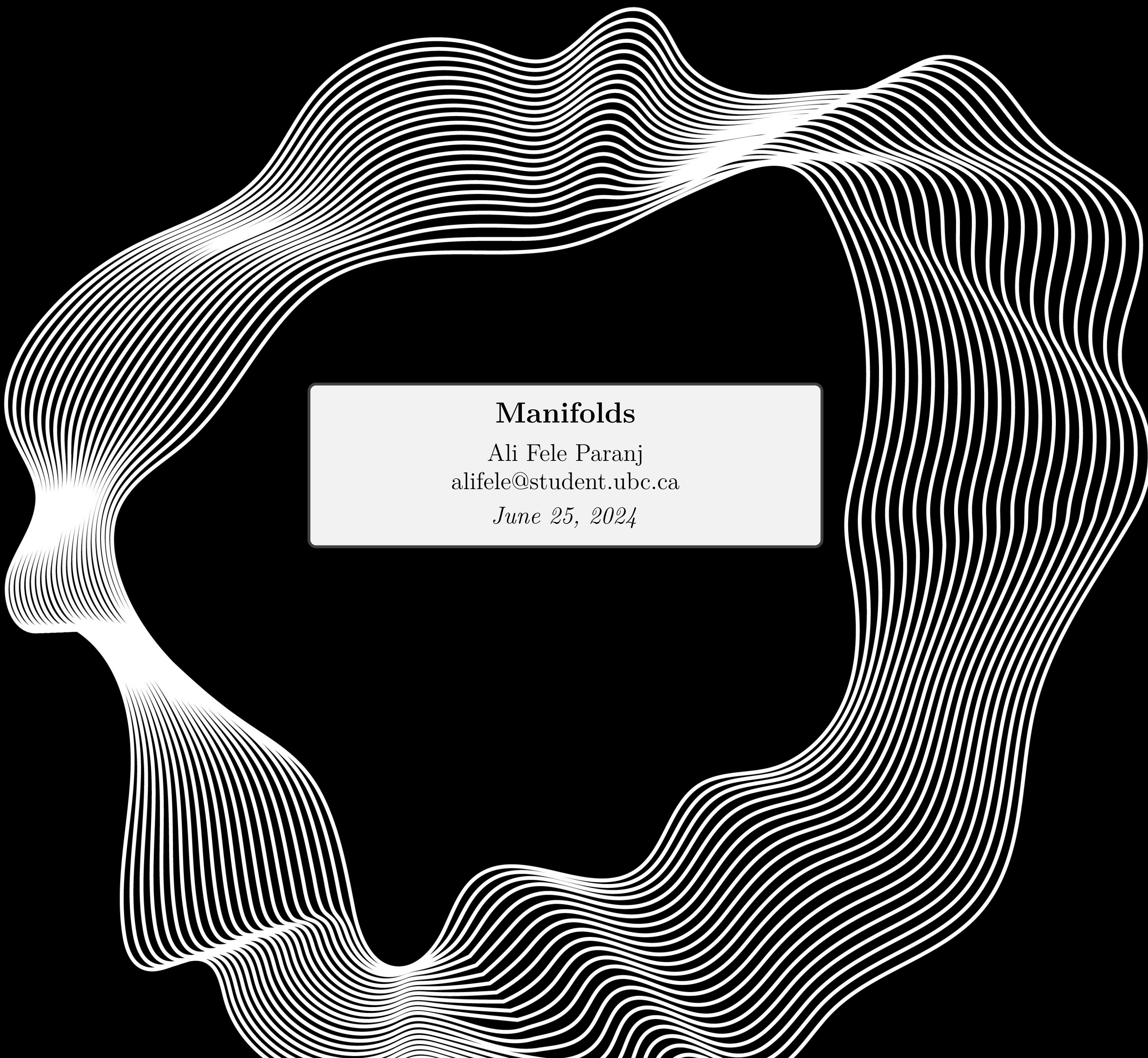


Manifolds

Ali Fele Paranj
alifele@student.ubc.ca

June 25, 2024

0. Contents

1 Euclidean Spaces **5**
1.1 Basic Notions and Definitions . . . 5
1.1.1 A Review on the algebraic structures . . . 5
1.1.2 Directional Derivative . . . 7
1.2 Tangent Spaces and Derivations . . . 8
1.3 Summary . . . 10
1.4 Solved Problems . . . 13

2 Manifolds **29**
2.1 Topological manifolds and Smooth manifolds . . . 29
2.2 Smooth maps on a manifold . . . 31
2.2.1 Smooth maps between manifolds . . . 32
2.3 Diffeomorphisms . . . 35
2.4 Partial Derivatives . . . 37
2.5 Inverse function theorem . . . 39
2.6 Quotients . . . 40
2.6.1 Continuity of a map on a Quotient . . . 41
2.7 Hausdorff and Second Countablity Conditions . . . 41
2.8 Real Projective Space . . . 43
2.9 Real Projective Plane . . . 44
2.10 Standard C^∞ Atlas on a Real Projective Space . . . 45
2.11 Summary . . . 45
2.12 Solved Problems . . . 49

3 Tangent Spaces **73**
3.1 Tangent Spaces . . . 73
3.1.1 Basis for the Tangent Space at a Point . . . 76
3.2 Solved Problems . . . 76

4 Appendix **77**
4.1 Change of Basis . . . 77

1. Euclidean Spaces

1.1 Basic Notions and Definitions

1.1.1 A Review on the algebraic structures

Here in this chapter I will be covering the details of some notions that was challenging for me do digest in the first read.

Definition 1.1 — Axioms of Group. Group is a set A along with a binary operation $* : A \times A \to A$ that satisfies the following properties. Let $a, b, c \in A$, then

- **Associativity**: $a * (b * c) = (a * b) * c$.
- **Identity element**: $\exists 1 \in A$ such that
$$1 * a = a * 1 = a.$$
- **Inverse element**: $\forall a \in A$ $\exists \hat{a} \in A$ such that
$$a * \hat{a} = \hat{a} * a = 1.$$

■ **Remark 1.1** A set along with a binary operation that does not satisfy any properties is called a **magma**. If the binary operation is only associative, then we are dealing with **semi-group**. If the binary operation has an identity element as well, then we call this algebraic structure as **monoid**.

Definition 1.2 — Axioms of Ring. A ring is a set R along with two operations $+ : R \times R \to R$ and $* : R \times R \to R$, where

- $(R, +)$ is an Abelian group.
- $(R, *)$ is a monoid.
- The operator $(*)$ has distributive (left and right) law over $(+)$ i.e.
$$a * (b + c) = (a * b) + (a * c), \qquad (b + c) * a = (b * a) + (c * a).$$
.

■ **Remark 1.2** **Field** is a ring where every non-zero element (i.e. inverse element in the $(R, +)$ group in the ring) has a multiplicative inverse.

Definition 1.3 — Axioms of Module. A **module** is a group M along with a ring R where the monoid of the ring acts on M (through scalar multiplication) (i.e. it satisfies the idenity and compatibility properties) and satisfies the distributive property. I.e.

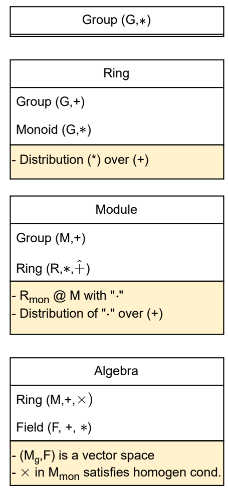
- **Compatibility of the monoid action**: $a, b \in R,\ u \in M$ then
$$a(bu) = (ab)u.$$
- **Identity of the monoid action**: Let 1 be the identity element of the ring R. Then $\forall u \in M$
$$1u = u1 = u.$$
- **Distribution law**: $a, b \in R$ and $u, v \in M$ then
  - $(a + b)u = au + bu$.
  - $a(u + v) = au + av$.

■ **Remark 1.3** A module (M, R) is called a **vector space**, if the **ring** R is a **field**.

Definition 1.4 — Axioms of Algebra. An Algebra over field F is a ring A that F acts on it (thus A has vector space structure as well), where the monoid operation of F (i.e. multiplication) satisfies the homogeneity property. I.e. for $r \in F$ and $u, v \in A$ we have
$$r(uv) = (ru)v = u(rv).$$

There are some important observations when combining different algebraic structures with each other to get a new one. The first is that when we combine two structures with different operators, then the operators need to satisfy the distributive laws. Also, note that when an algebraic structure (like group or monoid) acts on another algebraic structure, we need to have the identity and and compatibility conditions satisfied.

The following diagram shows how different algebraic structures are combined with each other to produce another structure.



Note that in the figure above, I have used some non-standard notations to make the figure concise. For instance, the expression "$R_{\mathbf{mon}}@M$ **with** $\cdot$" means that the monoid structure in the field R acts on the group M with the $(\cdot)$ symbol. Or the expression "$\times$ **in** $M_{\mathbf{mon}}$ **satisfies homogen cond.**" means the multiplication operation of the monoid structure inside the ring M satisfies the homogeneity condition (see the definition of the algebra in Definition 1.4). Finally, M_g means the group structure inside the ring M.

1.1.2 Directional Derivative

The notion of directional derivative is very central in generalization of the multi-variable calculus manifolds.

Definition 1.5 — Directional derivative. Let $f : U \to \mathbb{R}$ where $U \subset \mathbb{R}^n$ and $f \in C^\infty(U)$. Then we define a directional derivative at $p \in U$ and in the direction $v \in T_p(\mathbb{R}^n)$ as

$$D_v f\Big|_p = \lim_{t\to 0} \frac{f(p+tv) - f(p)}{t} = \frac{d}{dt}\Big|_{t=0} f(p+vt).$$

We denote the set of all directional derivatives at p by $\mathcal{D}(C_p^\infty(U))$.

■ **Remark 1.4** By the chain rule we have

$$D_v f\Big|_p = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(p).$$

Proposition 1.1 — Directional derivative is $R-$linear map satisfying the Leibniz rule. A directional derivative D_v at point p is a $\mathbb{R}-$linear operator that maps

$$f \in C_p^\infty \mapsto D_v f \in \mathbb{R}$$

such that satisfies the Leibniz rule,

$$D_v(fg) = D_v(f)g + f D_v(g).$$

Proof. Let $D_v \in \mathcal{D}(C_p^\infty(U))$. Then by the remark above we can write it as

$$D_v f\Big|_p = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(p).$$

Since each of the partial derivatives above are $\mathbb{R}-$linear and satisfy the Leibniz rule, then D_v inherits those properties as well. □

1.2 Tangent Spaces and Derivations

Definition 1.6 — Informal definition of tangent space. Let $p \in \mathbb{R}^n$. A tangent space at p denoted by $T_p(\mathbb{R}^n)$ is a linear space containing all of vectors emerging from p.

■ **Remark 1.5** Note that the definition above is an informal definition of the tangent space. For a more formal and technical definition, we can use the notion of curves, or the notion of manifolds. We wont' touch this level of technicality in the early chapters.

To distinguish between points in $\mathbb{R}^n$ and vectors in $T_p(\mathbb{R}^n)$ we denote a point $p \in \mathbb{R}^n$ by

$$p = (p^1, p^2, \cdots, p^n),$$

while for a vector $v \in T_p(\mathbb{R}^n)$ we write

$$v = \langle v_1, v_2, \cdots, v_n \rangle.$$

As we observed in Definition 1.5, we have a natural one-to-one correspondence between each $v \in T_p(\mathbb{R}^n)$ and a $D_v \in \mathcal{D}(C_p^\infty(U))$, i.e. these linear spaces are isomorphic.

Proposition 1.2 Let $p \in \mathbb{R}^n$. The set of all directional derivatives at p, i.e. $\mathcal{D}(C_p^\infty(U))$ is isomorphic to the tangent space at p i.e. $T_p(\mathbb{R}^n)$.

Proof. Proof follows immediately from the following natural association in the definition of the directional derivative.

$$D_v f\Big|_p \longleftrightarrow \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(p)$$

□

Definition 1.7 — Point derivations at a point. Let $p \in \mathbb{R}^n$, and U an open set containing p. A **derivation at** p or a **point-derivation of** $C_p^\infty(U)$ is a *linear* operator

$$D : C_p^\infty(U) \to \mathbb{R}$$

such that satisfies the Leibniz property, i.e. for $f, g \in C_p^\infty(U)$ we have

$$D(fg) = D(f)g - fD(g).$$

We denote the set of all such maps as $\mathbb{D}(C_p^\infty(U))$.

■ **Remark 1.6** We know that a directional derivative at p satisfies the Leibniz rule. Thus $\mathcal{D}(C_p^\infty(U)) \subset \mathbb{D}(C_p^\infty(U))$. On the other hand, we know that both $\mathcal{D}$ and $\mathbb{D}$ are linear spaces. So $\mathcal{D}(C_p^\infty(U))$ is in fact a *linear subspace* of $\mathbb{D}(C_p^\infty(U))$. Thus the zero of these two linear spaces match.

The following Lemma follows form the algebraic property of the point derivation.

Lemma 1.1 Let D be a point derivation of $C_p^\infty(U)$. Then $D(c) = 0$ for any constant function c.

Proof. Let c be a constant function, i.e. a real number. Since D is $R-$linear, then we can write $D(c) = cD(1)$. On the other hand, from the Leibniz property we can write

$$D(1) = D(1 \cdot 1) = D(1) + D(1) = 2D(1)$$

Thus $D(1) = 0$, which implies $D(c) = 0$. □

The following theorem is very important as it states that all of the point derivations are in fact the directional derivatives and vise-versa. This is a very interesting result, since we are in fact stating that an operator is a point derivative if and only if it satisfies an algebraic property. This means that we can abstract away all of the detailed limit processes in the definition of derivative, and replace that with an axiomatic requirement which is a purely algebraic property. We can see these kind of ideas, i.e. axiomatic generalization all over the mathematics. For instance, in the PDE theory, at some point we need to relax the definition of derivative and talk about the weak derivatives. To do so, we get an identity that derivatives satisfy and carefully use that identity to define the notion of weak derivatives.

Theorem 1.1 — The set of all point-derivations is isomorphic to the set of all directional derivatives. Let $p \in U \subset \mathbb{R}^n$. Then $\mathbb{D}(C_p^\infty(U))$ is isomorphic to $T_p(\mathbb{R}^n)$.

Proof. Let $\varphi : T_p(\mathbb{R}^n) \to \mathbb{D}(C_p^\infty(U))$ be a linear isomorphism between the linear spaces. We need to show that this map is surjective and bijective. To show the bijectivity, we use the fact that a linear map is bijective if and only if its kernel is a singleton. To find the kernel of the map, let need to find all points in $T_p(\mathbb{R}^n)$ that maps to the zero of $\mathbb{D}$. As we discussed in the remark of Definition 1.7, the zero $\mathbb{D}$ is the same as the zero directional derivative, i.e. $D_v = 0$. We need to prove that v is the zero vector, i.e. the zero of $T_p(\mathbb{R}^n)$. To do this, we apply the D_v to the coordinate functions

$$0 = D_v(x^i) = \sum_{j=1}^n v^j \frac{\partial x^i}{\partial x^j} = v^j$$

Thus $v = 0 \in T_p(\mathbb{R}^n)$, thus φ is injective. In other words, the injectivity follows immediately from $T_p(\mathbb{R}^n)$ being isomorphic to $\mathcal{D}$, and $\mathcal{D}$ being a linear subspace of $\mathbb{D}$.

To prove the surjectivity, let D be a point derivation at p, and let (f, V) be a representative of a germ in C_p^∞. Marking V smaller if necessary, we may assume that V is an open ball, hence star shaped. By Taylor's approximation theorem we know there exists C^∞ functions $g_i(x)$ such that

$$f(x) = f(p) + \sum_{i=1}^n (x^i - p^i) g_i(x), \qquad g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

From Lemma Lemma 1.1, we know that $D(f(p)) = 0$ as well as $D(p^i) = 0$. Thus we can write

$$D(f(x)) = \sum_{i=1}^n (D(x^i)g_i(x) + (p^i - p^i)D(g_i(x))) = \sum_{i=1}^n D(x^i)g_i(x) = \sum_{i=1}^n D(x^i)\frac{\partial f}{\partial x^i}(p).$$

This is in fact a directional derivative in the direction $v = \langle D(x^1), D(x^2), \cdots, D(x^n)\rangle$. So for every D in $\mathbb{D}$ we can find a vector in $T_p(\mathbb{R}^n)$. Thus φ is surjective. □

Observation 1.2.1 Let D be a point derivation of $C_p^\infty(U)$. This corresponds to the directional derivative at p corresponding to the vector

$$v = \langle D(x^1), D(x^2), \cdots, D(x^n)\rangle.$$

Observation 1.2.2 Let $p \in U \subset \mathbb{R}^n$. Then

$$T_p(\mathbb{R}^n) \equiv \mathcal{D}(C_p^\infty(U)) \equiv \mathbb{D}(C_p^\infty(U)),$$

i.e. they are all isomorphic linear spaces.

Because of the observation above, we identify the standard basis $\{e^1, e^2, \cdots, e^n\}$ with the partial derivatives $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^n}\}$. Thus we can write a vector $v \in T_p(\mathbb{R}^n)$ as

$$v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}.$$

1.3 Summary

Summary 1.1 — Don't confuse the naming conventions. Note the following different names that might be confusing in some cases!

- **k-linear function** $\equiv$ **k-tensor** $\equiv$ **multi-linear function**: All of these names are for functions $f : V^k \to \mathbb{R}$. We denote the set of all k-tensors defined on vector space V as $L_k(V)$. If $\alpha^1 \cdots \alpha^n$ are the dual basis for V, then a basis for $L_k(V)$ is

$$\mathcal{B}(L_k(V)) = \{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}_{\text{for all multi-indices } (i_1, \cdots, i_k)}$$

  Also we have

$$\dim L_k(V) = n^k$$

- **alternating k-tensors** $\equiv$ **k-covectors** $\equiv$ **multi-covectors of degree k**: All of these names are for function k-linear functions that are **alternating**. We denote the set of all k-covectors on V as $A_k(B)$. A basis for this set is

$$\mathcal{B}(A_k(V)) = \{\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}\}_{\text{for all multi-indices } (i_1, \cdots, i_k) \text{ in ascending order}}$$

  Also we have

$$\dim A_l(v) = \binom{n}{k}.$$

Summary 1.2 — Determining sign of permutation. Let $\sigma \in S_k$. There we can find the sign of the permutation by decomposing it into cycles, or counting its number of inversions. For instance, consider $\sigma \in S_9$ where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 9 & 3 & 7 & 8 & 5 & 6 & 1 \end{pmatrix}$$

(i) **Determining sign by cycle decomposition**. For σ we can write it as

$$\sigma = (1\ 2\ 4\ 3\ 9)(5\ 7)(6\ 8) = (1\ 9)(1\ 3)(1\ 4)(1\ 2)(5\ 7)(6\ 8).$$

Since σ has 6 transpositions in it, then its sign is **even**.

(ii) **Determining sign by counting the number of inversions**. For σ the inversions are

$$(2,1),(3,1),(4,3),(4,1),(5,1),(6,1),(7,5),(7,6),(7,1),$$
$$(8,5),(8,6),(8,1),(9,1),(9,3),(9,5),(9,6),(9,7),(9,8).$$

Since σ has 18 inversions, then it is sign is **even**.

Summary 1.3 Let $f \in A_k(V)$ and $g \in A_l(V)$. Then

$$f \wedge g = (-1)^{kl}(g \wedge f).$$

In particular, if f is odd-linear function, then

$$f \wedge f = 0.$$

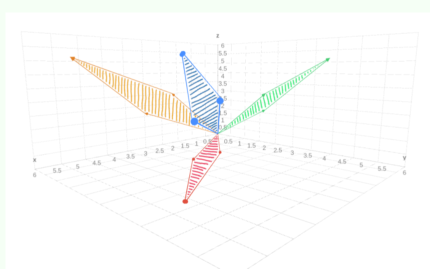
Summary 1.4 — Some intuition about the alternating functions. In this summary box, we will go through a concrete example to make some intuition about the alternating functions and their basis. Let $V = \mathbb{R}^3$, with standard basis $\{e_1, e_2, e_3\}$ and the dual basis $\{\alpha^1, \alpha^2, \alpha^3\}$. Let $f \in A_2(V)$. Then we know that we can decompose f in term of its basis

$$f = g_1\alpha^1 \wedge \alpha^2 + g_2\alpha^1 \wedge \alpha^3 + g_3\alpha^2 \wedge \alpha^3.$$

The key step in making the intuition is to observe that for any l 1-covectors $\omega^1, \cdots, \omega^l$ we have $(\omega^1 \wedge \cdots \wedge \omega^l)(v_1, \cdots, v_l) = \det\left[\omega^i(v_j)\right]$. Thus for the first term in the decomposition of $f(u,v)$ for $u, v \in \mathbb{R}^3$ we have

$$(\alpha^1 \wedge \alpha^2)(u,v) = \det\begin{pmatrix} \alpha^1(u) & \alpha^1(v) \\ \alpha^2(u) & \alpha^2(v) \end{pmatrix}.$$

This is the area covered by two vectors that we can get by projecting u, v onto the $x-y$ plane. With a similar argument, $\alpha^1 \wedge \alpha^3$ gives the area between the vectors when projected to the $x-z$ plane, and etc. The following figure summarizes these in a neat way.



In the figure above, we will get the red, green and orange volumes by applying $\alpha^1 \wedge \alpha^2$, $\alpha^2 \wedge \alpha^3$, $\alpha^1 \wedge \alpha^3$ on (u,v) respectively.

Summary 1.5 Consider the following $n \times n$ matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

The, determinant is defined to be

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\ a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}$$

Summary 1.6 Let V be a vector space, and $\alpha^1, \cdots, \alpha^k$ be 1-linear functions (i.e. 1-covectors) defined on V. Then we have

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \cdots, v_k) = \det \left[\alpha^i(v_j)\right].$$

In particular, For two 1-covectors we have

$$(\alpha^1 \wedge \alpha^2)(v_1, v_2) = \det \begin{pmatrix} \alpha^1(v_1) & \alpha^1(v_2) \\ \alpha^2(v_1) & \alpha^2(v_2) \end{pmatrix}$$

See Problem Problem 1.10 for the a detailed proof.

Summary 1.7 Let V be a vector space with dimension n, with $\{e_1, \cdots, e_n\}$ as its basis and $\{\alpha^1, \alpha^2, \cdots, \alpha^n\}$ as its dual basis. Let e$A_k(V)$ be the set of all *alternating* k-linear functions defined on V. Then we have

$$\dim A_k = \binom{n}{k}.$$

In particular, let $V = \mathbb{R}^n$ with standard basis $\{e_1, e_2, e_3\}$, and the corresponding dual basis $\{\alpha^1, \alpha^2, \alpha^3\}$. Then we have

$$\begin{aligned} &\mathbb{B}(A_0) = \{1 \in \mathbb{R}\} \\ &\mathbb{B}(A_1) = \{\alpha^1, \alpha^2, \alpha^3\}, \\ &\mathbb{B}(A_2) = \{\alpha^1 \wedge \alpha^2, \alpha^1 \wedge \alpha^3, \alpha^2 \wedge \alpha^3\}, \\ &\mathbb{B}(A_3) = \{\alpha^1 \wedge \alpha^2 \wedge \alpha^3\}. \end{aligned}$$

It is clear from the basis sets above that the dimensions of A_0, A_1, A_2 and A_3 are $1, 3, 3, 1$ respectively.

Also, observe that $A_k = 0$ for all $k > n$. This follows from the fact that $f \wedge f = 0$ for f a 1-covector.

Summary 1.8 Let V be n dimensional vector space with a basis $\{e_1, \cdots, e_n\}$, and $\{\alpha^1, \cdots, \alpha^n\}$ as its dual basis. Let $L_k(V)$ be the set of all k-linear functions defined on V. Then a basis for $L_k(V)$ is

$$\mathcal{B} = \{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$$

for all multi-indices $(i_1, \cdots, i_k)$. This shows that

$$\dim L_k(V) = n^k.$$

For instance, Let $V = \mathbb{R}^2$. Then for $L_1(V)$ we have

$$\mathcal{B}(L_1(V)) = \{\alpha^1, \alpha^2, \alpha^3\}.$$

Similarly, for $L_2(V)$ we have

$$\mathcal{B}(L_2(V)) = \{\alpha^1 \otimes \alpha^2, \alpha^1 \otimes \alpha^3, \alpha^2 \otimes \alpha^3, \alpha^2 \otimes \alpha^1, \alpha^3 \otimes \alpha^1, \alpha^3 \otimes \alpha^2, \alpha^1 \otimes \alpha^1, \alpha^2 \otimes \alpha^2, \alpha^3 \otimes \alpha^3\}$$

For a side by side comparison, compare it with the basis for $A_2(V)$ that is

$$\mathcal{B}(A_2(V)) = \{\alpha^1 \wedge \alpha^2, \alpha^2 \wedge \alpha^3, \alpha^1 \wedge \alpha^3\}.$$

Summary 1.9 Looking at the dimension $L_1(V), L_2(V), \cdots$, it reveals the fact that why most of the non-technical people are interested in interpreting tensors as multi-dimensional matrices! For instance, when $V = \mathbb{R}^2$ and $K = 3$, then $L_3(V)$ has dimension $2^3 = 8$ with the basis

$$\mathcal{B}(\mathbb{R}^2) = \begin{array}{l} \{\alpha^1 \otimes \alpha^1 \otimes \alpha^1, \alpha^1 \otimes \alpha^1 \otimes \alpha^2, \alpha^1 \otimes \alpha^2 \otimes \alpha^1, \alpha^1 \otimes \alpha^2 \otimes \alpha^2, \\ \alpha^2 \otimes \alpha^1 \otimes \alpha^1, \alpha^2 \otimes \alpha^1 \otimes \alpha^2, \alpha^2 \otimes \alpha^2 \otimes \alpha^1, \alpha^2 \otimes \alpha^2 \otimes \alpha^2\} \end{array}$$

It is very tempting to think of elements of $L_3(V)$ as $2 \times 2 \times 2$ matrix like structures. In the case of $L_2(V)$ we can identify $L_2(V)$ with the set of all 2×2 matrices.

Summary 1.10 — Determinant of a matrix is zero if its column vectors are linearly dependent. A k-tensor f is alternating if and only if $f(v_1, \cdots, v_k) = 0$ whenever two of the vectors $v_1, \cdots, v_k$ are equal. With a simple generalization, we can state a more general statement that f is alternating k-tensor if and only if $f(v_1, \cdots, v_k) = 0$ the vectors $v_1, \cdots, v_k$ are linearly dependent.

As an important (and simple!) Corollary for this statement, we can conclude that determinant of an $n \times n$ matrix (which is an alternating n-tensor) is zero when its column vectors are linearly dependent.

Summary 1.11 Let $U \subset \mathbb{R}^n$ open, and let $\{x_1, \cdots, x_n\}$ be standard basis for $\mathbb{R}^n$, and $\{x^1, \cdots, x^n\}$ the standard dual basis. Let $p \in U$. Then for $T_p(\mathbb{R}^n)$ and $T_p^*(\mathbb{R}^n)$ we have

$$\mathcal{B}(T_p(\mathbb{R}^n)) = \{\frac{\partial}{\partial x_1}\Big|_p, \cdots, \frac{\partial}{\partial x_n}\Big|_p\}, \qquad \mathcal{B}(T_p^*(\mathbb{R}^n)) = \{dx^1\Big|_p, \cdots, dx^n\Big|_p\}.$$

1.4 Solved Problems

■ **Problem 1.1 — Algebra structure on C_p^∞.** Define carefully addition, multiplication, and scalar multiplication in C_p^∞. Prove that addition in C_p^∞ is commutative.

Solution First, note that the elements of C_p^∞ are actually the equivalence classes, where two functions are equivalent if they both define the same germ.

- For the definition of the addition, we can use the point-wise addition of the functions. However, we need to check to see if this definition is well-defined (i.e. the result of the addition of two functions does not depend on the choice of representative of the equivalence class). Let $f_1, f_2, g_1, g_2 \in C_p^\infty$ where f_1 and f_2 define the same germ, and similarly for g_1 and g_2. Then, we claim that $f_1 + g_1$ define the same germ as $f_2 + g_2$. That is because for f_1, f_2 there is an open set U_1 containing p where $f_1(x) = f_2(x)\ \forall x \in U$. Similarly, there is an open set U_2 that contains p and for all $x \in U_2$ we have $g_1(x) = g_2(x)$. Let $W = U_1 \cap U_2$. Then on for all $x \in W$ we have $f_1(x) + f_2(x) = g_1(x) + g_2(x)$. Hence $f_1 + g_1$ defines the same germ as $f_2 + g_2$.

