Lecture Notes For: The Complex Analysis and Applications

Ali Fele Paranj
alifele@student.ubc.ca

December 14, 2023

The content of this lecture note will be mostly based on the course MATH 305 (Applied Complex Analysis) at UBC during Winter2, 2023 term. However, I have expanded the content and examples using the following text books as well:

- Fundamentals of Complex Analysis for Mathematics, Science and Engineering,(Third Edition) by E. Saff, A. Snider.
- Visual Complex Functions: An Introduction with Phase Portraits by Elias Wegert

1 Fundamentals

Complex numbers can be thought as an extension to the real number system in which we have a solution for the $x^2+1=0$ equation. There are some mathematically rigorous ways to construct the complex numbers from real numbers. However, those mathematically rigorous ways came out only recently (20th century) and there were not around when the first ideas of complex numbers were forming around 18th century. For that reason I have not discussed the detailed mathematical construction of the complex numbers here.

As discussed earlier, complex numbers system is a system in which we have a solution for the $x^2+1=0$ equation which is represented as $i=\sqrt{-1}$. A complex number is written like $z=a+bi$ in which $a,b\in\mathbb{R}$ and the set of all complex numbers is denoted as $\mathbb{C}$. It is easy to check that complex numbers still satisfy the *commutative*, *associative*, and *distributive* properties similar to the real numbers.

Definition: Set of Complex Numbers and Basic Definitions $\mathbb{C}$

- The set of complex numbers: The set of complex numbers $\mathbb{C}$ is:

$$\mathbb{C}=\{z=a+bi : a,b\in\mathbb{R}, i=\sqrt{-1}\}$$

In which we call a as the *real part* and b as the *complex part* of the complex number z and we show that as:

$$\text{Re}(z)=a$$
$$\text{Im}(z)=b$$

- Two complex numbers $z_1=a_1+b_1i$ and $z_2=a_2+b_2i$ are equal if:

$$\text{Im}(z_1)=\text{Im}(z_2)$$
$$\text{Re}(z_1)=\text{Re}(z_2)$$

- Polar representation of the complex numbers: A complex number $z=a+bi$ can be written as:

$$z=re^{i\varphi}$$

in which r is the modulus of z and φ is the argument of z which is defined as: $\varphi=\arctan(\frac{a}{b})$. Also we define the set Arg as:

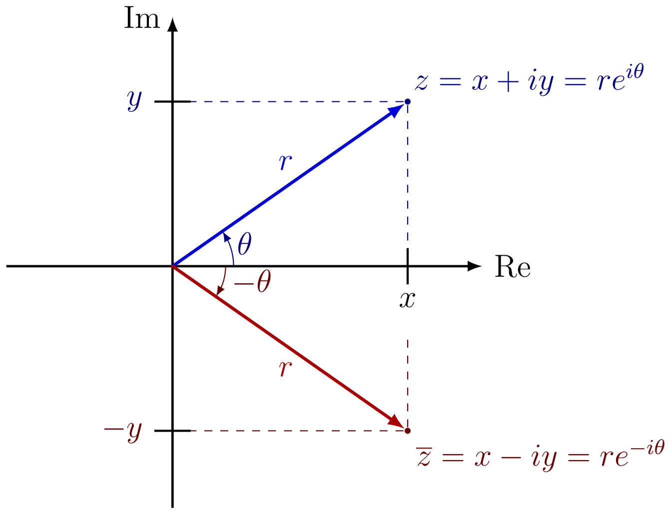


Figure 1.1: A summary of the polar and Cartesian representation of the complex numbers.

$$Arg(z) = \{arg(z) + 2\pi n : n \in \mathbb{Z}\}$$

- Complex conjugate: The complex conjugate of a complex number $z = a + bi$ is defined as:
$$\overline{z} = a - bi$$

- Modulus of a complex number: The modulus of a complex number $z = a + bi$ is defined as:
$$|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$$

The following figure summarizes the basic properties of the complex numbers discussed above.

Proposition: Fundamental Properties of Complex Numbers

Using the definition of the complex numbers, we can show that they satisfy the following properties.

