Lecture Notes For: Dynamical Systems

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Contents

1 Sporadic Calculus

In this section I will cover the basics of calculus which I have used throughout the text.

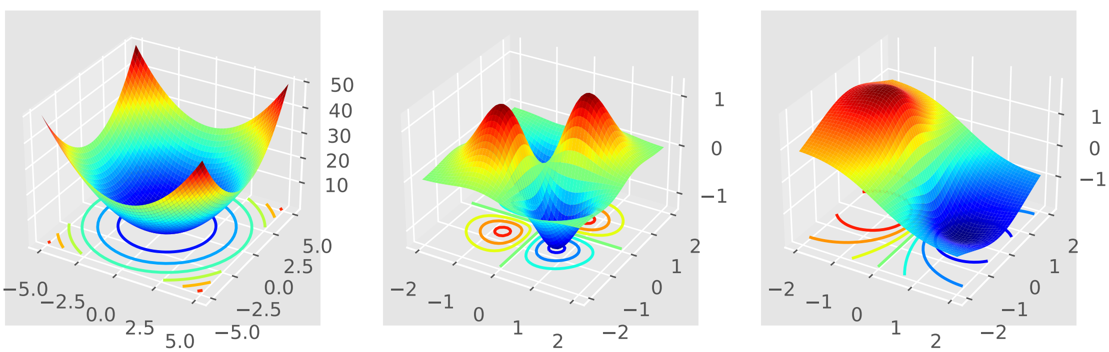
1.1 Level Curves

Here, I will focus my arguments to 2D scalar function and vector fields, as it is easier to imagine and also plot. However, it can easily been generalized to higher dimensions.

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a function. This function is called a scalar function, as it assigns an scalar value to each point in the $\mathbb{R}^2$ plane. This function can be visualized using it graph which is

$$\text{Graph}(F) = \{(x, y, z) \in \mathbb{R}^3 : z = F(x, y)\},$$

which is basically a surface in 3D. Consider the following plots which represent the graph of different scalar function.



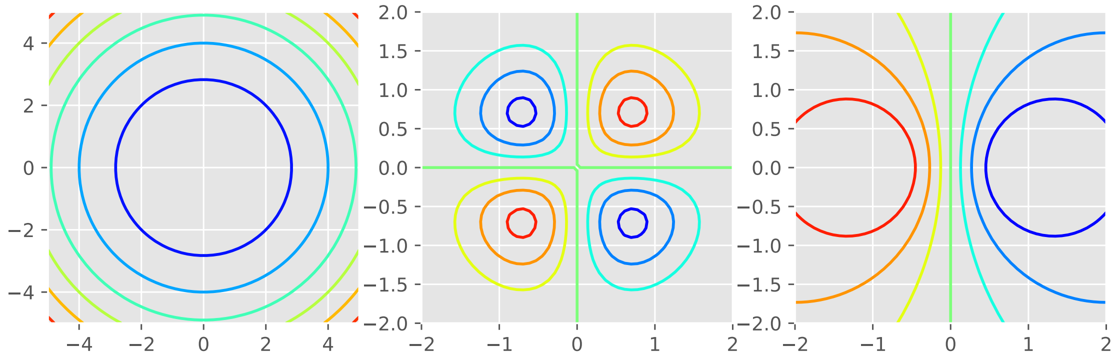
As we can see in the figure above, graph of a scalar function is not always very informative, as certain portions of the function might not be visible due to the projection of the 3d plot. Another idea is to use level curves of the function to represent it. Level curves of a scalar function is defined as

$$LC_c(F) = \{(x, y) \in \mathbb{R}^2 : F(x, y) = c, \ c \in \mathbb{R}\}.$$

Or as an alternative definition, a level curve of F is a path $\gamma : \mathbb{R} \to \mathbb{R}^2$ that satisfies

$$F(\gamma(t)) = c.$$

The following figure represents the level curves of the functions represented in the figure above.



The time derivative of the function $F \circ \gamma$ is zero. The time derivative of $F \circ \gamma$ is the directional derivative of F in the direction of $\gamma'(t)$ evaluated at $\gamma(t)$. Because

$$\frac{d}{dt}F(\gamma(t)) = \frac{\partial F}{\partial x}\Big|_{\gamma(t)}\gamma_1'(t) + \frac{\partial F}{\partial y}\Big|_{\gamma(t)}\gamma_2'(t) = \nabla F\Big|_{\gamma(t)} \cdot \gamma'(t) = D_{\gamma'(t)}F\Big|_{\gamma(t)}.$$

Thus a level curve of $F(x, y)$ passing through $p = (p_1, p_2) \in \mathbb{R}^2$, is a curve $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ that is the solution of the following initial value problem

$$F_x(\gamma(t))\gamma_1'(t) + F_y(\gamma(t))\gamma_2'(t) = 0, \qquad \gamma(0) = p.$$

Let $x = x(t) = \gamma_1(t)$ and $y = y(t) = \gamma_2(t)$. Then the equation above can be written as

$$F_x(x, y)x' + F_y(x, y)y' = 0, \qquad x(0) = p_1,\ y(0) = p_2.$$

This differential equation determines the level curve corresponding to $F(x, y) = F(p_1, p_2)$. We can write y in terms of x as

$$y = f(x) = (\gamma_2 \circ \gamma_1^{-1})(x)$$

Thus the time derivative of y can be written as $y' = \frac{dy}{dx}x'$. Assuming $x' \neq 0$ we can simplify the differential equation above as

$$\frac{df}{dx} = \frac{-F_x(x, y)}{F_y(x, y)}, \qquad f(p_0) = p_1.$$

Solving this initial value problem will determine the desired level curve.

