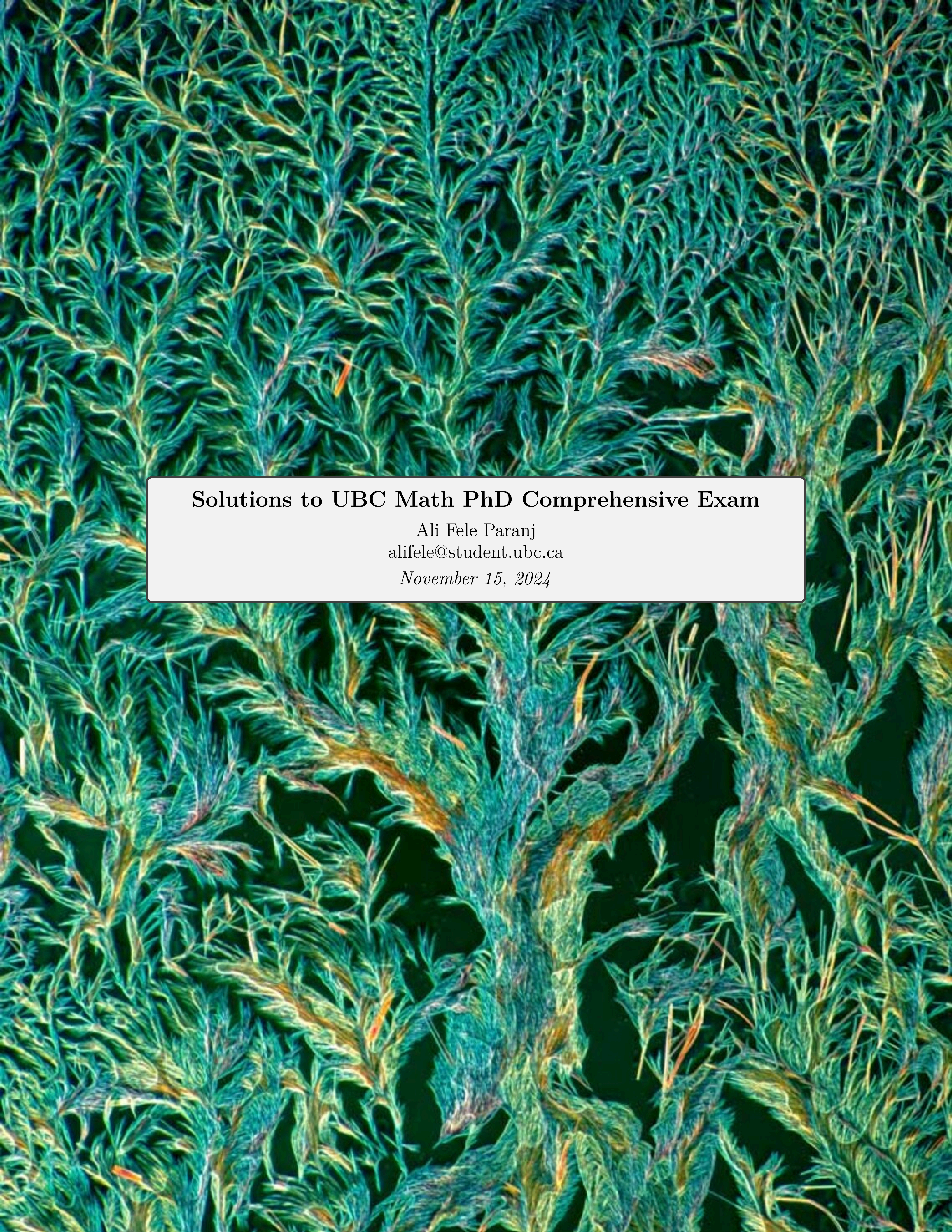

Solutions to UBC Math PhD Comprehensive Exam