1. Commutative property for the sum and product:

$$a + b = b + a \tag{1.1}$$
$$ab = ba \tag{1.2}$$

2. Associative property for the sum and product

$$a + (b + c) = (a + b) + c \tag{1.3}$$
$$a(bc) = (ab)c \tag{1.4}$$

3. Distributive property:

$$a(b + c) = ab + ac \tag{1.5}$$

Proof. All of the statements can be proved by writing complex numbers as $z = x + yi$ and substituting in the equations. □

Utilizing the basic definitions along with the mathematical logic, we can derive different properties for complex numbers. The following properties come in handy when solving problems:

Proposition: Basic Properties of Complex Numbers

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \tag{1.6}$$
$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2} \tag{1.7}$$
$$\overline{(\frac{z_1}{z_2})} = \frac{\overline{z_1}}{\overline{z_2}} \tag{1.8}$$
$$\overline{(\overline{z})} = z \tag{1.9}$$
$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2} \tag{1.10}$$
$$\mathrm{Im}(z) = \frac{z - \overline{z}}{2i} \tag{1.11}$$
$$(\overline{z})^k = \overline{(z^k)} \tag{1.12}$$
$$|z_1 z_2| = |z_1||z_2| \tag{1.13}$$

Proof. The proof for some of the properties:

- $|z_1 z_2| = |z_1||z_2|$: To show this we first need to raise the both sides to the power 2:

  $$|z_1 z_2|^2 = (z_1 z_2)\overline{(z_1 z_2)} = (z_1\overline{z_1})(z_2\overline{z_2}) = |z_1|^2|z_2|^2$$

  . So by taking the square root of the both sides we will arrive at: $|z_1 z_2| = |z_1||z_2|$

- $(\overline{z})^k = \overline{(z^k)}$: Let's start from the right hand side:

$$\overline{(z^k)} = \overline{(\underbrace{z * z * z * \ldots * z}_{\text{k times}})} = \underbrace{\overline{z} * \overline{z} * \overline{z} * \ldots * \overline{z}}_{\text{k times}} = (\overline{z})^k$$

□

Example: Ordering Property in Complex Numbers

Question. Show that we can not define any ordering property in the complex numbers.

Proof. Suppose that we *can* define ordering property in the complex numbers. So two cases might arise for the i. It will be $i > 0$ or $i < 0$.

- Let's assume that $i > 0$. Since $f(x) = x^2$ is a strictly increasing function, then by arising two sides of the inequality to the power of two we will have: $i^2 = -1 > 0$ which is a contradiction.
- Let's assume that $i < 0$. Since i is smaller than zero, we can multiply the two sides of the inequality by i but we should change the direction of the inequality sign. So we will have $i * i = -1 > 0$ which again arises a mathematical contradiction.

So we can conclude that we can not have any ordering property in the set of complex numbers. □

We can also have some important inequalities in the complex numbers that often come very handy in solving problems. The following box summarizes some of them.

Theorem: Important Inequalities in Complex Numbers

1. Basic Inequalities

$$\text{Re}(z) \leq |z|$$
$$\text{Im}(z) \leq |z|$$

2. Triangle Inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof.

- Triangle Inequality (proof number 1): Let's start by raising the right hand side to the power 2 and simplify the equation:

$$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + \overline{z_1}z_2$$

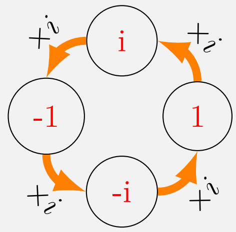
$\square$

Example: Cyclic Property of i

By using the basic property of the complex number $i = \sqrt{-1}$, for every $k \in \mathbb{Z}$ we can show that:

$$
\begin{aligned}
i^{4k} &= 1 \\
i^{4k+1} &= i \\
i^{4k+2} &= -1 \\
i^{4k+3} &= -i
\end{aligned}
$$

This also represent that fact that multiplying a complex number by i means a $\pi/2$ counterclockwise rotation. This becomes very clear if we consider the polar representation of complex numbers.



