Some thoughts on the stable matching problem

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September 12, 2025

1 Problem statement

Assume n applicants and n employers where each of the applicants and employers has their list of preference over the other group. And we want to match the applicants to the employers so that there is not blocking match. a_i matched to e_j is a blocking match if a_i matched to e' , then a_i will be happier with their new assignment, and so will be e'. And stable match is a matching that there is not blocking match.

Problem Statemnt 1. Let n be given. Let $P_A = (\sigma_i)_{i=1}^n$ and $P_E = (\tau_i)_{i=1}^n$ be the list of preferences of applicants and employers respectively, where $\sigma, \tau \in S_n$ be permutations. Then a matching $\mu \in S_n$ is stable if it does not have a blocking match.

Example 1. Let $n = 3$, and let

$$P_A = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad P_E = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$

where the corresponding permutations are shown as the rows of a matrix. In words, it is the same way of saying

$$\begin{aligned} a_1 &: e_3, e_1, e_2, \qquad & e_1 &: a_3, a_1, a_2 \\ a_2 &: e_2, e_1, e_3, \qquad & e_2 &: a_2, a_1, a_3 \\ a_3 &: e_1, e_3, e_2, \qquad & e_3 &: a_1, a_3, a_2 \end{aligned}$$

In this case, the matching $\sigma_A = (2, 3, 1)$, which says that a_1 chose e_2, a_3 chose e_3, and a_3 chose e_1. This is not stable matching. Because if we consider $(3, 2, 1)$, then a_1 gets e_3 (which was their preferred choice over e_2), and e_3 gets happier as well (because a_1 was their preferred choice over a_2).

As demonstrated in the example above, it gets quite complicated to argue if one matching is blocking or not. So instead, we can transform the given data of the problem into a notation that makes the argument easier.

Given $P_A = (\sigma_i)_i$ and $P_E = (\tau_j)_j$, consider

$$P_A^* = (\sigma_i^{-1})_i, \qquad P_H^* = (\tau_j^{-1})_j^T,$$

where what we really mean is P_A^* is a matrix that its i^{th} *row* is σ_i^{-1} and P_H^* is a matrix that its j^{th} *column* is τ_j^{-1}. For instance, for the example above

$$P_A^* = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{pmatrix} 2 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix} \end{array}$$

$$P_E^* = \begin{array}{c c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{pmatrix} 2 & 2 & 1 \\ 3 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix} \end{array}$$

The matrices P_A and P_E where merely a list of lists that were put on top of each other to look like a matrix, but P_A^* and P_E^* are qualified more to be called matrices. In words, $(P_A^*)_{ij}$ means the rank (order in the list) that the applicant a_i gave to the employer e_j, and $(P_E^*)_{ij}$ is the rank that e_j gave to applicant a_i. Now every possible solution, i.e. a permutation like in the example above, will be a selection of the elements of the matrices above in the following way: For instance $\sigma_A = (2, 3, 1)$ as one possible matching (which is unstable) in the question above, corresponds to

$$P_A^* = \begin{array}{c c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{pmatrix} 2 & \boxed{3} & 1 \\ 2 & 1 & \boxed{3} \\ \boxed{1} & 3 & 2 \end{pmatrix} \end{array}, \qquad P_E^* = \begin{array}{c c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{pmatrix} 2 & \boxed{2} & 1 \\ 3 & 1 & \boxed{3} \\ \boxed{1} & 3 & 2 \end{pmatrix} \end{array}$$