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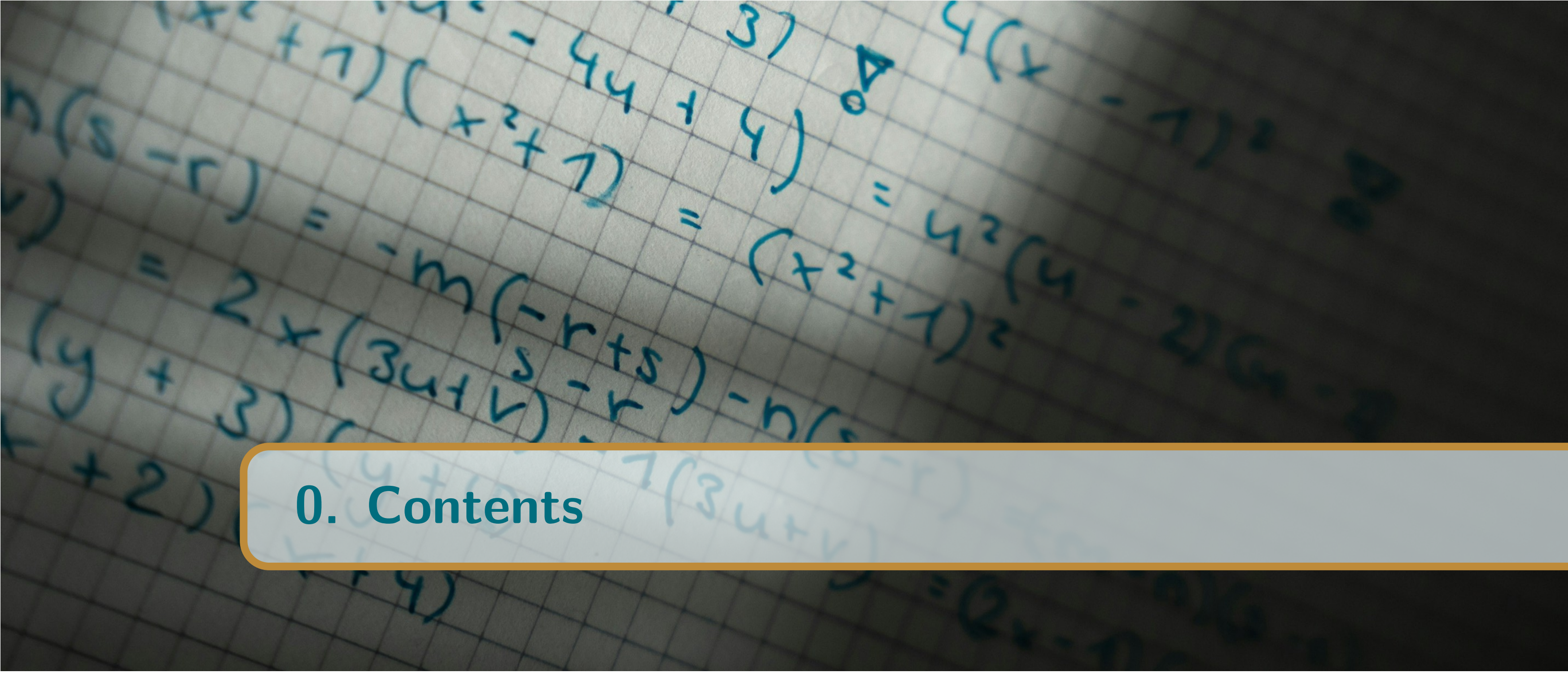
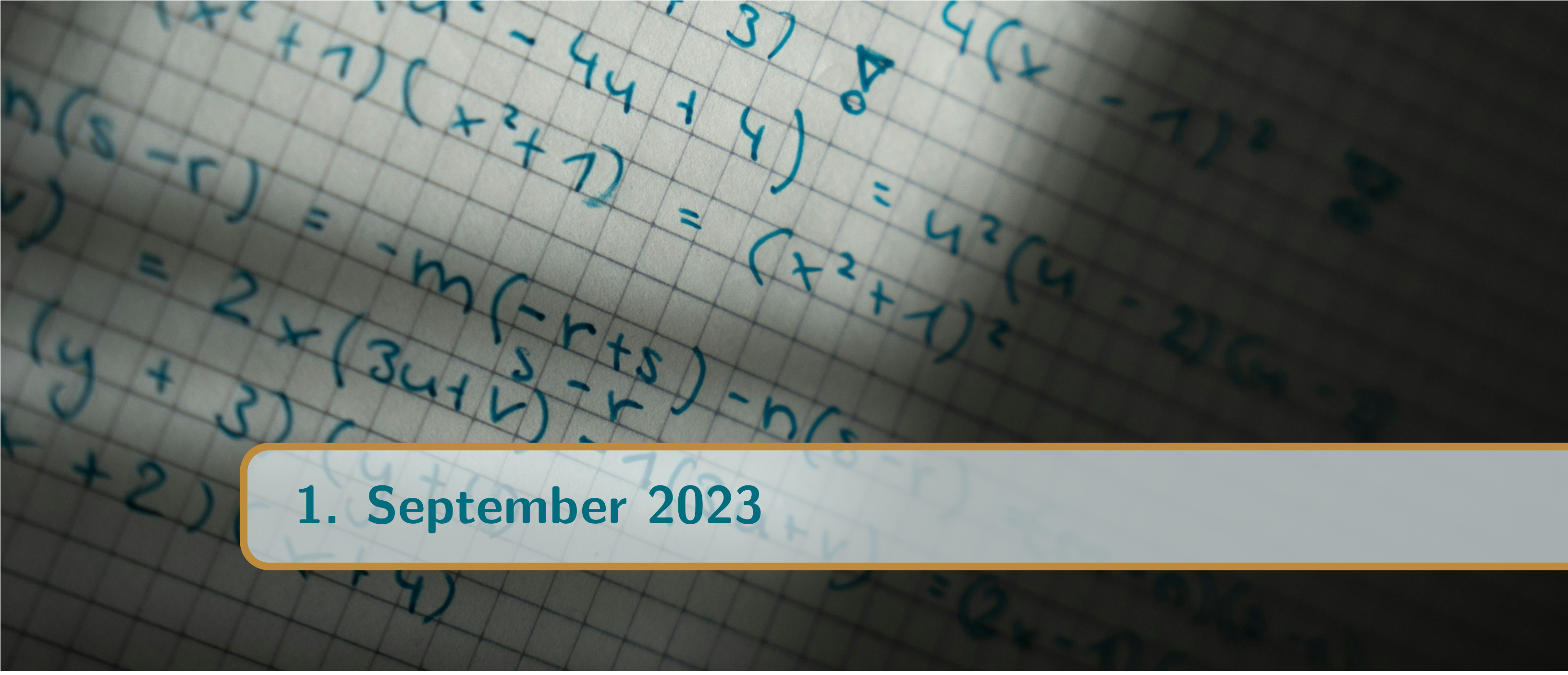


