Lecture Notes For: Probability and Stochastic Process

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0. Contents

1. Conditional Expectation

■ **Example 1.1 — Running example 1 (Inspired by Gordan Zitkovic lecture notes).** Throughout this chapter the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \{a, b, c, d, e, f\}$, $\mathbb{F} = \mathcal{P}(\Omega)$, and $\mathbb{P}$ uniform will be running example to demonstrate different notions in a tangible way. The following random variables defined as

$$X : (\Omega, \mathcal{F}) \to (I, \mathcal{I}), \qquad Y : (\Omega, \mathcal{F}) \to (I, \mathcal{I}), \qquad Z : (\Omega, \mathcal{F}) \to (I, \mathcal{I}),$$

where $I = \{1, \dots, 10\}$ and $\mathcal{I} = \mathcal{P}(I)$. will be in particular useful:

$$X = \begin{pmatrix} a & b & c & d & e & f \\ 1 & 3 & 3 & 5 & 5 & 7 \end{pmatrix}, \quad Y = \begin{pmatrix} a & b & c & d & e & f \\ 2 & 2 & 1 & 1 & 7 & 7 \end{pmatrix}, \quad Z = \begin{pmatrix} a & b & c & d & e & f \\ 3 & 3 & 3 & 3 & 2 & 2 \end{pmatrix}.$$

It is also important to describe $\sigma(X), \sigma(Y)$, and $\sigma(Z)$ explicitly. The atoms of $\sigma(X)$ will be the $X^{-1}(1) = \{a\}, X^{-1}(2) = \{b\}, X^{-1}(3) = \{c\}, X^{-1}(5) = \{d\}, X^{-1}(7) = \{e\}, X^{-1}(11) = \{f\}$. Thus $\sigma(X) = \mathcal{P}(\Omega)$. With a similar argument the atoms of $\sigma(Y)$ is $Y^{-1}(4) = \{a, b\}, Y^{-1}(4) = \{c, d\}, Y^{-1}(6) = \{e, f\}$. And finally, the atoms of Z will be $Z^{-1}(8) = \{a, b, c, d\}$, and $Z^{-1}(9) = \{e, f\}$. In summary

$$\begin{aligned} \text{Atom}(\sigma(X)) &= \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}\}, \\ \text{Atom}(\sigma(Y)) &= \{\{a, b\}, \{c, d\}, \{e, f\}\}, \\ \text{Atom}(\sigma(Z)) &= \{\{a, b, c, d\}, \{e, f\}\}. \end{aligned}$$

■

■ **Example 1.2 — Running example 2 (inspired from Nima Moshayedi's lecture notes).** $N \sim \text{Poisson}(\lambda)$. Consider a game, where we say that when $N = n$ we do n independent tossing of a coin where each time one obtains 1 with probability $p \in [0, 1]$ and 0 with probability $1 - p$. Define also S to be the random variable giving the total number of 1 obtained in the game. Therefore, if $N = n$ is given, we get that S has binomial distribution with parameters (p, n). ■

■ **Example 1.3 — Running example 3.** I will add some suitable random variable with density function $f_{X,Y}(x, y)$. The goal is to later calculate $f_{X|Y}(x, y)$ and $\mathbb{E}\left[X|Y\right]$, etc. TODO: Will be designed and added later. ■

We will be using the simple lemma below to demonstrate the main ideas of the conditional expectation.

Lemma 1.1 — Projection Lemma. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X a non-negative random variable. Then $\mathbb{E}[X] \in \mathbb{R}$ is the unique number that minimizes

$$\mathbb{E}\left[|X-n|^2\right]$$

over all choices for $n \in \mathbb{R}$.

Proof. By differentiating and setting equal to zero we will have

$$0 = \frac{d}{dn}\mathbb{E}\left[\frac{d}{dn}|X-n|^2\right] = \mathbb{E}\left[|X|\right] - \mathbb{E}\left[|n|\right].$$

So

$$\mathbb{E}[X] = n.$$

□

1.1 Conditional Probability (Discrete Case)

Summary 1.1 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $X, Y : \Omega \to I$ discrete random variables. Then the conditional expectation $\mathbb{E}\left[X|Y\right] : \Omega \to \mathbb{R}$:

(i) (Projective definition) is the unique $\sigma(Y)$ measurable random variable that minimizes $\mathbb{E}\left[|X' - X|^2\right]$ among all $\sigma(Y)$ measurable random variables X'. So $\mathbb{E}\left[X|Y\right]$ can be thought as the orthogonal projection of $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ to the subspace $L^2(\Omega, \sigma(Y), \mathbb{P})$.

(ii) (Alternative definition) is a random variable whose values are given as

$$\mathbb{E}\left[X|Y\right](\omega) = \sum_{n \in I} n\mathbb{P}(X = n | Y = Y(\omega)).$$

The nice thing about considering the conditional probability in the discrete case is the ability to do some explicit calculations.

Proposition 1.1 — Properties of conditional expectation. 1. Tower property: If $\sigma(Z) \subseteq^{\sigma} \sigma(Y)$ then

$$\mathbb{E}\left[\mathbb{E}\left[X|Y\right]|Z\right] = \mathbb{E}\left[X|Y\right].$$

2. Pulling out what is known: Let Z be $\sigma(Y)$ measurable. Then

$$\mathbb{E}\left[XZ|Y\right] = Z\mathbb{E}\left[X|Y\right].$$

.

■ **Example 1.4** In the running example in Example 1.1 calculate $\mathbb{E}\left[X|Y\right]$, $\mathbb{E}\left[X|Z\right]$, and explicitly verify if these random variables are $\sigma(Y)$ and $\sigma(Z)$ measurable (respectively). Then check the properties in Proposition 1.1. ■

Solution Recall that we had

$$X = \begin{pmatrix} a & b & c & d & e & f \\ 1 & 3 & 3 & 5 & 5 & 7 \end{pmatrix}, \quad Y = \begin{pmatrix} a & b & c & d & e & f \\ 2 & 2 & 1 & 1 & 7 & 7 \end{pmatrix}, \quad Z = \begin{pmatrix} a & b & c & d & e & f \\ 3 & 3 & 3 & 3 & 2 & 2 \end{pmatrix}.$$

Calculating $\mathbb{E}\left[X|Y\right]$: So we want to calculate $\mathbb{E}\left[X|Y\right]$. Let $\omega \in \Omega$. Then

$$\mathbb{E}\left[X|Y\right](\omega) = \sum_{n=0}^{\infty} n\mathbb{P}(X = n|Y = Y(\omega)).$$

When $\omega = a$ we have

$$\mathbb{E}\left[X|Y\right](a) = 1 \cdot \frac{\mathbb{P}(X = 1, Y = 2)}{\mathbb{P}(Y = 2)} + 3 \cdot \frac{\mathbb{P}(X = 3, Y = 2)}{\mathbb{P}(Y = 2)} = (1 + 3)/2 = 2.$$

With a similar argument we can calculate $\mathbb{E}\left[X|Y\right](b) = 2$. Let $\omega = c$. Then

$$\mathbb{E}\left[X|Y\right](c) = 3 \cdot \frac{\mathbb{P}(X = 3, Y = 1)}{\mathbb{P}(Y = 1)} + 5 \cdot \frac{\mathbb{P}(X = 5, Y = 1)}{\mathbb{P}(Y = 1)} = (3 + 5)/2 = 8.$$

With similar calculations we can see that

$$\mathbb{E}\left[X|Y\right] = \begin{pmatrix} a & b & c & d & e & f \\ 2 & 2 & 4 & 4 & 6 & 6 \end{pmatrix}.$$

Calculating $\mathbb{E}\left[X|Z\right]$: Similar to above, let $\omega = a$. Then

$$\mathbb{E}\left[X|Z\right](a) = 1 \cdot \frac{\mathbb{P}(X = 1, Z = 3)}{\mathbb{P}(Z = 3)} + 3 \cdot \frac{\mathbb{P}(X = 3, Z = 3)}{\mathbb{P}(Z = 3)} + 3 \cdot \frac{\mathbb{P}(X = 3, Z = 3)}{\mathbb{P}(Z = 3)} + 5 \cdot \frac{\mathbb{P}(X = 5, Z = 3)}{\mathbb{P}(Z = 3)} = (1+3+3+5)/4 = 3.$$

And with a similar computation we will get $\mathbb{E}\left[X|Y\right](e) = 6$. So we can write

$$\mathbb{E}\left[X|Z\right] = \begin{pmatrix} a & b & c & d & e & f \\ 3 & 3 & 3 & 3 & 6 & 6 \end{pmatrix}.$$

Measurability of $\mathbb{E}\left[X|Y\right], \mathbb{E}\left[X|Z\right]$: Recall the atoms of the σ-algebra generated by X, Y, Z as

$$\begin{aligned} \mathrm{Atom}(\sigma(X)) &= \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}\}, \\ \mathrm{Atom}(\sigma(Y)) &= \{\{a, b\}, \{c, d\}, \{e, f\}\}, \\ \mathrm{Atom}(\sigma(Z)) &= \{\{a, b, c, d\}, \{e, f\}\}. \end{aligned}$$

So it is clear that $\mathbb{E}\left[X|Y\right]$ is $\sigma(Y)$-measurable, while $\mathbb{E}\left[X|Z\right]$ is $\sigma(Z)$-measurable.
Verification of the projection interpretation. Recall Lemma 1.1. Then it immediately follows that the only function that assumes constant values on the atoms of $\sigma(Y)$ (or $\sigma(Z)$), hence $\sigma(Y)$-measurable (or $\sigma(Z)$-measurable) is the function that assumes the average value of Y (or Z) on the atoms of $\sigma(Y)$ (or $\sigma(Z)$) with the law $\mathcal{L}(\dot{|}A)$ (or $\mathcal{L}(\dot{|}B)$) where A is an atom of $\sigma(X)$ (or where B is an atom of $\sigma(Y)$).
Checking the Tower property. For an easier notation we will write $\tilde{X}_Y = \mathbb{E}\left[X|Y\right]$, and $\tilde{X}_Z = \mathbb{E}\left[X|Z\right]$. Then

$$\mathbb{E}\left[\mathbb{E}\left[X|Y\right]|Z\right](a) = \mathbb{E}\left[\tilde{X}_Y|Z\right] = (2 + 2 + 4 + 4)/4 = 3,$$

and similarly we can compute

$$\mathbb{E}\left[\mathbb{E}\left[X|Y\right]|Z\right] = \begin{pmatrix} a & b & c & d & e & f \\ 3 & 3 & 3 & 3 & 6 & 6 \end{pmatrix}.$$

