Abstract Algebra

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1. Abstract Algebra (G. T. Lee Book)

1.1 A Simple Example

1.1.1 Solved Problems

The following problems are from Gregory T. Lee abstract algebra book in SUMS.

∎ **Problem 1.1** In S_4 let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$. Calculate the followings.

(a) $\sigma\tau$

(b) $\tau\sigma$

(c) the inverse of σ

Solution (a)

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}.$$

(b)

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}.$$

(c)

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}.$$

∎ **Problem 1.2** In S_5, let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}$. Calculate the following.

(a) $\sigma\tau\sigma$

(b) $\sigma\sigma\tau$

(c) the inverse of σ

Solution   (a)

$$\sigma\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}.$$

(b)

$$\sigma\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}.$$

(c)

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix}.$$

■ **Problem 1.3** How many permutations are there in S_n? How many of those permutation satisfy $\alpha(2) = 2$?

Solution   There are n choices for $\alpha(1)$, $n-1$ choices for $\alpha(2)$, and so on. So there are in total $n!$ elements in S_n. Fixing the value of $\alpha(2) = 2$ will leave 4 possible values for $\alpha(1)$, 3 possible values for $\alpha(3)$, and so on. Thus there will be $4! = 24$ permutations satisfying $\alpha(2) = 2$.

■ **Problem 1.4** Let H be the set of all permutations $\alpha \in S_5$ satisfying $\alpha(2) = 2$. Which of the properties, closure, associativity, identity, and inverse does H enjoy under composition of functions?

Solution   Closure is satisfied: Let $\alpha, \beta \in H$. Then $\alpha(\beta(2)) = \alpha(2) = 2$ and also $\beta(\alpha(2)) = \beta(2) = 2$. Associativity is satisfied which follows from the axioms of the group. The identity of the group is in H, which is given by

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

Every element in H also has an inverse. Let $\alpha \in H$. Let $\tau \in S_5$ be its inverse. We have

$$\tau(2) = \tau(\alpha(2)) = e(2) = 2.$$

Thus $\tau \in H$.

■ **Problem 1.5** Consider the set of all functions from $\{1, 2, 3, 4, 5\}$ to $\{1, 2, 3, 4, 5\}$. Which of the properties, i.e. closure, associativity, identity, and inverse does this set enjoy under the composition of functions.

Solution   The composition of any two functions is a function, thus the set is closed under composition. The associativity follows from the properties of the function composition. The identity function is the function that maps every element to itself hence is in the set. But not every function necessarily has an inverse (injectivity, and surjectivity is needed to guarantee the inverse).

■ **Problem 1.6** Give group tables for the following additive groups

(a) $U(12)$,

(b) S_3.

Solution   (a)

$*$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

(b)

$*$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(2,0)$	$(2,1)$
$(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(2,0)$	$(2,1)$
$(0,1)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$	$(2,1)$	$(2,0)$
$(1,0)$	$(1,0)$	$(1,1)$	$(2,0)$	$(2,1)$	$(0,0)$	$(0,1)$
$(1,1)$	$(1,1)$	$(1,0)$	$(2,1)$	$(2,0)$	$(0,1)$	$(0,0)$
$(2,0)$	$(2,0)$	$(2,1)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(2,1)$	$(2,1)$	$(2,0)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$

■ **Problem 1.7** Give group tables from the following groups.

(a) $U(12)$.

(b) S_3.

Solution   (a) First observe that $U(12) = \{1, 5, 7, 11\}$. So

$*$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

(b) Call the following permutations as $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$, and σ_6 respectively

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$*$	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
σ_1	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
σ_2	σ_2	σ_1	σ_5	σ_6	σ_3	σ_4
σ_3	σ_3	σ_4	σ_1	σ_2	σ_6	σ_5
σ_4	σ_4	σ_3	σ_6	σ_5	σ_1	σ_2
σ_5	σ_5	σ_6	σ_2	σ_1	σ_4	σ_3