Lecture Notes For: $\mathbb{R}$eal Analysis

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Chapter 1

Sets and Mappings

Remark 1.0.1

Let $\mathbb{R}_+$ denote the real number greater than or equal to zero[a]. The we can view the association $x \mapsto x^2$ as a map from R to $\mathbb{R}_+$. When viewed so, the map is the surjective. Thus it is a reasonable convention not to identify this map with the map $f : \mathbb{R} \to \mathbb{R}$ defined by the same formula. To be completely accurate, we should therefore denote the set of arrival and the set of departure of the map into our notation, and for instance write

$$F_T^S : S \to T,$$

instead of our $f : S \to T$ notation. In practice, this notation is too clumsy, so that we omit the indices S, T. However, the reader should keep in mind the distinction between the maps

$$f_{\mathbb{R}_+}^{\mathbb{R}} : \mathbb{R} \to \mathbb{R}_+ \quad \text{and} \quad f_{\mathbb{R}}^{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$$

both defined by the association $x \mapsto x^2$. The first map is surjective, while the second one is not. Similarly the maps

$$f_{\mathbb{R}_+}^{R_+} : \mathbb{R}_+ \to \mathbb{R}_+ \quad \text{and} \quad f_{\mathbb{R}}^{\mathbb{R}_+} : \mathbb{R}_+ \to \mathbb{R}$$

defined by the same association $x \mapsto x^2$ are injective.

[a] This note is from Segel, undergraduate analysis.

Remark 1.0.2

Chapter 2

Topology of Real Numbers

2.1 Introduction and Some Historical Notes

In this section we will construct the set of real numbers from the integers. We will assume that we know the integers and its basic arithmetic properties. However the fact is that the set of integers can be constructed by the set of positive integers (natural number) that can be constructed using the concept of the cardinality of a set and the set of all subsets of a set.

Ancient Greek scientists knew how to construct the rational and irrational numbers (like $\sqrt{2}$) with a compass and straightedge. But they did not know how to construct the number π with that setting. This problem was known for them as *the problem of squaring a circle.* In 1666, Newton showed that π can be constructed with an infinite sum. It was in late 1600's that Newton and Leibniz had vague notions of "limit" and "infinity". It was until early 1800's that there were no rigorous mathematical definition of these concepts. For example stuff by Fourier (like infinite Fourier series) made Laplace and Lagrange very uneasy! The infinite and limit concepts were more like a toolbox that were working very well on certain physical problems (for example in solving the PDE for hear equation). Finally In the early 1800s, a revolution happened in making these concepts precise. For example works done by Cauchy in 1820's and Weierstrass and Riemann (1850's and 1860's) had a significant contribution on these concepts.

2.1.1 A little note about Leopold Kronecker

In the lecture note by Francis Su in youtube, he talks about this famous saying from Kronecker:

> God created the integers. All else is the work of man!

And Su continues explaining that Kronecker was a finitist (following the finitism school of thoughts). When I heard this discussion his argue with Cantor came in my mind. In the Wikipedia page of Cantor we read that Kronecker was calling him as a "scientific charlatan", a "renegade" and a "corrupter of youth". So there is a connection with him being a finitist and having serious arguments with Cantor. It is also very interesting for me that one of his contributions which is Kronecker delta function kind of works with integers both in its index and its output!

Strangely, the quote that I have written above by Kronecker was his reply to the Lindemann when he proved that the number pi is a transcendental number. It is believed that he said "this is a beautiful but proves nothing. transcendental numbers do not exist!!"

2.2 Constructing Rational Numbers

To construct the rational numbers from integers, we need to use the concept of relations on a set. I will not talk about the concept of relations here as it is covered in other lecture notes. The relation that is of our interest is called a **equivalence relation**. Equivalence relation is a relation that has three properties called *reflexivity*, *symmetry*, and *transitivity*. We can define the rational numbers as:

$$\mathcal{Q} = \{\frac{a}{b} | a, b \in \mathcal{Z}, b \neq 0\}.$$

The notation $\frac{\cdot}{\cdot}$ as a equivalence relation: the $\frac{a}{b}$ is a representation of the ordered pair (a, b). We say (a, b) (c, d) if and only if $ad = bc$. The relation is indeed a equivalence relation and this relation is in fact the equality relation for the rational numbers. For example $\frac{3}{5} = \frac{6}{10}$ because $3 * 10 = 6 * 5$.

