Lecture Notes For : Advanced Linear Algebra

Ali Fele Paranj
alifele@student.ubc.ca

5 février 2023

Chapitre 1

Matrices

1.1 What is a Matrix

A matrix is basically a notation convention that enables us to do some stuff more easily with a pencil and paper. A very similar concept to this is the long division algorithm for for dividing two integers. For example consider the following long division (in French-European style) that we are all familiar with

$$
\begin{array}{r|l}
198 & 12 \\ \cline{2-2}
-\,12 & 16{,}5 \\ \cline{1-1}
78 & \\
-\,72 & \\ \cline{1-1}
60 & \\
-\,60 & \\ \cline{1-1}
0 &
\end{array}
$$

So this notation and algorithms is to use some calculations more continent when is done by hand with a pen and paper. So the matrix notation can also be though as a computation convention. To make stuff more clear, consider the following example.

Example : Simple Pen and Paper Calculations

Consider V which is written as :

$$V = 2A + 3B + 4C$$

Given the following relation between A, B, and C, rewrite V in terms of x, y, and z.

$$
\begin{aligned}
A &= x + 2y + 3z \\
B &= 2x - y + z \\
C &= -x - y + z
\end{aligned}
$$

Solution 1.
To write V in terms of x, y, and z we write :

$$V = 2(x + 2y + 3z) + 3(2x - y + z) + 4(-x - y + z) \tag{1.1.1}$$

By arranging the terms using simple algebra we will have :

$$V = (2 + 6 - 4)x + (4 - 3 - 4)y + (6 + 3 + 4)z = 4x - 3y + 13z \tag{1.1.2}$$

Solution 2.
The calculations described in the first solution are not systematic. What I mean is that we started doing whatever we can do with you thinking about doing it in a more smart way that can also by systematically scaled to larger equations. This is where the matrices come into play. Matrices help us to do such calculations in a more algorithmic way (like the long division notation in which we do the calculations in a algorithmic way).

Let $\mathbb{B}$ be the set of all *objects* that the V is expanded in terms of and call this set as the *basis* set. So for $\mathbf{V} = 2A + 3B + 4C$ we have the basis

$$\mathbb{B}_1 = \{A, B, C\}.$$

We can arrange the coefficients of V in basis $\mathbb{B}_1$ in the following way :

$$V_{\mathbb{B}_1} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}_{\mathbb{B}_1}$$

We call it the coordinates of V in the basis $\mathbb{B}_1$. Since we want to write the vector V in terms of x, y, and z, we need to introduce the new basis $\mathbb{B}_2$ in the following way :

$$\mathbb{B}_2 = \{x, y, z\}$$

Since A, B, and C are expressed in terms of x, y, and z, we can arrange the coordinates of A, B, and C in the basis $\mathbb{B}_2$ in the following way :

$$L_{\mathbb{B}_1}^{\mathbb{B}_2} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2}$$

in which every column is the coefficients A, B, and C in the basis $\mathbb{B}_2$ respectively. Note the subscript and the superscripts of the matrix. This matrix means that its columns contains the coordinates of the basis $\mathbb{B}_1$ in the new basis $\mathbb{B}_2$. So when it is applied to any vector that is described in basis $\mathbb{B}_1$, we will get the components of that vector in the basis $\mathbb{B}_2$. In other words :

$$V_{\mathbb{B}_2} = L_{\mathbb{B}_1}^{\mathbb{B}_2} V_{\mathbb{B}_1}$$

$$V_{\mathbb{B}_2} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}_{\mathbb{B}_1} \tag{1.1.3}$$

Considering the basic operations introduced with matrix notation, this matrix equation can be written in two ways as described below :

$$V_{\mathbb{B}_2} = 2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_{\mathbb{B}_2} + 3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}_{\mathbb{B}_2} + 4 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}_{\mathbb{B}_2} \tag{1.1.4}$$

The equation above is equivalent to equation 1.1.1 but described in other way! Also the other way to write the matrix equation 1.1.3 is the following way in which we have used the matrix multiplication conventions :

$$V_{\mathbb{B}_2} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}_{\mathbb{B}_1} = \begin{pmatrix} (1*2)+(2*3)+(-1*4) \\ (2*2)+(-1*3)+(-1*4) \\ (3*2)+(1*3)+(1*4)) \end{pmatrix}_{\mathbb{B}_2} = \begin{pmatrix} 4 \\ -3 \\ 13 \end{pmatrix}_{\mathbb{B}_2} \tag{1.1.5}$$

which is essentially equivalent to the equation 1.1.2 but written in a different way.