And similarly we can compute

$$\mathbb{E}\left[X|Z\right] = \begin{pmatrix} a & b & c & d & e & f \\ 3 & 3 & 3 & 3 & 6 & 6 \end{pmatrix}.$$

In this case that the sample space and the random variables are finite, this property exactly translates to the fact that in order to compute the average of say n numbers, it is the same if we compute the average for the some disjoint sub-collections and then average those average values.

Checking the pull out property. Let W be a random variable that is $\sigma(Y)$ measurable, say given as

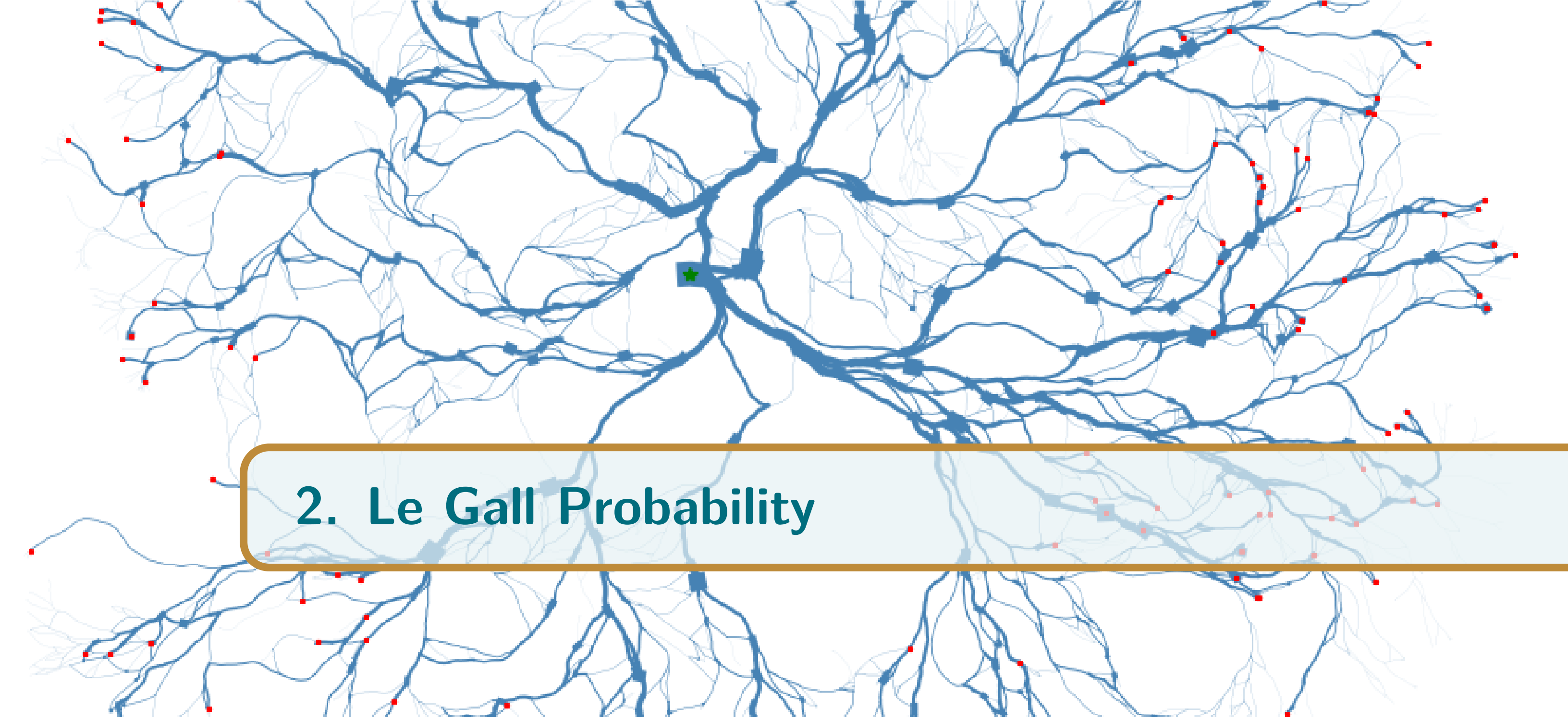
$$W = \begin{pmatrix} a & b & c & d & e & f \\ 5 & 5 & 6 & 6 & 7 & 7 \end{pmatrix}$$

Then

$$WX = \begin{pmatrix} a & b & c & d & e & f \\ 5 & 15 & 18 & 30 & 35 & 49 \end{pmatrix}$$

So we can compute

$$\mathbb{E}\left[WX|Y\right] = \begin{pmatrix} a & b & c & d & e & f \\ 10 & 10 & 24 & 24 & 42 & 42 \end{pmatrix}.$$



2. Le Gall Probability

In this chapter I will cover the proofs, problems, and etc that are left to the reader. I will also add some remarks as I read through the text. I will also add some theorems and their proofs that are hard to understand in the text but I have made some helpful remarks.

