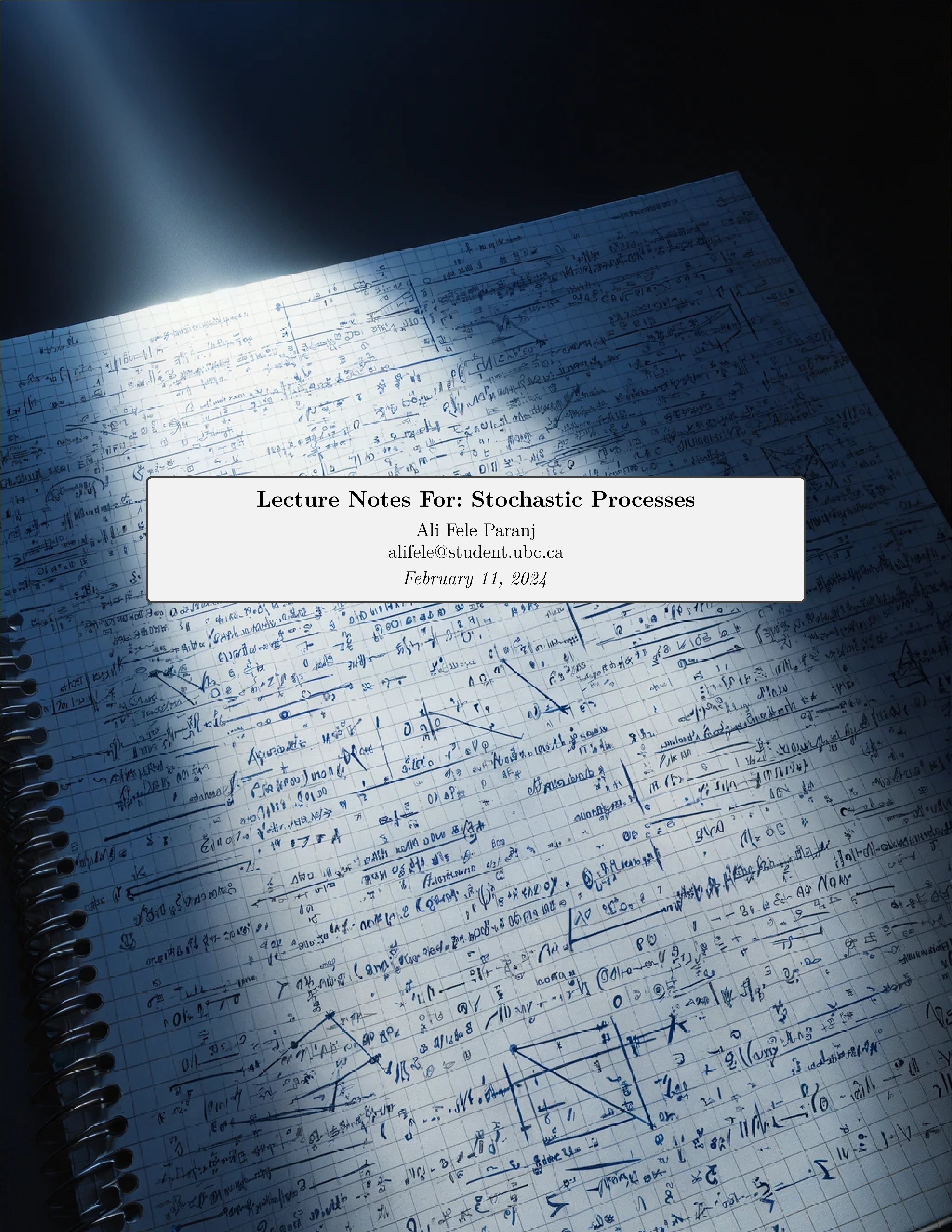

Lecture Notes For: Stochastic Processes

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0. Contents

1. Probability Theory

1.1 Foundamentals

The main concept in the field of statistics and probability is the set theory. Basically all we deal with the sets. The whole theroy of statistics can be built on that. Let's discuss some foundamental concepts in statistics and then build the theory.

1.1.1 Random Experiment

To understand the meaning of random experiment, do not over think! The first thing that comes into our minds when we hear the word "random experiment" is its definition! In a nutshell, random experiment is an experiment that its outcome is unkown to us. Like:

- Tossing two coin
- Rolling a dice
- Measuring the number of possible ReadWrite operations on a piece of EEPROM chip

Do not overthink about that. Yes we can go further and discuss stuff like "we can compute the exact movement of dice or coin so it is not random but determenistic" and etc. Here I will not touch the philosophical topics that are very deep and do not necessarily converge the a unified point of view!

The random experiments can be modeled and despite the fact that a random experiment is random, we can deduce many useful information from modeling that. To model a random experiment, we use three important concepts: sample space, events, probability. In the following section, we will discuss each of them in detail.

1.1.2 Sample Space

Definition 1.1 — Sample Space. Sample space Ω is simply a set that contains *all possible outcomes* of a random experiment/

For each of random experiments described above, we can define a sample space. For example:

- Ω of Tossing Tow Coins:
$$\Omega = \{HH, HT, TH, TT\}$$
- Ω of Rolling a Dice:
$$\Omega = \{1, 2, 3, 4, 5, 6\}$$
- Ω of Rolling Two Dices:
$$\Omega = \{(1, 1), (1, 2), \ldots, (1, 6), \ldots, (6, 6)\}$$
- Ω of Number of possible ReadWrite operations on a EEPROM chip:
$$\Omega = \mathbb{N}$$

1.1.3 Events

Definition 1.2 — Events. Event E is a set of outcomes of a random experiment and is the subset of sample space Ω.
$$E \in \Omega$$

For example for any of the sample spaces specified above, we can define so many possible events. In fact any set that is a subset of the sample space is a valid event of that sample space. For example:

- Tossing Three Coins
  - There are at least on Heads:
$$E = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$$
  - There are only two Tails:
$$E = \{TTH, THT, HTT\}$$
- Rolling Two Dices
  - The sum of two dices is 4:
$$E = \{(1, 3), (2, 2), (3, 1)\}$$
  - there are at least one prime number in the outcome:
$$E = \{(1, 2), (1, 3), (1, 5), (2, 1), (3, 1), (5, 1), (2, 2), (2, 3), (2, 5), \ldots, (5, 5)\}$$

Since we have define everything on the basics of set theory, then now we can correspond the everyday concepts to specific operations in the set theory.

■ **Example 1.1** The Mapping Between Everyday Language and Sets in the Theory of Probability

- At least one of two events $A, B \in \Omega$ happens: $E = A \cup B$.
- Tow events $A, B \in \Omega$ occures at the same time: $E = A \cap B$.
- Event $A \in \Omega$ does not happen: $E = \overline{A} = \Omega - A$.
- The event A happens but B does not happen: $E = A - B$.

■

In probability and statistics, we are dealing with three important concepts: sample space Ω, event E, and probability P.

Definition 1.3 — Disjoint events. If two events has no common elements (i.e. $A \cap B = \varnothing$) then we say that two events are *disjoint*. Basically, if two sets in the venn diagram has nothing is common they are considerent to be disjoint sets.

For example for the random experiment of tossing two coins, the events 1) both coins are heads: $A = \{HH\}$ and 2) both coins are tails: $B = \{TT\}$. Two events A, B are two disjoint events. **Two events being disjoing is NOT the same as being independent**. We will talk about independet events in future.

Note that since the events are basically sets, we can use theorems of set theory to solve the problems.

Theorem 1.1 — De Morgan's Laws. If A, B are two sets then:

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Proof. the proof is left as an excerise! □

1.1.4 Probability

The last foundamental ingeridient in modeling a random experiment, is to define a probability for each event. The probability should intuitively reflect how likely an event is probable to happen. This probability should satisfy some foundamental properties which are explained as follows.

Definition 1.4 — Axioms of probability (Kolmogorov axioms). Suppose that $A, B \in \Omega$ is an event and $\mathbb{P}$ is a probability function. Then $\mathbb{P}$ should satisfy the following properites:

1. $0 \leq \mathbb{P}(A) \leq 1$
2. $\mathbb{P}(\Omega) = 1$
3. For the events $E_1, E_2, ..., E_n \in \Omega$ that are mutually exclusive (i.e. disjoint events):

$$\mathbb{P}(\bigcup_i E_i) = \sum_i \mathbb{P}(E_i)$$

These axioms are called the foundamental axioms of probability and also the Kolmogorov axioms. We are free to define any kind of probabilty function that we want but it is important that 1) It should align with our common sense, 2) It should satisfy the Kolmogorov axioms.

Using the axioms above, we can observe and prove several interesting properties of the probability function. In the following box we have expressed some of them.

