Manifolds

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0. Contents

1. Euclidean Spaces

1.1 Basic Notions and Definitions

1.1.1 A Review on the algebraic structures

Here in this chapter I will be covering the details of some notions that was challenging for me do digest in the first read.

> **Definition 1.1 — Axioms of Group.** Group is a set A along with a binary operation $* : A \times A \to A$ that satisfies the following properties. Let $a, b, c \in A$, then
>
> - **Associativity**: $a * (b * c) = (a * b) * c$.
>
> - **Identity element**: $\exists 1 \in A$ such that
> $$1 * a = a * 1 = a.$$
>
> - **Inverse element**: $\forall a \in A \; \exists \hat{a} \in A$ such that
> $$a * \hat{a} = \hat{a} * a = 1.$$

■ **Remark 1.1** A set along with a binary operation that does not satisfy any properties is called a **magma**. If the binary operation is only associative, then we are dealing with **semi-group**. If the binary operation has an identity element as well, then we call this algebraic structure as **monoid**.

> **Definition 1.2 — Axioms of Ring.** A ring is a set R along with two operations $+ : R \times R \to R$ and $* : R \times R \to R$, where
>
> - $(R, +)$ is an Abelian group.
>
> - $(R, *)$ is a monoid.
>
> - The operator $(*)$ has distributive (left and right) law over $(+)$ i.e.
> $$a * (b + c) = (a * b) + (a * c), \qquad (b + c) * a = (b * a) + (c * a).$$
>
> .

■ **Remark 1.2 Field** is a ring where every non-zero element (i.e. inverse element in the $(R, +)$ group in the ring) has a multiplicative inverse.

Definition 1.3 — Axioms of Module. A **module** is a group M along with a ring R where the monoid of the ring acts on M (through scalar multiplication) (i.e. it satisfies the idenity and compatibility properties) and satisfies the distributive property. I.e.

- **Compatibility of the monoid action**: $a, b \in R$, $u \in M$ then

$$a(bu) = (ab)u.$$

- **Identity of the monoid action**: Let 1 be the identity element of the ring R. Then $\forall u \in M$
$$1u = u1 = u.$$

- **Distribution law**: $a, b \in R$ and $u, v \in M$ then

  - $(a + b)u = au + bu$.
  - $a(u + v) = au + av$.

■ **Remark 1.3** A module (M, R) is called a **vector space**, if the **ring** R is a **field**.

Definition 1.4 — Axioms of Algebra. An Algebra over field F is a ring A that F acts on it (thus A has vector space structure as well), where the monoid operation of F (i.e. multiplication) satisfies the homogeneity property. I.e. for $r \in F$ and $u, v \in A$ we have

$$r(uv) = (ru)v = u(rv).$$

There are some important observations when combining different algebraic structures with each other to get a new one. The first is that when we combine two structures with different operators, then the operators need to satisfy the distributive laws. Also, note that when an algebraic structure (like group or monoid) acts on another algebraic structure, we need to have the identity and and compatibility conditions satisfied.

The following diagram shows how different algebraic structures are combined with each other to produce another structure.

Group (G,∗)

Ring
Group (G,+)
Monoid (G,∗)
- Distribution (∗) over (+)

Module
Group (M,+)
Ring (R,∗,$\dot{+}$)
- R_{mon} @ M with "·"
- Distribution of "·" over (+)

Algebra
Ring (M,+,×)
Field (F, +, ∗)
- (M_g,F) is a vector space
- × in M_{mon} satisfies homogen cond.

Note that in the figure above, I have used some non-standard notations to make the figure concise. For instance, the expression "$R_{\mathbf{mon}}@M$ **with** ·" means that the monoid structure in the field R acts on the group M with the (·) symbol. Or the expression "× **in** $M_{\mathbf{mon}}$ **satisfies homogen cond.**" means the multiplication operation of the monoid structure inside the ring M satisfies the homogeneity condition (see the definition of the algebra in Definition 1.4). Finally, M_g means the group structure inside the ring M.

1.1.2 Directional Derivative

The notion of directional derivative is very central in generalization of the multi-variable calculus manifolds.

Definition 1.5 — Directional derivative. Let $f : U \to \mathbb{R}$ where $U \subset \mathbb{R}^n$ and $f \in C^\infty(U)$. Then we define a directional derivative at $p \in U$ and in the direction $v \in T_p(\mathbb{R}^n)$ as

$$D_v f\Big|_p = \lim_{t \to 0} \frac{f(p + tv) - f(p)}{t} = \frac{d}{dt}\Big|_{t=0} f(p + vt).$$

We denote the set of all directional derivatives at p by $\mathcal{D}(C_p^\infty(U))$.

■ **Remark 1.4** By the chain rule we have

$$D_v f\Big|_p = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(p).$$

Proposition 1.1 — Directional derivative is $R-$linear map satisfying the Leibniz rule. A directional derivative D_v at point p is a $\mathbb{R}-$linear operator that maps

$$f \in C_p^\infty \mapsto D_v f \in \mathbb{R}$$

such that satisfies the Leibniz rule,

$$D_v(fg) = D_v(f)g + fD_v(g).$$

Proof. Let $D_v \in \mathcal{D}(C_p^\infty(U))$. Then by the remark above we can write it as

$$D_v f \Big|_p = \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(p).$$

Since each of the partial derivatives above are $\mathbb{R}$−linear and satisfy the Leibniz rule, then D_v inherits those properties as well. $\square$

1.2   Tangent Spaces and Derivations

> **Definition 1.6 — Informal definition of tangent space.** Let $p \in \mathbb{R}^n$. A tangent space at p denoted by $T_p(\mathbb{R}^n)$ is a linear space containing all of vectors emerging from p.

■ **Remark 1.5** Note that the definition above is an informal definition of the tangent space. For a more formal and technical definition, we can use the notion of curves, or the notion of manifolds. We wont' touch this level of technicality in the early chapters.

To distinguish between points in $\mathbb{R}^n$ and vectors in $T_p(\mathbb{R}^n)$ we denote a point $p \in \mathbb{R}^n$ by

$$p = (p^1, p^2, \cdots, p^n),$$

while for a vector $v \in T_p(\mathbb{R}^n)$ we write

$$v = \langle v_1, v_2, \cdots, v_n \rangle.$$

As we observed in Definition 1.5, we have a natural one-to-one correspondence between each $v \in T_p(\mathbb{R}^n)$ and a $D_v \in \mathcal{D}(C_p^\infty(U))$, i.e. these linear spaces are isomorphic.

> **Proposition 1.2** Let $p \in \mathbb{R}^n$. The set of all directional derivatives at p, i.e. $\mathcal{D}(C_p^\infty(U))$ is isomorphic to the tangent space at p i.e. $T_p(\mathbb{R}^n)$.

Proof. Proof follows immediately from the following natural association in the definition of the directional derivative.

$$D_v f \Big|_p \longleftrightarrow \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(p)$$

$\square$

> **Definition 1.7 — Point derivations at a point.** Let $p \in \mathbb{R}^n$, and U an open set containing p. A **derivation at** p or a **point-derivation of** $C_p^\infty(U)$ is a *linear* operator
>
> $$D : C_p^\infty(U) \to \mathbb{R}$$
>
> such that satisfies the Leibniz property, i.e. for $f, g \in C_p^\infty(U)$ we have
>
> $$D(fg) = D(f)g - fD(g).$$
>
> We denote the set of all such maps as $\mathbb{D}(C_p^\infty(U))$.

■ **Remark 1.6** We know that a directional derivative at p satisfies the Leibniz rule. Thus $\mathcal{D}(C_p^\infty(U)) \subset \mathbb{D}(C_p^\infty(U))$. On the other hand, we know that both $\mathcal{D}$ and $\mathbb{D}$ are linear spaces. So $\mathcal{D}(C_p^\infty(U))$ is in fact a *linear subspace* of $\mathbb{D}(C_p^\infty(U))$. Thus the zero of these two linear spaces match.

The following Lemma follows form the algebraic property of the point derivation.

Lemma 1.1 Let D be a point derivation of $C_p^\infty(U)$. Then $D(c) = 0$ for any constant function c.

Proof. Let c be a constant function, i.e. a real number. Since D is $R-$linear, then we can write $D(c) = cD(1)$. On the other hand, from the Leibniz property we can write

$$D(1) = D(1 \cdot 1) = D(1) + D(1) = 2D(1)$$

Thus $D(1) = 0$, which implies $D(c) = 0$. $\qquad\square$

The following theorem is very important as it states that all of the point derivations are in fact the directional derivatives and vise-versa. This is a very interesting result, since we are in fact stating that an operator is a point derivative if and only if it satisfies an algebraic property. This means that we can abstract away all of the detailed limit processes in the definition of derivative, and replace that with an axiomatic requirement which is a purely algebraic property. We can see these kind of ideas, i.e. axiomatic generalization all over the mathematics. For instance, in the PDE theory, at some point we need to relax the definition of derivative and talk about the weak derivatives. To do so, we get an identity that derivatives satisfy and carefully use that identity to define the notion of weak derivatives.

Theorem 1.1 — The set of all point-derivations is isomorphic to the set of all directional derivatives.
Let $p \in U \subset \mathbb{R}^n$. Then $\mathbb{D}(C_p^\infty(U))$ is isomorphic to $T_p(\mathbb{R}^n)$.

Proof. Let $\varphi : T_p(\mathbb{R}^n) \to \mathbb{D}(C_p^\infty(U))$ be a linear isomorphism between the linear spaces. We need to show that this map is surjective and bijective. To show the bijectivity, we use the fact that a linear map is bijective if and only if its kernel is a singleton. To find the kernel of the map, let need to find all points in $T_p(\mathbb{R}^n)$ that maps to the zero of $\mathbb{D}$. As we discussed in the remark of Definition 1.7, the zero $\mathbb{D}$ is the same as the zero directional derivative, i.e. $D_v = 0$. We need to prove that v is the zero vector, i.e. the zero of $T_p(\mathbb{R}^n)$. To do this, we apply the D_v to the coordinate functions

$$0 = D_v(x^i) = \sum_{j=1}^n v^j \frac{\partial x^i}{\partial x^j} = v^j$$

Thus $v = 0 \in T_p(\mathbb{R}^n)$, thus φ is injective. In other words, the injectivity follows immediately from $T_p(\mathbb{R}^n)$ being isomorphic to $\mathcal{D}$, and $\mathcal{D}$ being a linear subspace of $\mathbb{D}$.

To prove the surjectivity, let D be a point derivation at p, and let (f, V) be a representative of a germ in C_p^∞. Marking V smaller if necessary, we may assume that V is an open ball, hence star shaped. By Taylor's approximation theorem we know there exists C^∞ functions $g_i(x)$ such that

$$f(x) = f(p) + \sum_{i=1}^n (x^i - p^i)g_i(x), \qquad g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

From Lemma Lemma 1.1, we know that $D(f(p)) = 0$ as well as $D(p^i) = 0$. Thus we can write

$$D(f(x)) = \sum_{i=1}^n (D(x^i)g_i(x) + (p^i - p^i)D(g_i(x))) = \sum_{i=1}^n D(x^i)g_i(x) = \sum_{i=1}^n D(x^i)\frac{\partial f}{\partial x^i}(p).$$

This is in fact a directional derivative in the direction $v = \langle D(x^1), D(x^2), \cdots, D(x^n) \rangle$. So for every D in $\mathbb{D}$ we can find a vector in $T_p(\mathbb{R}^n)$. Thus φ is surjective. $\qquad\square$

Observation 1.2.1 Let D be a point derivation of $C_p^\infty(U)$. This corresponds to the directional derivative at p corresponding to the vector

$$v = \langle D(x^1), D(x^2), \cdots, D(x^n) \rangle.$$

Observation 1.2.2 Let $p \in U \subset \mathbb{R}^n$. Then

$$T_p(\mathbb{R}^n) \equiv \mathcal{D}(C_p^\infty(U)) \equiv \mathbb{D}(C_p^\infty(U)),$$

i.e. they are all isomorphic linear spaces.

Because of the observation above, we identify the standard basis $\{e^1, e^2, \cdots, e^n\}$ with the partial derivatives $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^n}\}$. Thus we can write a vector $v \in T_p(\mathbb{R}^n)$ as

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}.$$

1.3   Summary

Summary ✎ 1.1 — Don't confuse the naming conventions. Note the following different names that might be confusing in some cases!