Example: Karatsuba Algorithm

Suppose that we want to multiply two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ in the computer and return $z = z_1 + z_2$. To do that we should get the imaginary and real part of two numbers and then return:

$$
\begin{aligned}
\mathrm{Re}(z) &= ac - bd \\
\mathrm{Im}(z) &= bc + ad
\end{aligned}
$$

So to multiply two complex numbers in the computer, we need to do 4 multiplications and 2 additions. So performing multiplication takes considerably more clock cycles in CPU. However we can do a trick to reduce the number of multiplications to 3 with a cost of some extra additions. To do so we need to calculate three intermediate variables each of which require one multiplication.

$$
\begin{aligned}
t_1 &= ac \\
t_2 &= bd \\
t_3 &= (a + b)(c + d)
\end{aligned}
$$

So we will have:

$$\begin{aligned}
\mathrm{Re}(z) &= t_1 - t_2 \\
\mathrm{Im}(z) &= t_3 - (t_1 + t_2)
\end{aligned}$$

So using the Karatsuba algorithm we will have 3 multiplications, and 5 addictions. There is a very interesting story behind this discovery by the Karatsuba in 1960 (when he was a 23-year-old graduate student) that he somehow proved the Kolmogorov's statement is wrong! You can read more on this in the Wikipedia page of Karatsuba algorithm.

Example: Complex Roots

Find all of the roots of the following equations

1. $z^4 - 16 = 0$

   Solution. Writing the complex number in the form of its polar representation will ease finding the roots of this equation significantly. So let's assume

   $$z = re^{i\varphi}$$

   Then by substituting in the equation we will have:

   $$r^4 e^{4i\varphi} = 16 = 16e^{2n\pi i}$$

   So we will have: $r = 2$ and $\varphi = \{n\pi/2 : n \in \mathbb{R}\} = \{0, \pi/2, \pi, 3\pi/2\}$. □

2. $\frac{z}{1-z} = 1 - 5i$

   Solution.

   $$\begin{aligned}
   z &= (1-z)(1-5i) \\
   &= 1 - 5i - z - 5iz \\
   &= \frac{1-5i}{2-5i} = \frac{(1-5i)(2+5i)}{29}
   \end{aligned}$$

   □

Proposition: Roots of Polynomial

if z_0 is the root of the following polynomial equation (in which $a_i \in \mathbb{R}$):

$$z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n = 0$$

then $\overline{z_0}$ is also the root of the equation.

Proof. This can easily be shown by taking the complex conjugate from both sides of the equation, and then using the fact that $(\overline{z})^k = \overline{(z^k)}$.

□

Example: Matrices with Complex Entries

TO BE COMPLETED

2 Planar Sets and Geometry

The interesting fact about the complex plane that I like the most is that we can construct different good looking (!) sets in the complex plain $\mathbb{C}$ with quite simple statements. For example the inequality $|z| \leq 9$ represents a circle in the complex plane centred at the origin and has a radius of 3.

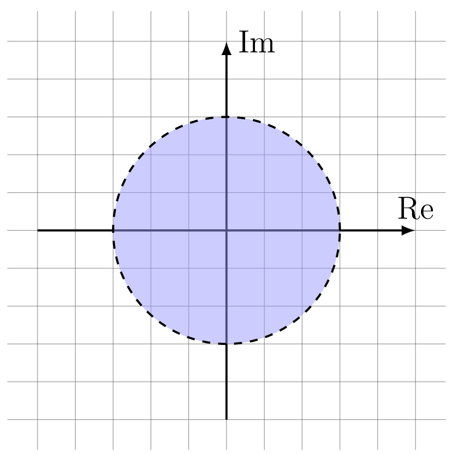


Figure 2.1: The planar set representing $|z| \leq 3$

Example:

Question. Derive the points in $\mathbb{C}$ that satisfy following conditions.

1. $|z+2| = |z-1|$

   Solution. There is not a single method that works the best for all of the problems. For this question, it is better to raise both sides of the eqution to the power 2.

$$
\begin{aligned}
|z+2|^2 &= |z-1|^2 \\
(z+2)\overline{(z+2)} &= (z-1)\overline{(z-1)} \\
(z+2)(\overline{z}+2) &= (z-1)(\overline{z}-1) \\
z\overline{z} + 2z + 2\overline{z} + 4 &= z\overline{z} - z - \overline{z} + 1 \\
z + 1 &= -\overline{z}
\end{aligned}
$$

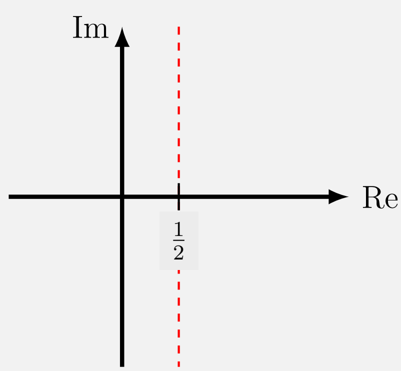
   So let's assume z is in the form $z = x + iy$. By inserting this in equation above we will get:

$$(x+1) + iy = -x + iy$$

   So:

$$
\begin{aligned}
x &= \frac{1}{2} \\
y &= \text{any real number}
\end{aligned}
$$