Example 1.1

We want to find the level curve of $F(x, y) = x^2 + y^2$ which passes through

$(1, 1) \in \mathbb{R}^2$. To do this, we need to solve the following initial value problem

$$\frac{dy}{dx} = -\frac{2x}{2y}, \qquad y(1) = 1.$$

By the method of separation of variables we will get

$$y^2 + x^2 = 1,$$

which is a circle passing through the origin and radius 1.

Note that we could do the whole business with following the concept of implicit differentiation (see a Calculus text book like Stewart).

2 Basic Definitions and Concepts

We will devote this section to basic definitions of dynamical systems and the core ideas. For this reason, this chapter, will be served as a main references for the definitions used anywhere in this note. Because of this nature, there might be a little consistency in the material in this section and the concepts will be scattered all over the place!

Definition 2.1. Stable set of an invariant set

Let S be in invariant set of a dynamical system, then it stable set $W^s(S)$, is the set of all states, whose orbits approach S forward in time. In other words

$$W^s(S) = \{x \in X : \; d(\varphi^t x, S) \to 0, \; as \; t \to +\infty\},$$

in which X is the state space, $d : X \times X \to \mathbb{R}$ is a metric function.

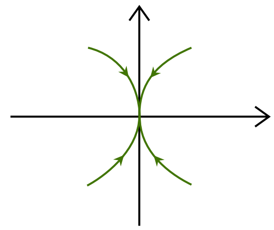
For instance, consider the following system

$$\dot{X} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} X,$$

where $X \in \mathbb{R}^2, \; 0 < \lambda_1 < \lambda_2$. The solution of this system is

$$X(t) = e^{At} X_0 = \begin{pmatrix} e^{-\lambda_1 t} & 0 \\ 0 & -\lambda_2 t \end{pmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.$$

For this system, the invariant sets are the origin $S_1 = (0, 0)$, $S_2 = \text{Span}\{(0, 1)^T\}$, and $S_3 = \text{Span}\{(1, 0)^T\}$. The following figure shows a basic phase portrait of this system.



Thus the stable sets of the invariant sets of this system will be

$$W^s(S_1) = \mathbb{R}^2, \qquad W^s(S_2) = \mathbb{R}^2, \qquad W^s(S_3) = \mathbb{R}^2.$$

With a similar logic, we can define the **unstable set** $W^u(S)$ of in invariant set, which is the set of all states whose orbits approach S *backward* in time.

Corollary: 2.1

The stable, and unstable sets of an invariant set, are invariant sets themselves.

Proof. Here we proof the statement for the unstable set only, however, the proof logic for both of them is similar. We proceed with the proof by contradiction. Assume that $W^s(S)$ is not a invariant set. This means that

$$\exists x_0 \in W^s(S),\ \exists t^* \in \mathbb{R},\ \text{s.t.}\ \ \varphi^{t^*} x_0 = z_0 \notin W^s(S).$$

So $\lim_{t\to\infty} d(\varphi^t z_0, S) \neq 0$. On the other hand, because of $\varphi^{s+t}x_0 = \varphi^s(\varphi^t x_0)$ we have

$$\varphi^{t^*} x_0 = z_0 \ \Leftrightarrow\ \varphi^{t-t^*}(\varphi^{t^*} x_0) = \varphi^{t-t^*} z_0 \ \Leftrightarrow\ \varphi^t x = \varphi^{t-t^*} z_0,$$

which implies that the distance between $\varphi^{t-t^*} z_0$ and the set S goes to zero, which contradicts our assumption. So we conclude that the stable set of an invariant set, is an invariant set itself.

□

Definition 2.2. Stable, Unstable and Center Subspace

Consider the dynamical system

$$\dot{x} = f(x) \approx [\mathcal{D}f](x^*)x,$$

in which $x \in X,\ f: X \to X$. Also $[\mathcal{D}f]$ represents Jacobian matrix evaluated at $x^* \in X$ which is an equilibrium point, i.e. $f(x^*) = 0$. The eigenspace (and possibly generalized eigenspace) $\text{Span}\{v_j^{[1]}, \cdots, v_j^{[m]}\}$ associated with the eigenvalue λ_j is

- stable subspace, if $\mathrm{Re}(\lambda_j) < 0$,
- unstable subspace, if $\mathrm{Re}(\lambda_j) > 0$,
- center subspace, if $\mathrm{Re}(\lambda_j) = 0$.

