

The background of the slide is a complex fractal pattern. It features a large, bright yellow circle in the upper half, which is surrounded by a dark, intricate fractal border. A thin green vertical line runs through the center of the circle. Below the circle, the fractal pattern continues, with a smaller yellow circle visible in the lower half. The overall effect is a symmetrical, mathematical design.

Complex Analysis

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1. Introduction

1.1 Holomorphic Functions

We start with a definition of the holomorphic functions.

Definition 1.1 — Holomorphic functions. Let $f : \Omega \rightarrow \mathbb{C}$ be a functions defined on the open set $\Omega \subset \mathbb{C}$. Then f is holomorphic at $z_0 \in \Omega$ if

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} = 0,$$

where $h \in \mathbb{C}$.

■ **Remark 1.1** Although the definition above resembles the definition of derivative for the real functions, but it is much more stronger. Later, we will see that the holomorphic functions are infinitely times differentiable, which is not necessarily true for the real differentiable functions. Also, we will see that every holomorphic function admits a power series, which is not again necessarily true in the case of real functions. I.e. there are real functions that are infinitely many times differentiable, but can not be expressed as a convergent power series. More on these later.

There are some very useful characterization of the holomorphic functions that comes in handy in making intuitions and proving some theorems in a more straight forward way. First, we will discuss the following observation.

Observation 1.1.1 — A bijection between complex numbers and 2×2 matrices. Observer the following multiplication between two complex variables:

$$(a + ib)(x + iy) = (ax - by) + (bx + ay).$$

This suggests that we can make the following one-to-one correspondence between the 2×2 matrices and complex numbers

$$a + ib \longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

The observation above is key to make some intuition about different notions in the complex analysis. For instance, for every complex map, we can associate it with a real map from \mathbb{R}^2 to \mathbb{R}^2 .

I.e. let $f = u + iv$. Then we can associate this with a real map $F(x, y) = (u(x, y), v(x, y))$. On the other hand, the differential of a map from \mathbb{R}^2 to \mathbb{R}^2 is a 2×2 matrix. However, in the case of complex maps, the complex derivative of a map at a point is again a complex number. This suggests that for the map $F(x, y)$ where its Jacobian is given by

$$JF = \begin{bmatrix} \partial u / \partial x & \partial u / \partial y \\ \partial v / \partial x & \partial v / \partial y \end{bmatrix},$$

in order JF to represent a complex number, we need to have

$$\partial u / \partial x = \partial v / \partial y, \quad \partial u / \partial y = -\partial v / \partial x.$$

This is known as the Cauchy-Riemann equations. Note that this is not a correct way to derive the Cauchy-Riemann equations, but it is just an intuitive way to see that C-R equations makes everything work smoothly.

Theorem 1.1 — Cauchy-Riemann Equations. Let $f : \Omega \rightarrow \mathbb{C}$ be complex differentiable. If $f = u + iv$ then

$$\partial u / \partial x = \partial v / \partial y, \quad \partial u / \partial y = -\partial v / \partial x.$$

Proof. Since f is complex differentiable, then

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

In particular, it does so for $h = h_1 + ih_2$ going to zero by $h_1 \rightarrow 0, h_2 = 0$, as well as $h_1 = 0, h_2 \rightarrow 0$. Then we will have

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}.$$

This implies the Cauchy-Riemann equations. □

Definition 1.2 The following differentiation operations comes in handy for some applications.

$$\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \left(\frac{\partial}{\partial x} - \frac{1}{i} \frac{\partial}{\partial y} \right)$$

The following characterization of the holomorphic functions is very useful.

Proposition 1.1 — A useful characterization of C-R equations (i.e. Holomorphic functions). Let f be a complex map. Then C-R is equivalent to $\partial f / \partial \bar{z} = 0$.

Proof. Observe that

$$\frac{\partial f}{\partial \bar{z}} = \left(\frac{\partial}{\partial x} - \frac{1}{i} \frac{\partial}{\partial y} \right) f.$$

Let $f = u + iv$. Then it is easy to check that $\partial f / \partial \bar{z} = 0$ if and only if we have

$$\partial u / \partial x = \partial v / \partial y, \quad \partial u / \partial y = -\partial v / \partial x.$$

□

Corollary 1.1 A complex map f is holomorphic at z_0 in its domain if and only if we have

$$f(z) = f(z_0) + L(z - z_0) + |z - z_0| \Psi(z - z_0),$$

for some $L \in \mathbb{R}$ and $\Psi(z - z_0) \rightarrow 0$ as $z \rightarrow z_0$.