It is very easy to show that this relation is an equivalence relation. However to check the transitivity property, we need to use the cancellation law. Keep in mind that we have not yet defied division for integers and the cancellation law is the next best thing to the division. The cancellation law for the integers is:

$$ab = ac, \quad a \neq 0 \quad \Rightarrow \quad b = c$$

So far we learned that we can construct the set of rational numbers like the following set:

$$\mathbb{Q} = \{\frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0\}.$$

However the question arise that what is the meaning of $\frac{a}{b}$. This is simply a representative of class of an equivalence defined on $\mathbb{Z} \times (\mathbb{Z} \backslash \{0\})$. The relation is defined as this:

> let $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0$. Then we write $(a, b) \sim (c, d)$ if and only if $ad = bc$. Then $\frac{a}{b}$ is an equivalence class such that:
>
> $$\frac{a}{b} = \{(c, d) \in \mathbb{Z} \times (\mathbb{Z} \backslash \{0\}) | (a, b) \sim (c, d)\}$$

As an example $\frac{1}{2} = \{(1, 2), (2, 4), (3, 6), \cdots\}$.

2.2.1 Defining addition for the rational numbers

So far we know how to add two integers but what does actually mean to add two rational numbers? We can throw any definitions that we want but we need to keep in mind that the definition should be well defined. In a sense that the definition does not depend on the representative of the class that we pick. For instance let's define the sum of rational numbers as:

> **A proposed definition for summation.** Let $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0$. Then let's define the summation of the rational numbers as the following:
>
> $$\frac{a}{b} + \frac{c}{d} = \frac{a + b}{c + d}.$$

The problem with the definition above is that it is not well defined, i.e. the result of the sum depends on the choice of representative for the class of interest. To illustrate that better let's do the following summation:

$$\frac{1}{2} + \frac{5}{3} = \frac{6}{5}$$

Now let's pick other representatives of the classes $\frac{1}{2}$ and $\frac{5}{3}$ which can be for instance $\frac{7}{14}$ and $\frac{10}{6}$. Now we expect to get a same result as before if we sum these two fractions:

$$\frac{7}{14} + \frac{10}{6} = \frac{17}{20}$$

It is clear that $\frac{17}{20}$ and $\frac{6}{5}$ are not equivalent. So if we define the summation in the specified way, then it is not well defined.

Also there is another problem. Defining the summation in this way will not extent the notion of sum for the integers. You can try summing $\frac{5}{1} + \frac{4}{1}$ and observe that the result is not the same as $5 + 4 = 9$.

Let's define that summation in the following way that is both well defined and also extends the notion of summation of the integers.

Definition 2.2.1. Defining summation for the rational numbers

Let $\frac{a}{b}$ and $\frac{c}{d}$ be two rational numbers. Then we define the summation for rational numbers as:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

To show that this definition is we ll defined, Let $\frac{a}{b}, \frac{c}{d}, \frac{a'}{b'}, \frac{c'}{d'}$ be rational numbers such that $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$. We need to show that

$$\frac{ad + bc}{bd} = \frac{a'd' + b'c'}{b'd'}$$

Proof. Since $(a, b) \sim (a', b')$, then we can write $ab' = a'b$ and similarly since $(c, d) \sim (c', d')$ then $cd' = c'd$. Since $b, b', d, d' \neq 0$, then we can multiply bb' to the both sides of the second equation and dd' to both sides of the first equation. Then we will have:

$$ab'dd' = a'bdd',$$
$$bb'cd' = bb'c'd.$$

By adding both sides of these equations then we will have:

$$ab'dd' + bb'cd' = a'bdd' + bb'c'd,$$
$$(b'd')(ad + bc) = (bd)(a'd' + b'c').$$

This clearly shows that $(ad + bc, bd) \sim (a'd' + b'c', b'd')$ hence

$$\frac{ad + bc}{bd} = \frac{a'd' + b'c'}{b'd'}$$

Now we can define the multiplication for the rational numbers.

