Lecture Notes For: Advanced Linear Algebra

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This lecture note is mostly based on the course MATH 257 Partial Differential Equations at UBC (2023). However I have also expanded the contents and examples from the following books.

1 Basics

Partial differential equations relate the partial derivatives of a function to each other. For example f can be a function of spacial coordinates (like x, y, z in the case of Cartesian coordinates), dynamical variable (like time), or any other kind of variables (like the space of genotypes g). For example suppose that $\Phi(x, y)$ represents the electric potential of a point charge. Such function should satisfy the Laplace equation:

$$\partial_{xx}\Phi + \partial_{yy}\Phi = 0$$

.

Note that the symbols ∂_{xx} and ∂_{yy} are short symbols for $\frac{\partial}{\partial x^2}$ and $\frac{\partial}{\partial y^2}$ respectively.

Definition: Order of PDE

The order of a PDE is the highest derivative that occurs in the equation.

Based on the definition above, the Laplace equation is a second order partial differential equation.

1.1 Classification of The Second Order PDEs

There are three categories of the second order PDEs that every other type of a second order PDE can be converted to one of these kinds. The most general type of a second order PDE can be written as:

$$A\partial_{xx}u + B\partial_{xy}u + C\partial_{yy}u + D\partial_{x}u + E\partial_{y}u + Fu = k \tag{1.1}$$

In which the coefficients are all a function of x, y (but not u in which case the PDE will be nonlinearx). Equation 1.1 can be summarized in a more compact form using the derivative operator L:

$$\mathrm{L}\,u = 0$$

in which:

$$\mathrm{L} = A\partial_{xx} + B\partial_{xy} + C\partial_{yy} + D\partial_{x} + E\partial_{y} + F$$

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The following table summarizes special categories of the lines second order PDEs that frequently occur in physical applications:

1.2 Intuitive Derivation of the Second Order PDEs

PDE	Analogous Quad Surf	Δ	Class	Application
$u_t = u_{xx}$	$T = x^2$	0	parabolioc	Diffusion - Heat Equation
$u_{tt} = u_{xx}$	$T^2 = x^2$	$\Delta > 0$	Hyperbolic	Wave Equation
$u_{xx} + u_{yy} = 0$	$x^2 + y^2 = 0$	$\Delta < 0$	Elliptic	Laplace
$u_{xx} + u_{yy} = c$	$x^2 + y^2 = k$			Poisson

Table 1: