Lecture Notes For: Advanced Linear Algebra

Ali Fele Paranj
alifele@student.ubc.ca

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Chapter 1

Vector Spaces

1.1 Vector Spaces

1.2 Dual Spaces

Stuff to be added here. According to the following list

- Examples of the Sadri hassani book for the stuff that can be considered as a linear functional.

  Show that the linear functionals can be expanded in terms of their basis. Choose an appropriate basis for that

  Show that the example linear functionals are actually linear

  Pullback of a linear transform

  Dual space as the row vectors

1.3 Multilinear Functions

1.3.1 Determinant Function

Chapter 2

Matrices

2.1 What is a Matrix

A matrix is basically a notation convention that enables us to do some stuff more easily with a pencil and paper. A very similar concept to this is the long division algorithm for for dividing two integers. For example consider the following long division (in French-European style) that we are all familiar with

$$
\begin{array}{r|l}
198 & 12 \\ \cline{2-2}
-\underline{12} & 16{,}5 \\
78 & \\
-\underline{72} & \\
60 & \\
-\underline{60} & \\
0 &
\end{array}
$$

So this notation and algorithms is to use some calculations more continent when is done by hand with a pen and paper. So the matrix notation can also be though as a computation convention. To make stuff more clear, consider the following example.

Example: Simple Pen and Paper Calculations

Consider V which is written as:

$$V = 2A + 3B + 4C$$

Given the following relation between A, B, and C, rewrite V in terms of x, y, and z.

$$A = x + 2y + 3z$$
$$B = 2x - y + z$$
$$C = -x - y + z$$

Solution 1.
To write V in terms of x, y, and z we write:

$$V = 2(x + 2y + 3z) + 3(2x - y + z) + 4(-x - y + z) \tag{2.1.1}$$

By arranging the terms using simple algebra we will have:

$$V = (2+6-4)x + (4-3-4)y + (6+3+4)z = 4x - 3y + 13z \tag{2.1.2}$$

Solution 2.
The calculations described in the first solution are not systematic. What I mean is that we started doing whatever we can do with you thinking about doing it in a more smart way that can also by systematically scaled to larger equations. This is where the matrices come into play. Matrices help us to do such calculations in a more algorithmic way (like the long division notation in which we do the calculations in a algorithmic way).

Let $\mathbb{B}$ be the set of all *objects* that the V is expanded in terms of and call this set as the *basis* set. So for $\mathbf{V} = 2A + 3B + 4C$ we have the basis

$$\mathbb{B}_1 = \{A, B, C\}.$$

We can arrange the coefficients of V in basis $\mathbb{B}_1$ in the following way:

$$V_{\mathbb{B}_1} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}_{\mathbb{B}_1}$$

We call it the coordinates of V in the basis $\mathbb{B}_1$. Since we want to write the vector V in terms of x, y, and z, we need to introduce the new basis $\mathbb{B}_2$ in the following way:

$$\mathbb{B}_2 = \{x, y, z\}$$

Since A, B, and C are expressed in terms of x, y, and z, we can arrange the coordinates of A, B, and C in the basis $\mathbb{B}_2$ in the following way:

$$L_{\mathbb{B}_1}^{\mathbb{B}_2} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2}$$

in which every column is the coefficients A, B, and C in the basis $\mathbb{B}_2$ respectively. Note the subscript and the superscripts of the matrix. This matrix means that its columns contains the coordinates of the basis $\mathbb{B}_1$ in the new basis $\mathbb{B}_2$. So when it is applied to any vector that is described in basis $\mathbb{B}_1$, we will get the components of that vector in the basis $\mathbb{B}_2$. In other words:

$$V_{\mathbb{B}_2} = L_{\mathbb{B}_1}^{\mathbb{B}_2} V_{\mathbb{B}_1}$$

$$V_{\mathbb{B}_2} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}_{\mathbb{B}_1} \tag{2.1.3}$$