Proof. In general, for any complex map f we can write

$$f(z) = f(z_0) + \alpha(z - z_0) + \beta \overline{(z - z_0)} + \Psi(z - z_0)|z - z_0|.$$

(To see this why, first observe that we can write every complex map as a map from \mathbb{R}^2 to \mathbb{R}^2 , and then do the change of variable $x = (z + \bar{z})/2, y = (z - \bar{z})(2i)$). However, from the proposition above, we conclude that $\beta = 0$ for all z_0 in the domain. This completes the proof. \square

Proposition 1.2 Let f be a holomorphic function on an open set. Then

$$f'(z_0) = \frac{\partial f}{\partial z} = 2 \frac{\partial u}{\partial z}.$$

Since f is holomorphic at z_0 , then the limit

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

converges for any path on which $h = h_1 + ih_2$ approaches zero, in particular $(h_1 \rightarrow 0, h_2 = 0)$ and $(h_1 = 0, h_2 \rightarrow 0)$. Thus we conclude

$$f'(z) = \frac{1}{2} \left(\frac{\partial f}{\partial x}(z_0) + \frac{1}{i} \frac{\partial f}{\partial y}(z_0) \right).$$

(To see this modify the terms of the limit in the definition of the complex differentiation and use the paths above for appropriate limits and conclude the identity above). Thus we can write $f = u + iv$ and using the C-R equations (since f is holomorphic) we conclude that

$$f'(z) = 2 \frac{\partial u}{\partial z}.$$

The following theorem is an important converse-like statement for the Cauchy-Riemann theorem.

Theorem 1.2 Let $f : \Omega \rightarrow \mathbb{C}$ be a function. Assume $f = u + iv$. If u, v are continuously differentiable on Ω , and satisfy the C-R equations, then f is holomorphic on Ω

Proof. Since u, v are continuously differentiable, then

$$u(x + h_1, y + h_2) - u(x, y) = \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 + |h| \Psi_1(h),$$

and similarly

$$v(x + h_1, y + h_2) - v(x, y) = \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 + |h| \Psi_2(h).$$

Since $f = u + iv$ then

$$f(z + h) - f(z) = \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 + i \frac{\partial v}{\partial x} h_1 + i \frac{\partial v}{\partial y} h_2 + |h| \Psi(h)$$

. Using C-R we can write

$$f(z + h) - f(z) = \left(\frac{\partial u}{\partial x} + \frac{1}{i} \frac{\partial u}{\partial y} \right) (h_1 + ih_2) + |h| \Psi(h).$$

Since

$$\frac{\partial u}{\partial x} + \frac{1}{i} \frac{\partial u}{\partial y} = 2 \frac{d}{dz} u = \frac{d}{dz} f,$$

then we conclude that f is holomorphic (follows from the characterization of holomorphic functions). \square

1.2 Second Review Notes

These notes are from my second review of the chapter 2 of Stein Complex analysis.

A function $f : \Omega \rightarrow \mathbb{C}$ is holomorphic at $z_0 \in \Omega$ if

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

exists. This is equivalent to the existence of a complex number $a \in \mathbb{C}$ such that

$$f(z_0 + h) = f(z_0) + ha + h\psi(h)$$

where $\psi(h) \rightarrow 0$ as $h \rightarrow 0$. Alternatively, instead of $h\psi(h)$ we can write $o(h)$.

1.2.1 A Clear Discussion on the Holomorphic Functions

First, note the following association

$$(a+ib)(h_1+ih_2) \longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix},$$

as well as

$$f(x+iy) = u(x,y) + iv(x,y) \longleftrightarrow F(x,y) = (u(x,y), v(x,y)),$$

where f is a complex function whereas F is a map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$. F is differentiable at $X \in \mathbb{R}^2$ if there exists a linear map (i.e. 2×2 matrix) J such that

$$F(X + \vec{h}) = F(X) + J\vec{h} + o(\|\vec{h}\|), \quad (1.2.1.I)$$

where $h = (h_1, h_2) \in \mathbb{R}^2$ and the linear map J is given by

$$J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}.$$

It is easy to check the following identity

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} a+d & b-c \\ c-b & a+d \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} a-d & -b-c \\ b+c & a-d \end{bmatrix} \begin{bmatrix} h_1 \\ -h_2 \end{bmatrix}.$$

