



Lecture Notes For: Measure Theory

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1. Le Gall Measure Theory - Chapter 1

1.1 Measure Spaces

In a nutshell, the idea of measure is to assign a real number to *some* subsets of a given set. However, we require this number assignment to be consistent in the sense that the number assigned to the disjoint union of sets is the sum of the numbers assigned to each set. This notion is quite similar to the notion of the homomorphisms of groups which preserves the group action.

■ **Remark** Because of some technical issues that are related to the axiom of choice, sometimes we do not consider defining a measure for all subsets of a given set. However, a σ -algebra has sufficient structure to define the notion of measure for them.

■ **Remark** — **Definition of Borel σ -algebra.** Often, the space for which its subsets we want to define a measure, is a topological space. We have a special name reserved for the smallest sigma algebra containing all the open sets. Let (Ω, \mathcal{T}) be a topological space. The smallest sigma algebra containing \mathcal{T} (or alternatively, the sigma algebra generated by all open sets \mathcal{T}), is called a Borel σ -algebra.

Proposition 1.1 Let $\{A_n\}$ be a collection of measurable sets. We have

$$\mu(\liminf_n A_n) \leq \liminf_n \mu(A_n).$$

Also

$$\mu(\limsup_n A_n) \geq \limsup_n \mu(A_n).$$

Using the fact that $\limsup_n \mu(A_n) \geq \liminf_n \mu(A_n)$ we can write

$$\mu(\liminf_n A_n) \leq \liminf_n \mu(A_n) \leq \limsup_n \mu(A_n) \leq \mu(\limsup_n A_n).$$

Proof. First, observe that

$$\bigcap_{k \geq n} A_k \subseteq A_m \quad \forall m \geq n.$$

By monotonicity

$$\mu\left(\bigcap_{k \geq n} A_k\right) \leq \mu(A_m) \quad \forall m \geq n.$$

So it follows that

$$\mu\left(\bigcap_{k \geq n} A_k\right) \leq \inf_{k \geq n} \mu(A_k).$$

Using the fact that $\bigcap_{k \geq n} A_k \uparrow \liminf_n A_n$, and thus

$$\lim_{n \rightarrow \infty} \mu\left(\bigcap_{k \geq n} A_k\right) = \mu(\liminf_n A_n)$$

we can take limit of both sides

$$\lim_{n \rightarrow \infty} \mu\left(\bigcap_{k \geq n} A_k\right) \leq \lim_{n \rightarrow \infty} \inf_{k \geq n} \mu(A_k).$$

Using the fact that. Then we will have

$$\mu(\liminf_n A_n) \leq \liminf_n \mu(A_n).$$

□

1.1.1 Measurable Functions

The following proposition comes very handy when checking that some function between two measure spaces is a measurable function. In a nutshell, to check to see a function from (Ω, \mathcal{F}) to (Γ, \mathcal{E}) is measurable, we do not need to check to see if the pre-image of all sets in \mathcal{E} is measurable. We only need to check this for a collection of sets that the σ -algebra that they generate is \mathcal{E} .

Proposition 1.2 Let (Ω, \mathcal{F}) and (Γ, \mathcal{E}) be two measure spaces and $F : \Omega \rightarrow \Gamma$ a function. Then F is a measurable function if there exists some subclass $C \subset \mathcal{E}$ such that $\mathcal{E} = \sigma(C)$ and $F^{-1}(B) \in \mathcal{F}$ for all $B \in C$.

Proof. To see this, let

$$\mathcal{G} = \{B \in \mathcal{E} : F^{-1}(B) \in \mathcal{F}\}.$$

By assumption $C \subset \mathcal{G}$. Since $F^{-1}(A^c) = (F^{-1}(A))^c$, $F^{-1}(\bigcup_n A_n) = \bigcup_n F^{-1}(A_n)$, and $F^{-1}(\bigcap_n A_n) = \bigcap_n F^{-1}(A_n)$, it is easy to check that \mathcal{G} is a σ -algebra. So $C \subseteq \mathcal{G}$. Because $\mathcal{E} = \sigma(C)$, then we conclude that $\mathcal{G} = \mathcal{E}$ and this completes the proof. □

One of the very important applications of the proposition above is the following Lemma.

Lemma 1.1 Let (E, \mathcal{A}) , (F_1, \mathcal{B}_1) and (F_2, \mathcal{B}_2) be measurable spaces and equip the product space $F_1 \times F_2$ with $\mathcal{B}_1 \otimes \mathcal{B}_2$ (that is defined to be $\sigma(\mathcal{B}_1 \times \mathcal{B}_2)$). Let $f_1 : E \rightarrow F_1$ and $f_2 : E \rightarrow F_2$ be measurable functions and define $f : E \rightarrow F_1 \times F_2$ where $f = (f_1, f_2)$. Then f is measurable if and only if f_1 and f_2 are measurable.