For instance, consider the linear dynamical system

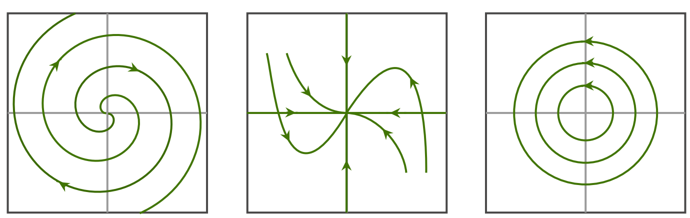
$$\dot{X} = AX,$$

where $X \in \mathbb{R}^6$, and the matrix A is given as

$$A = \begin{pmatrix} \lambda_1 & -\omega_1 & 0 & 0 & 0 & 0 \\ \omega_1 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega_2 \\ 0 & 0 & 0 & 0 & \omega_2 & 0 \end{pmatrix},$$

where $\lambda_j > 0$ for $j = 1, 2, 3$. Also $\omega_1 > 0, \omega_2 > 0$. Then the stable, unstable, and center subspaces associated with the origin will be like the following figure.

unstable subspace | stable subspace | center subspace



$$\begin{pmatrix} \lambda_1 & -\omega_1 \\ \omega_1 & \lambda_1 \end{pmatrix} \quad \begin{pmatrix} -\lambda_2 & 1 \\ 0 & -\lambda_3 \end{pmatrix} \quad \begin{pmatrix} 0 & -\omega_2 \\ \omega_2 & 0 \end{pmatrix}$$

3 Systems of Linear Differential Equations

In dealing with the systems of linear differential equations we encounter

$$\dot{x} = A(t)x, \tag{♫}$$

where $x = x(t) \in \mathbb{R}^n$. It turns out that this system of linear differential equations has n linearly independent solutions

$$\{x_1(t), x_2(t), \dots, x_n(t)\}.$$

It is important to note that the span of all of these solutions is a n-dimensional subspace of the all continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Thus any particular solution of the ODE can be expressed as the linear combination of the solutions stated above, i.e.

$$x(t) = \sum_{i=1}^{n} c_i x_i(t). \tag{1}$$

To show all of these ideas in a neat matrix form, we construct the matrix $\Psi(t)$ called a fundamental matrix (note that this is not unique) as

$$\Psi(t) = [x_1(t), x_2(t), \dots, x_n(t)],$$

that is a matrix whose jth column is $x_j(t)$. With this notation in hand we can express any particular solutions of the ODE system as

$$x(t) = \Psi(t)c. \tag{2}$$

Note that so far nothing serious is happening. What happened is that we just defined the matrix $\Psi(t)$ just to be able to write the equation (1) in a more fancy matrix notation (2).

So far, the functions $\{x_1(t), x_2(t), \dots, x_n(t)\}$ where acting like the basis of the space they span. However, we can find another basis that is more interesting. Consider the functions $\{X_1(t), X_2(t), \dots, X_n(t)\}$ where satisfy the (♫) as well as

$$X_1(t_0) = \hat{e}_1,\ X_2(t_0) = \hat{e}_2,\ \dots\ ,\ X_n(t_0) = \hat{e}_n,$$

where $\hat{e}_j$ is the standard basis of $\mathbb{R}^n$. Note that we can calculate these functions easily since by equation (2) we can write

$$X_j(t) = \Psi(t)c_j \implies c_j = \Psi^{-1}(t)X_j(t) \implies c_j = \Psi^{-1}(t_0)\hat{e}_j.$$

Thus any of $X_j(t)$ can be written as

$$X_j(t) = \Psi(t)\Psi^{-1}(t_0)\hat{e}_j$$

So we can now construct a new fundamental matrix $M(t, t_0)$ as

$$M(t, t_0) = [X_1(t), X_2(t), X_3(t), \dots, X_n(t)].$$

This fundamental matrix is useful when dealing with initial values problems, i..e

$$\dot{x} = A(t)x, \qquad x(t_0) = x_0. \tag{♪}$$

In this case the solution to initial values problem can be written as

$$x(t) = \varphi(t, t_0, x_0) = M(t, t_0)x_0.$$

That is simply because, since $M(t, t_0)$ is a fundamental matrix, then any solution can be written as $x(t) = M(t, t_0)c$, where $c \in \mathbb{R}^n$. Let $t = t_0$, then we will have $x_0 = x(t_0) = M(t_0, t_0)c = Ic = c$, thus $c = x_0$.