Thus we can write the Jacobi matrix as

$$Jh = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} u_x+v_y & u_y-v_x \\ v_x-u_y & u_x+v_y \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} u_x-v_y & -u_y-v_x \\ u_y+v_x & u_x-v_y \end{bmatrix} \begin{bmatrix} h_1 \\ -h_2 \end{bmatrix}.$$

Using the associations we had before we can write

$$\begin{aligned} \frac{1}{2} \begin{bmatrix} u_x + v_y & u_x - v_x \\ v_x - u_x & u_x + v_y \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} &\longleftrightarrow \frac{1}{2} \left(\frac{\partial}{\partial x}(u + iv) - i \left(\frac{\partial}{\partial y}(u + iv) \right) \right) (h_1 + ih_2) = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) (h_1 + ih_2) \\ &:= \frac{\partial f}{\partial z} h. \end{aligned}$$

Similarly

$$\frac{1}{2} \begin{bmatrix} u_x - v_y & -u_y - v_x \\ u_y + v_x & u_x - v_y \end{bmatrix} \begin{bmatrix} h_1 \\ -h_2 \end{bmatrix} \longleftrightarrow \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) (h_1 - ih_2) := \frac{\partial f}{\partial \bar{z}} \bar{h}$$

Thus we can write

$$\begin{aligned} F(X + \vec{h}) &= F(X) + J\vec{h} + o(\|\vec{h}\|) \\ &\quad \updownarrow \\ f(z + h) &= f(z) + \frac{\partial f}{\partial x} h + \frac{\partial f}{\partial \bar{z}} \bar{h} + o(|h|). \end{aligned}$$

Thus complex function $\Omega \rightarrow \mathbb{C}$ that is differentiable function is not necessarily holomorphic. However, it is holomorphic if and only if $\partial f / \partial \bar{z} = 0$.

This is a very interesting and clear characterization of the holomorphic functions. The condition $\partial f / \partial \bar{z} = 0$ implies

$$\boxed{f \text{ holomorphic}} \iff \boxed{\frac{\partial f}{\partial \bar{z}} = 0} \iff \boxed{\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}} \implies \boxed{u_x = v_y, \quad u_y = -v_x},$$

which is the Cauchy-Riemann equations for a holomorphic function.

1.3 Solved Problems

■ **Problem 1.1 — Convergence of complex variables.** Let $\{z_n = a_n + ib_n\}_{n \in \mathbb{N}}$ for some $a_n \in \mathbb{R}, b_n \in \mathbb{R}$ be a sequence of complex numbers. Show that z_n converges to $w = \alpha + i\beta$ if and only if $a_n \rightarrow \alpha$ and $b_n \rightarrow \beta$ as $n \rightarrow \infty$.

Proof. The proof is as follows

\Rightarrow Let otherwise. Without loss of generality, we can assume that a_n does not converge to α . Then $\exists \epsilon > 0$ such that $\forall N > 0$ we can find $n > N$ for which $|a_n - \alpha| > \epsilon$. This implies that

$$|(a_n - \alpha) + i(b_n - \beta)|^2 = |a_n - \alpha|^2 + |b_n - \beta|^2 > \epsilon^2$$

which implies

$$|z_n - w| = |(a_n - \alpha) + i(b_n - \beta)| > \epsilon$$

for some ϵ and for some $n > N$ for any choice of N . This is a contradiction, since implies z_n is not converging to w .

\Leftarrow Assume $a_n \rightarrow a$ and $b_n \rightarrow b$ as $n \rightarrow \infty$. Fix $\epsilon > 0$. Let N be large enough such that

$$|a_n - \alpha| < \epsilon^2/2, \quad |b_n - \beta| < \epsilon^2/2.$$

Then we can write

$$|(a_n - \alpha) + i(b_n - \beta)|^2 = |a_n - \alpha|^2 + |b_n - \beta|^2 < \epsilon^2.$$

This implies that z_n converges to w .

□

■ **Problem 1.2 — Completeness of \mathbb{C} .** Prove that the set of all complex numbers \mathbb{C} is complete.