Definition 2.2.2. Multiplication of the rational numbers

Let $\frac{a}{b}$ and $\frac{c}{d}$ be rational numbers. Then we define the multiplication for the rational numbers as:

$$\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$$

Similar to the last part, we can show that this definition is well defined. Namely we can show that for rational numbers $\frac{a}{b}, \frac{a'}{b'}, \frac{c}{d}, \frac{c'}{d'}$ that $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, we have $(ac, bd) \sim (a'c', b'd')$.

Proof. Since $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$ so $ab' = a'b$ and $cd' = c'd$. By multiplying both sides of the equation we will have:

$$a'bc'd = ab'cd',$$

which clearly shows that $(ac, bd) \sim (a'c', b'd')$ hence

$$\frac{ac}{bd} = \frac{a'c'}{b'd'}.$$

2.2.2 Does $\mathbb{Q}$ extends $\mathbb{Z}$?

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With the following correspondence (for $n \in \mathbb{Z}$)

$$\frac{n}{1} \leftrightarrow n,$$

we can show that the set $\{\frac{n}{1} | n \in \mathbb{Z}\}$ behaves exactly like the set of integers. In other words we say these two sets are isomorphic.

2.2.3 Orders in $\mathbb{Q}$

We know that the elements of $\mathbb{Z}$ are ordered (some elements are smaller or larger than the other ones). So the natural question that arise is that will this order be still valid for the points in $\mathbb{Q}$? To answer this question we need to rigorously define the order relation in $\mathbb{Z}$.

Definition 2.2.3. Definition of Order

An order on a set S is a relation $<$ satisfying:

- Low of trichotomy: $\forall x, y \in S$, the only one of the following statements are true

$$x < y, \quad x = y, \quad y < x$$

- Transitivity: For $x, y, z \in S$, $x < y$ and $y < z$ implies $x < z$.

Note that this is a general definition of order on a set and is not restricted to our usual definition of order between real numbers. However, we can define the notion of the "usual" order in $\mathbb{Z}$ like the following:

Definition 2.2.4. Order Relation on $\mathbb{Z}$

The order relation on $\mathbb{Z}$ denoted with the symbol $<$ is defined as the following. Let $a, b \in \mathbb{Z}$. We say $a < b$ if and only if $a - b$ a positive integer. The set of positive integers are defined as $\{1, 2, 3, 4, \cdots\}$.

Example 2.2.1. Dictionary Oder on $\mathbb{Z}$

As stated earlier, we can extend the definition of order. A **dictionary order** on $\mathbb{Z}^2$ is the relation $\lessdot$ such that for $(a_1, a_2), (b_1, b_2) \in \mathbb{Z}^2$ we write $(a_1, a_2) \lessdot (b_1, b_2)$ if and only if $(a_1 < b_1)$ and if $a_1 = b_1$, then $a_2 < b_2$.
For example, given this relation we can write: $(3, 4) \lessdot (5, 1), (1, 0) \lessdot (1, 10), (3, 1) \lessdot (3, 5)$

Definition 2.2.5. Positive rational numbers

We say the rational number $\frac{a}{b}$ is positive if the integer ab is positive.

Definition 2.2.6. Ordering of rational numbers

We say $\frac{a}{b} < \frac{c}{d}$ if $\frac{c}{d} - \frac{a}{b}$ is a positive rational number.

Given the ordering property of the rational numbers, we can look at the rational numbers with a new perspective.

2.2.4 $\mathbb{Q}$ Is a Field!

Field is one of many algebraic structures (like groups, rings, vector spaces, etc).

Definition 2.2.7. Field

A field is a set F along with two operations $+, \times$ that holds the following properties:

- (A_1) : The set F is closed under $+$.
- (A_2) : $+$ is commutative.
- (A_3) : $+$ is associative.
- (A_4) : Every element in F has a additive inverse
- (A_5) : Every element in F has a additive identity (call it 0)
- (M_1) : The set F is closed under $\times$.
- (M_2) : $\times$ is commutative.
- (M_3) : $\times$ is associative.
- (M_4) : Every element in F (except for the additive inverse) has an multiplicative inverse.
- (M_5) : Even element in F has an multiplicative identity (call it 1).
- (D_1) : The operator $\times$ distributes over $+$.

Example 2.2.2. $\mathbb{Q}$ is a field

Question. Show that the set of rational numbers is a field.

