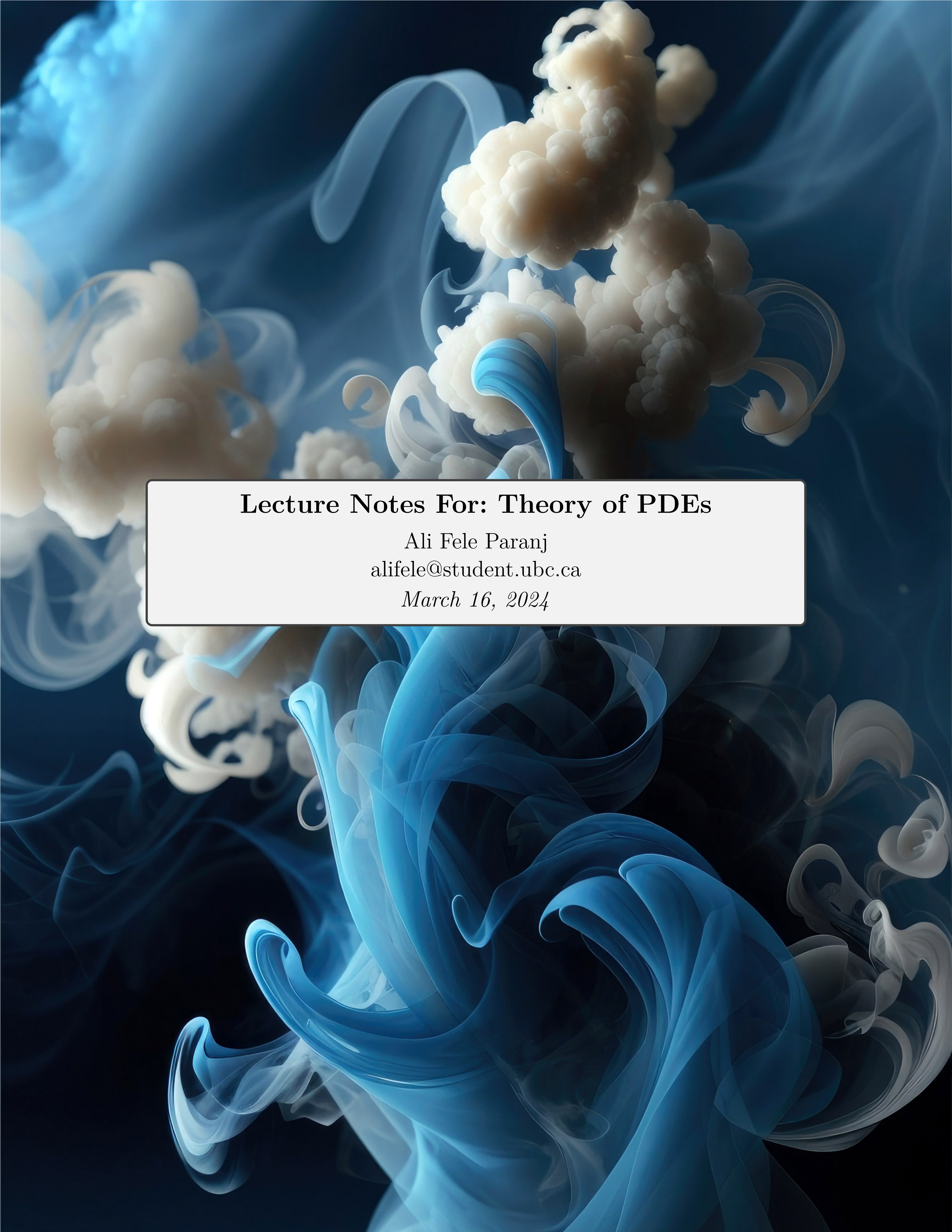

Lecture Notes For: Theory of PDEs

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0. Contents

Part I

Ordinary Differential Equations

1. Second Order Linear Differential Equations

We start with the definition of second order linear differential equation.

Definition 1.1 — General form of second order linear differential equation. The second order differential equation for function $y : (\alpha, \beta) \to \mathbb{R}$ is of the form

$$y'' + p(x)y' + q(x)y = f,$$

for some known functions p, q, f. If we consider the right-hand side be identically zero function,

$$y'' + p(x)y' + q(x)y = 0,$$

is called a homogeneous differential equation.

If we impose additional initiation conditions for the differential equation, then we will have an initial value problem.

Definition 1.2 — Initial value problem. The differential equation

$$y'' + p(x)y' + q(x)y = f, \qquad x \in (\alpha, \beta)$$

where p, q, f are known functions of x, along with the initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y'_0,$$

where $t_0 \in (\alpha, \beta)$, us called an initial value problem.

Theorem 1.1 — Wronskian Condition. Let y_1 and y_2 be two solutions of the homogeneous differential equation

$$y'' + py' + qy = 0.$$

The

$$y = c_1 y_1 + c_2 y_2,$$

is the general solution of the differential equation (i.e. every solution can be written in this form) if we have

$$(y_1 y_2' - y_1' y_2)(t) \neq 0 \qquad \forall t \in (\alpha, \beta).$$

Proof. Let y_1 and y_2 be two solutions of the differential equation and let $\varphi = c_1 y_1 + c_2 y_2$ be the general solution. Let y be any solution. Then at some time $t_0 \in (\alpha, \beta)$, it has the values $y(t_0) = y_0$ and $y'(0) = y_0'$. Then for the general solution we need to have

$$\varphi(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) = y_0, \qquad \varphi'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

which forms the following system of equations

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}.$$

The only way that we can find the coefficients c_1 and c_2 (that can depend on time t_0) for every $t_0 \in (\alpha, \beta)$ is when the matrix has non-zero determinant. This translates to

$$y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0) \neq 0 \qquad \forall t_0 \in (\alpha, \beta)$$

□

The theorem above is basically saying that if we can find two solutions for the linear second order ODE that has non-zero Wronskian, then every other solution of the differential equation will be a linear combination of these two solutions. This, in some sense, is the same as the linear independence.

Definition 1.3 — Wronskian of two functions. The Wronskian of two function y_1 and y_2 defined as a function

$$W[y_1, y_2] = y_1 y_2' - y_1' y_2.$$

The following Lemma shows the relation between the original linear second order homogeneous differential equation and the Wronskian.

Lemma 1.1 Let y_1 and y_2 be two solutions of

$$y'' + py' + qy = 0.$$

Then their Wronskian $W(t) := W[y_1, y_2](t)$ satisfies the following ODE.

$$W' + p(t)W = 0.$$

Proof. The Wronskian is defined to be

$$W = W[y_1, y_2] = y_1 y_2' - y_1' y_2.$$

Then calculating the first derivative will yield

$$W' = y_1 y_2'' - y_1'' y_2.$$

Since y_1 and y_2 satisfies the homogeneous ODE, then $y_1'' = -py_1' - y_1$ and $y_2'' = -py_2' - y_2$. Substituting in the equation above, we will get

$$W' = -p(y_1 y_2' - y_1' y_2) = -pW.$$

Thus we have shown that the Wronskian of two solutions satisfies $W' + pW = 0$. □

Now, based on the Lemma above, we can state and prove the following important proposition.

Proposition 1.1 — The Wronskian of two solutions is either always zero or non-zero. Let y_1 and y_2 be two solutions of

$$y'' + py' + qy = 0, \qquad t \in (\alpha, \beta)$$

Then the Wronskian of solutions is either identically zero in the interval (α, β) or non-zero for all $t \in (\alpha, \beta)$.

Proof. From the Lemma above, we know that the Wronskian $W(t) = W[y_1, y_2](t)$ satisfies

$$W' + pW = 0, \qquad t \in (\alpha, \beta).$$

Let $t_0 \in (\alpha, \beta)$. First, observe that the zero function, i.e. $O(t) \equiv 0$ for all $t \in (\alpha, \beta)$ is also a solution to the differential equation. If $W(t_0) = 0$, then by the existence-uniqueness theorem for the ODE that W satisfies, we conclude that W should also be identically zero in (α, β). Let $W(t)$ be non-zero for all $t \in (\alpha, \beta)$. Then we can write

$$\begin{aligned}
&W' + pW = 0. \\
&W'/W = -p. \\
&\frac{d}{dt}(\ln(W)) = -p. \\
&\int_{t_0}^{t} (\ln\big(W(\tau)\big))' \, d\tau = -\int_{t_0}^{t} p(\tau) \, d\tau. \\
&\ln\big(W(t)\big) - \ln\big(W(t_0)\big) = -\int_{t_0}^{t} p(\tau) \, d\tau. \\
&W(t) = W(t_0) \exp\left(-\int_{t_0}^{t} p(\tau) \, d\tau\right).
\end{aligned}$$

Observe that $\exp\big(-\int_{t_0}^{t} p(\tau) \, d\tau\big)$ is never zero. And since $W(t_0)$ is also non-zero $\forall t_0 \in (\alpha, \beta)$, then we can infer that $W(t)$ is either identically zero in (α, β) or always non-zero. □

As we will see in the following theorem, when the Wronskian of two solutions is zero, then we can infer some useful information about those solutions.

> **Proposition 1.2** Let y_1 and y_2 be two solutions of the following homogeneous linear second order ODE.
>
> $$y'' + py' + qy = 0, \qquad t \in (\alpha, \beta).$$
>
> If $W[y_1, y_2](t_0) = 0$ for some $t_0 \in (\alpha, \beta)$. Then one of the solutions is a constant multiple of the other.

Proof. Since y_1 and y_2 are solutions of the ODE provided in the proposition, then $W(t_0) = 0$ implies $W(t) = 0$ for all $t \in (\alpha, \beta)$. Then we will have

$$y_1 y_2' = y_1' y_2.$$

Assuming $y_1 y_2 \neq 0$ for all $t \in (\alpha, \beta)$, dividing both sides by $y_1 y_2$ we will get

$$y_2'/y_2 = y_1'/y_1 \implies \frac{d}{dt}(\ln\big(y_2(t)\big)) = \frac{d}{dt}(\ln\big(y_1(t)\big))$$

Integrating both sides we will have

$$\int_{t_0}^{t} \frac{d}{dt}(\ln\big(y_2(\tau)\big)) \, d\tau = \int_{t_0}^{t} \frac{d}{dt}(\ln\big(y_1(\tau)\big)) \, d\tau.$$

We will get

$$\ln\big(y_2(t)\big) - \ln\big(y_2(t_0)\big) = \ln\big(y_1(t)\big) - \ln\big(y_1(t_0)\big)$$

By rearranging the terms we will get

$$\ln\big(y_2(t)/y_1(t)\big) = \ln\big(y_2(t_0)/y_1(t_0)\big) = c.$$

Then this implies

$$y_2(t) = e^c y_1(t).$$

However, for the case where $y_1(t^*)y_2(t^*) = 0$ for some $t^* \in (\alpha, \beta)$, WLOG we can assume $y_1(t^*) = 0$ (otherwise, we can relabel the solutions). Since $W(t^*) = 0$, then we will get

$$y_1'(t^*)y_2(t^*) = 0. \implies y_1'(t^*) = 0 \vee y_2(t^*) = 0.$$

In the first case, we will have $y_1'(t^*) = 0$ and $y_1(t^*) = 0$ which from existence-uniqueness theorem we can conclude that $y_1(t) \equiv 0$ (thus y_1 is a trivially constant multiple of y_2). However, for the case where we assume $y_2(t^*) = 0$ I have not figure out yet how to prove. This is as a question in Braun, edition 4, section 2.1 problem 19. We somehow need to show that in this case we will have $y_2(t) = (y_2'(t^*)/y_1'(t^*))y_1$. $\square$

2. Systems of Ordinary Differential Equations

In this chapter we are going to study the solutions of n simultaneous first order differential equation in the form of

$$\dot{X} = F(t, X)$$

where $X : \mathbb{R} \to \mathbb{R}^n$, and F is a family of vector fields $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. In an expanded form, we can re-write this system of differential equations in the form

$$\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, x_2, \cdots, x_n), \\
\dot{x}_2 &= f_2(t, x_1, x_2, \cdots, x_n), \\
&\vdots \\
\dot{x}_n &= f_n(t, x_1, x_2, \cdots, x_n).
\end{aligned} \tag{1}$$

A solution of (1) is a parameterized curve in $\mathbb{R}^n$, or in other words, n functions $x_1, x_2, \cdots, x_n$ that satisfies the simultaneous system of ODEs as above. We also often impose initial condition

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \cdots \quad , x_n(t_0) = x_n^0.$$

We call system (1) along with the initial conditions as an initial value problem.

The following Lemma, reflects the importance of studying the system of first order differential equations.

Lemma 2.1 Every n-th order differential equation on a single variable y

$$y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_2 y'' + p_1 y' + p_0 y = g,$$

can be converted to a system of first order differential equations with the following change of variables

$$x_1 = y,\ x_2 = y',\ x_3 = y'',\ \cdots,\ x_n = y^{(n-1)}$$

which yields

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
&\vdots \\
\dot{x}_{n-1} &= x_n, \\
\dot{x}_n &= g - (p_{n-1}x_n + \cdots + p_2x_3 + p_1x_2 + p_0x_1)
\end{aligned}$$

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Part II

Partial Differential Equations

3. Introduction

3.1 Some Random Calculus

In this section I will cover some basic calculus concepts which will be used in the later chapters. I start with the Divergence theorem

Theorem 3.1 — Divergence theorem. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field defined on $\Omega \subset \mathbb{R}$. Then

$$\int_\Omega \nabla \cdot F dV = \int_{\partial\Omega} F \cdot d\mathbf{s}$$

By letting $F = \varphi\nabla\psi$, the theorem above helps us to define a similar notion to integration by parts in higher dimensions. To see this, first observe that

$$\nabla \cdot \varphi\nabla\psi = \nabla\varphi \cdot \nabla\psi + \varphi \underbrace{\nabla^2\psi}_{\nabla\cdot(\nabla\psi)} .$$

You can check this by simply writing the equation term by term. Then the divergence theorem for $\varphi\nabla\psi$ will be

$$\int_\Omega \nabla \cdot (\varphi\nabla\psi) dV = \int_\Omega (\nabla\varphi \cdot \nabla\psi + \varphi\nabla^2\varphi) dV = \int_{\partial\Omega} \varphi\nabla\psi \cdot d\mathbf{s}.$$

By rearranging the terms we will have

$$\int_\Omega \nabla\varphi \cdot \nabla\psi dV = \int_{\partial\Omega} \psi\nabla\varphi \cdot d\mathbf{s} - \int_\Omega \varphi\nabla \cdot (\nabla\varphi) dV$$

. To make this identity more similar to the integration by parts in 1D, let $F = \nabla\psi$. Then we will have the following theorem

Theorem 3.2 — Integration by parts in higher dimensions. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field, and $\varphi : \mathbb{R}^3 \to \mathbb{R}$ an scalar function, both defined on Ω. Then

$$\int_\Omega (F \cdot \nabla\varphi) dV = \int_{\partial\Omega} \varphi F \cdot d\mathbf{s} - \int_\Omega \varphi\nabla \cdot F dV.$$

By applying the divergence theorem on $F = \psi\nabla\varphi - \varphi\nabla\psi$, and using the identity $\nabla \cdot (\psi\nabla\varphi) = \nabla\psi \cdot \nabla\varphi - \psi\nabla^2\psi$, we can derive the Green's second identity.