■ **Problem 2.1** We define the discrete conditioning as

$$\mathbb{P}'(A) = \mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

where $A, B \in \mathcal{F}$ are measurable sets. Furthermore, we define the conditional expectation of a non-negative, or L^1 random variable X as

$$\mathbb{E}\left[X|B\right] := \frac{\mathbb{E}\left[\mathbb{1}_B X\right]}{\mathbb{P}(X)}.$$

Show that $\mathbb{E}\left[X|B\right] = \mathbb{E}_{\mathbb{P}'}\left[X\right]$.

Solution We first show this for a random variable given as

$$X = \alpha \mathbb{1}_B$$

for some $A \in \mathcal{F}$. Observe that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}'}\left[X\right] &= \mathbb{E}_{\mathbb{P}'}\left[\alpha \mathbb{1}_A\right] = \alpha \mathbb{E}_{\mathbb{P}'}\left[\mathbb{1}_A\right] \\
&= \alpha \mathbb{P}'(A) = \alpha \mathbb{P}(A|B) \\
&= \alpha \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \\
&= \alpha \frac{\mathbb{E}\left[\mathbb{1}_A \mathbb{1}_B\right]}{\mathbb{P}(B)} \\
&= \frac{\mathbb{E}\left[\alpha \mathbb{1}_A \mathbb{1}_B\right]}{\mathbb{P}(B)} \\
&= \frac{\mathbb{E}\left[X \mathbb{1}_B\right]}{\mathbb{P}(B)} = \mathbb{E}\left[X|B\right].
\end{aligned}$$

Using the linearity of the expectation, the conclusion above also follows for a simple random variable $X = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{B_i}$ for $B_i \in \mathcal{F}$ for all $i = 1, \ldots, n$. Let X be a non-negative random variable. We can construct a sequence of random variables $\{X_n\}$ such that $X_n \uparrow X$. From monotone convergence theorem we have $\mathbb{E}_{\mathbb{P}'}[X_n] \uparrow \mathbb{E}_{\mathbb{P}'}[X]$, as well as $\mathbb{E}[\mathbb{1}_B X_n] \uparrow \mathbb{E}[\mathbb{1}_B X]$ as $n \to \infty$. Since each X_n is a simple random variable we also have

$$\mathbb{E}_{\mathbb{P}'}[X_n] = \frac{\mathbb{E}[X_n \mathbb{1}_B]}{\mathbb{P}(B)}.$$

Letting $n \to \infty$ on both sides of the equation above we will have

$$\mathbb{E}_{\mathbb{P}'}[X] = \frac{\mathbb{E}[X \mathbb{1}_B]}{\mathbb{P}(B)}.$$

And finally for the case that $X \in L^1$ we can write $X = X^+ - X^-$ where X^+ and X^- are non-negative random variables and using the linearity of expectation it also follows that

$$\mathbb{E}_{\mathbb{P}'}[X] = \frac{\mathbb{E}[X \mathbb{1}_B]}{\mathbb{P}(B)}.$$

Definition 2.1 Let $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ be a random variable and $Y : (\Omega, \mathcal{A}) \to (E, 2^E)$ be a random variable that assumes its values in a countable set E. The conditional expectation of X knowing Y is defined as

$$\mathbb{E}\left[X|Y\right] = \varphi(Y),$$

where $\varphi : E \to \mathbb{R}$ is given as

$$\varphi(y) = \begin{cases} \mathbb{E}\left[X|Y = y\right] & y \in E' \\ 0 & y \notin E' \end{cases},$$

where $E' = \{y \in E : \mathbb{P}(Y = y) > 0\}$.

Alternatively, let $B_n = Y^{-1}(n)$ for $n \in Y$. Then $\mathcal{B} = \{B_n\}_{n \in E}$ is a countable partition for Ω. Let $\mathcal{B} \supset \mathcal{B}' = \{B \in \mathcal{B} : \mathbb{P}(B) > 0\}$, i.e. those sets in the partition that has value zero. Then the conditional expectation of X knowing Y is given as

$$\mathbb{E}\left[X|Y\right] = \sum_{B \in \mathcal{B}'} \mathbb{E}\left[X|B\right] \mathbb{1}_B.$$

where $\mathbb{E}\left[X|B\right] = \mathbb{E}[X \mathbb{1}_B] / \mathbb{P}(B)$ is as defined in Le Gall, page 228, first section of the page. Observe that with this definition it automatically follows that $X = 0$ on any $B \notin \mathcal{B}'$.

■ **Remark** Note that the value of $\varphi(y)$ when $y \notin E'$ can be any number and this will change $\mathbb{E}\left[X|Y\right]$ only on a set of measure zero. But it is convention to set it to be zero.