- For the scalar multiplication, we can use the notion of scalar multiplication in functions, and following an idea similar to the reasoning above, we can show that this definition is well-defined.

- For the multiplication on C_p^∞ we can use of the point-wise multiplication of functions as the definition, and with a similar reasoning to the one in item 1, we can show that this definition is well-defined.

For the commutativity of the addition on C_p^∞, we need to emphasis that it follows immediately from the commutativity of the point-wise addition of functions.

■ **Problem 1.2 — Vector space structure on derivations at a point.** Prove that the set of all point derivatives is closed under addition and scalar multiplication.

Solution Let D and D' be derivations at $p \in \mathbb{R}^n$, and define $\hat{D} = D + D'$. Let $f, g \in C_p^\infty$. Then we can write

$$\hat{D}(fg) = (D + D')(fg)$$

On the other hand we have

$$D(fg) = D(f)g + fD(g), \qquad D'(fg) = D'(f)g + fD'(g).$$

Adding two equations we will get

$$D(fg) + D'(fg) = (D(f) + D'(f))g + f(D(g) + D'(g))$$

Defining $\hat{D} = (D + D')(f) = D(f) + D'(f)$ we will get

$$\hat{D}(fg) = \hat{D}(f)g + f\hat{D}(g).$$

which shows that $\hat{D}$ is also a point derivation at p. For the scalar multiplication, let $r \in \mathbb{R}$ and define $\tilde{D} = rD$. By defining $(rD)(f) = rD(f)$, we can write

$$(rD)(fg) = rD(fg) = rD(f)g + rfD(g) = rD(f)g + frD(g),$$

which shows that rD also satisfies the Leibniz property, this it is a point derivation.

■ **Problem 1.3** Let A be an algebra over a field K. If D_1 and D_2 are derivations of A, show that $D_1 \circ D_2$ is not necessarily a derivation (it is if D_1 or $D_2 = 0$), but $D_1 \circ D_2 - D_2 \circ D_1$ is always a derivation of A.

Solution TO BE ADDED.

■ **Problem 1.4** Find the inversions in the permutation $\tau = (1\ 2\ 3\ 4\ 5)$.

Solution This permutation can be written as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

Thus the number of inversions can be written as $(2,1), (3,1), (4,1)$, and $(5,1)$.

■ **Problem 1.5** Let $f : V^k \to \mathbb{R}$ be a k-linear function defined on vector space V. Show that the following functions is alternating.

$$Af = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\ \sigma f.$$

Solution Let $\tau \in S_k$. Then

$$\tau(Af) = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\ (\tau\sigma)f = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \operatorname{sign}(\tau) \operatorname{sign}(\tau)\ (\tau\sigma)f = \operatorname{sign}(\tau) \sum_{\sigma \in S_k} \operatorname{sign}(\tau\sigma)\ (\tau\sigma)f$$

Note that since in the sum above σ runs through all permutations S_k, so does $\tau\sigma$. Thus we can write

$$\tau(Af) = \operatorname{sign}(\tau)(Af).$$

This shows that Af is alternating.

■ **Problem 1.6** Let $f : V^k \to \mathbb{R}$ be a k-linear function. Show that Sf given below is symmetric.

$$Sf = \sum_{\sigma \in S_k} \sigma f.$$

Solution Let $\tau \in S_k$. Then we

$$\tau(Sf) = \sum_{\sigma \in S_k} \tau\sigma f = Sf.$$

Note that the last equality above holds, since σ rums through all permutations S_k and so does $\tau\sigma$.

■ **Problem 1.7** If f is a 3-linear function on a vector space V and $v_1, v_2, v_3 \in V$, what is $(Af)(v_1, v_2, v_3)$?

Solution

$$(Af)(v_1,v_2,v_3) = f(v_1,v_2,v_3) - f(v_2,v_1,v_3) + f(v_3,v_1,v_2) - f(v_1,v_3,v_2) + f(v_2,v_3,v_1) - f(v_3,v_2,v_1)$$

■ **Problem 1.8** Show that the tensor product of multi-linear functions is associative: If f, g, and h are multi-linear functions on V, then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h).$$

Solution We start with the left hand side. I.e.

$$\begin{aligned}
((f \otimes g) \otimes h)(v_1 \cdots, v_{k+l+m}) &= (f \otimes g)(v_1, \cdots, v_{k+l}) h(k_{k+l+1}, \cdots, k_{k+l+m}) \\
&= \Big(f(v_1, \cdots, v_k) g(v_{k+1}, \cdots, v_{k+l})\Big) h(v_{k+l+1}, \cdots, v_{k+l+m}) \\
&= f(v_1, \cdots, v_k) \Big(g(v_{k+1}, \cdots, v_{k+l}) h(v_{k+l+1}, \cdots, v_{k+l+m})\Big) \\
&= (f \otimes (g \otimes h))(v_1, \cdots, v_{k+l+m}).
\end{aligned}$$

■ **Problem 1.9** Consider following two ways that we can express the sum in the wedge product formula. Let $f \in A_k(V)$ and $g \in A_l(V)$. Then the wedge product $f \wedge g$ can be written as

$$(f \wedge g)(v_1, \cdots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_k} \text{sign}(\sigma)\ \sigma f(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)}).$$

Or, alternatively, we can write it as $(k, l)-$shuffle

$$(f \wedge g)(v_1, \cdots, v_{k+l}) = \sum_{(k,l)-\text{shuffles } \sigma} \text{sign}(\sigma)\ \sigma f(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)}).$$

Now, as a concrete example, let $f, g \in A_2(V)$. Write $f \wedge g$ in these two forms.

Solution First, we want to use the sum on the all permutations.

$$\begin{aligned} 4(f \wedge g)(v_1, v_2, v_3, v_4) = {} & f(v_1, v_2)g(v_3, v_4) - f(v_2, v_1)g(v_3, v_4) + f(v_2, v_1)g(v_4, v_3) - f(v_1, v_2)g(v_4, v_3) \\ & -f(v_1, v_3)g(v_2, v_4) + f(v_3, v_1)g(v_2, v_4) - f(v_3, v_1)g(v_4, v_2) + f(v_1, v_3)g(v_4, v_2) \\ & +f(v_1, v_4)g(v_2, v_3) - f(v_4, v_1)g(v_2, v_3) + f(v_4, v_1)g(v_3, v_2) - f(v_1, v_4)g(v_3, v_2) \\ & +f(v_2, v_3)g(v_1, v_4) - f(v_3, v_2)g(v_1, v_4) + f(v_3, v_2)g(v_4, v_1) - f(v_2, v_3)g(v_4, v_1) \\ & -f(v_2, v_4)g(v_1, v_3) + f(v_4, v_2)g(v_1, v_3) - f(v_4, v_2)g(v_3, v_1) + f(v_2, v_4)g(v_3, v_1) \\ & +f(v_3, v_4)g(v_1, v_2) - f(v_4, v_3)g(v_1, v_2) + f(v_4, v_3)g(v_2, v_1) - f(v_3, v_4)g(v_2, v_1). \end{aligned}$$

Note that in every row, the functions are equal to each at (because f, g are alternating). Thus we have the factor of 4 not to count the redundant terms. However, we can do this sum using the (2,2)-shuffles. This means that we only keep the very first column, as their argument permutation is the same as all (2,2)-shuffles on 4 symbols. Thus we can write

$$\begin{aligned} (f \wedge g)(v_1, v_2, v_3, v_4) = {} & f(v_1, v_2)g(v_3, v_4) - f(v_1, v_3)g(v_2, v_4) + f(v_1, v_4)g(v_2, v_3) \\ & +f(v_2, v_3)g(v_1, v_4) - f(v_2, v_4)g(v_1, v_3) \\ & +f(v_3, v_4)g(v_1, v_2). \end{aligned}$$

Observation 1.4.1 How do we count the (2,2)-shuffles of $\{v_1, v_2, v_3, v_4\}$? We start by a vertical line where in its left side we put v_1, v_2 and in its right side we write v_3, v_4. Like the following table. This is already a shuffle (identity). To write the next shuffle, we keep v_1 in the first position, and write the next symbol whose its subscript is larger than 2 (i.e. v_3), and write the remaining symbols in the right hand side of the vertical line in increasing order. Then we continue this process, until there are no possible shuffles for its first element be v_1. Then we put v_2 in the first place and right next to it we write the next symbol that its subscript is larger than 2 (i.e. v_3), and we continue.

$$\begin{array}{r|l} v_1, v_2 & v_3, v_4 \\ v_1, v_3 & v_2, v_4 \\ v_1, v_4 & v_2, v_3 \\ & \\ v_2, v_3 & v_1, v_4 \\ v_2, v_4 & v_1, v_3 \\ & \\ v_3, v_4 & v_1, v_2 \end{array}$$

In order to find the sign of the each of these permutations, it is enough to extract the independent cycles. For instance, consider the following shuffle, corresponding to last row above.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

This permutation is the same as the permutation given by the cycle $\tau = (1\ 3)(2\ 4)$. To extract this we start with 1 and track where it goes, and we do this tracking until we get back to 1. Then we start this process with the remaining element until we get all of the cycles. This cycle decomposition of τ clearly shows that τ is an even permutation (can be written as 2 cycles). However, in the case of v_2, v_4, v_1, v_3, the corresponding permutation is

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}.$$

We can write this as $\tau = (1\ 2\ 4\ 3)$. As with every cycle, we can decompose this into smaller cycles (not necessarily independent) as $\tau = (1\ 3)(1\ 4)(1\ 2)$. This shows that this particular permutation is odd.

■ **Problem 1.10 — Wedge product of covectors and determinant.** Let V be a vector space and let $\beta^1, \cdots \beta^l \in A_1(V)$. Prove that

$$(\beta^1 \wedge \cdots \wedge \beta^l)(v_1, \cdots, v_l) = \det\left[\beta^i(v_j)\right]$$

Solution This follows simply from the definition. From the definition of wedge product we have

$$(\beta^1 \wedge \cdots \wedge \beta^l) = A\left[\beta^1(v_1)\cdots\beta^l(v_l)\right] = \sum_{\sigma\in S_l} \operatorname{sign}(\sigma)\beta^1(v_{\sigma(1)})\cdots\beta^l(v_{\sigma(l)}) = \det\left[\beta^i(v_j)\right]$$

where $\left[\beta^i(v_j)\right]$ is $l \times l$ square matrix.

■ **Problem 1.11** Verify

$$f \wedge g = (-1)^{kl}(g \wedge f)$$

for f and g being k and l linear maps correspondingly. Do this verification by considering $f \in A_2(V)$ and $g \in A_1(V)$.

Solution For $f \wedge g$ we can write (using $(2,1)$-shuffles)

$$(f \wedge g)(v_1, v_2, v_3) = f(v_1, v_2)g(v_3) - f(v_1, v_3)g(v_2) + f(v_2, v_3)g(v_1).$$

However, for $g \wedge f$, we can use $(1,2)$-shuffles to show

$$(g \wedge f)(v_1, v_2, v3) = g(v_1)f(v_2, v_3) - g(v_2)f(v_1, v_3) + g(v_3)f(v_1, v_2).$$

Evaluating two expressions above reveals that $f \wedge g = g \wedge f$.

■ **Problem 1.12 — Tensor product of covectors (from W. Tu).** Let $e_1, e_2, \cdots, e_n$ be a basis for vector space V and let $\alpha^1, \cdots, \alpha^n$ be its dual basis in $V^\vee$. Suppose $\left[g_{i,j}\right] \in \mathbb{R}^{n\times n}$ is an $n \times n$ matrix. Define a bilinear function $f : V \times V \to \mathbb{R}$ by

$$f(v, w) = \sum_{1\le i,j\le n} g_{i,j} v^i w^j$$

for $v = \sum v^i e_i$ and $w = \sum w^j e_j$ in V. Describe f in terms of the tensor products of α^i and α^j, $1 \le i, j \le n$.

Solution We know that $v^i = \alpha^i(v)$, and similarly $w^j = \alpha^j(w)$. So we can write

$$v^i w^j = \alpha^i(v)\alpha^j(w) = (\alpha^i \otimes \alpha^j)(v, w).$$

Thus the bilinear function can be written as

$$f = \sum_{1 \le i,j \le n} g_{i,j} \alpha^i \otimes \alpha^j.$$

This bilinear function is very similar to the notion of weighted inner product.

■ **Problem 1.13 — Hyperplanes(from W. Tu).** (a) Let V be a vector space of dimension n, and $f : V \to \mathbb{R}$ a nonzero linear functional. Show that $\dim \ker f = n - 1$. A linear subspace of V of dimension $n - 1$ is called a hyperplane in V.

(b) Show that a nonzero linear functional on a vector space V is determined up to multiplicative constant by its kernel, a hyperplane in V. In other words, if f and $g : V \to \mathbb{R}$ are nonzero linear functionals and $\ker f = \ker g$, then $g = cf$ for some constant $c \in \mathbb{R}$.

Solution (a) We can use the Rank-Nullity theorem for the linear function f. This theorem states that

$$\dim \operatorname{rank} f + \dim \ker f = \dim V.$$

We know that $\dim \operatorname{rank} f = 1$, and $\dim V = n$. This implies $\dim \ker f = n - 1$.

(b) Let f, g be two linear functionals on V that has the same kernel, which we call it the set K. Since K is $n - 1$ dimensional linear subspace of K, then we can find a basis for it, call it $\mathcal{B}_1 = \{\hat{e}_1, \cdots, \hat{e}_{n-1}\}$. Let $\hat{e}_n$ be a vector normal to $\mathcal{B}_1$. We can do all of these since V is n dimensional and we have just modified the basis vectors as described above. Let $z \in V$ where $z = \sum_{i=1}^n z_i \hat{e}_i$. Thus we have

$$f(z) = z_n f(\hat{e}_n), \qquad g(z) = z_n g(\hat{e}_n).$$

This is true because $f(\hat{e}_i) = g(\hat{e}_i) = 0$ for all $\hat{e}_i \in \mathcal{B}$. Since f, g are non-zero functional, then we can write

$$\frac{f(z)}{g(z)} = \frac{f(\hat{e}_n)}{g(\hat{e}_n)} = c \qquad \text{for some } c \in \mathbb{R}$$

Thus we can write

$$f = cg.$$

Note that c is determined by $\hat{e}_n$. $\hat{e}_n$ itself is determined by the kernel of f, g, i.e. the hyperplane mentioned above.

Observation 1.4.2 I have this weird observation that I think it must be wrong, but I don't know how! Consider the set of all linear functionals defined on $\mathbb{R}^2$. From duality, we know that this set (denoted by $L_1(\mathbb{R}^2)$) is isomorphic to $\mathbb{R}^2$. But here comes the strange argument that leads to the non-real observation. As we saw in the question above, every hyperplane (in this case every line that passes through the origin) determines a linear functional up to a constant multiplication. On the other hand, we know that the set of all liens passing through the origin is the real projective plane (which is in fact $L_1(\mathbb{R}^2)/\sim$ where $\sim$ identifies linear functionals that are constant multiple of each other), which definitely is not isomorphic to $\mathbb{R}^2$.

■ **Problem 1.14 — A basis for k-tensors (from W. Tu).** Let V be a vector space of dimension n with basis $e_1, \cdots, e_n$. Let $\alpha^1, \cdots, \alpha^n$ be the dual basis for $V^\vee$. Show that a basis for the space $L_k(V)$ of k-linear functions on V is $\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ for all multi-indices $(i_1, i_2, \cdots, i_k)$. In particular, this show that $\dim L_k(V) = n^k$.

Solution To show this, we need to show that the basis vectors are linearly independent, and spans all $L_k(v)$. To show the linear independence, consider the following linear combination of all basis vectors, where the total sum is set to zero.

$$\sum C_{i_1,\cdots,i_k}\alpha^{i_1}\otimes\cdots\otimes\alpha^{i_k}=0,$$

where the sum is done on all of the multi-indices $(i_1,\cdots,i_k)$. This sum implies that all of the coefficients are zero. We can see this by applying $(e_{j_1},\cdots,e_{j_k})$ to both sides. We have

$$(\alpha^{i_1}\otimes\cdots\otimes\alpha^{i_k})(e_{j_1},\cdots,e_{j_k})=0$$

if there is any mismatch between the multi-indices I, J. This implies $C_{j_1,\cdots,j_k}=0$ for all multi-indices. Thus we can conclude that the basis vectors are linearly independent.

To show that the basis vectors span all the space $L_k(V)$, let $f\in L_k(V)$ be any k-tensor. Define

$$g=\sum f(e_{i_1},\cdots,e_{i_k})\alpha^{i_1}\otimes\cdots\otimes\alpha^{i_k}.$$

We can see that

$$g(e_{j_1},\ldots,e_{j_k})=f(e_{j_1},\ldots,e_{j_k})\qquad\text{for all multi-indices}$$

From linearity we can deduce that $f=g$. Thus the basis vectors span the whole space.

■ **Problem 1.15 — A characterization of alternating k-vectors (from W. Tu).** Let f be a k-tensor on a vector space V. Prove that f is alternating if and only if f changes sign whenever two successive arguments are interchanged

$$f(\cdots,v_{i+1},v_i,\cdots)=-f(\cdots,v_i,v_{i+1},\cdots).$$

Solution The first direction follows immediately from the definition. I.e. assume f is alternating. Then sign of the permutation corresponding interchanging two successive arguments is a single transposition which has sign -1. Thus from definition we have $f(\cdots,v_{i+1},v_i,\cdots)=-f(\cdots,v_i,v_{i+1},\cdots)$.
For the converse, first observe that we can decompose arbitrary permutation to cycles, and then decompose cycles into transpositions. Eventually, we can decompose every transposition to transposition of adjacent elements. See the Lemma below

Lemma 1.2 — Decomposing arbitrary transposition to transposition of adjacent elements. Let $\tau=(k\ l)$ be an arbitrary transposition where $k<l$. Assume that k,l are m distance apart (i.e. $l-k=m$). Then we can write

$$(k\ l)=(k\ k+1)\cdot(k+1\ k+2)\cdots(l-1\ l)\cdot(l-2\ l-1)\cdots(k\ k+1).$$

Thus $(k\ l)$ can be decomposed to $2m-1$ adjacent transpositions.

Let $\sigma\in S_k$ be an arbitrary permutation. We can decompose it to the following permutations

$$\sigma=\tau_1\tau_2\cdots\tau_l,$$

where from the number of permutations we can see that $\text{sign}(\sigma)=(-1)^l$. Let τ_i be one of the transpositions above that interchanges two elements m distance apart. Then

$$\tau_i f=(-1)^{2m-1}f=-f.$$

This follows immediately from the Lemma above. Thus we can write

$$\sigma f=(-1)^l f=\text{sign}(\sigma)f.$$

This shows that f is an alternating k-tensor.

■ **Problem 1.16 — Another characterization of alternating k-tensors (from W. Tu).** Let f be a k-tensor on a vector space V. Prove that f is alternating if and only if

$$f(v_1, \cdots, v_k) = 0$$

whenever two of the vectors $v_1, \cdots, v_k$ are equal.

Solution We first start with the forward direction. Assume f is alternating. Thus for $\sigma \in S_k$ we have $\sigma f = \operatorname{sign}(\sigma) f$. Without loss of generality, we can assume $i < j$ and let $(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) \in V^k$ where $v_i = v_j$.Thus we can write

$$f(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = f(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k).$$

Furthermore, let $\tau = (v_i, v_j)$, i.e. a transposition that interchanges v_i with v_j. Hence

$$\begin{aligned} f(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) &= \tau f(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k) \\ &= -f(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k). \end{aligned}$$

This implies

$$f(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = f(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k) = 0$$

To show the converse, let $\sigma \in S_k$ be any permutation. We can decompose this permutation into transpositions, i.e. $\sigma = \tau_1 \tau_2 \cdots \tau_r$. Let τ be any of transpositions above that permutes $(i\ j)$. Let $(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) \in V^k$. Since V^k is a vector space we can write

$$(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) - (v_1, \cdots, v_i, \cdots, v_i, \cdots, v_k) - (v_1, \cdots, v_j, \cdots, v_j, \cdots, v_k) = -(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k).$$

Applying f to both sides, and using the hypothesis and k-linearity of f we will get

$$f(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = -f(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k) = -\tau f(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k).$$

Thus for each transposition in the decomposition $\sigma = \tau_1 \tau_2 \cdots \tau_r$ we will get a factor of -1. Thus

$$\sigma f = (-1)^r f = \operatorname{sign}(\sigma) f$$

■ **Problem 1.17 — Wedge product and scalars (from W. Tu).** Let V be a vector space. For $a, b \in \mathbb{R}$, $f \in A_k(V)$, and $g \in A_\ell(v)$, show that $af \wedge bg = (ab) f \wedge g$.

Solution This follows immediately from the definition of wedge product.

$$af \wedge bg = \frac{1}{k!\ell!} A\left(af \otimes bg\right) = \frac{ab}{k!\ell!} A(f \otimes g) = (ab) f \wedge g.$$

■ **Problem 1.18 — Transformation rule for a wedge product of covectors (from W. Tu).** Suppose two sets of covectors on a vector space V, $\beta^1, \cdots, \beta^k$ and $\gamma^1, \cdots, \gamma^k$ are related by

$$\beta^i = \sum_{j=1}^{k} a^i_j \gamma^j, \qquad i = 1, 2, \cdots, k,$$

for a $k \times k$ matrix $A = \left[a^i_j\right]$. Show that

$$\beta^1 \wedge \cdots \wedge \beta^k = (\det A) \gamma^1 \wedge \cdots \wedge \gamma^k.$$

Solution It will be helpful if we write the relation between β^i's and γ^js as a matrix multiplication. To do so we can write

$$\begin{pmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^k \end{pmatrix} = \begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_k^1 \\ a_1^2 & a_2^2 & \cdots & a_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^k & a_2^k & \cdots & a_k^k \end{pmatrix} \begin{pmatrix} \gamma^1 \\ \gamma^2 \\ \vdots \\ \gamma^k \end{pmatrix}$$

On the other hand, we know

$$\begin{aligned}
(\beta^1 \wedge \cdots \wedge \beta^k)(v_1, \cdots, v_k) &= \det \begin{pmatrix} \beta^1(v_1) & \beta^1(v_2) & \cdots & \beta^1(v_k) \\ \beta^2(v_1) & \beta^2(v_2) & \cdots & \beta^2(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \beta^k(v_1) & \beta^k(v_2) & \cdots & \beta^k(v_k) \end{pmatrix} \\
&= \det \left[\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_k^1 \\ a_1^2 & a_2^2 & \cdots & a_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^k & a_2^k & \cdots & a_k^k \end{pmatrix} \begin{pmatrix} \gamma^1(v_1) & \gamma^1(v_2) & \cdots & \gamma^1(v_k) \\ \gamma^2(v_1) & \gamma^2(v_2) & \cdots & \gamma^2(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^k(v_1) & \gamma^k(v_2) & \cdots & \gamma^k(v_k) \end{pmatrix} \right] \\
&= \det(A) \det \begin{pmatrix} \gamma^1(v_1) & \gamma^1(v_2) & \cdots & \gamma^1(v_k) \\ \gamma^2(v_1) & \gamma^2(v_2) & \cdots & \gamma^2(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^k(v_1) & \gamma^k(v_2) & \cdots & \gamma^k(v_k) \end{pmatrix}
\end{aligned}$$

Thus we can write

$$(\beta^1 \wedge \cdots \wedge \beta^k)(v_1, \cdots, v_k) = \det(A)(\gamma^1 \wedge \cdots \wedge \gamma^k)(v_1, \cdots, v_k).$$

■ **Problem 1.19 — Transformation rule for k-covectors (from W. Tu).** Let f be a k-covector on a vector space V (of say, dimension n). Suppose two sets of vectors $u_1, \cdots, u_k$ and $v_1, \cdots, v_k$ in V are related by

$$u_j = \sum_{i=1}^{k} a_j^i v_i, \qquad j = 1, \cdots, k,$$

for a $k \times k$ matrix $A = \left[a_j^i\right]$. Show that

$$f(u_1, \cdots, u_k) = (\det A) f(v_1, \cdots, v_k).$$

By

Solution Since f is a k-vector, then we can expand it in its basis

$$f = \sum_{I \in AM} C_I \alpha^I,$$

where AM is the set of all ascending multi-indices $I = (i_1, \cdots, i_k)$ for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Thus

$$f(u_1, \cdots, u_k) = \sum_{I \in AM} C_I \alpha^I(u_1, \cdots, u_k).$$

Let's focus on the term $I = (i_1, \cdots, i_k)$. For this term we can write

$$
\begin{aligned}
(\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k})(u_1, \cdots, u_k) &= \det \begin{pmatrix} \alpha^{i_1}(u_1) & \alpha^{i_1}(u_2) & \cdots & \alpha^{i_1}(u_k) \\ \alpha^{i_2}(u_1) & \alpha^{i_2}(u_2) & \cdots & \alpha^{i_2}(u_k) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{i_k}(u_1) & \alpha^{i_k}(u_2) & \cdots & \alpha^{i_k}(u_k) \end{pmatrix} \\
&= \det \left[\begin{pmatrix} \alpha^{i_1}(v_1) & \alpha^{i_1}(v_2) & \cdots & \alpha^{i_1}(v_k) \\ \alpha^{i_2}(v_1) & \alpha^{i_2}(v_2) & \cdots & \alpha^{i_2}(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{i_k}(v_1) & \alpha^{i_k}(v_2) & \cdots & \alpha^{i_k}(v_k) \end{pmatrix} \begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_k^1 \\ a_1^2 & a_2^2 & \cdots & a_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^k & a_2^k & \cdots & a_k^k \end{pmatrix} \right] \\
&= \det(A) \det \begin{pmatrix} \alpha^{i_1}(v_1) & \alpha^{i_1}(v_2) & \cdots & \alpha^{i_1}(v_k) \\ \alpha^{i_2}(v_1) & \alpha^{i_2}(v_2) & \cdots & \alpha^{i_2}(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{i_k}(v_1) & \alpha^{i_k}(v_2) & \cdots & \alpha^{i_k}(v_k) \end{pmatrix} \\
&= \det(A)(\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k})(v_1, \cdots, v_k)
\end{aligned}
$$

Thus we can write

$$f(u_1, \cdots, u_k) = \sum_{I \in AM} C_I \alpha^I(u_1, \cdots, u_k) = \det(A) \sum_{I \in AM} C_I \alpha^I(v_1, \cdots, v_k)$$

or

$$\boxed{f(u_1, \cdots, u_k) = \det(A) f(v_1, \cdots, v_k).}$$

□

■ **Problem 1.20 — Vanishing of a covector of top degree (from W. Tu).** Let V be a vector space of dimension n. Prove that if an n-covector ω vanishes on a basis $e_1, \cdots, e_n$ for V, then ω is the zero covector on V.

Solution First, observe that for a vector space of dimension n we have

$$\dim(A_n) = \binom{n}{n} = 1, \qquad \mathcal{B} = \{\alpha^1 \wedge \cdots \wedge \alpha^k\}.$$

Let $f \in A_n(V)$. Thus we can write

$$f = g\alpha^1 \wedge \cdots \wedge \alpha^n \qquad \text{for some } g \in \mathbb{R}.$$

On the other hand, we know

$$(\alpha^1 \wedge \cdots \wedge \alpha^n)(e_1, \cdots, e_n) = \det \left[\alpha^i(e_j) \right] = 1.$$

This $f(e_1, \cdots, e_n) = 0$ if and only if $g = 0$.

■ **Problem 1.21 — Linear independence of covectors (from W. Tu).** Let $\alpha^1, \cdots, \alpha^k$ be 1-covectors on a vector space V. Show that $\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$ if and only if $\alpha^1, \cdots, \alpha^k$ are linearly independent in the dual space $V^\vee$.

Solution For the forward direction, we want to show

$$\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0 \implies \alpha^1, \cdots, \alpha^k \text{ are linearly independent.}$$

To show this we will prove the contrapositive. Assume $\alpha^1, \cdots, \alpha^k$ are linearly dependent. Thus we can write one of them as a linear combination of others. With out loss of generality, assume

$$\alpha^1 = c_2\alpha^2 + \cdots + c_n\alpha^n,$$

where c_is are not all zero. Consider

$$\alpha^1 \wedge \cdots \wedge \alpha^k = c_2\alpha^2 \wedge \cdots \wedge \alpha^k + \cdots + c_k\alpha^k \wedge \cdots \wedge \alpha^k = 0.$$

The last equality is true since $\alpha^I = 0$ if the multi-indices I as at least two repetitive indices.