Considering the basic operations introduced with matrix notation, this matrix equation can be written in two ways as described below:

$$V_{\mathbb{B}_2} = 2\begin{pmatrix}1\\2\\3\end{pmatrix}_{\mathbb{B}_2} + 3\begin{pmatrix}2\\-1\\1\end{pmatrix}_{\mathbb{B}_2} + 4\begin{pmatrix}-1\\-1\\1\end{pmatrix}_{\mathbb{B}_2} \tag{2.1.4}$$

The equation above is equivalent to equation 2.1.1 but described in other way! Also the other way to write the matrix equation 2.1.3 is the following way in which we have used the matrix multiplication conventions:

$$V_{\mathbb{B}_2} = \begin{pmatrix}1 & 2 & -1\\2 & -1 & -1\\3 & 1 & 1\end{pmatrix}^{\mathbb{B}_2}_{\mathbb{B}_1}\begin{pmatrix}2\\3\\4\end{pmatrix}_{\mathbb{B}_1} = \begin{pmatrix}(1*2)+(2*3)+(-1*4)\\(2*2)+(-1*3)+(-1*4)\\(3*2)+(1*3)+(1*4))\end{pmatrix}_{\mathbb{B}_2} = \begin{pmatrix}4\\-3\\13\end{pmatrix}_{\mathbb{B}_2} \tag{2.1.5}$$

which is essentially equivalent to the equation 2.1.2 but written in a different way.

2.2 Linear Operators and Matrices

Consider the vector spaces $\mathcal{V}$ and $\mathcal{W}$ by are spanned with $\mathbb{B}_{\mathcal{V}} = \{|a_i\rangle\}_{i=1}^{N}$ and $\mathbb{B}_{\mathcal{W}} = \{|b_j\rangle\}_{i=1}^{M}$ respectively. The act of a linear operator $A \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ on the vector $|v\rangle \in \mathcal{V}$ can be specified by its act on the basis of that vector space $\mathbb{B}_{\mathcal{V}}$. Suppose that A acts on the basis vectors $|a_i\rangle$ as the following:

$$|w_i\rangle = A|a_i\rangle = \sum_{j=1}^{M} \rho_{ji}|b_j\rangle \tag{2.2.1}$$

in which ρ_{ji} are the components of $|w_i\rangle$ in the basis $\mathbb{B}_{\mathcal{W}}$ and can be organized as column vectors:

$$\underline{w_i}_{\mathbb{B}_{\mathcal{W}}} = \begin{pmatrix}\rho_{1i}\\\rho_{2i}\\\vdots\\\rho_{Mi}\end{pmatrix}_{\mathbb{B}_{\mathcal{W}}}$$

Now if A acts on the vector $|v\rangle = \sum_{i=1}^{N} \alpha_i |a_i\rangle$ can be written as:

$$|u\rangle = A|v\rangle = A\sum_{i=1}^{N}\alpha_i|a_i\rangle = \sum_{i=1}^{N}\alpha_i A|a_i\rangle = \sum_{i=1}^{N}\alpha_i\sum_{j=1}^{M}\rho_{ji}|b_j\rangle = \sum_{j=1}^{M}\underbrace{\sum_{i=1}^{N}\alpha_i\rho_{ji}}_{\eta_j}|b_j\rangle = \sum_{j=1}^{M}\eta_j|b_j\rangle$$

so the coordinates of the vector $|u\rangle = A|v\rangle$ will be:

$$\eta_j = \sum_{i=1}^{N} \alpha_i \rho_{ji} \tag{2.2.2}$$

which also can be written as:

$$\underline{u}_{\mathbb{B}_\mathcal{W}} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_j \end{pmatrix}_{\mathbb{B}_\mathcal{W}}$$

The above equation is very important and can be shown in matrix notation to make the idea more clear. To do this we need to build matrix $\mathrm{A}_{\mathbb{B}_\mathcal{V}}^{\mathbb{B}_\mathcal{W}}$ by arranging the components of $\underline{w_i}_{\mathbb{B}_\mathbf{W}}$ as the columns of $\mathrm{A}_{\mathbb{B}_\mathcal{V}}^{\mathbb{B}_\mathcal{W}}$:

$$\mathrm{A}_{\mathbb{B}_\mathcal{V}}^{\mathbb{B}_\mathcal{W}} = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{M1} & \rho_{M2} & \cdots & \rho_{MN} \end{pmatrix}_{\mathbb{B}_\mathcal{V}}^{\mathbb{B}_\mathcal{W}} \tag{2.2.3}$$

