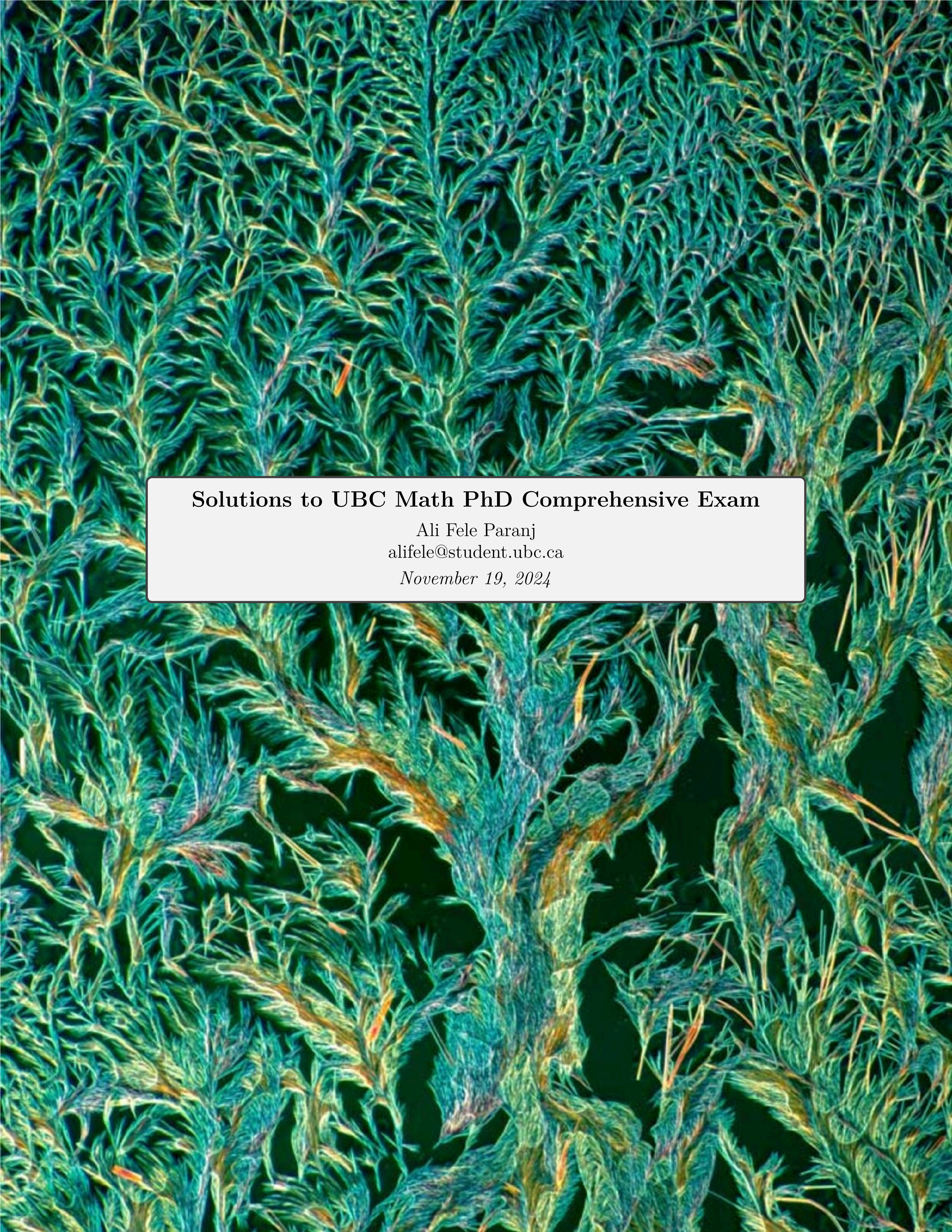
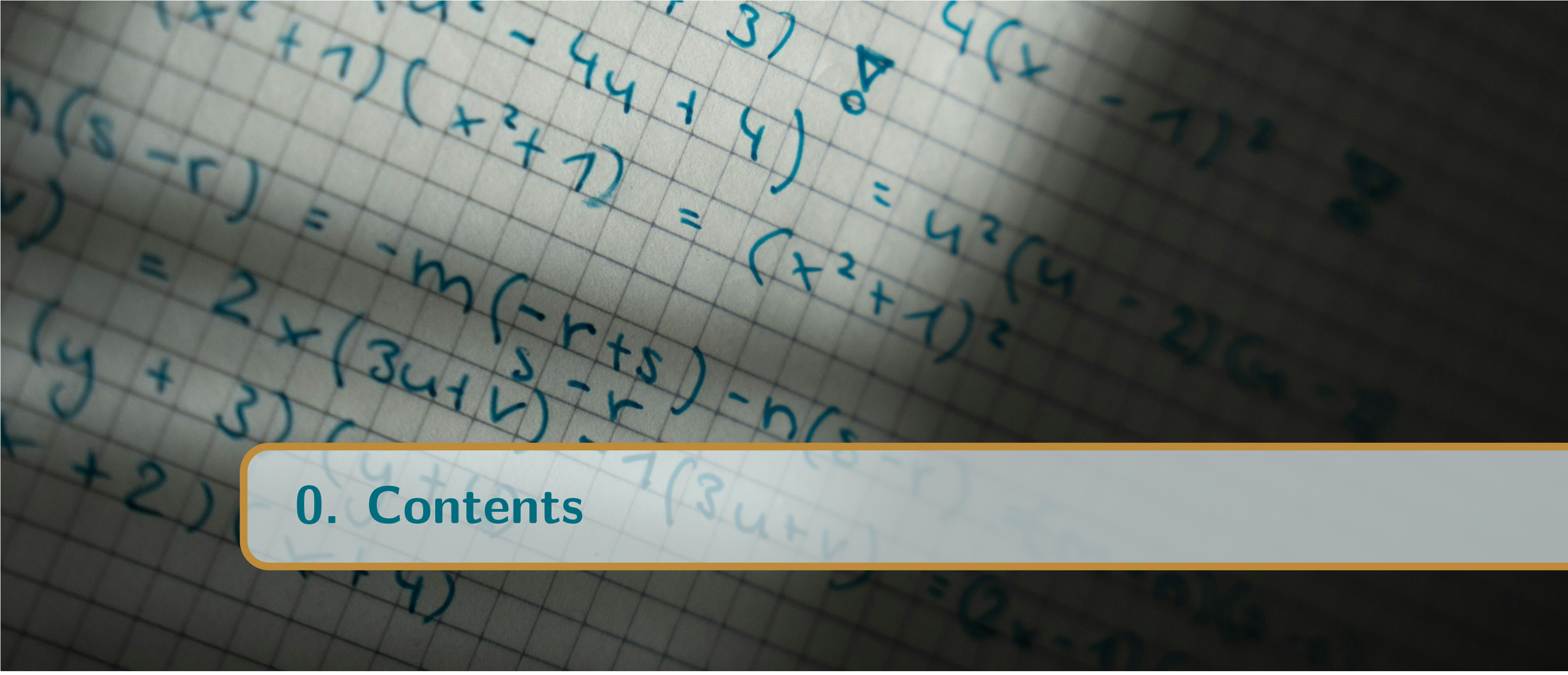