For the converse, we want to show

$$\alpha^1, \cdots, \alpha^k \text{ are linearly independent.} \implies \alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$$

Assume $\alpha^1, \cdots, \alpha^k$ are linearly independent. Then it can be extended as a basis for $V^\vee$ as as $\alpha^1, \cdots, \alpha^k, \cdots, \alpha^n$. Let $v_1, v_2, \cdots, v_n$ be the dual basis for V. Then

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \cdots, v_k) = \det\left[\alpha^i(v_j)\right] = 1 \neq 0.$$

Thus $\alpha^1 \wedge \cdots \wedge \alpha^k$ can not be a zero k-covector.

■ **Problem 1.22 — Exterior multiplication (from W. Tu).** Let α be a non-zero 1-covector and γ a k-covector on a finite dimensional vector space V. Show that $\alpha \wedge \gamma = 0$ if and only if $\gamma = \alpha \wedge \beta$ for some $(k-1)$-covector β on V.

Solution First, we prove the first direction. Let $\gamma \in A_k(V), \alpha \in A_1(V)$ such that $\gamma = \alpha \wedge \beta$ for some $\beta \in A_{k-1}(V)$. Then

$$\alpha \wedge \gamma = \alpha \wedge \alpha \wedge \beta = 0.$$

For the converse, let α and γ as before and we want to prove

$$\alpha \wedge \gamma = 0 \implies \gamma = \alpha \wedge \beta \text{ for some } \beta \in A_{k-1}(V).$$

We prove the contrapositive. Assume $\forall \beta \in A_{k-1}(V)$ we have $\gamma \neq \alpha \wedge \beta$. Then by wedge product of α to both sides we will get

$$\alpha \wedge \gamma \neq \alpha \wedge \alpha \wedge \beta = 0.$$

This completes the proof. □

■ **Problem 1.23 — A basis for 3-covectors (from W. Tu).** Let x^1, x^2, x^3, x^4 be the coordinates on $\mathbb{R}^4$ and p a point in $\mathbb{R}^4$. Write down a basis for the vector space $A_3(T_p(\mathbb{R}^4))$.

Solution First, observe that a basis for $T_p(\mathbb{R}^4)$ is $\{\frac{\partial}{\partial x^1}|_p, \cdots, \frac{\partial}{\partial x_4}|_p\}$, and for $T_p^*(\mathbb{R}^n)$ a standard basis is $\{dx_1|_p, \cdots, dx_4|_p\}$. Thus a standard basis for $A_3(T_p(\mathbb{R}^4))$ would be

$$\mathcal{B} = \{dx^1 \wedge dx^2 \wedge dx^3|_p,\ dx^1 \wedge dx^2 \wedge dx^4|_p,\ dx^1 \wedge dx^3 \wedge dx^4|_p,\ dx^2 \wedge dx^3 \wedge dx^4|_p\}.$$

■ **Problem 1.24 — Wedge product of a 2-form with a 1-form (from W. Tu).** Let Ω be a 2-form and τ a 1-form on $\mathbb{R}^3$. If X, Y, Z are vector fields on M find an explicit formula for $(\omega \wedge \tau)(X, Y, Z)$ in terms of the values of ω and τ on the vector fields X, Y, Z.

Solution From the definition of wedge product, and using the notion of $(2,1)$ shuffles, that are $(1<2,3),(1<3,2),(2<3,1)$ with respective signs $+,-,+$ we can write

$$(\omega\wedge\tau)(X,Y,Z)=(\omega\otimes\tau)(X,Y,Z)-(\omega\otimes\tau)(X,Z,Y)+(\omega\otimes\tau)(Y,Z,X),$$

or alternatively

$$(\omega\wedge\tau)(X,Y,Z)=\omega(X,Y)\tau(Z)-\omega(X,Z)\tau(Y)+\omega(Y,Z)\tau(X).$$

■ **Problem 1.25 — A closed 1-form on the punctured plane (from W. Tu).** Define a 1-form ω on $\mathbb{R}^2-\{0\}$ by

$$\omega=\frac{1}{x^2+y^2}(-ydx+xdy).$$

Show that ω is closed.

Solution define $f,g:\mathbb{R}^2-\{0\}\to\mathbb{R}$ as

$$f(x,y)=\frac{-y}{x^2+y^2},\qquad g(x,y)=\frac{x}{x^2+y^2}.$$

Thus we can write ω as $\omega=fdx+gdy$. The exterior derivative of ω will be

$$d\omega=(g_x-f_y)dx\wedge dy.$$

By a simple calculation we can observe

$$f_x(x,y)=\frac{y^2-x^2}{x^2+y^2},\qquad g_y(x,y)=\frac{y^2-x^2}{x^2+y^2}.$$

Thus we will have

$$d\omega=0.$$

■ **Problem 1.26 — A 1-form on $\mathbb{R}^3$ (from W. Tu).** Let ω be the 1-form $zdx-dz$, and let X be the vector field $y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$ on $\mathbb{R}^3$. Compute $\omega(X)$ and $d\omega$.

Solution To calculate $\omega(X)$ we can write

$$\omega(X)=X\omega=(y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y})(zdx-dz)=yz-x.$$

And to calculate $d\omega$ we have

$$d\omega=dz\wedge dx$$

■ **Problem 1.27 — A 2-form on $\mathbb{R}^3$ (from W. Tu).** At each point $p\in\mathbb{R}^3$, define a bilinear function ω_p on $T_p(\mathbb{R}^3)$ by

$$\omega_p(a,b)=\omega_p\left(\begin{bmatrix}a^1\\a^2\\a^3\end{bmatrix},\begin{bmatrix}b^1\\b^2\\b^3\end{bmatrix}\right)=p^3\det\begin{pmatrix}a^1&b^1\\a^2&b^2\end{pmatrix},$$

For tangent vectors $a,b\in T_p(\mathbb{R}^3)$, where p^3 is the third component of $p=(p^1,p^2,p^3)$. Since ω_p is an alternating bilinear function on $T_p(\mathbb{R}^3)$, ω is a 2-form on $\mathbb{R}^3$. Write ω in terms of the standard basis $dx^i\wedge dx^j$ at each point.

Solution In the definition of ω, we can replace determinant by the following wedge product

$$\omega_p=p^3dx^1\wedge dx^2.$$

■ **Problem 1.28 — Exterior calculus (from W. Tu).** Suppose the standard coordinates on $\mathbb{R}^2$ are called r and θ (this $\mathbb{R}^2$ is the (r,θ) plane not the (x,y) plane). If $x = r\cos\theta$ and $y = r\sin\theta$, calculate dx, dy, and $dx \wedge dy$ in terms of dr and $d\theta$.

Solution Using the relations $x = r\cos\theta$ and $y = r\sin\theta$, we can use the definition of exterior derivative to have

$$dx = \cos\theta\ dr - r\sin\theta d\theta, \qquad dy = \sin\theta dr + r\cos\theta\ d\theta.$$

We can write this in a more clean way as

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{bmatrix} dr \\ d\theta \end{bmatrix}.$$

By using the fact proved in Problem Problem 1.18 we will have

$$dx \wedge dy = \det(A) dr \wedge d\theta = r dr \wedge d\theta.$$

■ **Problem 1.29 — Exterior calculus (from W. Tu).** Suppose the standard coordinates on $\mathbb{R}^3$ are called ρ, φ and θ. If $x = \rho\cos\varphi\sin\theta$, $y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\theta$, calculate dx, dy, dz, and $dx \wedge dy \wedge dz$ in terms of $d\rho, d\varphi$, and $d\theta$.

Solution Similar to the problem above, using the definition of the exterior derivative we have

$$\begin{aligned} dx &= \cos\varphi\sin\theta\ d\rho - \rho\sin\varphi\sin\theta\ d\varphi + \rho\cos\varphi\cos\theta\ d\theta, \\ dy &= \sin\varphi\sin\theta\ d\rho + \rho\cos\varphi\sin\theta\ d\varphi + \rho\sin\varphi\cos\theta\ d\theta, \\ dz &= \cos\theta\ d\rho - \rho\sin\theta\ d\theta. \end{aligned}$$

or in matrix notation we have

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\theta & 0 & -\rho\sin\theta \end{pmatrix} \begin{bmatrix} d\rho \\ d\varphi \\ d\theta \end{bmatrix}.$$

From the fact proved in Problem Problem 1.18 we can write

$$dx \wedge dy \wedge dz = \det(A)\ d\rho \wedge d\theta \wedge d\varphi.$$

We now need to calculate $\det(A)$

$$\det(A) = -\rho^2\cos^2\theta\sin\theta - \rho^2\sin^3\theta = -\rho^2\sin\theta.$$

Thus we have

$$dx \wedge dy \wedge dz = (-\rho^2\sin\theta)\ d\rho \wedge d\varphi \wedge d\theta.$$

■ **Problem 1.30 — Wedge product (from W. Tu).** Let α be a 1-form and β a 2-form on $\mathbb{R}^3$. Then

$$\begin{aligned} \alpha &= a_1 dx^1 + a_2 dx^2 + a_3 dx^3, \\ \beta &= b_1 dx^1 \wedge dx^3 + b_2 dx^3 \wedge dx^1 + b_3 dx^1 \wedge dx^2. \end{aligned}$$

Simplify the expression $\alpha \wedge \beta$ as much as possible.

Solution By following the rules for the wedge product we will have

$$\alpha \wedge \beta = (a_2 b_2 + a_3 b_3 - a_2 b_1)\ dx^1 \wedge dx^2 \wedge dx^3$$

■ **Problem 1.31 — Wedge product and cross product (from W. Tu).** The correspondence between differential forms and vector fields on an open subset of $\mathbb{R}^3$ also makes sense point wise. Let V be a vector space of dimension 3 with basis e_1, e_2, e_3, and dual basis $\alpha^1, \alpha^2, \alpha^3$. To a 1-covector $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ on V, we associate the vector $v_\alpha = \langle a_1, a_2, a_3\rangle \in \mathbb{R}^3$. To the 2-covector

$$\gamma = c_1\alpha^1 \wedge \alpha^3 + c_2\alpha^3 \wedge \alpha^1 + c_3\alpha^1 \wedge \alpha^2,$$

on V, we associate the vector $v_\gamma = \langle c_1, c_2, c_3\rangle \in \mathbb{R}^3$. Show that under this correspondence, the wedge product of 1-covectors corresponds to the cross product of vectors in $\mathbb{R}^3$: if $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ and $\beta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$, then we have

$$v_{\alpha\wedge\beta} = v_\alpha \times v_\beta.$$

Solution Let $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$, and $\beta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$. Then for the wedge product we have

$$\alpha \wedge \beta = (a_1b_2 - a_2b_1)\alpha^1 \wedge \alpha^3 + (a_1b_3 - a_3b_1)\alpha^1 \wedge \alpha^3 + (a_2b_3 - a_3b_2)\alpha^3 \wedge \alpha^2.$$

We can associate this with vector $v_{\alpha\wedge\beta} = \langle a_1b_2 - a_2b_1, a_1b_3 - a_3b_1, a_2b_3 - a_3b_2\rangle \in \mathbb{R}^3$. On the other hand, we can write $v_\alpha \times v_\beta = \langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle = \langle a_1b_2 - a_2b_1, a_1b_3 - a_3b_1, a_2b_3 - a_3b_2\rangle$. This shows that

$$v_{\alpha\wedge\beta} = v_\alpha \times v_\beta.$$

■ **Problem 1.32 — Commutator of derivations and antiderivations (from W. Tu).** Let $A = \bigoplus_{k=-\infty}^{\infty} A^k$ be a graded algebra over a field K with $A^k = 0$ for $k < 0$. Let m be an integer. A superderivation of A of degree m is a K-linear map $D : A \to A$ such that for all k, $D(A^k) \in A^{k+m}$ and for all $a \in A^k$ and $b \in A^\ell$,

$$D(ab) = (Da)b + (-1)^{k\ell} aD(b).$$

If D_1 and D_2 are two superderivations of A of respective degrees m_1 and m_2, define their commutator to be

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{m_1m_2} D_2 \circ D_1.$$

Show that $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$. A superderivation is said to be even or odd depending on the parity of its degree. An even superderivation is a derivation; an odd superderivation is an antiderivation.

Solution To show that $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$ we need to show

$$[D_1, D_2](ab) \stackrel{?}{=} [D_1, D_2](a)b + (-1)^{k(m_1+m_2)} a[D_1, D_2](b).$$

Using the definition of $[D_1, D_2]$ we can expand this as

$$[D_1, D_2](ab) \stackrel{?}{=} \left[D_1(D_2(a)) - (-1)^{m_1m_2} D_2(D_1(a))\right] b + (-1)^{k(m_1+m_2)} a \left[D_1(D_2(b)) - (-1)^{m_1m_2} D_2(D_1(b))\right]. \tag{Q.1}$$

To show this, we simply start with $[D_1, D_2]$ and apply it on $a \in A_k, b \in A_\ell$ and use the definition of $[D_1, D_2]$ to simplify it and turn it into the format of equation $(Q.1)$.

$$\begin{aligned}
[D_1, D_2](ab) &= (D_1 \circ D_2)(ab) - (-1)^{m_1m_2}(D_2 \circ D_1)(ab) = D_1(D_2(ab)) - (-1)^{m_1m_2} D_2(D_1(ab)) \\
&= D_1(D_2(a))b + (-1)^{m_1(k+m_2)} D_2(a)D_1(b) + (-1)^{m_2k}\left[D_1(a)D_2(b) + (-1)^{m_1k} aD_1(D_2(b))\right] \\
&\quad - (-1)^{m_1m_2}\left[D_2(D_1(a))b + (-1)^{m_2(k+m_1)} D_1(a)D_2(b) + (-1)^{m_1k}\left[D_2(a)D_1(b) + (-1)^{m_2k} aD_2(D_1(b))\right]\right]
\end{aligned}$$

Now by matching the terms we can write

$$\begin{aligned}[D_1, D_2](ab) = &\left[D_1(D_2(a)) - (-1)^{m_1 m_2} D_2(D_1(a))\right] b + (-1)^{k(m_1+m_2)} a \left[D_1(D_2(b)) - (-1)^{m_1 m_2} D_2(D_1(b))\right] \\ &+ \underbrace{D_1(a) D_2(b) \left[(-1)^{m_2 k} - (-1)^{m_2(k+2m_1)}\right]}_{=0}.\end{aligned}$$

The last term is equal to zero since if $m_2 k$ is even, then $m_2 k + 2m_1 m_2$ is also even, which leads to $(-1)^{m_2 k} - (-1)^{m_2(k+2m_1)} = -1 + 1 = 0$. As the other case, if $m_2 k$ is even, then $m_2 k + 2m_1 m_2$ is also even, where we will have $(-1)^{m_2 k} - (-1)^{m_2(k+2m_1)} = 1 - 1 = 0$. This proves that $[D_1, D_2]$ is indeed an superderivative of order $m_1 + m_2$.

2. Manifolds

Very important note: Throughout this text, to avoid repeating some words like *smooth, and etc*, we often omit them and we let the context to reflect these notions. So we emphasis that throughout this text, unless otherwise specified, by a **manifold** we always mean a C^∞ manifold. An atlas of a chart on a smooth manifold means an atlas or a chart contained in the differentiable structure of the smooth manifold.

2.1 Topological manifolds and Smooth manifolds

We start with the definition of topological manifolds. We will later study the smooth manifolds that are of our main interest in this lecture. But first, we need to review some basic definition.

Definition 2.1 — Hausdorff topological space. A set A, along with a collection of subsets of A called $\mathcal{T} \subset 2^A$, i.e. $(A, \mathcal{T})$ is called a topological space if we have

(I) $\varnothing \in \mathcal{T}$ and $A \in \mathcal{T}$.

(II) For any infinite collection $\{A_\alpha\}_\alpha \subset \mathcal{T}$ we have

$$\bigcup_\alpha A_\alpha \in \mathcal{T}.$$

(III) For a finite collection $\{A_i\}_{i\in I}$ where $I = \{1, 2, \cdots, N\}$ for some $N \in \mathbb{N}$, we have

$$\bigcap_{i\in I} A_i \in \mathcal{T}.$$

(IV) $\forall x, y \in A$ there exists $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$ and we have $U \cap V = \varnothing$.

The following definition reviews the notion of second countable topological spaces.

Definition 2.2 — Second countable topological spaces. Let $(X, \mathcal{T})$ be a topological space. This space is second countable if there exists an at most countable collection of sets $\mathcal{B} \subset \mathcal{T}$ such that

$\forall U \in \mathcal{T}$ we have

$$U = \bigcup_{A \in \mathcal{B}} A.$$

I.e. any open set in $\mathcal{T}$ can be written as a union of sets in $\mathcal{B}$

The last piece of definition that we need is the notion of locally Euclidean spaces.

Definition 2.3 — Locally Euclidean topological spaces. Let $(X, \mathcal{T})$ be a topological space. We say X is locally Euclidean if $\forall p \in X$, there exists an open neighborhood $p \in U \in \mathcal{T}$ such that is homeomorphic to an *open* subset of $\mathbb{R}^n$. I.e. there is a homeomorphism $\varphi : U \to \mathbb{R}^n$. We call the pair $(U, \varphi : U \to \mathbb{R}^n)$ a *chart*, U a *coordinate neighborhood* or a **coordinate open set**, and φ a *coordinate map* or *coordinate system* on U.

Definition 2.4 — Topological manifolds. A topological manifold is a Hausdorff, second countable topological space that is locally Euclidean space. It is said to be of dimension n, if it is locally Euclidean of dimension n.

In the following section, we will review the notion of compatible charts which will be central to our study of manifolds. Assume we have two charts (U, φ) and (V, ψ). Since U, V are open in M (the manifold), then $U \cap V$ is open in U and V. Furthermore, since φ is a homeomorphism to an open subset of $\mathbb{R}^n$ then $\varphi(U \cap V)$ and $\psi(U \cap V)$ are also open. One way to show this, let $\varphi(U cap V) = B, \varphi(U) = A$, and $A \backslash B = C$. We know that A is open, and $B \cup C = A, B \cap C = \varnothing$. Then A being open implies B and C is open. Now we can make the following definition for C^∞-compatible maps.

Definition 2.5 — C^∞ compatible maps. Let M be a topological manifold, where (U, φ) and (V, ψ) are two charts. These charts are called C^∞-compatible if the maps

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V) \qquad \text{and} \qquad \psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V),$$

are C^∞. These two maps are called *transition functions* between charts.

■ **Remark 2.1** If two U, V in the definition above, i.e. $U \cap V = \varnothing$, then they are automatically compatible.

Definition 2.6 — C^∞ atlas. Let M be a topological manifold. The collection of charts $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}$ is called a C^∞ atlas, or simply an atlas if

- all of the charts are pairwise C^∞ compatible,
- the charts cover the whole manifold, i.e. $M = \bigcup_\alpha U_\alpha$.

Lemma 2.1 Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be an atlas for the manifold M. Let (V, ψ) and (W, σ) be two charts where both of them are compatible with the atlas $\mathcal{A}$. Then these two charts are compatible with each other.

Proof. We start by showing that $\psi \circ \sigma^{-1}$ is C^∞ on $V \cap W$. Let $p \in V \cap W$. Then $\exists \alpha \in I$ such that $p \in U_\alpha$ for the chart $(U_\alpha, \varphi_\alpha)$. Then

$$(\psi \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \sigma^{-1}) : \sigma(W \cap U_\alpha \cap V) \to \psi(W \cap U_\alpha \cap V)$$

is C^∞ in $\sigma(W \cap U_\alpha \cap V)$, because it is the composition of C^∞ maps. Call $\mathbb{B}_\alpha = W \cap U_\alpha \cap V$. Then what we have shows is simply $\forall p \in V \cap W$, there exists an open set $p \in \mathbb{B}_\alpha \subset V \cap W$ for $\alpha \in I$ and the map $\psi \circ \sigma^{-1}$ is C^∞ in $\mathbb{B}_\alpha$. Since this holds for every $p \in V \cap W$ this proves that $\psi \circ \sigma^{-1}$ is C^∞ on $\sigma(V \cap W)$. Similarly, we can show $\sigma \circ \psi^{-1}$ is C^∞ on $\psi(V \cap W)$, and this completes the proof. □

■ **Remark 2.2** Note that in an equality like $\psi \circ \sigma^{-1} = (\psi \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \sigma^{-1})$, two functions in the sides of the equality have different domains. So the equality sign here means that these two functions are equal on their common domain.

Definition 2.7 — Maximal atlas. Let $\mathcal{U}$ be an atlas for the manifold M. Then $\mathcal{U}$ is called maximal if it contains any other atlas of the manifold M. Or equivalently, if $\mathcal{M}$ is any other atlas containing $\mathcal{U}$ then $\mathcal{U} = \mathcal{M}$.

We can use the notion of the maximal atlas to define a smooth manifold.

Definition 2.8 — Smooth manifold. A topological manifold together with a maximal atlas is called a smooth manifold. The maximal atlas is also called a differentiable structure on M.

In practice, to show that a topological manifold is smooth, it is not necessary to exhibit a maximal atlas. Existence of any atlas will do so, as proposed by the following proposition.

Proposition 2.1 Any atlas of a locally euclidean space is contained in a *unique* maximal atlas.

Proof. Let $\mathfrak{U} = \{(U_\alpha, \varphi_\alpha)\}$ be any atlas for the locally Euclidean space M. Let $S = \{(V_i, \psi_i)\}$ denote the set of all charts compatible with $\mathfrak{U}$. Construct the set $\mathfrak{M} = \mathfrak{U} \cup S$. We claim that this set is a maximal atlas. Let (W, ψ) be a chart compatible with $\mathfrak{M}$. Then it should also be compatible with $\mathfrak{U}$, thus it is contained in the set S, i.e. $(W, \psi) \in S$, thus in $\mathfrak{M}$. This shows that the atlas $\mathfrak{M}$ is maximal.

To show the uniqueness, let $\mathfrak{M}'$ be another maximal atlas. Since $\mathfrak{M}'$ is compatible with $\mathfrak{U}$, thus it is also compatible with $\mathfrak{M}$. But due to the construction it is contained in the new atlas, thus $\mathfrak{M}' \subset \mathfrak{M}$. Thus the maximal atlas is unique. □

Considering the proposition above, we arrive at the following important observation.

Observation 2.1.1 — Showing a space is an smooth manifold. To show that a space is an smooth manifold, we just need to show that the space is

- a Hausdorff topological space that is also second countable,
- there exists any C^∞ atlas (not necessarily maximal).

2.2 Smooth maps on a manifold

In this section, we will study the notion of smooth maps between manifolds. We will use the notion of coordinate charts to transfer the notion of smooth maps from Euclidean spaces to manifolds. We start with the functions on manifolds.

Definition 2.9 — Smooth functions on manifolds. Let $f : M \to R$ be a function on a manifold. The function f is smooth at point $p \in M$ if there exists a chart (U, φ) such that $p \in U$ and the map

$$f \circ \varphi^{-1} \; : \; \varphi(U) \to \mathbb{R}$$

is smooth at p. The function f is said to be C^∞ on M if it is C^∞ at every point of M.

■ **Remark 2.3** Note that the smoothness of the function f in the definition above is independent of the local chart (U, ψ) that we choose. Let (V, ψ) be another local chart containing the point p. Then the map $f \circ \psi^{-1}$ is also smooth at p. Because

$$f \circ \psi^{-1} : (f \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1}).$$

We know that the map $(\varphi \circ \psi^{-1}$ is smooth at p (since the local charts in an atlas are compatible). Also $(f \circ \varphi^{-1})$ is smooth (we have assumed so). Thus $f \circ \psi^{-1}$ is smooth.

Proposition 2.2 — Smoothness of real valued functions. Let M be a manifold of dimension n with atlas $\mathfrak{U}$, and let $f : M \to \mathbb{R}$ a real-valued function on M. The following are equivalent:

(i) The function $f : M \to \mathbb{R}$ is C^∞.

(ii) The manifold M has an atlas such that for every chart (U, φ) in the atlas, $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is C^∞.

(iii) For every chart (V, ψ) on M, the function $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}$ is C^∞.

Proof. We prove a cyclic chain of implications $(ii) \implies (i) \implies (iii) \implies (ii)$.
$(ii) \implies (i)$: Let $p \in M$. From the assumption we know that there is an atlas with the desired property. Thus $\exists (U_\alpha, \varphi_\alpha)$ that contains p and the function $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is smooth. Thus $f : M \to \mathbb{R}$ is smooth at p, ans since the point p was arbitrary, then f is smooth on M.
$(i) \implies (iii)$: Let $p \in U$. From assumption we know that there is a coordinate open set U_α such that $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U) \to \mathbb{R}$ is smooth. Then we can write

$$f \circ \psi^{-1} = (f \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \psi^{-1})$$

which shows that $f \circ \psi^{-1}$ is smooth (look at the remark below).
$(iii) \implies (ii)$: This follows immediately from the definition. □

■ **Remark 2.4** Considering the "very important note" at the beginning of this chapter, we need to emphasis again that in part (iii), when we say *for every chart* (V, ψ) *on* M, we mean a chart in the unique differentiable structure of the manifold that contains the atlas $\mathfrak{U}$.

Observation 2.2.1 — From W. Tu. The smoothness conditions from the proposition above will be a recurrent motif through out the book. To prove the smoothness of an object, it is sufficient that a smoothness criterion hold on the charts of some atlas. Once the object is shown to be smooth, it then follows that the same smoothness criterion holds on every chart on the manifold.

Definition 2.10 — Pull back of a function. Let $F : N \to M$ be a a map between manifolds, and $\varphi : M \to \mathbb{R}$. Then the pullback of the function φ by F is denoted by $F * \varphi$ is defined as

$$F^*\varphi = \varphi \circ F.$$

■ **Remark 2.5** In this terminology, a function $f : M \to \mathbb{R}$ is smooth on a chart (U, φ) if and only if the pullback $(\varphi^{-1})^* f$ is smooth on the subset $\varphi(U)$ of Euclidean space.

2.2.1 Smooth maps between manifolds

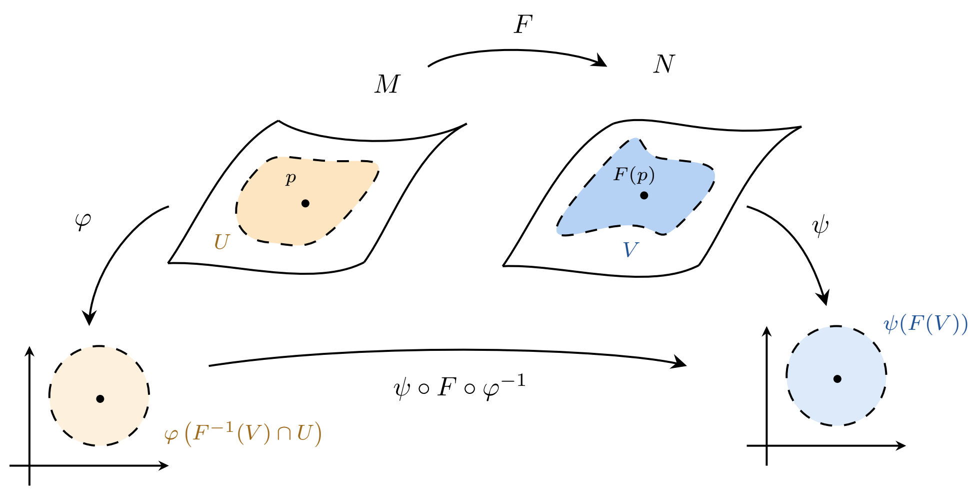
Here in this section we will discuss the smooth maps between manifolds. We will be able to recover the definition of the smooth functions on manifolds as a special case.

Definition 2.11 — Smooth maps between manifolds. Let M, N be manifolds with dimensions m, n respectively. A map $F : M \to N$ is said to be smooth at point $p \in M$, if there exists charts (U, φ) and (V, ψ) such that $p \in U$ and $F(p) \in V$, and the function

$$\psi \circ F \circ \varphi^{-1} : \mathbb{R}^m \supset \varphi(F^{-1}(V) \cup U) \to \mathbb{R}^n$$

is smooth at p. The map F is said to be smooth on manifold M if it is smooth at every point of the manifold.

■ **Remark 2.6** The following figure helps to digest the definition above.



■ **Remark 2.7** Since the Euclidean space is indeed a smooth manifolds, then we can recover the definition of smooth functions on manifolds form the definition above. Let M be a manifold, and $N = \mathbb{R}^n$ with the atlas $\{(\mathbb{R}^n, \mathbb{1} : \mathbb{R}^n \to \mathbb{R}^n)\}$, then we will have the notion of vector valued functions on manifold. By setting $n = 1$ we will recover the definition of smooth function on manifold.

In the following proposition we will be showing that the smoothness of the map is independent of the charts chosen, thus the smoothness of the map is well-defined.

Proposition 2.3 — Smoothness of maps between manifolds is well-defined. Suppose $F : N \to M$ is C^∞ at $p \in N$. If (U, φ) is any chart in N that contains p and (V, ψ) is any chart in M that contains $F(p)$, then $\psi \circ F \circ \varphi^{-1}$ is smooth at $\varphi(p)$.