Apart from being useful in solving initial values problems, $M(t, t_0)$ has the following properties as well. In fact, $M(t, t_0)$ is an example of a *flow operator*. Also, derivative of $M(t, t_0)$ is given by

$$\frac{d}{dt}M(t, t_0) = A(t)M(t, t_0).$$

That is because substituting $x(t) = M(t, t_0)x_0$ in (♪) will yield in

$$\frac{d}{dt}M(t, t_0)x_0 = A(t)M(t, t_0)x_0,$$
$$(\frac{d}{dt}M(t, t_0) - A(t)M(t, t_0))x_0 = 0.$$

Since this is true for all x_0, then

$$\frac{d}{dt}M(t, t_0) = A(t)M(t, t_0).$$

$\square$

4 Poincare Map

When dealing with dynamical systems that has periodic orbits, one useful machinery to study the stability of those orbits is to use Poincare map. We will explore the idea in the following example

Question 1. Consider the dynamical system described by the following ODE system.

$$\dot{x}_1 = x_1 - \omega x_2 - x_1^3 - x_1 x_2^2,$$
$$\dot{x}_2 = \omega x_1 + x_2 - x_1^2 x_2 - x_2^3,$$

in which $\omega > 0$. Analyze the behaviour of this system near the equilibrium points and the periodic orbits.

Answer.

The study the behaviour of the system near the equilibrium points, we first need to find them. So we solve the following equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By analyzing the ODE system we can infer that $X = (0,0)^T$ a solution of the equation above. So the point $X = (0,0)^T$ is an equilibrium point. To study the stability of this equilibrium point, we need to linearize the system near the equilibrium point.

$$F_X(X)|_{X=0} = \begin{pmatrix} (f_1)_{x_1} & (f_1)_{x_2} \\ (f_2)_{x_1} & (f_2)_{x_2} \end{pmatrix}_{X=0} = \begin{pmatrix} 1 & -\omega \\ \omega & 1 \end{pmatrix}$$

Since the trance and determinant are both positive, then the origin is unstable.

However, to analyze the stability of the periodic orbits, we need to find them in the first place. To find the periodic orbit in this example, it is more convenient to convert this ODE system to the polar coordinate in which we have

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta,$$

in which $r > 0$ and $\theta = \mathbb{S}^1$, in which $\mathbb{S}^1$ is the unit circle. Substituting in the ODE system and utilizing the chain rule, then we can write

$$\begin{aligned} \dot{r}\cos\theta - r\dot{\theta}\sin\theta &= r\cos\theta - r\omega\sin\theta - r^3\cos^3\theta - r^3\cos\theta\sin^2\theta, \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta &= r\omega\cos\theta + r\sin\theta - r^3\cos^2\theta\sin\theta - r^3\sin^3\theta. \end{aligned}$$

Multiplying the first equation in $\sin\theta$ and the second one in $\cos\theta$ and then subtracting them, And then multiplying the second equation in $\cos\theta$ and the first one in $\sin\theta$ and then adding the equations will yield in

$$\begin{aligned} \dot{r} &= r - r^3, \\ \dot{\theta} &= \omega. \end{aligned}$$

It is clear that

$$p^0(t) = \begin{bmatrix} r(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} 1 \\ \omega t \pmod{2}\pi \end{bmatrix},$$

a solution of the this ODE system (because, simply, it satisfies the differential equation). Also, this is a periodic orbit with period $T = 2\pi/\omega$, since

$$p^0(t + 2\pi/\omega) = \begin{bmatrix} 1 \\ (\omega t + 2\pi) \pmod{2}\pi \end{bmatrix} = \begin{bmatrix} 1 \\ \omega t \end{bmatrix}.$$

We can also write this periodic orbit in the original x_1 and x_2 coordinates

$$p^0_X(t) = \begin{bmatrix} r(t)\cos\big(\theta(t)\big) \\ t(t)\sin\big(\theta(t)\big) \end{bmatrix} = \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix}.$$

Note that we write $\sin(\theta)$ instead of $\sin\big(\theta \pmod{2}\pi\big)$, because they are the same and the modulu operator is kind of defined in the definition of sin and cos functions.

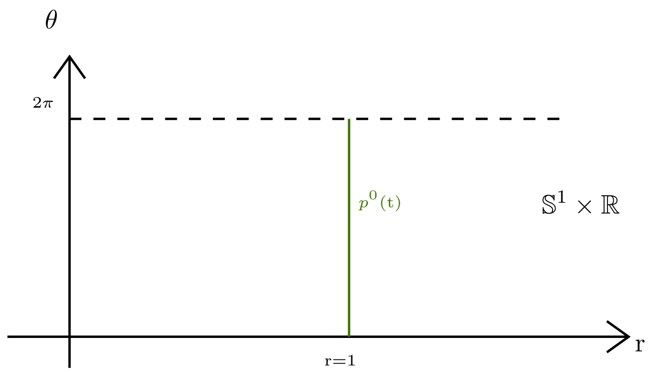


Figure 4.1: State space of the system in polar coordinates.

Now we can evaluate the stability of this periodic orbit in two ways: 1. staying in the polar coordinate, 2. translating stuff to the original rectangular coordinate.

Poincare map in the polar coordinate.

Drawing the phase space of this system in the polar coordinates makes to build intuition what is happening here.