Theorem 1.2 — Basic Properties of the Probability Function. Suppose that $\mathbb{P}$ is a probability function and $A, B \in \Omega$ are events of the sample space Ω. We can show that the probability function has the following properties:

1. $\mathbb{P}(\varnothing) = 0$
2. If $A \subset B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
3. $\mathbb{P}(\overline{A}) = 1 - \mathbb{P}(A)$.

4. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Proof. The properties can be proved using the basic set theory theorems.

1. Since $\varnothing$ is the complement of Ω, so these two sets are disjoint (i.e. $\varnothing \cap \Omega = \varnothing$). On the other hand from the set theory we know that $\varnothing \cup \Omega = \Omega$. So $\mathbb{P}(\varnothing \cup \Omega) = \mathbb{P}(\Omega)$. On the other hand, using the third axiom we can write: $\mathbb{P}(\varnothing \cup \Omega) = \mathbb{P}(\varnothing) + \mathbb{P}(\Omega)$. Comparing the two recent equations we can conclude that $\mathbb{P}(\varnothing) = 0$.

The proofs for 2,3,4 are left as a exercise. However, the solutions can be found in the book "Statistical Modeling and Computation by Kroese" chapter 1.

□

■ **Example 1.2 — Defining a simple probability function.** Let's define a probability function for the rolling n dice experiment that is both aligned with our common sense and also satisfy the Kolmogorov equations. Suppose that the Ω is the sample space and $E \in \Omega$ is an event. Then let's define:

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|}$$

in which the $|E|$ means the cardinality (number of elements) of the set E.

■

1.1.5 Isomorphism between random experiments

Often, there is this intuition that certain random experiments are really the same, although they might look very different from each other. For instance, consider two random experiments. In one, we are playing a dice successively and asking what is the probability that after 5 plays, 1 is not appeared. The second experiment is that we have 6 Urns and we place balls in them successively, i.e. at each step one ball is placed in one of the urns and the chance of a ball to end up in any of the urns in equal. These two experiment, although very different, but looks very similar. There is one way that we can formalize this wage intuition, and that is the notion of isomorphism between sets. We say two sets are isomorphic if there is a bijection between them. And the reason that the previously mentioned experiments feel the same is that the sample space Ω of these two experiments are in fact isomorphic.

1.2 Random Variables

Often, we are interested in the some measurements of the outcome of a random experiment rather than knowing the outcome it self. For instance, if the experiment of tossing two dice, we might be interested in asking the question if the sum of two dice is 6, and not concerned over whether the actual outcome was (3,3) or (2,4), etc. These quantities of interest are called random variables. The following definition put this into a more formal definition.

Definition 1.5 Let $(\Omega, E, \mathbb{P})$ be a probability space. Then a random variable X is a function $X : \Omega \to S$, where S called the state space.

■ **Remark** The state space S must have some properties, i.e. being measurable, etc. You can read more about this on the Wikipedia of random variables. Also, the state space S if often $\mathbb{R}$, or in the case of a discrete time Markov chain, S is a finite set (that can be the edge set of a graph).

Since the value of a random variable is determined by the outcomes of the random experiment, we can assign probabilities to the possible values of the random variable. We use the following notation for this purpose.

Definition 1.6 — Notation for probability of random variables. Let X be a random variable. Then we define event

$$E = \{X = a\} = \{\omega \in \Omega : X(\omega) = a\}.$$

Then the following notations are usually used interchangeably:

$$\mathbb{P}(X = a) = \mathbb{P}(\{X = a\})$$

both of which is simply $\mathbb{P}(E)$.

■ **Example 1.3** Let X be a random variable defined to be the sum of two fair dice. Then

$$\begin{aligned}
&\mathbb{P}(\{X = 2\}) = \mathbb{P}(\{(1,1)\}) = \frac{1}{36},\\
&\mathbb{P}(\{X = 3\}) = \mathbb{P}(\{(1,2),(2,1)\}) = \frac{2}{36},\\
&\mathbb{P}(\{X = 13\}) = \mathbb{P}(\varnothing) = 0.
\end{aligned}$$

■

■ **Example 1.4** Suppose that we toss a coin having probability p of coming up heads. We continue tossing the coin until we see a heads. Let the random variable N be the number of times we toss the coin. Describe this random variable.

Solution Although, we can always solve this kind of questions in an ad hoc way by just simply following our intuition, but it is always a best practice to try to fine tune our abstract thinking with our intuitive understandings in these kind of example. Then we can use of abstract thinking capability to solve problems that are almost impossible to address by solely depending on the intuition. So, it is a good idea to try to see how does the set Ω look like. The set Ω will be the set of all finite string of all T letters terminated with H. In other words

$$\Omega = \{H, TH, TTH, TTTH, TTTTH, \cdots\}.$$

Then the random variable $N : \Omega \to \mathbb{Z}$ is basically the length of the string. For instance, if $\omega = TTH \in \Omega$, then $N(\omega) = 3$. Let's calculate

$$\mathbb{P}(N = 3) = \mathbb{P}(\{\omega \in \Omega : N(\omega) = 3\}).$$

To solve this, we need to define appropriate events and then condition our probability on those events. Define F_n be the event where the n first outcomes are tails. For instance

$$F_1 = \{TH, TTH, TTTH, \cdots\},\; F_2 = \{TTH, TTTH, TTTTH, \cdots\},\; \cdots.$$

And let $E = \{N = 3\} = \{TTH\}$. Then we can condition $\mathbb{P}(E)$ on F_2

$$\mathbb{P}(E) = \mathbb{P}(E|F_2)\mathbb{P}(F_2) + \mathbb{P}(E|F_2^c)\mathbb{P}(F_2^c).$$

Note that $F_2^c = \{H, TH\}$, this $\mathbb{P}(E|F_2^c) = \mathbb{P}(E \cap F_2^c)/\mathbb{P}(F_2^c) = 0$. Now we need to determine $\mathbb{P}(F_2)$. Again, we can condition this event on F_1. Then we can write

$$\mathbb{P}(F_2) = \mathbb{P}(F_2|F_1)\mathbb{P}(F_1) + \mathbb{P}(F_2|F_1^c)\mathbb{P}(F_1^c).$$

with the same argument as above $\mathbb{P}(F_2|F_1^c) = 0$. Combining these equations we will get

$$\mathbb{P}(E) = \mathbb{P}(E|F_2)\mathbb{P}(F_2|F_1)\mathbb{P}(F_1).$$

Now these probabilities are easy to calculate which leads to the final answer

$$\mathbb{P}(E) = (1-p)(1-p)p.$$

And by induction we can conclude

$$\mathbb{P}(\{N = n\}) = (1-p)^n p.$$

■

■ **Example 1.5** Suppose that independent trials, each of which results in m possible outcomes with respective probabilities $p_1, p_2, \ldots, p_m$ such that $\sum_{i=1}^m p_i = 1$. Are continually performed. Let X be the number of trials needed until each outcome has occurred at least once. Describe the properties of this random variable.

Solution It is sometime a good idea to try to imagine what does the sample space look like. Let $\Sigma = \{s_1, s_2, s_3, \ldots, s_m\}$ be a set of m distinct symbols. Then each time we are continually performing the experiment, we are getting each of these symbols with corresponding probability p_m. Thus the sample space will be the set of all infinite sequences of these symbols. In other words

$$\Omega = \{\text{all infinite sequence of symbols from } \Sigma\}.$$

Then the random number $X(\omega)$ for $\omega \in \Omega$ is basically the length of the prefix string of ω in which any of the symbols in Σ has been occurred at least once.

■

1.2.1 Cumulative Distribution of Random Variable

The notion of the cumulative distribution of a random variable comes handy in most of the future calculations. Also, this distribution can be used to derive other notions of distributions what are extremely important in applications.

Definition 1.7 — Cumulative distribution. Let X be a random variable $X : \Omega \to \mathbb{R}$. Then the cumulative distrubition $F : \mathbb{R} \to \mathbb{R}$ is defined as

$$F(x) = \mathbb{P}(\{X \leq x\}).$$

Proposition 1.1 The cumulative distribution of a random variable has the following properties.

(i) $\mathbb{P}(a < X \leq b) = F(b) - F(a)$.

(ii) $F(x)$ is a non-decreasing function of x.

Proof. (i)

$$\mathbb{P}(\{a < X \leq b\}) = \mathbb{P}(\{X \leq b\} \cap \{X \leq a\}^c) = -\mathbb{P}(\Omega) + \mathbb{P}(\{X \leq b\}) + \underbrace{\mathbb{P}(\{X \leq a\}^c)}_{1-\mathbb{P}(\{X \leq a\})} = F(b) - F(a).$$

(ii) Let $b_1, b_2 \in \mathbb{R}$ and $b_1 \leq b_2$. Then $\{X \leq b_1\} \subseteq \{X \leq b_2\}$. This implies

$$\mathbb{P}(\{X \leq b_1\}) \leq \mathbb{P}(\{X \leq b_2\}) \implies F(b_1) \leq F(b_2).$$

This implies that $F(x)$ is a non-decreasing function.