Solution. We can start with finding the additive and multiplicative inverses and identities. It is obvious that:

- Additive identity: $\frac{0}{1}$.
- Additive inverse for $\frac{a}{b}$: $\frac{-a}{b}$.
- Multiplicative identity: $\frac{1}{1}$.
- Multiplicative inverse for $\frac{a}{b}$: $\frac{b}{a}$

Now we need to show that the conditions $A_1, A_2, A_3, M_1, M_2, M_3, D_1$ holds. Let $a, b \in \mathbb{Z}$. Then we know that $a+b$ and ab are also integers and are in $\mathbb{Z}$. So A_1, M_1 immediately follows from the definition of addition and multiplication for the rational numbers.

- A_2: We need to show that $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$ By following the addition defined for the rational numbers, we can write the expression for the LHS and RHS seperately and observe that those two are equal. So for LHS we have:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

- (A_3): We need to show $(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{a}{b} + (\frac{c}{d} + \frac{e}{f})$
  Following the definition of addition for the rational numbers, for the LHS we can write:

$$\frac{ad+bc}{bd} + \frac{e}{f} = \frac{fad + fbc + edb}{bdf}$$

  And for the RHS we can write:

$$\frac{a}{b} + \frac{cf+de}{df} = \frac{adf + bcf + bde}{bdf}$$

  Because of the associativity and commutativity properties of Z, we can conclude that RHS = LHS.

So we can observe that A_2, A_3, M_1, M_2 follows from the commutativity and associativity properties of the integers (which are considered as a ring).

2.2.5 Constructing the Real Numbers from Rational Numbers

The rational numbers extends the set of integers in a very useful way. But it turns out that there are many holes in the set of rational numbers; i.e. there are some real numbers (real in the sense that we can construct some length equal to it on a paper) but that does not belong to the set of rational numbers. A very famous example is $\sqrt{2}$ that has been known from the ancient Greek. This number is the hypotenuse of a right triangle with sides equal to 1 (using the Pythagorean theorem). However we can prove that this number is not rational number.

Proof. Let x be number s.t. $x^2 = 2$. We claim that this number can not be a rational number.

To show this, let's assume that x is rational. So we can write x as:

$$x = \{x = \frac{a}{b} | x^2 = 2, b \neq 0, (a, b) = 1\}$$

Note that we require $(a, b) = 1$ (i.e. relatively prime) as an extra condition since we know that in the class of equivalence with the representative $\frac{a}{b}$, there is an element $\frac{c}{d}$ such that (a, b) (c, d) (since (c, d) belongs to the $\frac{a}{b}$) and c, d are relatively prime. So we can write $x^2 = \frac{a^2}{b^2} = 2$. Then $a^2 = 2b^2$. We can easily show (by contrapositive) that if a^2 is even, then a is even as well. So for some integer k we can write $a = 2k$. Then $b^2 = 2k^2$, so b is also even. Hence for some integer l, $b = 2l$, and this is a contradiction because a, b are not relatively prime.

Now that we observed numbers like $\sqrt{x}$ are not rational, then we can say that the set of rational number $\mathbb{Q}$ does not have the **least upper bound property**.

Definition 2.2.8. Least Upper Bound Property

A set S is said to have the least upper bound property every nonempty subset of S that is bounded above (thus has an upper bound), has a least upper bound (i.e. supremum) as well.

It is clear from the definition that the set $\mathbb{Q}$ does not have a least upper bound property since the set

$$A = \{x \in \mathbb{Q} | x^2 < 2\}$$

has an upper bound (like 2) but does not have a least upper bound. This indicates the wholes present in the set of rational numbers. However, we can extend the idea of rational numbers in a way that contains the set of rational numbers as a subset and also fills in the gaps. We can do that in many ways one of which is the concept of Dedekind cuts. Here is the definition of a Dedekind cut:

Definition 2.2.9. Dedekind cut

A Dedekind cut α is a **subset of rational numbers** that has the following properties:

1. The set is not trivial (i.e. is not empty and does not contain all of the rationals),
2. is closed downwards. In other words $(x \in \alpha \wedge q \in \mathbb{Q}) \wedge q < x \Rightarrow q \in \alpha$, and
3. has no largest element. In other words: $\forall x \in \alpha, \exists r \in \alpha \quad s.t. \quad x < r$.