Theorem 3.3 — Green's second identity. Let $\varphi, \psi : \mathbb{R}^3 \to \mathbb{R}$ defined on some region Ω. Then

$$\int_\Omega (\psi\nabla^2\varphi - \varphi\nabla^2\psi) dV = \int_{\partial\Omega} (\psi\nabla\varphi - \varphi\nabla\psi) \cdot d\mathbf{s}.$$

3.2 Some Random Linear Algebra

Self-adjoint matrices

Definition 3.1 — Self adjoint matrix. A matrix $M : V \to V$ is self adjoint if and only if and only if

$$\langle v, Mu\rangle = \langle Mv, u\rangle$$

for $v, u \in V$. In words, $M = M^* = (\overline{M})^T$.

Proposition 3.1 The set of all basis sets for a vector space V is identical with the set of all non-singular and self-adjoint operators $M : V \to V$ up to matrix similarity

Proof. This is not a formal proof, but an informal discussion of the idea. Every self-adjoint matrix has a set of orthogonal eigenvectors that can be considered as a basis. □

3.3 Classification of The Second Order PDEs

Partial differential equations relate the partial derivatives of a function to each other. For example f can be a function of spacial coordinates (like x, y, z in the case of Cartesian coordinates), dynamical variable (like time), or any other kind of variables (like the space of genotypes g). For example suppose that $\Phi(x, y)$ represents the electric potential of a point charge. Such function should satisfy the Laplace equation:

$$\partial_{xx}\Phi + \partial_{yy}\Phi = 0.$$

Note that the symbols ∂_{xx} and ∂_{yy} are short symbols for $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$ respectively.

Definition 3.2 Order of PDE The order of a PDE is the highest derivative that occurs in the equation.

Based on the definition above, the Laplace equation is a second order partial differential equation.

There are three categories of the second order PDEs that every other type of a second order PDE can be converted to one of these kinds. The most general type of a second order PDE can be written as:

$$A\partial_{xx}u + B\partial_{xy}u + C\partial_{yy}u + D\partial_{x}u + E\partial_{y}u + Fu = k \tag{3.3.1}$$

In which the coefficients are all a function of x, y (but not u in which case the PDE will be nonlinearx). Equation 3.3.1 can be summarized in a more compact form using the derivative operator L:

$$\mathrm{L}\, u = 0,$$

in which:

$$\mathrm{L} = A\partial_{xx} + B\partial_{xy} + C\partial_{yy} + D\partial_{x} + E\partial_{y} + F$$

.

Because of the similarities of the equation 3.3.1 with the generic quadratic equation describing the conic sections, we call each class of second order PDEs with its corresponding conic section. The generic equation describing the conic sections is:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + K = 0. \tag{3.3.2}$$

All of the conic sections (ellipse, parabola, hyperbola) can be described with the equation 3.3.2 which is determined with the discriminant $\Delta = B^2 - 4AC$. for $\Delta = 0$, $\Delta > 0$, and $\Delta < 0$ the conic section will be **parabolic**, **hyperbolic**, and **elliptic** respectively. Table 3.1 summarizes special categories of the linear second order PDEs that frequently occur in physical applications.

3.3.1 Intuitive Derivation of the Second Order PDEs

The three classes of the second order linear PDEs in table 3.1 can be derived intuitively using the continuity law (conservation law) and the constitutive law that is determined by the nature of the problem which is the subject of the following sections.

PDE	Analogous conic sec.	Δ	Class	Application
$u_t = u_{xx}$	$T = x^2$	0	parabolioc	Diffusion - Heat Equation
$u_{tt} = u_{xx}$	$T^2 = x^2$	$\Delta > 0$	Hyperbolic	Wave Equation
$u_{xx} + u_{yy} = 0$	$x^2 + y^2 = 0$	$\Delta < 0$	Elliptic	Laplace
$u_{xx} + u_{yy} = c$	$x^2 + y^2 = k$			Poisson

Table 3.1: A summary of the three class of second order linear PDE.

Continuity Equation or Conservation Laws

The most important part of deriving the PDE equations is the continuity law or conservation law. This fact is imposed because of our common sense about nature. Suppose that we want to study the concentration of of a red ink in a infinitesimal cube. The continuity equation, in simple terms, state that the change of the concentration of the ink inside the infinitesimal cube is equal to the ink that has entered the cube from outside from its boundaries (we are assuming no source or sink of ink inside the cube). For instance, consider the infinitesimal box in figure 3.3.2. The change of the concentration of the ink inside the cube is $\frac{\partial c}{\partial t}$. Because we know that there are no sources or sinks of ink inside the cube, then the change in the concentration is equal to the amount that comes in and goes out from the boundaries of the box. To put this in numbers, we introduce the important vector quantity *flux* Φ. Flux is the amount of particles flow per unit area per unit time (see figure 3.3.1).

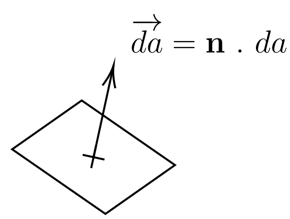


Figure 3.3.1: The dot product $\overrightarrow{\Phi}.\overrightarrow{da}$ is the amount of particles passing through the infinitesimal cross section in unit time.

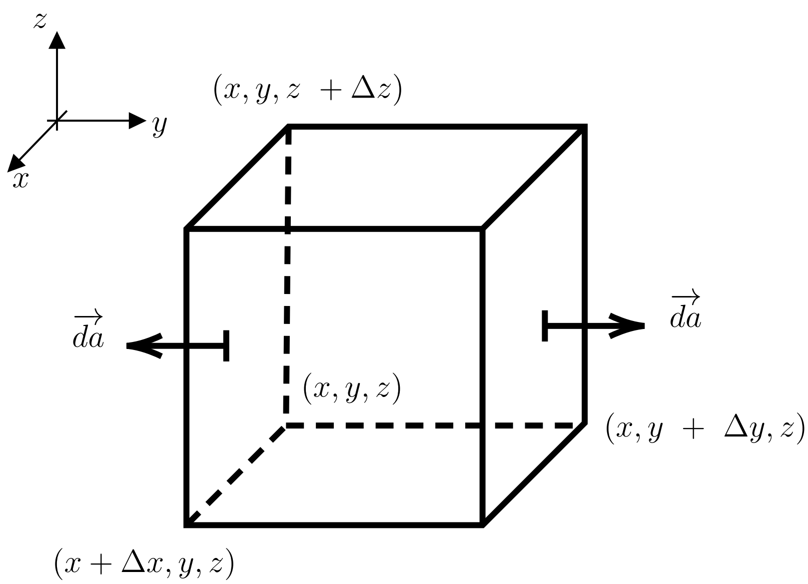


Figure 3.3.2: The infinitesimal cube for deriving the continuity equation

Let's get back to the infinitesimal cube in figure 3.3.2 and derive the continuity equation. We can start we tracking the net change in the number of particles inside the cube due to the flux in x direction:

$$\begin{aligned}
-\frac{dN_x}{dt} &= \Phi_x(x, y, z).(-dzdy) + \Phi_x(x + dx, y, z).(dzdy) \\
&= (\Phi_x(x + dx, y, z) - \Phi_x(x, y, z))dydz
\end{aligned}$$

Note that the negative sign in the LHS of the equation above is simply to match the meaning of the two sides of the equations. For instance, if the RHS of the equation above is positive, it means that the net change of the number of particles in the box in negative (meaning that particles are leaving the box) which is equivalent to $-\frac{dN}{dt}$. Similarly in the y and z direction:

$$\begin{aligned}
-\frac{dN_y}{dt} &= (\Phi_y(x, y + dy, z) - \Phi_y(x, y, z))dxdz \\
-\frac{dN_z}{dt} &= (\Phi_z(x, y, z + dz) - \Phi_z(x, y, z))dxdy
\end{aligned}$$

So the net change in the number of particle in the box per dt will be:

$$
\begin{aligned}
-\frac{dN}{dt} &= -(\frac{dN_x}{dt} + \frac{dN_y}{dt} + \frac{dN_z}{dt}) \\
&= (\Phi_x(x+dx,y,z) - \Phi_x(x,y,z))dydz + \\
&(\Phi_y(x,y+dy,z) - \Phi_y(x,y,z))dxdz + \\
&(\Phi_z(x,y,z+dz) - \Phi_z(x,y,z))dxdy
\end{aligned}
$$

By dividing the both sides of the above equation by the volume of the cube $dV = dxdydz$ we can write:

$$\frac{dc}{dt} = -(\partial_x \Phi_x + \partial_y \Phi_y + \partial_z \Phi_z) = -\nabla.\Phi$$

In which $c = N/V$ is the concentration, ∇ is the divergence operator, and Φ is the flux.

Definition 3.3 Continuity Equation The following important relation is known as the continuity equation(or conservation law):

$$\frac{dc}{dt} + \nabla.\Phi = 0 \tag{3.3.3}$$

in which Φ is the flux, c is the concentration, and ∇ is the divergence operator.

Conservation Law for Momentum

will be used to derive the wave equation

3.3.2 Constitutive Laws: Advection, Diffusion and Wave Equation

Having the continuity equation in hand makes the derivation PDEs very straight forward. We only need to insert the constitutive laws (which are enforced by the nature of the problem) in the continuity equations derived in the section above.

- Fick's Law $\rightarrow$ Constitutive law for diffusion
- Hook's law $\rightarrow$ Constitutive law for the wave equation

4. Ordinary Differential Equations

4.1 Introduction

Let's start by studying the following first order differential equation for u

$$u' + mu = f, \tag{1}$$

where $m \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. In fact, we are trying to find a function that its derivative plus some constant times the function itself produces the function f. One way to tackle this problem is to multiply both sides by e^{mx}. Then, we will get

$$e^{mx}u' + mue^{mx} = fe^{mx}, \quad x \in \mathbb{R}.$$

Now the trick is to do a similar thing as completing the square in algebra, with only difference that we are aiming at completing the derivative. In other words

$$(e^{mx}u(x))' = f(x)e^{mx}, \quad x \in \mathbb{R}.$$

Now by integrating both sides (or, equivalently, using the fundamental theorem of calculus) we will get

$$e^{mx}u(x) - u(0) = \int_0^x f(s)e^{ms}ds.$$

By a little rearrangement of the terms we will get

$$u(x) = ce^{-mx} + \int_0^1 f(s)e^{m(s-x)}ds, \quad c \in \mathbb{R}$$

Now we can see that on only there is one function that satisfies that particular relation imposed by the differential equation, there is in fact a continuum of functions satisfying the differential equations that are parameterized by $c \in \mathbb{R}$. It turns out that the set of all such functions, as a set, has some very interesting properties. Let's call this set $\mathcal{A}$. As we discussed above, the elements of this set is parameterized by $c \in \mathbb{R}$. We can also specify this set as

$$\mathcal{A} = \{u \mid u(x) = ce^{-mx} + \int_0^x f(s)e^{m(s-x)}ds \ \forall c \in \mathbb{R}\}$$

Note that we will assume that the function f is Lipschitz continuous. Then we will be sure that all of the possible solutions of the ODE above are in set $\mathcal{A}$. Studying this set more closely, reveals the fact that this set is actually a linear space, or, a vector space. This is true since

(A1) $u, v \in \mathcal{A} \implies u + v = v + u \in \mathcal{A}$.

(A2) $\forall u, v, r \in \mathcal{A}$ we have $u + (v + r) = v + (u + r)$.

(A3) $\mathcal{O} \in \mathcal{A}$ such that $v + \mathcal{O} = \mathcal{O} + v = v$ for all $v \in \mathcal{A}$.

(A4) $\forall v \in \mathcal{A},\ \exists u \in \mathcal{A}$ s.t. $v + u = u + v = \mathcal{O}$.

(M1) $\forall v \in \mathcal{A}$ we have $\mathcal{I} \cdot v = v \cdot \mathcal{I} = v$.

(M2) $\forall v \in \mathcal{A}$ we have $\mathcal{O} \cdot v = v \cdot \mathcal{O} = \mathcal{O}$

(M3) $\forall v \in \mathcal{A},\ a, b \in \mathbb{R}$ we have $a(bv) = (ab)v$

(D1) $\forall u, v \in \mathcal{A},\ a \in \mathbb{R}$ we have $a(u + v) = au + av$.

(D2) $\forall u \in \mathcal{A},\ a, b \in \mathbb{R}$ we have $(a + b)u = au + bu$.

Proof. All of the properties of this set follows immediately from the fact that $\mathbb{R}$ is a vector space. To see this, see the following proof for (A1). Let $u, v \in \mathcal{A}$. Then $\exists c_1, c_2 \in \mathbb{R}$ such that

$$u(x) = c_1e^{-mx} + \int_0^x f(s)e^{m(s-x)}ds, \quad v(x) = c_2e^{-mx} + \int_0^x f(s)e^{m(s-x)}ds.$$

Thus

$$(u + v)(x) = (c_1 + c_2)e^{-mx} + \int_0^x f(s)e^{m(s-x)}ds.$$

Since $c_1 + c_2 \in \mathbb{R}$, then $u + v \in \mathcal{A}$. □

So far, we have fined a set of functions $\mathcal{A}$ that solves the differential equation. However, now, we can ask for more requirement. For instance, we can ask for functions that satisfy certain initial conditions, i.e. $u(0) = u_0$. Or we can ask for functions that are defined on some interval, say $[0, 1]$ that satisfy certain boundary conditions like $u(0) + u'(1) = 3$, or $u(0) = u(1)$, etc. For the case of specifying an initial condition, say $x(0) = x_0$, we can find a unique $x \in \mathcal{A}$ that solves the ODE and satisfies this initial value problem (choose $x \in \mathcal{A}$ that has $c = x_0$).