   So the set of points satisfying this equation will be:



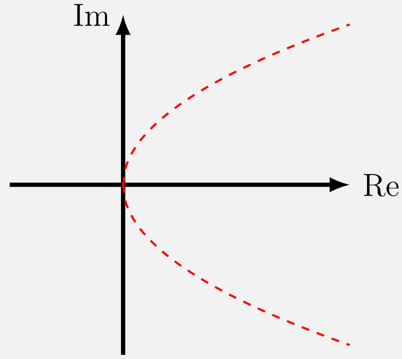
---

2. $|z-1| = \operatorname{Re} z + 1$

   Solution. For this question it is better to start with a general form of $z = x + iy$ and then substitute in the equation.

$$
\begin{aligned}
|(x-1) + yi| &= x + 1 \\
(x-1)^2 + y^2 &= (x+1)^2 \\
z^2 - 2x + 1 + y^2 &= x^2 + 2x + 1 \\
y^2 &= 4x
\end{aligned}
$$

So the points of complex plain that satisfy the equation will be like the curve:



2.1 * Discussion on the Argument of Complex Numbers

As discussed earlier, we can have Cartesian or polar representation of the complex number $z = x + iy$. The polar presentaion will be in the form $z = re^{i\varphi}$ in which r is the modulus of z and φ is so called the argument of the complex number. The modulus is defined as $r = \sqrt{x^2 + y^2} = |z|$. However, the argument definition is quite tricky.

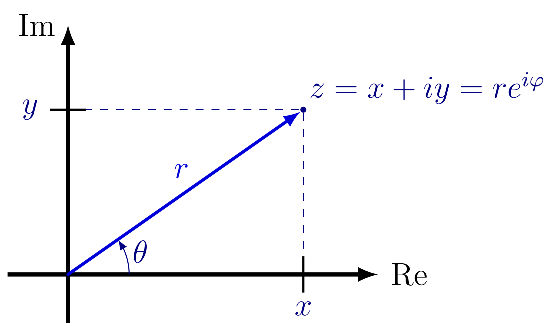


Figure 2.2: Polar and Cartesian representaion of complex number $z = x + iy$.

Based on the geometrical interpretation of the complex numbers, it feels natural that we define the argument using the inverse tangent function. $\varphi = \arctan(\frac{x}{y})$. But according to the figure 2.1, the range of this function is defined in $(-\pi/2, \pi/2)$ interval. So for the complex points in the first and fourth quadrant we get a meanungful φ value but for the points in the second and the third quadrant, the interpretation of the φ does not work well. This probelm can be solved by considering the signs of the Re and Im parts of the complex number or considering the $\cos(\theta) = \frac{x}{r}$ and $sin(\theta) = \frac{y}{r}$ seperatly. The latter works because sin and cos uniquely determine the position (and hence the argument) of a complex number. For the former, the following definition determines a unique argument even for the points in the second and the fourth quadrant.

$$arg(z) = \begin{cases} \arctan(y/x) + \pi/2(1 - \text{sgn}(x)) & \text{if } x \neq 0 \\ \pi/2 \ \text{sgn}(y) & \text{if } x = 0, y \neq 0 \\ \text{undefined} & \text{if } x = y = 0 \end{cases}$$

in which sgn(x) is the sign function that returns the sign of its argument.

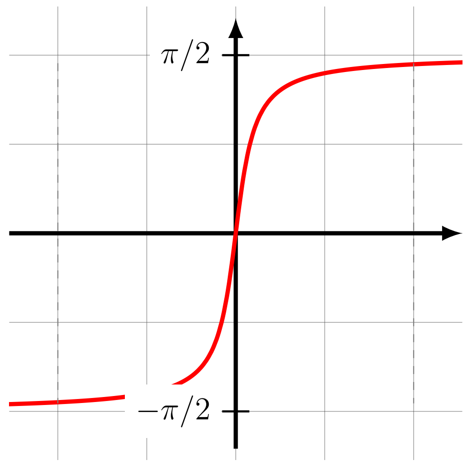


Figure 2.3: Plot of $y = \arctan(\theta)$ function.