Theorem 2.1 Let $Y : (\Omega, \mathcal{A}) \to (E, 2^E)$ be a random variable where E is a countable set. Let $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$. Then we have $\mathbb{E}\left[\left|\mathbb{E}\left[X|Y\right]\right|\right] \le \mathbb{E}\left[|X|\right]$, and thus

$$\mathbb{E}\left[X|Y\right] \in L^1(\Omega, \mathcal{A}, \mathbb{P}).$$

Furthermore, for every bounded $\sigma(Y)$-measurable random variable Z we have

$$\mathbb{E}[ZX] = \mathbb{E}\left[Z\mathbb{E}\left[X|Y\right]\right].$$

Proof. **Proof for $\mathbb{E}\left[X|Y\right] \in L^1(\Omega, \mathcal{A}, \mathbb{P})$:** I will provide two proves for this. The first one is an elaborated version of Le Gall's proof, and the second one is my proof based on the stuff you can read in the appendix of this note.

- **Proof (1)** For Le Gall's proof we we will have

$$\mathbb{E}\left[\left|\mathbb{E}\left[X|Y\right]\right|\right] = \sum_{y\in E'} \mathbb{P}(Y = y) \left|\mathbb{E}\left[X|Y = y\right]\right|.$$

  Note that for this step we used the definition of expectation for a random variable that takes countably many values (i.e. $\mathbb{E}\left[X|Y\right]$). Note that since Y takes countably many values, then by the definition of $\mathbb{E}\left[X|Y\right]$ (see Le Gall Definition 11.1), $\mathbb{E}\left[X|Y\right]$ also assumes countably many values. Furthermore, note that $E' = \{y \in E : \mathbb{P}(Y = y) > 0\}$. A different way to write this is to use the partition generated by Y. I.e. we can write

$$\mathbb{E}\left[\left|\mathbb{E}\left[X|Y\right]\right|\right] = \sum_{B\in\mathcal{B}'} \mathbb{P}(B) \left|\mathbb{E}\left[X|B\right]\right|$$

  which is exactly the same as the first expression replacing $\{Y = y\}$ with B and changing the summation index accordingly. So we will have

$$\begin{aligned}
\mathbb{E}\left[\left|\mathbb{E}\left[X|Y\right]\right|\right] &= \sum_{y\in E'} \mathbb{P}(Y = y) \left|\mathbb{E}\left[X|Y = y\right]\right| \\
&= \sum_{y\in E'} \mathbb{P}(Y = y) \left|\frac{\mathbb{E}\left[X\mathbb{1}_{\{Y=y\}}\right]}{\mathbb{P}(Y = y)}\right| \\
&= \sum_{y\in E'} \left|\mathbb{E}\left[X\mathbb{1}_{\{Y=y\}}\right]\right| \\
&\le \sum_{y\in E'} \mathbb{E}\left[|X|\,\mathbb{1}_{Y=y}\right] \\
&= \mathbb{E}\left[|X| \sum_{y\in E'} \mathbb{1}_{\{Y=y\}}\right] \\
&= \mathbb{E}\left[|X|\right].
\end{aligned}$$

- **Proof (2)** For this proof we will use the fact that since $\mathbb{E}\left[X|Y\right]$ is $\sigma(Y)$-measurable, then we can write it as

$$\mathbb{E}\left[X|Y\right] = \sum_{y\in E'} \mathbb{E}\left[X|Y = y\right] \mathbb{1}_{Y=y}$$

  or equivalently

$$\mathbb{E}\left[X|Y\right] = \sum_{B\in\mathbb{B}'} \mathbb{E}\left[X|B\right] \mathbb{1}_{B},$$

  where E' and $\mathcal{B}'$ are as in Definition 2.1. For simplicity for the notation we will write $\alpha_B = \mathbb{E}\left[X|B\right]$.

So we will have

$$\begin{aligned}
\mathbb{E}\left[\left|\mathbb{E}\left[X|Y\right]\right|\right] &= \mathbb{E}\left[\left|\sum_{B\in\mathcal{B}'}\alpha_B \mathbb{1}_B\right|\right] \\
&\leq \mathbb{E}\left[\sum_{B\in\mathcal{B}'}\left|\alpha_B\right| \mathbb{1}_B\right] \\
&= \sum_{B\in\mathcal{B}'}\mathbb{E}\left[\left|\alpha_B\right| \mathbb{1}_B\right] \\
&= \sum_{B\in\mathcal{B}'}\left|\alpha_B\right| \mathbb{P}(B) \\
&= \sum_{B\in\mathcal{B}'}\frac{\left|\mathbb{E}\left[X\mathbb{1}_B\right]\right|}{\mathbb{P}(B)}\mathbb{P}(B) \\
&\leq \sum_{B\in\mathcal{B}'}\mathbb{E}\left[\left|X\right| \mathbb{1}_B\right] \\
&= \mathbb{E}\left[\left|X\right|\sum_{B\in\mathcal{B}'}\mathbb{1}_B\right] \\
&= \mathbb{E}\left[\left|X\right|\right].
\end{aligned}$$