Now we can choose $\Sigma \subset \mathbb{R} \times \mathbb{S}^1$ on which $F(r, \theta) \neq 0$. We choose

$$\Sigma = \{(r, \theta) : \theta = 0 \pmod 2\}.$$

Obviously, in this subset of state space we have

$$F(r, \theta) = \begin{bmatrix} r - r^3 \\ \omega \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \forall (r, \theta) \in \Sigma.$$

Let $p_0^0 = p^0(t) = (1, 0)^T \in \Sigma$. Now let's define a coordinate $\xi \in \mathbb{R}$ for Σ, by

$$r = 1 + \xi.$$

With this definition $\xi = 0$ corresponds to the point $p_0^0 \in \Sigma$. Fix an arbitrary value of ξ_0. Now we are interested to know starting from $(1 + \xi_0, 0)$ as initial value of the dynamical system, when and where we will return to Σ. So we need to solve

$$\begin{aligned} \dot{r} &= r - r^3, \qquad r(0) = 1 + \xi_0, \\ \dot{\theta} &= \omega, \qquad \theta(0) = 0. \end{aligned}$$

We can use elementary methods to solve this initial value problem. So we will get

$$\begin{aligned} r(t, \xi_0) &= \frac{e^t}{\sqrt{(1+\xi)^{-2} - 1 + e^{2t}}}, \\ \theta(t) &= \omega t \pmod 2 \pi. \end{aligned}$$

From the definition of Σ, it is clear that the time of first return is when $\theta(T_0) = 2\pi$, so $T_0 = 2\pi/\omega$. Thus the value r in the first return will be $r_1 = r(T_0, \xi_0)$, and the value of ξ in the first return will be $\xi_1 = r_1 - 1$. So, we basically got

$$\xi_1 = r(2\pi/\omega, \xi_0).$$

Since the return time is always $T = 2\pi/\omega$, then we can conclude

$$\xi_{k+1} = r(2\pi/\omega, \xi_k).$$

This is the Poincare map $P : \mathbb{R} \to \mathbb{R}$ and we have

$$P(\xi) = \frac{e^{2\pi/\omega}}{\sqrt{(1+\xi)^{-2} - 1 + e^{4\pi/\omega}}} - 1.$$

We can analyze the fixed point of this map using linearization. First observe that $\xi = 0$ is a fixed point since $P(0) = 0$. With linearization argument at $\xi = 0$ we have

$$P_\xi(\xi)|_{\xi=0} = e^{-4\pi/\omega} < 1.$$

Thus the origin is a stable equilibrium point. This is also clear form the cobweb plot of this map

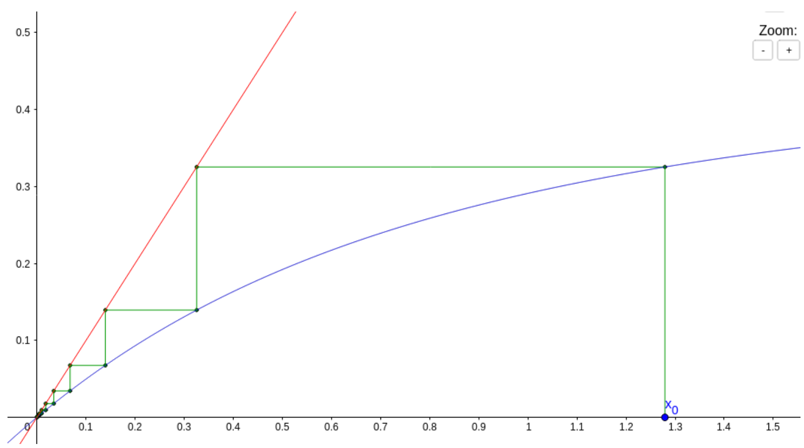


Figure 4.2: Cobweb plot of the poincare map.

5 Hamiltonian Systems and Lyapunov Function

5.1 Hamiltonian Systems

Assuming extra structures in a vector field leads to systems that are easier to analyze. One of such systems is the Hamiltonian systems. A Hamiltonian system is

$$\dot{x} = f(x)$$

where $x \in \mathbb{R}^{2s}$, and $x = (q, p)$ where $q = (q_1, \cdots, q_s) \in \mathbb{R}^s,\ p = (p_1, \cdots, p_s) \in \mathbb{R}^s$. Also, the vector field $f : \mathbb{R}^{2s} \to \mathbb{R}^{2s}]$ has a special property which is

$$f(x) = f(q, p) = (H_p(q, p), -H_q(q, p)) = (\frac{\partial}{\partial p_1}, \cdots, \frac{\partial}{\partial p_s}, -\frac{\partial}{\partial q_1}, \cdots, -\frac{\partial}{\partial q_s})H(q, p).$$

where $H : \mathbb{R}^{2s} \to R$ is the Hamiltonian function. In other words we have

$$\dot{q} = H_p(q, p), \qquad \dot{p} = -H_q(q, p).$$

Corollary: 5.1

If $x(t)$ is a solution of a Hamiltonian system, then $\frac{d}{dt}H(x(t)) = 0$, hence $H(x(t)) = c$ is a constant. Thus, all solutions remain on the level sets of the Hamiltonian function, which is known as conservation of energy.