Proof. For the $\boxed{\implies}$ direction, we know that f is measurable. So $f^{-1}(B) \in \mathcal{A}$ for $B \in \mathcal{B}_1 \otimes \mathcal{B}_2$. Let $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$, and define $B = B_1 \times B_2$. Then since $f^{-1}(B_1 \times B_2) = f_1^{-1}(B_1) \cap f_2^{-1}(B_2)$, and $f^{-1}(B_1 \times B_2) \in \mathcal{A}$, we conclude that $f_1^{-1}(B_1) \in \mathcal{A}$ and $f_2^{-1}(B_2) \in \mathcal{A}$. Thus f_1 and f_2 are both measurable.

For the $\boxed{\impliedby}$ direction, we know that f_1^{-1} and f_2^{-1} are measurable. So for any $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ we have $f^{-1}(B_1 \times B_2) = f_1^{-1}(B_1) \cap f_2^{-1}(B_2) \in \mathcal{A}$ because $f_1^{-1}(B_1) \in \mathcal{A}$ and $f_2^{-1}(B_2) \in \mathcal{A}$. By proposition above, we can conclude that f is measurable on $\sigma(\mathcal{B}_1 \times \mathcal{B}_2)$. □

The following two simple properties of inf and sup are very important in establishing very interesting results.

Lemma 1.2 — Simple properties of \inf and \sup . Let $\{a_n\}$ be a sequence of real numbers, and $c \in \mathbb{R}$. Then

$$\sup_n a_n \leq c \implies \forall n \in \mathbb{N} \ a_n \leq c,$$

and

$$\inf_n a_n \leq c \implies \exists n \in \mathbb{N} \text{ s.t. } a_n \leq c.$$

Proof. Proof follows immediately from the definition. \square

The following proposition makes is very important and demonstrates the beautiful interplay between \limsup (and \liminf) of numbers and sets.

Proposition 1.3 Let (E, \mathcal{A}) be a measurable space, and let $f : E \rightarrow \mathbb{R}$. Let $a \in \mathbb{R}$. Then

$$\{\limsup f_n \leq a\} = \limsup \{f_n \leq a\},$$

and

$$\{\liminf f_n \leq a\} = \liminf \{f_n \leq a\}.$$

Note the remark below for the notation convention used above.

Proof. Let $x \in \{\limsup f_n \leq a\}$. So $\limsup f_n(x) \leq a$. From definition $\limsup f_n(x) = \inf_n \sup_{k \geq n} f_k(x)$. Having $\inf_n \sup_{k \geq n} f_n(x) \leq a$ implies that $\exists n \in \mathbb{N}$ we have $\sup_{k \geq n} f_n(x) \leq a$. The second statement in turn implies that $\forall k \geq n$ we have $f_k(x) \leq a$. In summary $\limsup f_n(x) \leq a$ implies $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ we have $f_n(x) \leq a$. This is precisely

$$\{\limsup f_n\} = \bigcup_n \bigcap_{k \geq n} \{f_k \leq a\}.$$

For the second statement, let $x \in \{\liminf f_n \leq a\}$. Then $\liminf f_n(x) \leq a$. By definition $\sup_n \inf_{k \geq n} f_n(x) \leq a$. So $\forall n \in \mathbb{N}$ we have $\inf_{k \geq n} f_n(x) \leq a$, which further implies that $\exists k \geq n$ we have $f_k(x) \leq a$. This is precisely

$$\{\liminf f_n \leq a\} = \bigcap_n \bigcup_{k \geq n} \{f_k \leq a\}.$$

\square

■ **Remark** In the proposition above use the following notation which is frequently used in probability theory context.

$$\{f \leq a\} = \{x \in E : f(x) \leq a\}.$$

The Proposition and Lemma above result in the following very useful Corollary. Note that the co-domain of the measurable functions below is \mathbb{R} .

Corollary 1.1 Let $\{f_n\}$ be a sequence of measurable function from the measurable space (E, \mathcal{A}) to \mathbb{R} . Then

$$\inf_n f_n, \quad \sup_n f_n, \quad \liminf_n f_n, \quad \limsup_n f_n,$$

are all measurable functions.

Proof. We need to show that $\{f \leq a\} \in \mathcal{A}$ for all $a \in \mathbb{R}$. From proposition above

$$\{\limsup f_n \leq a\} = \limsup \{f_n \leq a\},$$

and

$$\{\liminf f_n \leq a\} = \liminf \{f_n \leq a\}.$$

So it follows that $\limsup_n f_n$ and $\liminf_n f_n$ are measurable functions. Furthermore, be the Lemma above

$$\{\inf_n f_n \leq a\} = \bigcup_n \{f_n \leq a\},$$

and

$$\{\sup_n f_n \leq a\} = \bigcap_n \{f_n \leq a\}.$$

Then it follows immediately that $\sup_n f_n$ and $\inf_n f_n$ are all measurable functions. \square

1.2 Solved Problems

■ **Problem 1.1 — Properties of measure, page 6.** Let (E, \mathcal{A}, μ) be a measure space. Prove the following properties of the measure function μ .