- **k-linear function $\equiv$ k-tensor $\equiv$ multi-linear function**: All of these names are for functions $f : V^k \to \mathbb{R}$. We denote the set of all k-tensors defined on vector space V as $L_k(V)$. If $\alpha^1 \cdots \alpha^n$ are the dual basis for V, then a basis for $L_k(V)$ is

$$\mathcal{B}(L_k(V)) = \{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}_{\text{for all multi-indices } (i_1, \cdots, i_k)}$$

  Also we have
$$\dim L_k(V) = n^k$$

- **alternating k-tensors $\equiv$ k-covectors $\equiv$ multi-covectors of degree k**: All of these names are for function k-linear functions that are **alternating**. We denote the set of all k-covectors on V as $A_k(B)$. A basis for this set is

$$\mathcal{B}(A_k(V)) = \{\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}\}_{\text{for all multi-indices } (i_1, \cdots, i_k) \text{ in ascending order}}$$

  Also we have
$$\dim A_l(v) = \binom{n}{k}.$$

Summary ✎ 1.2 — Determining sign of permutation. Let $\sigma \in S_k$. There we can find the sign of the permutation by decomposing it into cycles, or counting its number of inversions. For instance, consider $\sigma \in S_9$ where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 9 & 3 & 7 & 8 & 5 & 6 & 1 \end{pmatrix}$$

(i) **Determining sign by cycle decomposition**. For σ we can write it as

$$\sigma = (1\ 2\ 4\ 3\ 9)(5\ 7)(6\ 8) = (1\ 9)(1\ 3)(1\ 4)(1\ 2)(5\ 7)(6\ 8).$$

Since σ has 6 transpositions in it, then its sign is **even**.

(ii) **Determining sign by counting the number of inversions**. For σ the inversions are

$$(2,1),(3,1),(4,3),(4,1),(5,1),(6,1),(7,5),(7,6),(7,1),$$
$$(8,5),(8,6),(8,1),(9,1),(9,3),(9,5),(9,6),(9,7),(9,8).$$

Since σ has 18 inversions, then it is sign is **even**.

Summary ✒ 1.3 Let $f \in A_k(V)$ and $g \in A_l(V)$. Then

$$f \wedge g = (-1)^{kl}(g \wedge f).$$

In particular, if f is odd-linear function, then

$$f \wedge f = 0.$$

Summary ✒ 1.4 — Some intuition about the alternating functions. In this summary box, we will go through a concrete example to make some intuition about the alternating functions and their basis. Let $V = \mathbb{R}^3$, with standard basis $\{e_1, e_2, e_3\}$ and the dual basis $\{\alpha^1, \alpha^2, \alpha^3\}$. Let $f \in A_2(V)$. Then we know that we can decompose f in term of its basis

$$f = g_1 \alpha^1 \wedge \alpha^2 + g_2 \alpha^1 \wedge \alpha^3 + g_3 \alpha^2 \wedge \alpha^3.$$

The key step in making the intuition is to observe that for any l 1-covectors $\omega^1, \cdots, \omega^l$ we have $(\omega^1 \wedge \cdots \wedge \omega^l)(v_1, \cdots, v_l) = \det \left[\omega^i(v_j)\right]$. Thus for the first term in the decomposition of $f(u,v)$ for $u, v \in \mathbb{R}^3$ we have

$$(\alpha^1 \wedge \alpha^2)(u,v) = \det \begin{pmatrix} \alpha^1(u) & \alpha^1(v) \\ \alpha^2(u) & \alpha^2(v) \end{pmatrix}.$$

This is the area covered by two vectors that we can get by projecting u, v onto the $x - y$ plane. With a similar argument, $\alpha^1 \wedge \alpha^3$ gives the area between the vectors when projected to the $x - z$ plane, and etc. The following figure summarizes these in a neat way.

In the figure above, we will get the red, green and orange volumes by applying $\alpha^1 \wedge \alpha^2$, $\alpha^2 \wedge \alpha^3$, $\alpha^1 \wedge \alpha^3$ on (u,v) respectively.

Summary ✎ 1.5 Consider the following $n \times n$ matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

The, determinant is defined to be

$$\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \; a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

Summary ✎ 1.6 Let V be a vector space, and $\alpha^1, \cdots, \alpha^k$ be 1-linear functions (i.e. 1-covectors) defined on V. Then we have

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \cdots, v_k) = \det\left[\alpha^i(v_j)\right].$$

In particular, For two 1-covectors we have

$$(\alpha^1 \wedge \alpha^2)(v_1, v_2) = \det\begin{pmatrix} \alpha^1(v_1) & \alpha^1(v_2) \\ \alpha^2(v_1) & \alpha^2(v_2) \end{pmatrix}$$

See Problem 1.10 for the a detailed proof.

Summary ✎ 1.7 Let V be a vector space with dimension n, with $\{e_1, \cdots, e_n\}$ as its basis and $\{\alpha^1, \alpha^2, \cdots, \alpha^n\}$ as its dual basis. Let $eA_k(V)$ be the set of all *alternating* k-linear functions defined on V. Then we have

$$\dim A_k = \binom{n}{k}.$$

In particular, let $V = \mathbb{R}^n$ with standard basis $\{e_1, e_2, e_3\}$, and the corresponding dual basis $\{\alpha^1, \alpha^2, \alpha^3\}$. Then we have

$$\begin{aligned} \mathbb{B}(A_0) &= \{1 \in \mathbb{R}\} \\ \mathbb{B}(A_1) &= \{\alpha^1, \alpha^2, \alpha^3\}, \\ \mathbb{B}(A_2) &= \{\alpha^1 \wedge \alpha^2, \alpha^1 \wedge \alpha^3, \alpha^2 \wedge \alpha^3\}, \\ \mathbb{B}(A_3) &= \{\alpha^1 \wedge \alpha^2 \wedge \alpha^3\}. \end{aligned}$$

It is clear from the basis sets above that the dimensions of A_0, A_1, A_2 and A_3 are $1, 3, 3, 1$ respectively.

Also, observe that $A_k = 0$ for all $k > n$. This follows from the fact that $f \wedge f = 0$ for f a 1-covector.

Summary ✎ 1.8 Let V be n dimensional vector space with a basis $\{e_1, \cdots, e_n\}$, and $\{\alpha^1, \cdots, \alpha^n\}$ as its dual basis. Let $L_k(V)$ be the set of all k-linear functions defined on V. Then a basis for $L_k(V)$ is

$$\mathcal{B} = \{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$$

for all multi-indices $(i_1, \cdots, i_k)$. This shows that

$$\dim L_k(V) = n^k.$$

For instance, Let $V = \mathbb{R}^2$. Then for $L_1(V)$ we have

$$\mathcal{B}(L_1(V)) = \{\alpha^1, \alpha^2, \alpha^3\}.$$

Similarly, for $L_2(V)$ we have

$$\mathcal{B}(L_2(V)) = \{\alpha^1 \otimes \alpha^2, \alpha^1 \otimes \alpha^3, \alpha^2 \otimes \alpha^3, \alpha^2 \otimes \alpha^1, \alpha^3 \otimes \alpha^1, \alpha^3 \otimes \alpha^2, \alpha^1 \otimes \alpha^1, \alpha^2 \otimes \alpha^2, \alpha^3 \otimes \alpha^3\}$$

For a side by side comparison, compare it with the basis for $A_2(V)$ that is

$$\mathcal{B}(A_2(V)) = \{\alpha^1 \wedge \alpha^2, \alpha^2 \wedge \alpha^3, \alpha^1 \wedge \alpha^3\}.$$

Summary ✎ 1.9 Looking at the dimension $L_1(V), L_2(V), \cdots$, it reveals the fact that why most of the non-technical people are interested in interpreting tensors as multi-dimensional matrices! For instance, when $V = \mathbb{R}^2$ and $K = 3$, then $L_3(V)$ has dimension $2^3 = 8$ with the basis

$$\mathcal{B}(\mathbb{R}^2) = \begin{matrix} \{\alpha^1 \otimes \alpha^1 \otimes \alpha^1, \alpha^1 \otimes \alpha^1 \otimes \alpha^2, \alpha^1 \otimes \alpha^2 \otimes \alpha^1, \alpha^1 \otimes \alpha^2 \otimes \alpha^2, \\ \alpha^2 \otimes \alpha^1 \otimes \alpha^1, \alpha^2 \otimes \alpha^1 \otimes \alpha^2, \alpha^2 \otimes \alpha^2 \otimes \alpha^1, \alpha^2 \otimes \alpha^2 \otimes \alpha^2\} \end{matrix}$$

It is very tempting to think of elements of $L_3(V)$ as $2 \times 2 \times 2$ matrix like structures. In the case of $L_2(V)$ we can identify $L_2(V)$ with the set of all 2×2 matrices.

Summary ✎ 1.10 — Determinant of a matrix is zero if its column vectors are linearly dependent. A k-tensor f is alternating if and only if $f(v_1, \cdots, v_k) = 0$ whenever two of the vectors $v_1, \cdots, v_k$ are equal. With a simple generalization, we can state a more general statement that f is alternating k-tensor if and only if $f(v_1, \cdots, v_k) = 0$ the vectors $v_1, \cdots, v_k$ are linearly dependent.

As an important (and simple!) Corollary for this statement, we can conclude that determinant of an $n \times n$ matrix (which is an alternating n-tensor) is zero when its column vectors are linearly dependent.

Summary ✎ 1.11 Let $U \subset \mathbb{R}^n$ open, and let $\{x_1, \cdots, x_n\}$ be standard basis for $\mathbb{R}^n$, and $\{x^1, \cdots, x^n\}$ the standard dual basis. Let $p \in U$. Then for $T_p(\mathbb{R}^n)$ and $T_p^*(\mathbb{R}^n)$ we have

$$\mathcal{B}(T_p(\mathbb{R}^n)) = \{\frac{\partial}{\partial x_1}\Big|_p, \cdots, \frac{\partial}{\partial x_n}\Big|_p\}, \qquad \mathcal{B}(T_p^*(\mathbb{R}^n)) = \{dx^1\Big|_p, \cdots, dx^n\Big|_p\}.$$

1.4 Solved Problems

■ **Problem 1.1 — Algebra structure on C_p^∞.** Define carefully addition, multiplication, and scalar multiplication in C_p^∞. Prove that addition in C_p^∞ is commutative.

Solution First, note that the elements of C_p^∞ are actually the equivalence classes, where two functions are equivalent if they both define the same germ.

- For the definition of the addition, we can use the point-wise addition of the functions. How-

ever, we need to check to see if this definition is well-defined (i.e. the result of the addition of two functions does not depend on the choice of representative of the equivalence class). Let $f_1, f_2, g_1, g_2 \in C_p^\infty$ where f_1 and f_2 define the same germ, and similarly for g_1 and g_2. Then, we claim that $f_1 + g_1$ define the same germ as $f_2 + g_2$. That is because for f_1, f_2 there is an open set U_1 containing p where $f_1(x) = f_2(x) \ \forall x \in U$. Similarly, there is an open set U_2 that contains p and for all $x \in U_2$ we have $g_1(x) = g_2(x)$. Let $W = U_1 \cap U_2$. Then on for all $x \in W$ we have $f_1(x) + f_2(x) = g_1(x) + g_2(x)$. Hence $f_1 + g_1$ defines the same germ as $f_2 + g_2$.

- For the scalar multiplication, we can use the notion of scalar multiplication in functions, and following an idea similar to the reasoning above, we can show that this definition is well-defined.

- For the multiplication on C_p^∞ we can use of the point-wise multiplication of functions as the definition, and with a similar reasoning to the one in item 1, we can show that this definition is well-defined.

For the commutativity of the addition on C_p^∞, we need to emphasis that it follows immediately from the commutativity of the point-wise addition of functions.

■ **Problem 1.2 — Vector space structure on derivations at a point.** Prove that the set of all point derivatives is closed under addition and scalar multiplication.

Solution   Let D and D' be derivations at $p \in \mathbb{R}^n$, and define $\hat{D} = D + D'$. Let $f, g \in C_p^\infty$. Then we can write

$$\hat{D}(fg) = (D + D')(fg)$$

On the other hand we have

$$D(fg) = D(f)g + fD(g), \qquad D'(fg) = D'(f)g + fD'(g).$$

Adding two equations we will get

$$D(fg) + D'(fg) = (D(f) + D'(f))g + f(D(g) + D'(g))$$

Defining $\hat{D} = (D + D')(f) = D(f) + D'(f)$ we will get

$$\hat{D}(fg) = \hat{D}(f)g + f\hat{D}(g).$$

which shows that $\hat{D}$ is also a point derivation at p. For the scalar multiplication, let $r \in \mathbb{R}$ and define $\tilde{D} = rD$. By defining $(rD)(f) = rD(f)$, we can write

$$(rD)(fg) = rD(fg) = rD(f)g + rfD(g) = rD(f)g + frD(g),$$

which shows that rD also satisfies the Leibniz property, this it is a point derivation.

■ **Problem 1.3** Let A be an algebra over a field K. If D_1 and D_2 are derivations of A, show that $D_1 \circ D_2$ is not necessarily a derivation (it is if D_1 or $D_2 = 0$), but $D_1 \circ D_2 - D_2 \circ D_1$ is always a derivation of A.

Solution   TO BE ADDED.

