Manifolds

Ali Fele Paranj
alifele@student.ubc.ca

April 28, 2024

0. Contents

1. Euclidean Spaces

1.1 Basic Notions and Definitions

Here in this chapter I will be covering the details of some notions that was challenging for me do digest in the first read.

Definition 1.1 — Axioms of Group. Group is a set A along with a binary operation $* : A \times A \to A$ that satisfies the following properties. Let $a, b, c \in A$, then

- **Associativity**: $a * (b * c) = (a * b) * c$.
- **Identity element**: $\exists 1 \in A$ such that
$$1 * a = a * 1 = a.$$
- **Inverse element**: $\forall a \in A \; \exists \hat{a} \in A$ such that
$$a * \hat{a} = \hat{a} * a = 1.$$

■ **Remark** A set along with a binary operation that does not satisfy any properties is called a **magma**. If the binary operation is only associative, then we are dealing with **semi-group**. If the binary operation has an identity element as well, then we call this algebraic structure as **monoid**.

Definition 1.2 — Axioms of Ring. A ring is a set R along with two operations $+ : R \times R \to R$ and $* : R \times R \to R$, where

- $(R, +)$ is an Abelian group.
- $(R, *)$ is a monoid.
- The operator $(*)$ has distributive (left and right) law over $(+)$ i.e.
$$a * (b + c) = (a * b) + (a * c), \qquad (b + c) * a = (b * a) + (c * a).$$
.

■ **Remark** **Field** is a ring where every non-zero element (i.e. inverse element in the $(R, +)$ group in the ring) has a multiplicative inverse.

Definition 1.3 — Axioms of Module. A **module** is a group M along with a ring R where the monoid of the ring acts on M (through scalar multiplication) (i.e. it satisfies the idenity and compatibility properties) and satisfies the distributive property. I.e.

- **Compatibility of the monoid action**: $a, b \in R,\ u \in M$ then
$$a(bu) = (ab)u.$$
- **Identity of the monoid action**: Let 1 be the identity element of the ring R. Then $\forall u \in M$
$$1u = u1 = u.$$
- **Distribution law**: $a, b \in R$ and $u, v \in M$ then
  - $(a + b)u = au + bu$.
  - $a(u + v) = au + av$.

■ **Remark** A module (M, R) is called a **vector space**, if the **ring** R is a **field**.