

Lecture Notes For : Advanced Linear Algebra

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Chapitre 1

Matrices

1.1 What is a Matrix

A matrix is basically a notation convention that enables us to do some stuff more easily with a pencil and paper. A very similar concept to this is the long division algorithm for dividing two integers. For example consider the following long division (in French-European style) that we are all familiar with

$$\begin{array}{r|l} 198 & 12 \\ -12 & 16,5 \\ \hline 78 & \\ -72 & \\ \hline 60 & \\ -60 & \\ \hline 0 & \end{array}$$

So this notation and algorithms is to use some calculations more convenient when is done by hand with a pen and paper. So the matrix notation can also be thought as a computation convention. To make stuff more clear, consider the following example.

Example : Simple Pen and Paper Calculations

Consider V which is written as :

$$V = 2A + 3B + 4C$$

Given the following relation between A, B , and C , rewrite V in terms of x, y , and z .

$$A = x + 2y + 3z$$

$$B = 2x - y + z$$

$$C = -x - y + z$$

Solution 1.

To write V in terms of x, y , and z we write :

$$V = 2(x + 2y + 3z) + 3(2x - y + z) + 4(-x - y + z) \quad (1.1.1)$$

By arranging the terms using simple algebra we will have :

$$V = (2 + 6 - 4)x + (4 - 3 - 4)y + (6 + 3 + 4)z = 4x - 3y + 13z \quad (1.1.2)$$

Solution 2.

The calculations described in the first solution are not systematic. What I mean is that we started doing whatever we can do with you thinking about doing it in a more smart way that can also be systematically scaled to larger equations. This is where the matrices come into play. Matrices help us to do such calculations in a more algorithmic way (like the long division notation in which we do the calculations in a algorithmic way).

Let \mathbb{B} be the set of all *objects* that the V is expanded in terms of and call this set as the *basis* set. So for $\mathbf{V} = 2A + 3B + 4C$ we have the basis

$$\mathbb{B}_1 = \{A, B, C\}.$$

We can arrange the coefficients of V in basis \mathbb{B}_1 in the following way :

$$V_{\mathbb{B}_1} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}_{\mathbb{B}_1}$$

We call it the coordinates of V in the basis \mathbb{B}_1 . Since we want to write the vector V in terms of x, y , and z , we need to introduce the new basis \mathbb{B}_2 in the following way :

$$\mathbb{B}_2 = \{x, y, z\}$$

Since A, B , and C are expressed in terms of x, y , and z , we can arrange the coordinates of A, B , and C in the basis \mathbb{B}_2 in the following way :

$$L_{\mathbb{B}_1}^{\mathbb{B}_2} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2}$$

in which every column is the coefficients A, B , and C in the basis \mathbb{B}_2 respectively. Note the subscript and the superscripts of the matrix. This matrix means that its columns contains the coordinates of the basis \mathbb{B}_1 in the new basis \mathbb{B}_2 . So when it is applied to any vector that is described in basis \mathbb{B}_1 , we will get the components of that vector in the basis \mathbb{B}_2 . In other words :

$$V_{\mathbb{B}_2} = L_{\mathbb{B}_1}^{\mathbb{B}_2} V_{\mathbb{B}_1}$$

$$V_{\mathbb{B}_2} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}_{\mathbb{B}_1}$$

This matrix equation can be written in two ways :

$$V_{\mathbb{B}_2} = 2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_{\mathbb{B}_2} + 3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}_{\mathbb{B}_2} + 4 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}_{\mathbb{B}_2} \quad (1.1.3)$$

The equation above is equivalent to