■ **Problem 1.4** Find the inversions in the permutation $\tau = (1\ 2\ 3\ 4\ 5)$.

Solution   This permutation can be written as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

Thus the number of inversions can be written as $(2,1), (3,1), (4,1)$, and $(5,1)$.

■ **Problem 1.5** Let $f : V^k \to \mathbb{R}$ be a k-linear function defined on vector space V. Show that the following functions is alternating.

$$Af = \sum_{\sigma \in S_k} \text{sign}(\sigma) \, \sigma f.$$

Solution   Let $\tau \in S_k$. Then

$$\tau(Af) = \sum_{\sigma \in S_k} \text{sign}(\sigma) \, (\tau\sigma)f = \sum_{\sigma \in S_k} \text{sign}(\sigma) \text{sign}(\tau) \text{sign}(\tau) \, (\tau\sigma)f = \text{sign}(\tau) \sum_{\sigma \in S_k} \text{sign}(\tau\sigma) \, (\tau\sigma)f$$

Note that since in the sum above σ runs through all permutations S_k, so does $\tau\sigma$. Thus we can write

$$\tau(Af) = \text{sign}(\tau)(Af).$$

This shows that Af is alternating.

■ **Problem 1.6** Let $f : V^k \to \mathbb{R}$ be a k-linear function. Show that Sf given below is symmetric.

$$Sf = \sum_{\sigma \in S_k} \sigma f.$$

Solution   Let $\tau \in S_k$. Then we

$$\tau(Sf) = \sum_{\sigma \in S_k} \tau\sigma f = Sf.$$

Note that the last equality above holds, since σ rums through all permutations S_k and so does $\tau\sigma$.

■ **Problem 1.7** If f is a 3-linear function on a vector space V and $v_1, v_2, v_3 \in V$, what is $(Af)(v_1, v_2, v_3)$?

Solution

$$(Af)(v_1, v_2, v_3) = f(v_1, v_2, v_3) - f(v_2, v_1, v_3) + f(v_3, v_1, v_2) - f(v_1, v_3, v_2) + f(v_2, v_3, v_1) - f(v_3, v_2, v_1)$$

■ **Problem 1.8** Show that the tensor product of multi-linear functions is associative: If f, g, and h are multi-linear functions on V, then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h).$$

Solution   We start with the left hand side. I.e.

$$\begin{aligned} ((f \otimes g) \otimes h)(v_1 \cdots, v_{k+l+m}) &= (f \otimes g)(v_1, \cdots, v_{k+l})h(k_{k+l+1}, \cdots, k_{k+l+m}) \\ &= \Big(f(v_1, \cdots, v_k)g(v_{k+1}, \cdots, v_{k+l})\Big)h(v_{k+l+1}, \cdots, v_{k+l+m}) \\ &= f(v_1, \cdots, v_k)\Big(g(v_{k+1}, \cdots, v_{k+l})h(v_{k+l+1}, \cdots, v_{k+l+m})\Big) \\ &= (f \otimes (g \otimes h))(v_1, \cdots, v_{k+l+m}). \end{aligned}$$

■ **Problem 1.9** Consider following two ways that we can express the sum in the wedge product formula. Let $f \in A_k(V)$ and $g \in A_l(V)$. Then the wedge product $f \wedge g$ can be written as

$$(f \wedge g)(v_1, \cdots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_k} \text{sign}(\sigma) \ \sigma f(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)}).$$

Or, alternatively, we can write it as $(k, l)-$shuffle

$$(f \wedge g)(v_1, \cdots, v_{k+l}) = \sum_{(k,l)-\text{shuffles } \sigma} \text{sign}(\sigma) \ \sigma f(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)}).$$

Now, as a concrete example, let $f, g \in A_2(V)$. Write $f \wedge g$ in these two forms.

 Solution   First, we want to use the sum on the all permutations.

$$
\begin{aligned}
4(f \wedge g)(v_1, v_2, v_3, v_4) = \ & f(v_1, v_2)g(v_3, v_4) - f(v_2, v_1)g(v_3, v_4) + f(v_2, v_1)g(v_4, v_3) - f(v_1, v_2)g(v_4, v_3) \\
& - f(v_1, v_3)g(v_2, v_4) + f(v_3, v_1)g(v_2, v_4) - f(v_3, v_1)g(v_4, v_2) + f(v_1, v_3)g(v_4, v_2) \\
& + f(v_1, v_4)g(v_2, v_3) - f(v_4, v_1)g(v_2, v_3) + f(v_4, v_1)g(v_3, v_2) - f(v_1, v_4)g(v_3, v_2) \\
& + f(v_2, v_3)g(v_1, v_4) - f(v_3, v_2)g(v_1, v_4) + f(v_3, v_2)g(v_4, v_1) - f(v_2, v_3)g(v_4, v_1) \\
& - f(v_2, v_4)g(v_1, v_3) + f(v_4, v_2)g(v_1, v_3) - f(v_4, v_2)g(v_3, v_1) + f(v_2, v_4)g(v_3, v_1) \\
& + f(v_3, v_4)g(v_1, v_2) - f(v_4, v_3)g(v_1, v_2) + f(v_4, v_3)g(v_2, v_1) - f(v_3, v_4)g(v_2, v_1).
\end{aligned}
$$

Note that in every row, the functions are equal to each at (because f, g are alternating). Thus we have the factor of 4 not to count the redundant terms. However, we can do this sum using the (2,2)-shuffles. This means that we only keep the very first column, as their argument permutation is the same as all (2,2)-shuffles on 4 symbols. Thus we can write

$$
\begin{aligned}
(f \wedge g)(v_1, v_2, v_3, v_4) = \ & f(v_1, v_2)g(v_3, v_4) - f(v_1, v_3)g(v_2, v_4) + f(v_1, v_4)g(v_2, v_3) \\
& + f(v_2, v_3)g(v_1, v_4) - f(v_2, v_4)g(v_1, v_3) \\
& + f(v_3, v_4)g(v_1, v_2).
\end{aligned}
$$

> **Observation 1.4.1** How do we count the (2,2)-shuffles of $\{v_1, v_2, v_3, v_4\}$? We start by a vertical line where in its left side we put v_1, v_2 and in its right side we write v_3, v_4. Like the following table. This is already a shuffle (identity). To write the next shuffle, we keep v_1 in the first position, and write the next symbol whose its subscript is larger than 2 (i.e. v_3), and write the remaining symbols in the right hand side of the vertical line in increasing order. Then we continue this process, until there are no possible shuffles for its first element be v_1. Then we put v_2 in the first place and right next to it we write the next symbol that its subscript is larger than 2 (i.e. v_3), and we continue.
>
> $$
> \begin{array}{c|c}
> v_1, v_2 & v_3, v_4 \\
> v_1, v_3 & v_2, v_4 \\
> v_1, v_4 & v_2, v_3 \\
> \\
> v_2, v_3 & v_1, v_4 \\
> v_2, v_4 & v_1, v_3 \\
> \\
> v_3, v_4 & v_1, v_2 \\
> \end{array}
> $$
>
> In order to find the sign of the each of these permutations, it is enough to extract the independent cycles. For instance, consider the following shuffle, corresponding to last row

above.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

This permutation is the same as the permutation given by the cycle $\tau = (1\ 3)(2\ 4)$. To extract this we start with 1 and track where it goes, and we do this tracking until we get back to 1. Then we start this process with the remaining element until we get all of the cycles. This cycle decomposition of τ clearly shows that τ is an even permutation (can be written as 2 cycles). However, in the case of v_2, v_4, v_1, v_3, the corresponding permutation is

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}.$$

We can write this as $\tau = (1\ 2\ 4\ 3)$. As with every cycle, we can decompose this into smaller cycles (not necessarily independent) as $\tau = (1\ 3)(1\ 4)(1\ 2)$. This shows that this particular permutation is odd.

■ **Problem 1.10 — Wedge product of covectors and determinant.** Let V be a vector space and let $\beta^1, \cdots \beta^l \in A_1(V)$. Prove that

$$(\beta^1 \wedge \cdots \wedge \beta^l)(v_1, \cdots, v_l) = \det \left[\beta^i(v_j) \right]$$

Solution This follows simply from the definition. From the definition of wedge product we have

$$(\beta^1 \wedge \cdots \wedge \beta^l) = A \left[\beta^1(v_1) \cdots \beta^l(v_l) \right] = \sum_{\sigma \in S_l} \text{sign}(\sigma) \beta^1(v_{\sigma(1)}) \cdots \beta^l(v_{\sigma(l)}) = \det \left[\beta^i(v_j) \right]$$

where $\left[\beta^i(v_j) \right]$ is $l \times l$ square matrix.

■ **Problem 1.11** Verify

$$f \wedge g = (-1)^{k+l}(g \wedge f)$$

for f and g being k and l linear maps correspondingly. Do this verification by considering $f \in A_2(V)$ and $g \in A_1(V)$.

Solution For $f \wedge g$ we can write (using $(2,1)$-shuffles)

$$(f \wedge g)(v_1, v_2, v_3) = f(v_1, v_2)g(v_3) - f(v_1, v_3)g(v_2) + f(v_2, v_3)g(v_1).$$

However, for $g \wedge f$, we can use $(1,2)$-shuffles to show

$$(g \wedge f)(v_1, v_2, v3) = g(v_1)f(v_2, v_3) - g(v_2)f(v_1, v_3) + g(v_3)f(v_1, v_2).$$

Evaluating two expressions above reveals that $f \wedge g = g \wedge f$.

■ **Problem 1.12 — Tensor product of covectors (from W. Tu).** Let $e_1, e_2, \cdots, e_n$ be a basis for vector space V and let $\alpha^1, \cdots, \alpha^n$ be its dual basis in $V^\vee$. Suppose $\left[g_{i,j} \right] \in \mathbb{R}^{n \times n}$ is an $n \times n$ matrix. Define a bilinear function $f : V \times V \to \mathbb{R}$ by

$$f(v, w) = \sum_{1 \le i,j \le n} g_{i,j} v^i w^j$$

for $v = \sum v^i e_i$ and $w = \sum w^j e_j$ in V. Describe f in terms of the tensor products of α^i and α^j, $1 \le i, j \le n$.

Solution   We know that $v^i = \alpha^i(v)$, and similarly $w^j = \alpha^j(w)$. So we can write

$$v^i w^j = \alpha^i(v)\alpha^j(w) = (\alpha^i \otimes \alpha^j)(v, w).$$

Thus the bilinear function can be written as

$$f = \sum_{1 \le i,j \le n} g_{i,j}\alpha^i \otimes \alpha^j.$$

This bilinear function is very similar to the notion of weighted inner product.

■ **Problem 1.13 — Hyperplanes(from W. Tu).**   (a) Let V be a vector space of dimension n, and $f : V \to \mathbb{R}$ a nonzero linear functional. Show that $\dim \ker f = n - 1$. A linear subspace of V of dimension $n - 1$ is called a hyperplane in V.

(b) Show that a nonzero linear functional on a vector space V is determined up to multiplicative constant by its kernel, a hyperplane in V. In other words, if f and $g : V \to \mathbb{R}$ are nonzero linear functionals and $\ker f = \ker g$, then $g = cf$ for some constant $c \in \mathbb{R}$.

Solution   (a) We can use the Rank-Nullity theorem for the linear function f. This theorem states that

$$\dim \operatorname{rank} f + \dim \ker f = \dim V.$$

We know that $\dim \operatorname{rank} f = 1$, and $\dim V = n$. This implies $\dim \ker f = n - 1$.

(b) Let f, g be two linear functionals on V that has the same kernel, which we call it the set K. Since K is $n - 1$ dimensional linear subspace of K, then we can find a basis for it, call it $\mathcal{B}_1 = \{\hat{e}_1, \cdots, \hat{e}_{n-1}\}$. Let $\hat{e}_n$ be a vector normal to $\mathcal{B}_1$. We can do all of these since V is n dimensional and we have just modified the basis vectors as described above. Let $z \in V$ where $z = \sum_{i=1}^n z_i \hat{e}_i$. Thus we have

$$f(z) = z_n f(\hat{e}_n), \qquad g(z) = z_n g(\hat{e}_n).$$

This is true because $f(\hat{e}_i) = g(\hat{e}_i) = 0$ for all $\hat{e}_i \in \mathcal{B}$. Since f, g are non-zero functional, then we can write

$$\frac{f(z)}{g(z)} = \frac{f(\hat{e}_n)}{g(\hat{e}_n)} = c \qquad \text{for some } c \in \mathbb{R}$$

Thus we can write

$$f = cg.$$

Note that c is determined by $\hat{e}_n$. $\hat{e}_n$ itself is determined by the kernel of f, g, i.e. the hyperplane mentioned above.