3 Complex Maps

3.1 Linear Map

3.2 Inverse Map

3.3 Mobius Map

3.4 Quadratic Map

3.5 Exponential Map

4 Calculus for Complex variables

4.1 Limit

4.2 Continuity

4.3 Differentiability

5 Quick Facts

5.1 Analytic Functions

Fact 5.1.1 (Harmonic Function)**.** Let $f : D \to \mathbb{C}$ be a holomorphic function in the connected open set $D \subseteq \mathbb{C}$, where $f(x, y) = u(x, y) + iv(x, y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$. Then the function u, v are harmonic functions.

Proof. Since f is holomorphic in the open connected subset $D \subseteq \mathbb{C}$, then it means that the function u, v satisfy the Cauchy-Riemann equations, the first derivatives exists, and are continuous. As we will show later, in fact u, v are smooth function, so any higher derivative exists and is continuous. From Cauchy-Riemann we have

$$u_x = v_y, \quad u_y = -v_x.$$

Calculating the second derivatives will yield

$$u_{xx} = v_{yx}, \quad u_{yy} = -v_{xy}.$$

Note that since u, v are smooth, then $v_{xy} = v_{yx}$. Then we can conclude that

$$u_x x + u_y y = 0,$$

which indicates that u is a harmonic function (it satisfies the Laplace equation). The same reasoning works for v. $\square$

Fact 5.1.2 (Harmonic Conjugate Function)**.** Let $u : \Omega \to R$ be a harmonic function define on the connected open set $\Omega \subseteq \mathbb{R}^2$. Then $v : \Omega \to \mathbb{R}$ is a *harmonic conjugate* of u if and only if the function f of the complex variable $z := x + iy \in \Omega$ is holomorphic.

Example 5.1.1. Construct an analytic function whose real part is $u(x, y) = x^3 - 3xy^2 + y$.

Answer. Let $f(z) = u(x, y) + iv(x, y)$ holomorphic as required. Then it should satisfy the Cauchy-Riemann equations.

$$v_y = 3x^2 - 3y^2, \quad v_x = 6xy - 1.$$

Integrating the first expression yields in $v(x, y) = 3x^2y - y^3 + f(x)$ and differentiating it and comparing it with the second expression above yields $f(x) = -x + C$ for some $C \in \mathbb{R}$. Thus a complex conjugate of $u(x, y)$ will be

$$v(x, y) = 3x^2y + y^3 - x + C.$$

Fact 5.1.3. Let $f : \Omega \to \mathbb{C}$ where $\Omega \subseteq \mathbb{C}$ is an open connected set. Assume $f(x + iy) = u(x, y) + iv(x, y)$. If f is holomorphic at Ω, then v is a harmonic conjugate function of u. Further more, the level sets of these function are perpendicular to each other everywhere in the domain Ω.

Proof. let $(x_0, y_0) \in \Omega$. The gradient of u and v are given as

$$\nabla u = (u_x, u_y), \quad \nabla v = (v_x, v_y).$$

When evaluated at (x_0, y_0), they will be the vectors perpendicular to the corresponding level curves. We calculate the inner product of these vectors.

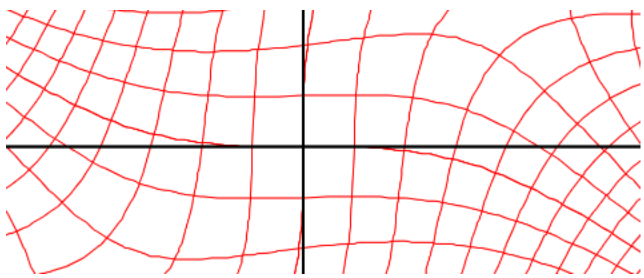
$$\nabla u(x_0, y_0) \cdot \nabla v(x_0, v_0) = u_x(x_0, y_0) y_x(x_0, y_0) + u_y(x_0, y_0) v_y(x_0, y_0)$$

Since f is holomorphic at Ω, then it satisfies the Cauchy-Riemann equations. Thus we get

$$\nabla u(x_0, y_0) \cdot \nabla v(x_0, v_0) = 0.$$

This implies that the level curves are perpendicular. □

The following figure shows this fact for the harmonic conjugate functions we calculated in Example 5.1.1.