Proof for $\mathbb{E}\left[ZX\right] = \mathbb{E}\left[Z\mathbb{E}\left[X|Y\right]\right]$. Since Z is $\sigma(Y)$-measurable, by Proposition 8.9 Le Gall, we can find a bounded function φ such that $Z = \varphi(Y)$. So we can write

$$\begin{aligned}
\mathbb{E}\left[Z\mathbb{E}\left[X|Y\right]\right] &= \mathbb{E}\left[\varphi(Y)\mathbb{E}\left[X|Y\right]\right] \\
&= \sum_{y\in E}\varphi(y)\mathbb{E}\left[X|Y=y\right]\mathbb{P}(Y=y) \\
&= \sum_{y\in E}\varphi(y)\mathbb{E}\left[X\mathbb{1}_{\{Y=y\}}\right] \\
&= \mathbb{E}\left[X\sum_{y\in E}\varphi(y)\mathbb{1}_{\{Y=y\}}\right] \\
&= \mathbb{E}\left[XZ\right],
\end{aligned}$$

where we have used the Fubini's theorem to interchange $\mathbb{E}$ with sum, and also we used the fact that $Z = \sum_{y\in E}\varphi(y)\mathbb{1}_{Y=y}$ (that follows from $Z = \varphi(Y)$).

□

■ **Problem 2.2** In the properties of conditional expectation of random variables in Le Gall (page 234-245), right after part (d) he concludes that using the property (d) for any non-negative random variable $(Y_n)_{n\in\mathbb{N}}$ we have

$$\mathbb{E}\left[\sum_n Y_n\middle|\mathcal{B}\right] = \sum_n \mathbb{E}\left[Y_n|\mathcal{B}\right].$$

Prove this!

Solution Since $(Y_n)_{n\in\mathbb{N}}$ is non-negative, then the sequence of partial sums form an increasing sequence $\sum_{n=1}^{m} Y_n = X_m \uparrow \sum_n Y_n$. Using the linearity of conditional expectation we can write

$$\mathbb{E}\left[\sum_{n=1}^{m} Y_n \Big| \mathcal{B}\right] = \sum_{n=1}^{m} \mathbb{E}\left[Y_n \Big| \mathcal{B}\right].$$

Taking the limit of both sides and using the property (d) we will have

$$\mathbb{E}\left[\sum_{n} Y_n \Big| \mathcal{B}\right] = \sum_{n} \mathbb{E}\left[Y_n | \mathcal{B}\right].$$

Proposition 2.1 — Properties of conditional expectation for integrable random variables. Let $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$. Then we have

(a) If X is $\mathcal{B}$-measurable, then $\mathbb{E}\left[X|\mathcal{B}\right] = X$.

(b) The mapping $X \mapsto \mathbb{E}\left[X|\mathcal{B}\right]$ is linear on $L^1(\Omega, \mathcal{A}, \mathbb{P})$.

(c) If $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$, then $\mathbb{E}\Big[\mathbb{E}\left[X|\mathcal{B}\right]\Big] = \mathbb{E}\left[X\right]$.

(d) If $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$, then $\Big|\mathbb{E}\left[X|\mathcal{B}\right]\Big| \leq \mathbb{E}\left[|X|\,|\mathcal{B}\right]$ a.s. and consequently, we have

$$\mathbb{E}\Big[\Big|\mathbb{E}\left[X|\mathcal{B}\right]\Big|\Big] \leq \mathbb{E}\left[|X|\right].$$

There for the mapping $X \mapsto \mathbb{E}\left[X|\mathcal{B}\right]$ is a contraction mapping of $L^1(\Omega, \mathcal{A}, \mathbb{P})$.

(e) Let $X, X' \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ where $X \geq X'$. Then it implies that

$$\mathbb{E}\left[X|B\right] \geq \mathbb{E}\left[X'|\mathcal{B}\right].$$

Proof. (a) By definition of conditional expectation for integrable random variables (see Theorem and Definition 11.3 Le Gall) the conditional expectation of X is a unique (up to a set of measure zero) $\mathcal{B}$-measurable random variable. On the other hand X is itself a $\mathcal{B}$-measurable random variable. So $\mathbb{E}\left[X|\mathcal{B}\right] = B$ almost surely.

(b) Let X, X' be two random variables and $\alpha, \beta \in \mathbb{R}$. Then for all random variable Z that is $\mathcal{B}$-measurable we have

$$\mathbb{E}\left[Z(\alpha X + \beta X')\right] = \mathbb{E}\Big[Z\mathbb{E}\left[\alpha X + \beta X'|\mathcal{B}\right]\Big].$$

Using the linearity of expectation, as well as the characterization of conditional expectation (11.1 Le Gall), for the LHS we have

$$\begin{aligned}\mathbb{E}\left[Z(\alpha X + \beta X')\right] = \alpha\mathbb{E}\left[ZX\right] + \beta\mathbb{E}\left[ZX'\right] &= \alpha\mathbb{E}\Big[Z\mathbb{E}\left[X|\mathcal{B}\right]\Big] + \beta\mathbb{E}\Big[Z\mathbb{E}\left[X'|\mathcal{B}\right]\Big] \\ &= \mathbb{E}\Big[Z(\alpha\mathbb{E}\left[X|\mathcal{B}\right] + \beta\mathbb{E}\left[X'|\mathcal{B}\right])\Big].\end{aligned}$$