Be Careful Here Consider the following initial value problem

$$\dot{x} = x^{2/3}, \qquad x(0) = 0.$$

Then at any open interval that the function $x(t)$ does not take the value 0, we can write the ODE as

$$\frac{\dot{x}}{x^{2/3}} = 1.$$

Now by integration we get the set of all solutions of the ODE

$$\mathcal{A} = \{x : [a,b] \to \mathbb{R} \ : \ x(t) = (\frac{x+c}{3})^3, \ x(t) \neq 0 \ \forall t \in [a,b]\}.$$

Clearly, the constant solution $x(t) \equiv 0$ doe not belong to the set $\mathcal{A}$. However, we can easily verify that $x(t) \equiv 0$ is a solution to the initial value problem. So when the function f is not descent enough (not Lipschitz continues in this case), then the set $\mathcal{A}$ does not contain all of the solutions for the initial value problem. So in conclusion, the initial value problem has two solutions $x(t) = t^3/3$ that belongs to $\mathcal{A}$ and $x(t) \equiv 0$ which is not in $\mathcal{A}$.
However, if the function f (RHS of the initial value problem) is Lipschitz continuous, then by the Picard iteration argument we can show that there is always a unique solution that can be achieved by the integration[a].

[a]see the chapter 1 of the book "Ordinary Differential Equations: Qualitative Theory" by Barreira.

However, in the case of specifying boundary conditions (boundary condition problems), like demanding $u(0) = u(1)$ the situation is not as clear as the initial value problem. Given that the RHS function is Lipschitz continuous and all of the solutions of the ODE lives in the set $\mathcal{A}$, then our task is basically look for functions in $\mathcal{A}$ (which is basically isomorphic to $\mathbb{R}$) to see which of them satisfy the boundary condition. Then we might find no solutions, or a unique solution, or more than one solution (note the similarity with finding the solutions of a linear system $AX = B$ that based on the characterizations of the matrix A we might have different scenarios for the solutions.)

Back to our example above, we find that the solutions of

$$u' + mu = f$$

are all in the set

$$\mathcal{A} = \{u \mid u(x) = ce^{-mx} + \int_0^x f(s)e^{m(s-x)}ds \ \forall c \in \mathbb{R}\}$$

So we can determine which functions in $\mathcal{A}$ satisfies our boundary condition. This leads to the following equation

$$c = ce^{-m} + \int_0^1 f(s)e^{m(s-1)}\ ds,$$

which implies

$$\boxed{c = \frac{1}{1-e^{-m}} \int_0^1 e^{m(s-1)} f(s)\ ds, \qquad m \neq 0}.$$

Thus when $m \neq 0$, we have a unique function in $\mathcal{A}$ that satisfies the boundary condition. However, when $m = 0$, then the above equation does not make sense. Then the function $u \in \mathcal{A}$ where

$$u(x) = c + \int_0^x f(s)\ ds.$$

To satisfy the boundary condition, we need to have

$$u(0) = u(1) \implies c = c + \int_0^1 f(s)ds \implies \boxed{\int_0^1 f(s)\ ds = 0}.$$

So in the case where $m = 0$, if $\int_0^1 f(s)ds = 0$, then we have infinitely many solutions for the boundary value problem. But if $\int_0^1 f(s)ds \neq 0$, then there is no solutions for the boundary value problem.

Summary 4.1 Consider the following ODE

$$u' + mu = f.$$

where f is Lipschitz continuous. Then the set of all solutions to this ODE is

$$\mathcal{A} = \left\{ u \mid u(x) = ce^{-mx} + \int_0^x f(s)e^{m(s-x)} ds\ \forall c \in \mathbb{R} \right\}.$$

Then for any initial value $u(0) = u_0$, we can find a unique $u \in \mathcal{A}$ that satisfies the initial value problem ($c = u_0$). However, for the boundary value problem $u(0) = u(1)$ we will have the following cases

- $m \neq 0$. Then there is a unique $u \in \mathcal{A}$ where

$$c = \frac{1}{1-e^{-m}} \int_0^1 e^{m(s-1)} f(s)\ ds$$

- $m = 0$. Then there are two cases

- $\int_0^1 f(s)ds = 0$. We will have infinite number of solutions $\forall c \in R$ for the BVP.
- $\int_0^1 f(s)ds \neq 0$. We will have no solutions for BVP.

We can think about the boundary value problem in a more systematic way. First note that all the functions $u : [0,1] \to \mathbb{R}$ that satisfies the boundary condition $u(0) = u(1)$ for a vector space (easy to check). Thus we can think about our boundary value condition in the following way

> Let $\mathcal{B} = u : \mathcal{C}^1([0,1], \mathbb{R}) : u(0) = u(1)$ be a Banach space. This is a linear space equipped with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$ (thus $\mathcal{B}$ is a Hilbert space). Let $Y = \mathcal{C}([0,1], \mathbb{R})$ equipped with the suprimum norm. Consider for any $m \in \mathbb{R}$, the linear operator $L_m : X \to Y$ define as $L_m u = u' + mu$. Given any $f \in Y$ find those $u \in \mathcal{B}$ such that $L_m u = f$.

If we have a unique solution, then we can write $u = L_m^{-1} f$ where L_m^{-1} is the inverse operator of L_m. So the uniqueness of the solution is the question invertibility of the operator L_m. This operator is the inverse of a differential operator, thus it is an integral operator. For the specific example that we solved above, we can easily calculate this integral operator. This inverse operator exists if we have a unique solution. So we will consider the boundary value problem we solved above when $m \neq 0$. Then we know that

$$u(x) = ce^{-mx} + \int_0^x e^{m(s-x)} f(s)\ ds, \qquad c = \frac{1}{1-e^{-m}} \int_0^1 e^{m(s-1)} f(s) ds.$$

By substituting the value of c we can write

$$u(x) = \frac{e^{-mx}}{1-e^{-m}} \int_0^1 e^{m(s-1)} f(s)\ ds + \int_0^x e^{m(s-x)} f(s)\ ds$$

To merge the integrals into a single integral, we write

$$u(x) = \frac{e^{-mx}}{1-e^{-m}} \int_0^1 e^{m(s-1)} f(s)\ ds + \int_0^1 H(s,x) e^{m(s-x)} f(s)\ ds$$

where $H(s,x)$ is step function with $H(s,x) = 1$ when $s < x$ and $H(s,x) = 0$ when $x < s$. Then we can write the integral above as

$$u(x) = \int_0^1 G(s,x) f(s)\ ds = L_m^{-1} f(t), \qquad G(s,x) = \frac{1}{1-e^{-m}} \begin{cases} e^{m(s-x)} & 0 < s < x < 1, \\ e^{m(s-x-1)} & 0 < x < s < 1. \end{cases}$$

thus for this specific boundary value problem we have

$$\boxed{L_m^{-1} f(t) = \int_0^1 G(s,x) f(s)\ ds}.$$

The function G is called the Green's function. The Green's function can give us the exact solution of certain boundary value problems, but the most important thing about the Green's function is that it contains lots of analytic and quantitative information about the solution that we can utilize even before solving the integral[1].

4.2 Construction of Green's function

Consider the following n dimensional linear boundary value problem

$$x'(t) = A(t)x(t) + f(t), \qquad t \in J = [a,b] \tag{2.1}$$

with the boundary condition

$$Bx(a) + Cx(b) = h,$$

where $n \in \mathbb{N}$, $a, b \in \mathbb{R}$ that $a < b$, $A \in \mathcal{L}^1(J, \mathcal{M}_{n\times n})$, $f \in \mathcal{L}^1(J, \mathbb{R}^n)$, $B, C \in \mathcal{M}_{\backslash\times\backslash}$, $h \in \mathbb{R}^n$, and $x \in \mathcal{AC}(J, \mathbb{R}^n)$. As usual, we denote by $\mathcal{L}^1$ the set of all Lebesgue integrable function on J and by $\mathcal{AC}(J, \mathbb{R}^n)$ the set of absolutely continuous functions on J. Note that n, a, b, f, A, B, C and h are known data of the problem.

Be Careful Here Note that we are not dealing with initial value problem, but the problem that we are looking at is boundary value problem. That is why our approach and the set of tools that we use might be different (although with some similarity). It is the nature of the problem that determines which tools we need to use. For example, for the boundary value problems, exploiting the vector space property of the set of solutions is very beneficial, whereas for the initial value problems, we exploit some other algebraic structures (the notion of flow that has a group structure) to tackle the problems. So, these two theories for boundary value and initial value problems is a very good example how to be aware of the useful structures in a problem and exploit them.

First, we study the structure of the set of solutions of the homogeneous problem, i.e. $f \equiv 0, h \equiv 0$. Let $\mathcal{H}$ be the set of all $\mathcal{C}^1([a,b], \mathbb{R}^n)$ functions that satisfy the boundary condition.

$$\mathcal{H} = \{f \in \mathcal{C}^1([a,b], \mathbb{R}^n) \ : \ Bf(a) + Cf(b) = 0\}.$$

Let $L : \mathcal{H} \to \mathcal{C}^1([a,b], \mathbb{R}^n)$ be an operator (differential operator) that for $u \in \mathcal{H}$ we have

$$Lu = u' - Au.$$

[1] To see some of these quantitative information from the Green's function see page 4 of "Greens function in the theory of ordinary differential equations" by Alberto Cabada.

Thus the set of all solutions to the homogeneous problem will be the kernel of the linear operator L, i.e. $u \in \mathcal{H}$ s.t. $Lu = 0$.

The following example demonstrates our discussion above on a more concrete example.

■ **Example 4.1** Consider the following boundary value problem

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}}_{A} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad t \in [0, 2\pi]$$

with boundary conditions given as

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{B} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x(2\pi) \\ y(2\pi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We can write this in a much more compact vector form

$$\Phi'(t) = A\Phi(t), \qquad B\Phi(0) + C\Phi(2\pi) = 0. \tag{E.2.1}$$

where $\Phi : [0, 2\pi] \to \mathbb{R}^2$. Observe that the solutions of this system of ODEs is the same as the solutions for the second order ODE $x'' = -\lambda x$. Denote the set of all differentiable functions that satisfy the boundary condition as $\mathcal{H}$.

$$\mathcal{H} = \{\Phi \in \mathcal{C}^1([0,1], \mathbb{R}^2) \ : \ B\Phi(0) + C\Phi(2\pi) = \mathcal{O}\}.$$

where $\mathcal{O}$ is a $\mathcal{C}^1$ function that is identically zero $\mathcal{O}(t) \equiv 0$. Define the differential operator $L : \mathcal{H} \to \mathcal{C}^1([0, 2\pi], \mathbb{R}^2)$ to be

$$L\Phi = \Phi' - A\Phi, \qquad \Phi \in \mathcal{H}.$$

Now Depending on the value of λ, the boundary value problem will have different answers.

(i) $\lambda < 0$. For this case the solution of the second order ODE $x'' = -\lambda x$ will be

$$x(t) = C_1 e^{kt} + C_2 e^{-kt}, \qquad k = \sqrt{-\lambda}.$$

So the set of all solutions to the ODE problem will be

$$\mathcal{S} = \left\{ \Phi \in \mathcal{C}^1([0, 2\pi], \mathbb{R}^2) : \Phi(t) = \begin{pmatrix} C_1 e^{kt} + C_2 e^{-kt} \\ kC_1 e^{kt} - kC_2 e^{-kt} \end{pmatrix} \quad \forall C_1, C_2 \in \mathbb{R} \right\}.$$

This set is a linear space with $\mathbb{R}$ as the underlying field and is spanned by two function e^{kt} and e^{-kt} with real coefficients, thus it is a two dimensional vector space. However, we need to determine which functions from the set $\mathcal{H}$ are the solutions for the boundary value problem. We can find this by applying the boundary conditions on the functions in $\mathcal{S}$ to see which one of them satisfies the boundary conditions. Thus we will have

$$C_1 - C_2 = 0, \qquad C_1 e^{2\pi k} - C_2 e^{-kt} = 0$$

which implies $C_1 = 0$, and $C_2 = 0$. Thus the kernel of the linear operator will be

$$\ker[L] = \{\Phi \in \mathcal{H} \ : \ \Phi \equiv 0\}.$$

(ii) $\lambda = 0$. For this case, the solution to the second order ODE will be

$$x(t) = C_1 + C_2 t, \qquad C_1, C_2 \in \mathbb{R}$$

So the set of all solutions for the original ODE will be

$$\mathcal{S} = \left\{ \Phi \in \mathcal{C}^1([0, 2\pi], \mathbb{R}^2) \ : \ \Phi(t) = \begin{pmatrix} C_1 + C_2 t \\ C_2 \end{pmatrix} \quad C_1, C_2 \in \mathbb{R} \right\}.$$

Which is again a two dimensional linear space. To find the solutions for the boundary value problem, we need to check to see which one of functions in $\mathcal{S}$ solves the boundary value problem. By applying the boundary conditions, we will get

$$C_2 = 0, \quad C_1 \in \mathbb{R}.$$

Thus the kernel of the operator L will be

$$\ker[L] = \left\{ \Phi \in \mathcal{H} \ : \ \Phi(t) = \begin{pmatrix} C_1 \\ 0 \end{pmatrix} \quad C_1 \in \mathbb{R} \right\}.$$

which is a one dimensional space spanned by the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(iii) $\lambda > 0$. For this case we will have

$$x(t) = C_1 \cos(kt) + C_2 \sin(kt), \qquad k = \sqrt{\lambda}.$$

Then the set of all solutions to the original system of ODEs will be

$$\mathcal{S} = \left\{ \Phi \in \mathcal{C}^1([0, 2\pi], \mathbb{R}^2) \ : \ \Phi(t) = \begin{pmatrix} C_1 \cos(kt) + C_2 \sin(kt) \\ -C_1 k \sin(kt) + C_2 k \cos(kt) \end{pmatrix} \quad C_1, C_2 \in \mathbb{R} \right\}.$$

Now we can impose the boundary condition, that will result in

$$C_2 = 0, \qquad \begin{cases} C_1 \in \mathbb{R} & k = n/2, \\ C_1 = 0 & k \neq n/2. \end{cases}$$

Thus when $k \neq n/2$ the kernel of the operator will be

$$\ker[L] = \{\Phi \in \mathcal{H} \ : \ \Phi \equiv 0\}.$$

While, when $k = n/2$, the kernel of the operator will be

$$\ker[L] = \left\{ \Phi \in \mathcal{H} \ : \ \Phi(t) = C_1 \begin{pmatrix} \cos\left(\frac{n}{2}t\right) \\ -\frac{n}{2}\sin\left(\frac{n}{2}t\right) \end{pmatrix} \right\}.$$

■

4.3 Sturm-Liouville theory

In this section, we will again utilize the algebraic properties of some set of functions (those that satisfy specific boundary condition) to solve the boundary value problems. We will use the ideas and notions from the finite dimensional linear algebra, tailored to our purpose.