□

2. Basics and Definitions

2.1 Solved Problems

■ **Problem 2.1 — From Ross.** Ben can talk a course in compute science or chemistry. If she takes the computer science course, then she will get A grade with probability $\frac{1}{2}$. If she takes the chemistry course, then she will get A grade with probability $\frac{1}{3}$. She decides to base her decision on the flip of a fair coin. What is the probability that she gets an A in chemistry?

Solution We define the following events

A: she will get an A grade.
CO: she will take the computer science course.
CH: she will take the chemistry course.

Then the question is basically asking for $\mathbb{P}(A \cap CH)$. We can compute it by

$$\mathbb{P}(A \cap CH) = \mathbb{P}(A|CH)\mathbb{P}(CH) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}.$$

■ **Problem 2.2** And urn contains seven black balls and five white balls. We draw two times from the urn. Given that the each ball has the same probability to be drawn, what is the probability that both balls drawn are black?

Solution This question nicely demonstrates the fact that there are many ways to define the event spaces, and not all of them are very useful in computing the desired probability. Define

E: two drawn balls are black.

The question is in fact asking $\mathbb{P}(E)$. But this even is not very useful in any progress with the solution. Thus we need to define some finer events

E_1: The first drawn ball is black.
E_2: The second drawn ball is black.

It is clear that $E = E_1 \cap E_2$. These two finer events allows us to compute the probability of interest given the data we have in our hand.

$$\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_2|E_1)\mathbb{P}(E_1) = \frac{6}{11} \cdot \frac{7}{12}$$

■ **Problem 2.3 — From Ross.** Three men at a party through their hats into the center of the room, and then, after mixing the hats, each pick a hat randomly. What is the probability if non of them get their own hat back.

Solution There are a million ways to tack a probability problem. We can construct a suitable sample space and then compute the probabilities explicitly, or we can use the properties of the probability function to computer the desired probability without any need to construct the sample space. Here, we will demonstrate two ways.

Solving the problem by utilizing the properties of the probability function. First we need to define some suitable events. There are again many ways to define event sets and each have their own pros and cons. We proceed with the following definition.

E_i: The person i "selects" his own hat.

Also, with this particular construction of the event sets, it is much more easier to compute the complementary probability of the desired probability first and then compute the desired one by simply subtracting it from 1. The complement of the event "no men gets his own hat back" is "at least one man gets his hat back" which is $\mathbb{P}(E_1 \cup E_2 \cup E_3)$. To compute the terms of this we first need to calculate $\mathbb{P}(E_i)$, $\mathbb{P}(E_i \cap E_j)$ where $i \neq j$ and also $\mathbb{P}(E_1 \cap E_2 \cap E_3)$. We know that $\mathbb{P}(E_i) = 1/3$ for $i = 1, 2, 3$. That is because it is equally likely he selects any of the hats at the center. For $\mathbb{P}(E_i \cap E_j)$ we can write

$$\mathbb{P}(E_i \cap E_j) = \mathbb{P}(E_i|E_j)\mathbb{P}(E_j) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$

In which we used the fact that $\mathbb{P}(E_i|E_j)$ is $\frac{1}{2}$ for distinct i, j. That is because given person j selects his hat correctly, then there are two possibilities for E_i to select his hat (he can pick the correct one or the wrong one). Lastly for $\mathbb{P}(E_1 \cap E_2 \cap E_3)$ we write

$$\mathbb{P}(E_1 \cap E_2 \cap E_3) = \mathbb{P}(E_1|E_2 \cap E_3)\mathbb{P}(E_2 \cap E_3) = \mathbb{P}(E_1|E_2 \cap E_3)\mathbb{P}(E_2|E_3)\mathbb{P}(E_3) = 1 \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$

Thus

$$\mathbb{P}(E_1 \cup E_2 \cup E_3) = (1) - (1/2) + (1/6) = \frac{4}{6}.$$

Then the probability of interest will be

$$\mathbb{P}(E) = 1 - \frac{4}{6} = \frac{1}{3}.$$

Solving by constructing a sample space. A suitable sample space for this problem can be the set of all permutations on three letters. This set is

$$\Omega = \{\begin{pmatrix} a & b & c \\ \boxed{a} & \boxed{b} & \boxed{c} \end{pmatrix}, \begin{pmatrix} a & b & c \\ \boxed{a} & c & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & a & \boxed{c} \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & \boxed{b} & a \end{pmatrix}\}.$$

Note that the elements in the box represents the fixed point of the permutation. The probability of interest is basically the number of permutations that has no fixed point. As it is clear from the set Ω, the probability is

$$\mathbb{P}(E) = \frac{2}{6} = \frac{1}{3}.$$

3. Markov Chain

NOTE TO MYSELF: I prefer to develop whole theory of discrete Markov chains by defining the state space to be the set of symbols Σ (at most countable). This is beneficial, because then the sample space of any Markov chain will be the set of all infinite sequences (strings) from the symbols from Σ. At some point in the future, I might rewrite this chapter, working consistently with Σ as the state space.

We start with the definition of a Markov Chain.

Notation Let $(X_n)_{n\geq 0}$ be a Markov chain on the state space S, $x \in S$, and let E be an event. Then

$$\mathbb{P}_x(E) = \mathbb{P}(E|X_0 = x).$$

The following proposition will be one of our main tools throughout the chapter.

Proposition 3.1 — Conditional expansion. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathfrak{F}$ be a finite collection of events $\mathfrak{F} = \{F_1, F_2, \cdots, F_n\}$ that partitions Ω. I.e.

(i) $F_i \cap F_j = \varnothing \qquad i \neq j$,

(ii) $\bigcap_i F_i = \Omega$.

Let $E \in \mathcal{F}$ be any nonempty event. Then we can write

$$\mathbb{P}(E) = \sum_i \mathbb{P}(E|F_i)\mathbb{P}(F_i).$$

Proof. Since $\mathfrak{F}$ partitions Ω and $E \neq \varnothing$, then $\{E \cap F_i\}_i$ is a partition of E. Thus

$$\mathbb{P}(E) = \mathbb{P}(\bigcup_i (E \cap F_i)) = \sum_i \mathbb{P}(E \cap F_i) = \sum_i \mathbb{P}(E|F_i)\mathbb{P}(F_i).$$

This completes the proof. $\square$

Proposition 3.2 — First step argument. Let $(X_n)_{n\geq 0}$ be a Markov chain on the state space S.

Let $x \in S$, and $W, Z \subset S$. Let B be any event. Then

$$\mathbb{P}_x(B) = \sum_{y:x\sim y} \mathbb{P}_y(B)P(x,y).$$

Proof. To prove the proposition above, we Let $E_i = \{X_0 = x, X_1 = y_i\}$ where $y_i \sim x$. So, in words, we say that the event E_i has occurred if $X_1 = y_i$. It is clear that $E_i \cap E_j = \varnothing$ where $i \neq j$. Thus $\bigcup_i (B \cap E_i) = B$. Thus

$$\mathbb{P}_x(B) = \sum_i \mathbb{P}_x(B \cap E_i) = \sum_i \mathbb{P}_x(B|E_i)\mathbb{P}_x(E_i).$$

In which $\mathbb{P}_x(E_i) = \mathbb{P}(E_i|X_0 = x) = \mathbb{P}(X_1 = y_i|X_0 = x) = P(x, y_i)$. Also

$$\mathbb{P}_x(B|E_i) = \mathbb{P}(B|X_1 = y_i, X_0 = x) = \mathbb{P}(B|X_1 = y_i) = \mathbb{P}_{y_i}(B),$$

. in which we have used the Markov property. Thus we can write

$$P_x(B) = \sum_i \mathbb{P}_{y_i}(B)P(x, y_i).$$

□

3.1 Solved Problems

■ **Example 3.1** An urn always contains 2 balls. Ball colors are red and blue. At each stage a ball is randomly chosen and then replaced by a new ball, which with probability 0.8 is the same color, and with probability 0.2 is the opposite color, as the ball it replaces. If initially both balls are red, find the probability that the fifth ball selected is red. [This question is from Ross]

Solution First, we need to translate this problem to a suitable Markov chain. There are many ways we can do so, each with its own pros and cons. The difference between all of these formulations come down to our choice for the state space (i.e. the co-domain of the random variable). For instance, we can assume that the state space is $S = \{RR, RB, BB\}$ that is the content of the Urn, or we can simply say that the state space is $S = \{0, 1, 2\}$ that is the number of red ball inside the Urn. Since these two sets are isomorphic (as there is a bijection between these two sets), but the actual choice depends on personal preference. Let's proceed with $S = \{0, 1, 2\}$. Then, we need to determine the transition matrix. We can do so by doing the first step argument. We start with $P(0, 0)$.