The set of real numbers can be defined using the idea of the Dedekind cuts in the following way:

Definition 2.2.10. The Set of Real Number

The set of real numbers denoted by $\mathbb{R}$ is the set of all cuts:

$$\mathbb{R} = \{\alpha : \alpha \text{ is a cut}\}.$$

So when we refer to the real number $\sqrt{2}$, it is a set that:

$$\sqrt{2} = \{q \in \mathbb{Q} : x^2 < 2\}.$$

Remember that this set (which is also a cut) itself had not sup in the set of rational numbers. However the set of all such cuts (that we denoted that set as the set of real number), will have the least upper bound property. For an instance:

$$\sup\{x \in \mathbb{R} : x^2 < 2\} = \{x \in \mathbb{Q} : x^2 < 2\}$$

We can define the addition and multiplication operations for these sets (cuts) in a proper way that extends the idea of addition and multiplication for rationals (thus integers). Also, we can show that the set of real numbers also posses the order relations ($\alpha < \beta \quad iff \quad \alpha \subset \beta$). So we can show that the set of rational numbers form an **ordered field**. In fact we can show that the set of rational numbers is the only ordered field with the least upper bound property and for any other ordered field there is a one-to-one correspondence (bijective) between its elements and the set of real numbers. As an instance, we can define the addition and multiplication for the real number as:

Definition 2.2.11. Addition and Multiplication for Real Numbers

Let α and β be two Dedekind cuts. Then:

$$\alpha + \beta = \{r + s : r \in \alpha, s \in \beta\},$$
$$\alpha * \beta = \{rs : r \in \alpha, s \in \beta\}$$

One of the important things to check when defining a **binary operator** on a set ($O : S \to S$) is to check if the result of the operation still is in the set. So it is a good practice to show that $\alpha + \beta$ and $\alpha * \beta$ are still considered as cuts.

2.2.6 Note that I need to add them to the main text

- In a field, just one element (that is the additive identity) should have no inverse (no any other thing).

Chapter 3

Hausdorff Topological Spaces

The following is a section of the great book "Mathematical Discovery" by Bruckner.

> Professional mathematicians must adhere to strict standards in their work. This entails providing precise definitions, even for seemingly familiar concepts. Such precision often requires the use of complex technical tools and methods. A mathematician must possess a clear understanding of fundamental concepts, such as the precise definition of a "curve," the mathematical interpretation of "traversing a curve with the inside to the left," the formal description of the number of "holes" in a pretzel, and the mathematical definition of area.
>
> **It's important to note that this level of rigor and precision is not typically present when a mathematician initially approaches a problem and begins working on a solution.** In the early stages, ideas tend to be more abstract and intuitive. The refinement and meticulousness become evident only in the final drafts of mathematical work.

Thus, we first need to have a discussion that show that the ideas behind the abstractions and generalizations are achievable by careful studying the mathematical objects already around us. This this section focuses to motivate the reader towards the more abstract concepts.

Thus we will discuss that the $\mathbb{R}^n$ along with the Euclidean distance has some special properties (which later will be generalized to the concept of metric spaces), and then we will see that the notion of Euclidean distance give rise to special sets called open ball which will give rise to the notion of open sets. We will study the properties of these open sets and later we will study what if we define the notion of open sets on its own (without any need to any particular metric) which will lead to the notions and ides of topological spaces.

3.1 Motivation

Consider the set $\mathbb{R}^k$ which is a k fold Cartesian product of our favorite set $\mathbb{R}$!. We can also extend the notion of Euclidean distance in $\mathbb{R}$ (which was simply $|x-y|$ for $x, y \in \mathbb{R}$) to $\mathbb{R}^k$ as follows

$$|x-y| = \sqrt{\sum_{i=1}^{k}(y_i - x_i)^2}, \qquad x = (x_1, \cdots, x_k),\ y = (y_1, \cdots, y_k) \in \mathbb{R}^k.$$

We can easily observe that the Euclidean distance is a function $d : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ that satisfies the following properties

(i) $|x-y| \geq 0, \quad |x-y| = 0 \Leftrightarrow x = y,$

(ii) $|x-y| = |y-x|,$

(iii) $|x-y| \leq |x-z| + |z-y|$

Later, We will study the generalized idea of such functions defined on a set which will give rise to the concept of metric spaces.