Proof. We need to calculate $\frac{d}{dt}H(x(t))$ via the chain rule.

$$\frac{d}{dt}H(x(t)) = \sum_{i=1}^{2s} \frac{\partial H}{\partial x_i}\frac{dx_i}{dt} = \sum_{i=1}^{s} \frac{\partial H}{\partial q_i}\frac{dq_i}{dt} + \sum_{i=1}^{s} \frac{\partial H}{\partial p_i}\frac{dp_i}{dt} = \sum_{i=1}^{s} -\frac{dp_i}{dt}\frac{dq_i}{dt} + \sum_{i=1}^{s} \frac{dq_i}{dt}\frac{dp_i}{dt} = 0$$

□

This is a very useful property of Hamiltonian systems. That is because we can easily draw the phase portrait as the set of all level sets of the Hamiltonian functions. Following two examples will help to illustrate this point.

Example 5.1

Consider the following non-Linear oscillator:

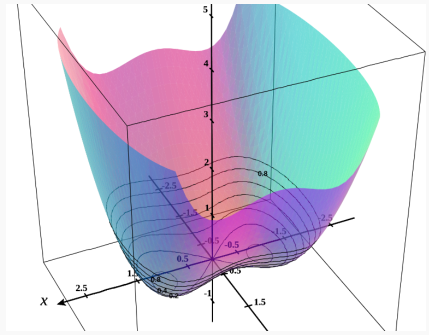
$$\ddot{u} - u + u^3 = 0.$$

To analyze this system, first we need to write it in the form of a system of first order ODEs. Let $q = u, p = \dot{u}$. Then we can write the system as

$$\dot{q} = v, \qquad \dot{p} = q - q^3.$$

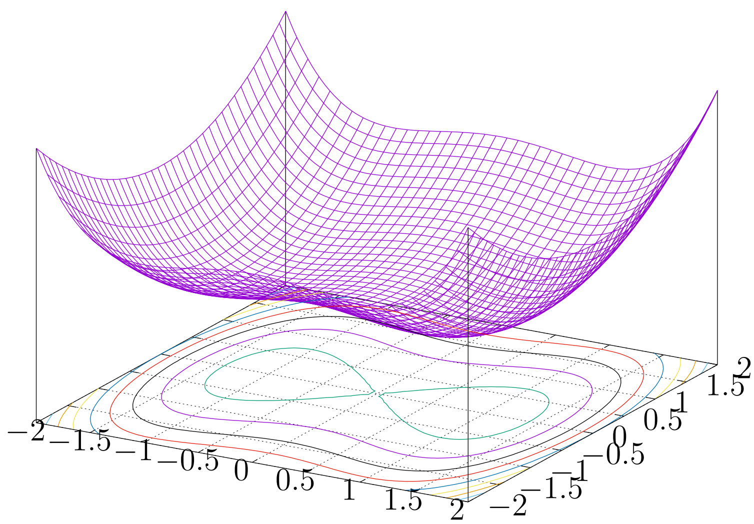
We can analyze this system using the level curves of the Hamiltonian function. We can find the Hamiltonian as

$$H(q, p) = \frac{1}{2}p^2 + \frac{1}{2}q^2 - \frac{1}{4}q^4.$$



The figure above shows the graph of this function. The orbits in the phase portrait is simply the level curves of this function.

The level curves of the dynamical system analyzed above worth more analysis. The following is the phase portrait of the dynamics



As we can see in the figure above, the equilibria $p^\circ = (0,0)$ is sort of special. By linearization argument at the origin, we see that the Jacobian matrix has two eigenvalues, one of which is positive and the other one is negative. Thus we conclude that the equilibria is unstable and is in fact a saddle point. However, we can see that some orbits emerge from it and comeback to it again! We call such orbits homoclinic orbits.

Definition 5.1.

Homoclinic orbits are the orbits that emerge from one equilibria point and return to it again. In other words, the points of homoclinic orbit approach the equilibrium point from which the orbit emerged, as $t \to \pm\infty$.
More formally, consider the continuous dynamical system

$$\dot{x} = f(x),$$

and assume there is an equilibrium point at x_0. A solution $x(t)$ is a homoclinic orbit if

$$x(t) \to x_0, \quad as \quad t \to \pm\infty.$$

Homoclinic orbits are in fact the intersection of stable and unstable manifolds (see Def 2.1) of an equilibrium point.

5.2 Lyapunov Function

Lyapunov functions can be thought of generalization of Hamiltonian systems. Lyapunov function are tools that enable us to determine the invariant sets of the dynamics as well as the stability of the equilibrium points. The concept behind using Lyapunov functions is that for every scalar function $F : X \to \mathbb{R}$, there is a corresponding natural vector field. This vector field is the gradient of the scalar function. The gradient determines the iso-curves of the scalar function, i.e. the curves on which the value of the scalar function is constant. For a scalar field chosen carefully, these level curves are also closed, thus we will have the notions like the interior of the level curves, etc. These regions (i.e. interior of the level curves) along with the gradient vector field can help in determining the behaviour of a dynamical system.

