



## Numerical Analysis

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# 1. Introduction and Background

## 1.1 Summary

**Summary 🦋 1.1 — Continuous functions on  $\Omega$  and  $\bar{\Omega}$ .** Let  $\Omega \subset \mathbb{R}^n$  be an open set.  $C(\Omega)$  denotes the set of all continuous functions defined on  $\Omega$ . Similarly,  $C(\bar{\Omega})$  denotes the set of all continuous functions defined on the closure of  $\Omega$ .

For any  $f \in C(\Omega)$  we have  $f \in C(\bar{\Omega})$ . However, for the converse, if  $g \in C(\bar{\Omega})$ , then if  $g$  is uniformly continuous and  $\Omega$  is bounded, then  $g$  can continuously be extended to  $\partial\Omega$ . Note that  $C(\Omega)$  functions can behave badly near  $\partial\Omega$ . For instance, consider the function  $f : (0, 1) \rightarrow \mathbb{R}$  given by  $f(x) = \sin(1/x)$ .

**Summary 🦋 1.2 — The space of continuous  $2\pi$  periodic functions.** Consider the space of continuous functions defined on  $\mathbb{R}$ , i.e.  $C(\mathbb{R})$ . An important subset of this set is  $C_p(2\pi)$  which is the set of all continuous  $2\pi$  periodic functions where for  $f \in C_p(2\pi)$  we have

$$f(x + 2\pi) = f(x), \quad x \in \mathbb{R}.$$

This set, is in one-to-one correspondence with the set of all continuous function defined from the manifold  $S^1$ , or equivalently  $\mathbb{R}/2\pi\mathbb{Z}$  to  $\mathbb{R}$ .

**Summary 🦋 1.3 — Basis for the set of polynomials.** Let  $\mathbb{P}_n$  denote the set of all polynomials defined on  $\mathbb{R}$  with degree less than or equal to  $n$ . Then a basis for this linear space is

$$\mathbb{B} = \{1, x, \dots, x^n\}.$$

Thus the dimension of this space is  $n + 1$ .

Now, consider a *linear subspace* of this space, the set of all polynomials that vanishes at 0 and 1 denoted by

$$\mathbb{P}_{n,0} = \{p \in \mathbb{P}_n \mid p(0) = p(1) = 0\}.$$

A basis for this linear subspace can be given as

$$\mathbb{B}_{n,0} = \{x(1-x), x^2(1-x), \dots, x^{n-1}(1-x)\}.$$

Thus the dimension of this linear subspace is  $\dim(\mathbb{P}_n) - 2$ . The difference two in the dimension comes from the fact that polynomials in  $\mathbb{P}_{n,0}$  vanished at two points of their domain. Thus the set of all polynomials of degree  $n$  that vanish at  $n$  points of their domain is a 1 dimensional linear subspace of  $\mathbb{P}_n$ .

**Summary** 🦋 **1.4 — Normed space  $\mathbb{R}^d$ .** Consider the linear space  $\mathbb{R}^d$ . Then the followings are the common norms for this space.

$$\|x\|_p = \left( \sum_{i=1}^d |x_i|^p \right)^{1/p}, \quad \text{for } 1 \leq p < \infty.$$

$$\|x\|_\infty = \max_{1 \leq i \leq d} |x_i|.$$

**Proposition 1.1** In  $\mathbb{R}^d$  we have for all  $x \in \mathbb{R}^d$

$$\|x\|_\infty \leq \|x\|_p \leq d^{1/p} \|x\|_\infty.$$

■ **Remark 1.1** As a simple corollary of the proposition above we can see

$$\|x\|_\infty = \lim_{p \rightarrow \infty} \|x\|_p.$$

**Proposition 1.2 — Equivalence of norms.** On a finite dimensional space all norms are equivalent.

■ **Remark 1.2** The proposition above dose not hold true on infinite dimensional spaces. In those space, some norms has more stronger sense of convergence than others.

**Summary** 🦋 **1.5 — Normed space  $C(\Omega)$ .** Let  $V = C[0, 1]$  denote the linear space of all continuous function defined on  $[0, 1]$ . Define the following norms

$$\|f\|_p = \left( \int_0^1 |f(x)|^p dx \right)^{1/p}, \quad 1 \leq p < \infty$$

$$\|f\|_\infty = \sup_{0 \leq x \leq 1} |f(x)|$$

The norm  $\|x\|_\infty$  is a natural norm for this space that is also called *uniform norm*.

**Proposition 1.3** For the norms given above we have

$$\|v\|_p \leq \|v\|_\infty \quad \forall v \in V.$$

This implies that the convergence under the uniform norm  $\|\cdot\|_\infty$  implies the convergence under the norm  $\|\cdot\|_p$ . Note that the converse is not true.

■ **Remark 1.3** A very good example to see the proposition above is  $f : [0, 1] \rightarrow \mathbb{R}$  given by

$$f(x) = \begin{cases} 1 - nx, & 0 \leq x \leq 1/n, \\ 0, & 1/n < x \leq 1. \end{cases}$$

**Proposition 1.4** Let  $\Omega \subset \mathbb{R}^d$  and let  $\bar{\Omega}$  denote its closure. Then the space  $C(\bar{\Omega})$  with the norm  $\|\cdot\|_\infty$  is a Banach space. However, this space is not a Banach space with  $\|\cdot\|_p$  for  $1 \leq p < \infty$ .

■ **Remark 1.4** The proposition above is true since the continuity of a sequence of functions persists under the uniform continuity. A good example for the second statement is the function  $f : [0, 1] \rightarrow \mathbb{R}$

$$f(x) = \begin{cases} 0, & 0 \leq x \leq 1/2 - 1/(2n), \\ n(x - 1/2 + 1/(2n)), & 1/2 - 1/(2n) \leq x \leq 1/2 + 1/(2n), \\ 1, & 1/2 + 1/(2n) \leq x \leq 1. \end{cases}$$

**Summary 🦋 1.6 — Normed space  $C^k(\Omega)$ .** Let  $\Omega \subset \mathbb{R}$ . The space  $C^k(\bar{\Omega})$  is the set of all  $k$  times continuously differentiable functions. Define the following metric on this space

$$\|f\|_{k,p} = \left( \sum_{i=1}^k \|f^{(i)}\|_p^p \right)^{1/p}, \quad 1 \leq p < \infty.$$

$$\|f\|_{k,\infty} = \max_{1 \leq i \leq k} \|f^{(i)}\|_\infty.$$

The natural basis for this space is  $\|f\|_{k,\infty}$ .

**Proposition 1.5 —  $C^k$  is a Banach space.** The space  $C^k$  is complete under the norm  $\|\cdot\|_{k,\infty}$ .

**Summary 🦋 1.7 — Completion of  $C(\bar{\Omega})$ .** The space  $C(\bar{\Omega})$  is not complete under the norm  $\|\cdot\|_p$  for  $1 \leq p < \infty$ . Its completion is the space of *Lebesgue integrable functions*  $L^p$ .

**Summary 🦋 1.8 — Completion of  $C^k(\bar{\Omega})$ .** The space  $C^k(\bar{\Omega})$  is not complete under the norm  $\|\cdot\|_{k,p}$  for  $1 \leq p < \infty$ . Its completion is the *Sobolev spaces*.

**Summary 🦋 1.9 — Norm changing the topology in action!** Consider the spaces  $V = C^1[0, 1] \subset$

$C[0, 1]$ , and  $W = C[0, 1]$ , and the linear operator

$$T = \frac{d}{dx} : V \rightarrow W.$$

Consider the same infinity norm  $\|\cdot\|_\infty$  for both  $V$  and  $W$ . Let  $\{f_n\}$  be a sequence of functions in  $V$  defined as

$$f_n(x) = \frac{1}{n} \sin(2^n \pi x).$$

It is evident that  $\|f_n\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ . However,  $\|f'_n\|_\infty \rightarrow \infty$  as  $n \rightarrow \infty$ . Geometrically, we can feel that the sequence  $\{f_n\}$  sort of moves towards the origin of the space  $(V, \|\cdot\|_\infty)$  while  $\{f'_n\}$  shoots to infinity in the space  $(W, \|\cdot\|_\infty)$ .