Now, we intuitively define the notion of "open ball" centered at $x \in \mathbb{R}^k$ with radius $r \in \mathbb{R}$ as follows

$$\mathcal{B}_x(r) = \{y \in \mathbb{R}^k : \ |x-y| < r\}.$$

Then we define a set $A \subseteq \mathbb{R}^k$ to be a open set such that for every element $x \in$, we can have an open ball $\mathcal{B}_x(r)$ for some $r \in \mathbb{R}$ which is contained in A. More formally we can write

$$\forall x \in A, \ \exists r > 0 \text{ s.t. } x \in \mathcal{B}_x(r) \subseteq \mathbb{R}^k.$$

Since this notion is a very central one (as we will find out later), for $\mathbb{R}^k$, we have the notion of the set of all open sets of $\mathbb{R}^k$, for which we write $\mathcal{T}$.

Then we can go a little bit beyond the immediate intuition and define the notion of the set of all neighborhoods of x as

$$\mathcal{N}(x) = \{S \subseteq \mathbb{R}^k : \ \exists u \in \mathcal{T}, \ x \in u \subseteq S\}.$$

It immediately follows from the definition that all open balls containing x (not necessarily containing x) are in $\mathcal{N}(x)$, along with other sets which satisfies the required property. We are now in a good shape to study the properties of the open sets $u \in \mathcal{T}$. We claim the followings are some of such properties (which as it will turn out are the central properties in some sense)

(i)
$$\varnothing, \ \mathbb{R}^k \in \mathcal{T}.$$

(ii)
$$\forall \mathcal{G} \subseteq \mathcal{T} \text{ we have } \bigcup_{g \in \mathcal{G}} g \in \mathcal{T}.$$

(iii)
$$U_1, \cdots, U_n \in \mathcal{T}, \ n \in \mathbb{N} \implies \bigcap_{i=1}^{n} U_i \in \mathcal{T}.$$

(iv)
$$\forall x, y \in \mathbb{R}^k, \ \exists U, V \in \mathcal{T} \text{ s.t. } x \in U, \ y \in V, \quad U \cap V = \varnothing.$$

Proof. TO BE ADDED. □

Since the notion of open sets is closely related with the distance function, thus there is no surprise that it can be used to express the ideas of convergence of a sequence in $\mathbb{R}^k$ (such a fundamental concept in analysis) with the new terminology. For instance, the following two statements are logically equivalent for $x_n \to \hat{x}$

(i) $\forall \epsilon > 0, \ \exists N > 0,$ s.t. $\forall n > N$ we have $|x_n - \hat{x}| < \epsilon$.

(ii) $\forall S \in \mathcal{N}(\hat{x}), \ \exists N > 0,$ s.t. $\forall n > N$ we have $x_n \in S$.

Proof. TO BE ADDED. □

3.2 Metric Spaces

Although $\mathbb{R}^k$ is a very useful set synthesized in a special way to meet most of our requirements (for instance the completeness arguments in the sense of Cauchy sequence, least upper bound property, and etc.) but not every set we encounter is $\mathbb{R}^k$. We can have sets that are globally very different than the "*flat*" $\mathbb{R}^k$, for instance $\mathbb{S}^1$ (unit circle), $\mathbb{S}^1 \times \mathbb{S}^1$ (a tours), etc. One of the main approaches in dealing with such structures is to "locally" convert it (in a useful way) to a collection (or atlas) of subsets of $\mathbb{R}^k$ and then work with the original "alien" set in an indirect way by focusing on these local images in $\mathbb{R}^k$. Apart from this approach, it is also useful to generalize the notions of distance in a set, which will enable us working with other classes of abstract structures without relying on $\mathbb{R}^k$, as some of them are way larger than $\mathbb{R}^k$. For instance, consider the set of all bounded function $f : [0, 1] \to \mathbb{R}$. This set has a cardinality that is bigger than the cardinality of continuum. Also, might want to work with sets that are discrete in nature, like $\mathbb{N}$ which their cardinality is less than $\mathbb{R}^k$. So relying on $\mathbb{R}^k$ for all purposes is not feasible, thus it might be a good idea to have the notion of distance between elements in set.

