



Topology

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1. Topological Spaces and Continuous Function

Definition 1.1 — Topology on a set. Let X be a set. A topology on X is a collection of subsets of X , called *open sets* and denoted by \mathcal{T} , that satisfies

- $X, \emptyset \in \mathcal{T}$,
- For an *arbitrary* collection of open sets $\{U_\alpha\}_{\alpha \in J}$ we have

$$\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}.$$

- For a *finite* collection of open sets $\{U_1, \dots, U_n\}$ for some $n \in \mathbb{N}$ we have

$$\bigcap_{i=1}^n U_i \in \mathcal{T}$$

Definition 1.2 — Finer and Coarse Topologies. Let X be a set and \mathcal{T}_1 and \mathcal{T}_2 two topologies. The we say the topology \mathcal{T}_1 is finer if we have $\mathcal{T}_1 \subset \mathcal{T}_2$. Or alternatively, we can call the topology \mathcal{T}_2 to be coarser.

Definition 1.3 — Bases for a topology. Let X be a set. A collection of subsets of X , denoted by \mathbb{B} , is called a basis for the topology on X if we have

- $\forall x \in X$ there exists $B \in \mathbb{B}$ such that $x \in B$.
- If for sets $B_1, B_2 \in \mathbb{B}$ we have $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathbb{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Definition 1.4 — Topology Generated by a Basis. Let X be a set and \mathbb{B} a basis for a topology. We define the topology \mathcal{T} generated by \mathbb{B} as follow: For any $U \in \mathcal{T}$ and $x \in U$ there exists $B \in \mathbb{B}$ such that $x \in B \subset U$.

■ **Remark 1.1** We can now check that if \mathcal{T} generated by \mathbb{B} is really a topology. We need to show that \mathcal{T} has the topology properties.

- The statement $\emptyset \in \mathcal{T}$ is vacuously true. To see $X \in \mathcal{T}$, Consider $x \in X$. Then from the definition of a basis $\exists B \in \mathbb{B}$ such that $x \in B$ and since X is the whole space, thus $x \in B \subset X$.

- Let $\{U_\alpha\}_{\alpha \in J}$ be an arbitrary collection of sets in \mathcal{T} . Consider $x \in \bigcup_\alpha U_\alpha$. Then there exists an index α such that $x \in U_\alpha$. Since $U_\alpha \in \mathcal{T}$ thus there exists $B \in \mathbb{B}$ such that $x \in B \subset U_\alpha$ which implies $x \in B \subset \bigcup_\alpha U_\alpha$, hence $\bigcup_\alpha U_\alpha \in \mathcal{T}$.
- To see that the intersection of a finite collection of sets in \mathcal{T} is also in \mathcal{T} , first if $U_1 \in \mathcal{T}$ and $U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$. That is because for $x \in U_1 \cap U_2$ we know that $\exists B_1, B_2 \in \mathbb{B}$ such that $x \in B_1 \cap B_2$. Thus from the definition of a basis $\exists B_3 \in \mathbb{B}$ such that $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$, hence $U_1 \cap U_2 \in \mathcal{T}$. Then by induction we can conclude that for any finite collection of sets in \mathcal{T} their intersection is also in \mathcal{T} .

Proposition 1.1 — Characterization of Open Sets in Terms of Basis. Let X be a set and \mathbb{B} a basis for a topology \mathcal{T} on X . Then \mathcal{T} is the collection of all unions of \mathbb{B} .

Proof. Let \mathcal{A} be the collection of all unions of \mathbb{B} . We need to show the equality of the sets

$$\mathcal{A} = \mathcal{T}.$$

Let $A \in \mathcal{A}$. Thus $A = \bigcup_\alpha B_\alpha$ for $B_\alpha \in \mathbb{B}$. Let $x \in A$. Then we can choose any B_α and we will have $x \in B_\alpha \subset A$. Thus $A \in \mathcal{T}$. To show the converse let $A \in \mathcal{T}$. Thus for any $x \in A$ we have $B_x \in \mathbb{B}$ such that $x \in B_x \subset A$. We can write

$$A = \bigcup_{x \in A} B_x.$$

Thus $A \in \mathcal{A}$. This completes the proof. \square

Be Careful Here!  1.0.1 — Be Aware of the Terminology. From the proposition above, we can say that a subset $U \subset X$ is open (i.e. $U \in \mathcal{T}$) if it can be written as union of sets in \mathbb{B} . This union, however, need not be unique. This is a very important difference with the notion of basis in linear algebra, where given a basis for a space, we can then write any vector in the space uniquely by the basis vectors.

