Lecture Notes For : Advanced Linear Algebra

Ali Fele Paranj
alifele@student.ubc.ca

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Chapitre 1

Matrices

1.1 What is a Matrix

A matrix is basically a notation convention that enables us to do some stuff more easily with a pencil and paper. A very similar concept to this is the long division algorithm for for dividing two integers. For example consider the following long division (in French-European style) that we are all familiar with

$$
\begin{array}{r|l}
198 & 12 \\ \cline{2-2}
-\underline{12} & 16{,}5 \\
78 & \\
-\underline{72} & \\
60 & \\
-\underline{60} & \\
0 &
\end{array}
$$

So this notation and algorithms is to use some calculations more continent when is done by hand with a pen and paper. So the matrix notation can also be though as a computation convention. To make stuff more clear, consider the following example.

Example : Simple Pen and Paper Calculations

Consider V which is written as :

$$V = 2A + 3B + 4C$$

Given the following relation between A, B, and C, rewrite V in terms of x, y, and z.

$$A = x + 2y + 3z$$
$$B = 2x - y + z$$
$$C = -x - y + z$$

Solution 1.
To write V in terms of x, y, and z we write :

$$V = 2(x + 2y + 3z) + 3(2x - y + z) + 4(-x - y + z) \tag{1.1.1}$$

By arranging the terms using simple algebra we will have :

$$V = (2 + 6 - 4)x + (4 - 3 - 4)y + (6 + 3 + 4)z = 4x - 3y + 13z \tag{1.1.2}$$

Solution 2.
The calculations described in the first solution are not systematic. What I mean is that we started doing whatever we can do with you thinking about doing it in a more smart way that can also by systematically scaled to larger equations. This is where the matrices come into play. Matrices help us to do such calculations in a more algorithmic way (like the long division notation in which we do the calculations in a algorithmic way).

Let $\mathbb{B}$ be the set of all *objects* that the V is expanded in terms of and call this set as the *basis* set. So for $\mathbf{V} = 2A + 3B + 4C$ we have the basis

$$\mathbb{B}_1 = \{A, B, C\}.$$

We can arrange the coefficients of V in basis $\mathbb{B}_1$ in the following way :

$$V_{\mathbb{B}_1} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}_{\mathbb{B}_1}$$

We call it the coordinates of V in the basis $\mathbb{B}_1$. Since we want to write the vector V in terms of x, y, and z, we need to introduce the new basis $\mathbb{B}_2$ in the following way :

$$\mathbb{B}_2 = \{x, y, z\}$$

Since A, B, and C are expressed in terms of x, y, and z, we can arrange the coordinates of A, B, and C in the basis $\mathbb{B}_2$ in the following way :

$$L_{\mathbb{B}_1}^{\mathbb{B}_2} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2}$$

in which every column is the coefficients A, B, and C in the basis $\mathbb{B}_2$ respectively. Note the subscript and the superscripts of the matrix. This matrix means that its columns contains the coordinates of the basis $\mathbb{B}_1$ in the new basis $\mathbb{B}_2$. So when it is applied to any vector that is described in basis $\mathbb{B}_1$, we will get the components of that vector in the basis $\mathbb{B}_2$. In other words :

$$V_{\mathbb{B}_2} = L_{\mathbb{B}_1}^{\mathbb{B}_2} V_{\mathbb{B}_1}$$

$$V_{\mathbb{B}_2} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}_{\mathbb{B}_1}$$

This matrix equation can be written in two ways as described below :

$$V_{\mathbb{B}_2} = 2\begin{pmatrix}1\\2\\3\end{pmatrix}_{\mathbb{B}_2} + 3\begin{pmatrix}2\\-1\\1\end{pmatrix}_{\mathbb{B}_2} + 4\begin{pmatrix}-1\\-1\\1\end{pmatrix}_{\mathbb{B}_2} \tag{1.1.3}$$

The equation above is equivalent to