First, we will have a quick review of self-adjoint matrices.

4.3.1 Self-adjoint Matrices

Let V and W be two finite dimensional vector spaces where $\dim W = m$ and $\dim V = n$. Let $\mathcal{B}_v = \{\mathbf{v}_i\}_{i=1}^n$ and $\mathcal{B}_w = \{\mathbf{w}_i\}_{i=1}^m$ be basis for these linear spaces. Let M be a linear operator between these to spaces, i.e. $M : V \to W$. By linearity we know that

$$\forall \mathbf{u}_1, \mathbf{u}_2 \in V \qquad M(a\mathbf{u}_1 + b\mathbf{u}_2) = aM(\mathbf{u}_1) + bM(\mathbf{u}_2).$$

Thus we know the effect of operator M on all vectors on V by just knowing the effect on the basis vectors of V. We can record these information in columns of a matrix as follows

$$\mathbf{M}_{j,i} = \langle \mathbf{w}_j, M\mathbf{v}_i \rangle.$$

Given a complex $n \times n$ matrix, the Hermitian conjugate of the adjoint of a matrix is defined to by the complex conjugate of the transpose of a matrix. I.e.

$$\mathbf{M}^* = \overline{\mathbf{M}^T}.$$

We call a matrix **self-adjoint** or **Hermitian** if

$$M^* = M.$$

Another way to put this is we call a matrix self-adjoint if it satisfies

$$\langle \mathbf{a}, \mathbf{M}\mathbf{b} \rangle = \langle \mathbf{M}\mathbf{a}, \mathbf{b} \rangle \qquad \forall \mathbf{a}, \mathbf{b} \in V.$$

This definition, although seems a little bit too abstract to be useful, will help us when working with differential operators. Because, we only need to have a notion of inner product on a space to work with this definition.

Proposition 4.1 Let $\mathbf{M}$ be a self-adjoint matrix. Then

(i) All of the eigenvalue of this matrix are real

(ii) All of the eigenvectors of this matrix are orthogonal to each other.

Proof.

(i) Let $\mathbf{v}_i$ be an eigenvector of $\mathbf{M}$. Then

$$\lambda \langle \mathbf{v}_i, \mathbf{v}_i \rangle = \langle \mathbf{v}_i, \lambda \mathbf{v}_i \rangle = \langle \mathbf{v}_i, \mathbf{M}\mathbf{v}_i \rangle = \langle \mathbf{M}\mathbf{v}_i, \mathbf{v}_i \rangle = \overline{\lambda} \langle \mathbf{v}_i, \mathbf{v}_i \rangle.$$

Since $\langle \mathbf{v}_i, \mathbf{v}_i \rangle \neq 0$ (from the properties of dot product given that the eigenvectors are not zero vectors), then we can conclude

$$\lambda = \overline{\lambda} \implies \lambda \in \mathbb{R}.$$

(ii) Let $\mathbf{v}_i$ and $\mathbf{v}_j$ be two eigenvectors with different eigenvalues, i.e. $\lambda_i \neq \lambda_j$. The we can write

$$\lambda_j \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \langle \mathbf{v}_i, \lambda_j \mathbf{v}_j \rangle = \langle \mathbf{v}_i, \mathbf{M}\mathbf{v}_j \rangle = \langle \mathbf{M}\mathbf{v}_i, \mathbf{v}_j \rangle = \overline{\lambda_i} \langle \mathbf{v}_i, \mathbf{v}_j \rangle.$$

Then $\lambda_i \neq \lambda_j$ implies $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$.

□

The proposition above is basically saying that any self-adjoint matrix defines a natural set of basis vectors for the linear space we are working with.

Observation 4.3.1 Every self-adjoint operator defines a natural set of orthogonal basis vectors which is the set of all of its eigenvectors.

This set of natural basis for the space helps us to solve the system of linear equations in a very convenient way, which will be very easy to generalize to more general type of operators acting on more general kind of linear spaces.

Observation 4.3.2 Let $\mathbf{Mu} = \mathbf{f}$ be a linear system of equations, where M is a self-adjoint matrix that is not singular, i.e. $\det(\mathbf{M}) \neq 0$. Let $\{v_i\}$ be the set of all eigenvectors of the operator, which also is a basis for the space. Then we can write

$$\mathbf{u} = \sum_{i=1}^{n} u_i \mathbf{v}_i, \qquad \mathbf{f} = \sum_{i=1}^{n} f_i \mathbf{v}_i.$$

Thus the linear system will be

$$\sum_{i=1}^{n} u_i \underbrace{\mathbf{M}\mathbf{v}_i}_{\lambda_i \mathbf{v}_i} = \sum_{i=1}^{n} f_i \mathbf{v}_i$$

Thus we will have

$$u_i = \frac{f_i}{\lambda_i}.$$

If M is singular, then either there is no solutions, or there is a non-unique solution, that depends on the choice of f.

4.3.2 Self-adjoint Differential Operators

Let $\mathcal{H}$ be the set of all continues function that has two derivatives and satisfy certain boundary condition given as below

$$\mathcal{H} = \left\{ f, g \in \mathcal{C}^2([a,b], \mathbb{R}) \ : \ \left[p(x)(\overline{f}' g - \overline{f} g') \right]_a^b = 0 \right\}, \tag{2.3.2.1}$$

where $p(x)$ is any *real* and once differential function in $\mathcal{C}^1([a,b], \mathbb{R})$. Example of such functions are provided as below.

■ **Example 4.2** For any $p \in \mathcal{C}^1([a,b], \mathbb{R})$ and given $b_1, b_2, c_1, c_2 \in \mathbb{R}$, all the functions $f \in \mathcal{C}^2([a,b], \mathbb{R})$ that satisfy

$$b_1 f'(a) + b_2 f(a) = 0, \qquad c_1 f'(b) + c_2 f'(a) = 0.$$

will form a set with the same structure as $\mathcal{H}$. ■

■ **Example 4.3** For a given $p(x)$ that satisfies $p(a) = p(b)$, All of the functions that are periodic, i.e. $f(a) = f(b)$ and $f'(a) = f'(b)$, will form a set with the same structure as $\mathcal{H}$. ■

Consider the following differential operator

$$\mathcal{L} f = \frac{d}{dx}(p(x)\frac{df}{dx}) - q(x) f, \tag{2.3.2.2}$$

where $p(x)$ the same as in the definition of the set $\mathcal{H}$ given at (2.3.2.1), and $q(x)$ is a real continouse function on $[a, b]$.

Proposition 4.2 The operator defined as (2.3.2.2) is a *self-adjoint operator* on $\mathcal{H}$ define by 2.3.2.1 with the following inner product

$$\langle f, g\rangle = \int_a^b \overline{f} g dx.$$

Proof. Let $f, g \in \mathcal{H}$. Then we will use the integration by parts twice.

$$\begin{aligned}
\langle f, \mathcal{L}g\rangle &= \int_a^b (\overline{f}(pg')' + qg)dx = \left[\overline{f}pg'\right]_a^b - \int_a^b p\overline{f}'g'dx + \int_a^b qgdx \\
&= \left[\overline{f}pg'\right]_a^b - \left[p\overline{f}'g\right]_a^b + \int_a^b g(p\overline{f}')'dx + \int_a^b qgdx \\
&= \underbrace{\left[\overline{f}pg' - p\overline{f}'g\right]_a^b}_{0} + \int_a^b g(p\overline{f}')' + qg\ dx \\
&= \langle \mathcal{L}f, g\rangle.
\end{aligned}$$

□

■ **Remark** Following the proof above reveals the reason we required that strange boundary condition for the function in $\mathcal{H}$!

■ **Remark** At first glance, the form of the second order differential operator for above might seem to be specific case of a general second order differential equation. But this is not since given the following general form of a linear second order differential operator

$$P(x)\frac{d^2}{dx^2} + R(x)\frac{d}{dx} - Q(x),$$

by assuming that $P(x) \neq 0$, we can divide through by $P(x)$ to obtain

$$\frac{d^2}{dx^2} + \frac{R(x)}{P(x)}\frac{d}{dx} - \frac{Q(x)}{P(x)} = e^{-\int_0^x R(t)/P(t)dt}\frac{d}{dx}(e^{\int_0^x R(t)/P(t)dt}\frac{d}{dx}) - \frac{Q(x)}{P(x)}.$$

Thus we can see that we can obtain the differential operator in the case of Strum-Liouville theorem from a general second order linear differential operator.

Now we are at a position of defining eigenvalue and eigenfunctions for differential operators.

Definition 4.1 — Eigenfunction of differential operator . $f \in \mathcal{H}$ is called an eigenfunction of $\mathcal{L}$ with eigenvalue λ and weight $w(x)$ if

$$\mathcal{L}f = \lambda w(x) f$$

■ **Remark** Note that the wight function is a non-negative function that has at most finitely many zeros on the domain $[a, b]$. Although it can be absorbed to the definition of $\mathcal{L}$, but it is always a good idea (and also convenient)to keep it explicit.

Definition 4.2 — Inner product. We define an inner product with weight $w(x)$ on set $\mathcal{H}$ as

$$\langle f, g\rangle_w = \int_a^b w(x)\overline{f}(x)g(x)\ dx.$$

The following result is one of the most important results that our theory gives so far.

Proposition 4.3 The eigenvalues of the Sturm-Liouville operator are *real*, and the eigenfunction with different eigenvalues are *orthogonal*. Thus given a self-adjoint operator, we can form an orthonoraml set $\{Y_1(x), Y_2(x), \cdots\}$ of it eigenfunctions by setting

$$Y_n = \frac{y_n}{\sqrt{\int_a^b |y_n|^2 w\ dx}},$$

where y_n are the un-normalized eigenfunction of $\mathcal{L}$.

Proof. The proof follows immediately from the proof for the finite dimensional vectors space, as in those proves we only utilized the properties of inner products. □

The following theorem is the ultimate goal behind all of the theory developed above.

Theorem 4.1 Given a set of function $\mathcal{H}$, that satisfy a particular form of boundary condition as below

$$\mathcal{H} = \left\{f, g \in \mathcal{C}^2([a, b], \mathbb{R})\ :\ \left[p(x)(\overline{f}'g - \overline{f}g')\right]_a^b = 0\right\}.$$

Then the operator $\mathcal{L} : \mathcal{H} \to \mathcal{C}([a, b], \mathbb{R})$ given as

$$\mathcal{L}f = \frac{d}{dx}(p(x)\frac{df}{dx}) - q(x)f,$$

is a self-adjoin operator, whose eigenvalues form a countably infinite sequence $\lambda_1, \lambda_2, \ldots,$ with $|\lambda_n| \to \infty$, and that the corresponding eigenfunctions form a complete basis for $\mathcal{H}$. That is $\forall f \in \mathcal{H}$ we can write

$$f(x) = \sum_{n=1}^{\infty} f_n Y_n(x), \quad \text{where} \quad f_n = \langle Y_n, f\rangle_w.$$

In fact, $\mathcal{H}$ is a complete linear metric space equipped with inner product, i.e. a Hilbert space.

Be Careful Here In the theorem above, we have claimed that the set $\mathcal{H}$ is a Hilbert space. It is immediate to show that this set is a linear space (vector space), but we have *not* yet proved that this set is complete. Also, we claimed that the set of eigenfunction of the Strum-Liouville operator form a complete basis which is not proved yet.

■ **Example 4.4 — Simple Laplace Equation.** We want to find the solutions of the following boundary value problem

$$\frac{d^2}{dx^2}f(x) = 0,\ x \in [-L, L], \quad \text{B.C.:} \quad f(-L) = f(L), \quad f'(-L) = f'(L).$$

Let the set $\mathcal{H}$ denote the set of all such functions

$$\mathcal{H} = \left\{ f \in \mathcal{C}^2([-L, L], \mathbb{R}) \ :\ f(-L) = f(L),\ f'(-L) = f'(L) \right\}.$$

The by the discussion above, the operator $\mathcal{L} = d^2/dx^2$ is a Sturm-Liouville self-adjoint operator where $p(x) \equiv 1$ and $q(x) \equiv 0$. Then we know that the set of all normalized eigenfunction of this operator forms a natural orthonormal basis for the space $\mathcal{H}$. We let the weight function $w(x)$ be identically zero, and we look for the eigenfunctions

$$\frac{d^2 f}{dx^2} = \lambda f.$$

Depending on the sign of λ, we will have different cases.