$$P(0,0) = \mathbb{P}(X_1 = 0|X_0 = 0) = \mathbb{P}(X_1 = 0|X_0 = 0, E_R)\underbrace{\mathbb{P}(E_R|X_0 = 0)}_{0} + \underbrace{\mathbb{P}(X_1 = 0|X_0 = 0, E_B)}_{0.8}\underbrace{\mathbb{P}(E_B|X_0 = 0)}_{1},$$

where E_R is the event at which a red balls is drawn from the Urn, while E_B is the event where a blue ball is drawn. The reason behind the values for the term above are very straight forward. For instance $\mathbb{P}(E_R|X_0 = 0) = 0$ because given the fact that number of red balls in the Urn is zero ($X_0 = 0$), then the probability that we draw a red ball is zero (as there is not red balls in the Urn). For the term $\mathbb{P}(X_1 = 0|X_0 = 0, E_B) = 0.8$, because given there is no red balls inside the urn, and also given the fact that the drawn ball is blue, the probability of ending up at the state $X_1 = 0$ (i.e. still no red balls) is that probability is that we replaced the drawn ball with a blue ball (same color) which has the probability 0.8. Similarly, we can calculate the first step transition probabilities.

$$P(0,1) = \mathbb{P}(X_1 = 1|X_0 = 0) = \mathbb{P}_0(X_1 = 1) = \mathbb{P}_0(X_1 = 1|E_R)\underbrace{\mathbb{P}_0(E_R)}_{0} + \underbrace{\mathbb{P}_0(X_1 = 1|E_B)}_{0.2}\underbrace{\mathbb{P}_0(E_B)}_{1} = 0.2,$$

$$P(0,2) = \mathbb{P}(X_1 = 2|X_0 = 0) = \mathbb{P}_0(X_1 = 2) = \mathbb{P}_0(X_1 = 2|E_R)\underbrace{\mathbb{P}_0(E_R)}_{0} + \underbrace{\mathbb{P}_0(X_1 = 2|E_B)}_{0}\underbrace{\mathbb{P}_0(E_B)}_{1} = 0,$$

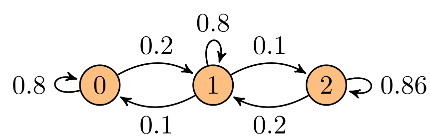
$$P(1,0) = \mathbb{P}(X_1 = 0|X_0 = 1) = \mathbb{P}_1(X_1 = 0) = \underbrace{\mathbb{P}_1(X_1 = 0|E_R)}_{0.2}\underbrace{\mathbb{P}_1(E_R)}_{0.5} + \underbrace{\mathbb{P}_1(X_1 = 0|E_B)}_{0}\underbrace{\mathbb{P}_1(E_B)}_{0.5} = 0.1.$$

$$P(1,1) = \mathbb{P}(X_1 = 1|X_0 = 1) = \mathbb{P}_1(X_1 = 1) = \underbrace{\mathbb{P}_1(X_1 = 1|E_R)}_{0.8}\underbrace{\mathbb{P}_1(E_R)}_{0.5} + \underbrace{\mathbb{P}_1(X_1 = 1|E_B)}_{0.8}\underbrace{\mathbb{P}_1(E_B)}_{0.5} = 0.8.$$

and so on. Then we will have the following transition matrix for this problem.

$$M = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{pmatrix}$$

with the following graph



Now, we need to compute the probability that the fifth ball drawn is red. This means that we have already drawn four balls, and now we want to draw the fifth one. So, we need to consider the 4 step transition matrix, i.e. M^4. Then

$$M^4 = \begin{pmatrix} 0.4872 & 0.4352 & 0.0776 \\ 0.2176 & 0.5648 & 0.2176 \\ 0.0776 & 0.4352 & 0.4872 \end{pmatrix}$$

Given that we have started with 2 red balls, then the probability of finding the Urn with 0 red balls is 0.0776, with 1 red ball is 0.4352, and with 2 red balls is 0.4872. So the probability that the next drawn balls is red is

$$\mathbb{P}(E_R) = \underbrace{\mathbb{P}(E_R|X_4 = 0)}_{0}\underbrace{\mathbb{P}(X_4 = 0)}_{0.0776} + \underbrace{\mathbb{P}(E_R|X_4 = 1)}_{0.5}\underbrace{\mathbb{P}(X_4 = 1)}_{0.4352} + \underbrace{\mathbb{P}(E_R|X_4 = 2)}_{1}\underbrace{\mathbb{P}(X_4 = 2)}_{0.4872} = 0.7048.$$

■

■ **Example 3.2 — Turning non-Markov processes to Markov-chain.** Suppose that whether or not it rains today depends on previous weather conditions through the last two days. Specifically, suppose that if it has rained for the past two days, then it will rain tomorrow with probability 0.7; if it rained today but not yesterday, then it will rain tomorrow with probability 0.5; if it rained yesterday but not today, then it will rain tomorrow with probability 0.4; if it has not rained in the past two days, then it will rain tomorrow with probability 0.2. [This question is from Ross]. Given that it rained on Monday and Tuesday, what is the probability that it will rain on Thursday?

Solution This random process is not a Markov chain, the value of the random variable at the next state, depends on two previous states. However, we can turn this into a Markov chain. Define the following states

RR: Rained yesterday and today.
$R\overline{R}$: Rained yesterday, but not today.
$\overline{R}R$: Not rained yesterday, but rained today.
$\overline{RR}$: Not rained yesterday and today.

Suppose that we are at state RR. Suppose that it rained yesterday and also today. Thus we are at state RR. If it rains tomorrow, then we will be still at state RR. That is because, That is because the yesterday of tomorrow is today! So if it rains tomorrow, since today (yesterday of tomorrow) was also rainy, thus if it rains tomorrow then we will stay at state RR. If it does not rain tomorrow, then we will get to state $\overline{R}R$. The following matrix is the transition matrix for this Markov chain

$$M = \begin{pmatrix} 0,7 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \end{pmatrix}$$

Now, to calculate the probability of raining on Thursday, given it rained on Monday and Tuesday, we first need to calculate the two step transition probability.

$$M^2 = \begin{pmatrix} \boxed{0.49} & 0.21 & \boxed{0.12} & 0.18 \\ 0.2 & 0.2 & 0.12 & 0.48 \\ 0.35 & 0.15 & 0.2 & 0.3 \\ 0.1 & 0.1 & 0.16 & 0.64 \end{pmatrix}$$

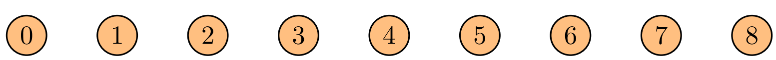
The probability to rain on Thursday is the sum of the boxed elements in the matrix above. So the desired probability is

$$p = 0.61.$$

■

■ **Example 3.3** Suppose that balls are successively distributed among 8 urns, with each ball being equally likely to be put in any of these urns. What is the probability that there will be exactly 3 nonempty urns after 9 balls have been distributed? [Question from Ross]

Solution Let the random variable X_n be the number of filled (non-empty) urns at step n. So the state space will be $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, which is represented in the following graph.



This picture is not yet complete and we need to include the transition probabilities. We will do so by the first step argument. First, observe that $P(0, 0) = 0$, because if we start with all of the urns empty, then after one step, we have put a ball somewhere, thus it is impossible to end up with zero filled urn. Similarly, $P(8, 8) = 1$, that is because if all of the urns are filled, then adding any new ball somewhere to any of the urns will keep the number of filled urns at 8. Then for $X_0 = n$, i.e. starting with n filled urns, we have

$$\mathbb{P}_n(X_1 = n - 1) = 0.$$

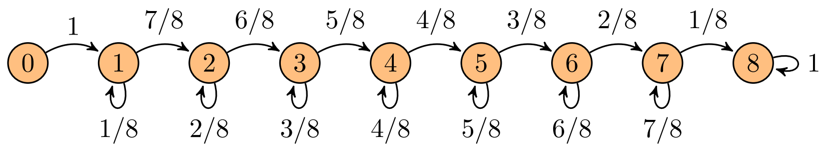
That is because starting with n filled urns, after doing one step, it is not possible to have less urns filled. I.e. after each step, we can either end up with more filled urns or the same number of filled urns. For $P(n, n)$, define the event E be the event of putting the ball in any of the filled urns. Thus E^c will be the probability of putting the ball at one of the empty urns.

$$\mathbb{P}_n(X_1 = n) = \underbrace{\mathbb{P}_n(X_1 = n|E)}_{1}\underbrace{\mathbb{P}_n(E)}_{n/8} + \underbrace{\mathbb{P}_n(X_1 = n|E^c)}_{0}\underbrace{\mathbb{P}_n(E^c)}_{(8-n)/8} = \frac{n}{8}.$$

Now for $\mathbb{P}(n, n+1)$ we can write

$$\mathbb{P}_n(X_1 = n+1) = \underbrace{\mathbb{P}_n(X_1 = n+1|E)}_{0}\underbrace{\mathbb{P}_n(E)}_{n/8} + \underbrace{\mathbb{P}_n(X_1 = n+1|E^c)}_{1}\underbrace{\mathbb{P}_n(E^c)}_{(8-n)/8} = 1 - \frac{n}{8}.$$

Thus the completed graph will be



The corresponding transition matrix will be

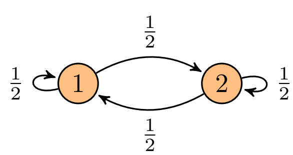
$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/8 & 7/8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/8 & 6/8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3/8 & 5/8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4/8 & 4/8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5/8 & 3/8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6/8 & 2/8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7/8 & 1/8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The probability that after 9 steps, there are exactly three empty urns is $(M^9)_{(0,3)}$, which is

$$p = (M^9)_{(0,3)} \approx 0.007572.$$

■

■ **Example 3.4** It is a good practice to derive the value of the transition probability of a simple Markov chain using the first principles. Consider the Markov chain representing a lamp that turns on with probability 1/2 and turns off with probability 1/2, and stays at the old state with probability 1/2. Thus we will have the following diagram for this Markov chain.