Proof. Since $F : N \to M$ is smooth at $p \in N$, then there are charts (G, γ) and (L, λ) (from the corresponding maximal atlases) such that $p \in G \subset N$ and $F(p) \in L \subset M$ and the function

$$(\lambda \circ F \circ \gamma^{-1}) : \gamma(F^{-1}(L) \cap G) \to \lambda(F(G))$$

is smooth at p. Consider the charts (U, φ) and (V, ψ) as above. Then the function

$$\psi \circ F \circ \varphi^{-1} = (\psi \circ \lambda^{-1}) \circ (\lambda \circ F \circ \gamma^{-1}) \circ (\gamma \circ \varphi^{-1}).$$

Note that the equality sign above merely means the functions are equal on their common domain, as the function in RHS and the function in LHS have different domains. We know that the coordinate maps ψ, λ, and γ, φ are compatible respectively. Thus the function $\psi \circ F \circ \varphi^{-1}$ is smooth at $\varphi(p)$. See the remark below for mode details. □

■ **Remark 2.8** In the proof above and in the equality

$$\psi \circ F \circ \varphi^{-1} = (\psi \circ \lambda^{-1}) \circ (\lambda \circ F \circ \gamma^{-1}) \circ (\gamma \circ \varphi^{-1})$$

thus equality sign does not indicate that these function are equal, but it just indicates that these two functions are equal in their common domain. To be more clear, for the function in the LHS we have

$$\psi \circ F \circ \varphi^{-1} : \varphi(F^{-1}(V) \cap U) \to \psi(F(U)).$$

But for the function in the LHS we have

$$(\psi \circ \lambda^{-1}) \circ (\lambda \circ F \circ \gamma^{-1}) \circ (\gamma \circ \varphi^{-1}) : \varphi(F^{-1}(L \cap V) \cap (G \cap U)) \to \psi(F(L \cap V))$$

Proposition 2.4 — Smoothness of maps in terms of charts. Let N and M be smooth manifolds, and $F : N \to M$ a continuous map. The following are equivalent:

(i) The map $F : N \to M$ is C^∞.

(ii) There are atlases $\mathfrak{U}$ for N and $\mathfrak{V}$ for M such that for every chart (U, φ) in $\mathfrak{U}$ and (V, ψ) in $\mathfrak{V}$ the map
$$\psi \circ F \circ \varphi^{-1} : \varphi(F^{-1}(V) \cap U) \to \mathbb{R}^m$$
is C^∞.

(iii) For every chart (U, φ) on N and (V, ψ) on M, the map
$$\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \mathbb{R}^m$$
is C^∞.

Proof. We will prove this by showing a cyclic chain of implications $(ii) \implies (i) \implies (iii) \implies (ii)$.

- $(ii) \implies (i)$. Let $p \in N$. Then by hypothesis there are charts (U, φ) and (V, ψ) such that $p \in U$ and $F(p) \in V$ and the map
$$\psi \circ F \circ \varphi^{-1} : \varphi(F^{-1}(V) \cap U) \to \mathbb{R}^m$$
is smooth at $\varphi(p)$, thus F is smooth at p. Since p is arbitrary, then F is smooth on N.
- $(i) \implies (iii)$. This follows immediately from Proposition 2.3.
- $(iii) \implies (ii)$. Since N, M are smooth manifolds, then they have maximal atlases. Let $\mathfrak{U}$ be the maximal atlas for N and $\mathfrak{V}$ be the maximal atlas for M. This (ii) follows immediately form (iii).

□

Proposition 2.5 — Composition of C^∞ maps. If $F : N \to M$ and $G : M \to P$ are C^∞ maps of manifolds, then the composite $G \circ F : N \to P$ is a smooth map.

Proof. We will demonstrate two proofs for this proposition to demonstrate what happens if we do not notice a possible level of abstraction. The first proof below is a very crude, hard-core, direct proof, that seems to be tough and easy to make mistakes, just because it is not encapsulating some of the details into another proposition. However, for the second proof, it will encapsulate some of the details into the results of the Proposition 2.4, which will allows us to do a more high level thinking.

proof 1. Let $p \in M$. Since F is smooth, then $\exists$ (U, γ) chart for M and $\exists$ (V, φ) chart for M such that $p \in U$ and $F(p) \in V$ and the map

$$\varphi \circ F \circ \gamma^{-1} : \gamma(F^{-1}(V) \cap U) \to \varphi(V)$$

is smooth at $\gamma(p)$. Also, since G is smooth, then at $F(p)$, there exists the charts (W, λ) for M and (Y, ψ) for P such that $F(p) \in W$ and $G(F(p)) \in Y$ and the map

$$\psi \circ G \circ \lambda^{-1} : \lambda(G^{-1}(Y) \cap W) \to \psi(Y)$$

is smooth at $\lambda(F(p))$. Also note that since the charts (W, λ) and (V, φ) for M are compatible (they both are chosen from the maxima atlas), then the maps

$$\varphi \circ \lambda^{-1} : \lambda(W \cap V) \to \varphi(W \cap V) \quad \text{and} \quad \lambda \circ \varphi^{-1} : \varphi(W \cap V) \to \lambda(W \cap V)$$

are both smooth. Now it is time to glue all of these pieces together to infer the smoothness of the composite function $G \circ F$. Based on these pieces, the function

$$\psi \circ (G \circ F) \circ \gamma^{-1} = (\psi \circ G \circ \lambda^{-1}) \circ (\lambda \circ \varphi^{-1}) \circ (\varphi \circ F \circ \gamma^{-1}) : \gamma(F^{-1}(G^{-1}(Y) \cap V \cap W) \cap U) \to \psi(G(F(p)))$$

is smooth at $\gamma(p)$. Since the point $p \in N$ was arbitrary, then the map $G \circ F$ is smooth on M.

proof 2. In this proof, we will hide much of the details of the proof 1 in higher level of abstractions encapsulated in Proposition 2.4. Since F is smooth, then ($(i) \implies (iii)$ of Proposition 2.4) for every chart (U, γ) for M and (V, φ) for M the map

$$\varphi \circ F \circ \gamma^{-1} : \gamma(F^{-1}(V) \cap U) \to \varphi(V)$$

is smooth.. Also, since G is smooth, then for every chart (Y, ψ) for P the map

$$\psi \circ G \circ \varphi^{-1} : \varphi(G^{-1}(Y) \cap V) \to \psi(Y)$$

is smooth. Since the composition of smooth real vector valued functions is smooth, then

$$\psi \circ (G \circ F) \circ \gamma^{-1} = (\psi \circ G \circ \varphi^{-1}) \circ (\varphi \circ F \circ \gamma^{-1})$$

is smooth. Since for every chart (U, γ) and (Y, φ) the function $\psi \circ (G \circ F) \circ \gamma^{-1}$ is smooth, then by $(iii) \implies (i)$ of Proposition 2.4 we proved that $G \circ F$ is smooth. ☐

2.3 Diffeomorphisms

We start with the definition of diffeomorphism.

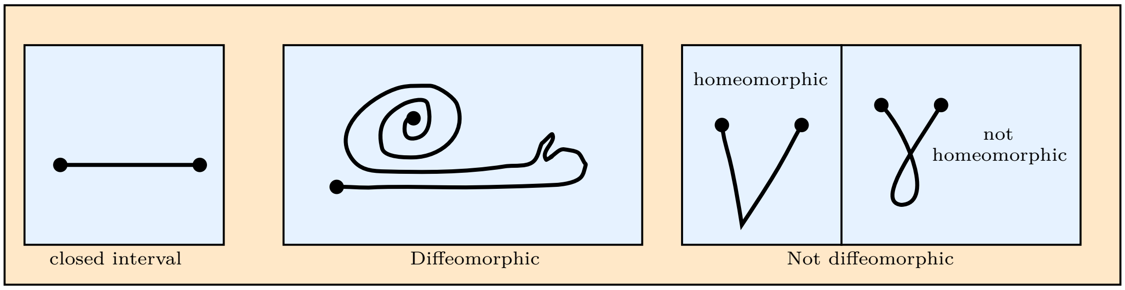
Proposition 2.6 — Diffeomorphism between manifolds. A map $F : M \to N$ between manifolds M and N is called a diffeomorphism if

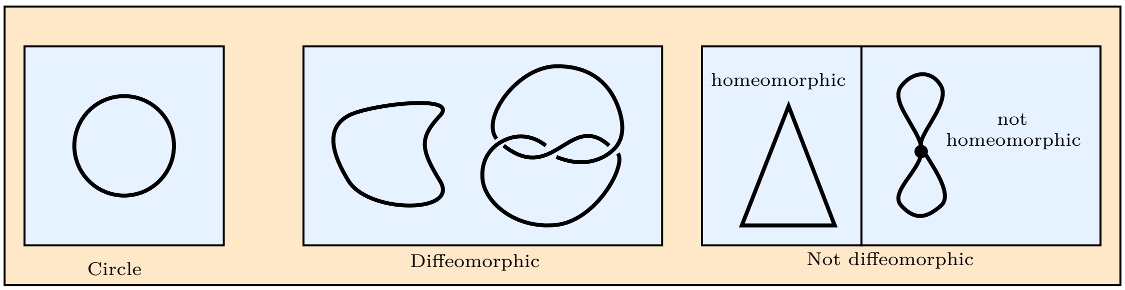
- F is a bijection,
- F is C^∞, and
- F^{-1} is also C^∞.

The following observation box summarizes the differences between homeomorphisms and diffeomorphisms.

Observation 2.3.1 — Difference between homeomorphisms and diffeomorphisms. homeomorphisms are isomorphisms in the category of topological spaces. However, the diffeomorphisms are isomorphisms in the category of smooth manifolds, where the isomorphism preserves the differential structures. The following figure (original idea from[a]) demonstrates some of these ideas.

[a] mathematical analysis lecture notes.





According to the two following propositions, in a *smooth* manifold, the coordinate maps are diffeomorphisms, and conversely, any diffeomorphism from an open subset of the manifold to an open subset of a Euclidean space can serve as a coordinate map.

Proposition 2.7 — Coordinate maps are diffeomorphisms. If (U, φ) is a chart on a manifold M of dimension n, then the coordinate map $\varphi : U \to \varphi(\varphi) \in \mathbb{R}^n$ is a diffeomorphism.

Proof. For the proof, we will use the notion of smooth maps between manifolds in a smart way! First, observe that since φ is a coordinate map, thus it is automatically a homeomorphism. Thus we just need to show that φ and φ^{-1} are smooth. Let U be a manifold with an atlas $\{(U, \varphi)\}$. Also, let $\varphi(U) \subset \mathbb{R}^n$ be a manifold (this is indeed a manifold as it is an open subset of a Euclidean space), with an atlas $\{(\varphi(U), \mathbb{1}_{\varphi(U)})\}$. In this point of view, the coordinate map φ is in fact a map between manifolds. This is smooth, since

$$\mathbb{1}_{\varphi(U)} \circ \varphi \circ \varphi^{-1} : \underbrace{\varphi(U \cap \varphi^{-1}(\varphi(U)))}_{\varphi(U)} \to \varphi(U)$$

is smooth, then by Proposition 2.4 $(ii) \implies (i)$, the map φ is smooth.

To show the smoothness of the map φ^{-1}, we observe that this is a map from the manifold with atlas $\{(\varphi(U), \varphi^{-1})\}$ to the manifold with atlas $\{(U, \varphi)\}$. Since the following map

$$\varphi \circ \varphi^{-1} \circ \mathbb{1}_{\varphi(U)} : \varphi(U) \to \varphi(U)$$

is smooth (since it is just the identity map), then by Proposition 2.4 $(ii) \implies (i)$ the map φ^{-1} is smooth.

Putting these two results together, we have shown that the coordinate map φ is indeed a diffeomorphism from U to $\mathbb{R}^n$. $\square$

The following proposition is the converse of the proposition above.

Proposition 2.8 Let U be an open subset of a manifold M of dimension n. If $F : U \to F(U) \subset \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$, then (F, U) is a chart in the differentiable structure of M.

Proof. Since F is a diffeomorphism, then it is certainly a homeomorphism. Thus we just need to show that (U, F) is a chart, i.e. compatible with the maximal atlas of the manifold (note that since M is an smooth manifold, then there exists a maximal atlas). Let $p \in U$ and let (V, ψ) be a chart that contains p. Then consider the function

$$F \circ \psi^{-1} : \psi(U \cap V) \to F(U \cap V) \qquad \psi \circ F^{-1} : F(U \cap V) \to \psi(U \cap V).$$

By the proposition above, we know that since φ is a coordinate chart of a smooth manifold, then φ and φ^{-1} are both smooth. Then the functions above are a composition of smooth functions, thus they are smooth as well. This proves that (U, F) is compatible with the chart (V, ψ). Since this chart was arbitrary, then it is compatible with the whole atlas, thus (U, F) is a chart. Also, by the maximality of the atlas, (U, F) is contained in the atlas. $\square$

Now using the notion of smooth maps between manifolds, we can define a Lie group.

Definition 2.12 — Lie group. A *Lie group* is a C^∞ manifold G having a group structure such that the multiplication map

$$\mu : G \times G \to G$$

and the inverse map

$$\iota : G \to G, \qquad \iota(x) = x^{-1},$$

are both C^∞.

■ **Remark 2.9** Similar to the definition above, a topological group is a topological space having a group structure such that the multiplication and the inverse maps are both continuous. Note that a topological group is required to be a topological space, but not a topological manifold.

The followings are some examples of Lie groups. We can easily check the group operation and the inverse map are both smooth by using the definitions that we have had above.

(i) The Euclidean space $\mathbb{R}^n$ is a Lie group under addition.

(ii) The set $\mathbb{C}^\times$ of nonzero complex numbers is a Lie group under multiplications.

(iii) Subsequently, the unit circle in $\mathbb{C}^\times$ is a Lie group under multiplication.

2.4 Partial Derivatives

In this section we will define the notion of partial derivatives of the maps between manifolds. We will use the notion of coordinate charts for this definition. Let N be a manifold of dimension n, and let $(r^1, \cdots, r^n)$ be the standard coordinate function on $\mathbb{R}^n$. For a chart (U, φ) on the manifold, φ is a vector valued function on manifold. We denote its coordinate functions as x^i where $x^i = r^i \circ \varphi$. This is just a fancy (but very useful) way of writing the i-th coordinate of the vector valued function φ. Now we can define the directional derivatives for the functions on manifolds.

Definition 2.13 — Directional derivative. Let N be a manifold, and $f : N \to \mathbb{R}$ an smooth function. Let $p \in U$ and (U, φ) any chart containing p, where $\varphi = (x^1, \cdots, x^n)$. We define the direction derivative of f at p as

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial (f \circ \varphi^{-1})}{\partial r^i}(\varphi(p)) = \frac{\partial}{\partial r^i}\Big|_{\varphi(p)}(f \circ \varphi^{-1}).$$

■ **Remark 2.10** In the definition above, we can re-write it as

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial f}{\partial x^i}(\underbrace{\varphi^{-1}(\varphi(p))}_{p}) = \frac{\partial (f \circ \varphi^{-1})}{\partial r^i}(\varphi(p)).$$

We can now write

$$\frac{\partial (f \circ \varphi^{-1})}{\partial r^i} = \frac{\partial f}{\partial x^i} \circ \varphi^{-1} = [\varphi^{-1}]^*[\frac{\partial f}{\partial x^i}],$$

where shows that $\frac{\partial (f \circ \varphi^{-1})}{\partial r^i}$ is the pull back of the function $\frac{\partial f}{\partial x^i}$ when viewed as a function on the manifold. Since the pull back of $\frac{\partial f}{\partial x^i}$ is smooth (as $f \circ \varphi^{-1}$ is smooth), then $\frac{\partial f}{\partial x^i}$ is smooth when viewed as a real valued function on the manifold.

Proposition 2.9 Suppose $(U, x^1, \cdots, x^n)$ is a chart on a manifold. Then we have

$$\frac{\partial x^i}{\partial x^j} = \delta^i_j.$$

Proof. We view x^i as a real valued function on the manifold. For a point p in the manifold, we can use the definition of partial derivative to write

$$\frac{\partial x^i}{\partial x^j}(p) = \frac{\partial (x^i \circ \varphi^{-1})}{\partial r^j} = \frac{\partial (r^i \circ \varphi \circ \varphi^{-1})}{\partial r^j} = \frac{\partial r^i}{\partial r^j} = \delta^i_j.$$

Note that we have used the fact that $\varphi = (x^1, \cdots, x^n)$. □

Using the notion of partial derivatives, we can define the notion of Jacobian matrix for the maps between manifolds.

Definition 2.14 — Jacobian matrix of maps between manifolds. Let $F : N \to M$ be a smooth map, and let $(U, \varphi) = (U, x^1, \cdots, x^n)$ and $(V, \psi) = (V, y^1, \cdots, y^m)$ be charts on N and M respectively such that $F(U) \subset V$. Denote by

$$F^i = y^i \circ F = r^i \circ \psi \circ F : U \to \mathbb{R}$$

the i-th component of F in the chart (V, ψ). The following matrix

$$J = \left[\frac{\partial F^i}{\partial x^j}\right]$$

is called the Jacobian matrix of F relative to the charts given as above. In case M, N have the same dimension, the determinant of this matrix is called the Jacobian determinant of F

relative to the two charts. The Jacobian determinant is also written as

$$\det(J) = \frac{\partial(F^1, \cdots, F^n)}{\partial(x^1, \cdots, x^n)}.$$

2.5 Inverse function theorem

As we studied before, on a manifold N of dimension n, a diffeomorphism $f : U \to \mathbb{R}^n$ is indeed a coordinate map. In this section, we want to study the conditions under which a smooth map between manifolds can be upgraded to a local diffeomorphism. For this end, we will use a slight generalization of the inverse function theorem in the Euclidean spaces.

Definition 2.15 — Inverse function theorem in Euclidean spaces. Let $F : W \to \mathbb{R}^n$ where $W \subset \mathbb{R}^n$ be a smooth map. We say this function is a *local diffeomorphism* or locally invertible at point p, if there exists an open set $U \subset W$ containing p such that the Jacobian determinant is non-zero at p, i.e.

$$\det\left[\frac{\partial F^i}{\partial r^j}\right](p) \neq 0.$$

We will skip proving this theorem here and we will just use it to prove the generalized version of this theorem for smooth maps between manifolds.

Definition 2.16 — Inverse function theorem for manifolds. Let $F : N \to M$ be a smooth map between two manifolds of the same dimension, and $p \in N$. Suppose for some charts $(U, \varphi) = (U, x^1, \cdots, x^n)$ about p in N and $(V, \psi) = (V, y^1, \cdots, y^n)$ about $F(p)$ in M, $F(U) \subset V$. Set $F^i = y^i \circ F$. Then F is locally invertible at p if and only if its Jacobian determinant $\det\big(\partial F^i/\partial x^j\big)(p)$ is nonzero.

Proof. First, observe the proof structure in the observation box below. We just need to show two following bi-directional implications.

$$F \text{ is locally invertible} \iff \psi \circ F \circ \varphi^{-1} \text{ is locally invertible.}$$

and also

$$\det\left[\frac{\partial(\psi \circ F \circ \varphi^{-1})^i}{\partial r^j}\right](\varphi(p)) \neq 0 \iff \det\left[\frac{\partial F^i}{\partial x^j}\right](p) \neq 0.$$

For the first bi-directional implication since φ, ψ are coordinate maps, thus are invertible (they are bijective) and from the composition of functions it follows that the bi-directional implication is true.

For the second bi-directional implication we start with the forward direction. We can write

$$\frac{\partial(\psi \circ F \circ \varphi^{-1})^i}{\partial r^j} = \frac{\partial(r^i \circ \psi \circ F \circ \varphi^{-1})}{\partial r^j} = \frac{\partial(y^i \circ F \circ \varphi^{-1})}{\partial r^j} = \frac{\partial(F^i \circ \varphi^{-1})}{\partial r^j} = \frac{\partial F^i}{\partial x^j}.$$

This proves the forward direction. We can do this starting from the last equation which proves the backward direction. □

Observation 2.5.1 — Some hidden treasures!. When proving the theorem above, we can figure out what should be the structure of the proof, even if we can not demonstrate the proof steps completely. For me, there is some sense of dimensional analysis if physics in what I am saying. By looking only at the ingredients of the problem/theorem/etc, we can figure out what is the possible way that we can combine those ingredients to achieve to a full proof. The following figure reveals this structure for the proof.

$$\boxed{F \text{ is locally invertible}} \iff \boxed{\psi\circ F\circ\varphi^{-1} \text{ is locally invertible}} \Leftrightarrow \boxed{\det\left[\frac{\partial\left(\psi\circ F\circ\varphi^{-1}\right)^i}{\partial r^j}\right]\neq 0} \iff \boxed{\det\left[\frac{\partial F^i}{\partial x^j}\right]\neq 0}$$

using the inverse function theorem in $\mathbb{R}^n$

Corollary 2.1 Let N be a manifold of dimension n. A set of n smooth functions $F^1,\dots,F^n$ defined on a coordinate neighborhood $(U,x^1,\cdots,x^n)$ of a point $p\in N$ forms a coordinate system about p if and only if the Jacobian determinant

$$\left[\frac{\partial F^i}{\partial x^j}\right](p)\neq 0.$$

Proof. define the vector valued function as $F=(F^1,\cdots,F^n)$. Then we have

$$\begin{aligned}
&\det\left[\partial F^i/\partial x^j\right]\neq 0.\\
\iff\ & F \text{ is locally invertible.}\\
\iff\ & \text{There is a neighborhood } W \text{ containing } p \text{ such that } F:W\to F(W)\subset\mathbb{R}^n \text{ is a diffeomorphism.}\\
\iff\ & (W,F) \text{ is a coordinate system in the maximal atlas of } N.
\end{aligned}$$

□

2.6 Quotients

The quotient construction is a process of simplification. Starting with an equivalence relation on a set, we identify each equivalence class to a point. If the original set is a topological space, then it is always possible to give the quotient space a topology so that the natural projection map (i.e. $\pi:S\to S/\sim\ ,\ x\mapsto[x]$) becomes continuous. However, if the original set is a manifold, then it is not always true that the quotient set is also manifold, as the conditions like being Hausdorff or second countable might break.

Definition 2.17 — Quotient topology. Let $(X,\mathcal{T})$ be a topological space, and $\sim$ an equivalence relation with π as the natural projection map. A set $U\in X/\sim$ is open, if and only if $\pi^{-1}(U)\in\mathcal{T}$, i.e. open.

■ **Remark 2.11** In the definition above, clearly the empty set $\varnothing$ and the whole quotient space $X/\sim$ are open. That is because $\pi^{-1}(\varnothing)=\varnothing\in\mathcal{T}$ (since the equivalence relation $\sim$ partitions the set), and $\pi^{-1}(X/\sim)=X\in\mathcal{T}$. Also, for any map, in particular π we have

$$\pi^{-1}\Big(\bigcup_\alpha U_\alpha\Big)=\bigcup_\alpha\pi^{-1}(U_\alpha),\qquad \pi^{-1}\Big(\bigcap_i U_i\Big)=\bigcap_i\pi^{-1}(U_i)$$

where $\pi^{-1}(U)$ means the pre-image of the set U under the map φ. This shows that an arbitrary union of the open sets in the quotient space is open, and also a finite intersection of open sets of the quotient space is also open. Thus the space $(X/\sim,\mathcal{T}')$ where $U\in\mathcal{T}'$ if and only if $\pi^{-1}(U)\in\mathcal{T}$

is a topological space.
Also Note that since π^{-1} maps open sets of $X/\sim$ to the open sets of X, then π is automatically (i.e. by design) a continuous function.

2.6.1 Continuity of a map on a Quotient

Let S be a topological space and $\sim$ and equivalence relation on this set. Give the quotient set $S/\sim$ the quotient topology. Then a function $f : S \to Y$ from S to another topological space, where it is constant for all elements in the same equivalence class. This map then induces a map on $S/\sim$, i.e. $\bar{f} : S/\sim \to Y$ where

$$\bar{f}([p]) = f(p) \qquad \text{for } p \in S.$$

or in other words

$$\bar{f}(\pi(p)) = f(p) \qquad \text{or equivalently} \qquad \bar{f} \circ \pi = f.$$

Proposition 2.10 The induced map $\bar{f} : S/\sim \to Y$ is continuous if and only if the map $f : S \to Y$ is continuous.

Solution First, we will show that continuity of $\bar{f}$ implies the continuoity of f. Let $U \in \mathcal{T}_Y$. Since $\bar{f}$ is continuous, then $\bar{f}^{-1}(U) \in \mathcal{T}_{S/\sim}$. Furthermore, since π is continuous then $\pi^{-1}(\bar{f}^{-1}(U)) \in \mathcal{T}_S$, thus $(\bar{f} \circ \pi)$ is continuous. Since $f = \bar{f} \circ \pi$, then f is also continuous. As an alternative prove, we could start with the fact that $f = \bar{f} \circ \pi$, and since $\bar{f}$ and π are both continuous, then so is f.

For the converse, we want to show that the continuity of f implies the continuity of f^{-1}. Let $V \in \mathcal{T}_Y$. Since f is continuous, then $f^{-1}(V) \in \mathcal{T}_S$. On the other hand, we have $f^{-1}(V) = (\pi^{-1} \circ \bar{f}^{-1})(V)$. Since π^{-1} sends open sets to open sets, then $\bar{f}(V) \in \mathcal{T}_{S/\sim}$. Thus $\bar{f}$ is continuous.

Definition 2.18 — Identification of a subset to a point. Let S be a topological space and $A \subset S$. Let $\sim$ be s relation on S where

$$x \sim x \qquad \forall x \in S,$$

and also

$$x \sim y \qquad \forall x, y \in A.$$

This is an equivalence relation on S. We call the quotient space $S/\sim$ is obtained from S by identifying the set A to a point.

2.7 Hausdorff and Second Countablity Conditions

The quotient construction often does not preserve the second countablity and Hausdorff property of a topological space. We start with the following proposition.

Proposition 2.11 Let X be a topological space, and $\sim$ an equivalence relation on X. If $X/\sim$ is Hausdorff then equivalent class $[p]$ of any $p \in X$ is closed in X.

Proof. In a Hausdorff space, every singleton set is closed. let $p \in X$. Then $\{\pi(p)\}$ is closed in $X/\sim$. Since π is continuous, then the pre image $\pi^{-1}(\{\pi(p)\}) = [p]$ is closed in X. This completes the proof. □

■ **Remark 2.12** In the proof above, we claimed that in a Hausdorff space, every singleton set is closed. To see this, let $p \in S$ a Hausdorff space. Consider the singleton $\{p\}$ and its complement, i.e. $S - \{p\}$. We claim that set $S - \{p\}$ is open. To show this, let $U \in \mathcal{T}$ where $p \in U$. Since S is

Hausdorff, then there exists an open set $V \in \mathcal{T}$ such that contains any point $q \in S - \{p\}$ and do not overlap with U. This $q \in V \subset S - \{p\}$. Thus for every point $S - \{p\}$ we can find an open set containing q that is contained in the set $S - \{p\}$.

Definition 2.19 — Open equivalence relation. Let $\sim$ be an equivalence relation defined on a set. This equivalence relation is said to be open, if $\pi(U)$ is open for some open set U

■ **Remark 2.13** We can give a useful characterization of open equivalence relations. Let U be an open set. Then $\pi(U)$ being open means that $\pi^{-1}(\pi(U))$ is also open. Thus an equivalence relation is open if and only if

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x]$$

is open.

Theorem 2.1 Let S be a topological space, and $\sim$ an open equivalence relation defined on S. Then $S/\sim$ is Hausdorff if and only if the graph R of the relation is closed in $S \times S$.

Proof. First, consider the following chain of bidirectional implications.

R is closed in $S \times S$
$\Leftrightarrow$ $S \times S - R$ is closed in $S \times S$
$\Leftrightarrow$ For all $(x, y) \in (S \times S) - R$, there exist an open cell $U \times V \ni (x, y)$ such that $U \times V \cap R = \varnothing$.
$\Leftrightarrow$ For all pair $x, y \in S$ where $x \not\sim y$, there exists open sets $U \ni x$ and $V \ni y$ such that no element of U is equivalent to an element of V.
$\Leftrightarrow$ For all two points in $[x] \neq [y] \in S/\sim$ there exists open sets $U \ni x, V \ni y$ open in S such that $\pi(U) \cap \pi(V) = \varnothing$.