Fact 5.1.4 (Harmonic Conjugacy is Not Symmetric)**.** If v is a harmonic conjugate of u, then it means that $f(x + iy) = u(x, y) + iv(x, y)$ is holomorphic at the open, connected disk of definition. Then this implies that $if(x + iy) = -v + iu$ is holomorphic as well. Thus we conclude that u is a harmonic conjugate of $-v$.

Fact 5.1.5 (The parallel between Picard iteration and Newton's method)**.** As we know from Galois theory, there are no closed form formula for the roots of polynomials of order 5 and higher. On the other hand, from the fundamental theorem of algebra we know that such roots exists and the number of roots is in fact the same as the degree of polynomials (multiplicity counted). However, in general, to find the roots of the degree 5 or higher polynomials we use the methods like the Newton's method. There is a beautiful parallel between this and the Picard iteration in finding the solution of an ODE. In both cases, we use an iterative approach to find the solution of a particular equation.

Fact 5.1.6. A Taylor expansion about 0 is called a Maclaurin form or Maclaurin series.

5.2 Polynomials and Rational Functions

Fact 5.2.1. Every non-constant polynomial with complex coefficients has at least one zero in $\mathbb{C}$. It immediately follows that every polynomial of degree n denoted by $P_n(z)$ with complex coefficients has exactly n zeros in $\mathbb{C}$ (counting multiplicities). Thus we can factor every polynomial as

$$P_n(z) = a_n(z - z_1)^{d_1}(z - z_2)^{d_2} \dots (z - z_k)^{d_k},$$

where $d_1 + d_2 + \cdots + d_k = n$.

Then when we say the polynomial $P_n(z)$ has zero of order m, it means that we can write it as

$$P_n(z) = (z - z_0)^m Q(z),$$

where $Q(z)$ is a degree $n - m$ polynomial such that $Q(z_0) \neq 0$.

Fact 5.2.2. We can use the following trick to expand any polynomial about any $z_0 \in \mathbb{C}$. For instance, we want to express the polynomial $P_3(z) = -2z^3 - 4z^2 + 10z + 12$ in terms of powers of $(z - 1)$. To do this we can write $P_3(z) = a_0 + a_1(z - 1) + a_2(z - 1)^2 + a_3(z - 1)^3$, and it only remains to determine the coefficients. It immediately reveals that

$$\begin{aligned} P_3(1) &= a_0, \\ P_3'(1) &= a_1, \\ P_3''(1) &= 2a_2, \\ P_3'''(1) &= 2 \cdot 3a_3. \end{aligned}$$

thus we can easily calculate the unknown coefficients.

Fact 5.2.3 (Poles of a Function)**.** A rational function is a function that the ratio of two polynomials. I.e.

$$R_{r,s}(z) = \frac{P_r(z)}{Q_s(z)}.$$

The rational function is not defined at the zeros of the denominator. However, we can cancel out any common zeros in the denominator and numerator and then arrive at the following expression

$$F(z) = \frac{a_m(z - z_1)(z - z_2) \cdots (z - z_m)}{b_n(z - \xi_1)(z - \xi_2) \cdots (z - \xi_n)}$$

Note that the common zeros of the numerator and the denominator are canceled. Then the set $\{\xi_1, \xi_2, \cdots, \xi_n\}$ are the poles of the rational function F. However, if we want to also count the multiplicities, we can write

$$F(z) = \frac{a_m(z - z_1)^{d_1}(z - z_2)^{d_2} \cdots (z - z_k)^{d_k}}{b_n(z - \xi_1)^{t_1}(z - \xi_2)^{t_2} \cdots (z - \xi_l)^{t_l}},$$

from which we can infer that for example ξ_1 is a pole of degree t_1, etc.

In a nutshell, the zero of order m of denominator is the pole of degree m of the rational function. Thus it immediately follows that if ξ_1 is the pole of order t_1 of $F(z)$, then we can write

$$H(z) = (z - \xi_1)^{t_1} F(z),$$

in which we have

$$\lim_{z \to \xi_1} H(z) \neq 0 < \infty.$$

Fact 5.2.4 (Partial Fraction Decomposition)**.** The rational functions $R_{m,n} = \frac{P_m(z)}{Q_n(z)}$ can be written as partial fractions. Note that when $m > n$, we first need to do the long division and then do the partial fraction. The following example makes this more clear.