> **Observation 1.4.2** I have this weird observation that I think it must be wrong, but I don't know how! Consider the set of all linear functionals defined on $\mathbb{R}^2$. From duality, we know that this set (denoted by $L_1(\mathbb{R}^2)$) is isomorphic to $\mathbb{R}^2$. But here comes the strange argument that leads to the non-real observation. As we saw in the question above, every hyperplane (in this case every line that passes through the origin) determines a linear functional up to a constant multiplication. On the other hand, we know that the set of all liens passing through the origin is the real projective plane (which is in fact $L_1(\mathbb{R}^2)/ \sim$ where $\sim$ identifies linear functionals that are constant multiple of each other), which definitely is not isomorphic to $\mathbb{R}^2$.

■ **Problem 1.14 — A basis for k-tensors (from W. Tu).** Let V be a vector space of dimension n with basis $e_1, \cdots, e_n$. Let $\alpha^1, \cdots, \alpha^n$ be the dual basis for $V^\vee$. Show that a basis for the space $L_k(V)$ of k-linear functions on V is $\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ for all multi-indices $(i_1, i_2, \cdots, i_k)$. In particular, this show that $\dim L_k(V) = n^k$.

Solution To show this, we need to show that the basis vectors are linearly independent, and spans all $L_k(v)$. To show the linear independence, consider the following linear combination of all basis vectors, where the total sum is set to zero.

$$\sum C_{i_1, \cdots, i_k} \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k} = 0,$$

where the sum is done on all of the multi-indices $(i_1, \cdots, i_k)$. This sum implies that all of the coefficients are zero. We can see this by applying $(e_{j_1}, \cdots, e_{j_k})$ to both sides. We have

$$(\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k})(e_{j_1}, \cdots, e_{j_k}) = 0$$

if there is any mismatch between the multi-indices I, J. This implies $C_{j_1, \cdots, j_k} = 0$ for all multi-indices. Thus we can conclude that the basis vectors are linearly independent.

To show that the basis vectors span all the space $L_k(V)$, let $f \in L_k(V)$ be any k-tensor. Define

$$g = \sum f(e_{i_1}, \cdots, e_{i_k}) \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}.$$

We can see that

$$g(e_{j_1}, \ldots, e_{j_k}) = f(e_{j_1}, \ldots, e_{j_k}) \qquad \text{for all multi-indices}$$

From linearity we can deduce that $f = g$. Thus the basis vectors span the whole space.

■ **Problem 1.15 — A characterization of alternating k-vectors (from W. Tu).** Let f be a k-tensor on a vector space V. Prove that f is alternating if and only if f changes sign whenever two successive arguments are interchanged

$$f(\cdots, v_{i+1}, v_i, \cdots) = -f(\cdots, v_i, v_{i+1}, \cdots).$$

Solution The first direction follows immediately from the definition. I.e. assume f is alternating. Then sign of the permutation corresponding interchanging two successive arguments is a single transposition which has sign -1. Thus from definition we have $f(\cdots, v_{i+1}, v_i, \cdots) = -f(\cdots, v_i, v_{i+1}, \cdots)$.

For the converse, first observe that we can decompose arbitrary permutation to cycles, and then decompose cycles into transpositions. Eventually, we can decompose every transposition to transposition of adjacent elements. See the Lemma below

> **Lemma 1.2 — Decomposing arbitrary transposition to transposition of adjacent elements.** Let $\tau = (k\ l)$ be an arbitrary transposition where $k < l$. Assume that k, l are m distance apart (i.e. $l - k = m$). Then we can write
>
> $$(k\ l) = (k\ k+1) \cdot (k+1\ k+2) \cdots (l-1\ l) \cdot (l-2\ l-1) \cdots (k\ k+1).$$
>
> Thus $(k\ l)$ can be decomposed to $2m - 1$ adjacent transpositions.

Let $\sigma \in S_k$ be an arbitrary permutation. We can decompose it to the following permutations

$$\sigma = \tau_1 \tau_2 \cdots \tau_l,$$

where from the number of permutations we can see that $\text{sign}(\sigma) = (-1)^l$. Let τ_i be one of the transpositions above that interchanges two elements m distance apart. Then

$$\tau_i f = (-1)^{2m-1} f = -f.$$

This follows immediately from the Lemma above. Thus we can write

$$\sigma f = (-1)^l f = \text{sign}(\sigma) f.$$

This shows that f is an alternating k-tensor.

■ **Problem 1.16 — Another characterization of alternating k-tensors (from W. Tu).** Let f be a k-tensor on a vector space V. Prove that f is alternating if and only if

$$f(v_1, \cdots, v_k) = 0$$

whenever two of the vectors $v_1, \cdots, v_k$ are equal.

Solution   We first start with the forward direction. Assume f is alternating. Thus for $\sigma \in S_k$ we have $\sigma f = \text{sign}(\sigma)f$. Without loss of generality, we can assume $i < j$ and let $(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) \in V^k$ where $v_i = v_j$. Thus we can write

$$f(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = f(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k).$$

Furthermore, let $\tau = (v_i, v_j)$, i.e. a transposition that interchanges v_i with v_j. Hence

$$f(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = \tau f(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k)$$
$$= -f(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k).$$

This implies

$$f(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = f(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k) = 0$$

To show the converse, let $\sigma \in S_k$ be any permutation. We can decompose this permutation into transpositions, i.e. $\sigma = \tau_1 \tau_2 \cdots \tau_r$. Let τ be any of transpositions above that permutes $(i\ j)$. Let $(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) \in V^k$. Since V^k is a vector space we can write

$$(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) - (v_1, \cdots, v_i, \cdots, v_i, \cdots, v_k) - (v_1, \cdots, v_j, \cdots, v_j, \cdots, v_k) = -(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k).$$

Applying f to both sides, and using the hypothesis and k-linearity of f we will get

$$f(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = -f(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k) = -\tau f(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k).$$

Thus for each transposition in the decomposition $\sigma = \tau_1 \tau_2 \cdots \tau_r$ we will get a factor of -1. Thus

$$\sigma f = (-1)^r f = \text{sign}(\sigma)f$$

■ **Problem 1.17 — Wedge product and scalars (from W. Tu).** Let V be a vector space. For $a, b \in \mathbb{R}$, $f \in A_k(V)$, and $g \in A_\ell(v)$, show that $af \wedge bg = (ab)f \wedge g$.

Solution   This follows immediately from the definition of wedge product.

$$af \wedge bg = \frac{1}{k!\ell!} A\left(af \otimes bg\right) = \frac{ab}{k!\ell!} A(f \otimes g) = (ab)f \wedge g.$$

■ **Problem 1.18 — Transformation rule for a wedge product of covectors (from W. Tu).** Suppose two sets of covectors on a vector space V, $\beta^1, \cdots, \beta^k$ and $\gamma^1, \cdots, \gamma^k$ are related by

$$\beta^i = \sum_{j=1}^{k} a_j^i \gamma^j, \qquad i = 1, 2, \cdots, k,$$

for a $k \times k$ matrix $A = \left[a_j^i\right]$. Show that

$$\beta^1 \wedge \cdots \wedge \beta^k = (\det A)\gamma^1 \wedge \cdots \wedge \gamma^k.$$

Solution It will be helpful if we write the relation between β^i's and γ^js as a matrix multiplication. To do so we can write

$$
\begin{pmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^k \end{pmatrix} = \begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_k^1 \\ a_1^2 & a_2^2 & \cdots & a_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^k & a_2^k & \cdots & a_k^k \end{pmatrix} \begin{pmatrix} \gamma^1 \\ \gamma^2 \\ \vdots \\ \gamma^k \end{pmatrix}
$$

On the other hand, we know

$$
(\beta^1 \wedge \cdots \wedge \beta^k)(v_1, \cdots, v_k) = \det \begin{pmatrix} \beta^1(v_1) & \beta^1(v_2) & \cdots & \beta^1(v_k) \\ \beta^2(v_1) & \beta^2(v_2) & \cdots & \beta^2(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \beta^k(v_1) & \beta^k(v_2) & \cdots & \beta^k(v_k) \end{pmatrix}
$$

$$
= \det \left[\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_k^1 \\ a_1^2 & a_2^2 & \cdots & a_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^k & a_2^k & \cdots & a_k^k \end{pmatrix} \begin{pmatrix} \gamma^1(v_1) & \gamma^1(v_2) & \cdots & \gamma^1(v_k) \\ \gamma^2(v_1) & \gamma^2(v_2) & \cdots & \gamma^2(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^k(v_1) & \gamma^k(v_2) & \cdots & \gamma^k(v_k) \end{pmatrix} \right]
$$

$$
= \det(A) \det \begin{pmatrix} \gamma^1(v_1) & \gamma^1(v_2) & \cdots & \gamma^1(v_k) \\ \gamma^2(v_1) & \gamma^2(v_2) & \cdots & \gamma^2(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^k(v_1) & \gamma^k(v_2) & \cdots & \gamma^k(v_k) \end{pmatrix}
$$

Thus we can write

$$
(\beta^1 \wedge \cdots \wedge \beta^k)(v_1, \cdots, v_k) = \det(A)(\gamma^1 \wedge \cdots \wedge \gamma^k)(v_1, \cdots, v_k).
$$

■ **Problem 1.19 — Transformation rule for k-covectors (from W. Tu).** Let f be a k-covector on a vector space V (of say, dimension n). Suppose two sets of vectors $u_1, \cdots, u_k$ and $v_1, \cdots, v_k$ in V are related by

$$
u_j = \sum_{i=1}^k a_j^i v_i, \qquad j = 1, \cdots, k,
$$

for a $k \times k$ matrix $A = \left[a_j^i \right]$. Show that

$$
f(u_1, \cdots, u_k) = (\det A) f(v_1, \cdots, v_k).
$$

By

Solution Since f is a k-vector, then we can expand it in its basis

$$
f = \sum_{I \in AM} C_I \alpha^I,
$$

where AM is the set of all ascending multi-indices $I = (i_1, \cdots, i_k)$ for $1 \le i_1 < i_2 < \cdots < i_k \le n$. Thus

$$
f(u_1, \cdots, u_k) = \sum_{I \in AM} C_I \alpha^I(u_1, \cdots, u_k).
$$

Let's focus on the term $I = (i_1, \cdots, i_k)$. For this term we can write

$$(\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k})(u_1, \cdots, u_k) = \det \begin{pmatrix} \alpha^{i_1}(u_1) & \alpha^{i_1}(u_2) & \cdots & \alpha^{i_1}(u_k) \\ \alpha^{i_2}(u_1) & \alpha^{i_2}(u_2) & \cdots & \alpha^{i_2}(u_k) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{i_k}(u_1) & \alpha^{i_k}(u_2) & \cdots & \alpha^{i_k}(u_k) \end{pmatrix}$$

$$= \det \left[\begin{pmatrix} \alpha^{i_1}(v_1) & \alpha^{i_1}(v_2) & \cdots & \alpha^{i_1}(v_k) \\ \alpha^{i_2}(v_1) & \alpha^{i_2}(v_2) & \cdots & \alpha^{i_2}(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{i_k}(v_1) & \alpha^{i_k}(v_2) & \cdots & \alpha^{i_k}(v_k) \end{pmatrix} \begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_k^1 \\ a_1^2 & a_2^2 & \cdots & a_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^k & a_2^k & \cdots & a_k^k \end{pmatrix} \right]$$

$$= \det(A) \det \begin{pmatrix} \alpha^{i_1}(v_1) & \alpha^{i_1}(v_2) & \cdots & \alpha^{i_1}(v_k) \\ \alpha^{i_2}(v_1) & \alpha^{i_2}(v_2) & \cdots & \alpha^{i_2}(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{i_k}(v_1) & \alpha^{i_k}(v_2) & \cdots & \alpha^{i_k}(v_k) \end{pmatrix}$$

$$= \det(A)(\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k})(v_1, \cdots, v_k)$$

Thus we can write

$$f(u_1, \cdots, u_k) = \sum_{I \in AM} C_I \alpha^I(u_1, \cdots, u_k) = \det(A) \sum_{I \in AM} C_I \alpha^I(v_1, \cdots, v_k)$$

or

$$\boxed{f(u_1, \cdots, u_k) = \det(A) f(v_1, \cdots, v_k).}$$

$\square$

■ **Problem 1.20 — Vanishing of a covector of top degree (from W. Tu).** Let V be a vector space of dimension n. Prove that if an n-covector ω vanishes on a basis $e_1, \cdots, e_n$ for V, then ω is the zero covector on V.

 Solution   First, observe that for a vector space of dimension n we have

$$\dim(A_n) = \binom{n}{n} = 1, \qquad \mathcal{B} = \{\alpha^1 \wedge \cdots \wedge \alpha^k\}.$$

Let $f \in A_n(V)$. Thus we can write

$$f = g \alpha^1 \wedge \cdots \wedge \alpha^n \qquad \text{for some } g \in \mathbb{R}.$$

On the other hand, we know

$$(\alpha^1 \wedge \cdots \wedge \alpha^n)(e_1, \cdots, e_n) = \det \left[\alpha^i(e_j) \right] = 1.$$

This $f(e_1, \cdots, e_n) = 0$ if and only if $g = 0$.