Definition 3.2.1. Metric Space

A metric space is simply (X, d) in which X is a set and $d : X \times X \to \mathbb{R}$ is a function called metric that satisfies the following properties

(i) $d(x, y) \geq 0, \quad d(x, y) = 0 \Leftrightarrow x = y.$

(ii) $d(x, y) = d(y, x).$

(iii) $d(x, y) \leq d(x, z) + d(z, y).$

In which $x, y, z \in X$.

Now we can easily see that all of the notions like open balls, open sets, and etc. which we defined for $\mathbb{R}^k$ can also be defined for a metric space. We can define different metrics on a particular set based on our demands. In fact there are infinitely many ways to come up with a metric function. **One of our main tasks in studying metric spaces is to show that there are some properties of a metric space that are independent of a particular defined metric**. As we will see later, this will give rise to more abstract construct called topological spaces.

Definition 3.2.2. Open Ball in $\mathbb{R}^k$

Let (X, d) be a metric space. An open ball centered at $x \in X$ with radius r is the set

$$\mathcal{B}_r(x) = \{y \in X : d(y, x) < r\},$$

Definition 3.2.3. Open Set in $\mathbb{R}^k$

Let (X, d) be a metric space and let $U \subseteq X$. U is open if

$$\forall x \in X, \ \exists \mathcal{B}_r(x) \text{ s.t. } \mathcal{B}_r(x) \subseteq U.$$

We denote the set of all open sets of X as $\mathcal{T}$.

A very useful intuition about open sets is that we can move around any points of the set (sufficiently small) and still be in the set. In other words, we can perturb the points of an open set (a sufficiently small perturbation) and still remain in the set.

Definition 3.2.4. Neighborhood of x

Let (X, d) be a metric space. Then the set of all neighborhoods of x is

$$\mathcal{N}(x) = \{S \in \mathcal{P}(X) : \exists U \in \mathcal{T} \text{ s.t. } x \in U \subseteq S\}.$$

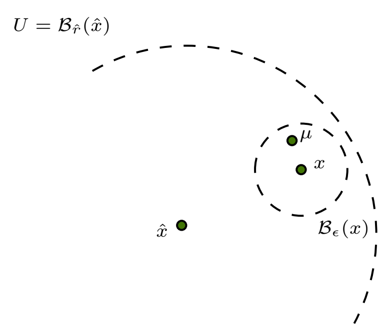
In other words, A neighborhood of $x \in X$, is the collection of all subsets of X, such that contains an open set containing x.

Remark 3.2.1

An open ball is an open set. This is not a tautological statement. The word "open" in the notion of open ball, has nothing to do with the word "open" in the notion of open set. however, we can show that an open ball is indeed an open set, thus deserves the name "open". In more accurate language

Let $\hat{x} \in X$, $\hat{r} \in \mathbb{R}$, and define $U = \mathcal{B}_{\hat{r}}(\hat{x})$. Then U is an open set.

Proof. The proof of remark above can be facilitated by considering the following diagram.



Considering the visual idea, we can proceed with the proof. Let $x \in U$. Then let $r^* = r - d(x, \hat{x})$ and $\epsilon < r^*$. This implies that $d(x, \hat{x}) = \hat{r} - r^*$. We claim that $\mathcal{B}_\epsilon(x) \subseteq U$. Indeed, let $\mu \in \mathcal{B}_\epsilon(x)$. By definition $d(\mu, x) < \epsilon < r^*$. Thus

$$d(\hat{x}, \mu) \leq d(\hat{x}, x) + d(x, \mu) < (\hat{r} - r^*) + r^* = \hat{r}.$$

Thus we showed that $\mu \in \mathcal{B}_\epsilon(x)$ implies $\mu \in \mathcal{B}_{\hat{r}}(\hat{x})$. Thus we can conclude that $\mathcal{B}_\epsilon(x) \subseteq \mathcal{B}_{\hat{r}}(\hat{x})$, for any choice of x. Thus U is an open set. $\square$

Following our arguments in the motivation section, we argued that the open sets of $\mathbb{R}^k$ satisfy some properties. If you read the proof closely, we used no facts very special about the Euclidean distance other than the properties described in the definition of a metric space. Thus it is not a surprise if we observe that those properties also hold for a general metric space.