Now what remains to show is to show that the last statement (call it statement *) is equivalent to being Hausdorff. If $*$ is true, then since π is open, then $S/\sim$ being Hausdorff follows immediately. For the converse, if $S/\sim$ Hausdorff, then for $[x] \neq [y] \in S/\sim$ we can find open sets A, B containing $[x], [y]$ respectively where $A \cap B = \varnothing$. From the surjectivity of π we know that $\pi(\pi^{-1}(A)) = A$. Thus let $U = \pi^{-1}(A)$ and $V = \pi^{-1}(B)$, thus the statement (*) follows. □

A very simple use of the theorem above is that consider a case where the equivalence relation is simply the equality. Then we will have the following corollary.

Corollary 2.2 — Hausdorff characterization. Let S be a topological space. This space is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \in S \times S\}$ is closed in $S \times S$.

■ **Remark 2.14** Surprisingly, we can prove Theorem 2.1 using the corollary above. See Problem 2.21.

Similar to our procedure above, we can find some conditions under which a quotient space of a second countable space is still second countable. This criterion is given in the proposition below.

Proposition 2.12 — Second Countability of a Quotient Space . Let X be a second countable topological space, and let $\sim$ be an equivalence relation. The quotient space $X/\sim$ is second countable of $\sim$ is an *open equivalence relation*, i.e. π is an open map.

Proof. We need to show that $X/\sim$ posses a countable basis for its open sets. We claim that if $\mathcal{B} = \{B_\alpha\}$ is countable basis for X, then $\{\pi(B_\alpha)\}$ is a countable basis for $X/\sim$. To show this let

$U \subset X/\sim$ be an open set. By continuoity of π^{-1} the set $\pi^{-1}(U)$ is also open in X. Thus we can write

$$\pi^{-1}(U) \subset \bigcup_{\alpha \in I} B_\alpha.$$

Then we can write

$$U \subset \pi(\bigcup_{\alpha \in I} B_\alpha) = \bigcup_{\alpha \in I} \pi(B_\alpha).$$

Thus $\{\pi(B_\alpha)\}$ is indeed a countable basis for $X/\sim$. This completes the proof. □

2.8 Real Projective Space

Here in this section we will review one of the very interesting manifolds that we can construct by quotient of another manifold. We start with the definition.

Definition 2.20 — Real Projective Space. Consider the set $\mathbb{R}^{n+1} - \{0\}$. Define the equivalence relation $\sim$ as follows

$$x \sim y \qquad \Longleftrightarrow \qquad x = ty \quad \text{for some } t \in \mathbb{R}$$

where $x, y \in \mathbb{R}^n - \{0\}$. The quotient set under this equivalence relation is called the real projective space or $\mathbb{R}P^n$.

■ **Remark 2.15** When I was writing this, it came to me what do we need to generalize this notion of projective plane to any vector space. Because the only ingredients that we use in this definition is the vector space properties of the underlying set.

Geometrically, the real projective plane can be tough of the set of all point passing through the origin. On the other hand, we know that every line passing through the origin hits the unit sphere S^n in exactly two antipodal points. The following proposition reveals the connections between the some quotient structure of unit spheres and the real projective space.

Proposition 2.13 Define an equivalence relation on S^n by identifying the antipodal points. I.e.

$$x \sim y \quad \Longleftrightarrow \quad x = \pm y.$$

The quotient set $S^n/\sim$ is homeomorphic to the real projective space.

Proof. To show this we need to find homeomorphisms between these two sets. Define $f : \mathbb{R}^{n+1} - \{0\} \to S^n$ be

$$f(x) = \frac{x}{\|x\|}$$

where $\|x\| = (\sum_i x_i^2)^{1/2}$ is the modulus of the point $x = (x_1, \cdots, x_n)$. This function is continuous since the denominator is never zero and both nominator and denominator are continuous functions (note that $\|x\|$ is a composition of smooth functions). Consider the following diagram.

$$\begin{tikzcd}
\mathbb{R}^{n+1} - \{0\} \arrow[r, "f"] \arrow[d, "\pi_1"'] & S^n \arrow[d, "\pi_2"] \\
\mathbb{R}P^n \arrow[r, "\bar{f}"'] & S^n/\sim
\end{tikzcd}$$

The function $\bar{f} : \mathbb{R}P^n \to S^n/\sim$ is a function induced by f given by

$$\bar{f}([x]) = \left[\frac{x}{\|x\|}\right] \in S^n/\sim .$$

This function is well defined as it dose not depend on the representative of the equivalence class that we choose. Since for any other element in the equivalence class we will have

$$\bar{f}([tx]) = \left[\frac{tx}{\|tx\|}\right] = \left[\frac{tx}{|t|\|x\|}\right] = \left[\pm\frac{x}{\|x\|}\right] = \left[\frac{tx}{\|tx\|}\right] = \bar{f}([x]).$$

Since the function

$$\pi_2 \circ f$$

is continuous, then the function $\bar{f}$ is also continuous. Next, define $g : S^n \to R^{n+1} - \{0\}$ given by $g(x) = x$. This map induces a map $\bar{g} : S^n/\sim \to \mathbb{R}P^n$ given by $g([x]) = [x]$. By the same argument as above, $\bar{g}$ is continuous and well defined. It just remains to show that these two maps are inverses of each other. To show this we have

$$(\bar{f} \circ \bar{g})([x]) = \bar{f}([x]) = \left[\frac{x}{\|x\|}\right] = [x],$$

and also

$$(\bar{g} \circ \bar{f})([x]) = \bar{g}(\left[\frac{x}{\|x\|}\right]) = \left[\frac{x}{\|x\|}\right] = [x].$$

Thus the spaces $\mathbb{R}P^n$ and $S^n/\sim$ are homeomorphic to each other. □

2.9 Real Projective Plane

Here in this section we will focus on the real projective plane, which is the set of all lines passing through origin in $\mathbb{R}^3$. Although imagining an sphere with identified antipodal points are much easier than imagining the set of all line passing through the origin, but it is still hard to imagine the final geometrical and global properties of such a set. So we need more simplification.

Proposition 2.14 Let $S^2/\sim$ be a unit sphere that its antipodal points are identified. This set is homeomorphic to the closed upper hemisphere where its antipodal points on the equator are identified.

Proof. The proof is given as the solution to Problem 2.19. □

Now, we can go further and simplify this space more.

Proposition 2.15 Let $H^2/\sim$ be the quotient space adopted from the closed upper hemisphere by identifying the antipodal points on its equator. Also, let $D^2/\sim$ be the quotient space adopted from the closed unit disk D^2 by identifying the antipodal points on its boundary circle. Then $H^2/\sim$ and $D^2/\sim$ are homeomorphic.

Proof. We need to find a homeomorphism between $H^2/\sim$ and $D^2/\sim$. Consider the projection map

$$f : H^2 \to D^2 \qquad (x, y, z) \mapsto (x, y).$$

Consider the following commutative diagram.

$$\begin{tikzcd} H^2 \arrow[r, "f"] \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2\circ f"] & D^2 \arrow[d, "\pi_2"] \\ H^2/\sim \arrow[r, "\bar{f}"'] & D^2/\sim \end{tikzcd}$$

The map $\pi_2 \circ f$ is continuous and assumes a constant value for all of points in its domain that are in a same equivalence class. Thus this induces a continuous map $\bar{f} : H^2/\sim \to D^2/\sim$. To find the inverse for this function, consider

$$g : D^2 \to S^2 \qquad (x, y) \mapsto (x, y, \sqrt{1 - (x^2 + y^2)}).$$

Consider the following commutative diagram.

$$\begin{tikzcd} D^2 \arrow[r, "g"] \arrow[d, "\pi_2"'] \arrow[dr, "\pi_1\circ g"] & H^2 \arrow[d, "\pi_1"] \\ D^2/\sim \arrow[r, "\bar{g}"'] & H^2/\sim \end{tikzcd}$$

The map $\pi_1 \circ g$ is continuous and assumes a constant value for all points in its domain that are in the same equivalence class. Thus it induces a map $\bar{g}$ that is continuous.
We now need to show that $\bar{f}$ and $\bar{g}$ are inverses of each other. To see this consider the map

$$\bar{f} \circ \bar{g} : H^2/\sim \to H^2/\sim .$$

We can write

$$\bar{f}(\bar{g}([x, y])) = \bar{f}([x, y, \sqrt{1 - (x^2 + y^2)}]) = [x, y].$$

With a similar argument for $\bar{g} \circ \bar{f}$ we can show that these two functions are inverses of each other, thus we could find the explicit homeomorphism. □

In summary, we have the following chain of homeomorphisms

$$\mathbb{R}P^2 \xrightarrow{\sim} S^2/\sim \xrightarrow{\sim} H^2/\sim \xrightarrow{\sim} D^2/\sim$$

2.10 Standard C^∞ Atlas on a Real Projective Space

2.11 Summary

Summary 2.1 — A useful thing!. Let N be a manifold of dimension n, and (U, φ) a local coordinate system containing $p \in N$. Then φ can be viewed as a vector valued real function $\varphi : U \to \mathbb{R}^n$. Let r^i be the standard coordinate functions of $\mathbb{R}^n$. Then the i-th of φ can be written as.

$$x^i = r^i \circ \varphi.$$

Summary 2.2 Let S be a manifold, and $\sim$ an equivalence relation. Then $S/\sim$ is

- Second countable if the equivalence relation is open.
- Hausdorff if the graph of the equivalence relation is *open* and its graph is closed in $S \times S$.

Summary 2.3 — Two ways to show that two quotient spaces are homeomorphic. In general, in order to show that two spaces are homeomorphic, we need to find an homeomorphism between the spaces (a continuous bijection with continuous inverse). In particular, if two spaces are quotient spaces, it is often easier to work with the original spaces (that are not necessarily homeomorphic). In this summary box I will talk about two important ways that we can do this.

1. **Direct method**. In this method we directly find the homeomorphism. Consider showing that the real projective space $\mathbb{R}P^n$ is homeomorphic to $S^n/\sim$ where the equivalence relation $\sim$ identifies the antipodal points in the unit sphere. The followings are the steps that we take

   - Find $f : \mathbb{R}^n - \{0\} \to S^n$ where $\pi_2 \circ f$ assumes a constant value for all of the points in the same equivalence relation defined on $\mathbb{R}^{n+1} - \{0\}$. This induces a map $\bar{f} : \mathbb{R}P^n \to S^n/\sim$. From Proposition 2.10 it follows that $\bar{f}$ is continuous iff $\pi^2 \circ f$ is continuous (consider the following commutative diagram).

$$\begin{tikzcd} \mathbb{R}^{n+1} - \{0\} \arrow[r, "f"] \arrow[d, "\pi_1"'] & S^n \arrow[d, "\pi_2"] \\ \mathbb{R}P^n \arrow[r, "\bar{f}"'] & S^n/\sim \end{tikzcd}$$

     Also note that since the domain of $\bar{f}$ is equivalence classes, we need to make sure that the value of the function does not depend on the specific representative that we choose for a particular equivalence class. I.e. we need to show that the function is well defined. (I think this is the same as the fact that this function assumes a constant value for all of the elements in the same equivalence class).

   - We need to find $g : S^n \to \mathbb{R}^n - \{0\}$ that induces a map $\bar{g} : S^n/\sim \to \mathbb{R}^n - \{0\}/\sim$. Again, using Proposition 2.10, $\bar{g}$ is continuous if and only if $\pi_1 \circ g$ is continuous. Similar to what we did for $\bar{f}$ we need to show that $\bar{g}$ is indeed well defined.
   - As the last step, we need to check if $\bar{f}$ and $\bar{g}$ are indeed inverses of each other, i.e. their composition leads to identity maps on the corresponding domains.

   Also look at the proof of Proposition 2.15 for a similar argument.

2. **Using compactness arguments**. Some times it is hard to find the homeomorphism directly. In this case we follow our approach as in Problem 2.18 and Problem 2.19, as well as the following important proposition

   Proposition 2.16 A bijection from a compact space to a Hausdorff space is a homeomorphism.

Summary 2.4 There are some points in the proposition above that I will highlight by this summary box. First, note that the composition of two function, possibly one or both of them discontinuous, can be a continuous function (see this). Thus in the summary box above, neither of f or g need to be continuous. The function that need to be continuous are $\pi_2 \circ f$ and $\pi_1 \circ g$.

Summary 2.5 Let S be a topological space and G be a topological group. Consider the right action of the group G on S via the map

$$\alpha : S \times G \to S, \qquad \alpha(s, g) = sg$$

If the right action if *continuous* (i.e. the map α is continuous) Then for $g \in G$ the map

$$\alpha_g : S \to S$$

is a *homeomorphism*. See Problem 2.23 for the proof.

Summary 2.6 — Showing a space is an smooth manifold. To show that a space is an smooth manifold, we just need to show that the space is

- a Hausdorff topological space that is also second countable,
- there exists any C^∞ atlas (not necessarily maximal).

Summary 2.7 — A confusing thing about the equivalence classes. Define $\sim$ on $\mathbb{R}$ as

$$x \sim y \qquad \text{iff} \qquad x - y = 2n\pi \quad \text{for some } n \in \mathbb{Z}.$$

Then consider the following sets in $\mathbb{R}/\sim$.

$$V_1 = \{[t] \mid t \in (-\pi, \pi)\}, \qquad V_2 = \{[t] \mid t \in (0, 2\pi)\}.$$

In words, what we really mean is that V_1 contains the equivalence classes whose representatives comes from the set $(-\pi, \pi)$. Similarly for V_2, we say that it contains the equivalence classes whose representatives comes from the set $(0, 2\pi)$. We define the following coordinate maps

$$\begin{aligned} \varphi_1 : V_1 &\to \mathbb{R} \qquad & \varphi_2 : V_2 &\to \mathbb{R} \\ [t] &\mapsto t & [t] &\mapsto t \end{aligned}$$

Note that although the functions φ_1 and φ_2 look similar, but they are different functions as they have different domains. These two functions both return the representative of the equivalence classes in their argument. Note that we said *the representative* and not *a representative*) because of the design of their domain. For instance, let's parse the following operation

$$\varphi_1([23.5\pi]) = -0.5\pi.$$

Since the domain of φ_1 is the equivalence classes whose representatives comes from the set $(-\pi, \pi)$, then we need to find a representative for $[23.5\pi]$ that lives in $(-\pi, \pi)$. I.e. $[23.5\pi] = [1.5\pi] = [-0.5\pi]$.

Summary 2.8 — Strategy to show an equivalence relation is open. Let S be a topological space and $\sim$ be an equivalence relation on S. This equivalence relation can also be given by a group action of G acting on S. I.e. we can write

$$x \sim y \qquad \text{iff} \qquad x = yg \quad \text{for some } g \in G.$$

Then to show that the equivalence relation $\sim$ is open, or equivalently, to show that the projection map $\pi : S \to S/G$ is an open map, then we need to show for $U \subset S$ open we have

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} Ug$$

is open. By Lemma 2.2, this set is open if and only if the group action is continuous. See Problem 2.26 for a detailed example.

Summary 2.9 — Grassmannian. Grassmannian $G(k,n)$ is a generalization of the real projective plane. $G(k,n)$ defined to be the set all of k-planes passing through the origin as the linear subspaces of $\mathbb{R}^n$.

The Grassmannian $G(k,n)$ is in one-to-one correspondence with $F(k,n)/GL(k,\mathbb{R})$, where $F(k,n)$ is the set of all $n \times k$ matrices with rank k.

As a concrete example, consider $G(2,4)$. The open coordinate sets are given as $U_{ij} = V_{ij}/GL(2,\mathbb{R})$, where
$$V_{ij} = \{A \in F(2,4) \mid A_{ij} \text{ is nonsingular}\}.$$

The corresponding coordinate maps are induced by the maps

$$\tilde{\varphi}_{ij} : V_{ij} \to \mathbb{R}^{2\times 2}.$$

where for instance $\tilde{\varphi}_{12}(A) = A_{34}A_{12}^{-1}$. See Problem 2.26 for a detailed discussion.

2.12 Solved Problems

■ **Problem 2.1 — A C^∞ atlas on a circle (From W. Tu).** construct a C^∞ atlas for the unit circle S^1.

Solution The unit circle C^1 can be described as a set of points $S^1 = \{e^{it} | t \in [0, 2\pi]\}$. Let U_1 and U_2 be two subsets of S^1 described as

$$U_1 = \{e^{it} | t \in (-\pi, \pi)\}, \qquad U_2 = \{e^{it} | t \in (0, 2\pi)\}.$$

Consider the functions $\varphi_\alpha : U_\alpha \to \mathbb{R}$ for $\alpha = 1, 2$ given by

$$\begin{aligned} \varphi_1(e^{it}) &= t, \qquad -\pi < t < \pi, \\ \varphi_2(e^{it}) &= t, \qquad 0 < t < 2\pi. \end{aligned}$$

These functions are in fact different branches of the complex logarithm function $1/i \log(z)$, thus homeomorphisms onto their respective images. Thus $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ is an atlas for S^1. To demonstrate the compatibility of these charts, we need to first calculate $U_1 \cap U_2$. This set has two connected components, i.e. $U_1 \cap U_2 = A \sqcup B$ where $\sqcup$ is used to demonstrate the disjoint union of A, B. Explicitly, we can write

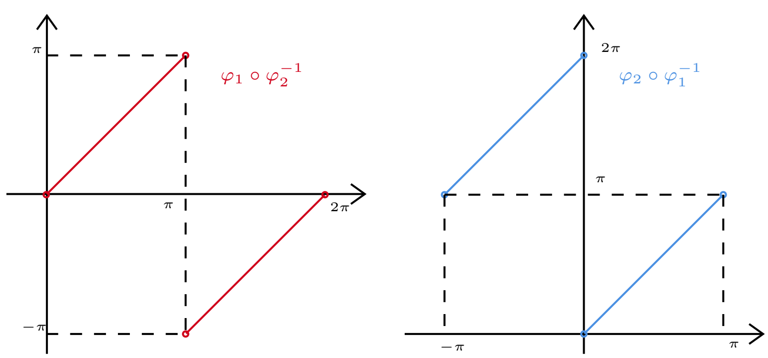
$$A = \{e^{it} | t \in (-\pi, 0)\}, \qquad B = \{e^{it} | t \in (0, \pi)\}.$$

First, we start with the function $\varphi_1 \circ \varphi_2^{-1} : \underbrace{\varphi_2(U_1 \cap U_2)}_{(0,\pi)\sqcup(\pi,2\pi)} \to \underbrace{\varphi_1(U_1 \cap U_2)}_{(-\pi,0)\sqcup(0,\pi)}$. For this function we have

$$(\varphi_1 \circ \varphi_2^{-1})(t) = \begin{cases} t & t \in (0, \pi), \\ t - 2\pi & t \in (\pi, 2\pi). \end{cases}$$

Similarly, for $\varphi_2 \circ \varphi_1^{-1} : \underbrace{\varphi_1(U_1 \cap U_2)}_{(-\pi,0)\sqcup(0,\pi)} \to \underbrace{\varphi_2(U_1 \cap U_2)}_{(0,\pi)\sqcup(\pi,2\pi)}$ we can write

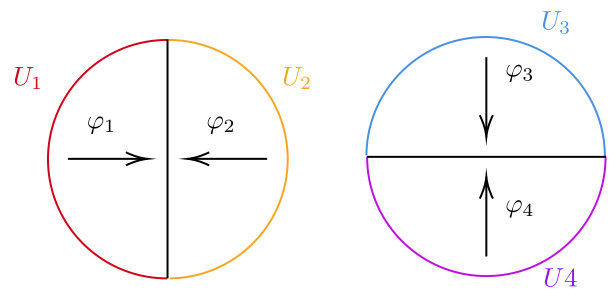
$$(\varphi_2 \circ \varphi_1^{-1})(t) = \begin{cases} t + 2\pi & t \in (0, \pi), \\ t & t \in (\pi, 2\pi). \end{cases}$$



Observation 2.12.1 I was thinking about my solution to the problem above, and I thought it is wrong, as I was thinking that the function φ_1 is not homeomorphism as it is not continuous. But the point that I was missing is that this function is indeed continuous on its domain and

the point of discontinuity (i.e. $x = \pi$) is not in the domain.

■ **Problem 2.2** Another C^∞ atlas on a circle In the previous problem, we constructed an atlas for a unit circle siting in the complex plane. In this problem we are going to construct a different atlas for a unit circle siting in the $x - y$ plane. The following diagram are the charts for this unit circle. Write these charts explicitly and check if they are pairwise compatible.



Solution The explicit formulas for the charts depicted above is as following

$$(U_1, \varphi_1 : U_1 \to \mathbb{R}), (U_2, \varphi_2 : U_2 \to \mathbb{R}), (U_3, \varphi_3 : U_3 \to \mathbb{R}), (U_4, \varphi_4 : U_4 \to \mathbb{R}),$$

where we have

$$\varphi_1(x, y) = y, \qquad \varphi_2(x, y) = y, \qquad \varphi_3(x, y) = x, \qquad \varphi_4(x, y) = x.$$

Note that although some of the functions above might have a same formula, but they are different functions as they have different domains. To show that these functions are pairwise compatible, we start by noting that since $U_1 \cap U_2 = \varnothing$, thus (U_1, φ_1) and (U_2, φ_2) are compatible. With the same reasoning, the charts (U_3, φ_3) and (U_4, φ_4) are compatible. Now, we want to show that (U_1, φ_1) is compatible with (U_3, φ_3). We need to show that

$$\varphi_1 \circ \varphi_3^{-1} : \underbrace{\varphi_3(U_1 \cap U_3)}_{(-1,0)} \to \underbrace{\varphi_1(U_1 \cap U_3)}_{(0,1)} \quad \text{and} \quad \varphi_3 \circ \varphi_1^{-1} : \underbrace{\varphi_1(U_1 \cap U_3)}_{(0,1)} \to \underbrace{\varphi_3(U_1 \cap U_3)}_{(-1,0)}$$

are C^∞. To write them explicitly, we have

$$(\varphi_1 \circ \varphi_3^{-1})(x) = \varphi_1(x, \sqrt{1 - x^2}) = \sqrt{1 - x^2}.$$

Also

$$(\varphi_3 \circ \varphi_1^{-1})(x) = \varphi_3(-\sqrt{1 - x^2}, x) = -\sqrt{1 - x^2}.$$

We can see that both of these functions are C^∞ in their domain. Now, for the charts (U_1, φ_1) and (U_4, φ_4) need to show

$$\varphi_1 \circ \varphi_4^{-1} : \underbrace{\varphi_4(U_1 \cap U_4)}_{(-1,0)} \to \underbrace{\varphi_1(U_1 \cap U_4)}_{(-1,0)} \qquad \text{and} \qquad \varphi_4 \circ \varphi_1^{-1} : \underbrace{\varphi_1(U_1 \cap U_4)}_{(-1,0)} \to \underbrace{\varphi_4(U_1 \cap U_4)}_{(-1,0)}$$

are C^∞ in their domain. Explicitly, we have

$$(\varphi_1 \circ \varphi_4^{-1})(x) = -\sqrt{1 - x^2}, \qquad (\varphi_4 \circ \varphi_1^{-1})(x) = -\sqrt{1 - x^2}.$$

With the same strategy, we can show that this collection of charts indeed makes a C^∞ atlas for the unit circle in $x - y$ plane.

■ **Problem 2.3 — The real line with two origins (from W. Tu).** Let A and B be two points not on the real line $\mathbb{R}$. Consider the set $S = (\mathbb{R} - \{0\}) \cup \{A, B\}$. For nay two positive real numbers c, d, define

$$I_A(-c, d) = (-c, 0) \cup \{A\} \cap (0, d)$$

and similarly for $I_B(-c, d)$, with B instead of A. Define a topology on S as follows: On $(\mathbb{R} - \{0\})$, use the subspace topology inherited from $\mathbb{R}$, with open intervals as a basis. A basis of neighborhoods at A is the set $\{I_A(-c, d) \mid c, d > 0\}$; similarly, a basis of neighborhoods at B is $\{I_B(-c, d) \mid c, d > 0\}$.

(a) Prove that the map $h : I_A(-c, d) \to (-c, d)$ defined by

$$\begin{aligned} h(x) &= x \qquad \text{for } x \in (-c, 0) \cup (0, d), \\ h(A) &= 0 \end{aligned}$$

is a homeomorphism.

(b) Show that S is locally Euclidean and second countable, but not Hausdorff.

Solution (a) We need to show that h is one-to-one, onto, and continuous, with continuous inverse. To show being one-to-one, let $x, y \in I_A(-c, d)$ such that $x \neq y$ and possibly one of them equal to A. If none is equal to A, then h is the identity map which is one-to-one. However, if one of them is equal to A, let's say $x = A$, then $h(x) = 0$ where $h(y) \in (-c, d) \cup (0, d)$, thus $h(y) \neq 0$. This proves that h is indeed one to one. To show that the function is onto, let $z \in (-c, d)$. Then if $z = 0$ we have $h(A) = z$, and if $z \neq 0$, we have $h(z) = z$. Thus the function h is a bijection.

As the second step, we need to show that this function is continuous with continuous inverse. From definition of continuoity, we just need to show that both h and h^{-1} maps opens to opens (because for continuous function the pre-image of every open set is an open set; and to show that the inverse of the function is also continuous we need to show that the image of every open is also open). Let $U = (a, b) \subset I_A(-c, d)$ be an open set in the topology of S. If $A \notin (a, b)$, then the image of this set under h is (a, b) which is open in $(-c, d)$. But if $A \in (a, b)$, then the image of this set under the map h is $(a, 0) \cup (0, b) \cup \{0\} = (a, b)$, which is also open. Thus h^{-1} is continuous. To show the continuity of h^{-1}, let $(a, b) \subset (c, d)$. If $0 \notin (a, b)$, then pre-image of this set under the map f is (a, b) that is open in S. However if $0 \in (a, b)$, then the pre-image of this set under f is the set $(-a, 0) \cup \{A\} \cup (0, b)$ which is indeed open in S as we can construct this with the basis if opens at A.

(b) First, we show that S is locally Euclidean. To show this let $p \in S$. If $p \neq A$ and $p \neq B$, then we choose an open set $U = (a, b)$ containing p that does not contain neither of A and B. Then U is homeomorphic to (a, b) with the identity map. However if $p = A$, we choose any open $U = (a, b)$ containing p. Then $(U = (a, b), I_A(a, b))$ is a local chart. For the case where $p = B$, we can find a suitable chart with the same reasoning as for A. Thus we have shown that S is locally Euclidean.

To show that S is second countable, let $\mathcal{B}$ be a basis for $\mathbb{R} - \{0\}$. Since $\mathbb{R}$ is second countable, then $\mathcal{B}$ is countable. Let $\mathbb{B} = \mathcal{B} \cup I_A(-c, d) \cup I_B(-c, d)$ is a countable basis for S for some $c, d > 0$. This shows that S is also second countable.

However, this space is not Hausdorff. To show this consider the points A, B. We can not find any two open U, V such that $A \in U$ and $B \in V$ and we have $U \cap V = \varnothing$. Or equivalently, for all open sets U, V such that $A \in U$ and $B \in V$ we have $U \cap V \neq \varnothing$. That is because from the basis of the open neighborhoods at A, B we have $U = I_A(-c, d)$ and $V = I_B(-e, f)$ for some $c, d, e, f > 0$. It is clear that $U \cap V \neq 0$, thus S is not Hausdorff.

■ **Problem 2.4 — A sphere with a hair (from W. Tu) .** A fundamental theorem of topology, the theorem on invariance of dimension, states that if two nonempty open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are homeomorphic, then $n = m$. Prove that the sphere with a hair in $\mathbb{R}^3$ is not locally Euclidean at q (the point that the hair attaches to the sphere). Hence it cannot be a topological manifold.

Solution We will proceed with the proof by contradiction. Assume that the ball with hair at point q is homeomorphic to some open set in $\mathbb{R}^n$. Then q has an open neighborhood U homeomorphic to an open ball $B := \mathbb{B}(0, \epsilon) \in \mathbb{R}^n$, with q mapping to 0. We can restrict this homeomorphism to a homeomorphism $U - \{q\} \to B - \{0\}$. Now $B - \{0\}$ is either connected if $n \geq 2$ or has two connected components if $n = 1$, where $U - \{q\}$ has two connected components where one component is a 1 dimensional manifold (the hair) and the other component is a 2 dimensional manifold (the sphere). The only case where $B - \{0\}$ has two connected components is when $n = 1$. But because of the invariance of dimension principle, the sphere (2 dimensional manifold) can not be homeomorphic to a one dimensional manifold. Thus there is no such a homeomorphism between the hairy ball and $\mathbb{R}^n$ and the hairy ball is not locally Euclidean in q.