■ **Problem 1.21 — Linear independence of covectors (from W. Tu).** Let $\alpha^1, \cdots, \alpha^k$ be 1-covectors on a vector space V. Show that $\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$ if and only if $\alpha^1, \cdots, \alpha^k$ are linearly independent in the dual space $V^\vee$.

Solution For the forward direction, we want to show

$$\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0 \implies \alpha^1, \cdots, \alpha^k \text{ are linearly independent.}$$

To show this we will prove the contrapositive. Assume $\alpha^1, \cdots, \alpha^k$ are linearly dependent. Thus we can write one of them as a linear combination of others. With out loss of generality, assume

$$\alpha^1 = c_2 \alpha^2 + \cdots + c_n \alpha^n,$$

where c_is are not all zero. Consider

$$\alpha^1 \wedge \cdots \wedge \alpha^k = c_2 \alpha^2 \wedge \cdots \wedge \alpha^k + \cdots + c_k \alpha^k \wedge \cdots \wedge \alpha^k = 0.$$

The last equality is true since $\alpha^I = 0$ if the multi-indices I as at least two repetitive indices.

For the converse, we want to show

$$\alpha^1, \cdots, \alpha^k \text{ are linearly independent.} \implies \alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$$

Assume $\alpha^1, \cdots, \alpha^k$ are linearly independent. Then it can be extended as a basis for $V^\vee$ as as $\alpha^1, \cdots, \alpha^k, \cdots, \alpha^n$. Let $v_1, v_2, \cdots, v_n$ be the dual basis for V. Then

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \cdots, v_k) = \det\left[\alpha^i(v_j)\right] = 1 \neq 0.$$

Thus $\alpha^1 \wedge \cdots \wedge \alpha^k$ can not be a zero k-covector.

■ **Problem 1.22 — Exterior multiplication (from W. Tu).** Let α be a non-zero 1-covector and γ a k-covector on a finite dimensional vector space V. Show that $\alpha \wedge \gamma = 0$ if and only if $\gamma = \alpha \wedge \beta$ for some $(k-1)$-covector β on V.

Solution First, we prove the first direction. Let $\gamma \in A_k(V), \alpha \in A_1(V)$ such that $\gamma = \alpha \wedge \beta$ for some $\beta \in A_{k-1}(V)$. Then

$$\alpha \wedge \gamma = \alpha \wedge \alpha \wedge \beta = 0.$$

For the converse, let α and γ as before and we want to prove

$$\alpha \wedge \gamma = 0 \implies \gamma = \alpha \wedge \beta \text{ for some } \beta \in A_{k-1}(V).$$

We prove the contrapositive. Assume $\forall \beta \in A_{k-1}(V)$ we have $\gamma \neq \alpha \wedge \beta$. Then by wedge product of α to both sides we will get

$$\alpha \wedge \gamma \neq \alpha \wedge \alpha \wedge \beta = 0.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

■ **Problem 1.23 — A basis for 3-covectors (from W. Tu).** Let x^1, x^2, x^3, x^4 be the coordinates on $\mathbb{R}^4$ and p a point in $\mathbb{R}^4$. Write down a basis for the vector space $A_3(T_p(\mathbb{R}^4))$.

Solution First, observe that a basis for $T_p(\mathbb{R}^4)$ is $\{\frac{\partial}{\partial x^1}|_p, \cdots, \frac{\partial}{\partial x_4}|_p\}$, and for $T_p^*(\mathbb{R}^n)$ a standard basis is $\{dx_1|_p, \cdots, dx_4|_p\}$. Thus a standard basis for $A_3(T_p(\mathbb{R}^4))$ would be

$$\mathcal{B} = \{dx^1 \wedge dx^2 \wedge dx^3|_p, \ dx^1 \wedge dx^2 \wedge dx^4|_p, \ dx^1 \wedge dx^3 \wedge dx^4|_p, \ dx^2 \wedge dx^3 \wedge dx^4|_p\}.$$

■ **Problem 1.24 — Wedge product of a 2-form with a 1-form (from W. Tu).** Let Ω be a 2-form and τ a 1-form on $\mathbb{R}^3$. If X, Y, Z are vector fields on M find an explicit formula for $(\omega \wedge \tau)(X, Y, Z)$ in terms of the values of ω and τ on the vector fields X, Y, Z.

Solution   From the definition of wedge product, and using the notion of $(2, 1)$ shuffles, that are $(1 < 2, 3), (1 < 3, 2), (2 < 3, 1)$ with respective signs $+, -, +$ we can write

$$(\omega \wedge \tau)(X, Y, Z) = (\omega \otimes \tau)(X, Y, Z) - (\omega \otimes \tau)(X, Z, Y) + (\omega \otimes \tau)(Y, Z, X),$$

or alternatively

$$(\omega \wedge \tau)(X, Y, Z) = \omega(X, Y)\tau(Z) - \omega(X, Z)\tau(Y) + \omega(Y, Z)\tau(X).$$

■ **Problem 1.25 — A closed 1-form on the punctured plane (from W. Tu).** Define a 1-form ω on $\mathbb{R}^2 - \{0\}$ by

$$\omega = \frac{1}{x^2 + y^2}(-ydx + xdy).$$

Show that ω is closed.

Solution   define $f, g : \mathbb{R}^2 - \{0\} \to \mathbb{R}$ as

$$f(x, y) = \frac{-y}{x^2 + y^2}, \qquad g(x, y) = \frac{x}{x^2 + y^2}.$$

Thus we can write ω as $\omega = fdx + gdy$. The exterior derivative of ω will be

$$d\omega = (g_x - f_y)dx \wedge dy.$$

By a simple calculation we can observe

$$f_x(x, y) = \frac{y^2 - x^2}{x^2 + y^2}, \qquad g_y(x, y) = \frac{y^2 - x^2}{x^2 + y^2}.$$

Thus we will have

$$d\omega = 0.$$

■ **Problem 1.26 — A 1-form on $\mathbb{R}^3$ (from W. Tu).** Let ω be the 1-form $zdx - dz$, and let X be the vector field $y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on $\mathbb{R}^3$. Compute $\omega(X)$ and $d\omega$.

Solution   To calculate $\omega(X)$ we can write

$$\omega(X) = X\omega = (y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y})(zdx - dz) = yz - x.$$

And to calculate $d\omega$ we have

$$d\omega = dz \wedge dx$$

■ **Problem 1.27 — A 2-form on $\mathbb{R}^3$ (from W. Tu).** At each point $p \in \mathbb{R}^3$, define a bilinear function ω_p on $T_p(\mathbb{R}^3)$ by

$$\omega_p(a, b) = \omega_p \left(\begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix}, \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix} \right) = p^3 \det \begin{pmatrix} a^1 & b^1 \\ a^2 & b^2 \end{pmatrix},$$

For tangent vectors $a, b \in T_p(\mathbb{R}^3)$, where p^3 is the third component of $p = (p^1, p^2, p^3)$. Since ω_p is an alternating bilinear function on $T_p(\mathbb{R}^3)$, ω is a 2-form on $\mathbb{R}^3$. Write ω in terms of the standard basis $dx^i \wedge dx^j$ at each point.

Solution   In the definition of ω, we can replace determinant by the following wedge product

$$\omega_p = p^3 dx^1 \wedge dx^2.$$

■ **Problem 1.28 — Exterior calculus (from W. Tu).** Suppose the standard coordinates on $\mathbb{R}^2$ are called r and θ (this $\mathbb{R}^2$ is the (r, θ) plane not the (x, y) plane). If $x = r \cos \theta$ and $y = r \sin \theta$, calculate dx, dy, and $dx \wedge dy$ in terms of dr and $d\theta$.

Solution Using the relations $x = r \cos \theta$ and $y = r \sin \theta$, we can use the definition of exterior derivative to have

$$dx = \cos \theta \ dr - r \sin \theta d\theta, \qquad dy = \sin \theta dr + r \cos \theta \ d\theta.$$

We can write this in a more clean way as

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \begin{bmatrix} dr \\ d\theta \end{bmatrix}.$$

By using the fact proved in Problem 1.18 we will have

$$dx \wedge dy = \det(A) dr \wedge d\theta = r dr \wedge d\theta.$$

■ **Problem 1.29 — Exterior calculus (from W. Tu).** Suppose the standard coordinates on $\mathbb{R}^3$ are called ρ, φ and θ. If $x = \rho \cos \varphi \sin \theta$, $y = \rho \sin \varphi \sin \theta$, and $z = \rho \cos \theta$, calculate dx, dy, dz, and $dx \wedge dy \wedge dz$ in terms of $d\rho, d\varphi$, and $d\theta$.

Solution Similar to the problem above, using the definition of the exterior derivative we have

$$dx = \cos \varphi \sin \theta \ d\rho - \rho \sin \varphi \sin \theta \ d\varphi + \rho \cos \varphi \cos \theta \ d\theta,$$
$$dy = \sin \varphi \sin \theta \ d\rho + \rho \cos \varphi \sin \theta \ d\varphi + \rho \sin \varphi \cos \theta \ d\theta,$$
$$dz = \cos \theta \ d\rho - \rho \sin \theta \ d\theta.$$

or in matrix notation we have

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{pmatrix} \cos \varphi \sin \theta & -\rho \sin \varphi \sin \theta & \rho \cos \varphi \cos \theta \\ \sin \varphi \sin \theta & \rho \cos \varphi \sin \theta & \rho \sin \varphi \cos \theta \\ \cos \theta & 0 & -\rho \sin \theta \end{pmatrix} \begin{bmatrix} d\rho \\ d\varphi \\ d\theta \end{bmatrix}.$$

From the fact proved in Problem 1.18 we can write

$$dx \wedge dy \wedge dz = \det(A) \ d\rho \wedge d\theta \wedge d\varphi.$$

We now need to calculate $\det(A)$

$$\det(A) = -\rho^2 \cos^2 \theta \sin \theta - \rho^2 \sin^3 \theta = -\rho^2 \sin \theta.$$

Thus we have

$$dx \wedge dy \wedge dz = (-\rho^2 \sin \theta) \ d\rho \wedge d\varphi \wedge d\theta.$$

■ **Problem 1.30 — Wedge product (from W. Tu).** Let α be a 1-form and β a 2-form on $\mathbb{R}^3$. Then

$$\alpha = a_1 dx^1 + a_2 dx^2 + a_3 dx^3,$$
$$\beta = b_1 dx^1 \wedge dx^3 + b_2 dx^3 \wedge dx^1 + b_3 dx^1 \wedge dx^2.$$

Simplify the expression $\alpha \wedge \beta$ as much as possible.

Solution By following the rules for the wedge product we will have

$$\alpha \wedge \beta = (a_2 b_2 + a_3 b_3 - a_2 b_1) \ dx^1 \wedge dx^2 \wedge dx^3$$

■ **Problem 1.31 — Wedge product and cross product (from W. Tu).** The correspondence between differential forms and vector fields on an open subset of $\mathbb{R}^3$ also makes sense point wise. Let V be a vector space of dimension 3 with basis e_1, e_2, e_3, and dual basis $\alpha^1, \alpha^2, \alpha^3$. To a 1-covector $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ on V, we associate the vector $v_\alpha = \langle a_1, a_2, a_3 \rangle \in \mathbb{R}^3$. To the 2-covector

$$\gamma = c_1\alpha^1 \wedge \alpha^3 + c_2\alpha^3 \wedge \alpha^1 + c_3\alpha^1 \wedge \alpha^2,$$

on V, we associate the vector $v_\gamma = \langle c_1, c_2, c_3 \rangle \in \mathbb{R}^3$. Show that under this correspondence, the wedge product of 1-covectors corresponds to the cross product of vectors in $\mathbb{R}^3$: if $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ and $\beta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$, then we have

$$v_{\alpha \wedge \beta} = v_\alpha \times v_\beta.$$

Solution   Let $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$, and $\beta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$. Then for the wedge product we have

$$\alpha \wedge \beta = (a_1b_2 - a_2b_1)\alpha^1 \wedge \alpha^3 + (a_1b_3 - a_3b_1)\alpha^1 \wedge \alpha^3 + (a_2b_3 - a_3b_2)\alpha^3 \wedge \alpha^2.$$

We can associate this with vector $v_{\alpha \wedge \beta} = \langle a_1b_2 - a_2b_1, a_1b_3 - a_3b_1, a_2b_3 - a_3b_2 \rangle \in \mathbb{R}^3$. On the other hand, we can write $v_\alpha \times v_\beta = \langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle a_1b_2 - a_2b_1, a_1b_3 - a_3b_1, a_2b_3 - a_3b_2 \rangle$. This shows that

$$v_{\alpha \wedge \beta} = v_\alpha \times v_\beta.$$

■ **Problem 1.32 — Commutator of derivations and antiderivations (from W. Tu).** Let $A = \bigoplus_{k=-\infty}^{\infty} A^k$ be a graded algebra over a field K with $A^k = 0$ for $k < 0$. Let m be an integer. A superderivation of A of degree m is a K-linear map $D : A \to A$ such that for all k, $D(A^k) \in A^{k+m}$ and for all $a \in A^k$ and $b \in A^\ell$,

$$D(ab) = (Da)b + (-1)^{k\ell}aD(b).$$

If D_1 and D_2 are two superderivations of A of respective degrees m_1 and m_2, define their commutator to be

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{m_1 m_2}D_2 \circ D_1.$$

Show that $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$. A superderivation is said to be even or odd depending on the parity of its degree. An even superderivation is a derivation; an odd superderivation is an antiderivation.