The argument here focuses on $\mathbb{R}^2$, as it is easier to visualize. However, the general idea can easily be extended to higher dimensions. Let a dynamical system be defined as

$$\dot{x} = f(x),$$

where $x \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$. Also, let $L : \mathbb{R}^2 \to \mathbb{R}$ be a scalar function. The gradient of this scalar is a vector field ∇L. If for a region $U \subseteq \mathbb{R}^2$ we have

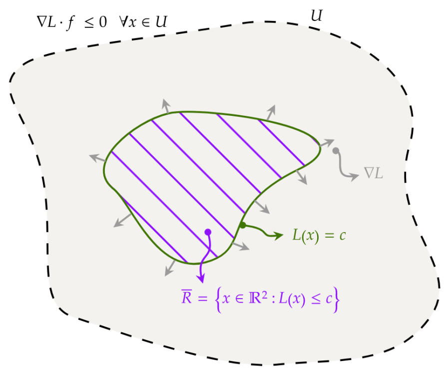
$$\nabla L\Big|_x \cdot f(x) \leq 0 \quad \forall\, x \in U,$$

And the compact (closed and bounded) set

$$\overline{R} = \{x \in \mathbb{R}^2 : \; L(x) \leq c, \; c \in \mathbb{R}\}$$

be contained in U, then $\overline{R}$ is a *positively invariant* set. Considering the figure below, this fact can be justified intuitively.

Here are some intuitive explanations of this fact



- $\nabla L \cdot f$ evaluated at point $p \in \mathbb{R}^2$ is in fact the directional derivative of the scalar field at point p along the direction determined by f. Thus if $\nabla L \cdot f \leq 0$ in some region, then the value of L along any path whose tangent is determined by $f(x)$ (which is in fact the solution of the ODE) will decrease or remain constant. Thus any such path whose one of its points is in $\overline{R}$ will remain in $\overline{R}$.

- $\nabla L \cdot f$ determines the projection of f on ∇L at each point $x \in \mathbb{R}^2$. Since $\nabla L \cdot \gamma(t) = 0$ for any level curve $\gamma(t)$ for which $L(\gamma(t)) = c$, then $\nabla L \cdot f \leq 0$ means that any path whose tangent is determined by $f(x)$ (i.e. solutions of the ODE) won't leave the region $\overline{R}$.

- We can treat this fact more formally which is easier to follow. Let $x(t)$ be the solution of the ODE provided above, thus $\dot{x}(t) = f(x(t))$. Then by the chain rule we can write
$$\frac{d}{dt} L(x(t)) = \nabla L \Big|_{x(t)} \cdot f(x(t)).$$
Then $\nabla L \cdot f \leq 0$ in a region U means that the value of $L(x(t))$ can not increase. Thus if a compact region $\overline{R} = \{x \in \mathbb{R}^2 : L(x) \leq c, \; c \in \mathbb{R}\}$ is contained at U, then the solution $x(t)$ can not leave this region since the the value of $L(x(t))$ can not increase for $t \to \mathbb{R}$.

Such a smartly chosen scalar function is called a Lyapunov function. Other than determining positively invariant sets, Lyapunov function can also help to determine the stability of the equilibrium points of the dynamics.

Proposition: 5.1

Consider the dynamical system described by

$$\dot{x} = f(x),$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $L : \mathbb{R}^n \to \mathbb{R}$ be a Lyapunov function such that $\nabla L \cdot f \leq 0$ in a region $U \subseteq \mathbb{R}^n$. Then

- If $\overline{R} = \{x \in \mathbb{R}^n : \ L(x) \leq c, \ c \in \mathbb{R}\}$ is compact and $\overline{R} \subseteq U$ then

$$\boxed{\overline{R} \text{ is positively invariant}}$$

- If p^0 is an equilibrium point, and $L(p^0)$ is an *isolated local minimum* of L, then

$$\boxed{p^0 \text{ is Lyapunov stable.}}$$

- If **in addition to the condition above**, we have $\nabla F \cdot f < 0$ in $U \backslash \{p^0\}$, then

$$\boxed{p^0 \text{ is stable.}}$$

In the following example we explore this useful tool.