Proof. The convergence of a complex number is equivalent to the convergence of its real and imaginary parts. Since \mathbb{R} is complete, it follows that \mathbb{C} is also complete. □

2. Cauchy Theorem and Cauchy Integral Formula

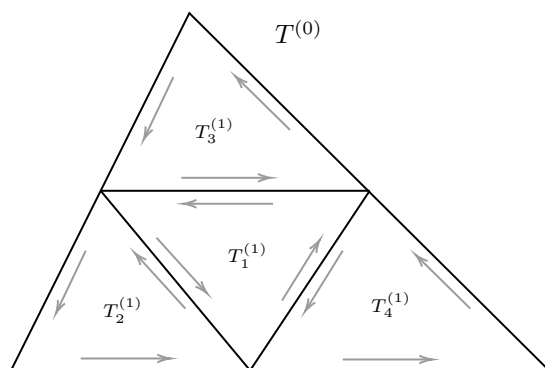
We start this chapter by the Goursat's theorem which in nutshell states that for a holomorphic function on an open region, its contour integral on a triangle contained in the open region is zero. We will use this idea later on to prove the Cauchy theorem.

Theorem 2.1 — Goursat's theorem. Let $\Omega \subset \mathbb{C}$ be an open set and $T \subset \Omega$ be a triangle that its interior is also contained in Ω . Then

$$\int_T f(z) dz = 0$$

whenever f is a holomorphic function on Ω .

Proof. To see the proof, we will use the following argument. Consider the triangle below. Starting



with the triangle $T^{(0)}$, we connect the mid-points of the edges and then we will get four *congruent* triangles. By a simple geometrical argument we can see that the radius and the diameter of the triangles $T^{(1)}_j$ for $j = 1, 2, 3, 4$ is half of the radius and perimeter of the original triangles. I.e. By doing the same process we will get smaller and smaller triangles at each step and at step n we have

$$d(n) = 2^{-n} d^{(0)}, \quad p^n = 2^{-n} p^{(0)}.$$

TODO: TO BE WRITTEN

□

2.1 Solved Problems

■ **Problem 2.1 — From Stein.** Suppose f is continuously complex differentiable on Ω , and $T \subset \Omega$ is a triangle whose interior is also contained in Ω . Apply Green's theorem to show that

$$\int_T f(z)dz = 0.$$

This provides a proof of Goursat's theorem under the additional assumption that f' is continuous. *Hint: Green's theorem says that if (F, G) is a continuously differentiable vector field, then*

$$\int_T Fdx + Gdy = \int_{\text{int}(T)} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dxdy.$$

For appropriate F and G , one can then use the Cauchy-Riemann equations.

Solution Let T be parameterized by piece-wise smooth $\gamma : [0, 1] \rightarrow \mathbb{R}$, given by $\gamma(t) = x(t) + iy(t)$. Let $f = u + iv$. By the definition of the Contour integral

$$\begin{aligned} \int_T f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b (u(x, y) + iv(x, y))(x'(t) + iy'(t))dt \\ &= \int_a^b ((u + iv)x'(t) + (iu - v)y'(t))dt = \int_T fdx + gdy, \end{aligned}$$

where

$$f(u, v) = u + iv, \quad g(u, v) = iu - v.$$

Using Green's theorem

$$\int_T fdx + gdy = \int_{T^\circ} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy$$

where T° denotes the interior of the triangle. Observe that

$$\frac{\partial f}{\partial y} = u_y + iv_y, \quad \frac{\partial g}{\partial x} = iu_x - v_x.$$

Thus

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = -(u_y + v_x) + (u_x - v_y) = 0,$$

where we have used the fact that the Cauchy-Riemann equations hold (as f is complex differentiable.) Thus

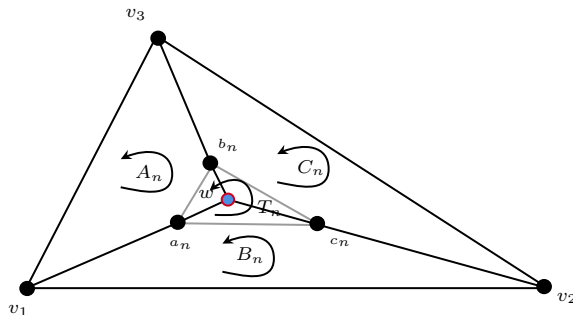
$$\int_T f(z)dz = 0.$$

Observation 2.1.1 In the question above, using the fact that f is complex differentiable, *plus* the extra regularity condition, that f' is continuous, we can simply use the Green's theorem to prove a variant of the Goursat's theorem. The reason that we need the extra regularity is that for the Green's theorem, we need the partial derivatives of the components of the vector field to be continuous.