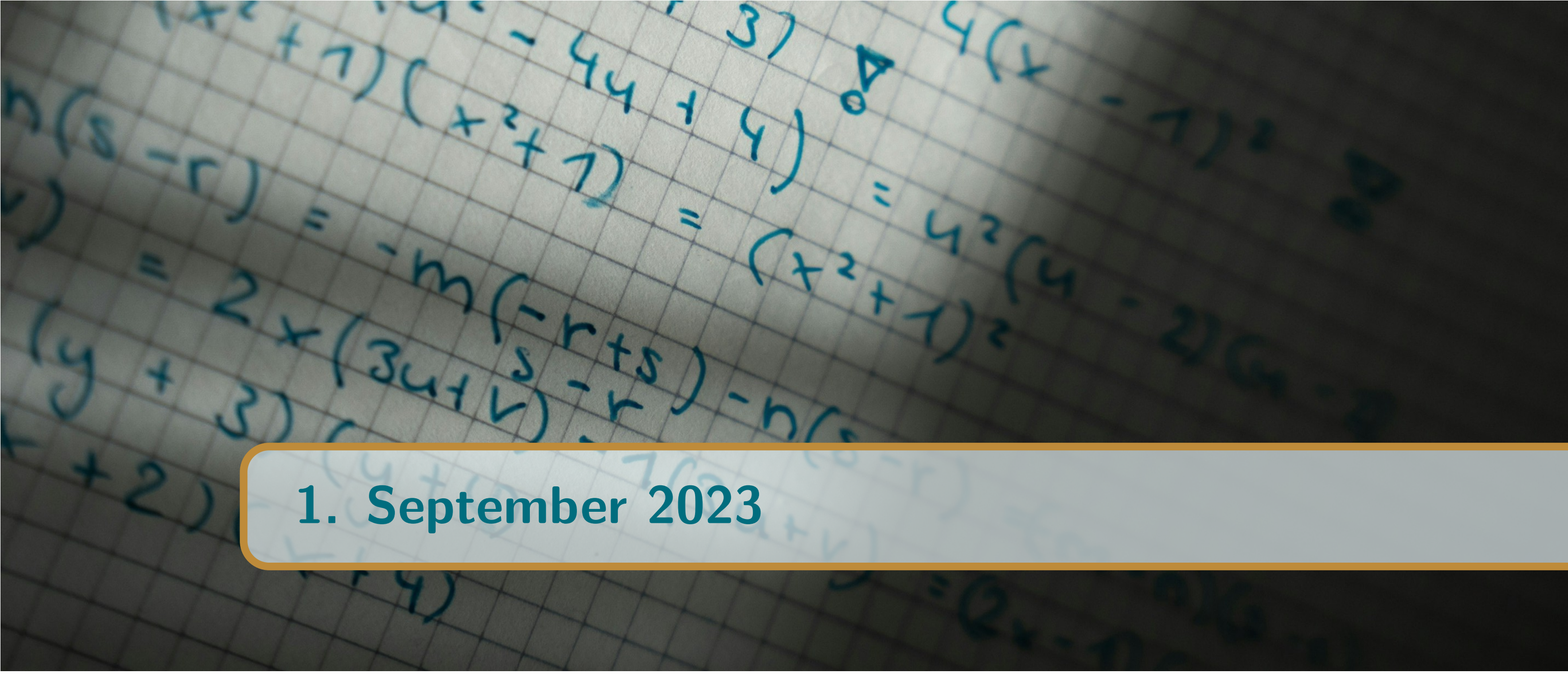
Solutions to UBC Math PhD Comprehensive Exam