Example 5.2

In this example we want to analyze the damped planar pendulum system described by

$$\ddot{\theta} + \delta\dot{\theta} + \sin\theta = 0, \quad \theta \in \mathbb{S}^1.$$

To analyze this system, we first need to write it in the form of a system of ODEs. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$, where $x = (x_1, x_2) \in \mathbb{S}^1 \times \mathbb{R}$. Then we can write

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\delta x_2 - \sin x_1, \end{aligned}$$

The equilibria points are

$$p^0_{[1]} = (0 \pmod{2\pi}, 0), \quad p^0_{[2]} = (\pi \pmod{2\pi}, 0).$$

Now by the linearization argument we can determine the stability of these equilibria. The Jacobian matrix is

$$[Df] = \begin{pmatrix} 0 & 1 \\ -\cos x_1 & -\delta \end{pmatrix}.$$

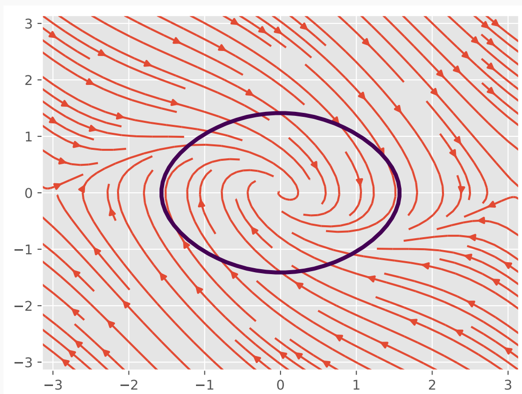
Evaluating this matrix at the equilibria points we will get

$$[Df](p^0_{[1]}) = \begin{pmatrix} 0 & 1 \\ -1 & -\delta \end{pmatrix}, \quad [Df](p^0_{[2]}) = \begin{pmatrix} 0 & 1 \\ 1 & -\delta \end{pmatrix}$$

For $p^0_{[1]}$ we have $\Delta > 0$ and $\sigma < 0$, thus $\lambda^1_1 \le \lambda^1_2 < 0$ which implies that $p^0_{[1]}$ is stable equilibrium and a hyperbolic sink. However, for $p^0_{[2]}$ we have $\Delta < 0$ and $\sigma < 0$ which implies $\lambda^2_1 < 0 < \lambda^2_2$, implying the equilibrium is a hyperbolic saddle.
Let $L = \frac{1}{2}x_2^2 - \cos(x_1)$ be a Lyapunov function. Note that this is in fact the Hamiltonian of a simple harmonic oscillator (whose ODE is $\ddot{\theta} + \sin\theta = 0$). This is a Lyapunov function since

$$\nabla L \cdot f = (\sin(x_1), x_2) \cdot (x_2, -\delta x_2 - \sin(x_1)) = -\delta x_2^2 \le 0.$$

Let $\overline{R} = \{x \in \mathbb{R}^2 : L(x) \le 0\}$. The following figure shows the boundary of the set $\overline{R}$. As we can visually see, all of the arrows determined by the vector field f are pointing towards the interior of this region, which is also reflected by $\nabla L \cdot f \le 0$ inside any open ball containing $\overline{R}$. Thus we can conclude that $\overline{R}$ is **positively invariant**.



Furthermore, since $p^0_{[1]}$ is an isolated local minimum for $L(x)$ (check $\nabla L\Big|_{x=(0,0)} = 0$), thus we can conclude that $p^0_{[1]}$ is Lyapunov stable.

6 Bifurcation Theory

Here in this section we will explore the ideas of the bifurcation theory in the dynamical systems. This is a very important topic since it can give explanation for some sudden change that we see in the nature (for example the type I and type II phase transition in the thermodynamics). We will study them with the following classification.

6.0.1 A Quick Review

Consider the following family of dynamical systems parameterized by one parameter α.

$$\dot{x} = f(x, \alpha),$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $x, \alpha \in \mathbb{R}$. And assume $f(x_0, \alpha_0) = 0$, thus for $\alpha = \alpha_0$ the point x_0 is an equilibrium point. Now we can do the linearization at the equilibrium point and determine the type of stability. Thus after linearization we will have

$$\dot{x} = f_x(x_0, \alpha_0)(x - x_0).$$

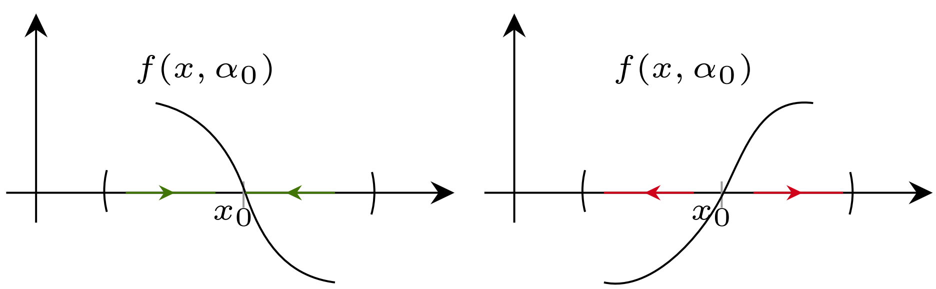
By letting $\xi = x - x_0$ we can now write

$$\dot{\xi} = f_x(0, \alpha_0)\xi.$$

This is a very simple ODE whose solution is the exponential function. Thus we will have

$$\begin{aligned} f_x(0, \alpha_0) < 0 &\implies x_0 \text{ is stable.} \\ f_x(0, \alpha_0) > 0 &\implies x_0 \text{ is unstable.} \end{aligned}$$

Another way of seeing this is the following figure



6.1 Fold (Saddle-Note) Bifurcation