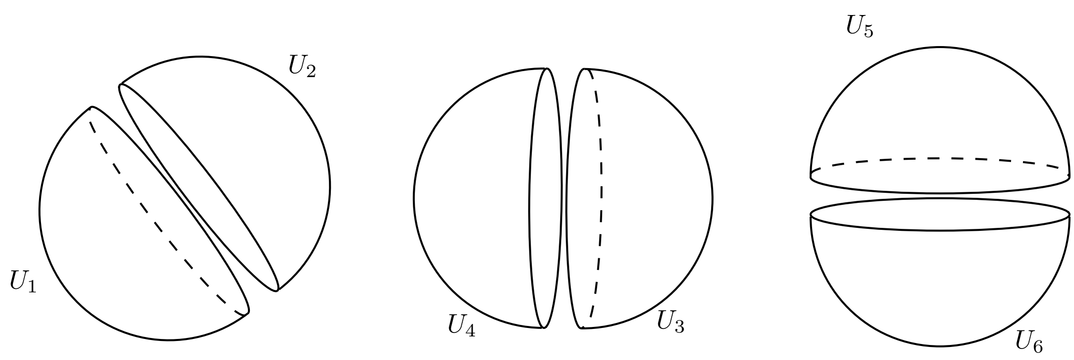
■ **Problem 2.5 — Charts on a sphere (from W. Tu).** Let S^2 be the unit sphere, i.e.

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1\}$$

in $\mathbb{R}^3$. Define in S^2 the six charts corresponding to the six hemispheres (from the front, rear, right, left, upper, and lower hemispheres) as in the figure.

$$\begin{aligned}
U_1 &= \{(x, y, z) \in S^2 \mid x > 0\}, & \varphi_1(x, y, z) &= (y, z), \\
U_2 &= \{(x, y, z) \in S^2 \mid x < 0\}, & \varphi_2(x, y, z) &= (y, z), \\
U_3 &= \{(x, y, z) \in S^2 \mid y > 0\}, & \varphi_3(x, y, z) &= (x, z), \\
U_4 &= \{(x, y, z) \in S^2 \mid y < 0\}, & \varphi_4(x, y, z) &= (x, z), \\
U_5 &= \{(x, y, z) \in S^2 \mid z > 0\}, & \varphi_5(x, y, z) &= (x, y), \\
U_6 &= \{(x, y, z) \in S^2 \mid z < 0\}, & \varphi_6(x, y, z) &= (x, y).
\end{aligned}$$

Note that although some of the functions above might look similar (like φ_3 and φ_4) but they are in fact different functions as they have different domains. Show that $\varphi_1 \circ \varphi_4^{-1}, \varphi_4 \circ \varphi_1^{-1}$ is C^∞ on $\varphi_4(U_1 \cap U_4), \varphi_1(U_1 \cap U_4)$ respectively. Do the same same analysis for $\varphi_6 \circ \varphi_1^{-1}$.



Solution For the set $U_1 \cap U_4$ we have

$$U_1 \cap U_4 = \{(x, y, z) \in S^2 \ : \ x > 0 \text{ and } y < 0\}.$$

Thus we will have

$$\varphi_4(U_1 \cap U_4) = \{(x, z) \in \mathbb{R}^2 \ : \ x^2 + z^2 \leq 1, \ x > 0\}.$$

Thus we can write

$$(\varphi_1 \circ \varphi_4^{-1})(\langle x, z\rangle) = \varphi_1(\langle x, -\sqrt{1-(x^2+y^2)}, z\rangle) = \langle -\sqrt{1-(x^2+y^2)}, z\rangle$$

This is indeed a C^∞ vector valued function, since each component is a C^∞ function. Now to evaluate $\varphi_4 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_4) \to \varphi_4(U_1 \cap U_4)$ we need to first evaluate the set $\varphi_1(U_1 \cap U_4)$. For this set we have

$$\varphi_1(U_1 \cap U_4) = \{(z, y) \in \mathbb{R}^2 \ : \ z^2 + y^2 \leq 1, y < 0\}.$$

Then we can write

$$(\varphi_4 \circ \varphi_1^{-1})(\langle y, z\rangle) = \varphi_4(\langle \sqrt{1-(y^2+z^2)}, y, z\rangle) = \langle \sqrt{1-(y^2+z^2)}, z\rangle.$$

This is indeed a C^∞ vector valued function.

To evaluate the function $\varphi_6 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_6) \to \varphi_6(U_1 \cap U_6)$, we first need to determine the domain of this function. First observe that

$$U_1 \cap U_6 = \{(x, y, z) \in S^2 \ : \ x > 0 \text{ and } y < 0\},$$

which is the same as $U_1 \cap U_4$. Then for the domain of the function of interest we can write

$$\varphi_1(U_1 \cap U_6) = \{(y, z) \in \mathbb{R}^2 : \ z^2 + y^2 \leq 1 \text{ and } y < 0\},$$

$$(\varphi_6 \circ \varphi_1^{-1})(\langle y, z\rangle) = \varphi_6(\langle \sqrt{1-(y^2+z^2)}, y, z\rangle) = \langle \sqrt{1-(y^2+z^2)}, y\rangle.$$

■ **Problem 2.6 — Existence of a coordinate neighborhood (from W. Tu).** Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be the maximal atlas on manifold M. For any open set U in M and a point $p \in U$, prove the existence of a coordinate open set U_α such that $p \in U_\alpha \subset U$.

Solution Since $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ is an atlas, then for $p \in U$ given as above, we can find some $\alpha_1 \in I$ such that $p \in U_{\alpha_1}$. Consider the open set $W = U_{\alpha_1} \cap U$. The chart $(W, \varphi_{\alpha_1}|_W)$ is in the atlas (since it is maximal), i.e. $\exists \alpha \in I$ such that $(U_\alpha, \varphi_\alpha) = (W, \varphi_{\alpha_1}|_W)$. This completes the proof.

■ **Problem 2.7 — An atlas for a product manifold (from W. Tu).** Prove the following proposition.

Proposition 2.17 If $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ are C^∞ atlases for the manifold M and N of dimensions m and n, respectively, then the collection

$$\{(U_\alpha \times V_i, \varphi_\alpha \times \psi_i)\}$$

where

$$\varphi_\alpha \times \psi_i \ : \ U_\alpha \times V_i \to \mathbb{R}^m \times \mathbb{R}^n$$

of charts is a C^∞ atlas on $M \times N$. Therefore, $M \times N$ is a C^∞ manifold of dimension $m + n$.

Solution In order to show that the collection $\mathfrak{U} = \{(U_\alpha \times V_i, \varphi_\alpha \times \psi_i)\}$ is an atlas for $M \times N$, we need to show that charts are pairwise compatible as well as the set covering the whole space. To show that the chart covers the whole space $M \times N$, let $(p_1, p_2) \in M \times N$. Then $p_1 \in M$ and $p_2 \in N$. Then there are two coordinate open sets such that $p_1 \in U_{\alpha_1}$ and $p_2 \in V_{i_1}$. Thus the coordinate open set $U_{\alpha_1} \times V_{i_1}$ contains the point (p_1, p_2).

To show that two any two charts in the collection $\mathfrak{U}$ are compatible, let $(U_{\alpha_1} \times V_{i_1}, \varphi_{\alpha_1} \times \psi_{i_1})$ and $(U_{\alpha_2} \times V_{i_2}, \varphi_{\alpha_2} \times \psi_{i_2})$ be two charts. We claim that the corresponding coordinate maps are C^∞. This follows directly from the fact the each component of the map these coordinate maps are C^∞.

■ **Problem 2.8 — Smoothness of a projection map (from W. Tu).** Let M and N be manifolds and $\pi : M \times N \to M$, $\pi(p,q) = p$ the projection to the first factor. Prove that π is a C^∞ map.

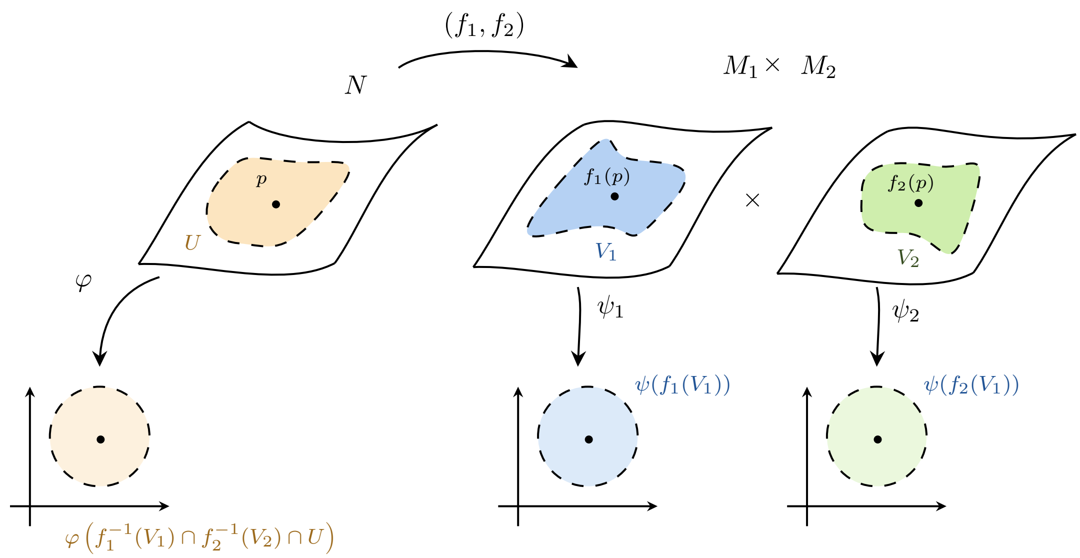
Solution Let $p \in M$ and $q \in N$, thus $(p,q) \in M \times N$. Let φ and ψ be two coordinate maps such that $\varphi(p) = x$ and $\psi(q) = y$. Thus we can write $(\varphi,\psi)(p,q) = (x,y)$. Consider the function

$$(\varphi \circ \pi \circ (\varphi \times \psi)^{-1})(x,y) = \varphi(\pi(p,q)) = \varphi(p) = x.$$

The function above is a $\mathbb{C}^\infty$ map from $\mathbb{R}^{n+m}$ to $\mathbb{R}^n$ (assuming M, N are m and n dimensional manifolds). This proves that the projection map π is smooth.

■ **Problem 2.9 — Smoothness of a map tp a Cartesian product (from W. Tu).** Let M_1, M_2, and N be manifolds of dimensions m_1, m_2 and n respectively. Prove that a map $(f_1, f_2) : N \to M_1 \times M_2$ is smooth if and only if $f_i : N \to M$ for $i = 1,2$ are both smooth.

Solution Consider the following picture.



For the first direction, we will show that smoothness of $(f_1, f_2) : N \to M_1 \times M_2$ implies the smoothness of $f_1 : N \to M_1$ and $f_2 : N \to M_2$. As depicted in the picture above, let φ be a coordinate map for N and ψ_1, ψ_2 be coordinate maps for M_1, M_2 respectively. Since (f_1, f_2) is smooth, then $(\psi_1 \times \psi_2) \circ (f_1, f_2)$ is smooth. Let $p \in N$, then

$$((\psi_1 \times \psi_2) \circ (f_1, f_2) \circ \varphi^{-1})(\varphi(p)) = ((\psi_1 \times \psi_2) \circ (f_1, f_2))(p) = (\psi_1 \times \psi_2)(f_1(p), f_2(p)) = (\psi_1(f_1(p)), \psi_2(f_2(p))).$$

This is a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^{m_1+m_2}$. Thus the components of the function are also smooth.

For the converse, we need to show that the smoothness of f_1 and f_2 implies the smoothness of (f_1, f_2). Similar to the argument above, the smoothness of (f_1, f_2) follows immediately from the smoothness of the components.

■ **Problem 2.10 — Smooth functions on unit circle (W. Tu).** We have studied before that the unit circle S^1 defined by $x^2 + y^2 = 1$ in $\mathbb{R}^2$ is a C^∞ manifold. Prove that a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ defined on $\mathbb{R}^2$ restricts to a C^∞ function on S^1.

Solution Consider the following inclusion map

$$i : S^1 \to \mathbb{R}^2$$

defined as $i(p) = (x(p), y(p))$, where x, y are the standard coordinate functions. The restriction of f to manifold will be

$$f|_{S^1} = f \circ i.$$

To show that $f|_{S^1}$ is smooth, we just need to show that the inclusion map is smooth. To show this we need to show that the components of this vector valued function is smooth. We start by showing that x is smooth. We use the same charts as in Problem 2.2. Let $p \in S^1$. If $p \in U_3 \cap U_4$ then we have

$$\mathbb{1}_{(0,1)} \circ x \circ \varphi_3^{-1} = \mathbb{1}_{(0,1)} : (-1,1) \to \mathbb{R}^2, \qquad \mathbb{1}_{(0,1)} \circ x \circ \varphi_4^{-1} = \mathbb{1}_{(0,1)} : (-1,1) \to \mathbb{R}^2,$$

which are both identity maps, thus smooth. So x is smooth on $U_3 \cap U_4$. For U_1 we have

$$\mathbb{1}_{(0,1)} \circ x \circ \varphi_1^{-1}(x) = -\sqrt{1-x^2}, \qquad \text{for } x \in \varphi_1(U_1) = (-1,1)$$

thus x is smooth on U_1 as well. For U_2 we have

$$\mathbb{1}_{(0,1)} \circ x \circ \varphi_2^{-1}(x) = \sqrt{1-x^2}, \qquad \text{for } x \in \varphi_2(U_2) = (-1,1)$$

thus x is smooth on U_2 as well. We can use a similar strategy to show that the coordinate function y is also smooth.

■ **Problem 2.11** The general linear group $\mathrm{GL}(n, \mathbb{R})$ is the set of all real valued matrices with non-zero determinant under matrix multiplication. In other words

$$\mathrm{GL}(n, \mathbb{R}) = \{A = \left[a_{ij}\right] \in \mathbb{R}^{n \times n} \mid \det(A) \neq 0\}.$$

We can see this as an open subset of $\mathbb{R}^{n \times n}$, thus it is a manifold. Show that this forms a Lie group.

Solution Let A, B be two matrices in the manifold. Then the i, j element of AB is

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

which is a polynomial in the coordinates of A and B, thus it is smooth. To show that the inverse is also smooth, for any function A in the manifold we have

$$A^{-1} = \frac{1}{\det(A)} \cdot (-1)^{i+j}((i,j)\text{- minor of } A).$$

The (i, j)-minor of matrix A is the determinant of the sub matrix by deleting the i-th row and jth column, which is again a polynomial in the coordinates of A and B, thus smooth (given that $\det(A) \neq 0$).

■ **Problem 2.12 — Jacobian matrix os a transition map (form W. Tu).** Let $(U, \varphi) = (U, x^1, \cdots, x^n)$ and $(V, \psi) = (V, y^1, \cdots, y^n)$ be overlapping charts on a manifold M. The transition map $\psi \circ \varphi^{-1}$: $\varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$. Show that its Jacobian matrix $J(\psi \circ \varphi^{-1})$ at $\varphi(p)$ is the matrix $[\partial y^i / \partial x^j]$ of partial derivatives at p.

Solution From the definition of the Jacobian matrix we can write

$$J(\psi \circ \varphi^{-1}) = \frac{\partial(\psi \circ \varphi^{-1})^i}{\partial r^j} = \frac{\partial(r^i \circ \psi \circ \varphi^{-1})}{\partial r^j} = \frac{\partial(y^i \circ \varphi^{-1})}{\partial r^j} = \frac{\partial y^i}{\partial x^j}.$$

Observation 2.12.2 — Some mnemonics for the Jacobian matrix. Here I introduce a symbolic mnemonic of remembering the form of the Jacobian matrix of a smooth map between manifolds. Let $F : N \to M$ a smooth map between manifolds. Since this function is from N to M, then the Jacobian matrix will be of the form

$$J(F) = [\partial F^i/\partial x^j]$$

where x^j a local coordinate in N and $F^i = y^i \circ F$ where y^i is a local coordinate in M. What we mean by local coordinates here is that for a point p on the manifold for which we want to calculate the Jacobian matrix, there are charts $(U, x^1 \cdots, x^n)$ and $(V, y^1, \cdots, y^n)$ on N and M respectively such that $p \in U$ and $F(U) \subset V$.

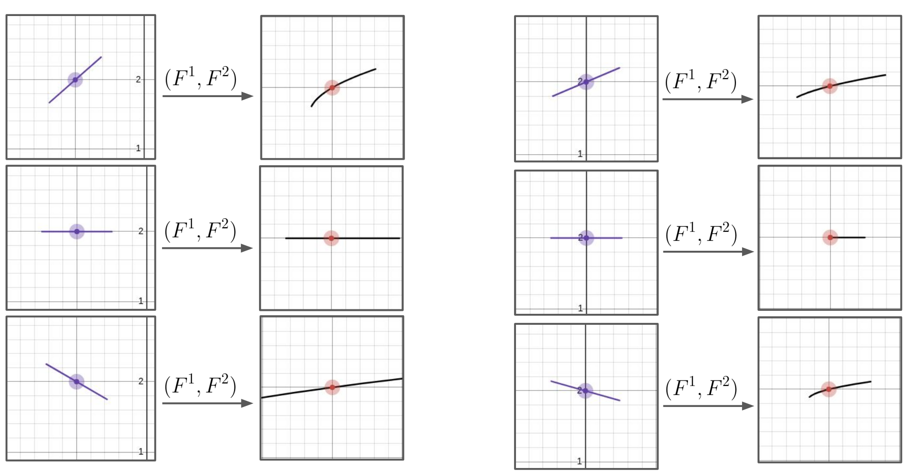
As another example, in the question above, since the function $\psi \circ \varphi^{-1}$ is defined from $\mathbb{R}^n \to \mathbb{R}^n$, then the Jacobian of this function starts with coordinates of $\mathbb{R}^n$, i.e. r^i.

■ **Problem 2.13 — From W. Tu.** Find all points in $\mathbb{R}^2$ in a neighborhood of which the functions given by $x^2 + y^2 - 1$ and y can serve as a local coordinate system.

Solution Define $F^1(x, y) = x^2 + y^2 - 1$ and $F^2(x, y) = y$. Then the pair (F^1, F^2) is locally invertible, thus can serve as a local diffeomorphism to $\mathbb{R}^2$, thus a coordinate map if the Jacobian determinant $[\partial F^i/\partial x^i]$ is not zero. I.e.

$$\det \begin{pmatrix} 2x & 2y \\ 0 & 1 \end{pmatrix} = 2x \neq 0 \implies x \neq 0.$$

Thus the function $F = (F^1, F^2)$ can act as a local coordinate map everywhere except for on the points on the y axis. You can see why this happens in the plots below. As you can see, any path that is not crossing the y axis with zero vertical velocity is diffeormorphically mapped to an smooth curve. But when the curve passes through the y axis with a zero vertical velocity, then the curve folds on itself when mapped by F, thus F fails be locally invertible and thus fails to be a coordinate map. Thus for any point that is not on the y axis, we can find an open set small enough that does not overlap with the y axis. But there is no such an open ball for the points on the y axis. You can try the online plotting tool that I have configured to generate the following plots here.



■ **Problem 2.14 — Differentiable structure on $\mathbb{R}$.** Let $\mathbb{R}$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \varphi = \mathbb{1} : \mathbb{R} \to \mathbb{R})$, and let $\mathbb{R}'$ be the real line with the differentiable structure given by the maxima atlas of the chart $(\mathbb{R}, \psi : \mathbb{R} \to \mathbb{R})$, where $\psi(x) = x^{1/3}$.

(a) Show that these two differentiable structures are distinct.

(b) Show that there is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$. (*Hint:* The identity map $\mathbb{R} \to \mathbb{R}$ is not the desired diffeomorphism; in fact, this map is not smooth).

Solution (a) Let M_1 be the atlas containing $(\mathbb{R}, \varphi)$ and M_2 the atlas containing $(\mathbb{R}, \psi)$. Assume $M_1 = M_2$. Then $(\mathbb{R}, \varphi)$ should be compatible with $(\mathbb{R}, \psi)$, i.e. the functions

$$\psi \circ \varphi^{-1} = \mathbb{R} \to \mathbb{R}, \qquad \varphi \circ \psi^{-1} : \mathbb{R} \to \mathbb{R},$$

are smooth. This is not true since $(\psi \circ \varphi^{-1})(x) = x^{1/3}$ that is not differentiable at $x = 0$. So $M_1 \neq M_2$ and these two atlases are distinct.

(b) For a more clear demonstration, the manifold with atlas generated by $(\mathbb{R}, \varphi = \mathbb{1} : \mathbb{R} \to \mathbb{R})$ the manifold R, and call the other manifold the manifold R'. Define the map F between manifolds

$$F : R \to R'$$

as $F(p) = p^3$ for $p \in R$. The inverse of this map will be $F^{-1}(p) = p^{1/3}$. To show this map is a diffeomorphism, we need to show that the following

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(F(U)), \qquad \varphi \circ F^{-1} \circ \varphi^{-1} : \varphi(V) \to \varphi(F^{-1}(V)),$$

are smooth. In equations above, $(U, \varphi : x \mapsto x)$ is some coordinate system (i.e. chart) on R whereas $(V, \psi : x \mapsto x^1/3)$ is a coordinate system on R'. Let $\varphi(x) = x \in \varphi(U \cap V)$. Then

$$(\psi \circ F \circ \varphi^{-1})(x) = \psi(F(x)) = \psi(x^3) = x,$$

which is the identity map and is smooth. Furthermore

$$(\varphi \circ F^{-1} \circ \varphi^{-1})(x) = \varphi(F^{-1}(x^3)) = \psi(x) = x,$$

which is again the identity map and is smooth. Thus the map F is a diffeomorphism between the manifolds.

■ **Problem 2.15 — The smoothness of an inclusion map (From L. Tu).** Let M and N be manifolds and let q_0 be a point in N. Prove that the inclusion map $i_{q_0} : M \to M \times N$, $i_{q_0}(p) = (p, q_0)$, is C^∞.

Solution Let $p \in M$ and let (U, ψ), (V, φ) coordinate charts for M and N respectively where $x \in U$ and $q_0 \in V$. Then the chart $(U \times V, \psi \times \varphi)$ will be a coordinate chart for $M \times N$ where $(p, q_0) \in U \times V$. To show the smoothness of i_{q_0} we need to show that the following

$$(\psi \times \varphi) \circ i_{q_0} \circ \psi^{-1} : \psi(U) \to (\psi \times \varphi)(i_{q_0}(U)).$$

Let $\psi(p) = x$. Then

$$((\psi \times \varphi) \circ i_{q_0} \circ \psi^{-1})(x) = ((\psi \times \varphi) \circ i_{q_0})(p) = (\psi \times \varphi)(p, q_0) = (x, \varphi(q_0)).$$

Thus $(\psi \times \varphi) \circ i_{q_0} \circ \psi^{-1}$ maps $x \mapsto (x, \varphi(q_0))$ where $\varphi(q_0) \in \mathbb{R}^n$ a constant. This is smooth since each components is smooth.

■ **Problem 2.16 — Group of automorphisms of a vector space (from L. Tu).** Let V be a finite-dimensional vector space over $\mathbb{R}$, and $GL(V)$ the group of all linear automorphisms of V. Relative to an ordered basis $e = (e_1, \cdots, e_n)$ for V, a linear automorphism $L \in GL(V)$ is represented by a matrix $\left[a^i_j\right]$ defined by

$$L(e_j) = \sum_i a^i_j e_i.$$

The map

$$\varphi_e : GL(V) \to GL(n, \mathbb{R}), \qquad L \mapsto \left[a^i_j\right],$$

is a bijection with an open subset of $\mathbb{R}^{n\times n}$ that makes $GL(V)$ into a smooth manifolds, which we denote temporarily by $GL(V)_e$. If $GL(V)_u$ is the manifolds structure induces from another ordered basis $u = (u_1, \cdots, u_n)$ for V, show that $GL(V)_e$ is the same as $GL(V)_u$.

Solution To show that these manifolds are the same, we will show that the underlying set of these manifolds are the same. Thus we need to show $L \in GL(V)_e \implies L \in GL(V)_u$ and $L \in GL(V)_u \implies L \in GL(V)_e$. We start with the first implication. Let $L \in GL(V)_e$. Then we can write

$$\begin{bmatrix} L(e_1) \\ \vdots \\ L(e_n) \end{bmatrix} = A^T \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

For some matrix $A = [a^i_j]$. Thus $\varphi_e(L) = A \in GL(n, \mathbb{R})$. Let $u = (u_1, \cdots, u_n)$ be another basis for V. By the change of basis matrix (which is non-singular and invertible) we can write

$$\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = B^T \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

for some invertible matrix $B = [b^l_k]$. Since B is $n \times n$ invertible, thus $B \in GL(n, \mathbb{R})$. From the linearity of L we have

$$\begin{bmatrix} L(e_1) \\ \vdots \\ L(e_n) \end{bmatrix} = B^T \begin{bmatrix} L(u_1) \\ \vdots \\ L(u_n) \end{bmatrix}, \qquad \begin{bmatrix} L(u_1) \\ \vdots \\ L(u_n) \end{bmatrix} = (B^T)^{-1} \begin{bmatrix} L(e_1) \\ \vdots \\ L(e_n) \end{bmatrix}.$$

Thus we can write

$$\begin{bmatrix} L(u_1) \\ \vdots \\ L(u_n) \end{bmatrix} = (B^T)^{-1} A^T B^T \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

Thus we can write

$$(L(u_1) \ \cdots \ L(u_n)) = BAB^{-1}(u_1 \ \cdots \ u_n)$$

Thus we can write

$$\varphi_u(L) = BAB^{-1} = B\varphi_e(L)B^{-1} \in GL(n, \mathbb{R}),$$

where the equality is true since $B \in GL(n, \mathbb{R})$ and $\varphi_e(L) \in GL(n, \mathbb{R})$. Thus $\varphi_u(L) \in GL(n, \mathbb{R})$. This implies $L \in GL(V)_u$. To show the second implication, we can use the similar idea as above.

■ **Problem 2.17 — Local coordinate systems (from L. Tu).** Find all points in $\mathbb{R}^3$ in a neighborhood of which the functions $x, x^2 + y^2 + z^2 - 1$, and z can serve as a local coordinate system.

Solution Define

$$F^1(x,y,z) = x, \qquad F^2(x,y,z) = x^2 + y^2 + z^2 - 1, \qquad F^3(x,y,z) = z.$$

Then the function $F = (F^1, F^2, F^3)$ can be a local coordinate map in a local coordinate system if its Jacobian determinant is nonzero. Thus we need to have

$$\det \begin{pmatrix} \partial F^1/\partial x & \partial F^1/\partial y & \partial F^1/\partial z \\ \partial F^2/\partial x & \partial F^2/\partial y & \partial F^2/\partial z \\ \partial F^3/\partial x & \partial F^3/\partial y & \partial F^3/\partial z \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 \\ 2x & 2y & 2z \\ 0 & 0 & 1 \end{pmatrix} = 2y \neq 0.$$

Thus $F = (F_1, F_2, F_3)$ is locally invertible, thus can be a local diffeomorphism to $\mathbb{R}^3$, thus it can be a coordinate char for all points on the manifold where $y \neq 0$.

■ **Problem 2.18** Let I be the unit closed interval $I = [0,1]$ and $I/\sim$ the quotient space obtained from I by identifying the two points $\{0,1\}$ to a point. Denote by S^1 the unit circle in the complex plane. Show that I is homeomorphic to S^1.

Solution As we have discussed in the summary box Summary 2.3, we will use the second method to show that an isomorphism exists between these two sets. Consider the function $f : I \to \mathbb{C}$ given by $f(t) = e^{2i\pi t}$. Thus function assumes the same value for the points $\{0,1\}$ in its domain that are in the same equivalence class as well. Thus this induces a map $\bar{f} : I/\sim \to S^1$. Consider the following commutative diagram.

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d, "\pi"'] & S^1 \\ I/\sim \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Since f is continues, then by Proposition 2.10 $\bar{f}$ is also continuous. Because

- $\bar{f}$ is continuous (from our reasoning above).
- $\bar{f}$ is a bijection (note that f is not a bijection but $\bar{f}$ is a bijection).
- $I/\sim$ is compact (since it is a continuous image of a compact set I).
- S^1 is Hausdorff.

Then by the proposition embedded in Summary 2.3 we conclude that $\bar{f}$ is indeed a homeomorphism.