However, if we change the norm of space  $V$  to the standard norm of  $C^1[0, 1]$ , i.e.

$$\|f\|_\infty^1 = \max\{\|f\|_\infty, \|f'\|_\infty\},$$

then we can see that the sequence  $\{f\}$  is not moving towards the origin, but shoots off to the infinity of the space  $(V, \|\cdot\|_\infty^1)$ . From this example it is clear that how norm induces topology. A sequence that originally was moving towards the origin in one topology, shoots off to the infinity in another topology.

**Summary 1.10 — Continuity of differentiation operator.** According to the summary box above, the differentiation operator

$$T_1 = \frac{d}{dx} : C^1[0, 1] \subset (C[0, 1], \|\cdot\|_\infty) \rightarrow (C[0, 1], \|\cdot\|_\infty)$$

is not continuous, but the operator

$$T_2 = \frac{d}{dx} : (C^1[0, 1], \|\cdot\|_\infty^1) \rightarrow (C[0, 1], \|\cdot\|_\infty)$$

is continuous.

## 1.2 Solved Problems

■ **Problem 1.1 — The space of solutions of an ODE (from Atkinson).** Show that the set of all continuous solutions of the differential equation  $u''(x) + u(x) = 0$  is a finite-dimensional linear space. Is the set of all continuous solutions of  $u''(x) + u(x) = 1$  is a linear space?

**Solution** Denote the set of all solutions for the ODE  $u'' + u = 0$  as

$$S = \{f \in C(\mathbb{R}) \mid f'' + f = 0\}.$$

We claim that  $S$  is a linear space. Because

- Closed under addition: Let  $f, g \in S$ . Then  $f'' + f = 0$  and  $g'' + g = 0$ . From the linearity of derivative it follows that  $(f + g)'' + (f + g) = 0$ , hence  $f + g \in S$ .
- Existence of zero element: The function  $g \equiv 0$  is in  $S$ .
- Existence of inverse element: Let  $f \in S$ . Then  $f'' + f = 0$ . Multiplying both sides by  $-1$  we will get  $(-f)'' + (-f) = 0$ . Thus  $-f \in S$ .
- Closed under scalar multiplication: Let  $f \in S$ . Then  $f'' + f = 0$ . Multiplying both sides by  $a \in \mathbb{R}$  we will get  $(af)'' + (af) = 0$ . Thus  $af \in S$ .



- Commutativity, associativity, distributivity, and follows from the same properties for the addition of functions.

To show that the dimension of this linear space is finite, consider two solutions  $u_1, u_2 \in S$  such that their Wronskian is non-zero. I.e.

$$W(t) = \det \begin{pmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{pmatrix} \neq 0.$$

For the particular ODE given in this question, we can consider  $u_1(t) = \cos(t)$  and  $u_2(t) = \sin(t)$ . Take any solution  $v \in S$ . Assume  $v(0) = a$  and  $v'(0) = b$ . Consider  $w(t) = pu_1(t) + qu_2(t)$  where  $p, q \in \mathbb{R}$  chosen such that  $v(0) = w(0)$  and  $v'(0) = w'(0)$ . Since both  $w, v$  are solutions of the ODE, then from the existence-uniqueness theorem, it follows that  $v(t) = w(t)$ . This shows that we can write every solution of the ODE in terms of  $u_1$  and  $u_2$ . Thus  $S$  is a linear space of dimension 2.

The continuous solutions of the ODE  $u'' + u = 1$  is not a linear space as it does not contain the zero element. However, we can show that this is an affine space.

■ **Problem 1.2 — Linear space (from Atkinson).** When is the set  $\{v \in C[0, 1] \mid v(0) = a\}$  a linear space?

**Solution** This set is a linear space only when  $a = 0$ . Otherwise, this set can not contain the zero function (to be served as the zero element of the vector space). Also, if  $a \neq 0$ , then this set will not be closed under addition and scalar multiplication.

■ **Problem 1.3 — Zero vector and linear independence (from Atkinson).** Show that in any linear space  $V$ , a set of vectors is always linearly dependent if one of the vectors is zero.