In this example, the state space is $S = \{0, 1\}$, and the sample space is

$$\Omega = \{(x_1, x_2, \cdots) : x_i \in S\}$$

which is basically the set of all sequences of one's and zero's. Given this, the random variables $(X_n)_n$ defined t be

$$X_n(\omega) = x_n,$$

where $\omega \in \Omega$ and x_n is the n-th letter in ω. Intuitively speaking, we know that

$$P(1,0) = \mathbb{P}(X_{n+1} = 1|X_n = 0) = \frac{1}{2}.$$

However, here we want to derive that number more explicitly by working directly with the elements of the probability space. First, we need to determine the event associated with $X_{n+1} = 1$. This is the event that has elements where the $n + 1$-th position is 1. I.e.

$$E = \{(x_1, x_2, \cdots, x_n, 1, x_{n+2}, \cdots) : x_i \in S\}.$$

Similarly, we have

$$F = \{(x_1, x_2, \cdots, x_{n-1}, 0, x_{n+1}, \cdots) : x_i \in S\}.$$

So we have

$$\mathbb{P}(X_{n+1} = 1 | X_n = 0) = \mathbb{P}(E|F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)} = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F \cap E) + \mathbb{P}(F \cap E^c)} = \frac{\frac{1}{|\Omega|}}{\frac{1}{|\Omega|} + \frac{1}{|\Omega|}} = \frac{1}{2}.$$

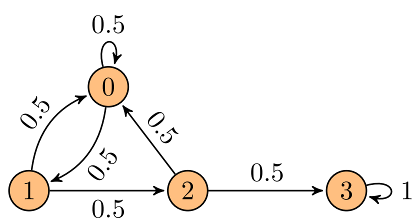
Note that $\mathbb{P}(E \cap F) = \frac{1}{|\Omega|}$, since out of many combinations of the sequence of zeros and ones, there is one one sequence whose n-th place is 0 and $n + 1$-th place is 1. Furthermore, $\mathbb{P}(F \cap E^c) = \frac{1}{|\Omega|}$ as there is only one string where its n-th and $(n + 1)$-th string are both zero. ■

■ **Example 3.5** In a sequence of independent flips of a fair coin, let N denote the number of flips until there is a run of three consecutive heads. Find

(a) $\mathbb{P}(N \leq 8)$,

(b) $\mathbb{P}(N = 8)$.

Solution Let X_n denote the number of consecutive heads at step n. For instance for the outcome $\omega \in \Omega$ where $\omega = HTTHTTHHHTTHT \ldots$, $X_2(\omega) = 0$ since the second symbol is T thus there is no consecutive heads. But $X_4(\omega) = 1$, as there is one consecutive heads at step 4. Lastly $X_9(\omega) = 3$, since there is three consecutive heads at step 9. This Markov chain will have the following transition diagram.



The transition probabilities are simply computed by the first step argument. For instance, for $P(0, 1)$ we have

$$\mathbb{P}_0(X_1 = 1) = \underbrace{\mathbb{P}_0(X_1 = 1|H)}_{1}\underbrace{\mathbb{P}_0(H)}_{1/2} + \underbrace{\mathbb{P}_0(X_1 = 1|T)}_{0}\underbrace{\mathbb{P}_0(T)}_{1/2},$$

where H is the event that the flipped coin is heads and $H^c = T$. The transition matrix for this Markov chain will be

$$M = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since the state 3 is an absorbing state, then if we get there we will be there for the rest of our life! Thus the probability that the random walker has got there for $N \leq 8$ is simply $(M^8)_{(0,3)}$. Then

$$\mathbb{P}(N \leq 8) = 0.4180.$$

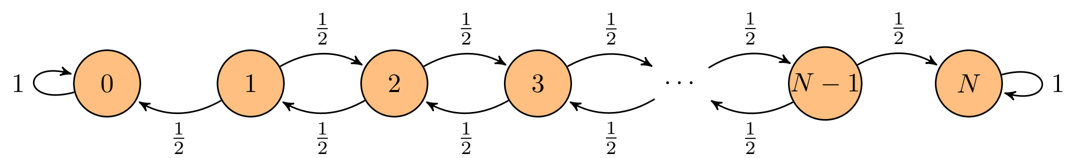
Now for part (b), the probability that the random walker has arrived at the state 3 right at the step 8, is

$$\mathbb{P}(N=8)=\mathbb{P}(N\le 8)-\mathbb{P}(N\le 7)=0.0508.$$

■

■ **Example 3.6 — Gambler's Ruin.** Suppose Alice and Bob have in total of N coins. Alice and Bob play a game with a fair coin. When Alice wins, gets a coin from Bop, and vise versa. What is the probability that Alice wins if she starts with $0\le a\le N$ coins.

Solution There are many ways to tackle a probability problem like this and the solution presented here is not the only way to find the solution to this problem. We want to model this with Markov chain whose state space is $\{0,1,2,\cdots,N\}$. Thus X_n represents the fortune of Alice after playing the games for n times.



Let p_a be the probability of Alice wining if she starts with a coins. First, observe that $p_0=0$ and $p_N=1$. Let E denote that event of Alice wining the whole game. Also, let F_1 be the event in which she looses the first game and F_2 the event in which she wins the first game. Then

$$p_a=\mathbb{P}_a(E)=\underbrace{\mathbb{P}_a(E|F_1)}_{\mathbb{P}(E|F_1,X_0=a)}\ \mathbb{P}(F_1)+\underbrace{\mathbb{P}_a(E|F_1^c)}_{\mathbb{P}(E|F_1^c,X_0=a)}\ \mathbb{P}(F_1^c)$$

(note that this identity is actually true for any set F_1, but here F_1 is the specific event explained above). The probability that she looses or wins the first game is $\frac{1}{2}$. Also, observe that $\mathbb{P}_a(E|F_1)=p_{a+1}$ (since if she wins the first game she will have one more coin) and $\mathbb{P}_a(E|F_1^c)=p_{a-1}$. Thus

$$p_a=\frac{1}{2}p_{a+1}+\frac{1}{2}p_{a-1}.$$

Now we can solve this recurrent equation with the characterization polynomial which is $2=X+1/X$ or $X^2-2X+1=(X-1)^2=0$. Thus the characteristic polynomial has a double root. Thus

$$p_a=(Aa+B)(1)^a=Aa+B.$$

Since $p_0=0,\ p_N=1$, then it turns out that

$$p_a=\frac{a}{N}.$$

■

■ **Example 3.7 — Gambler's Ruin with Draw.** Let Alice and Bob play Rock-Paper-Scissors. If Alice and Bob has a total of N coins, and at each play, the winner gets one coin from the loser, what is the probability that Alice will win the game if he starts with a coins. When they draw, then they repeat the game (or equivalently, they play another game without any coins exchange).

Solution We need to do a first step analysis similar to what we did before. Let E be the event that Alice wins the whole game, and the event $F=F_{-1}\cup F_0\cup F_1$ where

F_{-1}: Alice loses the first game,
F_0: Alice draws the first game,
F_1: Alice wins the first game.