Observation 2.12.3 In the problem above, one might ask why we did not find the homeomorphism explicitly as we usually do in other problems. The answer is that we can still find the explicit homeomorphism but in doing so, we will face some difficulties (not in all of the problems though). For instance, lets try to find a continuous map $\bar{g} : S^1 \to I/\sim$ that is the inverse of $\bar{f}$. We might arrange the sets and maps in the following form to be able to use the theorems we have had in this chapter (like Proposition 2.10). Consider the following diagram.

$$\begin{tikzcd} S^1 \arrow[r, "g"] \arrow[d, "id"'] \arrow[dr, "\pi\circ g" description] & I \arrow[d, "\pi"] \\ S^1 \arrow[r, "\bar{g}"'] & I/\sim \end{tikzcd}$$

The appropriate function to choose is $g : S^1 \to I$ which is the complex logarithm function with its branch cut on the positive real axis. But this function is clearly discontinuous on the positive real axis. Although this does not meant that the map $\pi \circ g$ is not continuous, but to show that this map is continuous we need to use the basic definitions to prove it and we can not use any higher level theorems (at least I am not aware of any). Then when we prove that $\pi \circ g$ is continuous, since it assumes a constant value on all of its points in the domain that are in the same equivalence class (the equivalence class here is the trivial equality relation, where each point has relation with only itself), then it induces a map $\bar{g}$ where its continuoity follows immediately from the continuoity of $\pi \circ g$.

If you be sharp enough, you will understand that the last part of the reasoning is really redundant (i.e. concluding the continuoity of $\bar{g}$ from $\pi \circ g$). That is simply because $\bar{g}$ is exactly the same as $\pi \circ g$ (can you see this from the diagram?)

■ **Problem 2.19** Let S^2 be the unit sphere and $S^2/\sim$ the quotient set obtained from S^2 by identifying the antipodal points to a point. Also, let H^2 be the closed upper hemisphere and $H^2/\sim$ be the quotient space obtained from H^2 by identifying the *antipodal points on its equator to a point*. Show that these two spaces are homeomorphisms.

Solution We will use the same approach as the question above. Let $i : H^2 \to S^2$ be the inclusion map. This map is continuous. Consider the commuting diagram below

$$\begin{tikzcd} H^2 \arrow[r, "i"] \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2 \circ i"] & S^2 \arrow[d, "\pi_2"] \\ H^2/\sim \arrow[r, "\bar{i}"'] & S^2/\sim \end{tikzcd}$$

Since π_2 and i are both continuous maps, then $\pi_2 \circ i$ is also continuous. Also, note that this map assumes a constant value for all of the points in its domain that are in the same equivalence class. Thus it induces a map $\bar{i}$. Since $\pi_2 \circ i$ is continuous, then $\bar{i}$ is also continuous. We can also easily show that $\bar{i}$ is a bijection (although i was not a bijective map). Furthermore, since

- $H^2/\sim$ is compact (since it is the continuous image of a compact set),
- $S^2/\sim$ is Hausdorff (I will show below)

from the proposition in Summary 2.3 we prove that $\bar{i}$ is indeed an homeomorphism.

Showing that $S^2/\sim$ is Hausdorff. It follows form Theorem 2.1 that $S^2/\sim$ is Hausdorff if and only if the equivalence relation is open and the graph of the equivalence relation is closed in $S^2 \times S^2$. Form Remark 2.13 it follows that $\sim$ is open if and only if $\pi^{-1}(\pi(U))$ is open in S^2 for all open $U \subset S^2$. Let $U \subset S^2$ open. Then

$$\pi^{-1}(\pi(U)) = U \cap \{(x, y, z) \in \mathbb{R}^3 : -(x, y, z) \in U\}.$$

Since both of sets above are open, thus $\pi^{-1}(\pi(U))$ is open.

To show that the graph of the equivalence relation is closed in $S^2 \times S^2$ we have

$$\begin{aligned} R &= \{(x, y) \in S^2 \times S^2 \ : \ x \sim y\} = \{(x, y) \in S^2 \times S^2 \ : \ x = \pm y\} \\ &= \{(x, \pm x) : x \in S^2\} = \{(x, x) : x \in S^2\} \cup \{(x, -x) : x \in S^2\}. \end{aligned}$$

On the other hand, from Corollary 2.2 we know that since S^2 is Hausdorff, then $\{(x, x) : x \in S^2\}$ is closed (as well as $\{(x, -x) : x \in S^2\}$). Thus R is closed in $S^2 \times S^2$.

■ **Problem 2.20 — Image of the inverse image of a map (from W. Tu).** let $f : X \to Y$ be a map of sets, and let $B \subset Y$. Prove that $f(f^{-1}(B)) = B \cap f(X)$. There fore if f is surjective, then $f(f^{-1}(B)) = B$.

Solution We basically want to show that the two following sets are equal

$$f(f^{-1}(B)) = B \cap f(X).$$

If $f(f^{-1}(B)) = \varnothing$, then it follows immediately that $f(f^{-1}(B)) = B \cap f(X) = \varnothing$. For that case that $f(f^{-1}(B)) \neq \varnothing$, let $y \in f(f(B)^{-1})$. Note that $f(f^{-1}(B)) \neq \varnothing \implies f^{-1}(B) \neq \varnothing$. This immediately implies that $y \in f(X)$. Also we can write

$$y \in f(f^{-1}(B)) \implies f^{-1}(y) \in f^{-1}(B) \implies y \in B.$$

Thus we get $y \in B \cap f(X)$. For the converse, let $y \in B \cap f(X)$. Then we can write

$$f^{-1}(y) \in f^{-1}(B \cap f(X)) = f^{-1}(B) \cap X = f^{-1}(B).$$

This implies

$$y \in f(f^{-1}(B)).$$

Now as a simple corollary, if f is surjective, then $f(X) = Y$, which implies

$$f(f^{-1}(B)) = B \cap Y = B.$$

■ **Problem 2.21 — Closedness of the diagonal of a Hausdorff space (from W. Tu).** Deduce Theorem 2.1 from Corollary 2.2.

Note. You might be quite confused as in the text we used the corollary mentioned above as an immediate result of the theorem above (which is the case in most of the cases). But in this case, the corollary has enough strength that can prove the original theorem.

Solution Let S be a topological space and $\sim$ an open equivalence relation. We start with the forward direction. I.e. we want to show that $S/\sim$ is Hausdorff implies that the graph of $\sim$ is closed in $S \times S$. Consider the diagonal of $S/\sim$ where

$$\Delta = \{(x, x) \in S/\sim \times S/\sim\}.$$

We know that Δ is closed in $S/\sim \times S/\sim$, as $S/\sim$ is assumed to be Hausdorff and Corollary 2.2 implies that Δ is closed in $S/\sim S/\sim$. Also, consider the graph of the equivalence relation

$$R = \{(x, y) \in S \times S \mid x \sim y\} = \{(x, y) \in S \times S \mid \pi(x) = \pi(y)\}.$$

This suggest that Δ is the image of R under the map $\Pi : S \times S \to S/\sim \times S/\sim$ that maps $(x, y) \in S \times S \mapsto (\pi(x), \pi(y))$. So we can write

$$\Delta = \Pi(R).$$

Or equivalently, we can state that R is the pre-image of Δ under that map Π, i.e. $R = \Pi^{-1}(\Delta)$. Since that components of the map Π are continuous, thus the pre-image of open sets are opens sets (or equivalently, the pre-image of closed sets are closed sets). Since Δ is closed in $S/\sim \times S/\sim$, then it implies that R is closed in $S \times S$.

To prove the converse, we need to show closedness of R in $S \times S$ implies $S/\sim$ is Hausdorff. First note that Δ is the image of R under the map Π, i.e. $\Delta = \Pi(R)$. Since the equivalence relation is open, then π maps open sets in its domain to the open sets in its pre-domain (or equivalently, it maps closed sets to closed sets.). So does Π. Since R is closed in $S \times S$, then Δ is closed in $S/\sim \times S/\sim$. Then from Corollary 2.2 it follows that $S/\sim$ is Hausdorff.

■ **Problem 2.22 — Quotient of a sphere with antipodal points identified (from W. Tu).** Let S^n be the unit sphere centered at the origin in $\mathbb{R}^{n+1}$. Define an equivalence relation $\sim$ on S^n by identifying antipodal points:

$$x \sim y \iff x = \pm y, \qquad x, y \in S^n.$$

(a) Show that $\sim$ is an open equivalence relation.

(b) Apply Theorem 2.1 and Corollary 2.2 to prove that the quotient space $S^n/\sim$ is Hausdorff, without making use of the homeomorphism $\mathbb{R}P^n \simeq S^n/\sim$.

Solution In some parts of our solution for Problem 2.19 we proved this for the case $n = 2$. To prove it for any $n \in \mathbb{N}$ we will use a pretty much same approach.

(a) By Remark 2.13 we need to show that for every $U \subset S^n$ the set $\pi^{-1}(\pi(U))$ is open in S^n. Let $U \subset S^n$ be an open set. Then

$$\pi^{-1}(\pi(U)) = U \cup \{x \in \mathbb{R}^{n+1} \mid -x \in U\}.$$

Since both sets in the RHS are open, then $\pi^{-1}(\pi(U))$ is open in S^n. Thus the equivalence relation is open.

(b) We need to show that $S^n/\sim$ is Hausdorff. Since the equivalence relation is open (proved in part (a) above), then by Theorem 2.1 we just need to show that the graph of equivalence relation is closed in $S^n \times S^n$. For the graph of this equivalence relation we have

$$R = \{(x,y) \in S^n \times S^n \mid x \sim y\} = \{(x,y) \in S^n \times S^n \mid x = \pm y\} = \{(x,x) \mid x \in S^n\} \cup \{(x,-x) \mid x \in S^n\}.$$

Since S^n is Hausdorff (under the subspace topology), then by Corollary 2.2 the two sets in the RHS above are both closed, thus making R closed, hence $S^n/\sim \times S^n/\sim$ is Hausdorff.

Be Careful Here! 2.12.1 In my proof above, Since S^n is Hausdorff, then by Corollary 2.2 it implies that Δ is closed. In the proof above I also proved that $\{(x,-x) \mid x \in S^n\}$ is closed, but I have not proved it.

■ **Problem 2.23 — Orbit space of a continuous group action (from W. Tu).** Suppose a right action of a topological group G on a topological space S is continuous; this simply means that the map $\alpha : S \times G \to S$ describing the action is continuous. Define two points x, y of S to be equivalent if they are in the same orbit; i.e. there is an element $g \in G$ such that $y = xg$. Let S/G be the quotient space; it is called the orbit space of the action. Prove that the projection map $\pi : S \to S/G$ is an open map.

Observation 2.12.4 In the problem above, note and appreciate (!) that the group G is stated to be a topological group. If G was not a topological group, then we were not able to talk about the notion of continuoity of the right group action. The continuoity of the right group action implies that for any open set $U \subset S$ we have $\alpha^{-1}(U) \subset S \times G$ is open.

Solution First, we need to prove the following Lemma.

Lemma 2.2 Let S be a topological space and G be a topological group. Consider the right action of the group G on S via the map

$$\alpha : S \times G \to S, \qquad \alpha(s,g) = sg$$

If the right action if *continuous* (i.e. the map α is continuous) Then for $g \in G$ the map

$$\alpha_g : S \to S$$

is a *homeomorphism.*

Proof. We need to do the following three steps

- We need to show that α_g is a bijection that is continuous with a continuous inverse. α_g is indeed inverse with the inverse given by $\alpha_{g^{-1}}$, where $g^{-1} \in G$ is the inverse element of $g \in G$ (note that since G is group then the inverse exists). Observe that

$$\alpha_g(\alpha_{g^{-1}}(x)) = xg^{-1}g = x, \qquad \alpha_{g^{-1}}(\alpha_g(x)) = xgg^{-1} = x.$$

- We need to show that α_g is continuous. Since the map α is continuous, and α_g is just a restriction of the α for a fixed g, then α_g is also continuous.

  Alternatively, we can show the continuoity of α_g by showing that the pre-image of any open set $U \subset S$ is open under the subspace topology of $S \times \{g\}$ (where its open sets will be of the form $\mathcal{T}_S \times \{g\}$).

- We need to show that α_g^{-1} is continuous. Similar to our reasoning above, since α is continuous and α_g^{-1} is a restriction of α to the constant g^{-1}, α_g^{-1} is also continuous.

□

To show that π is an open map, by Remark 2.13 we need to show that for any open $U \in S$ the set $\pi^{-1}(\pi(U))$ is open in S. Let $U \subset S$ be any open set. Then

$$\pi^{-1}(\pi(U)) = \bigcup_{g\in G} Ug.$$

From the Lemma we prove above, for any fixed g, then map $\alpha_g : S \to S$ is a homeomorphism. Thus $\forall g \in G$ the set Ug is open in S. It follows immediately that $\pi^{-1}(\pi(U))$ is open as well.

■ **Problem 2.24 — Quotient of $\mathbb{R}$ by $2\pi\mathbb{Z}$ (from L. Tu).** Let the additive group $2\pi\mathbb{Z}$ act on $\mathbb{R}$. on the right by $x \cdot 2\pi n = x + 2\pi n$, where n is an integer. Show that the orbit space $\mathbb{R}/2\pi\mathbb{Z}$ is a smooth manifold.

Solution To show that $X = \mathbb{R}/2\pi\mathbb{Z}$ is a smooth manifold, we need to show that X is a *Hausdorff* and *second countable topological space.* Furthermore, we need to exhibit a *maximal atlas.*

Showing that π is an open map. We first need to show that the projection map $\pi : \mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$ or equivalently $\pi : \mathbb{R} \to \mathbb{R}/\sim$ is an open map. Note that the equivalence relation $\sim$ is defined as $x, y \in \mathbb{R}$ are equivalent, i.e. $x \sim y$ if $y = x \cdot g$ for some $g \in 2\pi\mathbb{Z}$ or equivalently, $y = x + 2\pi n$ for some $n \in \mathbb{Z}$. Since the group action map $\alpha(x, n) = x + 2\pi n$ is continuous, then by Lemma 2.2 we conclude that the projection map π is open.

Showing that $\mathbb{R}/\sim$ is second countable. Since π is an open map, then by Proposition 2.12 we can conclude that $\mathbb{R}/\sim$ is second countable.

Showing that $\mathbb{R}/\sim$ is Hausdorff. Since π is an open map, to show that $\mathbb{R}/\sim$ is Hausdorff, by Theorem 2.1 we need to show that the graph of the equivalence relation R is closed in $\mathbb{R} \times \mathbb{R}$. The graph of this equivalence relation is

$$\begin{aligned} R &= \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \sim y\} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x \cdot g \text{ for some } g \in 2\pi\mathbb{Z}\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x + 2\pi n \text{ for some } n \in \mathbb{Z}\}. \end{aligned}$$

This R is the set of all lines in $\mathbb{R}^2$ with slop 1 with y-intercepts equal to $2\pi n$ for some $n \in \mathbb{Z}$. This set is indeed closed in $\mathbb{R}^2$ as its complement is open in $\mathbb{R}^2$. We can prove the openness of R^c by first writing at as

$$R^c = \bigcup_{n\in\mathbb{Z}} S_n$$

where S_n is the open set between two lines with slope 1 and intercepts $2\pi(n)$ and $2\pi(n+1)$. Since R^c is union of open sets, thus it is open.
Exhibiting a maximal atlas See the box below.

Be Careful Here! 2.12.2 For the last part of the question above, to show that $\mathbb{R}/2\pi\mathbb{Z}$ is a smooth manifolds, I need to exhibit a maximal atlas (at least I think I need to do that!). To do this, I have a feeling that I need first to find any atlas, and then by using Summary 2.6 we can show that the manifold is smooth. But, instead of designing a smooth atlas, I think[a] I can show that this manifold is smooth by showing that it is homeomorphic to S^1 which is indeed a smooth manifold. Calling this homeomorphism as f, and considering the atlas $\{(U_\alpha, \varphi_\alpha)\}$ as an atlas for S^1, then the atlas for $\mathbb{R}/2\pi\mathbb{Z}$ will be $\{(f^{-1}(U_\alpha)), \varphi_\alpha \circ f\}$.

[a] See the box below.

Be Careful Here! 2.12.3 — Your thought in the box above is not actually true!. In the Be-CareFul box above, I said I think I can show that the manifold is smooth by showing that it is homeomorphic to another manifold. This is not exactly true! Note that in the category of topological spaces the isomorphisms are the homeomorphisms, but in the category of smooth manifolds, the isomorphisms are the diffeomorphisms, not homeomorphisms. Thus in the Be-CareFul box above we actually need to show that the manifold of interest is *diffeomorphic* to S^1, not homeomorphic. The good news is that my proof below does not break apart. Surprisingly it is the same procedure in one of the exercises of L. Tu, which I have solved in Problem 2.25.

(The old and vague solution) Showing that $\mathbb{R}/2\pi\mathbb{Z} \simeq S^1$. For this we need to find a homeomorphism. Consider the map f

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[d, "\pi"'] & S^1 \subset \mathbb{C} \\ \mathbb{R}/2\pi\mathbb{Z} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

where $x \mapsto e^{2\pi i x}$. This map assumes a constant values for all of the elements in the same equivalence class. But it induces the map $\bar{f}$, and the diagram above commutes. Continuoity of f implies the continuoity of $\bar{f}$. Although f has no inverse (since it is not one to one), but we can find an inverse for $\bar{f}$ that is given by

$$\begin{aligned} \bar{f}^{-1} &: S^1 \to \mathbb{R} \\ e^{i\theta} &\mapsto \theta \mod 2\pi. \end{aligned}$$

Thus $\bar{f}$ is a bijection. Since $\mathbb{R}/2\pi\mathbb{Z}$ is compact (continuous image of a compact set), and S^1 is Hausdorff (with the subspace topology of $\mathbb{C}$), by Summary 2.3 we conclude that $\mathbb{R}/2\pi\mathbb{Z} \simeq S^1$.

The new and correct solution. For this, we present a C^∞ compatible chart for this manifold. First, consider the following open sets on the manifold

$$V_1 = \{[t] \mid t \in (-\pi, \pi)\}, \qquad V_2 = \{[t] \mid t \in (0, 2\pi)\}.$$

In words, what we really mean is that V_1 contains the equivalence classes whose representatives comes from the set $(-\pi, \pi)$. Similarly for V_2, we say that it contains the equivalence classes whose representatives comes from the set $(0, 2\pi)$. We define the following coordinate maps

$$\begin{aligned} \varphi_1 : V_1 &\to \mathbb{R} & \varphi_2 : V_2 &\to \mathbb{R} \\ [t] &\mapsto t & [t] &\mapsto t \end{aligned}$$

Note that although the functions φ_1 and φ_2 look similar, but they are different functions as they have different domains. These two functions both return the representative of the equivalence classes in their argument. Note that we said *the representative* and not *a representative*) because of the design of their domain. For instance, let's parse the following operation

$$\varphi_1([23.5\pi]) = -0.5\pi.$$

Since the domain of φ_1 is the equivalence classes whose representatives comes from the set $(-\pi, \pi)$, then we need to find a representative for $[23.5\pi]$ that lives in $(-\pi, \pi)$. I.e. $[23.5\pi] = [1.5\pi] = [-0.5\pi]$.
We claim that the collection $\{(V_1, \varphi_1), (V_2, \varphi_2)\}$ is a C^∞ compatible atlas. To show this we simply need to show that the coordinate charts are compatible. We start with

$$\varphi_1 \circ \varphi_2^{-1} : \varphi_2(V_1 \cap V_2) \to \varphi_1(V_1 \cap V_2).$$

Note that

$$V_1 \cap V_2 = \{[t] \mid t \in (0, \pi) \cup (\pi, 2\pi)\}$$

or equivalently

$$V_1 \cap V_2 = \{[t] \mid t \in (-\pi, 0) \cup (0, \pi)\}.$$

Thus the function $\varphi_1 \circ \varphi_2^{-1}$ will be

$$\varphi_1 \circ \varphi_2^{-1} : (0, \pi) \cap (\pi, 2\pi) \to \mathbb{R}.$$

and is given by

$$(\varphi_1 \circ \varphi_2^{-1})(t) = \begin{cases} t & t \in (0, \pi), \\ t - 2\pi & t \in (\pi, 2\pi). \end{cases}$$

Similarly we can write

$$\varphi_2 \circ \varphi_1^{-1} : \underbrace{\varphi_1(V_1 \cap V_2)}_{(-\pi,0)\cap(0,\pi)} \to \varphi_2(V_1 \cap V_2),$$

is $\varphi_2 \circ \varphi_1^{-1}$ is given by

$$(\varphi_2 \circ \varphi_1^{-1})(t) = \begin{cases} t + 2\pi & t \in (-\pi, 0), \\ t & t \in (0, \pi). \end{cases}$$

Since both of these functions are C^∞ in their corresponding domain, then it follows that the exhibited collection of coordinate charts is indeed a C^∞ atlas.

■ **Problem 2.25 — The circle as a quotient space.**

(a) Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha=1}^2$ be the atlas of the circle S^1 in Problem 2.1, and let $\bar{\varphi}_\alpha$ be the map φ_α followed by the projection $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$. On $U_1 \cap U_2 = A \sqcup B$, since φ_1 and φ_2 differ by an integer multiple of 2π, $\bar{\varphi}_1 = \bar{\varphi}_2$. Therefore, $\bar{\varphi}_1$ and $\bar{\varphi}_2$ piece together to give a well-defined map $\bar{\varphi} : S^1 \to \mathbb{R}/2\pi\mathbb{Z}$. Prove that $\bar{\varphi}$ is C^∞.

(b) The complex exponential $\mathbb{R} \to S^1, t \mapsto e^{it}$, is constant on each orbit of the action of $2\pi\mathbb{Z}$. Therefore, there is an induced map $F : \mathbb{R}/2\pi\mathbb{Z} \to S^1, F([t]) = e^{it}$. Prive that F is C^∞.

(c) Prove that $F : \mathbb{R}/2\pi\mathbb{Z} \to S^1$ is a diffeomorphism.

Solution

(a) We define the map $\bar{\varphi} : S^2 \to \mathbb{R}/2\pi\mathbb{Z}$ as following

$$\bar{\varphi}(x) = \begin{cases} (\pi \circ \varphi_1)(x) & x \in U_1, \\ (\pi \circ \varphi_2)(x) & x \in U_2. \end{cases}$$

Considering the atlas $\{(V_\alpha, \psi_\alpha)\}_{\alpha=1}^2$ for $\mathbb{R}/2\pi\mathbb{Z}$ that we constructed in Problem 2.24, we can show that $\bar{\varphi}$ is C^∞ by simply showing that it is a smooth map between manifolds (i.e. using Definition 2.11). Thus we need to show that the following maps (in their corresponding domains)

$$\begin{aligned}
&\psi_1 \circ \bar{\varphi} \circ \varphi_1^{-1} : \varphi_1(U_1 \cap \bar{\varphi}^{-1}(V_1)) \to \psi_1(\bar{\varphi}(U_1)), \\
&\psi_1 \circ \bar{\varphi} \circ \varphi_2^{-1} : \varphi_2(U_2 \cap \bar{\varphi}^{-1}(V_1)) \to \psi_1(\bar{\varphi}(U_2)), \\
&\psi_2 \circ \bar{\varphi} \circ \varphi_1^{-1} : \varphi_1(U_1 \cap \bar{\varphi}^{-1}(V_2)) \to \psi_2(\bar{\varphi}(U_1)), \\
&\psi_2 \circ \bar{\varphi} \circ \varphi_2^{-1} : \varphi_2(U_2 \cap \bar{\varphi}^{-1}(V_2)) \to \psi_2(\bar{\varphi}(U_2)).
\end{aligned}$$

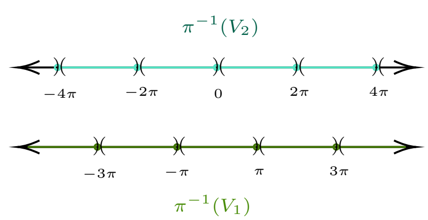
are smooth. To avoid multiple references to Problem 2.1 we write the following definitions here.

$$U_1 = \{e^{it} | t \in (-\pi, \pi)\}, \qquad U_2 = \{e^{it} | t \in (0, 2\pi)\}.$$

Now we can analyze all of these maps one by one. First, observe that

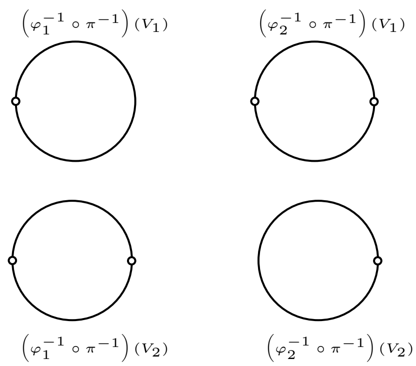
$$\pi^{-1}(V_1) = \bigcup_{n \in \mathbb{Z}} (-\pi + 2n\pi, \pi + 2n\pi), \qquad \pi^{-1}(V_2) = \bigcup_{n \in \mathbb{Z}} (2n\pi, 2\pi + 2n\pi).$$

which are demonstrated in the following diagrams. On the other hand we have



$$\begin{aligned}
U_1 \cap \bar{\varphi}^{-1}(V_1) &= (\varphi_1^{-1} \circ \pi^{-1})(V_1), \quad & U_2 \cap \bar{\varphi}^{-1}(V_1) &= (\varphi_2^{-1} \circ \pi^{-1})(V_1), \\
U_1 \cap \bar{\varphi}^{-1}(V_2) &= (\varphi_1^{-1} \circ \pi^{-1})(V_2), \quad & U_2 \cap \bar{\varphi}^{-1}(V_2) &= (\varphi_2^{-1} \circ \pi^{-1})(V_2).
\end{aligned}$$

The image of these maps are shown in the figure below



Thus we will have

$$\begin{aligned}
\varphi_1(U_1 \cap \bar{\varphi}^{-1}(V_1)) &= (-\pi, \pi),\\
\varphi_2(U_2 \cap \bar{\varphi}^{-1}(V_1)) &= (0, \pi) \cup (\pi, 2\pi),\\
\varphi_1(U_1 \cap \bar{\varphi}^{-1}(V_2)) &= (-\pi, 0) \cup (0, \pi),\\
\varphi_2(U_2 \cap \bar{\varphi}^{-1}(V_2)) &= (0, 2\pi).
\end{aligned}$$

Now we can re-write the maps of interest as

$$\begin{aligned}
&\psi_1 \circ \bar{\varphi} \circ \varphi_1^{-1} : (-\pi, \pi) \to \psi_1(\bar{\varphi}(U_1)),\\
&\psi_1 \circ \bar{\varphi} \circ \varphi_2^{-1} : (0, \pi) \cup (\pi, 2\pi) \to \psi_1(\bar{\varphi}(U_2)),\\
&\psi_2 \circ \bar{\varphi} \circ \varphi_1^{-1} : (-\pi, 0) \cup (0, \pi) \to \psi_2(\bar{\varphi}(U_1)),\\
&\psi_2 \circ \bar{\varphi} \circ \varphi_2^{-1} : (0, 2\pi) \to \psi_2(\bar{\varphi}(U_2)).
\end{aligned}$$

which are given by

$$(\psi_1 \circ \bar{\varphi} \circ \varphi_1^{-1})(t) = (\psi_1 \circ (\pi \circ \varphi_1) \circ \varphi_1^{-1})(t) = \psi_1(\pi(t)) = \psi_1([t]) = t, \qquad t \in (-\pi, \pi).$$

$$(\psi_1 \circ \bar{\varphi} \circ \varphi_2^{-1})(t) = (\psi_1 \circ (\pi \circ \varphi_2) \circ \varphi_2^{-1})(t) = \psi_1(\pi(t)) = \varphi_1([t]) = \begin{cases} t & t \in (0, \pi), \\ t - 2\pi & t \in (\pi, 2\pi). \end{cases}$$

$$(\psi_2 \circ \bar{\varphi} \circ \varphi_1^{-1})(t) = (\psi_2 \circ (\pi \circ \varphi) \circ \varphi_1^{-1})(t) = \psi_2(\pi(t)) = \psi_2([t]) = \begin{cases} t + 2\pi & t \in (-\pi, 0), \\ t & t \in (0, \pi). \end{cases}$$

$$(\psi_2 \circ \bar{\varphi} \circ \varphi_2^{-1})(t) = (\psi_2 \circ (\pi \circ \varphi_2) \circ \varphi_2^{-1})(t) = \psi_2(\pi(t)) = \psi_2(t) = t, \qquad t \in (0, 2\pi).$$

All of the maps above are C^∞. This shows that $\bar{\psi}$ is indeed C^∞.