Ali Fele Paranj
alifele@student.ubc.ca

November 19, 2024



0. Contents



1. September 2023

1.1 Algebra

■ **Problem 1.1** (a) Prove that any module over $\mathbb{Z}[i]$ is a direct sum of a free module and a torsion module. Is the same true for modules over $\mathbb{Z}[\sqrt{-5}]$?

(b) Let $I = 3 + 2i$ be the ideal in $\mathbb{Z}[i]$ generated by the element $3 + 2i$. Describe the quotient $\mathbb{Z}[i]/I$.

1.2 Linear Algebra

■ **Problem 1.2** (a) Find a lower triangular matrix L such that $LL^T = A$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 5 & 2 & 2 \\ 0 & 2 & 2 & 1 \\ 2 & 2 & 1 & 6 \end{pmatrix}$$

(b) Compute $\det(A)$.

(c) Find the volume in $\mathbb{R}^4$ of the set $S_A = \{x \in \mathbb{R}^4 : x^T A x \leq 1\}$.

Solution (a) Let L be

$$L = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix}$$

So $A = LL^T$ we will have

$$A = \begin{pmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} & l_{11}l_{41} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} & l_{21}l_{41} + l_{22}l_{42} \\ l_{11}l_{31} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 & l_{31}l_{41} + l_{32}l_{42} + l_{33}l_{43} \\ l_{41}l_{11} & l_{41}l_{21} + l_{42}l_{22} & l_{41}l_{31} + l_{42}l_{32} + l_{43}l_{33} & l_{41}^2 + l_{42}^2 + l_{43}^2 + l_{44}^2 \end{pmatrix}$$

By solving the equations formed by the first row we have

$$l_{11} = 1,\ l_{21} = 1,\ l_{31} = 0,\ l_{41} = 2.$$

Furthermore

$$l_{22} = 2,\ l_{32} = 1,\ l_{33} = 1,$$

and lastly

$$l_{42} = 0, l_{43} = 1, l_{44} = 1.$$

So the matrix L will be

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

(b) We will expand relative to the last column as it has more zeros.

$$\det(L) = \det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Again, expanding relative to the last column we will have

$$\det(L) = \det \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = 2.$$

(c) First, we find a suitable linear transformation that can map the volume to another volume for which we know how to calculate its volume. Consider the linear transformation $x = L^{-T}y$. Since $x^T Ax = (y^T L^{-1})LL^T(L^{-T}x) = y^T y$, thus the set $x^T Ax \le 1$ will be mapped to the 4-ball (4 dimensional ball). So its volume will be

$$|x^T Ax \le 1| = \det(L)|y^T y \le 1| = 2|S^4|.$$

(Note: The volume of the 4-ball is $\pi^2/2$.)