1.2 Change of Basis Matrix

As we discussed earlier, the matrix representation of a linear operator $\mathrm{A} \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ depends on the choice of basis $\mathbb{B}_{\mathcal{V}}$ and $\mathbb{B}_{\mathcal{W}}$ which are the basis of vector spaces $\mathcal{V}$ and $\mathcal{W}$ respectively. Now suppose that that in the vector spaces $\mathcal{V}$, the elements are described in basis $\mathbb{B}_1 = \{|e_i\rangle\}_{i=1}^{N}$ but we want to change it to the basis $\mathbb{B}_2 = \{\left|e'{}_j\right\rangle\}_{i=1}^{N}$. We need to know the relation between these two basis that is assumed to be the following :

$$|e_i\rangle = \sum_{j=1}^{N} \rho_{ji} \left|e'_j\right\rangle \tag{1.2.1}$$

Consider the vector $|v\rangle$ that is described in the basis $\mathbb{B}_1$ in the following way :

$$|v\rangle = \sum_{i=1}^{N} \alpha_i \, |e_i\rangle \tag{1.2.2}$$

$$|v\rangle = \sum_{i=1}^{N} \alpha_i \, |e_i\rangle$$

The coefficients of the expansion are called the *coordinates* of $|v\rangle$ in the basis $\mathbb{B}_1$ and can be shown like :

$$\underline{v}_{\mathbb{B}_1} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix}_{\mathbb{B}_1} \tag{1.2.3}$$

Now we can use the equation 1.2.1 to replace $|e_i\rangle$ in 1.2.2 with $\left|e'_j\right\rangle$:

$$|v\rangle = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \rho_{ji} \left|e'_j\right\rangle = \sum_{j=1}^{N}\sum_{i=1}^{N} \rho_{ji}\alpha_i \left|e'_j\right\rangle = \sum_{j=1}^{N} \eta_j \left|e'_j\right\rangle \tag{1.2.4}$$

in which $\eta_j = \sum_{i=1}^{N} \rho_{ji}\alpha_i$ is the coordinates of $|v\rangle$ in the new basis $\mathbb{B}_2$:

$$\underline{v}_{\mathbb{B}_2} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{pmatrix}_{\mathbb{B}_2}$$

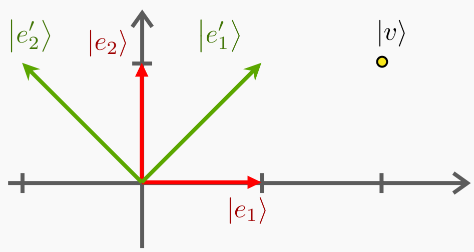
The equation 1.2.4 can be written in the following matrix equation :

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{pmatrix}_{\mathbb{B}_2} = \underbrace{\begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{N1} & \rho_{N2} & \cdots & \rho_{NN} \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2}}_{R} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix}_{\mathbb{B}_1} \tag{1.2.5}$$

The matrix R is called the change of basis matrix.

Example : Change of Basis

Consider the following Cartesian plane :



The vectors of the plain can be expressed using any arbitrary basis two of which are provided here as an example :

$$\mathbb{B}_1 = \{|e_1\rangle, |e_2\rangle\} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbb{B}_1}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathbb{B}_1}\}$$

$$\mathbb{B}_2 = \{|e'_1\rangle, |e'_2\rangle\} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbb{B}_2}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathbb{B}_2}\} = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\mathbb{B}_1}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}_{\mathbb{B}_1}\}$$

It is clear that the vector $|v\rangle$ can be expanded like :

$$|v\rangle = 2\,|e_1\rangle + |e_2\rangle \tag{1.2.6}$$

So we can write :

$$\underline{v}_{\mathbb{B}_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}_{\mathbb{B}_1}$$

in which $\underline{v}_{\mathbb{B}_1}$ is the coordinates of $|v\rangle$ in the basis $\mathbb{B}_1$. Now suppose that we want to fine the coordinates of $|v\rangle$ in the basis $\mathbb{B}_2$. To do that we need to write the $|e_1\rangle$ and $|e_2\rangle$ in terms of $|e'_1\rangle$ and $|e'_2\rangle$ (i.e. find the coordinates of elements of $\mathbb{B}_1$ in the basis $\mathbb{B}_2$) :

$$\begin{aligned} |e_1\rangle &= |e'_1\rangle - |e'_2\rangle \\ |e_2\rangle &= |e'_1\rangle + |e'_2\rangle \end{aligned} \tag{1.2.7}$$

This can be written in the column vector format :

$$\underline{e_1}_{\mathbb{B}_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}_{\mathbb{B}_2} , \quad \underline{e_1}_{\mathbb{B}_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\mathbb{B}_2}$$

By arranging these columns into the columns of matrix we will get the change of basis matrix :

$$\mathrm{R} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2}$$

By inserting 1.2.7 in 1.2.6 we can get the expanded form of $|v\rangle$ in the new basis $\mathbb{B}_2$.

$$|v\rangle = 2(|e'_1\rangle - |e'_2\rangle) + (|e'_1\rangle + |e'_2\rangle) = 3\,|e'_1\rangle - |e'_2\rangle$$

The above calculations can also be done by applying the change of basis matrix R on the coordinates of $|v\rangle$ in $\mathbb{B}_1$, i.e. $\underline{v}_{\mathbb{B}_1}$:

$$\underline{v}_{\mathbb{B}_2} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2} \begin{pmatrix} 2 \\ 1 \end{pmatrix}_{\mathbb{B}_1} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}_{\mathbb{B}_2}$$