(b) With a similar approach as above, using Definition 2.11 we need to show that the following maps are smooth.

$$\begin{aligned}
&\varphi_1 \circ F \circ \psi_1^{-1} : \psi_1(V_1 \cap F^{-1}(U_1)) \to \varphi_1(F(U_1))\\
&\varphi_1 \circ F \circ \psi_2^{-1} : \psi_2(V_2 \cap F^{-1}(U_1)) \to \varphi_1(F(U_2))\\
&\varphi_2 \circ F \circ \psi_1^{-1} : \psi_1(V_1 \cap F^{-1}(U_2)) \to \varphi_2(F(U_1))\\
&\varphi_2 \circ F \circ \psi_2^{-1} : \psi_2(V_2 \cap F^{-1}(U_2)) \to \varphi_2(F(U_2))
\end{aligned}$$

First, observe that

$$
\begin{aligned}
&V_1 \cap F^{-1}(U_1) = \{[t] \mid t \in (-\pi,\pi)\} \implies \psi_1(V_1 \cap F^{-1}(U_1)) = (-\pi,\pi),\\
&V_2 \cap F^{-1}(U_1) = \{[t] \mid t \in (0,\pi)\cup(\pi,2\pi)\} \implies \psi_2(V_2 \cap F^{-1}(U_1)) = (0,\pi)\cup(\pi,2\pi),\\
&V_1 \cap F^{-1}(U_2) = \{[t] \mid t \in (-\pi,0)\cup(0,\pi)\} \implies \psi_1(V_1 \cap F^{-1}(U_2)) = (-\pi,0)\cup(0,\pi),\\
&V_2 \cap F^{-1}(U_2) = \{[t] \mid t \in (0,2\pi)\} \implies \psi_2(V_2 \cap F^{-1}(U_2)) = (0,2\pi).
\end{aligned}
$$

Thus we can re-write the map definitions

$$
\begin{aligned}
&\varphi_1 \circ F \circ \psi_1^{-1} : (-\pi,\pi) \to \varphi_1(F(U_1))\\
&\varphi_1 \circ F \circ \psi_2^{-1} : (0,\pi)\cup(\pi,2\pi) \to \varphi_1(F(U_2))\\
&\varphi_2 \circ F \circ \psi_1^{-1} : (-\pi,0)\cup(0,\pi) \to \varphi_2(F(U_1))\\
&\varphi_2 \circ F \circ \psi_2^{-1} : (0,2\pi) \to \varphi_2(F(U_2))
\end{aligned}
$$

These maps are given by

$$
\begin{aligned}
&(\varphi_1 \circ F \circ \psi_1^{-1})(t) = \varphi_1(e^{it}) = t, \qquad t \in (-\pi,\pi).\\
&(\varphi_1 \circ F \circ \psi_2^{-1})(t) = \begin{cases} t & t \in (0,\pi),\\ t-2\pi & t \in (\pi,2\pi).\end{cases}\\
&(\varphi_2 \circ F \circ \psi_1^{-1})(t) = \begin{cases} t+2\pi & t \in (-\pi,0),\\ t & t \in (0,\pi).\end{cases}\\
&(\varphi_2 \circ F \circ \psi_2^{-1})(t) = \varphi_2(e^{it}) = t, \qquad t \in (-\pi,\pi).
\end{aligned}
$$

All of the functions above are C^∞. Thus F is C^∞.

(c) To show that $F : \mathbb{R}/2\pi\mathbb{Z} \to S^1$ is a diffeomorphism it is enough to show that $\bar{\varphi} : S^1 \to \mathbb{R}/2\pi\mathbb{Z}$ is inverse of it. Consider the following composition of maps

$$F \circ \bar{\varphi} : S^1 \to S^1.$$

For the points $x \in U_1 \subset S^1$, since $U_1 = \{e^{it} \mid t \in (-\pi,\pi)\}$ we have

$$(F \circ \pi \circ \varphi_1)(x) = (F \circ \pi \circ \varphi_1)(\underbrace{e^{it}}_{t\in(-\pi,\pi)}) = (F(\pi(t))) = F([t]) = e^{it}.$$

Similarly, for the points $x \in U_2 \subset S^1$, since $U_2 = \{e^{it} \mid t \in (0,2\pi)\}$ we have

$$(F \circ \pi \circ \varphi_2)(x) = (F \circ \pi \circ \varphi_2)(\underbrace{e^{it}}_{t\in(0,2\pi)}) = F(\pi(t)) = F([t]) = e^{it}.$$

For the converse, consider the following map

$$\bar{\varphi} \circ F : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}.$$

Let $[t] \in \mathbb{R}/2\pi\mathbb{Z}$. Then we have

$$(\bar{\varphi} \circ F)([t]) = \bar{\varphi}(e^{it}) = \pi(t + 2n\pi) = [t] \qquad \text{for some } n \in \mathbb{Z}.$$

This shows that F is indeed a diffeomorphism between manifolds.

Summary 2.10 In the question above, we proved (by finding an explicit diffeomorphism) that $\mathbb{R}/2\pi\mathbb{Z}$ is diffeomorphic to $S^1 \subset \mathbb{C}$.

■ **Problem 2.26 — The Grassmannian $G(k, n)$.** The Grassmannian $G(k, n)$ is the set of all k-planes through the origin in $\mathbb{R}^n$. Such a k-plane is a linear subspace of dimension k of $\mathbb{R}^n$ and has a basis consisting of k linearly independent vectors $a_1, \cdots, a_k$ in $\mathbb{R}^n$. IT is therefore completely specified by an $n \times k$ matrix $A = [a_1 \ \cdots \ a_k]$ of rank k, where the rank of a matrix A, denoted by $\operatorname{rank} A$ is defined to be the number of linearly independent columns of A. This matrix is called a matrix representative of the k-plane.

Two bases $a_1, \cdots, a_k$ and $b_1, \cdots, b_k$ determine the same k-planes if there is a change of basis matrix $g = [g_{ij}] \in GL(k, \mathbb{R})$ such that

$$\begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix} = g^T \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix},$$

or equivalently

$$b_j = \sum_i g_{ij} a_j,$$

or in matrix notation

$$B = Ag$$

where B, A are matrices whose columns are the vectors $a_1, \cdots, a_k$ and $b_1, \cdots, b_k$ respectively.

Let $F(k, n)$ denote the set of all $n \times k$ matrices of rank k, topologized as a subspace of $R^{n\times k}$. Thus $F(k, n)$ is an open subset of $R^{n\times k}$.

Let the group $GL(k, \mathbb{R})$ act on $F(k, n)$ on the right by $A \cdot g = Ag$ where the right hand side is the usual matrix multiplication. Consider the orbit space $F(k, n)/GL(k, \mathbb{R})$. Equivalently, we can show this space as the quotient space $F(k, n)/\sim$ where the equivalence relation $\sim$ is given as

$$A \equiv B \qquad \text{iff} \qquad A = Bg \quad \text{for some } g \in GL(n, \mathbb{R}).$$

There is a bijection between the Grassmannian $G(k, n)$ and the orbit space $f(k, n)/GL(k, \mathbb{R})$. We given the Grassmannian $G(k, n)$ the quotient topology on $F(k, n)/GL$.

(a) Show that $\sim$ is an open equivalence relation.

(b) Prove that the Grassmannian $G(k, n)$ is second countable.

(c) Let $S = F(k, n)$. Prove that the graph R in $S \times S$ of the equivalence relation $\sim$ is closed. (*Hint*: Two matrices $A = [a_1 \ \cdots \ a_k]$ and $B = [b_1 \ \cdots \ b_k]$ are equivalent $\Leftrightarrow$ every column of B is a linear combination of the columns of $A \Leftrightarrow \operatorname{rank}[A\ B] \le k \Leftrightarrow$ all $(k + 1) \times (k + 1)$ minors of $[A\ B]$ are zero).

(d) Prove that the Grassmannian $G(n, k)$ is Hausdorff.

Now we want to find a C^∞ atlas on the Grassmannian $G(k, n)$. For simplicity, we specialize to $G(2, 4)$. For any 4×2 matrix A, let A_{ij} be the 2×2 sub matrix consisting of its ith row and jth row. Define

$$V_{ij} = \{A \in F(2, 4) \mid A_{ij} \text{ is nonsingular}\}.$$

For any $A \in V_{ij}$, since $\det\big(A_{ij}\big) \neq 0$ (i.e. it is nonsingular), then we can perturb it small enough and by continuoity of the determinant operator the nonsingularity of A_{ij} persists. This shows that V_{ij} is indeed an open subset of $F(2, 4)$. Alternatively, we can show this by arguing that since the complement of V_{ij} is defined by vanishing of $\det\big(A_{ij}\big)$, we conclude that V_{ij} is an open subset of $F(2, 4)$.

(e) Prove that if $A \in V_{ij}$, then $Ag \in V_{ij}$ for any nonsingular matrix $g \in GL(2, \mathbb{R})$.

Define $U_{ij} = V_{ij}/\sim$. Since $\sim$ is an open equivalence relation, U_{ij} is an open subset of $G(2, 4)$. For $A \in V_{12}$

$$A \sim AA_{12}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} I \\ A_{34}A_{12}^{-1} \end{bmatrix}.$$

This shows that the matrix representative of a 2-plane in U_{12} have a canonical form B where B_{12} is the identity matrix.

(f) Show that the map $\tilde{\varphi}_{12} : V_{12} \to \mathbb{R}^{2\times 2}$,

$$\tilde{\varphi}_{12}(A) = A_{34}A_{12}^{-1}$$

induces a homeomorphism $\varphi_{12} : U_{12} \to \mathbb{R}2 \times 2$.

(g) Define similarly homeomorphism $\varphi_{ij} : U_{ij} \to \mathbb{R}^2$. Compute $\varphi_{12} \circ \varphi_{23}^{-1}$, and show that it is C^∞.

(h) Show that $\{U_{ij} \mid 1 \le i \le j \le 4\}$ is an open cover of $G(2, 4)$ and that $G(2, 4)$ is a smooth manifold.

Solution

(a) According to Lemma 2.2 the projection map $\pi : F(k, n) \to F(k, n)/GL$ is an open map if and only if the group action map $\alpha : F(k, n) \times G \to F(k, n)$ is continuous. Let $A \in F(k, n)$ and $g \in G$. Then $B = \alpha(A, g)$. For the element b_{kl} of B we have

$$b_{kl} = \sum_i A_{ki} g_{il}$$

which is a polynomial. Thus α is a continuous map. Now according to Lemma 2.2 the equivalence relation $\sim$ is open.

(b) From Proposition 2.12 it follows that since $\sim$ is an open equivalence relation, then quotient space $G(k, n) = F(k, n)/\sim$ is second countable.

(c) To show that R is closed by showing that it complement is open. Let $x \in R^c$. Then $x = [A\ B]$ for some matrices $A, B \in S$ where the matrix $[A\ B]$ has at least on $(k + 1) \times (k + 1)$ that is not zero. Then by continuoity of determinant operation, for small enough $\epsilon > 0$ we can find an open ball $\mathbb{B}_\epsilon$ centered at $[A\ B] \in \mathbb{R}^{n\times 2k}$ such that for all $y \in \mathbb{B}_\epsilon$ the non-zero minor persists. This shows that R^c is open which implies the closedness of R.

(d) Since $\sim$ is an open equivalence relation and also the graph of the equivalence relation is closed in $S \times S$, then by Theorem 2.1 it follows that $G(k, n)$ is Hausdorff.

(e) First, observe that

$$(Ag)_{ij} = A_{ij}g.$$

Let $A \in V_{ij}$. Then from definition we know that $\det\big(A_{ij}\big) \ne 0$. Since $(Ag)_{ij} = A_{ij}g$ and $\det\big((Ag)_{ij}\big) = \det\big(A_{ij}\big)\det(g)$, then from nonsingularity of g it follows that $\det\big((Ag)_{ij}\big) \ne 0$. Hence $Ag \in V_{ij}$.

(f) Consider the following diagram.

$$\begin{tikzcd} V_{12} \arrow[r, "\tilde{\varphi}_{12}"] \arrow[d, "\pi"'] & \mathbb{R}^{2\times 2} \\ U_{12} \arrow[ur, "\varphi_{12}"'] & \end{tikzcd}$$

First observe that $\tilde{\varphi}_{12}$ assumes a constant value for all the points in its domain that are in the same equivalence class. That is because if for $A, B \in V_{12}$ we have $A \sim B$, then $\exists g \in GL(2, \mathbb{R})$ such that $A = Bg$. Thus

$$A_{34}A_{12}^{-1} = \tilde{\varphi}_{12}(A) = \tilde{\varphi}_{12}(Bg) = (Bg)_{34}(Bg)_{12}^{-1} = B_{34}gg^{-1}B_{12}^{-1} = B_{34}B_{12}^{-1}.$$

Also, since $\tilde{\varphi}_{12}$ is continuous, then it follows that the induced map φ_{12} is also continuous.

Now, we need to find an inverse for this map. Consider the map

$$\psi_{12} : \mathbb{R}^{2\times 2} \to U_{12}, \ M \mapsto \begin{bmatrix} I \\ M \end{bmatrix}.$$

This inclusion map is indeed continuous. Now we need to show that ψ and φ are inverse maps. Consider

$$\psi \circ \varphi : U_{12} \to U_{12}.$$

Let $A \in U_{12}$. Then we know that there exist a canonical representation of A as

$$A \sim \begin{bmatrix} I \\ A_{34}A_{12}^{-1} \end{bmatrix}.$$

Then

$$\psi(\varphi(\begin{bmatrix} I \\ A_{34}A_{12}^{-1} \end{bmatrix})) = \psi(A_{34}A_{12}^{-1}) = \begin{bmatrix} I \\ A_{34}A_{12}^{-1} \end{bmatrix}$$

For the converse, consider the map

$$\varphi \circ \psi : R^{2\times 2} \to \mathbb{R}^{2\times 2}.$$

Let $M \in \mathbb{R}^{2\times 2}$. Then

$$(\varphi \circ \psi)(M) = \varphi_{23}(\begin{bmatrix} I \\ M \end{bmatrix}) = M.$$

This shows that ψ and φ are inverse maps and this completes the proof.

(g) For this map we have

$$\varphi_{12} \circ \varphi_{23}^{-1} : \varphi(U_{12} \cap U_{23}) \to \mathbb{R}^{2\times 2}.$$

Note that the set $U_{12} \cap U_{23}$ is defined as the quotient space of $V_{12} \cap V_{23}$ where

$$V_{12} \cap V_{23} = \{A \in F(2,4) \mid \det(A_{12}) \neq 0 \text{ and } \det(A_{23}) \neq 0\}.$$

For any matrix in $A \in U_{12}$ it will have a canonical form

$$A \sim \begin{bmatrix} I \\ A_{34}A_{23}^{-1} \end{bmatrix}$$

and similarly for $B \in U_{23}$

$$B \sim \begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1 \\ c & d \end{bmatrix}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = B_{14} B_{23}^{-1}$$

Let $M \in \varphi(U_{12} \cap U_{23})$. Then

$$(\varphi_{12} \circ \varphi_{23}^{-1})(M) = \varphi_1 \begin{bmatrix} M_{11} & M_{12} \\ 1 & 0 \\ 0 & 1 \\ M_{21} & M_{22} \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} M_{11} & M_{21} \\ 1 & 0 \end{pmatrix}^{-1}.$$

The result of the matrix multiplication above is a polynomial in terms of the components, thus it is C^∞.

(h) First observe that V_{ij} for $1 \leq i \leq j \leq 4$ are open subsets of $F(2,4)$. For instance, if $A \in V_{12}$, then A_{12} is nonsingular. From the continuity of the determinant operator, the nonsignularity persists under small enough perturbation. Also, since the projection map π is an open map (shown in part (a)), then U_{ij} are open subsets of $G(2,4)$. Now we need to show that all U_{ij} covers the whole space. Since there is a bijection between $G(2,4)$ and $F(2,4)/GL(2,\mathbb{R})$ (as we discussed in the statement of the problem), thus for any $x \in G(2,4)$ there exists $A \in U_{ij}$ for some $1 \leq i \leq j \leq 4$. Thus the set $\{U_{ij} \mid 1 \leq i \leq j \leq 4\}$ is an open cover for $G(2,4)$. Also from what we showed above, the atlas

$$\{(U_{ij}, \varphi_{ij}) \mid 1 \leq i \leq j \leq 4\}$$

is C^∞ compatible. Thus $G(2,4)$ is a smooth manifold.

3. Tangent Spaces

3.1 Tangent Spaces

For several reasons, the notion of tangent space is more easily developed using algebraic view point rather than the geometric one. Then we can easily construct our geometric intuition from using the developed theory. We start with the notion of derivation.

Definition 3.1 — Derivation at a point. Let M be a manifold and p a point of manifold. Let $C_p^\infty(M)$ denote the germ of C^∞ functions at p. Derivation at p is a linear map

$$D : C_p^\infty(M) \to \mathbb{R}$$

such that for $f, g \in C_p^\infty(M)$ we have

$$D(fg) = D(f)g + fD(g).$$

The definition above is a purely algebraic one.

■ **Remark 3.1** The set of all point derivations at p forms a vector space.

Definition 3.2 — Tangent space. Let M be a manifold and $p \in M$. The tangent space at p denoted by $T_p(M)$ is the vector space of all point derivations at p.

In the following proposition we will see a very familiar point derivation.

Proposition 3.1 — Partial derivative $\partial/\partial x^i|_p$ is a tangent vector. Let M be a manifold and $p \in M$. Let (U, φ) be a coordinate chart containing p and $f : M \to \mathbb{R}$ a function defined at a neighborhood of p. By definition we have

$$\frac{\partial}{\partial x^i}\Big|_p f = \frac{\partial}{\partial r^i}\Big|_{\varphi(p)} (f \circ \varphi^{-1})$$

where $x^i = r^i \circ f$ with r^i as the standard coordinate function of $\mathbb{R}^n$. The map $\frac{\partial}{\partial x^i}\Big|_p$ is a point derivation at p, thus a tangent vector.

Proof. First observe that $\partial/\partial x_i|_p$ is indeed a linear map from $C_p^\infty(M)$ to $\mathbb{R}$. Let $f, g \in C_p^\infty(U)$

and $(U, x^1, \cdots, x^n)$ a coordinate chart. Then we have

$$\begin{aligned}\frac{\partial}{\partial x^i}\Big|_p (fg) &= \frac{\partial}{\partial r^i}\Big|_{\varphi(p)} ((fg)\circ\varphi^{-1}) = \frac{\partial}{\partial r^i}\Big|_{\varphi(p)} (f\circ\varphi^{-1}\cdot g\circ\varphi^{-1}) \\ &= \frac{\partial}{\partial r^i}\Big|_{\varphi(p)} (f\circ\varphi^{-1})\cdot g(p) + f(p)\cdot\frac{\partial}{\partial r^i}\Big|_{\varphi(p)} (g\circ\varphi^{-1}) \\ &= (\frac{\partial}{\partial x^i}\Big|_p f)g + f(\frac{\partial}{\partial x^i}\Big|_p g).\end{aligned}$$

□

Definition 3.3 — Differential of a smooth map. Let $F : N \to M$ be a smooth map between manifolds. This map induces a *linear* map called the *differential of F*

$$F_* : T_pN \to T_{F(p)}M$$

where for $X_P \in T_pN$ we have

$$(F_*(X_p))f = X_p(f\circ F)$$

where f is a germ at p.

Proposition 3.2 Let $F : N \to M$. It differential F_* is a derivation at $F(p)$ for $p \in N$.

Proof. See Problem 3.1 □

Differential of a map between Euclidean spaces

To see what kind of an object is the differential of a map, we calculate it explicitly on Euclidean spaces. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a map and let $x^1, \cdots, x^n$ be the standard coordinate functions of $\mathbb{R}^n$ while $y^1, \cdots, y^m$ be the standard coordinate functions of $\mathbb{R}^m$. Let $p \in \mathbb{R}^n$. since F_* is a linear map between two vector spaces, then we can write it as a matrix once we fix the basis for the spaces. The basis for $T_p(\mathbb{R}^n)$ is the set $\{\partial/\partial x^i|_p\}_i$, while for $T_p(\mathbb{R}^m)$ is $\{\partial/\partial y^j|_p\}_j$. To get the matrix representation of F_* we need to find its effect on the basis vectors

$$F_*(\frac{\partial}{\partial x^j}\Big|_p) = \sum_k a_j^k \frac{\partial}{\partial y^k}\Big|_{F(p)}.$$

By applying it on y_i we will get

$$a_j^i = F_*(\frac{\partial}{\partial x^j}\Big|_p) y^i = \frac{\partial}{\partial x^j}\Big|_p (y^i\circ F) = \frac{\partial}{\partial x^j}\Big|_p F^i.$$

This is precisely the Jacobian matrix of F evaluated at p.

The Chain Rule

A nice thing behind this algebraic point of view is that we can now write the notion of chain rule in a much simpler form. The following propositions captures this ides.

Proposition 3.3 — The chain rule. Let $F : N \to M$ and $G : M \to P$ be maps between manifolds, and $p \in N$. Then the differential of F at p and G at $F(p)$ are linear maps

$$T_pN \xrightarrow{F_{*,p}} T_pM \xrightarrow{G_{*,F(p)}} T_pP.$$

Then we have
$$(G \circ F)_{*,p} = G_{*,F(p)} F_{*,p}.$$

Proof. Let $X_p \in T_p(N)$. From definition we have for some $f : G \to \mathbb{R}$ we have
$$((G \circ F)_{*,p} X_p) f = X_p(f \circ G \circ F) = X_p((f \circ G) \circ F) = (F_{*,p} X_p)(f \circ G).$$
Let $Y_q = F_{*,p} X_p$ where $q = F(p)$. Then
$$Y_q(f \circ G) = (G_{*,q} Y_q) f.$$
This implies that
$$(G \circ F)_{*,p} = G_{*,q} F_{*,p}$$
and this completes the proof. □

■ **Example 3.1** Let $F : \mathbb{R} \to \mathbb{R}^3$ and $G : \mathbb{R}^3 \to \mathbb{R}$. Then the differential of the composite function $G \circ F$ is the matrix multiplication of the differential of each map. I.e.
$$(G \circ F)_* = G_* F_* = (G_x \; G_y \; G_z) \cdot \begin{pmatrix} F_1' \\ F_2' \\ F_3' \end{pmatrix} = G_x F_1' + G_y F_2' + G_z F_3' = \frac{\partial G}{\partial x} \frac{dF_1}{dt} + \frac{\partial G}{\partial y} \frac{dF_2}{dt} + \frac{\partial G}{\partial z} \frac{dF_3}{dt},$$
which is the same chain rule in calculus. ■

Writing the chain rule in this form not only is visually appealing, but also we can deduce two important Corollaries from it as follows.

Corollary 3.1 — Diffeomorphism of Manifolds vs. Isomorphism of Tangent Spaces. Let $F : N \to M$ be a diffeomorphism of manifolds and $p \in N$. Its differential F_* is an isomorphism of tangent spaces $T_p N$ and $T_{F(p)} M$.

Proof. F being a diffeomorphism of manifolds, it implies that we have $G : M \to N$ such that $G \circ F = \mathbb{1}_N$ and $F \circ G = \mathbb{1}_M$. Now using Proposition 3.3 we can write
$$(G \circ F)_* = G_* \circ F_* = (\mathbb{1}_N)_* = \mathbb{1}_{T_p N}$$
$$(F \circ G)_* = F_* \circ G_* = (\mathbb{1}_M)_* = \mathbb{1}_{T_{F(p)} M}.$$
This shows that F_* and G_* are the right and left inverses of each other, hence an isomorphism of vector spaces. □

Corollary 3.2 — Invariance of Dimension. Let $U \subset \mathbb{R}^n$ and $V \subset R^m$ be open subsets that are diffeomorphic. Then $m = n$.

Proof. Let $F : U \to V$ be a diffeomorphism and $p \in U$. Then F_* is an isomorphism of tangent spaces $T_p U$ and $T_{F(p)} V$, thus of the same dimension. Since in a Euclidean space the tangent space has the same dimension as the space itself (by definition and the way that we construct a basis for tangent space using the standard coordinate vectors) then this implies that $n = m$. □

Corollary 3.3 — Tangent Space Has the Same Dimension as Manifold. Let M be a manifold of dimension n and $p \in M$. Then $T_p(M)$ also has dimension n.

Proof. Let (U, φ) be a coordinate chart containing the point p. Since M is assumed to be a smooth manifold, then the coordinate map φ automatically upgrades to a diffeomorphism between U and $\mathbb{R}^n$. Thus the tangent spaces $T_p(U)$ and $T_{\varphi(p)}(\mathbb{R}^n)$ are isomorphic. This implies that $T_p(U)$ is also of dimension n. □

3.2 Basis for the Tangent Space at a Point

3.3 Solved Problems

■ **Problem 3.1 — The differential of a map.** Check that $F_*(X_p)$ is a derivation at $F(p)$ and that $F_* : T_pN \to T_{F(p)}M$ is a linear map.

Solution We start with showing that $F_*(X_p)$ is a derivation at $F(p)$. By definition we have

$$\begin{aligned}
[(F_*(X_p))(f \cdot g)](F(p)) &= [X_p((f \cdot g) \circ F)](p) = [X_p((f \circ F) \cdot (g \circ F))](p) \\
&= [X_p(f \circ F)](p) \cdot g(F(p)) + f(F(p)) \cdot [X_p(g \circ F)](p) \\
&= [(F_*(X_p))f](F(p)) \cdot g(F(p)) + f(F(p)) \cdot [(F_*(X_p))g].
\end{aligned}$$

This shows that $F_*(p)$ is indeed a derivation at $F(p)$. Furthermore, to prove the linearity, let $X_p, Y_p \in T_p(N)$. Then

$$(F_*(X_p + Y_p))f = (X_p + Y_p)(f \circ F) = X_p(f \circ F) + Y_p(f \circ F) = (F_*(X_p))f + (F_*(Y_p))f.$$

4. Appendix

4.1 Change of Basis

Let V be a vector space of dimension n, and $\mathbb{B}_1 = \{e_i\}_{i=1}^n$ be a basis. Then for $\alpha \in V$ we can write it uniquely as

$$\alpha = \sum_i \alpha^i e_i.$$

Once we know a particular basis, we can do the following association

$$e_i \longleftrightarrow \begin{bmatrix} 0 \\ \vdots \\ \underbrace{1}_{i\text{–th element}} \\ \vdots \\ 0 \end{bmatrix} \qquad \alpha \longleftrightarrow \vec{a} = \begin{bmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ \alpha^{n-1} \\ \alpha^n \end{bmatrix}$$

And we define the transpose operator to be $\vec{a}^T = (\alpha^1 \ \cdots \ \alpha^n)$. Then we can write the unique decomposition of α to the basis vectors as

$$\alpha = (\alpha_1 \ \cdots \ \alpha_n) \begin{bmatrix} e^1 \\ \vdots \\ e^n \end{bmatrix}$$

where these symbols act with the usual multiplication of vectors. Since these representations of elements of the vector space by column vectors depends on the particular basis that we choose, then then natural question is that how these symbols will change if we change the basis. Assume that the old basis is changed to the new $\mathbb{B}_2 = \{e'_i\}_{i=1}^n$ where

$$\begin{bmatrix} e'_1 \\ e'_2 \\ \vdots \\ e'_n \end{bmatrix} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1N} \\ r_{21} & r_{22} & \cdots & r_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ r_{N1} & r_{N2} & \cdots & r_{NN} \end{pmatrix}^T \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = R^T \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

To find the coordinates of α in the new basis we can write

$$\alpha = \underbrace{(\alpha_1 \ \cdots \ \alpha_n)(R^T)^{-1}}_{(\beta_1 \ \cdots \ \beta_2)} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

Thus the the relation between then new coordinate of α and the old coordinate is

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = R \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

To observer how does linear map representation in new coordinates change consider the equation

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = M \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

where the representation of column vectors and the matrix M is in the basis $\mathbb{B}_1$. If we change it to a new basis by the change of basis matrix R, then for the new coordinates we will have

$$R \begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix} = MR \begin{bmatrix} y'_1 \\ \vdots \\ y'_n \end{bmatrix}.$$

This implies

$$\boxed{\begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix} = \underbrace{R^{-1}MR}_{M'} \begin{bmatrix} y'_1 \\ \vdots \\ y'_n \end{bmatrix}}.$$

Summary 4.1 — Change of basis. In a nutshell, Let V is a vector space of dimension N. Then after specifying a basis $\mathbb{B}_1 = \{e_i\}$, we can represent the elements of vector space as column vectors. If we want to change the basis to a new basis $\mathbb{B}_2 = \{e'_i\}$ with the change of basis matrix R (whose $j-$th) column contains the coordinates of e'_i in terms of $\mathbb{B}_1$ basis vectors. Then the relation between new coordinates and the old coordinates is

$$\underbrace{\begin{bmatrix} x_1 \\ \vdots \\ v_n \end{bmatrix}_{\mathbb{B}_1}}_{\text{represented according to } \mathbb{B}_1} = R \overbrace{\begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix}}^{\text{represented according to } \mathbb{B}_2}_{\mathbb{B}_2}.$$

Also if

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}_{\mathbb{B}_1} = M \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{\mathbb{B}_2}$$

then

$$\begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix}_{\mathbb{B}_2} = R^{-1} M R \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}.$$