Proposition: 3.2.1

Let (X, d) be a metric space. Then the open sets determined by d satisfy the following properties

(i) $X, \varnothing \in \mathcal{T}$.

(ii) $\mathcal{G} \subseteq \mathcal{T} \implies \bigcup_{g \in \mathcal{G}} g \in \mathcal{T}$.

(iii) $U_1, \ldots, U_n \in \mathcal{T} \implies \bigcap_{i=1}^{n} U_i \in \mathcal{T}$.

(iv) $\forall x, y \in X,\ \exists U, V \in \mathcal{T}$ s.t. $x \in U,\ y \in\ V,\ U \cap V = \varnothing$.

Proof. TO BE ADDED. □

3.2.1 Convergence in Metric Spaces

So far, we only hand the notion of sequence in sets like $\mathbb{R}, \mathbb{Z}, \mathbb{Q}$, etc. and we developed the notion of convergence of a sequence by $\epsilon - N$ business. Remember that a sequence is a fundamental concept which enables us to discover the word that is formed at the infinity! For a detailed discussion see my opinion piece titled by "What the Hell is Analysis".

The great thing about metric spaces is that we can now have the notion of sequence and also

3.2.2 UNDER CONSTRUCTION

Definition 3.2.5. The Usual Topology on $\mathbb{R}^k$

The set $\mathcal{T} = \{U \subseteq \mathbb{R}^k : U \text{ is an open set}\}$, is called the usual "topology" on $\mathbb{R}^k$.

Note that we will cover the notion of topology on a set later, but the purpose of this definition is just to keep in mind that $\mathcal{T}$ is the set of all open sets of $\mathbb{R}^k$. From this notion, here comes the important definition of a neighborhood of a set.

The way that we define a neighborhood of a point as above, is to emphasis that there is no pressure to restrict the notion of neighborhood to open balls only. In fact, any subset of $\mathbb{R}^k$ containing $x \in \mathbb{R}^k$, that can contains an open set (not necessarily an open ball) who contains x is a neighborhood of point x. The following corollary put this broad definition into a good use.

Corollary: 3.2.1

Let $S \in \mathcal{N}(x), x \in \mathbb{R}^k$. Then $\exists \mathcal{B}_r(x)$ for some $r > 0$, such that $x \in \mathcal{B}_r(x) \subseteq S$.

Based on this corollary that follows immediately from the definition of neighborhood, we can conclude that whenever we are given with $S \in \mathcal{N}(x)$ for $x \in \mathbb{R}^k$, then we can always find an open ball centered at x with sufficiently small radius.

Using all of these notions and definitions, we can now generalize the idea of convergence of a sequence

Proposition: 3.2.2

Converges $x_n \to \hat{x}$ in $\mathbb{R}^k$ can be expressed equivalently as

(a) $\forall \epsilon > 0,\ \exists N > 0\ :\ \forall n > N,\ x_n \in \mathcal{B}_\epsilon(\hat{x}).$ *or* $\forall \mathcal{B}_\epsilon(\hat{x}),\ \exists N > 0\ :\ \forall n > N, x_n \in \mathcal{B}_\epsilon(\hat{x}).$

(b) $\forall S \in \mathcal{N}(\hat{x}),\ \exists N > 0\ :\ \forall n > N,\ x_n \in S.$

Proof. Since the statements (a) and (b) are equivalent, then we need to proof the both ways. Both proofs are straight and can be deduced by following the definitions.

(a) $\Longrightarrow$ (b) : Given $S \in \mathcal{N}(\hat{x})$, $\exists \mathcal{B}_r(\bar{x})$ for $r > 0$ sufficiently small. Let $r = \epsilon$. Since (a) is true, then $\exists N > 0$ such that $\forall n > N$ we have $x_n \in \mathcal{B}_\epsilon(\hat{x}) \subseteq S$.

(b) $\Longrightarrow$ (a) : Given $\epsilon > 0$, let $S = \mathcal{B}_\epsilon(\hat{x})$. Then since (b) is true, then $\exists N > 0$ suc that $\forall n > N$ we have $x_n \in S$, thus we conclude $x_n \in \mathcal{B}_\epsilon(\hat{x})$.

$\square$