So the equation 2.2.2 can be written as:

$$\boxed{\underline{u}_{\mathbb{B}_\mathcal{W}} = \mathrm{A}_{\mathbb{B}_\mathcal{V}}^{\mathbb{B}_\mathcal{W}} \, \underline{v}_{\mathbb{B}_\mathcal{V}}} \tag{2.2.4}$$

Note that a matrix does not have any meaning by itself and is just a collection of numbers. To correspond any matrix with a linear operator $A \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, we need to fix the basis vectors $\mathbb{B}_\mathcal{V}$ and $\mathbb{B}_\mathcal{W}$.

2.3 Change of Basis

As we discussed earlier, the matrix representation of a linear operator $\mathrm{A} \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ depends on the choice of basis $\mathbb{B}_\mathcal{V}$ and $\mathbb{B}_\mathcal{W}$ which are the basis of vector spaces $\mathcal{V}$ and $\mathcal{W}$ respectively. Now suppose that that in the vector spaces $\mathcal{V}$, the elements are described in basis $\mathbb{B}_1 = \{|e_i\rangle\}_{i=1}^N$ but we want to change it to the basis $\mathbb{B}_2 = \{\left|e'_j\right\rangle\}_{i=1}^N$. We need to know the relation between these two basis that is assumed to be the following:

$$|e_i\rangle = \sum_{j=1}^{N} \rho_{ji} \left|e'_j\right\rangle \tag{2.3.1}$$

Consider the vector $|v\rangle$ that is described in the basis $\mathbb{B}_1$ in the following way:

$$|v\rangle = \sum_{i=1}^{N} \alpha_i \, |e_i\rangle \tag{2.3.2}$$

$$|v\rangle = \sum_{i=1}^{N} \alpha_i |e_i\rangle$$

The coefficients of the expansion are called the *coordinates* of $|v\rangle$ in the basis $\mathbb{B}_1$ and can be shown like:

$$\underline{v}_{\mathbb{B}_1} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix}_{\mathbb{B}_1} \tag{2.3.3}$$

Now we can use the equation 2.3.1 to replace $|e_i\rangle$ in 2.3.2 with $\left|e'_j\right\rangle$:

$$|v\rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \rho_{ji} \left|e'_j\right\rangle = \sum_{j=1}^{N} \sum_{i=1}^{N} \rho_{ji} \alpha_i \left|e'_j\right\rangle = \sum_{j=1}^{N} \eta_j \left|e'_j\right\rangle \tag{2.3.4}$$

in which $\eta_j = \sum_{i=1}^{N} \rho_{ji} \alpha_i$ is the coordinates of $|v\rangle$ in the new basis $\mathbb{B}_2$:

$$\underline{v}_{\mathbb{B}_2} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{pmatrix}_{\mathbb{B}_2}$$

The equation 2.3.4 can be written in the following matrix equation:

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{pmatrix}_{\mathbb{B}_2} = \underbrace{\begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{N1} & \rho_{N2} & \cdots & \rho_{NN} \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2}}_{R} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix}_{\mathbb{B}_1} \tag{2.3.5}$$

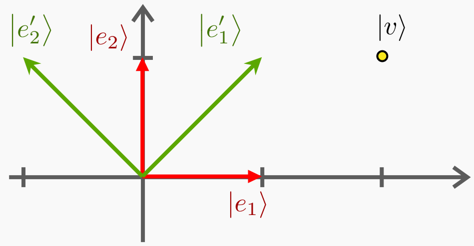
That can also be written in a more compact form:

$$\boxed{\underline{v}_{\mathbb{B}_2} = \mathrm{R}_{\mathbb{B}_1}^{\mathbb{B}_2} \, \underline{v}_{\mathbb{B}_1}} \tag{2.3.6}$$

The matrix R is called the change of basis matrix.