It is clear that $\mathbb{P}(F) = 1$, since the components are mutually disjoint. Thus $E \cap F_{-1}$, $E \cap F_0$, $E \cap F_1$ are also mutually disjoints where. Thus we can write

$$\mathbb{P}_a(E) = \mathbb{P}_a(E \cap F_{-1}) + \mathbb{P}_a(E \cap F_0) + \mathbb{P}_a(E \cap F_1) = \mathbb{P}_a(E|F_{-1})\mathbb{P}_a(F_{-1}) + \mathbb{P}_a(E|F_0)\mathbb{P}_a(F_0) + \mathbb{P}_a(E|F_1)\mathbb{P}_a(F_1).$$

Since the game is fair we know

$$\mathbb{P}_a(F_{-1}) = \mathbb{P}_a(F_0) = \mathbb{P}_a(F_1) = \frac{1}{3}.$$

Furthermore, we know

$$\mathbb{P}_a(E|F_{-1}) = p_{a-1}, \qquad \mathbb{P}_a(E|F_0) = p_a, \qquad \mathbb{P}_a(E|F_1) = p_{a+1}.$$

Thus the first step analysis will lead to the following identity.

$$\mathbb{P}_a(E) = p_a = \frac{1}{3}(p_{a-1} + p_a + p_{a+1}),$$

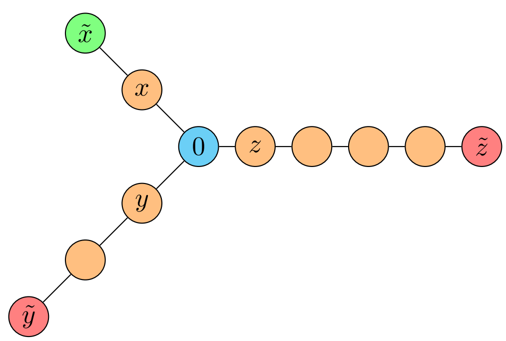
which after simplification becomes

$$2p_a = p_{a-1} + p_{a+1},$$

which is the same recursive formula we got in the previous example. So the possibility of the draw, will not change the behaviour of the system.

■

■ **Example 3.8** Consider the a simple random walker on the following graph. Let $B = \{T_{\tilde{x}} < T_{\{\tilde{z},\tilde{y}\}}\}$. Compute the probability $\mathbb{P}_0(B)$.



Solution This problem is simply asking what is the probability that we hit $\tilde{x}$ state before hitting any of $\tilde{y}$ or $\tilde{z}$ states, given the fact that the random walker starts from the state 0. To keep unnecessary details out of the way, we have only labeled the vertices that we will use in our analysis. We will have the following notation to simplify the solution

$$p_v = \mathbb{P}_v(B),$$

where v is any vertex in the graph. Note that starting at 0, i.e. $X_0 = 0$, then going to any of the states x, y, or z, are mutually disjoint events, and the probability of the union of these events is one. With our first time step analysis (see Proposition 3.2) we can write

$$\mathbb{P}_0(B) = \frac{1}{3}(p_x + p_y + p_z).$$

Now we need to analyze each of terms in the RHS. Let's start with p_z. Consider two events $\{T_0 < T_{\tilde{z}}\}$ and $\{T_0 > T_{\tilde{z}}\}$, where the first time is the event where the random walker hits the 0 state before hitting the $\tilde{z}$ step first, and the second one is the vice versa. These two events are disjoint and the probability of the union is 1. Thus we write the conditional expansion of p_z based on these events

$$p_z = \mathbb{P}_z(B) = \mathbb{P}_z(B|T_0 < T_{\tilde{z}})\mathbb{P}_z(T_0 < T_{\tilde{z}}) + \mathbb{P}_z(B|T_0 > T_{\tilde{z}})\mathbb{P}_z(T_0 > T_{\tilde{z}}).$$

We know that $\mathbb{P}_z(B|T_0 > T_{\tilde{z}}) = \mathbb{P}(B|X_0 = z, X_i = \tilde{z})$ for some $i > 0$. From Markov property it follows that

$$\mathbb{P}(B|X_0 = z, X_i = \tilde{z}) = \mathbb{P}(B|X_i = \tilde{z}) = \mathbb{P}(B|X_0 = \tilde{z}) = p_{\tilde{z}}.$$

Also $\mathbb{P}_z(B|T_0 < T_{\tilde{z}}) = \mathbb{P}_0(B) = p_0$ by the Markov property. Lastly, $\mathbb{P}_z(T_0 < T_{\tilde{z}})$ is determined by the Gambler's ruin method we say before, which is basically

$$\mathbb{P}_z(T_0 < T_{\tilde{z}}) = \frac{5}{4}, \qquad \mathbb{P}_z(T_0 > T_{\tilde{z}}) = \frac{1}{5}.$$

By doing the same kind of analysis for p_x as well as p_y we will get

$$p_z = \frac{4}{5}p_0, \qquad p_y = \frac{2}{3}p_0, \qquad p_x = \frac{1}{2}p_0 + \frac{1}{2}.$$

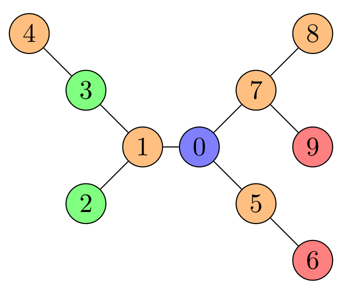
Now by substituting in the identity we got from the first time step argument, we can fine that

$$p_0 = \frac{15}{31},$$

And this completes our solution for the problem.

■

■ **Example 3.9** Consider the graph $\gamma = (V, E)$ drawn below. Set $Z = \{2, 3\}$, and $W = \{6, 9\}$. Compute $\mathbb{P}_0(T_Z < T_W)$. In colors: we start at blue, win if we reach green, and lose of we reach red.



■

Solution As always, we start with our powerful tool in hand, which is the first step argument (which is basically a special form of the more general conditional expansion). We start with first step argument at state 0. We will get

$$\mathbb{P}_0(B) = \frac{1}{3}(\mathbb{P}_1(B) + \mathbb{P}_7(B) + \mathbb{P}_5(B)),$$

and now we need to analyze each of the terms in the right hand side. We start with $\mathbb{P}_5(B)$ which is the most straight forward one. As we saw in the last example, we can analyze this state with a conditional expansion on the two disjoint events, whose union probability is 1. Let those two events be $\{T_6 < T_0\}$ (where the random walker hits the state 6 before hitting the state 0), and $\{T_6 > T_0\}$, where the random walker hits the state 0 before hitting the state 6. Thus the expansion will be

$$\mathbb{P}_5(B) = \mathbb{P}_5(B|T_6 < T_0)\mathbb{P}_5(T_6 < T_0) + \mathbb{P}_5(B|T_6 > T_0)\mathbb{P}_5(T_6 > T_0).$$

We know that if we hit the state 6 before 0, we have no chance to hit any of the green states (we will lose). Thus

$$\mathbb{P}_5(B|T_6 < T_0) = 0.$$

And from the Gambler's ruin we know that $\mathbb{P}_5(T_6 > T_0) = 1/2$, and from the Markov property we know that $\mathbb{P}_5(B|T_6 > T_0) = \mathbb{P}_0(B)$, because the conditional probability $\mathbb{P}_5(B|T_6 > T_0)$ is basically stating what is the probability of B happening, if we start from 5 and $X_i = 0$ for some i in the future. Thus

$$\mathbb{P}_5(B) = \frac{\mathbb{P}_0(B)}{2}.$$

Now, we need to analyze the term $\mathbb{P}_1(B)$. Again, at this step, we do another first step analysis.

$$\mathbb{P}_1(B) = \frac{1}{3}(\underbrace{\mathbb{P}_3(B)}_{=1} + \underbrace{\mathbb{P}_2(B)}_{=1} + \mathbb{P}_0(B)) = \frac{2 + \mathbb{P}_0(B)}{3}.$$

Note that from the assumption, we know that if we reach any of green states, then we are declared winner, that is why we have $\mathbb{P}_3(B) = \mathbb{P}_2(B) = 1$. Now it only remains to analyze the term $\mathbb{P}_7(B)$. Again, similar to the case above, we do a first time step argument

$$\mathbb{P}_7(B) = \frac{1}{3}(\mathbb{P}_0(B) + \underbrace{\mathbb{P}_8(B)}_{=\mathbb{P}_7(B)} + \underbrace{\mathbb{P}_9(B)}_{=0}) \implies \mathbb{P}_7(B) = \frac{\mathbb{P}_0(B)}{2}.$$

Note that $\mathbb{P}_8(B) = \mathbb{P}_7(B)$ by a first stem analysis when starting at the state 8. Putting all of these terms back to the original identity we derived the first, we can conclude that

$$p_0 = \mathbb{P}_0(B) = \frac{2}{5}.$$

3.2 Solved Problems

■ **Problem 3.1** The French roulette game has slots numbered from 0 to 36. The slot 0 is green, Among the slots from 1 to 36, 18 are black and 18 are red. Alex goes to a casino to play roulette. Their strategy is to always bet "red". They start with 50 coins, play 1 coin each turn, and stop when reaching 100 or getting broke.

(a) What is the probability that Alex reaches 100?

(b) How many coins should Alex start with to have about 50% chance to reach 100?