Example 5.2.1. Do partial fraction decomposition for the following rational function.

$$F(z) = \frac{2z^4 + 7z^3 + 10z^2 + 6z + 1}{z^3 + 3z^2 + 3z + 1}.$$

Answer. First we need to do the long division, since the degree of the numerator is larger than the degree of denominator. After doing the long division we will have

$$F(z) = 2z + 1 + \frac{z^2 + 2z + 1}{z^3 + 3z^2 + 3z + 1}.$$

Now we can do the partial fraction decomposition on the second term. We can write it as

$$P(z) = \frac{z^2 + 2z + 1}{(z+1)^3} = \frac{A_0^{(1)}}{(z-1)^3} + \frac{A_1^{(1)}}{(z-1)^2} + \frac{A_2^{(1)}}{(z-1)}$$

Now by some linear algebra and solving the system of equations we will get

$$P(z) = \frac{1}{(z-1)^3} + \frac{-1}{(z-1)^2} + \frac{1}{(z-1)}.$$

Finally, we can write

$$F(z) = 2z + 1 + \frac{1}{(z-1)^3} + \frac{-1}{(z-1)^2} + \frac{1}{(z-1)}.$$

Later, we will see that this is in fact the Laurent expansion of the function $F(z)$.

Fact 5.2.5 (Trick For Partial Fraction Decomposition)**.** We can do partial fraction decomposition in a more elegant way. Complex analysis is a place that geometry, analysis and algebra meet each other. As we replaced the algebraic operations in expanding a polynomial about an arbitrary point, by analysis and calculating the derivatives, there is no surprise that such a substitution is possible here as well. Also, we can see this blend of analysis and algebra as a hint that **analysis has a very clean algebraic manifestation in the space of polynomials.** The following example makes this link more clear.

Example 5.2.2. Do partial fraction decomposition for the following rational function.

$$F(z) = \frac{2z^4 + 7z^3 + 10z^2 + 6z + 1}{z^3 + 3z^2 + 3z + 1}.$$

Answer. As we saw in the question above,we can write $F(z)$ as

$$F(z) = 2z + 1 + \frac{z^2 + 2z + 1}{z^3 + 3z^2 + 3z + 1}.$$

Denote the last term as

$$P(z) = \frac{z^2 + 2z + 1}{z^3 + 3z^2 + 3z + 1} = \frac{z^2 + 2z + 1}{(z + 1)^3}.$$

Now the trick is as follows. Assume we have the partial fraction in the following from

$$P(z) = \frac{A_0^{(1)}}{(z + 1)^3} + \frac{A_1^{(1)}}{(z + 1)^2} + \frac{A_2^{(1)}}{(z + 1)}.$$

Thus the problem is now to find the values of the coefficients. We can it easily by multiplying both sides at $(z + 1)^3$. Then we will have

$$Q(z) = (z + 1)^3 P(z) = A_0^{(1)} + A_1^{(1)}(z + 1) + A_2^{(1)}(z + 1)^2.$$

It immediately follows that

$$\begin{aligned}
A_0^{(1)} &= \lim_{z \to -1} Q(z), \\
A_1^{(1)} &= \lim_{z \to -1} \frac{d}{dz} Q(z), \\
A_1^{(2)} &= \lim_{z \to -1} \frac{1}{2!} \frac{d^2}{dz^2} Q(z).
\end{aligned}$$

Note that we have used the notion of the limit in the above equations (instead of direct substitution) because $P(z)$ and thus $Q(z)$ is not defined at the point $z = -1$.

Fact 5.2.6. If $P(z) = a_n(z - z_1)^{d_1}(z - z_2)^{d_2} \cdots (z - z_r)^{d_r}$, then the partial fraction expansion of the logarithmic derivative P'/P is given by

$$\frac{P'(z)}{P(z)} = \frac{d_1}{z - z_1} + \frac{d_2}{z - z_2} + \cdots + \frac{d_r}{z - z_r}.$$

Proof. It immediately follows from the generalization of the derivative rule $(fgh)' = f'gh + fg'h + fgh'$. □

Fact 5.2.7. If $P(z)$ is any polynomial defined on the complex plane, then all of the zeros of $P'(z)$ lie in the convex hull of the zeros of $P(z)$.

Proof. See the questions 18,19,20, and 21 of Exercise 3.1, Snider book. □

---

6 Preparation for the Exam

6.1 Complex Integration

There are many ways to evaluate a complex integral, and in this section we will review some of them.