Example: Change of Basis

Consider the following Cartesian plane:



The vectors of the plain can be expressed using any arbitrary basis two of which are provided here as an example:

$$\mathbb{B}_1 = \{|e_1\rangle, |e_2\rangle\} = \{\begin{pmatrix}1\\0\end{pmatrix}_{\mathbb{B}_1}, \begin{pmatrix}0\\1\end{pmatrix}_{\mathbb{B}_1}\}$$
$$\mathbb{B}_2 = \{|e_1'\rangle, |e_2'\rangle\} = \{\begin{pmatrix}1\\0\end{pmatrix}_{\mathbb{B}_2}, \begin{pmatrix}0\\1\end{pmatrix}_{\mathbb{B}_2}\} = \{\begin{pmatrix}1\\1\end{pmatrix}_{\mathbb{B}_1}, \begin{pmatrix}-1\\1\end{pmatrix}_{\mathbb{B}_1}\}$$

It is clear that the vector $|v\rangle$ can be expanded like:

$$|v\rangle = 2\,|e_1\rangle + |e_2\rangle \tag{2.3.7}$$

So we can write:

$$\underline{v}_{\mathbb{B}_1} = \begin{pmatrix}2\\1\end{pmatrix}_{\mathbb{B}_1}$$

in which $\underline{v}_{\mathbb{B}_1}$ is the coordinates of $|v\rangle$ in the basis $\mathbb{B}_1$. Now suppose that we want to fine the coordinates of $|v\rangle$ in the basis $\mathbb{B}_2$. To do that we need to write the $|e_1\rangle$ and $|e_2\rangle$ in terms of $|e_1'\rangle$ and $|e_2'\rangle$ (i.e. find the coordinates of elements of $\mathbb{B}_1$ in the basis $\mathbb{B}_2$):

$$\begin{aligned}|e_1\rangle &= |e_1'\rangle - |e_2'\rangle\\ |e_2\rangle &= |e_1'\rangle + |e_2'\rangle\end{aligned} \tag{2.3.8}$$

This can be written in the column vector format:

$$\underline{e_1}_{\mathbb{B}_2} = \begin{pmatrix}1\\-1\end{pmatrix}_{\mathbb{B}_2}, \quad \underline{e_1}_{\mathbb{B}_2} = \begin{pmatrix}1\\1\end{pmatrix}_{\mathbb{B}_2}$$

By arranging these columns into the columns of matrix we will get the change of basis matrix:

$$\mathrm{R} = \begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2}$$

By inserting 2.3.8 in 2.3.7 we can get the expanded form of $|v\rangle$ in the new basis $\mathbb{B}_2$.

$$|v\rangle = 2(|e_1'\rangle - |e_2'\rangle) + (|e_1'\rangle + |e_2'\rangle) = 3\,|e_1'\rangle - |e_2'\rangle$$

The above calculations can also be done by applying the change of basis matrix R on the coordinates of $|v\rangle$ in $\mathbb{B}_1$, i.e. $\underline{v}_{\mathbb{B}_1}$:

$$\underline{v}_{\mathbb{B}_2} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2} \begin{pmatrix} 2 \\ 1 \end{pmatrix}_{\mathbb{B}_1} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}_{\mathbb{B}_2}$$

The idea behind the change of basis matrix and the matrices corresponding to linear operator are very similar and this similarity can lead to confusions. For example the equations 2.2 and 2.3.1 are very similar but are very different at the same time. In the former case the vector $|v\rangle$ changes to another vector $|w\rangle$ but in the latter case the vector $|v\rangle$ remains unchanged and only gets represented with new coordinates in the new basis. So when we can know that a matrix changes the basis only and keeps the vector unchanged or maps the vector into another vector? The following example would clarify the differences.

Example: Linear Operators Matrices vs. Change of Basis Matrices

Question.
Consider the matrix A

$$\mathrm{A} = \begin{pmatrix} -1 & -1 \\ -3 & 3 \end{pmatrix}$$

What is the effect of this matrix on a arbitrary vector $|v\rangle = \alpha_1 |a_1\rangle + \alpha_2 |a_2\rangle$?