Solution (a) Let B be the event $B = \{T_{100} < T_0\}$ and we are looking for $\mathbb{P}_a(B)$ where $0 \leq a \leq 100$ and indicates the number of coins we are starting with. First observe that

- $p_0 = 0$: Since if we start with zero coins we are already broken and the game is over.
- $p_{100} = 1$: Since if we start with 100 coins then we won the game and the game is finished.

To compute the probability for intermediate values of a, we do the first step argument. Let WF be the event where Alex wins the first bet, and LF the event where Alex loses the first bet. Then we can write

$$p_a = \mathbb{P}_a(B) = \mathbb{P}_a(B|WF)\mathbb{P}_a(WF) + \mathbb{P}_a(B|LF)\mathbb{P}_a(LF).$$

Since there are 18 red spots, then the chance to win the first bet is

$$\mathbb{P}_a(WF) = \frac{18}{37}.$$

and since there are 19 non-red spots in total, then the chance to win is

$$\mathbb{P}_a(LF) = \frac{19}{37}.$$

Also, from Markov property, we know that

$$\mathbb{P}_a(B|WF) = p_{a+1}, \qquad \mathbb{P}_a(B|LF) = p_a.$$

Thus the first step argument formula will be

$$p_a = \frac{18}{37}p_{a+1} + \frac{19}{37}p_{a-1} \implies \boxed{37p_a = 18p_{a+1} + 19p_{a-1}}.$$

The characteristic equation for the recursive equation is

$$37 = 18x + \frac{19}{x} \implies \boxed{18x^2 - 37x + 19 = 0}.$$

We can write it as $(x-1)(18x-19) = 0$. Thus the roots will be

$$r_1 = 1, \qquad r_2 = \frac{19}{18}.$$

So

$$p_a = A + Br_2^a.$$

To fine A and B we use the fact $p_0 = 0$, and $p_{100} = 1$. Then $A = -B$, and $A = 1/(1 - r_2^{100})$. Thus

$$\boxed{p_a = \frac{1 - r_2^a}{1 - r_2^{100}}}.$$

(b) We basically need to compute find a for which $p_a = 1/2$. Thus we need to solve for a

$$\frac{1 - r_2^a}{1 - r_2^{100}} = \frac{1}{2}.$$

After some algebra we will find

$$\boxed{a = \frac{\ln\left(\frac{1+r_2^{100}}{2}\right)}{\ln(r_2)} \approx 87.26}.$$

Thus we need to start with at least 88 coins to have a 50% chance of winning.

□

■ **Problem 3.2** There are 6 coins on a table, each showing heads (H) or tails (T). In each step we

- Select uniformly one of the coins.
- If it is heads, toss it and replace on the table (with random side).
- If it sis tails, toss it. If it comes up heads, leave it at that. If it comes up tails, toss it a second time, and leave the result as it is. Let X_n be the number of heads showing after n such steps. Answer the following questions
  (a) Determine the transition probabilities for this Markov chain.
  (b) Draw the transition diagram and write the transition matrix.
  (c) What is $\mathbb{P}(X_2 = 4 | X_0 = 5)$?

Solution (a) To compute the transition probabilities, we need to perform the first step analysis. Let the events

$$I = \{X_1 = a+1\}, \qquad S = \{X_1 = a\}, \qquad D = \{X_1 = a-1\},$$

where $0 \le a \le 6$ is the number of heads. So to compute the transition probabilities, we need to compute

$$P(a, a+1) = \mathbb{P}_a(I), \qquad P(a,a) = \mathbb{P}_a(S), \qquad P(a, a-1) = \mathbb{P}_a(D).$$

We start with $\mathbb{P}_a(I)$. Let ST be the event where the selected coin is tails, and SH be the event where the selected coin is heads. These two events are disjoint and the probability of their union is 1, thus

$$\mathbb{P}_a(I) = \underbrace{\mathbb{P}_a(I|SH)}_{\text{see Eq (2.I.1)}} \underbrace{\mathbb{P}_a(SH)}_{\frac{a}{6}} + \underbrace{\mathbb{P}_a(I|ST)}_{\text{see Eq (2.I.2)}} \underbrace{\mathbb{P}_a(ST)}_{\frac{6-a}{6}}. \tag{2.I}$$

Note that if we start with a coins heads, then the chance we choose a random coin from the table and find it heads is $\frac{a}{6}$, hence $\mathbb{P}_a(SH) = \frac{a}{6}$, and $\mathbb{P}_a(ST) = \frac{6-a}{6}$. Now we need to expand the remaining terms with appropriate conditioning. Let TT be the event where we toss a coin and find it tails and TH be the event where we toss a coin and find it heads. Thus we can write

$$\mathbb{P}_a(I|SH) = \underbrace{\mathbb{P}_a(I|SH, TH)}_{0} \underbrace{\mathbb{P}_a(TS)}_{\frac{1}{2}} + \underbrace{\mathbb{P}_a(I|SH, TT)}_{0} \underbrace{\mathbb{P}_a(TT)}_{\frac{1}{2}}. \tag{2.I.1}$$

Note that $\mathbb{P}_a(TT) = \mathbb{P}_a(TH) = \frac{1}{2}$, since the coin tossing is fair. Also, note that $\mathbb{P}_a(I|SH, TH) = \mathbb{P}_a(I|SH, TT) = 0$ since if we select a heads, and then toss it, finding it either heads or tails will not increase the total number of heads on the table. Similarly, for the other term in (2.1) we have

$$\mathbb{P}_a(I|ST) = \underbrace{\mathbb{P}_a(I|ST, TH)}_{1} \underbrace{\mathbb{P}_a(TH)}_{\frac{1}{2}} + \underbrace{\mathbb{P}_a(I|ST, TT)}_{\text{see Eq (2.I.3)}} \underbrace{\mathbb{P}_a(TT)}_{\frac{1}{2}}. \tag{2.I.2}$$

Now we need to expand the remaining terms in the equation above.

$$\mathbb{P}_a(I|ST, TT) = \underbrace{\mathbb{P}_a(I|ST, TT, TH)}_{1} \mathbb{P}_a(TH) + \underbrace{\mathbb{P}_a(I|ST, TT, TT)}_{0} \mathbb{P}_a(TT) = \frac{1}{2}. \tag{2.I.3}$$

Putting all together we can write

$$\boxed{P(a, a+1) = \mathbb{P}_a(I) = \frac{6-a}{8}}.$$

Similarly, we can compute other transition probabilities. For instance for $\mathbb{P}_a(S)$ we can write

$$\mathbb{P}_a(S) = \underbrace{\mathbb{P}_a(S|SH)}_{\text{see Eq (2.S.1)}}\,\underbrace{\mathbb{P}_a(SH)}_{\frac{a}{6}} + \underbrace{\mathbb{P}_a(S|ST)}_{\text{see Eq (2.S.2)}}\,\underbrace{\mathbb{P}_a(ST)}_{\frac{6-a}{6}}. \tag{2.S}$$

and for the remaining terms we can write

$$\mathbb{P}_a(S|SH) = \underbrace{\mathbb{P}_a(S|SH,TH)}_{1}\,\underbrace{\mathbb{P}_a(TS)}_{\frac{1}{2}} + \underbrace{\mathbb{P}_a(S|SH,TT)}_{0}\,\underbrace{\mathbb{P}_a(TT)}_{\frac{1}{2}}, \tag{2.S.1}$$

and

$$\mathbb{P}_a(S|ST) = \underbrace{\mathbb{P}_a(S|ST,TH)}_{0}\,\underbrace{\mathbb{P}_a(TH)}_{\frac{1}{2}} + \underbrace{\mathbb{P}_a(S|ST,TT)}_{\text{see Eq (2.S.3)}}\,\underbrace{\mathbb{P}_a(TT)}_{\frac{1}{2}}. \tag{2.S.2}$$

And for the remaining term above

$$\mathbb{P}_a(S|ST,TT) = \underbrace{\mathbb{P}_a(S|ST,TT,TH)}_{0}\,\mathbb{P}_a(TH) + \underbrace{\mathbb{P}_a(S|ST,TT,TT)}_{1}\,\mathbb{P}_a(TT) = \frac{1}{2}. \tag{2.S.3}$$

and by putting all together we will get

$$\boxed{P(a,a) = \mathbb{P}_a(S) = \frac{6+a}{24}}.$$

Finally, since $\mathbb{P}_a(I \cup S \cup D) = 1$, and I, S, D are mutually disjoint, we can write

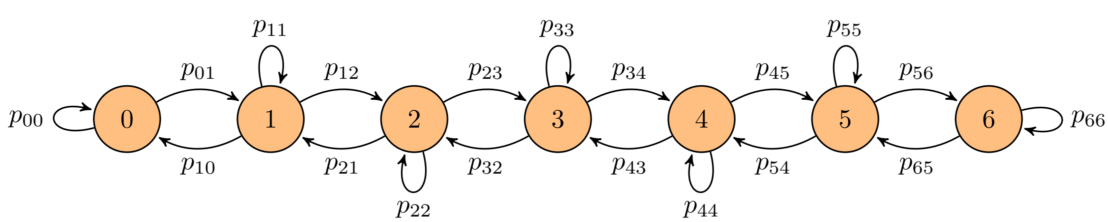
$$\mathbb{P}_a(D) = 1 - (\mathbb{P}_a(I) + \mathbb{P}_a(S)),$$

hence

$$\boxed{P(a,a-1) = \mathbb{P}_a(D) = \frac{a}{12}}.$$

so the transition probabilities are as calculated.