Solution   To show that $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$ we need to show

$$[D_1, D_2](ab) \stackrel{?}{=} [D_1, D_2](a)b + (-1)^{k(m_1+m_2)}a[D_1, D_2](b).$$

Using the definition of $[D_1, D_2]$ we can expand this as

$$[D_1, D_2](ab) \stackrel{?}{=} \left[D_1(D_2(a)) - (-1)^{m_1 m_2}D_2(D_1(a)) \right] b + (-1)^{k(m_1+m_2)}a \left[D_1(D_2(b)) - (-1)^{m_1 m_2}D_2(D_1(b)) \right].$$
$$(Q.1)$$

To show this, we simply start with $[D_1, D_2]$ and apply it on $a \in A_k, b \in A_\ell$ and use the definition of $[D_1, D_2]$ to simplify it and turn it into the format of equation $(Q.1)$.

$$[D_1, D_2](ab) = (D_1 \circ D_2)(ab) - (-1)^{m_1 m_2}(D_2 \circ D_1)(ab) = D_1(D_2(ab)) - (-1)^{m_1 m_2}D_2(D_1(ab))$$

$$= D_1(D_2(a))b + (-1)^{m_1(k+m_2)}D_2(a)D_1(b) + (-1)^{m_2 k}\left[D_1(a)D_2(b) + (-1)^{m_1 k}aD_1(D_2(b)) \right]$$

$$- (-1)^{m_1 m_2}\left[D_2(D_1(a))b + (-1)^{m_2(k+m_1)}D_1(a)D_2(b) + (-1)^{m_1 k}\left[D_2(a)D_1(b) + (-1)^{m_2 k}aD_2(D_1(b)) \right] \right]$$

Now by matching the terms we can write

$$[D_1, D_2](ab) = \left[D_1(D_2(a)) - (-1)^{m_1 m_2} D_2(D_1(a))\right] b + (-1)^{k(m_1 + m_2)} a \left[D_1(D_2(b)) - (-1)^{m_1 m_2} D_2(D_1(b))\right]$$
$$+ \underbrace{D_1(a) D_2(b) \left[(-1)^{m_2 k} - (-1)^{m_2(k + 2m_1)}\right]}_{=0}.$$

The last term is equal to zero since if $m_2 k$ is even, then $m_2 k + 2m_1 m_2$ is also even, which leads to $(-1)^{m_2 k} - (-1)^{m_2(k + 2m_1)} = -1 + 1 = 0$. As the other case, if $m_2 k$ is even, then $m_2 k + 2m_1 m_2$ is also even, where we will have $(-1)^{m_2 k} - (-1)^{m_2(k + 2m_1)} = 1 - 1 = 0$. This proves that $[D_1, D_2]$ is indeed an superderivative of order $m_1 + m_2$.

2. Manifolds

Very important note: Throughout this text, to avoid repeating some words like *smooth, and etc*, we often omit them and we let the context to reflect these notions. So we emphasis that throughout this text, unless otherwise specified, by a **manifold** we always mean a C^∞ manifold. An atlas of a chart on a smooth manifold means an atlas or a chart contained in the differentiable structure of the smooth manifold.

2.1 Topological manifolds and Smooth manifolds

We start with the definition of topological manifolds. We will later study the smooth manifolds that are of our main interest in this lecture. But first, we need to review some basic definition.

Definition 2.1 — Hausdorff topological space. A set A, along with a collection of subsets of A called $\mathcal{T} \subset 2^A$, i.e. $(A, \mathcal{T})$ is called a topological space if we have

(I) $\varnothing \in \mathcal{T}$ and $A \in \mathcal{T}$.

(II) For any infinite collection $\{A_\alpha\}_\alpha \subset \mathcal{T}$ we have

$$\bigcup_\alpha A_\alpha \in \mathcal{T}.$$

(III) For a finite collection $\{A_i\}_{i \in I}$ where $I = \{1, 2, \cdots, N\}$ for some $N \in \mathbb{N}$, we have

$$\bigcap_{i \in I} A_i \in \mathcal{T}.$$

(IV) $\forall x, y \in A$ there exists $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$ and we have $U \cap V = \varnothing$.

The following definition reviews the notion of second countable topological spaces.

Definition 2.2 — Second countable topological spaces. Let $(X, \mathcal{T})$ be a topological space. This space is second countable if there exists an at most countable collection of sets $\mathcal{B} \subset \mathcal{T}$ such that

$\forall U \in \mathcal{T}$ we have

$$U = \bigcup_{A \in \mathcal{B}} A.$$

I.e. any open set in $\mathcal{T}$ can be written as a union of sets in $\mathcal{B}$

The last piece of definition that we need is the notion of locally Euclidean spaces.

Definition 2.3 — Locally Euclidean topological spaces. Let $(X, \mathcal{T})$ be a topological space. We say X is locally Euclidean if $\forall p \in X$, there exists an open neighborhood $p \in U \in \mathcal{T}$ such that is homeomorphic to an *open* subset of $\mathbb{R}^n$. I.e. there is a homeomorphism $\varphi : U \to \mathbb{R}^n$. We call the pair $(U, \varphi : U \to \mathbb{R}^n)$ a *chart*, U a *coordinate neighborhood* or a **coordinate open set**, and φ a *coordinate map* or *coordinate system* on U.

Definition 2.4 — Topological manifolds. A topological manifold is a Hausdorff, second countable topological space that is locally Euclidean space. It is said to be of dimension n, if it is locally Euclidean of dimension n.

In the following section, we will review the notion of compatible charts which will be central to our study of manifolds. Assume we have two charts (U, φ) and (V, ψ). Since U, V are open in M (the manifold), then $U \cap V$ is open in U and V. Furthermore, since φ is a homeomorphism to an open subset of $\mathbb{R}^n$ then $\varphi(U \cap V)$ and $\psi(U \cap V)$ are also open. One way to show this, let $\varphi(U cap V) = B, \varphi(U) = A$, and $A \backslash B = C$. We know that A is open, and $B \cup C = A, B \cap C = \varnothing$. Then A being open implies B and C is open. Now we can make the following definition for C^∞-compatible maps.

Definition 2.5 — C^∞ compatible maps. Let M be a topological manifold, where (U, φ) and (V, ψ) are two charts. These charts are called C^∞-compatible if the maps

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V) \qquad \text{and} \qquad \psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V),$$

are C^∞. These two maps are called *transition functions* between charts.

■ **Remark 2.1** If two U, V in the definition above, i.e. $U \cap V = \varnothing$, then they are automatically compatible.

Definition 2.6 — C^∞ atlas. Let M be a topological manifold. The collection of charts $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}$ is called a C^∞ atlas, or simply an atlas if

- all of the charts are pairwise C^∞ compatible,

- the charts cover the whole manifold, i.e. $M = \bigcup_\alpha U_\alpha$.

Lemma 2.1 Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be an atlas for the manifold M. Let (V, ψ) and (W, σ) be two charts where both of them are compatible with the atlas $\mathcal{A}$. Then these two charts are compatible with each other.

Proof. We start by showing that $\psi \circ \sigma^{-1}$ is C^∞ on $V \cap W$. Let $p \in V \cap W$. Then $\exists \alpha \in I$ such that $p \in U_\alpha$ for the chart $(U_\alpha, \varphi_\alpha)$. Then

$$(\psi \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \sigma^{-1}) : \sigma(W \cap U_\alpha \cap V) \to \psi(W \cap U_\alpha \cap V)$$

is C^∞ in $\sigma(W \cap U_\alpha \cap V)$, because it is the composition of C^∞ maps. Call $\mathbb{B}_\alpha = W \cap U_\alpha \cap V$. Then

what we have shows is simply $\forall p \in V \cap W$, there exists an open set $p \in \mathbb{B}_\alpha \subset V \cap W$ for $\alpha \in I$ and the map $\psi \circ \sigma^{-1}$ is C^∞ in $\mathbb{B}_\alpha$. Since this holds for every $p \in V \cap W$ this proves that $\psi \circ \sigma^{-1}$ is C^∞ on $\sigma(V \cap W)$. Similarly, we can show $\sigma \circ \psi^{-1}$ is C^∞ on $\psi(V \cap W)$, and this completes the proof. $\qquad\square$

■ **Remark 2.2** Note that in an equality like $\psi \circ \sigma^{-1} = (\psi \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \sigma^{-1})$, two functions in the sides of the equality have different domains. So the equality sign here means that these two functions are equal on their common domain.

> **Definition 2.7 — Maximal atlas.** Let $\mathcal{U}$ be an atlas for the manifold M. Then $\mathcal{U}$ is called maximal if it contains any other atlas of the manifold M. Or equivalently, if $\mathcal{M}$ is any other atlas containing $\mathcal{U}$ then $\mathcal{U} = \mathcal{M}$.

We can use the notion of the maximal atlas to define a smooth manifold.

> **Definition 2.8 — Smooth manifold.** A topological manifold together with a maximal atlas is called a smooth manifold. The maximal atlas is also called a differentiable structure on M.

In practice, to show that a topological manifold is smooth, it is not necessary to exhibit a maximal atlas. Existence of any atlas will do so, as proposed by the following proposition.

> **Proposition 2.1** Any atlas of a locally euclidean space is contained in a *unique* maximal atlas.

Proof. Let $\mathfrak{U} = \{(U_\alpha, \varphi_\alpha)\}$ be any atlas for the locally Euclidean space M. Let $S = \{(V_i, \psi_i)\}$ denote the set of all charts compatible with $\mathfrak{U}$. Construct the set $\mathfrak{M} = \mathfrak{U} \cup S$. We claim that this set is a maximal atlas. Let (W, ψ) be a chart compatible with $\mathfrak{M}$. Then it should also be compatible with $\mathfrak{U}$, thus it is contained in the set S, i.e. $(W, \psi) \in S$, thus in $\mathfrak{M}$. This shows that the atlas $\mathfrak{M}$ is maximal.

To show the uniqueness, let $\mathfrak{M}'$ be another maximal atlas. Since $\mathfrak{M}'$ is compatible with $\mathfrak{U}$, thus it is also compatible with $\mathfrak{M}$. But due to the construction it is contained in the new atlas, thus $\mathfrak{M}' \subset \mathfrak{M}$. Thus the maximal atlas is unique. $\qquad\square$

Considering the proposition above, we arrive at the following important observation.

> **Observation 2.1.1 — Showing a space is an smooth manifold.** To show that a space is an smooth manifold, we just need to show that the space is
>
> - a Hausdorff topological space that is also second countable,
>
> - there exists any C^∞ atlas (not necessarily maximal).

2.2 Smooth maps on a manifold

In this section, we will study the notion of smooth maps between manifolds. We will use the notion of coordinate charts to transfer the notion of smooth maps from Euclidean spaces to manifolds. We start with the functions on manifolds.

> **Definition 2.9 — Smooth functions on manifolds.** Let $f : M \to R$ be a function on a manifold. The function f is smooth at point $p \in M$ if there exists a chart (U, φ) such that $p \in U$ and the map
> $$f \circ \varphi^{-1} \ : \ \varphi(U) \to \mathbb{R}$$
> is smooth at p. The function f is said to be C^∞ on M if it is C^∞ at every point of M.

■ **Remark 2.3** Note that the smoothness of the function f in the definition above is independent of the local chart (U, ψ) that we choose. Let (V, ψ) be another local chart containing the point p. Then the map $f \circ \psi^{-1}$ is also smooth at p. Because

$$f \circ \psi^{-1} : (f \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1}).$$

We know that the map $(\varphi \circ \psi^{-1}$ is smooth at p (since the local charts in an atlas are compatible). Also $(f \circ \varphi^{-1})$ is smooth (we have assumed so). Thus $f \circ \psi^{-1}$ is smooth.

> **Proposition 2.2 — Smoothness of real valued functions.** Let M be a manifold of dimension n with atlas $\mathfrak{U}$, and let $f : M \to \mathbb{R}$ a real-valued function on M. The following are equivalent:
>
> (i) The function $f : M \to \mathbb{R}$ is C^∞.
>
> (ii) The manifold M has an atlas such that for every chart (U, φ) in the atlas, $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is C^∞.
>
> (iii) For every chart (V, ψ) on M, the function $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}$ is C^∞.