Thus we have

$$\mathbb{E}\Big[Z\mathbb{E}\left[\alpha X + \beta X'|\mathcal{B}\right]\Big] = \mathbb{E}\Big[Z(\alpha\mathbb{E}\left[X|\mathcal{B}\right] + \beta\mathbb{E}\left[X'|\mathcal{B}\right])\Big].$$

This implies

$$\mathbb{E}\left[\alpha X + \beta X'|\mathcal{B}\right] = \alpha\mathbb{E}\left[X|\mathcal{B}\right] + \beta\mathbb{E}\left[X'|\mathcal{B}\right] \qquad a.s.$$

(c) Observe that constant random variable $X \equiv 1$ is a $\mathcal{B}$-measurable random variable. Using the characterization (11.2 Le Gall) it implies that

$$\mathbb{E}[X] = \mathbb{E}\left[\mathbb{E}\left[X|\mathcal{B}\right]\right].$$

Equivalently, we can use the characterization (11.1 Le Gall) where we take $B = \Omega$. Thus

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{1}_\Omega X] = \mathbb{E}\left[\mathbb{1}_\Omega \mathbb{E}\left[X|\mathcal{B}\right]\right] = \mathbb{E}\left[\mathbb{E}\left[X|\mathcal{B}\right]\right].$$

(d) First, observe that for any random variable $X \geq 0$ implies $\mathbb{E}\left[X|\mathcal{B}\right] \geq 0$. For $X \in L^(\Omega, \mathcal{A}, \mathbb{P})$ we can write

$$X = X^+ - X^-.$$

Thus we can write

$$\left|\mathbb{E}\left[X|\mathcal{B}\right]\right| = \left|\mathbb{E}\left[X^+|\mathcal{B}\right] + \mathbb{E}\left[X^-|\mathcal{B}\right]\right| \leq \left|\mathbb{E}\left[X^+|\mathcal{B}\right]\right| + \left|\mathbb{E}\left[X^-|\mathcal{B}\right]\right| = \mathbb{E}\left[X^+|\mathcal{B}\right] + \mathbb{E}\left[X^-|\mathcal{B}\right] = \mathbb{E}\left[|X|\,|\mathcal{B}\right].$$

The second part (the mapping being a contraction) follows immediately

$$\left\|\mathbb{E}\left[X|\mathcal{B}\right]\right\|_1 = \mathbb{E}\left[\left|\mathbb{E}\left[X|\mathcal{B}\right]\right|\right] \leq \mathbb{E}\left[\mathbb{E}\left[|X|\,|\mathcal{B}\right]\right] = \mathbb{E}\left[|X|\right] = \|X\|.$$

(e) We again use the fact that $X \geq 0$ implies $\mathbb{E}\left[X|\mathcal{B}\right] > 0$ (see Lemma below for the proof). The using the linearity of conditional expectation we have

$$0 \leq \mathbb{E}\left[(X - X')|\mathcal{B}\right] = \mathbb{E}\left[X|\mathcal{B}\right] - \mathbb{E}\left[X'|\mathcal{B}\right].$$

This implies that

$$\mathbb{E}\left[X|\mathcal{B}\right] \geq \mathbb{E}\left[X'|\mathcal{B}\right].$$

□

In the proof above, we used the fact $X \geq 0$ implies $\mathbb{E}\left[X|\mathcal{B}\right] \geq 0$. We prove this fact in the following Lemma.

Lemma 2.1 Let $X \geq 0$ be in $L^1(\Omega, \mathcal{A}, \mathbb{P})$, and $\mathcal{B}$ any sub σ-algebra. Then $\mathbb{E}\left[X|\mathcal{B}\right] \geq 0$ almost surely.

Proof. We simply use the characterization (11.1 Le Gall). Let $B \in \mathcal{B}$. Then

$$\mathbb{E}[\mathbb{1}_B X] = \mathbb{1}_{\mathcal{B}} \mathbb{E}\left[\mathcal{X}|\mathcal{B}\right].$$

We know the LHS is non-negative for all $B \in [\mathcal{B}]$. This implies

$$\mathbb{E}\left[\mathbb{1}_{\mathcal{B}} \mathbb{E}\left[\mathcal{X}|\mathcal{B}\right]\right] \geq 0, \quad \forall B \in \mathcal{B}.$$

Then it implies that

$$\mathbb{E}\left[X|\mathcal{B}\right] \geq 0.$$

□

Summary 2.1 — Conditional expectation. Le Gall starts developing the theory of conditional expectation by first considering the discrete conditioning. Then he generalizes it to conditioning with respect to some σ-algebra where the theory is developed for L^1 functions. Finally, he derives the notion of conditional expectation generally for any non-negative random variable that can be unbounded and even not integrable. However, for general random variables, we need the condition of integrablity L^1 to make the reasoning above for X^+ and X^- separately.

Proposition 2.2 Let X, Y be real random variable and assume that Y is $\mathcal{B}$-measurable. Then

$$\mathbb{E}\left[YX|\mathcal{B}\right] = Y\mathbb{E}\left[X|\mathcal{B}\right]$$

provided X and Y are both non-negative or X and XY are both integrable.