■ **Problem 1.3** Let P_n be the n+1-dimensional space of polynomials of degree n with real coefficients, and let $< \cdot, \cdot >$ be the inner product defined as

$$< p, q >= \int_{-1}^{1} p(x)q(x)\ dx.$$

(a) Find an orthogonal basis $\{u_0, u_1, u_2\}$ for P_2 such that $u_j \in P_j$, and $u_j(1) > 0$.

(b) Using the basis in part (a), express the operator $F[p] := \int_{-1}^{1} p(x)dx$ acting on P_2 as a 1×3 matrix.

(c) Using the basis in part (a), express the derivative operator $D[p] := \frac{d}{dx}p(x)$ as a 3×3 matrix.

Solution (a) We start with the Gram-Schmidt orthogonalization. Let $\{c, x, x^2\}$ be the basis vectors that we want to orthogonalize. Let $u_0 = c$. Then the first vector will be $\hat{u}_0 = u_0/\|u_0\| = 1/\sqrt{2}$. To find the second vector we have

$$u_1 = x- < x, \hat{u}_0 > \hat{u}_0 = x - \frac{1}{2}\int_{-1}^{1} x\ dx = x.$$

So the second normal vector will be $\hat{u}_1 = u_1/\|u_1\| = \sqrt{\frac{3}{2}}x$. To find the third vector we have

$$\begin{aligned} u_2 &= x^2 - (< x^2, \hat{u}_0 > \hat{u}_0 + < x^2, \hat{u}_1 > \hat{u}_1) \\ &= x^2 - (\frac{1}{2}\int_{-1}^{1} x^2 dx + \frac{3}{2}\int_{-1}^{1} x^3 dx) \\ &= x^2 - (1/3 + 0) = x^2 - 1/3. \end{aligned}$$

So the third normal vector will be $\hat{u}_2 = u_2/\|u_2\| = 3/2\sqrt{\frac{5}{2}}(x^2 - 1/3)$. However, since the question has not asked for normal basis vectors, for the following sections of the question we will use the following orthogonal (not orthonormal) basis vectors

$$u_1 = 1, \quad u_1 = x, \quad u_2 = x^2 - \frac{1}{3}.$$

(b) It is enough to see what is the effect of this operator on the basis vectors

$$F[u_0] = \int_{-1}^{1} 1dx = 2, \quad F[u_1] = \int_{-1}^{1} xdx = 0, \quad F[u_2] = \int_{-1}^{1} (x^2 - 1/3)dx = 2/3 - 2/3 = 0.$$

So the matrix representation of this operator will be

$$F = \begin{pmatrix} 2 & 0 & 0 \end{pmatrix}.$$

(c) Similarly to the solution above,

$$D[u_0] = 0, \quad D[u_1] = 1, D[u_2] = 2x,$$

so the matrix representation will be

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

■ **Problem 1.4** Recall that an orthogonal projection matrix is a matrix P that satisfies

$$P^2 = P, \quad P = P^T.$$

Suppose P is an $n \times n$ projection matrix with rank$(P) = k$. In the following, I_m denotes the $m \times m$ identity matrix.

(a) List all the eigenvalues of P, including multiplicity. Be sure to justify your reasoning.

(b) Show that $P = AA^T$ for some $n \times k$ matrix A such that $A^T A = I_k$.

(c) Suppose P_1, P_2 are two $n \times n$ projection matrices with rank k. Show that there exists an $n \times n$ orthonormal matrix U (i.e. such that $U^T U = I_n, UU^T = I_n$) such that $P_2 = UP_1U^T$.