■ **Problem 2.2 — From Stein.** Let Ω be an open subset of \mathbb{C} and let $T \subset \Omega$ be a triangle whose interior is also contained in Ω . Suppose that f is a function holomorphic in Ω except possibly at a point $w \in T^\circ$, where T° is the interior of the triangle. Prove that if f is bounded near w , then

$$\int_T f(z)dz = 0.$$

Solution Consider the following figure where we connect the point $w \in T^\circ$ to the edges. On



each of the line segments (v_i, w) we choose the points a_n, b_n, c_n respectively for $i = 1, 2, 3$ such that $a_n, b_n, c_n \rightarrow w$ as $n \rightarrow \infty$. By the cancellation of the integral on the paths that have been traversed twice on opposite directions we can write

$$\int_T f(z)dz = \int_{A_n} f(z)dz + \int_{B_n} f(z)dz + \int_{C_n} f(z)dz + \int_{T_n} f(z)dz,$$

where A_n, B_n , and C_n are the tetragons as shown above, and T_n is the central triangle. Observe that the integration over each tetragon can further be broken down into two integrals on two triangles, where by applying the Goursat's theorem it evaluates to zero. Thus the only remaining term will be

$$\int_T f(z)dz = \int_{T_n} f(z)dz.$$

We can write

$$\left| \int_T f(z)dz \right| = \left| \int_{T_n} f(z)dz \right| \leq \max_{z \in T_n} |f(z)| \ell(T_n).$$

Since the function f is bounded around w then $\max_{z \in T_n} |f(z)| \leq C$ for some $C \in \mathbb{R}$ and $\ell(T_n) \rightarrow 0$ as $n \rightarrow \infty$, we will have

$$\int_T f(z)dz = 0.$$

■ **Problem 2.3 — From Stein.** Suppose $f : \mathbb{D} \rightarrow \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z, w \in \mathbb{D}} |f(z) - f(w)|$ of the image of f satisfies

$$2|f'(0)| \leq d.$$

Moreover, it can be shown that the equality holds precisely when f is linear, $f(z) = a_0 + a_1 z$. *Hint: You can use the fact that $2f'(0) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi) - f(-\xi)}{\xi^2} d\xi$ whenever $0 < r < 1$.*

Solution Using the Cauchy integral formula we can write

$$f'(0) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi)}{\xi^2} d\xi = - \int_{|\xi|=r} \frac{f(-\xi)}{\xi^2} d\xi,$$

where the second equality holds by a simple change of variable. Then we can write

$$2f'(0) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi) - f(-\xi)}{\xi^2} d\xi.$$

Then

$$\begin{aligned} 2|f'(0)| &= \frac{1}{2\pi i} \left| \int_{|\xi|=r} \frac{f(\xi) - f(-\xi)}{\xi^2} d\xi \right| \\ &\leq \frac{1}{2\pi i} \int_{|\xi|=r} \frac{|f(\xi) - f(-\xi)|}{|\xi^2|} d\xi \\ &\leq \frac{d}{2\pi} \int_{|\xi|=r} \frac{1}{|\xi|^2} d\xi \\ &\leq \frac{d}{2\pi} \frac{2\pi r}{r^2} = d/r. \end{aligned}$$

This for all $r \in (0, 1)$ we have $2|f'(0)| \leq d/r$. This implies

$$2|f'(0)| \leq \inf_{r \in (0,1)} \frac{d}{r} = d.$$

■ **Problem 2.4 — From Stein.** If f is holomorphic function on the strip $-1 < y < 1$ and $x \in \mathbb{R}$ with

$$|f(z)| \leq A(1 + |z|)^\eta, \quad \eta \text{ a fixed real number}$$

for all z in that strip, show that for each integer $n \geq 0$ there exists $A_n \geq 0$ so that

$$|f^{(n)}(x)| \leq A_n(1 + |x|)^\eta, \quad \text{for all } x \in \mathbb{R}.$$

Hint: Use the Cauchy inequalities.