(i) $\lambda > 0$. Then the solution of the ODE above will be

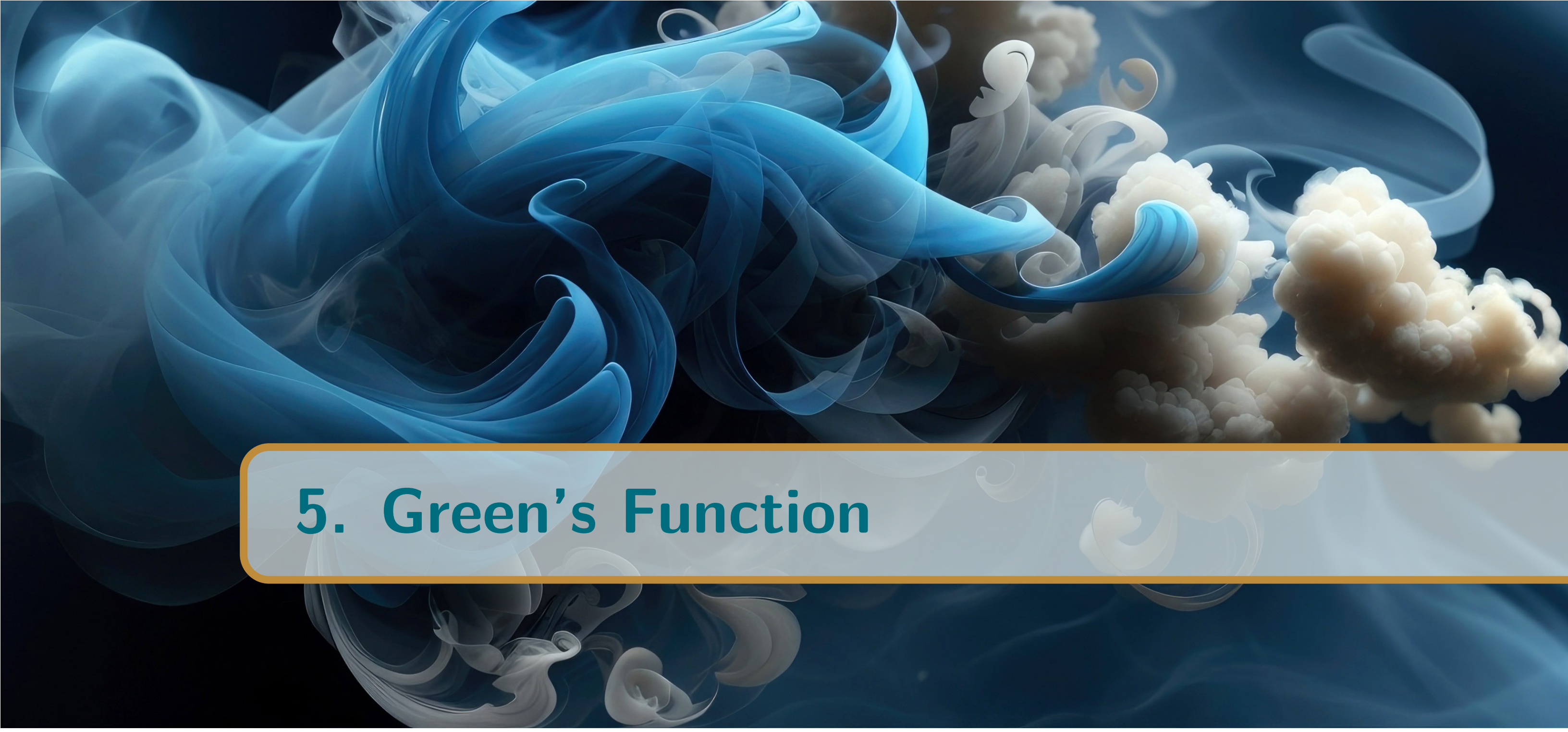
$$f(x) = C_1 e^{k(x-L)} + C_2 e^{k(x+L)},$$

where $k = \sqrt{\lambda}$, and the reason for the $x-L$ and $x+L$ term at the exponents are for convenience in checking the boundary conditions. By demanding boundary condition, it turns out that the only solutions that lies in $\mathcal{H}$ is the trivial solution. Thus we can not find a non-trivial eigenfunction for the case where $\lambda > 0$.

(ii) $\lambda < 0$. TO BE COMPLETED

■

■ **Example 4.5 — Hermit's Equation.** content ■



5. Green's Function

In my opinion, Green's function is nothing other than expressing the solution of a problem in a nice and clean way. So I would say it acts more like a level of abstraction thing that helps to express the solution of a particular problem in a special way that can be used for more higher level thinking for further development, that otherwise would be very hard. I other words, I believe Green's function is basically looking at a solution from a different perspective. I know that all of these statements are pretty wage, but you might get some kind of similar feeling after reading this chapter!

5.1 Green's function in linear algebra

Consider the following linear equation

$$Mu = f,$$

where M is a *symmetric* $n \times n$ matrix with $\det(M) \neq 0$, and $u, v \in \mathbb{R}^n$. Our problem is to find u given the known matrix M and f. The most simple way to approach this problem is basically

$$u = M^{-1} f.$$

To be honest, this is not a solution at all! This expression literally stating $Mu = f$ in an equivalent way. It is basically saying the vector u can be calculated by the act of the inverse mapping M^{-1} on the input function f. Now we can seek different ways to find this inverse mapping, which we are not discussing here.

Instead, we are going to discuss another very simple approach, that turns out the be generalization to infinite dimensional spaces as well! The strategy is basically utilizing the fact that every non-singular symmetric function defines a natural set of orthogonal basis vectors (which are actually the eigenvectors of the matrix)for the space. Let

$$\mathbb{B} = \{v_i\}_{i=1}^n$$

be the set of all *normalized* eigenvectors of M, which corresponding eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_i\}$ (note that these eigenvalues need not to be all distinct). All these things wants to say that we have a unique factorization of every vector in the space in terms of these eigenvectors. So we can write

$$\mathbf{u} = \sum_{i=1}^n \hat{u}_i \mathbf{v}_i, \qquad \mathbf{f} = \sum_{i=1}^n \hat{f}_i \mathbf{v}_i.$$

Now, simply by substituting the terms in $Mu = f$, and matching the terms we will get

$$\hat{u}_i = \frac{\hat{f}_i}{\lambda_i}.$$

Voila! we solved the system of equations, and the solution basically is

$$\mathbf{u} = \sum_{i=1}^n \frac{\hat{f}_i}{\lambda_i} v_i.$$

At this stage, you might finish you job and head towards the home being proud of yourself in solving this problem. But, we can do just a little bit modification to the expression above and view things from a different angle, which turns out the be very useful when considering certain problems in infinite dimensions (like ODEs and PDEs). We know that the coefficients $\hat{f}_i$ are basically the projection of the vector f on the basis vector v_i. I.e. $\hat{f}_i = \langle v_i, f\rangle = \sum_{k=1}^n (v_i)_k f_k$. Let's insert this in the equation above

$$\mathbf{u} = \sum_{i=1}^n \frac{1}{\lambda_i} (\sum_{k=1}^n (v_i)_k f_k) v_i.$$

or

$$\mathbf{u} = \sum_{i=1}^n \sum_{k=1}^n \frac{(v_i)_k v_i}{\lambda_i} f_k \implies \boxed{\mathbf{u} = \sum_{i=1}^n \sum_{k=1}^n G_{i,k} f_k}.$$

Where I call matrix G the Green's matrix! By staring at the equation above, you will find out that that matrix G has a very simple structure.

$$G = \begin{pmatrix} | & | & | & & | \\ \frac{v_1}{\lambda_1} & \frac{v_2}{\lambda_2} & \frac{v_3}{\lambda_3} & \cdots & \frac{v_n}{\lambda_n} \\ | & | & | & & | \end{pmatrix} \begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix}$$

Note that the eigenvectors are assumed to be normalized.

6. Functional Analysis

Here in this chapter we will cover some of the basics of the functional analysis and its applications to PDEs. Later, we will also all of these theories developed for numerical analysis of PDEs with finite element methods.

6.1 Introduction

Differential equations are the centeral part of the applied mathematics. Ordinary differential equations (ODEs) and partial differential equations (PDEs) are two major types of the differential equations. The nice thing about ODEs is that they can be formulated as a finite dimensional system, and different methods and mathematical tools ce be used to analyse these systems (like the notion of flow that is discussed in the dynamical systems course, Sturm-Lioiville theorem, Green's functions, etc). However, PDEs required their own language. The state space of a PDE is a Banach space, and the PDE itself can be seen as a combination of operators between Banach spaces, and the solutions oftern arise as weak or weak* limits in those Banach spaces.

Definition 6.1 — Banach Space. A Banach space $(X, \|\cdot\|_X)$ is a *complete normed* space.

■ **Remark** By definition, a Banach space is a vector space. That is because norm is defined for vector spaces.

■ **Remark** Vector spaces has the spacial element zero with its special properties. In an intuitive level, then norm of an element in a Banach space is basically how far it is located from the zero element (which we call it the origin). This intuitive description is based on the fact that every norm induces a metric.

■ **Example 6.1** On of the very important Banach spaces is $\mathbb{R}^n$ where $n \in \mathbb{N}$. Let

$x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Then the followings are some norms that we can define on this space. ■

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|,$$
$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2},$$
$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

The following definitions will be very useful.

Definition 6.2 Let X be a banach space. Then a subset $Y \subset X$ is dense of $\overline{Y} = X$. A Banach space that contains a dense and countable subset is called *separable*. For instance, $\mathbb{R}^n$ is separable, since $\mathbb{Q}^n$ is a dense subset that is countable.

6.1.1 Continuous and Differentiable Functions

We start with the following definition.

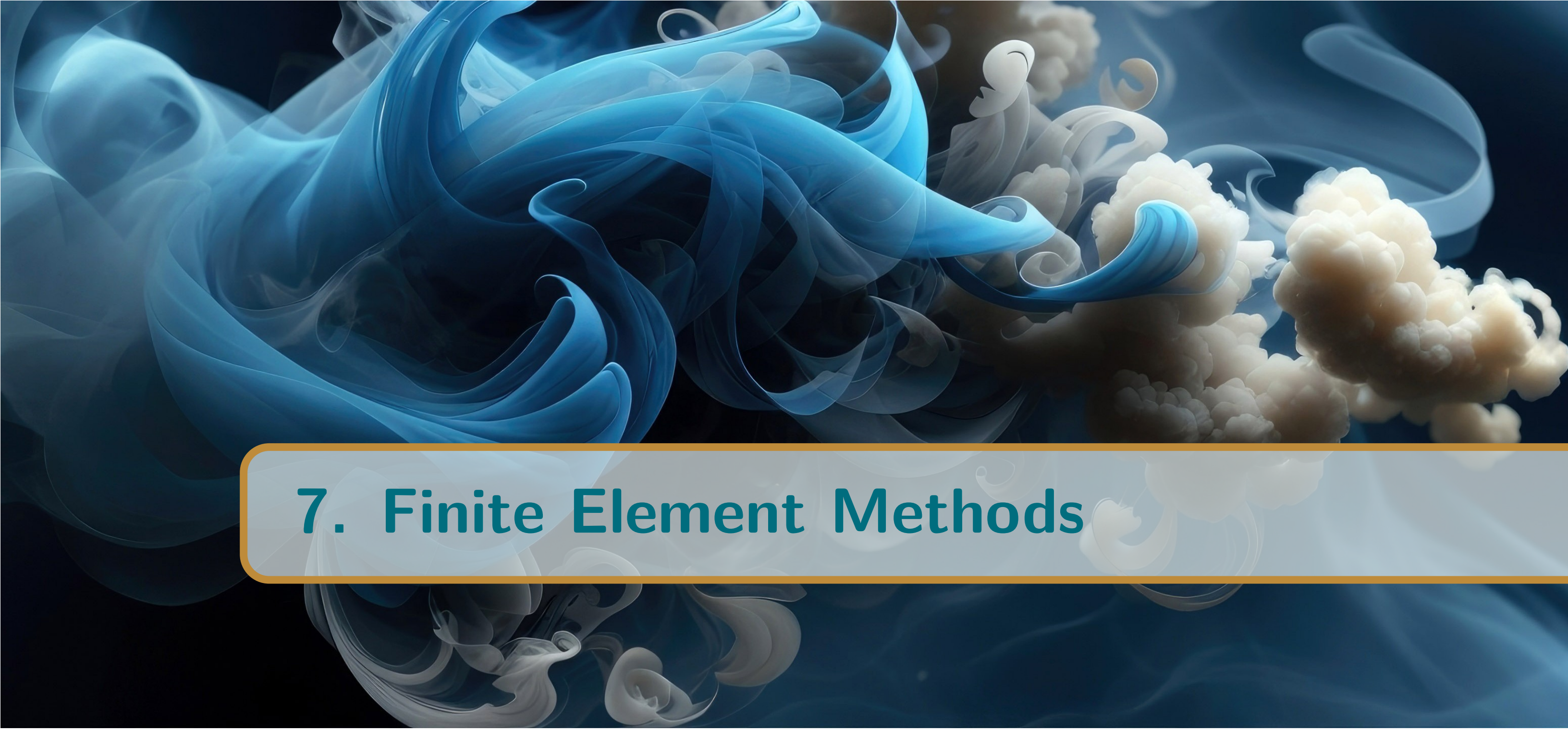
Definition 6.3 Let $\Omega \subset \mathbb{R}^n$ be a given set. If Ω is bounded, we introduce the domain boundary $\partial\Omega$ and its closure $\overline{\Omega}$. Then we define the following set of continuous functions.

$$\mathcal{C}^0(\Omega) = \{f : \Omega \to \mathbb{R} : \text{continuous}\},$$
$$\mathcal{C}^0(\overline{\Omega}) = \left\{f : \overline{\Omega} \to \mathbb{R} : \text{continuous}\right\},$$
$$\mathcal{C}^0_b(\Omega) = \left\{f \in \mathcal{C}^0(\Omega) : \text{bounded}\right\}.$$

■ **Remark** If Ω is *bounded*, then $\mathcal{C}^0(\Omega) = \mathcal{C}^0_b(\Omega)$.

6.2 Wokring Area

Observation 6.2.1 I have a feeling that the set of all continuous functions $f : \Omega \to \mathbb{R}$ has a dense subset that is the set of all smooth functions $f : \overline{\Omega} \to \mathbb{R}$. In other words $\mathcal{C}^\infty(\overline{\Omega})$ is a dense subset of $\mathcal{C}^0(\Omega)$. We can basically approximate every continuous function by a sequence of smooth functions. These smooth functions are constructed by mollifing the original continuous function by bump functions (smooth functions with compact support) with width ϵ.



7. Finite Element Methods

In this chapter, we will cover the basics of the theory of finite element methods.

7.1 Basics and Notations

For convenience in notation for partial derivatives, we introduce the following notation using the multi-index.

Definition 7.1 — Definition of multi-index. Let $n \in \mathbb{N}$. Then the following m-tuple

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{R}^n$$

is called a multi-index, with its length defined as

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

Now we can have a very compact notation for the partial derivatives of a function using the notion of multi-index.

Definition 7.2 — Partial derivatives with multi-index. Let $\Omega \subset \mathbb{R}^n$, and $f : \Omega \to \mathbb{R}$ be m-times continuously differentiable. Then we have

$$D^\alpha f = \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1}\Big(\frac{\partial}{\partial x_2}\Big)^{\alpha_2}\cdots\Big(\frac{\partial}{\partial x_n}\Big)^{\alpha_n} f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}\,\cdots\partial x_n^{\alpha_n}} f.$$

Note that since the function is m-times differentiable, then the above definition is defined for $|\alpha| < m$.