Fact 6.1.1 (Integration by Substitution)**.** We can do a contour integration by substitution method similar to what we had in the real analysis.

$$\int_\gamma f(z)dz = \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)dt,$$

where $\gamma : \mathbb{R} \to \mathbb{C}$ is a path in the complex plane parameterized by $t \in [t_0, t_1]$.

This method is the most general method of evaluating the integrals. However, it comes with the generality, the cumbersomeness of this method to evaluate some integrals. It is very easy to get some very messy integrals with this method that is almost impossible to solve analytically. That is why in the future steps we will develop some more theory that will enable us to evaluate the integrals more easily.

Fact 6.1.2 (Fundamental Theorem of Calculus)**.** To evaluate the integral

$$\int_\gamma f(z)dz,$$

if the path γ lies in an open connected region Ω in which f is continuous and has an anti-derivative $F(z)$ (i.e. $F(z)$ is holomorphic at Ω such that $F'(z) = f(z)$) then we have

$$\int_\gamma f(z)dz = F(z_1) - F(z_0),$$

where z_1 and z_0 are the starting and the finishing points of the path.

Then it follows immediately, that if f is continuous in a open, connected region Ω and posses an anti-derivative, then the contour integral around every closed loop is zero. I.e., for any closed contour γ we have

$$\int_\gamma f(z)dz = 0.$$

Fact 6.1.3 (Cauchy's integral theorem)**.** Cauchy's integral theorem (do not confuse with the Cauchy's integral formula!) can be stated in two different ways, from which I prefer the first one.

The first form of Cauchy's integral theorem: Let f be analytic at some open connected region, and γ_1 and γ_2 two *loops* that can be continuously deformed to each other. Then we have

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

The second form of Cauchy's integral theorem: Let f be analytic at some open, *simply* connected region, and γ any loop in that region. Then

$$\int_\gamma f(z)dz = 0.$$

Example 6.1.1. Evaluate the integral

$$\int_\gamma (z - z_0)^n dz,$$

where γ is any simple,closed loop winding around z_0 in counterclockwise direction.

Answer. When $n \geq 0$, then $(z - z_0)^n$ is entire, thus by the Cauchy's integral theorem the integral is zero. However, for $n < 0$, $f(z)$ is only analytic in the punctured plane, punctured at z_0. Thus Cauchy's integral formula can not make any statements about the value of the integral. However, for $n \leq -2$, the integrand is continuous and has anti-derivative. Thus its integral on any closed contour evaluates to zero. Finally, for $n = -1$, we can follow two methods: 1. To evaluate the integral using the direct method (which is very hard as we don't know the contour explicitly), or using the Fundamental theorem of calculus. We can break the path into two pieces, and on each piece we can find a branch of *Log* function that is analytic. Thus we can calculate the integral, which will be $2\pi i$.

Fact 6.1.4 (Cauchy's Integral Formula)**.** Let γ be a simple, closed, positively oriented loop in an open, simply connected region Ω and $z_0 \in \Omega$. If f is analytic in this region, then

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz.$$

Also, we have

$$f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^2} dz.$$

And in general

$$f^n(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

Fact 6.1.5 (Evaluating integrals with Lemma 1)**.** Let $P(z)$ be a polynomial of degree at least 2. The

$$\int_\gamma \frac{1}{P(z)} dz = 0,$$

for γ a positively oriented closed path.

Example 6.1.2. Instead of proof, we want to show case the reasoning in a specific example. We want to evaluate the integral

$$I = \int_\gamma \frac{1}{z^2(z-1)^3} dz,$$

where γ is define as $|z| = R > 2$.

Answer. First, note that all of the zeros of the polynomial fall inside the closed loop γ, thus we can easily inflate the loop to go to infinity. To take use of this, we first approximate the integral

$$|\int_\gamma \frac{1}{z^2(z-1)^3}dz| \leq \max_{z\in\gamma*}|\frac{1}{z^2(z-1)^3}|L(\gamma) = \frac{2\pi R}{\min_{z\in\gamma*}|z^2(z-1)^3|} = \frac{2\pi}{R(R-1)^3}.$$

Now letting $R \to \infty$, by the squeeze theorem we have

$$|I| = 0 \implies \int_\gamma \frac{1}{z^2(z-1)^3}dz$$