Lecture Notes For: Probability and Stochastic Process

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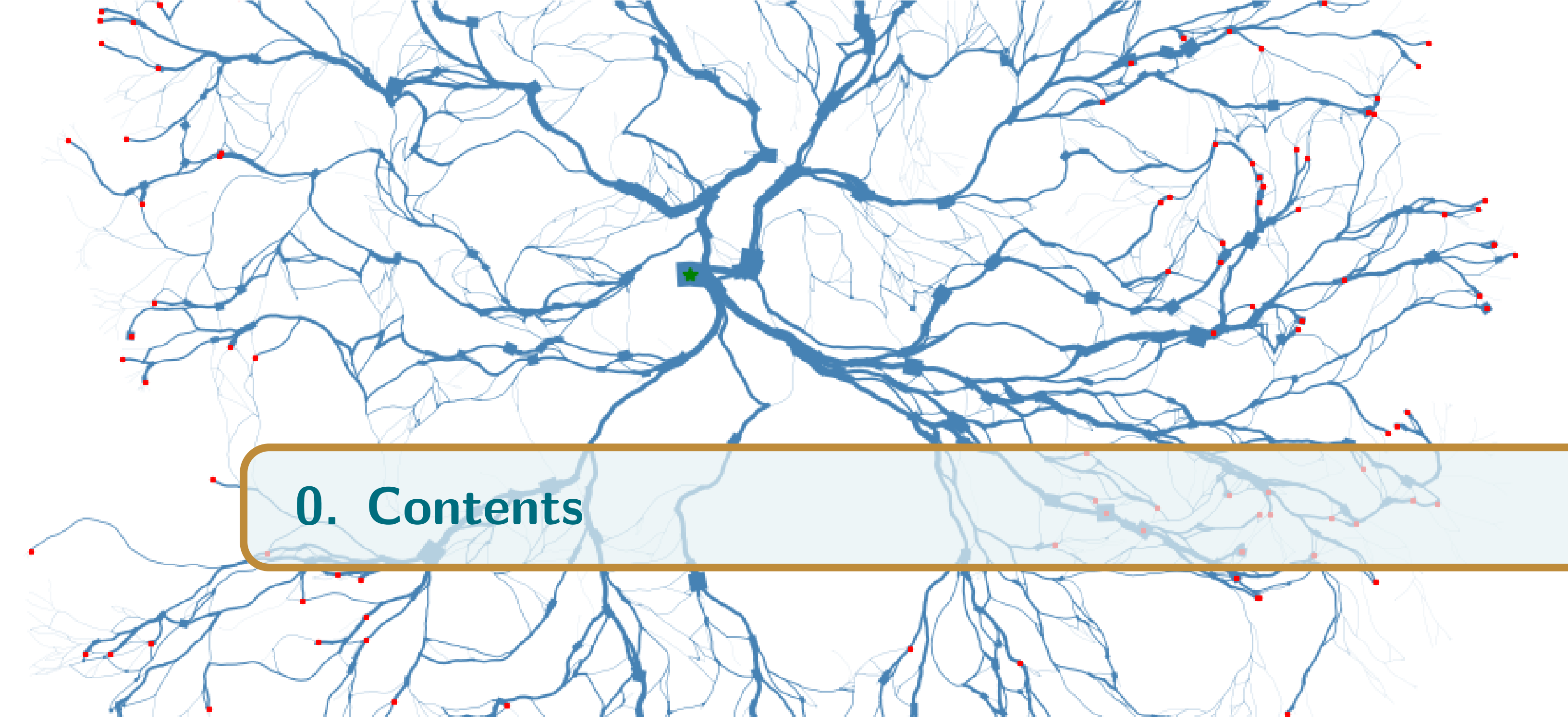

0. Contents

1. Conditional Expectation

Throughout this chapter the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \{a, b, c, d, e, f\}$, $\mathbb{F} = \mathcal{P}(\Omega)$, and $\mathbb{P}$ uniform will be running example to demonstrate different notions in a tangible way. The following random variables will be in particular useful.

$$X = \begin{pmatrix} a & b & c & d & e & f \\ 1 & 2 & 3 & 5 & 7 & 11 \end{pmatrix}, \quad Y = \begin{pmatrix} a & b & c & d & e & f \\ 1 & 1 & 4 & 4 & 6 & 6 \end{pmatrix}, \quad Z = \begin{pmatrix} a & b & c & d & e & f \\ 8 & 8 & 8 & 8 & 9 & 9 \end{pmatrix}.$$

It is also important to describe $\sigma(X), \sigma(Y)$, and $\sigma(Z)$ explicitly. The atoms of $\sigma(X)$ will be the $X^{-1}(1) = \{a\}, X^{-1}(2) = \{b\}, X^{-1}(3) = \{c\}, X^{-1}(5) = \{d\}, X^{-1}(7) = \{e\}, X^{-1}(11) = \{f\}$. Thus $\sigma(X) = \mathcal{P}(\Omega)$. With a similar argument the atoms of $\sigma(Y)$ is $Y^{-1}(4) = \{a, b\}, Y^{-1}(4) = \{c, d\}, Y^{-1}(6) = \{e, f\}$. And finally, the atoms of Z will be $Z^{-1}(8) = \{a, b, c, d\}$, and $Z^{-1}(9) = \{e, f\}$. In summary

$$\begin{aligned} \mathrm{Atom}(\sigma(X)) &= \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}\}, \\ \mathrm{Atom}(\sigma(Y)) &= \{\{a, b\}, \{c, d\}, \{e, f\}\}, \\ \mathrm{Atom}(\sigma(Z)) &= \{\{a, b, c, d\}, \{e, f\}\}. \end{aligned}$$

■ **Example 1.1 — from Nima Moshayedi's lecture notes.** Let $N \sim \mathrm{Poisson}(\lambda)$. Consider a game, where we say that when $N = n$ we do n independent tossing of a coin where each time one obtains 1 with probability $p \in [0, 1]$ and 0 with probability $1 - p$. Define also S to be the random variable giving the total number of 1 obtained in the game. Therefore, if $N = n$ is given, we get that S has binomial distribution with parameters (p, n). compute

(a) $\mathbb{E}\Big[S|X\Big]$.

(b) $\mathbb{E}\Big[X|S\Big]$.

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