■ **Example 7.1** Suppose that $n = 3$, and $\Omega \subset \mathbb{R}^3$. Define $f : \Omega \to \mathbb{R}$. Then we have

$$\sum_{|\alpha|=3} D^\alpha f = \Big(\frac{\partial^3}{\partial x_1^3} + \frac{\partial^3}{\partial x_2^3} + \frac{\partial^3}{\partial x_3^3} + \frac{\partial^3}{\partial x_1^2 \partial x_2^1} + \frac{\partial^3}{\partial x_1^1 \partial x_2^2} + \frac{\partial^3}{\partial x_2^1 \partial x_3^2} + \frac{\partial^3}{\partial x_2^2 \partial x_3^1} + \frac{\partial^3}{\partial x_1^1 \partial x_3^2} + \frac{\partial^3}{\partial x_1^2 \partial x_3^1} + \frac{\partial^3}{\partial x_1^1 \partial x_2^1 \partial x_3^1}\Big) f.$$

This example shows how we can simply avoid writing 10 terms using a appropriately designed notation. ■

7.1.1 Space of Continuously Differentiable Functions

Now we review some basic definitions about the function spaces.

Definition 7.3 — Space of continuous functions. Let $\Omega \subset \mathbb{R}^n$, an open region. The space of all continuous functions defined on this region is denoted by $\mathcal{C}^0(\Omega)$ and defined as follows

$$\mathcal{C}^0(\Omega) = \{ f : \Omega \to \mathbb{R} \ : \ f \text{ is continuous} \}.$$

Also, let $\mathcal{C}^0(\overline{\Omega})$ denote the set of all functions $f \in \mathcal{C}^0(\overline{\Omega})$ that can be extended from Ω to a continuous function defined on $\overline{\Omega}$.

■ **Remark** Note that since $\overline{\Omega}$ compact, then all of the functions in $\mathcal{C}^0(\overline{\Omega})$ are bounded. That is true because all the continuous functions defined on a compact set are uniformally continuous, thus bounded.

■ **Example 7.2** Let $\Omega = (0, 1)$. Then $f : \Omega \to \mathbb{R}$, where $f(x) = 1/x$ is in $\mathcal{C}^0(\Omega)$ but not in $\mathcal{C}^0(\overline{\Omega})$. I.e.

$$f \in \mathcal{C}^0(\Omega), \qquad f \notin \mathcal{C}^0(\overline{\Omega}).$$

■

Proposition 7.1 — Vector space structure of the space of continuous functions. Let $\Omega \subset \mathbb{R}^n$. The space $\mathcal{C}^0(\Omega)$ is a vector space. A suitable norm for this space is defined as follows

$$\|f\| = \|f\|_\infty = \sup_{x \in \Omega} |f(X)|,$$

where $f \in \mathcal{C}^0(\Omega)$.

Lemma 7.1 Let $f \in \mathcal{C}^0(\overline{\Omega})$. Then

$$\|f\|_\infty = \sup_{x\in\Omega} |f(x)| = \max_{x\in\Omega} |f(x)|.$$

Proof. A continuous function defined on a compact set attains its maximum and minimum in the set. □

Now we can define the space of k-times continuously differentiable functions.

Definition 7.4 — The space of k-times continuously differentiable functions. Let $\Omega \subset \mathbb{R}^n$. Then we define the space of k-times continuously differentiable functions defined on Ω as

$$\mathcal{C}^k(\Omega) = \left\{ f \in \mathcal{C}^0(\Omega) \ : \ D^\alpha f \in \mathcal{C}^0(\Omega) \quad \text{for } |\alpha| \le k \right\}.$$

Similar to the space of all continuous functions, the space of all k-times continuously differentiable functions also has a vector space structure as reflected by the following lemma.

Lemma 7.2 Let $\Omega \subset \mathbb{R}^n$. Then $\mathcal{C}^k(\Omega)$ is a vector space. The following norm is a useful norm for this space.

$$\|f\|_{\mathcal{C}^k} = \sum_{|\alpha|\le k} \|D^\alpha f\|_\infty.$$

■ **Remark** We can of course come up with many other norms, some of them as good as the norm defined above.

$$\|f\|_{\mathcal{C}^k} = \Big(\sum_{|\alpha|\le k} \|D^\alpha f\|_\infty^p \Big)^{1/p}.$$

And if we let $p \to \infty$ then

$$\|f\|_{\mathcal{C}^k} = \sup_{|\alpha|\le k} \|D^\alpha f\|_\infty = \max_{|\alpha|\le k} \|D^\alpha f\|_\infty.$$

Note that we say these norms are good in a sense that it makes the space of interest a complete normed vector space, i.e. a Banach space.

Observation 7.1.1 Considering the remark above, $\mathcal{C}^k(\Omega)$ kind of resembles the Euclidean space $\mathbb{R}^n$ in the sense of extending a norm from $\mathbb{R}$ to $\mathbb{R}^k$.

The following example demonstrates all of these definitions in a more concrete example.

■ **Example 7.3** Let $\Omega \subset \mathbb{R}^2$, open and bounded. The the space of all one time continuously differentiable functions is

$$\mathcal{C}^1(\Omega) = \left\{ f \in \mathcal{C}^0(\Omega) \ : \ \partial_x f \in \mathcal{C}^0(\Omega) \text{ and } \partial_y f \in \mathcal{C}^0(\Omega) \right\}.$$

And the corresponding suitable norm for this space is

$$\|f\|_{\mathcal{C}^k} = \|f\|_\infty + \left\| \frac{\partial f}{\partial x} \right\|_\infty + \left\| \frac{\partial f}{\partial y} \right\|_\infty .$$

■

7.1.2 Space of Lebesgue Integrable Functions

Here, we define some spaces of functions that are basically characterizing the growth and decay rate of functions. The following definition makes this more clear.

Definition 7.5 Let $\Omega \subset \mathbb{R}^n$, and let $p \geq 1$ be a real number. Then we define

$$\tilde{L}_p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \ : \ \Big(\int_\Omega |f(x)|^p dx \Big)^{1/p} < \infty \right\}$$

The following proposition is very crucial for our up coming discussions.

Proposition 7.2 Let $\Omega \subset \mathbb{R}^n$, and $p \geq 1$. Then $(\tilde{L}_p(\Omega)/\sim, \|\cdot\|_{L_p})$ is a normed vector space, where $\sim$ is a equivalence relation defined on the set $\tilde{L}_p(\Omega)$ identifying the functions that are equal almost everywhere. Further more the norm is defined as

$$\|f\|_{L_p} = \Big(\int_\Omega |f(x)|^p dx \Big)^{1/p}.$$

We denote

$$L_p(\Omega) := \tilde{L}_p(\Omega)/\sim$$

■ **Remark** There are some notes to emphasis here.
1. We defined the $L_p(\Omega)$ to be the quotient set $\tilde{L}_p(\Omega)/\sim$. The reason for that is because the $\|\cdot\|_{L^p}$, when defined on $\tilde{L}_p$ is no longer a norm, as it is not positive definite. I.e., the norm of a function in $\tilde{L}_p$ can be zero, while the function is not identically zero, but is zero almost everywhere, i.e. on a set of measure zero. That is why we need to identify the functions that are equal almost everywhere.
2. To check to see see if the norm defined as above satisfies the properties of norm, we can use the Minkowski inequality to show that it satisfies the triangle inequality. However, for the spacial case $p = 2$ we can show this by using the Cauchy-Schwartz inequality. Yes! we can define inner product when $p = 2$ and utilize the properties of Cauchy Schwartz inequality.

Proposition 7.3 — Hölder's inequality. Let $u \in L_p(\Omega)$ and $v \in L_q(\Omega)$ where $1/p + 1/q = 1$. Then we have

$$\|uv\|_{L^1} = \|u\|_{L^p}\|v\|_{L^q}$$

The following theorem is among the central theorems for this chapter.

Theorem 7.1 — L_p space is a Banach space. Let $\Omega \subset \mathbb{R}$. Then the space $L_p(\Omega)$ (where $p \geq 1$) is a complete normed vector space, i.e. Banach space.

When we let $p \to \infty$, we will get the space $L_\infty(\Omega)$, another interesting L_p space that has characterizations quite similar to $\mathcal{C}^0$.

Proposition 7.4 — L_∞ space.. Consider the space $L_p(\Omega)$. If we let $p \to \infty$, we will get the space $L_\infty(\Omega)$ that is defined as

$$L_\infty(p) = \{f : \Omega \to \mathbb{R} \ : \ f \text{ is finite bounded almost everywhere}\},$$

and the corresponding norm for this space is

$$\|f\|_{L_\infty} = \operatorname{ess.sup}_{x\in\Omega} |f(x)|.$$

This space is also a Banach space.

For $L_p(\Omega)$ when $p = 2$ then something interesting happens, and the space $L_2(\Omega)$ has so many useful features that is the horse power of the modern PDE analysis, among many other applications.

Proposition 7.5 — L^2 is a Hilbert space. $L_2(\Omega)$ is a Hilbert space (a Banach space equipped with an inner product) with the following inner product

$$(u, v) = \int_\Omega u(x)v(x)dx.$$

Clearly

$$\|u\|_{L_2} = \sqrt{(u, u)}.$$

Since the inner product defined on $L_2(\Omega)$ an actual inner product (look at this tautology!), then it satisfies the Cauchy-Schwartz inequality.

Proposition 7.6 Consider the Hilbert space $L_2(\Omega)$. Then for all $u, v \in L_2(\Omega)$ we have

$$|(u, v)| \leq \|u\|_{L^2}\|v\|_{L^2}$$

■ **Remark** Among many other applications, we can use the Cauchy-Schwartz inequality to show that the L_2 norm satisfies the triangle inequality. To show this, we begin with

$$\|u+v\|_{L_2}^2 = (u+v, u+v) = \|u\|_{L_2}^2 + \|v\|_{L_2}^2 + 2(u,v) \le (\|u\|_{L_2}^2 + \|v\|_{L_2}^2)^2.$$

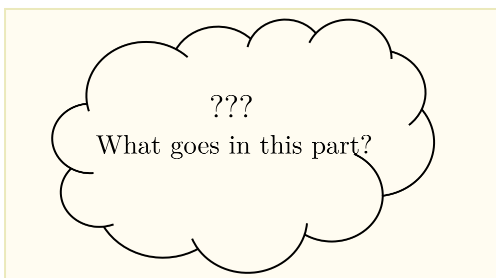
Then this implies

$$\|u+v\|_{L_2} \le \|u\|_{L_2} + \|v\|_{L_2}.$$

7.1.3 Sobolev Spaces

So far, given a domain $\Omega \subset \mathbb{R}^n$, we have introduced the notion of the space of functions defined on the domain to group them in different categories based on the similar properties that they posses. For instance, we group all the functions (defined on Ω) that are continuous into the set $\mathcal{C}^0(\Omega)$ and grouped all the functions that are square integrable into $L_2(\Omega)$ and etc. If I want to visualize these space and highlight the relation between them, I would suggest the following diagram (assume Ω is bounded).

$$\begin{array}{ccccccccc} L_1(\Omega) & \supset & L_2(\Omega) & \supset & \cdots & \supset L_\infty(\Omega) \supset & C^0(\Omega) & & \\ & & & & & & \cup & & \\ & & & & & & C^1(\Omega) & & \\ & & & & & & \cup & & \\ & & & & & & C^2(\Omega) & & \\ & & & & & & \cup & & \\ & & & & & & \vdots & & \\ & & & & & & \cup & & \\ & & & & & & C^\infty(\Omega) & \supset & C_c^\infty(\Omega) \end{array}$$



Then the notion of the weak derivatives, and the Sobolev spaces come into play. We can define the notion of weak derivative for the functions that do not posses derivative in the classical sense, like the the function corresponding to the mapping $x \mapsto |x|$.

For simplicity, we focus on the first derivative and try to generalize that notion. To do this, we need to focus on the set of functions that are continuously differentiable, i.e. $\mathcal{C}^0(\Omega)$. Let $f \in \mathcal{C}^0(\Omega)$. Then this function satisfies the following integration by parts identity

$$\boxed{\int_\Omega f\ u'\ dx = -\int_\Omega f'\ u\ dx, \qquad \forall u \in \mathcal{C}_c^\infty(\Omega).}$$

Where $\mathcal{C}_c^\infty(\Omega)$ is the set of all smooth functions with compact support. The identity above holds because

$$\int_\Omega f\ u'\ dx = \underbrace{\big[f\ u\big]_{\partial\Omega}}_{0} - \int_\Omega f'\ u\ dx.$$

Note that the boundary above is zero since the function u has compact support, thus it is zero at the boundary. Now using this identity, we can extend the notion of derivatives to the functions that do not posses derivatives in the classic sense. This is the intuitive idea behind the notion of Sobolev spaces.

Definition 7.6 — Sobolev Spaces. Let $\Omega \subset \mathbb{R}^n$. Then we define $W^{k,p}$ to be the space of functions in $L_p(\Omega)$ that posses k weak derivatives that belong to $L_p(\Omega)$. In other words, for k a non-negative integer, we have

$$W^{k,p} = \left\{ u \in L_p(\Omega) \ : \ u_\alpha \in L_p(\Omega), \ |\alpha| \leq k \right\},$$

Where u_α defined to be

$$u_\alpha = v \quad \iff \quad \int_\Omega u \ D^\alpha f \ dx = (-1)^{|\alpha|} \int_\Omega v f \ dx, \quad \forall f \in \mathcal{C}_c^\infty(\Omega).$$

Or in a more compact notation we can write

$$u_\alpha = v \quad \iff \quad (u, D^\alpha f) = (v, f).$$

So we can use the following equivalent definition for the Sobolev spaces.

$$W^{k,p} = \left\{ u \in L_p(\Omega) \ : \ \exists v \in L_p(\Omega) \text{ s.t. } (u, D^\alpha f) = (v, f) \quad \forall f \in \mathcal{C}_c^\infty(\Omega) \right\}.$$

Note that we sometimes, when there is not ambiguity, we denote the weak derivative of u with u_α and $D^\alpha u$ interchangeably.

Be Careful Here Consider the following definition of $W^{k,p}$

$$W^{k,p} = \left\{ u \in L_p(\Omega) \ : \ \exists v \in L_p(\Omega) \text{ s.t. } (u, D^\alpha f) = (v, f) \quad \forall f \in \mathcal{C}_c^\infty(\Omega) \right\}.$$

A careful reader might raise the question that how does one know that the integral $(u, D^\alpha f)$ exists given $u \in L_p(\Omega)$ and $D^\alpha f \in \mathcal{C}_c^\infty(\Omega)$? To show this we utilize the Hölder's inequality. First, observe that $\mathcal{C}_c^\infty(\Omega) \subset L_p(\Omega)$ for all $p \geq 1$, and in particular $\mathcal{C}_c^\infty(\Omega) \subset L_q(\Omega)$ where $1/q = 1 - 1/p$. Thus using the Hölder's inequality we can write

$$(u, D^\alpha f) = \int_\Omega |u \ D^\alpha f| \ dx \leq \|u\|_{L_p} \|D^\alpha f\|_{L_q} < \infty.$$

The following proposition shows that the Sobolev spaces are actually normed vector spaces.