Answer.
Case 1.
You might answer "it depends on the vector $|v\rangle$". But the truth is that this question is not a correct question. As emphasized before, a matrix by itself is just a collection of numbers and does not have any meaning. By fixing the vector spaces and the corresponding basis vectors, then the matrix A gets its meaning and corresponds to the a linear operator $A \in \mathcal{L}(\mathcal{V}, \mathcal{V})$. Let's assume:

$$\mathbb{B}_1 = \{|a_1\rangle, |a_2\rangle\} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbb{B}_1}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathbb{B}_1}\}$$
$$\mathbb{B}_2 = \{|b_1\rangle, |b_2\rangle\} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbb{B}_2}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathbb{B}_2}\} = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\mathbb{B}_1}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}_{\mathbb{B}_1}\}$$

So the vector $|v\rangle$ can be written as:

$$|v\rangle = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbb{B}_1} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathbb{B}_1}$$

Then the matrix

$$\mathrm{A}_{\mathbb{B}_1}^{\mathbb{B}_2} = \begin{pmatrix} -1 & -1 \\ -3 & 3 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2}$$

has the following interpretation:

The operator A maps the basis vectors in the following way

$$\begin{pmatrix}1\\0\end{pmatrix}_{\mathbb{B}_1} \xrightarrow{A} \begin{pmatrix}-1\\-3\end{pmatrix}_{\mathbb{B}_2} = -1\begin{pmatrix}1\\1\end{pmatrix}_{\mathbb{B}_1} - 3\begin{pmatrix}-1\\1\end{pmatrix}_{\mathbb{B}_1} = \begin{pmatrix}2\\-4\end{pmatrix}_{\mathbb{B}_1}$$
$$\begin{pmatrix}0\\1\end{pmatrix}_{\mathbb{B}_1} \xrightarrow{A} \begin{pmatrix}-1\\3\end{pmatrix}_{\mathbb{B}_2} = -1\begin{pmatrix}1\\1\end{pmatrix}_{\mathbb{B}_1} 3\begin{pmatrix}-1\\1\end{pmatrix}_{\mathbb{B}_1} = \begin{pmatrix}-4\\2\end{pmatrix}_{\mathbb{B}_1}$$

So $A\left|v\right\rangle$ will be:

$$A\left|v\right\rangle = \alpha_1\begin{pmatrix}-1\\-3\end{pmatrix}_{\mathbb{B}_2} + \alpha_2\begin{pmatrix}-1\\3\end{pmatrix}_{\mathbb{B}_2}$$

Case 2.
Let's consider the following sets as the basis

$$\mathbb{B}_1 = \{\left|a_1\right\rangle, \left|a_2\right\rangle\} = \{\begin{pmatrix}1\\0\end{pmatrix}_{\mathbb{B}_1}, \begin{pmatrix}0\\1\end{pmatrix}_{\mathbb{B}_1}\}$$
$$\mathbb{B}_2 = \{\left|b_1\right\rangle, \left|b_2\right\rangle\} = \{\begin{pmatrix}1\\0\end{pmatrix}_{\mathbb{B}_2}, \begin{pmatrix}0\\1\end{pmatrix}_{\mathbb{B}_2}\} = \{\begin{pmatrix}-1/2\\-1/2\end{pmatrix}_{\mathbb{B}_1}, \begin{pmatrix}-1/6\\1/6\end{pmatrix}_{\mathbb{B}_1}\}$$

Then the matrix

$$\mathrm{A}_{\mathbb{B}_1}^{\mathbb{B}_2} = \begin{pmatrix}-1 & -1\\-3 & 3\end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2}$$

has the following interpretation:

The operator A maps the basis vectors in the following way:

$$\begin{pmatrix}1\\0\end{pmatrix}_{\mathbb{B}_1} \xrightarrow{A} \begin{pmatrix}-1\\-3\end{pmatrix}_{\mathbb{B}_2} = -1\begin{pmatrix}-1/2\\-1/2\end{pmatrix}_{\mathbb{B}_1} - 3\begin{pmatrix}-1/6\\1/6\end{pmatrix}_{\mathbb{B}_1} = \begin{pmatrix}1\\0\end{pmatrix}_{\mathbb{B}_1}$$
$$\begin{pmatrix}0\\1\end{pmatrix}_{\mathbb{B}_1} \xrightarrow{A} \begin{pmatrix}-1\\3\end{pmatrix}_{\mathbb{B}_2} = -1\begin{pmatrix}-1/2\\-1/2\end{pmatrix}_{\mathbb{B}_1} 3\begin{pmatrix}-1/6\\1/6\end{pmatrix}_{\mathbb{B}_1} = \begin{pmatrix}0\\1\end{pmatrix}_{\mathbb{B}_1}$$

So the matrix $\mathrm{A}_{\mathbb{B}_1}^{\mathbb{B}_2}$ acts as a change of basis matrix. That is because the A does not change map the basis vectors in $\mathbb{B}_1$ to different vector.

The following proposition box summarizes the observations of the discussed example:

Proposition: Change of Basis Vector

The matrix $\mathrm{A}_{\mathbb{B}_1}^{\mathbb{B}_2}$ acts as a change of basis vector if the set of basis vectors $\mathbb{B}_2 = \{|e_i\rangle\}_{i=1}^{N}$ are the columns of $(\mathrm{A}_{\mathbb{B}_1}^{\mathbb{B}_2})^{-1}$ which is the inverse of $\mathrm{A}_{\mathbb{B}_1}^{\mathbb{B}_2}$.

2.3.1 Similarity Transformation

In the last section we studied that the basis transformation matrix $\mathrm{R}_{\mathbb{B}_1}^{\mathbb{B}_2}$ describes the components of a vector $|v\rangle$ in a new basis, which is achieved using the following relation

$$\underline{v}_{\mathbb{B}_2} = \mathrm{R}_{\mathbb{B}_1}^{\mathbb{B}_2}\,\underline{v}_{\mathbb{B}_1}$$

in which the $\underline{v}_{\mathbb{B}_1}$ and $\underline{v}_{\mathbb{B}_2}$ are the coordinates of $|v\rangle$ in the basis $\mathbb{B}_1$ and $\mathbb{B}_2$ respectively. Now consider an operator $A \in \mathrm{End}(\mathcal{V})$ that maps $|v\rangle$ to $|w\rangle = A\,|v\rangle$. The components of $|w\rangle$ in the basis $\mathbb{B}_1$ can be calculated using:

$$\underline{w}_{\mathbb{B}_1} = \mathrm{A}_{\mathbb{B}_1}^{\mathbb{B}_1}\,\underline{v}_{\mathbb{B}_1}$$

On the other hand the components of vector $|v\rangle$ and $|w\rangle$ in the basis $\mathbb{B}_2$ can be calculated by applying the basis transformation matrix on the components of the mentioned vectors. In other words:

$$\begin{aligned}\underline{v}_{\mathbb{B}_2} &= \mathrm{R}_{\mathbb{B}_1}^{\mathbb{B}_2}\,\underline{v}_{\mathbb{B}_1}\\ \underline{w}_{\mathbb{B}_2} &= \mathrm{R}_{\mathbb{B}_1}^{\mathbb{B}_2}\,\underline{w}_{\mathbb{B}_1}\end{aligned} \tag{2.3.9}$$

So the question that naturally arise is that what is the matrix $\mathrm{A}_{\mathbb{B}_2}^{\mathbb{B}_2}$ such that:

$$\underline{w}_{\mathbb{B}_2} = \mathrm{A}_{\mathbb{B}_2}^{\mathbb{B}_2}\,\underline{v}_{\mathbb{B}_2} \tag{2.3.10}$$

The matrix $\mathrm{A}_{\mathbb{B}_2}^{\mathbb{B}_2}$ can easily be found using the following logic. Let's start with act of basis transformation matrix on the coordinates of $|w\rangle$ in $\mathbb{B}_1$:

$$\underline{w}_{\mathbb{B}_2} = \mathrm{R}_{\mathbb{B}_1}^{\mathbb{B}_2}\,\underline{w}_{\mathbb{B}_1} = \mathrm{R}_{\mathbb{B}_1}^{\mathbb{B}_2}\,\mathrm{A}_{\mathbb{B}_1}^{\mathbb{B}_1}\,\underline{v}_{\mathbb{B}_1} = \mathrm{R}_{\mathbb{B}_1}^{\mathbb{B}_2}\,\mathrm{A}_{\mathbb{B}_1}^{\mathbb{B}_1}(\mathrm{R}_{\mathbb{B}_1}^{\mathbb{B}_2})^{-1}\underline{v}_{\mathbb{B}_2}$$