Solution (a) The projection matrix P has only two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0$. The eigenvectors corresponding to λ_2 eigenvalue are the basis vectors of the null-space of P (that has dimension $n - k$), and the eigenvectors corresponding to λ_1 are the basis vectors of the orthogonal sub-space to the null space of P. Thus λ_2 has multiplicity $n - k$ and λ_1 has multiplicity k.

(b) Since P is symmetric, then we can choose the eigenvectors to form an orthogonal basis. Let $U = [U_1\ U_2]$ be the matrix that its columns are the eigenvectors and U_1 are the one with eigenvalue 1 while U_2 are the one with eigenvalue 0. Note that since the columns in U are orthogonal, then $U^{-1} = U^T$. Let D be the diagonal matrix of P, then we can write

$$P = UDU^T = [U_1\ U_2] \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} = U_1 U_1^T.$$

Note that $U_1^T U_1 = I_k$ since U is orthogonal matrix.

(c) From part (b) we know that there are matrices A_1, A_2 such that

$$P_1 = A_1 A_1^T, \qquad P_2 = A_2 A_2^T.$$

Define $U = A_1 A_2^T$. Then

$$P_1 = A_1 A_1^T = A_1(A_2^T A_2 A_2^T A_2) A_1^T = (A_1 A_2^T) A_2 A_2^T (A_2 A_1^T) = U P_2 U^T.$$

1.3 Real Analysis

■ **Problem 1.5** Let S be the part of the paraboloid $z = 2 - x^2 - y^2$ above the cone $z = \sqrt{x^2 + y^2}$, with upward orientation. Let

$$F = (\tan\sqrt{z}) + \sin\left(y^3\right)\hat{i} + e^{-x^2}\hat{j} + z\hat{k}.$$

Evaluate the flux integral $\iint_S F \cdot dS$.

Solution TODO: Final answer to be added.

■ **Problem 1.6** Let $f(x) = \sum_{n=1}^{\infty} \sin(nx) x^n$ for those x for which the series converges. Note that this is NOT a power series.

(a) Show that f is defined and continuous on $(-1, 1)$.

(b) Shoat the f is differentiable and that f' is continuous on $(-1, 1)$.

Solution (a) We use the Weierstrass M-test to show that the series converges uniformally on $(-1, 1)$. To see this let $r \in (0, 1)$. Then on $[-r, r]$ we have $|\sin(nx)x^n| \le r^n$. Since $\sum r^n < \infty$ (by the geometric series) then by Weierstrass M-test the series $\sum_n \sin(nx)x^n$ converges on $[-r, r]$ uniformly and absolutely for all $r \in (0, 1)$. This implies uniform and absolute converges on $(-1, 1)$. To show the continuity of the series, observe that $\sin(nx)x^n$ is continuous for all n. Thus this continuity carries over through the uniform converges.

(b) First, observe that each term in the sum is continuously differentiable. Denote the n-th term by $f_n(x)$, then we will have

$$f_n(x) = n\cos(nx)x^n + nx^{n-1}\sin(nx) = nx^{n-1}(x\cos(nx) + \sin(nx)).$$

For $x \in (-1, 1)$, the term inside the parenthesis will be

$$|x\cos nx + \sin nx| \le |x||\cos nx| + |\sin nx| \le 2.$$

Let $r \in (0, 1)$ and $x \in [-r, r]$, then we will have

$$nx^{n-1} \le nr^{n-1} \quad \forall x \in [-r, r].$$

Observe that the series $\sum_n nr^{n-1}$ converges for $r \in (-1,1)$ (you can see this easily by the ratio test). So for $r \in (0,1)$

$$f_k(x) \leq M_k \qquad \forall x \in (-r, r).$$

for some $M_k > 0$ (to be precise $M_k = 2kr^{k-1}$) where $\sum_k M_k < \infty$. So by the Weierstrass M-test $\sum_k f_k'(x)$ converges uniformly on any compact subset $[-r, r]$ for $r \in (0,1)$, and because each term is continuous, the f' is also continuous on $(-1,1)$. So far have observed that $\sum_k f_k(x)$ converges on compact subsets of $(-1,1)$, $\sum_k f_k'(x)$ converges on compact subsets of $(-1,1)$, and f_k is continuously differentiable. So we can do a term by term differentiation.

■ **Problem 1.7** Let $\{x_n\}$ be a sequence of positive real numbers, and define

$$\alpha = \liminf_{n\to\infty} \frac{x_{n+1}}{x_n}, \qquad \beta = \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

Note that $\alpha = \infty$ and $\beta = \infty$ may occur.