(b) The transition diagram is plotted below.



And the transition matrix is

$$M = \begin{pmatrix} 1/4 & 3/4 & 0 & 0 & 0 & 0 & 0 \\ 1/12 & 7/24 & 5/8 & 0 & 0 & 0 & 0 \\ 0 & 1/6 & 1/3 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 3/8 & 3/8 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 5/12 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 5/12 & 11/24 & 1/8 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

(c) $\mathbb{P}(X_2 = 4|X_0 = 5)$ is the second transition probability $P_2(5,4)$. To compute this, we need to fine the element in the 6-th row and 5-th column in the M^2 matrix, which is basically the inner product between the vectors formed by the 6-th row and the 5-th column.

$$P_2(5,4) = (\frac{5}{12})^2 + \frac{11}{24} \cdot \frac{5}{12} = \frac{35}{96}$$

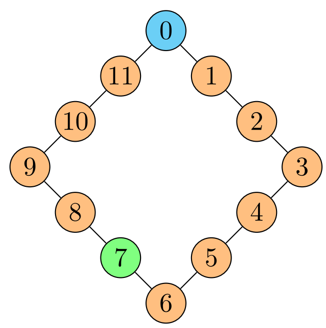
which after simplification becomes

$$\boxed{P_2(5,4) = \frac{35}{96}}.$$

□

■ **Problem 3.3** A clock is broken. It has only one hand which moves every hour either clockwise with probability 1/2 or counter-clockwise with probability 1/2 (the numbers are from 0 to 11 and the hand moves by one full hour when it moves). Assume it starts at 0. What is the probability that it reaches 7 before coming back to 0 for the first time?

Solution First, let's draw the graph representing the state space of the random variable of interest.



Define the event B be $B = \{T_0^+ > T_7\}$. We are interested in finding $\mathbb{P}_0(B)$. Now we can perform the first step argument as follows

$$p_0 = \frac{1}{2}(p_1 + p_{11}). \tag{3.1}$$

Then we analyze each term in the right hand side of the equation above. For p_1 we have

$$\mathbb{P}_1(B) = \underbrace{\mathbb{P}_1(B|T_0 > T_7)}_{1}\underbrace{\mathbb{P}_1(T_0 > T_7)}_{1/5} + \underbrace{\mathbb{P}_1(B|T_0 < T_7)}_{0}\underbrace{\mathbb{P}_1(T_0 < T_7)}_{6/7} = \frac{1}{5}.$$

Note that $\mathbb{P}_1(B|T_0 > T_7) = 1$ since it literally means the random walker reaches 7 before 0. Also $\mathbb{P}_1(B|T_0 < T_7) = 0$ since the event B is conditioned on reaching 0 before 7, which is clearly 0. The term $\mathbb{P}_1(T_0 > T_7)$ is computed using the Gambler's ruin analysis. Similarly, for the p_{11} term we have

$$\mathbb{P}_{11}(B) = \underbrace{\mathbb{P}_{11}(B|T_0 > T_7)}_{1}\underbrace{\mathbb{P}_{11}(T_0 > T_7)}_{1/7} + \underbrace{\mathbb{P}_{11}(B|T_0 < T_7)}_{0}\mathbb{P}_{11}(T_0 < T_7) = \frac{1}{7}.$$

The rationale behind the values of the terms are the same as the ones discussed above. Now we can substitute everything in (3.1)

$$\boxed{p_0 = \frac{1}{2}(\frac{12}{35}) = \frac{6}{35}}.$$

■ **Problem 3.4** The Fibonacci sequence is the sequence $(F_n)_{n\geq 0}$ defined by $F_0 = 0, F_1 = 1$ and

$$F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 0.$$

Find a general formula for F_n

Solution First, we construct the characteristic polynomial of the sequence. From the recursive formula we can write

$$X^2 = X + 1 \implies \boxed{X^2 - X - 1 = 0}.$$

The roots of the equation is

$$r_1, r_2 = \frac{1 \pm \sqrt{5}}{2}.$$

Now the general formula will be

$$F_n = Ar_1^n + Br_2^n.$$

To find the coefficients, we utilize the first two terms

$$0 = A + B, \qquad 1 = \frac{1}{2}(A + B) + \frac{\sqrt{5}}{2}(A - B).$$

This system of equations implies that

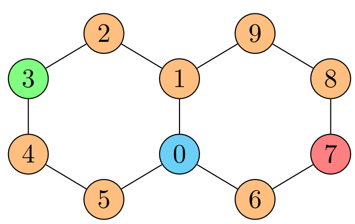
$$A = \frac{1}{\sqrt{5}}, \qquad B = \frac{-1}{\sqrt{5}}.$$

Thus the general formula will be

$$\boxed{F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - \frac{1-\sqrt{5}}{2})^n)}.$$

□

■ **Problem 3.5** Let (X_n) be the simple random walk on the following graph. Compute $\mathbb{P}_0(T_3 < T_7)$.



Solution For a much more simpler solution, let's define the two following notations

$$B = \{T_3 < T_7\}, \qquad p_v = \mathbb{P}_v(B).$$

Then, by first step argument at state 0, we can write

$$p_0 = \frac{1}{3}(p_5 + p_6 + p_1). \tag{5.1}$$

Now we need to evaluate each of the terms in the right hand side. We start with p_5.

$$p_5 = \mathbb{P}_5(B) = \underbrace{\mathbb{P}_5(B|T_3 < T_0)}_{1}\underbrace{\mathbb{P}_5(T_3 < T_0)}_{1/3} + \underbrace{\mathbb{P}_5(B|T_3 > T_0)}_{p_0}\underbrace{\mathbb{P}_5(T_3 > T_0)}_{2/3} = \frac{1}{3} + \frac{2}{3}p_0.$$

note that $\mathbb{P}_5(B|T_3 < T_0) = 1$, since if we get to state 3, before getting to state 0, then it means that we have reached the state 3 before reaching the state 7, thus the event B occurs with probability 1. Also $\mathbb{P}_5(T_3 < T_0) = 1/3$ from the Gambler's ruin. Furthermore $\mathbb{P}_5(B|T_3 > T_0) = p_0$ by using the Markov property, and finally $\mathbb{P}_5(T_3 > T_0) = 2/3$ by the Gambler's ruin.
Now, we need to evaluate the term p_6. To analyze this term, we will do a first step argument starting at this point

$$p_6 = \mathbb{P}_6(B) = \frac{1}{2}(\underbrace{p_7}_{0} + p_0) = \frac{p_0}{2}.$$

Note that $p_7 = 0$, since then the event B has not occurred.
Finally, we need to analyze the term p_1. Again, by first step argument on this state we have

$$p_1 = \frac{1}{3}(p_0 + p_9 + p_2).$$

By doing a analysis on p_9 similar to the one we did for 5, we can write

$$p_9 = \mathbb{P}_9(B) = \underbrace{\mathbb{P}_9(B|T_7 < T_1)}_{0}\mathbb{P}_9(T_7 < T_1) + \underbrace{\mathbb{P}_9(B|T_7 > T_1)}_{p_1}\underbrace{\mathbb{P}_9(T_7 > T_1)}_{2/3} = \frac{2}{3}p_1.$$

The rationale behind the values for each term in the equation above, is exactly the same as in analyzing the terms of p_5.
Now, we analyze the term p_2 by performing another first step analysis, similar to the one we did for state 6.

$$p_2 = \frac{1}{2}(\underbrace{p_3}_{1} + p_1) = \frac{1}{2}(1 + p_1).$$

Now we can calculate p_1 in terms of p_0 which turns out to be

$$p_1 = \frac{6}{11}p_0 + \frac{3}{11}.$$

Now we insert all of the terms in the equation (5.1) to get

$$\begin{aligned}
3p_0 &= \frac{1}{3} + \frac{2}{3}p_0 + \frac{1}{2}p_0 + \frac{6}{11}p_0 + \frac{3}{11} \\
\implies 3p_0 - \frac{113}{66}p_0 &= \frac{40}{33} \\
\implies p_0 &= \frac{66}{85} \cdot \frac{40}{33} = \frac{16}{17} \\
\implies & \boxed{p_0 = \frac{16}{17}}.
\end{aligned}$$

□