0. Contents



1. September 2023

1.1 Algebra

Problem 1.1 (a) Prove that any module over $\mathbb{Z}[i]$ is a direct sum of a free module and a torsion module. Is the same true for modules over $\mathbb{Z}[\sqrt{-5}]$?

(b) Let $I = 3 + 2i$ be the ideal in $\mathbb{Z}[i]$ generated by the element $3 + 2i$. Describe the quotient $\mathbb{Z}[i]/I$.

1.2 Linear Algebra

Problem 1.2 (a) Find a lower triangular matrix L such that $LL^T = A$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 5 & 2 & 2 \\ 0 & 2 & 2 & 1 \\ 2 & 2 & 1 & 6 \end{pmatrix}$$

(b) Compute $\det(A)$.

(c) Find the volume in $\mathbb{R}^4$ of the set $S_A = \{x \in \mathbb{R}^4 : x^T A x \leq 1\}$.

Solution (a) Let L be

$$L = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix}$$

So $A = LL^T$ we will have

$$A = \begin{pmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} & l_{11}l_{41} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} & l_{21}l_{41} + l_{22}l_{42} \\ l_{11}l_{31} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 & l_{31}l_{41} + l_{32}l_{42} + l_{33}l_{43} \\ l_{41}l_{11} & l_{41}l_{21} + l_{42}l_{22} & l_{41}l_{31} + l_{42}l_{32} + l_{43}l_{33} & l_{41}^2 + l_{42}^2 + l_{43}^2 + l_{44}^2 \end{pmatrix}$$