Proof. First we assume that X, Y are both non-negative. Then using the characterization conditional expectation for non-negative random variables (see 11.3 Le Gall), for any Z a $\mathcal{B}$-measurable random variable we can write

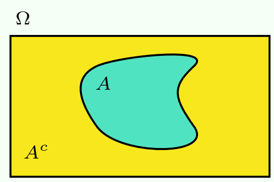
$$\mathbb{E}\left[Z\mathbb{E}\left[XY|\mathcal{B}\right]\right] = \mathbb{E}\left[XYZ\right] = \mathbb{E}\left[(ZY)X\right] = \mathbb{E}\left[ZY\mathbb{E}\left[X|\mathcal{B}\right]\right],$$

where we used the fact that YZ is $\mathcal{B}$-measurable (since both of them are $\mathcal{B}$-measurable). Again using the characterization of conditional expectation for non-negative random variables (11.3 Le Gall) we can write

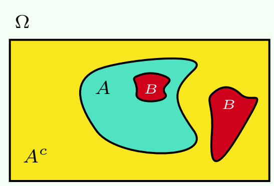
$$\mathbb{E}\left[XY|\mathcal{B}\right] = Y\mathbb{E}\left[X|\mathcal{B}\right].$$

Now it remains to show this for the case when X and XY are both integrable random variables (i.e. are L^1). Then we can have the same argument as above for $X = X^+ - X^-$ as well as $Y = Y^+ - Y^-$. $\square$

Summary 2.2 — Some notes on independent events. Let $A \in \mathcal{A}$ be an event. We can depict this event as the following Venn diagram.



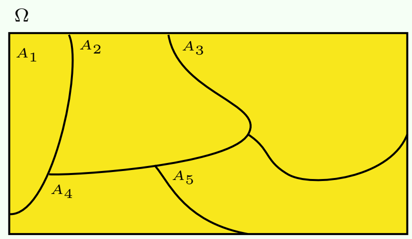
Let $B \in \mathcal{A}$ with $\mathbb{P}(B) = p$. Then B is independent of A if and only if B occupies p proportion of A and its complement A^c. I.e. the part of B that is in A should have $\mathbb{P}(A \cap B) = p\mathbb{P}(A)$ and the portion of B that is in A^c should also have $\mathbb{P}(A^c \cap B) = p\mathbb{P}(A^c)$ (this is literally the definition of independence). For instance, if $\mathbb{P}(B) = 0.2$, then B should be such that occupies 20% of A and 20% of A^c. A possible Venn diagram is



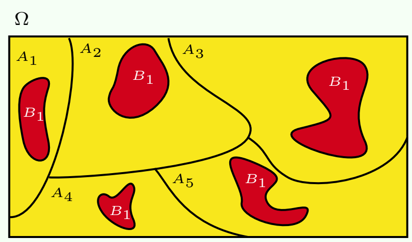
Now to understand the meaning of independent random variables we first need to understand the notion of independent σ-algebras. Considering the summary above it is very easy to characterize this notion intuitively. Consider the following summary for more details.

Summary 2.3 — Independence of σ-algebras. The independence of two σ-algebras is easiest to understand to understand when considering the atoms of σ-algebras. One might argue that not all σ-algebras will have such simple atoms. But the general philosophy of a theory in mathematics (for me at least) is to simply say "**that** weird infinite dimensional and non imaginable object is (axiomatically) the same thing as **this** simple object that you can imagine and work with". That is why we will provide only the intuitive explanation here and the whole theory that we will be developing will make sure that a similar kind of behaviour is also true for very complex situations that we are not even close the imagine them, let alone have arguments about them.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ are two σ-algebras with atoms $\mathcal{A}_1 = \{B_j\}$ and $\mathcal{A}_2 = \{A_i\}$. For instance, $\mathcal{A}_2$ can be depicted as below



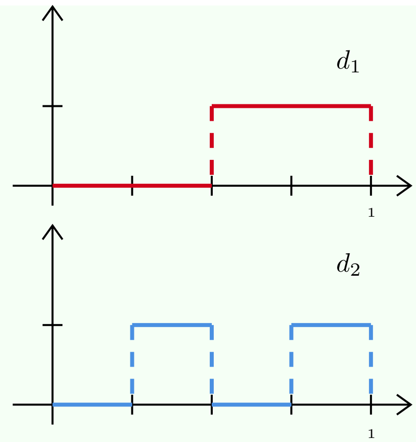
Then $\mathcal{B}_1$ is independent of $\mathcal{B}_2$ if every atom of $\mathcal{B}_1$ occupies the $\mathbb{P}(B_i)$ proportion of each A_i. For instance if $\mathbb{P}(B_1) = 0.3$ then $\mathbb{P}(B_1 \cap A_i) = 0.3\mathbb{P}(A_i)$ for all A_i. For instance we can depict this as the follows.



Another way to write this is for every $B \in \mathcal{B}_1$ to have

$$\mathbb{E}\left[\mathbb{1}_B | \mathcal{B}_2\right] = \mathbb{P}(B).$$

And we say that two random variables X, Y are independent, if their generating σ-algebras are independent. For example, let d_n and d_m be two dyadic functions. If $n \neq m$ then d_n and d_m are independent.


d_1
1
d_2
1