Theorem 7.2 — Sobolev spaces are normed vector spaces. Let $\Omega \subset \mathbb{R}^n$. Then $(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$ is a normed vector space. For $u \in W^{k,p}(\Omega)$ we have

$$\|u\|_{W^{k,p}(\Omega)} = \Big(\sum_{|\alpha|\le k} \|D^\alpha u\|^p_{L_p(\Omega)} \Big)^{1/p}.$$

Also we define the semi-norm (this is semi-norm since it is not positive definite) as

$$|u|_{W^{k,p}(\Omega)} = \Big(\sum_{|\alpha|= k} \|D^\alpha u\|^p_{L_p(\Omega)} \Big)^{1/p}.$$

Thus the norm can be written as

$$\|u\|_{W^{k,p}(\Omega)} = \Big(\sum_{i=1}^{k} |u|^p_{W^{i,p}(\Omega)} \Big)^{1/p}.$$

For the special case where $p = 2$, the Sobolev space $W^{k,2}(\Omega)$ will have more interesting features as summarized bellow.

Proposition 7.7 — $W^{k,2}(\Omega)$ is a Hilbert space. Let $\Omega \subset \mathbb{R}^n$. Then $W^{k,2}(\Omega)$ is a Hilbert space with the following inner product

$$(u, v)_{W^{k,2}(\Omega)} = \sum_{|\alpha|\le k} (D^\alpha u, D^\alpha v).$$

We usually denote this space as

$$H^k := W^{k,2}(\Omega).$$

The definitions and notations above are for the general setting. However, throughout this notes we will mostly be working with $H_1(\Omega)$ and $H^2(\Omega)$. So in the following example we will go through the definitions for these special cases to train our eye for those notations.

■ **Example 7.4 — Special cases of Sobolev spaces.** Let $\Omega \subset \mathbb{R}^n$. Then $H^1(\Omega)$ and $H^2(\Omega)$ are the set of all square integrable functions that posses first and second weak derivatives respectively.
We start with $H^1(\Omega)$ which is defined to be

$$H^1(\Omega) = \left\{ f \in L_2(\Omega) \ : \ \Big(\frac{\partial f}{\partial x_i}\Big) \in L_2(\Omega), \ i = 1, 2, \cdots, n \right\},$$

$$\|u\|_{H^1(\Omega)} = \Big(\|f\|^2_{L_2(\Omega)} + \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|^2_{L_2(\Omega)} \Big)^{1/2},$$

$$|u|_{H^1(\Omega)} = \Big(\sum_{i=1}^{n} \Big\| \frac{\partial f}{\partial x_i} \Big\|_{L_2(\Omega)}^2 \Big)^{1/2}$$

Similarly, for $H^2(\Omega)$ we can write

$$\begin{aligned} H^2(\Omega) =& \{ f : \Omega \to \mathbb{R} \ : (\frac{\partial f}{\partial x_i}) \in L_2(\Omega), \ i = 1, 2, \ldots, n, \\ & \qquad (\frac{\partial^2 f}{\partial x_i \partial x_j}) \in L_2(\Omega), \ i, j = 1, 2, \ldots, n \}. \end{aligned}$$

$$\|u\|_{H^2(\Omega)} = \Big(\|u\|_{L_2(\Omega)}^2 + \sum_{i=1}^{n} \Big\| \frac{\partial f}{\partial x_i} \Big\|_{L_2(\Omega)}^2 + \sum_{i,j=1}^{n} \Big\| \frac{\partial^2 f}{\partial x_i \partial x_j} \Big\|_{L_2(\Omega)} \Big)^{1/2},$$

and for the semi-norm we have

$$|u|_{H^2(\Omega)} = \Big(\sum_{i,j=1}^{n} \Big\| \frac{\partial^2 f}{\partial x_i \partial x_j} \Big\|_{L_2(\Omega)} \Big)^{1/2}.$$

■

The following definition introduces an important special Sobolev space.

Definition 7.7 — Definition of $H_0^1(\Omega)$. The Sobolev space $H_0^1(\Omega)$ is defined to be the closure of $\mathcal{C}_c^\infty(\Omega)$ under $\|\cdot\|_{H^1(\Omega)}$ norm. I.e. for any sequence $\{u_n\}$ in $\mathcal{C}_c^\infty(\Omega)$ that converges to u^* under $\|\cdot\|_{H^1(\Omega)}$, then $u^* \in H_0^1$.

■ **Remark** For instance, Let $\Omega = [-1, 1]$. Then the following function is in $H_0^1(\Omega)$.

$$f(x) = \begin{cases} 1 - |x| & x \in [-1/2, 1/2], \\ 0 & \text{else.} \end{cases}$$

The following proposition gives us some intuitive characterization of $H_0^1(\Omega)$.

Proposition 7.8 Let $\Omega \subset \mathbb{R}^n$ an open set with a sufficiently smooth boundary. Then

$$H_0^1(\Omega) = \Big\{ f \in H^1(\Omega) \ : \ f(\partial\Omega) = 0 \Big\}.$$

We conclude this section with the important inequality of Poincaré-Friedrichs inequality.

Theorem 7.3 — Poincaré-Friedrichs inequality. Let $\Omega \subset \mathbb{R}^n$. Then $\exists C_\star \in \mathbb{R}$ that depends on the geometry of the domain Ω and for all $u \in H_0^1(\Omega)$ we have

$$\int_\Omega |u(x)|^2 \, dx \leq C_\star \sum_{i=1}^{n} \int_\Omega \Big| \frac{\partial u}{\partial x_i} \Big|^2 \, dx, \qquad \forall u \in H_0^1.$$

Or equivalently we can write this in a more compact notation as follows

$$\|u\|_{L_2(\Omega)} \leq \tilde{C}_\star |u|_{H^1}.$$

Proof. To demonstrate the simplicity of the proof for such an un-intuitive result, we start with assuming $\Omega = [a, b] \subset \mathbb{R}$. Let $u \in H_0^1(\Omega)$. Then we can write

$$u(x) = u(a) + \int_a^x u'(\xi)\ d\xi = \int_a^x u'(\xi)\ d\xi, \qquad a \leq x \leq b$$

Note that since $u \in H_0^1(\Omega)$, then u is zero at the boundary. Now we can write

$$\int_a^b |u(x)|^2 dx = \int_a^b |\int_a^x u'(\xi)\ d\xi|^2 d\xi.$$

We can simplify the inner integral using the Cauchy-Schwartz inequality

$$\int_a^x u'(\xi)\ d\xi \leq (x-a)^{1/2} (\int_a^x |u'(\xi)|^2\ d\xi)^{1/2}$$

Also, note that we can write

$$\int_a^x |u'(\xi)|^2\ d\xi \leq \int_a^b |u'(\xi)|^2\ d\xi.$$

This is true since the integrand is always positive. Putting the pieces together we can not write

$$\int_a^b |u(x)|^2 dx = \int_a^b |\int_a^x u'(\xi)\ d\xi|^2 d\xi \leq \frac{(b-a)^2}{2} \int_a^b |u'(\xi)|^2\ d\xi.$$

For another simple case $\Omega = [a, b] \times [c, d] \subset \mathbb{R}^2$, the proof will be pretty much analogous to the proof above, with a difference that we will need to find an upper bound for $\int_a^b \int_c^d |u(x, y)|^2 dy dx$ twice (one with respect to the u_x and the other one with respect to u_y). Also, note that we will need to use the identity that for every $a, b > 0$, $ab/(a + b)$ is an upper bound for both a and b. Eventually we can show that

$$\int_\Omega |u(x, y)|^2\ dxdy \leq \underbrace{(\frac{2}{(d-c)^2} + \frac{2}{(b-a)^2})^{-1}}_{C_\star} \int_\Omega (|u_x|^2 + |u_y|^2) dxdy$$

□

■ **Example 7.5** Let $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. Then we have

$$\int_\Omega |u(x, y)|^2 dxdy \leq \frac{1}{4} \int_\Omega (|u_x(x, y)|^2 + |u_y(x, y)|^2)\ dxdy \qquad \forall u \in H_0^1(\Omega)$$

■

■ **Example 7.6** Let $\Omega = [0,1] \subset \mathbb{R}$. Then we have

$$\int_\Omega |u(x,y)|^2 dx \leq \frac{1}{2} \int_\Omega |u_x(x,y)|^2 \, dx \qquad \forall u \in H_0^1(\Omega)$$

■

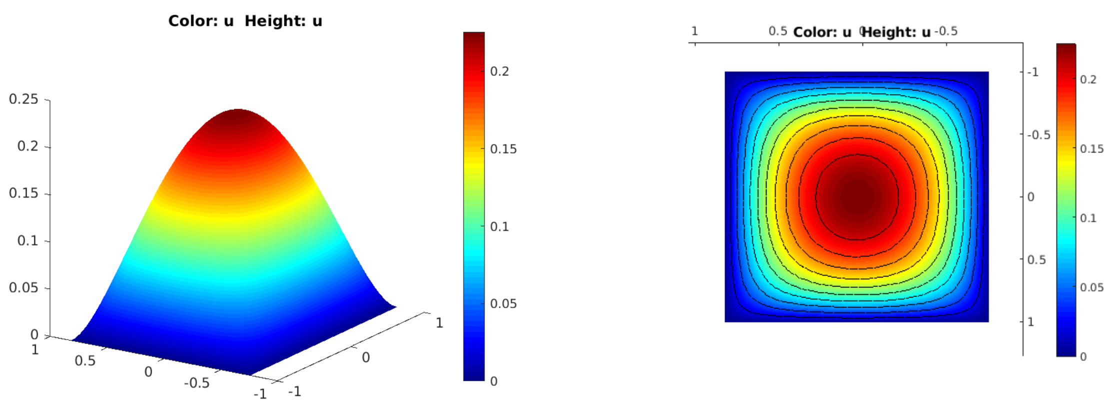
7.2 PDEs and the Weak Solutions

7.2.1 Introduction and Motivation

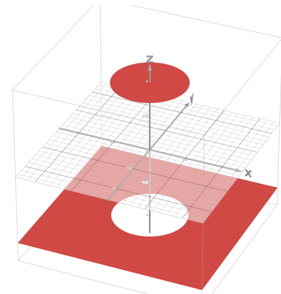
Let $\Omega \subset \mathbb{R}^2$ and consider a boundary value problem where we want to find $u : \Omega \to \mathbb{R}$ that satisfies

$$-\Delta u = f, \qquad u = 0 \quad \text{on } \partial\Omega.$$

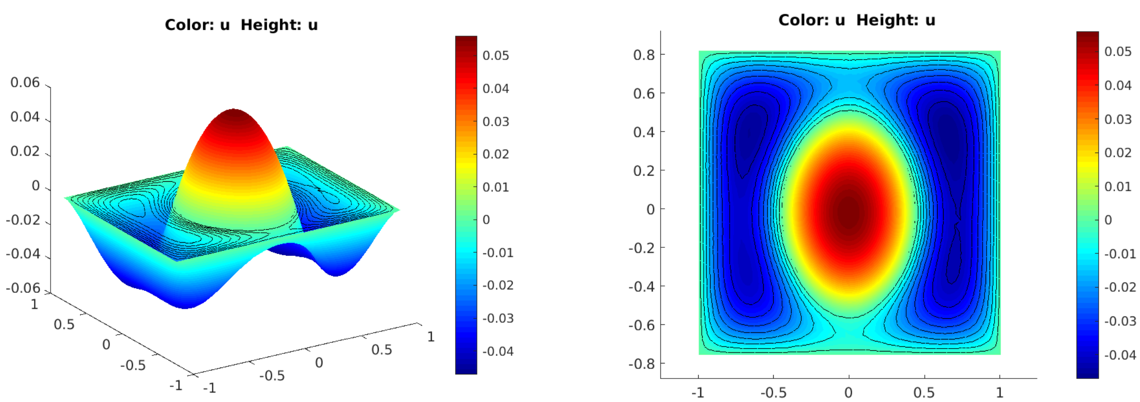
Given $f \in \mathcal{C}^0(\overline{\Omega})$ is continuous, then the classical theory of PDEs guarantees that there exists $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ that satisfies the PDF above, as well as the boundary condition (this solution is called a **classical solution**). For instance, the solution for $f = 1$ is shown as below.



But what if the function f is not continuous? Does that imply that a solution does not exists? However, we can observe solutions for such problems in experimental setting. Thus we need to weaken the requirements in order to study this problem in a more general setting. For instance, if the function f is given as



where
$$f(x,y) = \operatorname{sign}(\frac{1}{2} - \sqrt{x^2+y^2}).$$
By numerical solution, we find that the solution u is as demonstrated in the following figure.



In this section of the notes, we will develop a theory to study such "weak" solutions.

7.2.2 Weak Solutions to Elliptic Problems

General formulation of elliptic problem

Here, we consider the following general case of the elliptic problems. Let $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary. Then an elliptic problem is a partial differential equation, and a boundary condition, where the differential equation is

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial u}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f, \qquad x \in \Omega \tag{5.2.1}$$

or in a much more compact vector notation, we can write

$$-\nabla \cdot (A^T \nabla u) + B \cdot \nabla u + cu = f,$$

where

$$b_i \in \mathcal{C}^0(\overline{\Omega}), \qquad c \in \mathcal{C}^0(\overline{\Omega}),$$
$$f \in \mathcal{C}^0(\overline{\Omega}), \qquad a_{ij} \in \mathcal{C}^1(\overline{\Omega}) \tag{5.2.1.C}$$

Also, we require that there must exist a positive constant $c \in \mathbb{R}$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \tilde{c}\sum_{i=1}^{n} \xi_i^2 \quad \forall \xi \in \mathbb{R}^n,\ x \in \Omega. \tag{7.2.1}$$

The condition above is called the uniform ellipticity condition and the equation (5.2.1) is called an elliptic PDE. Now we can impose different boundary conditions on the PDE to formulate our boundary value problem. The choices are

(i) $u = g$ on $\partial\Omega$. This is known as a Dirichlet boundary condition.