Proposition 1.2 — Basis of a Given Topology. Let (X, \mathcal{T}) be a given topological spaces. A collection of open sets \mathcal{C} is a basis for the topology if $\forall U \in \mathcal{T}$ and $x \in U$ there exists $C \in \mathcal{C}$ such that $x \in C \subset U$.

Proof. This proof has two steps. First, we need to show that the collection \mathcal{C} is indeed a basis, and then we need to show that the topology that this basis generates is the same as the topology of the set itself.

Showing that \mathcal{C} is a basis. To show the first property of being a basis, we need to show that $\forall x \in X$ we can find an element of \mathcal{C} that contains x . Since $X \in \mathcal{T}$ and by hypothesis we can find $C \in \mathcal{C}$ such that $x \in C \subset X$, thus \mathcal{C} has the first property of being a basis. For the second property, we need to show that if $x \in C_1 \cap C_2$ then there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$. Let $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C}$. Since C_1, C_2 are open sets, then $C_1 \cap C_2$ is also open. Thus by hypothesis there exists $C \in \mathcal{C}$ such that $x \in C \subset C_1 \cap C_2$. This completes the proof that the collection \mathcal{C} is indeed a basis.

Showing that the basis \mathcal{C} generate the topology \mathcal{T} . To show this, we can directly use [Definition 1.4](#) or use the characterization in [Proposition 1.1](#), where we will use the latter. I.e. it suffices to show that for any given $U \in \mathcal{T}$ we can write it as a union of basis sets. Let $U \in \mathcal{T}$ and $x \in U$. Then by hypothesis we know that $\exists C_x \in \mathcal{C}$ such that $x \in C_x \subset U$. Thus we can write U as

$$U = \bigcup_{x \in U} C_x,$$

which completes the proof. \square

Proposition 1.3 — Comparing Topologies in Terms of Their Bases. Let X be a set and $\mathcal{B}, \mathcal{B}'$ be bases for the topologies \mathcal{T} and \mathcal{T}' on X respectively. Then the followings are equivalent.

- (i) \mathcal{T}' is finer than \mathcal{T} .
- (ii) For any $x \in X$ and any basis element $B \in \mathcal{B}$ containing x , there exists a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Proof. We will have two parts

$(i) \implies (ii)$ By [Definition 1.2](#) hypothesis implies $\mathcal{T} \subset \mathcal{T}'$. Let $x \in X$ and $B \in \mathcal{B}$ by any basis element that contains it. Since the topology \mathcal{T} is generated by \mathcal{B} , thus $B \in \mathcal{T}$. By hypothesis we have $B \in \mathcal{T}'$. Since \mathcal{B}' generates the topology \mathcal{T}' , thus there exists some $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

\square

1.1 Solved Problems

■ **Problem 1.1 — Finite Complement Topology.** Let X be a set and let \mathcal{T}_f be the collection of all subsets U of X such that $X - U$ is either finite or is all of X . Show that \mathcal{T}_f is a topology on X .

Solution First, observe that since X is the whole space, for any subset $U \subset X$ we have $X - U = U^c$. So we can write \mathcal{T}_f as

$$\mathcal{T}_f = \{U \subset X \mid U^c = X \text{ or } U^c \text{ is finite}\}.$$

We need to check if \mathcal{T}_f has the properties of topology.

- It is immediate that $\emptyset \in \mathcal{T}_f$ and $X \in \mathcal{T}_f$.
- Let $\{A_\alpha\}_{\alpha \in I}$ be an arbitrary collection where $A_\alpha \in \mathcal{T}_f$ for all $\alpha \in I$. Let $C = \bigcup_{\alpha} A_\alpha$. For C to be in \mathcal{T}_f , C^c needs to be X or finite.

$$C^c = \bigcap_{\alpha} A_\alpha^c.$$

Since all A_α are in \mathcal{T}_f , then A_α^c are X or finite. Thus C^c is also X or finite, hence $C \in \mathcal{T}_f$

- Let $\{A_1, \dots, A_n\}$ be finite collection of open sets. Consider $D = \bigcap_i A_i$. Since

$$D^c = \bigcup_i A_i^c,$$

and A_i^c are finite or X , then D^c is also finite or X , hence $D \in \mathcal{T}_f$.

■ **Problem 1.2** Construct a set and give it a finite complement topology.

Solution Consider the set

$$X = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \cup \{0\}.$$

The collection of sets $\mathcal{T} = \{A_n\}_{n \in \mathbb{N}}$, where

$$A_n = X - \left\{ \frac{1}{1}, \frac{1}{2}, \dots, \frac{1}{n} \right\},$$

is a finite complement topology.