(a) Prove that if $\beta < 1$, the sequence $\{x_n\}$ converges.

(b) Prove that if $\alpha > 1$, the sequence $\{x_n\}$ diverges.

(c) Give an example of a convergent sequence $\{x_n\}$ for which $\alpha = 1/2$.

(d) Give an example of a divergent sequence $\{x_n\}$ for which $\beta = 1$.

Solution (a) We choose $\epsilon > 0$ small enough such that $r = \beta + \epsilon < 1$. Since β is the limsup of the sequence x_{n+1}/x_n, then we know that $\exists N \in \mathbb{N}$ such that $\forall n > N$ we have $x_{n+1}/x_n < r$. Calling $x_n = C$ we see that the each term in sequence $x_N, x_{N+1}, \ldots$ will be dominated by $C, rC, r^2C, \ldots$. Since the latter sequence is summable (since $r < 1$), the former is summable as well. This implies that $\sum_n x_n$ converges.

(b) We choose $\epsilon > 0$ small enough such that $r = \alpha - \epsilon > 1$. So there exists $N \in \mathbb{N}$ such that $\forall n > N$ we have $x_{n+1}/x_n > r$. Calling $x_n = C$ we see that each term in the sequence $C, rC, r^2C, \ldots$ is dominated by $x_n, x_{n+1}, x_{n+2}, \ldots$. Since the former diverges, the latter diverges as well. This implies that $\sum_n x_n$ diverges.

(c) Let $\{x_n\}$ be the geometric series $1, r, r^2, \cdots$ with $r = 1/2$. Then $\alpha = r$ and we know that the series $\sum_n r^n$ converges to 2.

(d) One classic example is $x_n = 1/n$, the harmonic series.

1.4 Complex Analysis

■ **Problem 1.8** (a) Find

$$\int_C \left(\frac{z}{(z-1)(z^2+1)} + \frac{e^z}{z-3i} \right) dz,$$

where C is the counterclockwise oriented circle centered at $(0,0)$ of radius 2.

(b) Find all values of $z \in \mathbb{C}$ such that $f(z) = 2(x^3 - 3xy^2 + y) + i(3yx^2 - y^3)$ is analytic at z.

Solution (a) We use the generalized Cauchy theorem. To state the theorem, let Ω be an open set containing a closed curve and its interior and f be a holomorphic function on Ω except at poles $z_1, \cdots, z_n$ inside the closed curve, then

$$\int_\gamma f(z)dz = 2\pi i(\sum_{i=1}^{n} \mathrm{res}_{z_i} f).$$

Observe that the integrand has 4 residues, three of which lies inside the closed curve C, i.e. $z_1 = 1, z_2 = i, z_3 = -i$. We now need to calculate the residue of the integrand at these points. For $z_1 = 1$ we have

$$\mathrm{res}_{z_1} f = \lim_{z\to 1}(z-1)f(z) = \frac{1}{2}.$$

Similarly for $z_2 = i$

$$\mathrm{res}_{z_2} f = \lim_{z\to i}(z-i)f(z) = \frac{1}{2(i-1)}.$$

And finally for $z_3 = -i$

$$\mathrm{res}_{z_2} f = \lim_{z\to i}(z-i)f(z) = \frac{-1}{2(i+1)}.$$

So the sum of residues at the poles inside C is zero. This implies that the integral evaluates to zero.

(b) We will use the converse of the Cauchy-Riemann equations, i.e. we demand the partial derivatives to exist and be continuous, and the Cauchy-Riemann equations to hold. The partial derivatives of u, v are continuous. So we only demand the C-R equations to hold.

$$u_x = v_y, \qquad u_y = -v_x.$$

Observe that $u_x = 2(3x^2 - 3y^2), u_y = 2(-6xy + 1), v_x = 6xy, v_y = 3x^2 - 3y^2$. Thus we need to have

$$2(3x^2 - 3y^2) = 3x^2 - 3y^2, \qquad -12xy + 2 = -6xy.$$

The first equation results in $x^2 = y^2$ and the second equation results in $xy = 1/3$. So f is holomorphic at only the points where $x = y = \pm\frac{1}{\sqrt{3}}$. I.e. $z = \pm\frac{1}{\sqrt{3}}(1+i)$.