By comparing the last term in the equation above with the equation 2.3.10 we can conclude

$$\boxed{\mathrm{A}_{\mathbb{B}_2}^{\mathbb{B}_2} = \mathrm{R}_{\mathbb{B}_1}^{\mathbb{B}_2}\,\mathrm{A}_{\mathbb{B}_1}^{\mathbb{B}_1}(\mathrm{R}_{\mathbb{B}_1}^{\mathbb{B}_2})^{-1}} \tag{2.3.11}$$

This is called a **similarity transformation** on $\mathrm{A}_{\mathbb{B}_1}^{\mathbb{B}_1}$ and $\mathrm{A}_{\mathbb{B}_2}^{\mathbb{B}_2}$ is said to be **similar** to $\mathrm{A}_{\mathbb{B}_1}^{\mathbb{B}_1}$

Observation: Matrix Representation Depends on the Choice of Matrix

Note that in the similarity transformation

$$\mathrm{A}_{\mathbb{B}_2}^{\mathbb{B}_2} = \mathrm{R}_{\mathbb{B}_1}^{\mathbb{B}_2}\,\mathrm{A}_{\mathbb{B}_1}^{\mathbb{B}_1}(\mathrm{R}_{\mathbb{B}_1}^{\mathbb{B}_2})^{-1}$$

both $\mathrm{A}_{\mathbb{B}_2}^{\mathbb{B}_2}$ and $\mathrm{A}_{\mathbb{B}_1}^{\mathbb{B}_1}$ correspond to the single operator $A \in \mathrm{End}(\mathcal{V})$. This is another important observation that the matrix representation of a liner operator *depends on the choice of basis.*

Although we will have a more thorough discussion on the spectral value decomposition in the later sections and chapters, however, utilizing the similarity transformation, we can derive the spectral value decomposition intuitively.

Example: Intuitive Spectral Value Decomposition

Matrix $\Lambda \rightarrow$ diagonal matrix with eigenvalues of linear operator A
Matrix V $\rightarrow$ basis transformation matrix with columns containing the coordinates of the eigenvectors in the basis $\mathbb{B}_1 = \{|e_i\rangle\}_{i=1}^{N}$

$$\mathrm{A} = \mathrm{V}\,\Lambda\,\mathrm{V}^{-1}$$

Explain the idea behind this

Chapter 3

Proof Collections

Here in this section I will collect some of the relevant proofs for the linear algebra.

Theorem:

Let U, V be linear spaces and let $A : U \to V$ be a linear operator. Then A is injective if and only if its kernel is a singleton (i.e. a set with just one element).

Proof. For the first direction, we assume the map is injective and we prove that the kernel is singleton. First, observe that $0 \in U$ is in the kernel (from the linearity). I.e. $A(0) = 0$. Thus $\ker(A)$ has at least one element. We proceed by contradiction. Let $x \in U$ such that $x \in \ker(A)$ that is not equal to 0, i.e. $x \neq 0$. Since $x \in \ker(A)$ then $A(x) = 0 = A(0)$. From linearity $A(x - 0) = 0 = A(0)$. Since A is injective, then $x - 0 = 0$ which implies $x = 0$ that contradicts our assumption that $x \neq 0$. Thus $\ker(A) = \{0\}$.

For the converse, we want to prove $\ker(A)$ being a singleton implies A is injective. We proceed with proving the contrapositive. If A is not injective, then $\exists x, y \in U, x \neq y$ such that $f(x) = f(y)$. From linearity $f(x - y) = 0$. Thus $x - y \neq 0$ is in the kernel. Thus $\ker(A)$ contains at least two elements, i.e. $0, x - y$. $\square$