(ii) $D_v u = \nabla u \cdot v = g$ on $\partial\Omega$, where D_v is the directional derivative of u in the direction v which is the unit outward normal vector to $\partial\Omega$. This is known as a Neumann boundary condition.

(iii) $D_v u + u = g$. This is called a Robin boundary condition.

Formulating the Weak Form For Homogeneous Dirichlet Boundary Value Problem

Here, we focus on solving the homogeneous Dirichlet boundary value problem which is

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial u}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f, \qquad x \in \Omega$$
$$u = 0 \qquad \text{on } \partial\Omega \tag{5.2.$\clubsuit$}$$

where a_{ij}, b_i, c, f are as in (5.2.1.C). A function $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ is called a **classic solution** to the Dirichlet boundary value problem ($u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ in the case of Neumann problem which will be discussed later). Note that the boundary of the domain Ω should be smooth enough in order for the classical solution to exist. However, we want to weaken this requirement and find a more general class of solutions. To do this, we first need to find some property, or some identity that holds for the classic solution and extend that for more general solutions (similar to what we did for extending the notion of derivatives and making the notion of weak derivative).

Let $v \in \mathcal{C}_c^1(\Omega)$ (the set of continuously differentiable functions defined on Ω that has compact support). Multiplying both sides of the PDE by v and then using integration by parts for the first term, and also observing that v is zero in the boundary (since it has a compact support) then we can write

$$\sum_{i,j=1}^{n}\int_{\Omega} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\ \mathrm{d}x+\sum_{i=1}^{n}\int_{\Omega} b_i(x)\frac{\partial u}{\partial x_i}v\ \mathrm{d}x+\int_{\Omega} c(x)uv\ \mathrm{d}x=\int_{\Omega} fv\ \mathrm{d}x \qquad \forall v\in \mathcal{C}_c^1(\Omega)$$

The equation above is the satisfied by a classical solution, and we have only the first derivative of the solution in the identity above. Thus this is a good place to extend the notion of the solution. In order for the identity above to make sense, we need to make sure the integrals exists, which is true when considering the classic solution. That is because all of the integrands are continuous functions, thus they exists. However, we can use Cauchy-Schwartz inequality to obtain

$$\left|\int_{\Omega} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\ \mathrm{d}x\right|^2 \le \left(\int_{\Omega} |a_{ij}\frac{\partial u}{\partial x_i}|^2\ \mathrm{d}x\right)\left(\int_{\Omega} |\frac{\partial v}{\partial x_j}|^2\ \mathrm{d}x\right)$$

$$\left|\int_{\Omega} b_i(x)\frac{\partial u}{\partial x_i}v\ \mathrm{d}x\right|^2 \le \left(\int_{\Omega} |v|^2\ \mathrm{d}x\right)\left(\int_{\Omega} |b_i(x)\frac{\partial u}{\partial x_i}|^2\ \mathrm{d}x\right)$$

Thus the identity satisfied by the classic solution also makes sense if we have much more relaxed requirement on u and v, namely just requiring

$$u\in H_0^1(\Omega),\qquad v\in H_0^1,\qquad f\in L_2(\Omega)$$

Also, since that since no derivative is performed on a_{ij}, then we can relax the condition on a_{ij} to be $a_{ij}\in L_2(\Omega)$, as well as the smoothness conditions on b_i and c that can be relaxed to $b_i, c\in L_\infty(\Omega)$. Note that a_{ij}, b_i, c should still be essentially bounded in order for $b_i\partial u/\partial x_i$, $a_{ij}\partial u/\partial x_i$, and cu be square integrable.
Now we can formulate the weak problem as follows.

> Find $u\in H_0^1$ such that $\forall v\in H_0^1$ it satisfies
>
> $$\sum_{i,j=1}^{n}\int_{\Omega} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\ \mathrm{d}x+\sum_{i=1}^{n}\int_{\Omega} b_i(x)\frac{\partial u}{\partial x_i}v\ \mathrm{d}x+\int_{\Omega} c(x)uv\ \mathrm{d}x=\int_{\Omega} fv\ \mathrm{d}x \quad x\in\Omega,\ \forall v\in H_0^1(\Omega)$$
>
> where $a_{ij}, b_i, c\in L_\infty(\Omega)$, and $f\in L_2(\Omega)$.

This motivates the following definition.

Definition 7.8 — Weak Solution of Elliptic Dirichlet Homogeneous Boundary Value Problem. Let $a_{ij}, b_i, c_i\in L_\infty(\Omega)$, and $f\in L_2(\Omega)$. Then a function $u\in H_0^1(\Omega)$ that $\forall v\in H_0^1(\Omega)$ satisfies

$$\sum_{i,j=1}^{n}\int_{\Omega} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\ \mathrm{d}x+\sum_{i=1}^{n}\int_{\Omega} b_i(x)\frac{\partial u}{\partial x_i}v\ \mathrm{d}x+\int_{\Omega} c(x)uv\ \mathrm{d}x=\int_{\Omega} fv\ \mathrm{d}x \quad x\in\Omega$$

is called a weak solution to (5.2.2.♣). Note that all of the derivatives should be considered in the weak sense.

■ **Remark** Clearly, if u is a classic solution, then it is also a weak solution, but because of the relaxed conditions on the weak formulation, a weak solution is not necessarily a classic solution.

Now it remains to show the existence uniqueness of the solution to the weak formulation and the corresponding requirements.

Existence and Uniqueness of The Weak Solutions

In order to use the theoretical machinery of functional analysis, first we need to express our problem in a more useful way. Let $a : H_0^1 \times H_0^1 \to \mathbb{R}$ be a bi-linear form define as

$$a(u, v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, \mathrm{d}x + \sum_{i=1}^{n} \int_\Omega b_i(x) \frac{\partial u}{\partial x_i} v \, \mathrm{d}x + \int_\Omega c(x) uv \, \mathrm{d}x,$$

and $l : H_0^1 \to \mathbb{R}$ a linear functional given as

$$l(v) = \int_\Omega f v \, \mathrm{d}x.$$

Then we can re-write our Dirichlet boundary condition as the following alternative notation.

Definition 7.9 — Alternative Notation For The Weak Solution of Elliptic Dirichlet Boundary Value Problem. An alternative notation for Definition 7.8 is given as: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = l(v) \qquad \forall v \in H_0^1(\Omega)$$

The notation above is very useful, as we can now use Lax-Milgram theorem to discuss the existence and uniqueness of the boundary value problem under study. First, we need to state the Lax-Milgram theorem.

Theorem 7.4 — Lax-Milgram Theorem. Let V be a Hilbert space, $a : V \times V \to \mathbb{R}$ a bi-linear form, $l : V \to \mathbb{R}$ a linear functional that satisfies the following properties.

(i) Coercivity: $\exists c_0 > 0$ such that

$$a(v, v) \geq c_0 \|v\|_V^2 \quad \forall v \in V$$

(ii) Continuity: $\exists c_1 > 0$ such that

$$|a(w, v)| < c_1 \|w\|_V \|v\|_V \quad \forall w, v \in V$$

(iii) $\exists c_2 > 0$ st.

$$|l(v)| < c_2 \|v\|_V \quad \forall v \in V$$

Then there exists a unique $u \in V$ that satisfies

$$a(u, v) = l(v) \quad \forall v \in V.$$

Very are now just one step from showing the existence and uniqueness of the solution the weak formulation derived above. We just need to show that by letting $V = H_0^1(\Omega)$, the bi-linear form a and the functional l have the required properties by the Lax-Milgram theorem.

Theorem 7.5 The elliptic problem with Homogeneous Dirichlet boundary values in Definition 7.8 has a unique weak solution. I.e. $\exists u \in H_0^1(\Omega)$ such that $\forall v \in H_0^1(\Omega)$ we have

$$a(u, v) = l(v).$$

Proof. For the proof, we will use the Lax-Milgram theorem with $V = H_0^1$ and $\|\cdot\|_V = \|\cdot\|_{H^1}$. We need to show that the bi-linear form $a(u, v)$ and the linear form $l(v)$ satisfies the required properties.

(i) Showing that a is continuous, i.e. we need to find c_1 such that $|a(u, v)| \leq \|u\|_{H_1} \|v\|_{H_1}$. To establish this, we will use the following inequalities.

$$\left| \int_\Omega f \ \mathrm{d}x \right| \leq \int_\Omega |f| \ \mathrm{d}x$$

$$\left| \int_\Omega f \ g \ \mathrm{d}x \right| \leq \max_{x \in \Omega} |f| \left| \int_\Omega g \ \mathrm{d}x \right|$$

$$(f, g)_{L^2} = \left| \int_\Omega f \ g \ \mathrm{d}x \right| \leq \left(\int_\Omega f \ \mathrm{d}x \right)^{1/2} \left(\int_\Omega g \ \mathrm{d}x \right)^{1/2} = \|f\|_{L^2} \|g\|_{L^2}.$$

The last inequality is simply the Cauchy-Schwartz inequality. Note that, it is always a good idea to encapsulate the details of objects in smaller notations and use higher level of abstractions to proceed! Let

$$\hat{c} = \max\{\max_{x \in \Omega} \max_{ij} a_{ij}, \ \max_{x \in \Omega} \max_{i} b_i, \ \max_{x \in \Omega} c\}$$

Then we can write

$$
\begin{aligned}
|a(u,v)| &= \left| \sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \ \mathrm{d}x + \sum_{i=1}^{n} \int_\Omega b_i(x) \frac{\partial u}{\partial x_i} v \ \mathrm{d}x + \int_\Omega c(x) uv \ \mathrm{d}x \right| \\
&\le \hat{c} \left(\sum_{i,j=1}^{n} \int_\Omega \left| \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \right| \ \mathrm{d}x + \sum_{i=1}^{n} \int_\Omega \left| \frac{\partial u}{\partial x_i} v \right| \ \mathrm{d}x + \int_\Omega |uv| \ \mathrm{d}x \right) \\
&\le \hat{c} \left(\sum_{i,j=1}^{n} \|\partial_i u\|_{L_2} \|\partial_j v\|_{L_2} + \sum_{i=1}^{n} \|\partial_i u\|_{L_2} \|v\|_{L_2} + \|u\|_{L_2} \|v\|_{L_2} \right) \\
&\le \hat{c} \left(\sum_{i,j=1}^{n} \|\partial_i u\|_{L_2} \|\partial_j v\|_{L_2} + \sum_{i=1}^{n} \|\partial_i u\|_{L_2} \|v\|_{L_2} + \sum_{j=1}^{n} \|\partial_j v\|_{L_2} \|u\|_{L_2} + \|u\|_{L_2} \|v\|_{L_2} \right) \\
&\le \hat{c} \left(\|v\|_{L_2} + \sum_{j=1}^{n} \|\partial_j v\|_{L_2} \right) \left(\|u\|_{L_2} + \sum_{i=1}^{n} \|\partial_i u\|_{L_2} \right) \\
&\boxed{\le} \ 2n\hat{c} \left(\|v\|_{L_2}^2 + \sum_{j=1}^{n} \|\partial_j v\|_{L_2}^2 \right)^{1/2} \left(\|u\|_{L_2}^2 + \sum_{i=1}^{n} \|\partial_i u\|_{L_2}^2 \right)^{1/2} \\
&\le c \|v\|_{H_1} \|u\|_{H_1}.
\end{aligned}
$$

Note that in the argument above, I have highlighted the last inequality as red. That is because I don't know how this step works. This proof is provided at FEM lecture notes, and is called a "further majorization".

(ii) To show the coercivity of a, we will use the weird looking ellipticity condition Equation 7.2.1 to simplify the first term in $a(u,u)$. Also we need to relax the smoothness condition on b_i for the moment as we will see in the proof. To start the proof we can write

$$
\begin{aligned}
a(u,u) &= \underbrace{\sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \ \mathrm{d}x}_{\ge \tilde{c} \sum_{i=1}^{n} \int_\Omega (\partial_i u)^2} + \sum_{i=1}^{n} \int_\Omega b_i(x) \overbrace{\frac{\partial u}{\partial x_i} u}^{1/2(\partial_i u^2)} \ \mathrm{d}x + \int_\Omega c(x) u^2 \ \mathrm{d}x \\
&\ge \tilde{c} \sum_{i=1}^{n} \int_\Omega (\partial_i u)^2 \ \mathrm{d}x + \sum_{i=1}^{n} (\underbrace{\int_{\partial\Omega} 1/2 b_i u^2 \ \mathrm{d}x}_{0} - \int_\Omega 1/2 u^2 (\partial_i b_i) \ \mathrm{d}x) + \int_\Omega c(x) u^2 \ \mathrm{d}x \\
&= \tilde{c} \sum_{i=1}^{n} \int_\Omega (\partial_i u)^2 \ \mathrm{d}x + \int_\Omega u^2 (c - \sum_{i=1}^{n} \partial_i b_i / 2) \ \mathrm{d}x
\end{aligned}
$$

Thus if we have

$$\sum_{i=1}^{n} \partial_i b_i/2) \geq 0,$$

then we can establish

$$a(u,u) \geq \tilde{c} \sum_{i=1}^{n} \int_{\Omega} (\partial_i u)^2 \, \mathrm{d}x$$

And by applying the Poincaré inequality Theorem 7.3, we can get

$$a(u,u) \geq \frac{\tilde{c}}{\tilde{C}_{\star}} \|u\|_{L^2}^2 = \hat{c}\|u\|_{L^2}^2$$

and as we had before

$$a(u,u) \geq \tilde{c}\|\nabla u\|_{L^2}^2$$

Thus by choosing $c_0 = \frac{1}{\hat{c}+\tilde{c}}$ we can write

$$a(u,u) \geq c_0(\|u\|_{L_2}^2 + \|\nabla u\|_{L_2}^2) = c_0\|u\|_{H_1}^2$$

This proves the coercivity of the bi-linear form.

Now, what remains to show is the linearity and continuity of the linear form $l(u)$. To show this we use the Cauchy-Schwartz inequality as follows

$$\left|l(v)\right| = \left|\int_{\Omega} uv \, \mathrm{d}x\right| \leq \|u\|_{L_2}\|v\|_{L_2} \leq \|u\|_{L_2}\|v\|_{H_1} = c_2\|v\|_{H_1}.$$

The linearity of $l(v)$ follows immediately from the linear property of integrals. Thus by using the Lax-Milgram theorem, we can establish the existence and uniqueness of the solution.

□

7.3 Approximation of Elliptic Problems

7.4 Working Area

Things to add: Example for weak derivative, completing the diagram for the geometry of the function spaces.