Proof. We prove a cyclic chain of implications $(ii) \implies (i) \implies (iii) \implies (ii)$.
$(ii) \implies (i)$: Let $p \in M$. From the assumption we know that there is an atlas with the desired property. Thus $\exists (U_\alpha, \varphi_\alpha)$ that contains p and the function $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is smooth. Thus $f : M \to \mathbb{R}$ is smooth at p, ans since the point p was arbitrary, then f is smooth on M.
$(i) \implies (iii)$: Let $p \in U$. From assumption we know that there is a coordinate open set U_α such that $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U) \to \mathbb{R}$ is smooth. Then we can write

$$f \circ \psi^{-1} = (f \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \psi^{-1})$$

which shows that $f \circ \psi^{-1}$ is smooth (look at the remark below).
$(iii) \implies (ii)$: This follows immediately from the definition. $\square$

■ **Remark 2.4** Considering the "very important note" at the beginning of this chapter, we need to emphasis again that in part (iii), when we say *for every chart (V, ψ) on M*, we mean a chart in the unique differentiable structure of the manifold that contains the atlas $\mathfrak{U}$.

> **Observation 2.2.1 — From W. Tu.** The smoothness conditions from the proposition above will be a recurrent motif through out the book. To prove the smoothness of an object, it is sufficient that a smoothness criterion hold on the charts of some atlas. Once the object is shown to be smooth, it then follows that the same smoothness criterion holds on every chart on the manifold.

> **Definition 2.10 — Pull back of a function.** Let $F : N \to M$ be a a map between manifolds, and $\varphi : M \to \mathbb{R}$. Then the pullback of the function φ by F is denoted by $F * \varphi$ is defined as
>
> $$F^* \varphi = \varphi \circ F.$$

■ **Remark 2.5** In this terminology, a function $f : M \to \mathbb{R}$ is smooth on a chart (U, φ) if and only if the pullback $(\varphi^{-1})^* f$ is smooth on the subset $\varphi(U)$ of Euclidean space.

## 2.2.1 Smooth maps between manifolds

Here in this section we will discuss the smooth maps between manifolds. We will be able to recover the definition of the smooth functions on manifolds as a special case.

Definition 2.11 — Smooth maps between manifolds. Let M, N be manifolds with dimensions m, n respectively. A map $F : M \to N$ is said to be smooth at point $p \in M$, if there exists charts (U, φ) and (V, ψ) such that $p \in U$ and $F(p) \in V$, and the function

$$\psi \circ F \circ \varphi^{-1} : \mathbb{R}^m \supset \varphi(F^{-1}(V) \cup U) \to \mathbb{R}^n$$

is smooth at p.

■ **Remark 2.6** The following figure helps to digest the definition above.
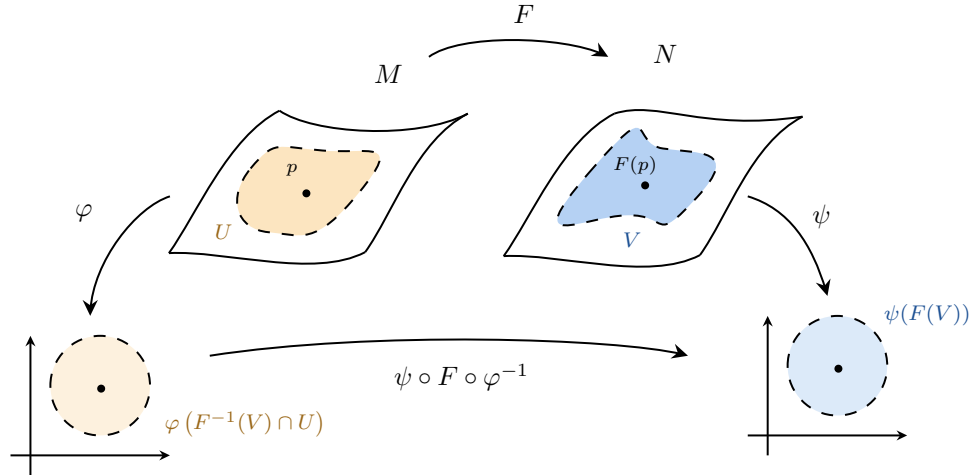
■ **Remark 2.7** Since the Euclidean space is indeed a smooth manifolds, then we can recover the definition of smooth functions on manifolds form the definition above. Let M be a manifold, and $N = \mathbb{R}^n$ with the atlas $\{(\mathbb{R}^n, \mathbb{1} : \mathbb{R}^n \to \mathbb{R}^n)\}$, then we will have the notion of vector valued functions on manifold. By setting $n = 1$ we will recover the definition of smooth function on manifold.

In the following proposition we will be showing that the smoothness of the map is independent of the charts chosen, thus the smoothness of the map is well-defined.

Proposition 2.3 — Smoothness of maps between manifolds is well-defined. Suppose $F : N \to M$ is C^∞ at $p \in N$. If (U, φ) is any chart in N that contains p and (V, ψ) is any chart in M that contains $F(p)$, then $\psi \circ F \circ \varphi^{-1}$ is smooth at $\varphi(p)$.

Proof. newEnv   □

2.3 Solved Problems

■ **Problem 2.1 — A C^∞ atlas on a circle (From W. Tu).** construct a C^∞ atlas for the unit circle S^1.

 Solution   The unit circle C^1 can be described as a set of points $S^1 = \{e^{it}|t \in [0, 2\pi]\}$. Let U_1 and U_2 be two subsets of S^1 described as

$$U_1 = \{e^{it}|t \in (-\pi, \pi)\}, \qquad U_2 = \{e^{it}|t \in (0, 2\pi)\}.$$

Consider the functions $\varphi_\alpha : U_\alpha \to \mathbb{R}$ for $\alpha = 1, 2$ given by

$$\varphi_1(e^{it}) = t, \qquad -\pi < t < \pi,$$
$$\varphi_2(e^{it}) = t, \qquad 0 < t < 2\pi.$$

These functions are in fact different branches of the complex logarithm function $1/i \log(z)$, thus homeomorphisms onto their respective images. Thus $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ is an atlas for S^1. To demonstrate the compatibility of these charts, we need to first calculate $U_1 \cap U_2$. This set has two connected components, i.e. $U_1 \cap U_2 = A \sqcup B$ where $\sqcup$ is used to demonstrate the disjoint union of A, B. Explicitly, we can write
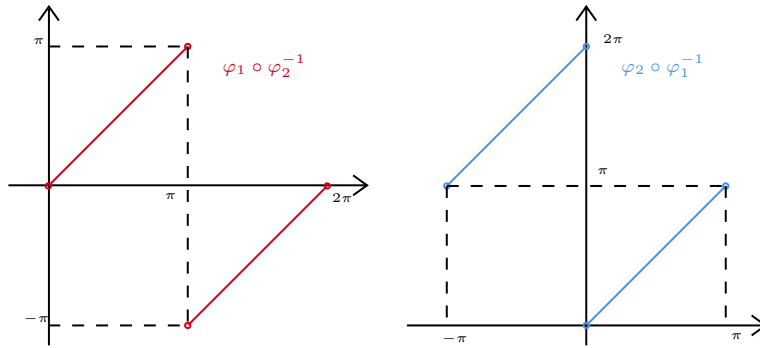
$$A = \{e^{it}|t \in (-\pi, 0)\}, \qquad B = \{e^{it}|t \in (0, \pi)\}.$$

First, we start with the function $\varphi_1 \circ \varphi_2^{-1} : \underbrace{\varphi_2(U_1 \cap U_2)}_{(0,\pi)\sqcup(\pi,2\pi)} \to \underbrace{\varphi_1(U_1 \cap U_2)}_{(-\pi,0)\sqcup(0,\pi)}$. For this function we have

$$(\varphi_1 \circ \varphi_2^{-1})(t) = \begin{cases} t & t \in (0, \pi), \\ t - 2\pi & t \in (\pi, 2\pi). \end{cases}$$

Similarly, for $\varphi_2 \circ \varphi_1^{-1} : \underbrace{\varphi_1(U_1 \cap U_2)}_{(-\pi,0)\sqcup(0,\pi)} \to \underbrace{\varphi_2(U_1 \cap U_2)}_{(0,\pi)\sqcup(\pi,2\pi)}$ we can write

$$(\varphi_2 \circ \varphi_1^{-1})(t) = \begin{cases} t + 2\pi & t \in (0, \pi), \\ t & t \in (\pi, 2\pi). \end{cases}$$

Observation 2.3.1 I was thinking about my solution to the problem above, and I thought it is wrong, as I was thinking that the function φ_1 is not homeomorphism as it is not continuous. But the point that I was missing is that this function is indeed continuous on its domain and

the point of discontinuity (i.e. $x = \pi$) is not in the domain.

■ **Problem 2.2** Another C^∞ atlas on a circle In the previous problem, we constructed an atlas for a unit circle siting in the complex plane. In this problem we are going to construct a different atlas for a unit circle siting in the $x - y$ plane. The following diagram are the charts for this unit circle. Write these charts explicitly and check if they are pairwise compatible.
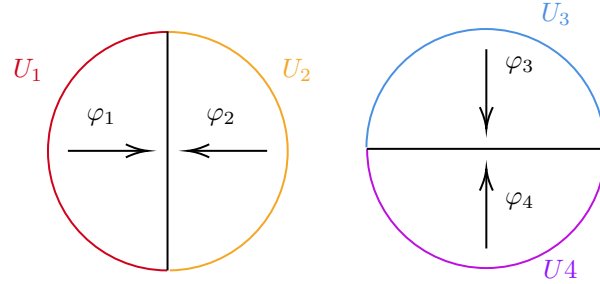
Solution The explicit formulas for the charts depicted above is as following

$$(U_1, \varphi_1 : U_1 \to \mathbb{R}), (U_2, \varphi_2 : U_2 \to \mathbb{R}), (U_3, \varphi_3 : U_3 \to \mathbb{R}), (U_4, \varphi_4 : U_4 \to \mathbb{R}),$$

where we have

$$\varphi_1(x, y) = y, \qquad \varphi_2(x, y) = y, \qquad \varphi_3(x, y) = x, \qquad \varphi_4(x, y) = x.$$

Note that although some of the functions above might have a same formula, but they are different functions as they have different domains. To show that these functions are pairwise compatible, we start by noting that since $U_1 \cap U_2 = \varnothing$, thus (U_1, φ_1) and (U_2, φ_2) are compatible. With the same reasoning, the charts (U_3, φ_3) and (U_4, φ_4) are compatible. Now, we want to show that (U_1, φ_1) is compatible with (U_3, φ_3). We need to show that

$$\varphi_1 \circ \varphi_3^{-1} : \underbrace{\varphi_3(U_1 \cap U_3)}_{(-1,0)} \to \underbrace{\varphi_1(U_1 \cap U_3)}_{(0,1)} \quad \text{and} \quad \varphi_3 \circ \varphi_1^{-1} : \underbrace{\varphi_1(U_1 \cap U_3)}_{(0,1)} \to \underbrace{\varphi_3(U_1 \cap U_3)}_{(-1,0)}$$

are C^∞. To write them explicitly, we have

$$(\varphi_1 \circ \varphi_3^{-1})(x) = \varphi_1(x, \sqrt{1 - x^2}) = \sqrt{1 - x^2}.$$

Also

$$(\varphi_3 \circ \varphi_1^{-1})(x) = \varphi_3(-\sqrt{1 - x^2}, x) = -\sqrt{1 - x^2}.$$

We can see that both of these functions are C^∞ in their domain. Now, for the charts (U_1, φ_1) and (U_4, φ_4) need to show

$$\varphi_1 \circ \varphi_4^{-1} : \underbrace{\varphi_4(U_1 \cap U_4)}_{(-1,0)} \to \underbrace{\varphi_1(U_1 \cap U_4)}_{(-1,0)} \quad \text{and} \quad \varphi_4 \circ \varphi_1^{-1} : \underbrace{\varphi_1(U_1 \cap U_4)}_{(-1,0)} \to \underbrace{\varphi_4(U_1 \cap U_4)}_{(-1,0)}$$

are C^∞ in their domain. Explicitly, we have

$$(\varphi_1 \circ \varphi_4^{-1})(x) = -\sqrt{1 - x^2}, \qquad (\varphi_4 \circ \varphi_1^{-1})(x) = -\sqrt{1 - x^2}.$$

With the same strategy, we can show that this collection of charts indeed makes a C^∞ atlas for the unit circle in $x - y$ plane.