Solution Let $x \in \mathbb{R}$. This point is in the strip where f is holomorphic. Consider the set $|z - x| = R$ that is a circle with radius R centered at x . Then So using the Cauchy inequalities

$$|f^{(n)}(x)| \leq \frac{n!}{R^n} \sup_{|z-x|=R} |f(z)| \leq \frac{An!}{R^n} \sup_{|z-x|=R} (1 + |z|)^\eta \leq \frac{An!}{R^n} (1 + |x + R|)^\eta \leq \frac{An!}{R^n} (1 + |x| + R)^\eta.$$

By letting $R = 1/2$ and using the fact that $3/2 + |x| \leq 3/2(1 + |x|)$, we will get

$$|f^{(n)}(x)| \leq \frac{An!}{(1/2)^n} \cdot \left(\frac{3}{2}\right)^\eta (1 + |x|)^\eta = A_n(1 + |x|)^\eta$$

3. Solutions for Gamelin Complex Analysis

Theorem 3.1 — Runge's Approximation Theorem. Let K be a compact subset of \mathbb{C} contained in an open set D . If f is analytic (holomorphic) on D , then it can be approximated on K uniformly by *rational* functions with poles off K .

There is some flexibility regarding the location of the poles of the approximating rational functions. This is captured by the following Lemma.

Lemma 3.1 Let K be a compact subset of \mathbb{C} , and U be a *connected open* subset of $\mathbb{C}^* \setminus K$, and let $z_0 \in U$. Every *rational* function f with poles at U can be approximated uniformly on K with *rational* functions with poles at z_0 .

3.1 Approximation Theorems

■ **Problem 3.1** Show that any analytic function $f(z)$ on a domain D can be approximated normally on D by a sequence of rational functions that are analytic on D .

Solution Let $K \subset D$ be any compact subset of D . By [Theorem 3.1](#) f can be approximated uniformly on K by rational functions with poles off K . Let $U = \mathbb{C}^* \setminus K$, and let $z_0 \in U \cap (\overline{D})^c$ (where \overline{D} is the closure of D). By [Lemma 3.1](#) each of the approximating functions above can be approximated by rational functions with poles at z_0 . Since $z_0 \in (\overline{D})^c$, then all of these approximating rational functions are holomorphic (analytic on D). So f can be approximated normally (on every compact subset of D) with rational functions that are holomorphic on D .

■ **Problem 3.2** Show that there is a sequence of polynomials $\{p_n(z)\}$ such that $p_n(z) \rightarrow 1$ if $\operatorname{Re}(z) > 0$, $p_n(z) \rightarrow 0$ if $\operatorname{Re}(z) = 0$, and $p_n(z) \rightarrow -1$ if $\operatorname{Re}(z) < 0$.

Solution TODO: TO BE ADDED.

■ **Remark 3.1** I have not been able to solve the question above yet. The followings are my attempts

- I was thinking what if we use some conformal map to map the Left-Right half planes to the Interior-Exterior of the unit disk where the imaginary axis maps to the boundary of the disk.

This is easily possible by simply rotating the Riemann sphere by 90 degrees around the y axis. I am not sure if this mapping will help.

- The function $\tanh(z) = (e^z - e^{-z})/(e^z + e^{-z})$ has a very interesting behaviour. Define $f_n(z) = \tanh(nz)$. Then for the Left half plane $f_n(z) \rightarrow -1$ while for the Right half plane $f_n(z) \rightarrow 1$. But on the imaginary axis the function oscillates. Not sure if some modifications of this function will help.

■ **Problem 3.3** Let $\{z_j\}$ be a sequence of distinct points in a domain D that accumulates on ∂D , and let $\{w_j\}$ be a sequence of complex numbers. Show that there is an analytic function $f(z)$ on D such that $f(z_j) = w_j$ for all j . The sequence $\{z_j\}$ is called an interpolating sequence for analytic functions on D .

4. Tips and Tricks

Observation 4.0.1 The transformation $1/z$ corresponds to the rotation of the Riemann sphere by 180 degrees around the x axis.

■ **Remark 4.1** These remarks are temporarily.

- normal convergence is the uniform convergence in compact subsets.
- When a compact set K is contained in an open set D , then the boundary of D , i.e. ∂D is uniformly away from the boundary of K (i.e. ∂K).