- (a) If $A \subset B$ then $\mu(A) \leq \mu(B)$. If in addition $\mu(A) < \infty$, then

$$\mu(B \setminus A) = \mu(B) - \mu(A).$$

- (b) **Principle of inclusion-exclusion:** Let $A, B \in \mathcal{A}$, then

$$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B).$$

- (c) **Continuity from below:** Let $\{A_n\}$ be a collection of sets that $A_n \uparrow A$, meaning $A_1 \subset A_2 \subset \dots$ and we have $A = \bigcup_n A_n$. Then

$$\mu(A_n) \rightarrow \mu(A) \quad \text{as } n \rightarrow \infty.$$

In other words

$$\mu\left(\bigcup_n A_n\right) = \lim_n \mu(A_n).$$

- (d) **Continuity from above:** Let $\{A_n\}$ be a collection of sets that $A_n \downarrow A$, meaning $A_1 \supset A_2 \supset \dots$ and we have $A = \bigcap_n A_n$. Then

$$\mu(A_n) \rightarrow \mu(A) \quad \text{as } n \rightarrow \infty.$$

- (e) **Sub-additivity:** Let $\{A_n\}$ be a collection of sets in \mathcal{A} . Then

$$\mu\left(\bigcup_n A_n\right) \leq \sum_n \mu(A_n).$$

Solution (a) First, observe that we can write

$$B = A \cup (B \cap A^c).$$

Since the union above is disjoint, then by σ -additivity we have

$$\mu(B) = \mu(A) + \mu(B \cap A^c).$$

Since measure is always positive, then it follows that

$$\mu(B) \geq \mu(A).$$

In particular, observing that $B \cap A^c = B \setminus A$ and using the fact that $\mu(A) < \infty$, then

$$\mu(B \setminus A) = \mu(B) - \mu(A).$$

(b) We can write

$$A \cup B = A \cup (B \setminus (A \cap B)).$$

Since the union above is disjoint, we have

$$\mu(A \cup B) = \mu(A) + \mu(B \setminus (A \cap B)) = \mu(A) + \mu(B) - \mu(A \cap B).$$

(c) Let $B_1 = A_1$ and define $B_2 = A_2 \setminus A_1$, $B_3 = A_3 \setminus A_2$, and etc. Notice that

$$A = \bigcup_n B_n.$$

By sigma additivity of μ we have

$$\mu(A) = \sum_n \mu(B_n)$$

Observe that $\mu(A_N) = \sum_{n=1}^N \mu(B_n)$. Thus

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n).$$

(d) This follows easily by using the fact that $A_n^c \uparrow A^c$ and $A^c = \bigcup A_n^c$. Thus by continuity from below

$$\mu(A^c) = \lim_{n \rightarrow \infty} \mu(A_n^c).$$

We can write

$$1 - \mu(A) = \lim_{n \rightarrow \infty} (1 - \mu(A_n)) = 1 - \lim_{n \rightarrow \infty} \mu(A_n),$$

where we have used the fact that $\lim_{n \rightarrow \infty} \mu(A_n)$ exists. Because by assumption $A_1 \supset A_2 \supset A_3 \cdots$ and by the monotonicity of the measure $\mu(A_n)$ is a positive decreasing sequence, thus bounded from below. So it converges.

(e) To see this define

$$B_n = A_n \setminus \left(\bigcup_{k=1}^{n-1} A_k \right).$$

Notice that $\bigcup_n A_n = \bigcup_n B_n$ and $B_n \subseteq A_n$. Also note that B_n s are disjoint. So

$$\mu\left(\bigcup_n A_n\right) = \mu\left(\bigcup_n B_n\right) = \sum_n \mu(B_n) \leq \sum_n \mu(A_n).$$

■ **Remark** Probability style proof of item (2) above: Consider

$$1 - \prod_i (1 - \mathbb{1}_{A_i}) = \sum_i \mathbb{1}_{A_i} - \sum_{i < j} \mathbb{1}_{A_i} \mathbb{1}_{A_j} + \sum_{i < j < k} \mathbb{1}_{A_i} \mathbb{1}_{A_j} \mathbb{1}_{A_k} - \cdots$$

Noting the fact that $1 - \mathbb{1}_A = \mathbb{1}_{A^c}$, $\mathbb{1}_{A_i} \mathbb{1}_{A_j} = \mathbb{1}_{A_i \cap A_j}$, $\mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A)$, and applying the expected value to both sides we will get

$$\mathbb{E}\left[1 - \prod_i (1 - \mathbb{1}_{A_i})\right] = \sum_i \mu(A_i) - \sum_{i < j} \mu(A_i \cap A_j) + \sum_{i < j < k} \mu(A_i \cap A_j \cap A_k) - \cdots \pm \mu(A_1 \cap \cdots \cap A_n).$$

Notice that the left hand side is

$$\mathbb{E}\left[1 - \prod_i (1 - \mathbb{1}_{A_i})\right] = \mathbb{E}\left[1 - \prod_i \mathbb{1}_{A_i^c}\right] = \mathbb{E}\left[1 - \mathbb{1}_{\bigcap_i A_i^c}\right] = \mathbb{E}\left[\mathbb{1}_{\bigcup_i A_i}\right] = \mu(A_1 \cup \cdots \cup A_n).$$