**Solution** Let  $\{u_1, u_n, f\}$  be a collection of vectors where  $f$  is the zero vector. Let  $\alpha_1 = \cdots = \alpha_n = 0$  and  $\beta \neq 0$  and consider the sum

$$\alpha_1 u_1 + \cdots + \alpha_n u_n + \beta f = 0.$$

There is one non-zero coefficient  $\beta$ , thus the collection of vectors are linearly dependent.

■ **Problem 1.4 — Unique expansion in terms of basis vectors (from Atkinson).** Let  $\{v_1, \dots, v_n\}$  be a basis of an  $n$ -dimensional space  $V$ . Show that for any  $v \in V$ , there are scalars  $\alpha_1, \dots, \alpha_n$  such that

$$v = \sum_{i=1}^n \alpha_i v_i,$$

and the scalars  $\alpha_1, \dots, \alpha_n$  are uniquely determined by  $v$ .

**Solution** Let  $\mathbb{B} = \{v_1, \dots, v_n\}$  be a basis and let  $v \in V$  be any vectors. Since  $\mathbb{B}$  is a basis, then by definition the vectors  $v_1, \dots, v_n$  are

- (I) linearly independent, and
- (II) spans the whole space.

(II) implies the existence of the scalars  $\alpha_1 \cdots \alpha_n$  such that

$$v = \sum_{i=1}^n \alpha_i v_i.$$

Furthermore, (I) implies the uniqueness of these scalars. To see this we will use the proof by contradiction. Consider the  $\beta_1, \dots, \beta_n$  where we have  $\beta_i \neq \alpha_i$  at least for one  $1 \leq i \leq n$ . Then

$$v = \sum_{i=1}^n \alpha_i v_i, \quad v = \sum_{i=1}^n \beta_i v_i.$$

Subtracting these two expressions we will get

$$0 = \sum_{i=1}^n (\alpha_i - \beta_i) v_i.$$

Since  $\alpha_i \neq \beta_i$  for at least one index  $i$ . From the definition of linear dependence, this implies that the collection of vectors in  $\mathbb{B}$  is linearly dependent that contradicts (I) which is a contradiction.

■ **Problem 1.5 — Cartesian product of linear spaces (from Atkinson).** Assume  $U$  and  $V$  are finite dimensional linear spaces, and let  $\{u_1, \dots, u_n\}$  and  $\{v_1, \dots, v_m\}$  be bases for them, respectively. Using these bases, create a basis for  $W = U \times V$ . Determine  $\dim W$ .

**Solution** Consider the following basis for  $U$  and  $V$

$$\mathbb{B}_U = \{u_1, \dots, u_n\}, \quad \mathbb{B}_V = \{v_1, \dots, v_m\}.$$

Construct the sets

$$\mathcal{B}_U = \{(u_i, 0_V) \mid u_i \in \mathbb{B}_U, 0_V \in V\}, \quad \mathcal{B}_V = \{(0_U, v_i) \mid v_i \in \mathbb{B}_V, 0_U \in U\}.$$

Then the following collection will be a basis for  $U \times V$ .

$$\mathbb{B}_{U \times V} = \mathcal{B}_U \cup \mathcal{B}_V.$$

The linearindependentness of the vectors in  $\mathbb{B}_{U \times V}$  follows immediately from the linearindependentness of  $\mathbb{B}_1$  and  $\mathbb{B}_2$ . Similarly, it follows immediately from the spanning property of  $\mathbb{B}_U$  and  $\mathbb{B}_V$  that  $\mathbb{B}_{U \times V}$  spans the whole space  $U \times V$ . This construction reveals that the space  $U \times V$  has dimension  $n + m$ .



## 2. Side Notes

### 2.1 Errors in the book “Finite Difference Computing with PDEs” by Hans Petter

- Equation (3.29): The forcing term is not present (maybe they have somehow assumes that this is zero but according to (3.40) this is unlikely).
- Equation (3.31): The forcing terms has no  $\Delta t$  coefficient.
- Equation (3.40): Comparing this with equation (3.31) reveals that the right hand side is not correct.
- Section (3.3.7): The very first equation in this section.  $Dxu$  should read  $D_x u$
- Page 256: I would prefer to see the more standard notation for the “mapsto” which is  $(0, 0) \mapsto 0$  instead of simple arrow.