By solving the equations formed by the first row we have

$$l_{11} = 1,\ l_{21} = 1,\ l_{31} = 0,\ l_{41} = 2.$$

Furthermore

$$l_{22} = 2,\ l_{32} = 1,\ l_{33} = 1,$$

and lastly

$$l_{42} = 0, l_{43} = 1, l_{44} = 1.$$

So the matrix L will be

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

(b) We will expand relative to the last column as it has more zeros.

$$\det(L) = \det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Again, expanding relative to the last column we will have

$$\det(L) = \det \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = 2.$$

(c) First, we find a suitable linear transformation that can map the volume to another volume for which we know how to calculate its volume. Consider the linear transformation $x = L^{-T}y$. Since $x^T Ax = (y^T L^{-1})LL^T(L^{-T}x) = y^T y$, thus the set $x^T Ax \leq 1$ will be mapped to the 4-ball (4 dimensional ball). So its volume will be

$$|x^T Ax \leq 1| = \det(L)|y^T y \leq 1| = 2|S^4|.$$

(Note: The volume of the 4-ball is $\pi^2/2$.)

■ **Problem 1.3** Let P_n be the $n+1$-dimensional space of polynomials of degree n with real coefficients, and let $< \cdot, \cdot >$ be the inner product defined as

$$< p, q >= \int_{-1}^{1} p(x)q(x)\ dx.$$

(a) Find an orthogonal basis $\{u_0, u_1, u_2\}$ for P_2 such that $u_j \in P_j$, and $u_j(1) > 0$.

(b) Using the basis in part (a), express the operator $F[p] := \int_{-1}^{1} p(x)dx$ acting on P_2 as a 1×3 matrix.

(c) Using the basis in part (a), express the derivative operator $D[p] := \frac{d}{dx}p(x)$ as a 3×3 matrix.

Solution (a) We start with the Gram-Schmidt orthogonalization. Let $\{c, x, x^2\}$ be the basis vectors that we want to orthogonalize. Let $u_0 = c$. Then the first vector will be $\hat{u}_0 = u_0/\|u_0\| = 1/\sqrt{2}$. To find the second vector we have

$$u_1 = x- < x, \hat{u}_0 > \hat{u}_0 = x - \frac{1}{2}\int_{-1}^{1} x\ dx = x.$$

So the second normal vector will be $\hat{u}_1 = u_1/\|u_1\| = \sqrt{\frac{3}{2}}x$. To find the third vector we have

$$\begin{aligned} u_2 &= x^2 - (< x^2, \hat{u}_0 > \hat{u}_0 + < x^2, \hat{u}_1 > \hat{u}_1) \\ &= x^2 - (\frac{1}{2}\int_{-1}^{1} x^2 dx + \frac{3}{2}\int_{-1}^{1} x^3 dx) \\ &= x^2 - (1/3 + 0) = x^2 - 1/3. \end{aligned}$$

So the third normal vector will be $\hat{u}_2 = u_2/\|u_2\| = 3/2\sqrt{\frac{5}{2}}(x^2 - 1/3)$. However, since the question has not asked for normal basis vectors, for the following sections of the question we will use the following orthogonal (not orthonormal) basis vectors

$$u_1 = 1, \quad u_1 = x, \quad u_2 = x^2 - \frac{1}{3}.$$

(b) It is enough to see what is the effect of this operator on the basis vectors

$$F[u_0] = \int_{-1}^{1} 1 dx = 2, \quad F[u_1] = \int_{-1}^{1} x dx = 0, \quad F[u_2] = \int_{-1}^{1} (x^2 - 1/3) dx = 2/3 - 2/3 = 0.$$

So the matrix representation of this operator will be

$$F = \begin{pmatrix} 2 & 0 & 0 \end{pmatrix}.$$

(c) Similarly to the solution above,

$$D[u_0] = 0, \quad D[u_1] = 1, D[u_2] = 2x,$$

so the matrix representation will be

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

■ **Problem 1.4** Recall that an orthogonal projection matrix is a matrix P that satisfies

$$P^2 = P, \quad P = P^T.$$

Suppose P is an $n \times n$ projection matrix with $\text{rank}(P) = k$. In the following, I_m denotes the $m \times m$ identity matrix.