■ **Problem 2.3 — The real line with two origins (from W. Tu).** Let A and B be two points not on the real line $\mathbb{R}$. Consider the set $S = (\mathbb{R} - \{0\}) \cup \{A, B\}$. For nay two positive real numbers c, d, define

$$I_A(-c, d) = (-c, 0) \cup \{A\} \cap (0, d)$$

and similarly for $I_B(-c, d)$, with B instead of A. Define a topology on S as follows: On $(\mathbb{R} - \{0\})$, use the subspace topology inherited from $\mathbb{R}$, with open intervals as a basis. A basis of neighborhoods at A is the set $\{I_A(-c, d) \mid c, d > 0\}$; similarly, a basis of neighborhoods at B is $\{I_B(-c, d) \mid c, d > 0\}$.

(a) Prove that the map $h : I_A(-c, d) \to (-c, d)$ defined by

$$h(x) = x \qquad \text{for } x \in (-c, 0) \cup (0, d),$$
$$h(A) = 0$$

is a homeomorphism.

(b) Show that S is locally Euclidean and second countable, but not Hausdorff.

Solution   (a) We need to show that h is one-to-one, onto, and continuous, with continuous inverse. To show being one-to-one, let $x, y \in I_A(-c, d)$ such that $x \neq y$ and possibly one of them equal to A. If none is equal to A, then h is the identity map which is one-to-one. However, if one of them is equal to A, let's say $x = A$, then $h(x) = 0$ where $h(y) \in (-c, d) \cup (0, d)$, thus $h(y) \neq 0$. This proves that h is indeed one to one. To show that the function is onto, let $z \in (-c, d)$. Then if $z = 0$ we have $h(A) = z$, and if $z \neq 0$, we have $h(z) = z$. Thus the function h is a bijection.

As the second step, we need to show that this function is continuous with continuous inverse. From definition of continuoity, we just need to show that both h and h^{-1} maps opens to opens (because for continuous function the pre-image of every open set is an open set; and to show that the inverse of the function is also continuous we need to show that the image of every open is also open). Let $U = (a, b) \subset I_A(-c, d)$ be an open set in the topology of S. If $A \notin (a, b)$, then the image of this set under h is (a, b) which is open in $(-c, d)$. But if $A \in (a, b)$, then the image of this set under the map h is $(a, 0) \cup (0, b) \cup \{0\} = (a, b)$, which is also open. Thus h^{-1} is continuous. To show the continuity of h^{-1}, let $(a, b) \subset (c, d)$. If $0 \notin (a, b)$, then pre-image of this set under the map f is (a, b) that is open in S. However if $0 \in (a, b)$, then the pre-image of this set under f is the set $(-a, 0) \cup \{A\} \cup (0, b)$ which is indeed open in S as we can construct this with the basis if opens at A.

(b) First, we show that S is locally Euclidean. To show this let $p \in S$. If $p \neq A$ and $p \neq B$, then we choose an open set $U = (a, b)$ containing p that does not contain neither of A and B. Then U is homeomorphic to (a, b) with the identity map. However if $p = A$, we choose any open $U = (a, b)$ containing p. Then $(U = (a, b), I_A(a, b))$ is a local chart. For the case where $p = B$, we can find a suitable chart with the same reasoning as for A. Thus we have shown that S is locally Euclidean.

To show that S is second countable, let $\mathcal{B}$ be a basis for $\mathbb{R} - \{0\}$. Since $\mathbb{R}$ is second countable, then $\mathcal{B}$ is countable. Let $\mathbb{B} = \mathcal{B} \cup I_A(-c, d) \cup I_B(-c, d)$ is a countable basis for S for some $c, d > 0$. This shows that S is also second countable.

However, this space is not Hausdorff. To show this consider the points A, B. We can not find any two open U, V such that $A \in U$ and $B \in V$ and we have $U \cap V = \varnothing$. Or equivalently, for all open sets U, V such that $A \in U$ and $B \in V$ we have $U \cap V \neq \varnothing$. That is because from the basis of the open neighborhoods at A, B we have $U = I_A(-c, d)$ and $V = I_B(-e, f)$ for some $c, d, e, f > 0$. It is clear that $U \cap V \neq 0$, thus S is not Hausdorff.

■ **Problem 2.4 — A sphere with a hair (from W. Tu)** . A fundamental theorem of topology, the theorem on invariance of dimension, states that if two nonempty open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are homeomorphic, then $n = m$. Prove that the sphere with a hair in $\mathbb{R}^3$ is not locally Euclidean at q (the point that the hair attaches to the sphere). Hence it cannot be a topological manifold.

Solution We will proceed with the proof by contradiction. Assume that the ball with hair at point q is homeomorphic to some open set in $\mathbb{R}^n$. Then q has an open neighborhood U homeomorphic to an open ball $B := \mathbb{B}(0, \epsilon) \in \mathbb{R}^n$, with q mapping to 0. We can restrict this homeomorphism to a homeomorphism $U - \{q\} \to B - \{0\}$. Now $B - \{0\}$ is either connected if $n \geq 2$ or has two connected components if $n = 1$, where $U - \{q\}$ has two connected components where one component is a 1 dimensional manifold (the hair) and the other component is a 2 dimensional manifold (the sphere). The only case where $B - \{0\}$ has two connected components is when $n = 1$. But because of the invariance of dimension principle, the sphere (2 dimensional manifold) can not be homeomorphic to a one dimensional manifold. Thus there is no such a homeomorphism between the hairy ball and $\mathbb{R}^n$ and the hairy ball is not locally Euclidean in q.

■ **Problem 2.5 — Charts on a sphere (from W. Tu).** Let S^2 be the unit sphere, i.e.

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1\}$$

in $\mathbb{R}^3$. Define in S^2 the six charts corresponding to the six hemispheres (from the front, rear, right, left, upper, and lower hemispheres) as in the figure.

$$
\begin{aligned}
U_1 &= \{(x, y, z) \in S^2 \mid x > 0\}, & \varphi_1(x, y, z) &= (y, z), \\
U_2 &= \{(x, y, z) \in S^2 \mid x < 0\}, & \varphi_2(x, y, z) &= (y, z), \\
U_3 &= \{(x, y, z) \in S^2 \mid y > 0\}, & \varphi_3(x, y, z) &= (x, z), \\
U_4 &= \{(x, y, z) \in S^2 \mid y < 0\}, & \varphi_4(x, y, z) &= (x, z), \\
U_5 &= \{(x, y, z) \in S^2 \mid z > 0\}, & \varphi_5(x, y, z) &= (x, y), \\
U_6 &= \{(x, y, z) \in S^2 \mid z < 0\}, & \varphi_6(x, y, z) &= (x, y).
\end{aligned}
$$

Note that although some of the functions above might look similar (like φ_3 and φ_4) but they are in fact different functions as they have different domains. Show that $\varphi_1 \circ \varphi_4^{-1}, \varphi_4 \circ \varphi_1^{-1}$ is C^∞ on $\varphi_4(U_1 \cap U_4), \varphi_1(U_1 \cap U_4)$ respectively. Do the same same analysis for $\varphi_6 \circ \varphi_1^{-1}$.
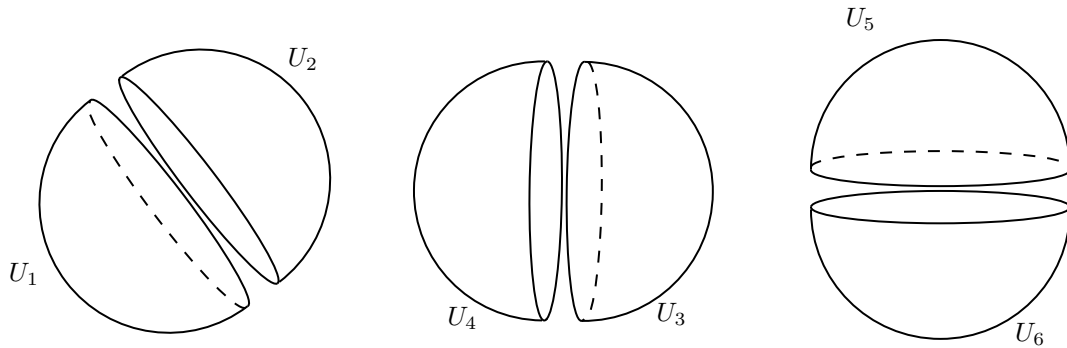
Solution For the set $U_1 \cap U_4$ we have

$$U_1 \cap U_4 = \{(x, y, z) \in S^2 \ : \ x > 0 \text{ and } y < 0\}.$$

Thus we will have

$$\varphi_4(U_1 \cap U_4) = \{(x, z) \in \mathbb{R}^2 \ : \ x^2 + z^2 \leq 1, \ x > 0\}.$$

Thus we can write

$$(\varphi_1 \circ \varphi_4^{-1})(\langle x, z \rangle) = \varphi_1(\langle x, -\sqrt{1 - (x^2 + y^2)}, z \rangle) = \langle -\sqrt{1 - (x^2 + y^2)}, z \rangle$$

This is indeed a C^∞ vector valued function, since each component is a C^∞ function. Now to evaluate $\varphi_4 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_4) \to \varphi_4(U_1 \cap U_4)$ we need to first evaluate the set $\varphi_1(U_1 \cap U_4)$. For this set we have

$$\varphi_1(U_1 \cap U_4) = \{(z, y) \in \mathbb{R}^2 \ : \ z^2 + y^2 \leq 1, y < 0\}.$$

Then we can write

$$(\varphi_4 \circ \varphi_1^{-1})(\langle y, z \rangle) = \varphi_4(\langle \sqrt{1 - (y^2 + z^2)}, y, z \rangle) = \langle \sqrt{1 - (y^2 + z^2)}, z \rangle.$$

This is indeed a C^∞ vector valued function.

To evaluate the function $\varphi_6 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_6) \to \varphi_6(U_1 \cap U_6)$, we first need to determine the domain of this function. First observe that

$$U_1 \cap U_6 = \{(x, y, z) \in S^2 \ : \ x > 0 \text{ and } y < 0\},$$

which is the same as $U_1 \cap U_4$. Then for the domain of the function of interest we can write

$$\varphi_1(U_1 \cap U_6) = \{(y, z) \in \mathbb{R}^2 : \ z^2 + y^2 \leq 1 \text{ and } y < 0\},$$

$$(\varphi_6 \circ \varphi_1^{-1})(\langle y, z \rangle) = \varphi_6(\langle \sqrt{1 - (y^2 + z^2)}, y, z \rangle) = \langle \sqrt{1 - (y^2 + z^2)}, y \rangle.$$

■ **Problem 2.6 — Existence of a coordinate neighborhood (from W. Tu).** Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be the maximal atlas on manifold M. For any open set U in M and a point $p \in U$, prove the existence of a coordinate open set U_α such that $p \in U_\alpha \subset U$.

Solution Since $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ is an atlas, then for $p \in U$ given as above, we can find some $\alpha_1 \in I$ such that $p \in U_{\alpha_1}$. Consider the open set $W = U_{\alpha_1} \cap U$. The chart $(W, \varphi_{\alpha_1}|_W)$ is in the atlas (since it is maximal), i.e. $\exists \alpha \in I$ such that $(U_\alpha, \varphi_\alpha) = (W, \varphi_{\alpha_1}|_W)$. This completes the proof.

■ **Problem 2.7 — An atlas for a product manifold (from W. Tu).** Prove the following proposition.

> **Proposition 2.4** If $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ are C^∞ atlases for the manifold M and N of dimensions m and n, respectively, then the collection
>
> $$\{(U_\alpha \times V_i, \varphi_\alpha \times \psi_i)\}$$
>
> where
>
> $$\varphi_\alpha \times \psi_i \ : \ U_\alpha \times V_i \to \mathbb{R}^m \times \mathbb{R}^n$$
>
> of charts is a C^∞ atlas on $M \times N$. Therefore, $M \times N$ is a C^∞ manifold of dimension $m + n$.

Solution   In order to show that the collection $\mathfrak{U} = \{(U_\alpha \times V_i, \varphi_\alpha \times \psi_i)\}$ is an atlas for $M \times N$, we need to show that charts are pairwise compatible as well as the set covering the whole space. To show that the chart covers the whole space $M \times N$, let $(p_1, p_2) \in M \times N$. Then $p_1 \in M$ and $p_2 \in N$. Then there are two coordinate open sets such that $p_1 \in U_{\alpha_1}$ and $p_2 \in V_{i_1}$. Thus the coordinate open set $U_{\alpha_1} \times V_{i_1}$ contains the point (p_1, p_2).

To show that two any two charts in the collection $\mathfrak{U}$ are compatible, let $(U_{\alpha_1} \times V_{i_1}, \varphi_{\alpha_1} \times \psi_{i_1})$ and $(U_{\alpha_2} \times V_{i_2}, \varphi_{\alpha_2} \times \psi_{i_2})$ be two charts. We claim that the corresponding coordinate maps are C^∞. This follows directly from the fact the each component of the map these coordinate maps are C^∞.