  $\overline{A^c} \subseteq (A^\circ)^c$ .

- (g) We assume that  $A$  is open. Then for any  $x \in A$  we have  $x \in A \subseteq A$ . Thus  $A$  is a neighborhood of  $x$ . For the converse, we assume that  $A$  is a neighborhood of all of its points. So for any  $x \in A$  there exists  $V_x \in \mathcal{T}$  such that  $x \in V \subseteq A$ .  $A$  can be written as  $A = \bigcup_x V_x$  where  $V_x$  is as above. This  $A$  is open.
- (h) Let  $x \in A_1^\circ$ . Then  $\exists V \in \mathcal{T}$  such that  $x \in V \subseteq A_1$ . From assumption we also have  $x \in V \subseteq A_2$ . This implies that  $x \in A_2^\circ$ . For the second statement, let  $x \in \overline{A_1}$ .
- (i) to be added.
- (j) to be added.
- (k) to be added.
- (l)  $\Rightarrow$ . We want to show  $\overline{A} \subseteq A \cup A'$ . We will prove by contrapositive. I.e. we equivalently prove  $A^c \cap (A')^c \subseteq \overline{A}^c$ . Let  $x \in A^c \cap (A')^c$ . This implies that  $x \notin A$  as well as  $x \notin A'$ . So  $\exists U \in \mathcal{T}$  such that  $A \cap U = \emptyset$  (note that we both used  $x \notin A$  and  $x \notin A'$ ). Thus  $x \in U \subseteq A^c$ . This implies  $x \in (A^c)^\circ = \overline{A}^c$ .
- $\Leftarrow$ . We know that  $A \subseteq \overline{A}$ . So it suffices to show  $A' \subseteq \overline{A}$ . It is easier to prove the contrapositive, i.e.  $(\overline{A})^c \subseteq (A')^c$  or equivalently  $(A^c)^\circ \subseteq (A')^c$ . Let  $x \in (A^c)^\circ$ . This implies  $\exists U \in \mathcal{T}$  such that  $x \in U \subseteq A^c$ . So  $A \cap (U \setminus \{x\}) = \emptyset$ , thus  $x \notin A'$ , or equivalently  $x \in (A')^c$ .
- (m)  $\Rightarrow$ . Assume  $A$  is closed. Then  $A = \overline{A}$ . Using above we will have  $\overline{A} = A \cup A'$  it implies that  $A' \subseteq A$ .
- $\Leftarrow$ . Assume  $A' \subseteq A$ . Then from above  $\overline{A} = A \cup A'$  it implies that  $\overline{A} = A$ , hence  $A$  is closed.  $\square$

■ **Remark** In item (e), by taking the complement from both sides we will have

$$\overline{A} = ((A^c)^\circ)^c$$

## 1.1 Sporadic Notes

In this section I will include the notes that do not fit with the current layout of the document and will be added later when I start writing the corresponding sections.

**observation 1.1** The subspace topology is the weakest topology that makes the inclusion map continuous. Let  $(X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ . The topological space  $(A, \mathcal{T}_A)$  is a topological space and  $\mathcal{T}_A$  is called the subspace topology. Define the inclusion map

$$\iota : A \rightarrow X.$$

The subspace topology is the weakest topology for which  $\iota$  is continuous. Let  $U \in \mathcal{T}$ . Then  $\iota^{-1}(U) = A \cap U \in \mathcal{T}_A$ . **TODO:** I above I just showed that under the subspace topology, the inclusion map is continuous. However, I also need to show that  $\mathcal{T}_A$  is the smallest such topology.





## 2. Solution Manual Folland

### 2.1 Elements of Functional Analysis

■ **Problem 2.1** If  $X$  is a normed vector space over  $K$  ( $= \mathbb{R}$  or  $\mathbb{C}$ ), then show that addition and scalar multiplication are continuous from  $X \times X$  and  $K \times X \rightarrow X$ . Moreover, the norm is continuous from  $X$  to  $[0, \infty)$ . In fact  $|||x| - |y|| \leq \|x - y\|$ .