■ **Problem 1.9** Find the domain of analyticity of $f(z) = \sqrt{\log(z+1) - \frac{\pi}{2}i}$, where the square root is given by the principal branch and $\log(z)$ is the principal branch of log function.

Solution We need to consider the branch cuts of the log function and the square root function. For the latter, if $w = \log(z+1) - \frac{\pi}{2}i$ the branch cut is where $w \leq 0$. Using the fact that $\log(z+1) = \ln|z+1| + i\arg(z)$, we need to have

$$\ln|z+1| \leq 0 \qquad \arg(z) = \pi/2.$$

The first equation above implies that $|z+1| \leq 1$ and the second equation implies that $z = -1 + it$ for $t \geq 0$. So the branch cut for square root will be the set $\{-1 + it : t \in [0,1]\}$. Furthermore the branch cut of the function $\log(z+1)$ is the set $\{t : t \leq -1\}$. So the domain of analyticity of the function f will be

$$\{-1 + it : t \in [0,1]\} \cup \{t : t \leq -1\}.$$

■ **Problem 1.10** Suppose that f is an analytic function on $H = \{z \in \mathbb{C} : \Re(z) \leq 0\}$ with

$$f(-1) = f'(-1) = 0 \qquad \text{and} \qquad f''(-1) = \frac{i}{2}.$$

(a) Show that $g(z) = f(z)/(z+1)^2$ is analytic on H. Find the residue of g at -1 and the residue of $f(z)/(z+1)^3$ at -1.

(b) Suppose $|f(z)| \leq \frac{1}{2}|z+1|^2$. Show that $|f(-3/2)| \leq 9/80$.

Solution (a) Since f is holomorphic on H, then at every point of H it has a power series. In particular

$$f(z) = (z-1) + (z-1)f'(z) + (z-1)^2 f''(z)/2 + (z-1)^3 f'''(z)/6 + \cdots .$$

Since $f(-1) = f'(-1) = 0$, we have

$$f(z) = (z-1)^2 h(z)$$

for some holomorphic function h that is non-vanishing close to $z = -1$. So

$$g(z) = f(z)/(z+1)^2 = h(z).$$

The residue of $g(z)$ at $z = -1$ is zero. However, for $f(z)/(z+1)^3$ we can write

$$f(z)/(z+1)^3 = h(z)/(z+1).$$

The residue of this function at $z = -1$ is calculated by

$$\lim_{z \to -1} (z+1)h(z)/(z+1) = h(-1).$$

To calculate $h(-1)$ observe that

$$f''(z) = 2h(z) + \text{other terms with factor } (z-1).$$

So we will have $f''(-1) = 2h(-1) = i/2$. So $h(-1) = i/4$. So the residue of the function above at $z = -1$ is $i/4$.

(b) See the remark below. TODO: Final answer to be added.

■ **Remark** I was not able to solve the problem. But I have a feeling that I need to use the Cauchy's estimate for the n-th derivative for some appropriate chosen radius R

$$|f^{(n)}(z)| \leq \frac{n!}{R^n} M, \qquad M = \sup_{z \in C} |f(z)|,$$

where C is a disk centered at z with radius R.

1.5 September 2022

■ **Problem 1.11** Find the shortest distance from x to $U = \operatorname{span}\{u_1, u_2\} \subseteq \mathbb{R}^4$ where

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \qquad x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Solution First, we form a matrix that its column spaces is the same as U. I.e.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}.$$

We can write $x = x_U + x_\perp$. Since x_U is in the column space, then $\exists c \in U$ such that $Ac = x_U$. However, by definition we know that $x_\perp$ is in $U_\perp$. Thus it belongs to the null space of A^T. I.e. $A^T x_\perp = 0$. We can write $A^T(x - x_U) = A^T(x - Ac) = 0$.

$$A^T Ac = A^T x.$$

This is the systems of equations

$$\begin{bmatrix} 4 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

By manipulating the augmented matrix we find that

$$c_1 = 1/2, \qquad c_2 = 1/4.$$

So

$$x_U = Ac = \frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix},$$

and

$$x_\perp = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

So the shortest distance from x to U is

$$\|x_\perp\| = \frac{\sqrt{2}}{2}.$$

■ **Problem 1.12** Let A be a real 3×3 matrix and suppose that the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

are the eigenvectors of A. Show that A is symmetric.