Manifolds

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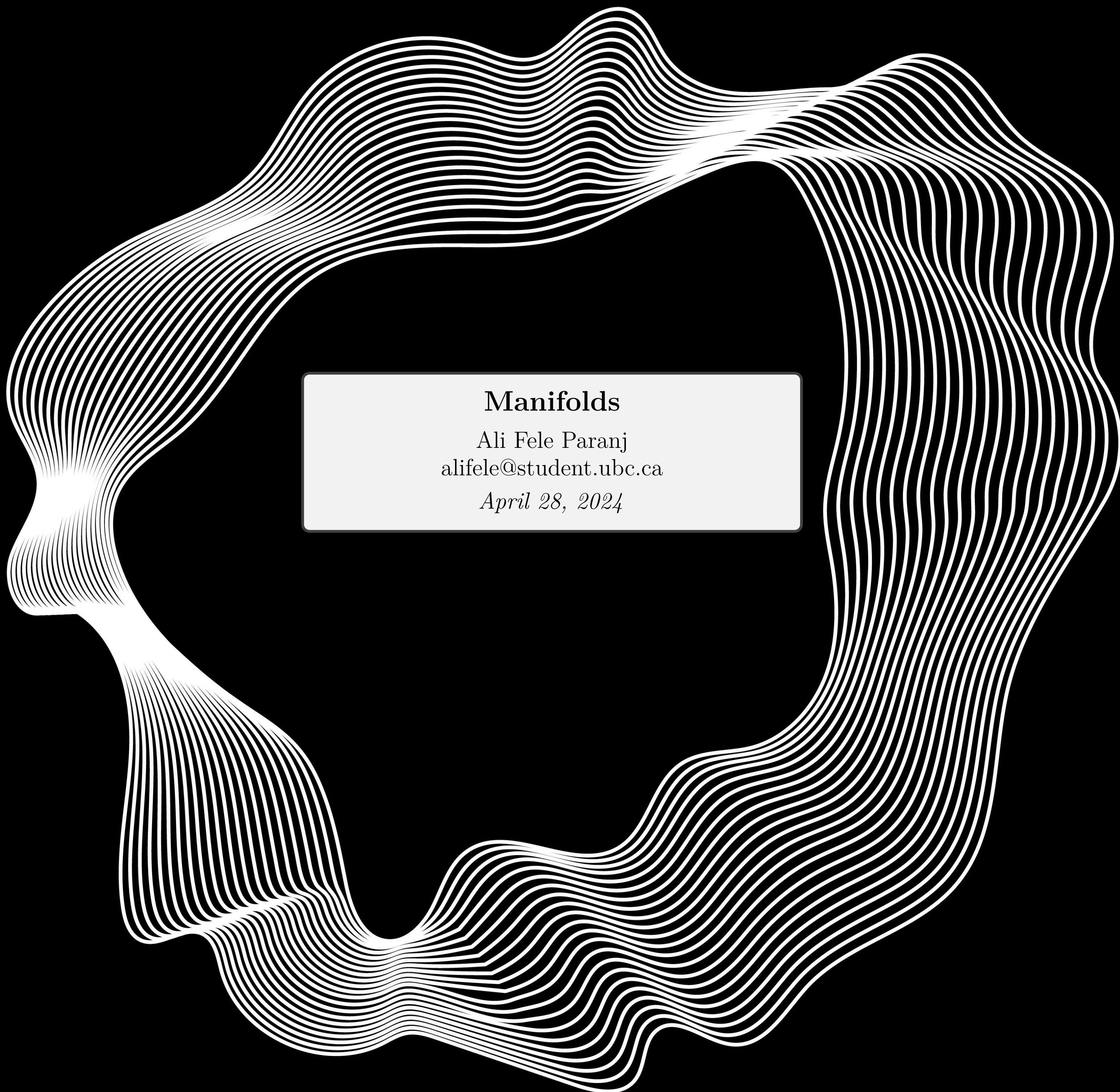

0. Contents

1. Euclidean Spaces

1.1 Basic Notions and Definitions

Here in this chapter I will be covering the details of some notions that was challenging for me do digest in the first read.

Definition 1.1 — Axioms of Group. Group is a set A along with a binary operation $* : A \times A \to A$ that satisfies the following properties. Let $a, b, c \in A$, then

- **Associativity**: $a * (b * c) = (a * b) * c$.
- **Identity element**: $\exists 1 \in A$ such that

$$1 * a = a * 1 = a.$$

- **Inverse element**: $\forall a \in A\ \exists \hat{a} \in A$ such that

$$a * \hat{a} = \hat{a} * a = 1.$$

■ **Remark** A set along with a binary operation that does not satisfy any properties is called a **magma**. If the binary operation is only associative, then we are dealing with **semi-group**. If the binary operation has an identity element as well, then we call this algebraic structure as **monoid**.

Definition 1.2 — Axioms of Ring. A ring is a set R along with two operations $+ : R \times R \to R$ and $* : R \times R \to R$, where

- $(R, +)$ is an Abelian group.
- $(R, *)$ is a monoid.
- The operator $(*)$ has distributive (left and right) law over $(+)$ i.e.

$$a * (b + c) = (a * b) + (a * c), \qquad (b + c) * a = (b * a) + (c * a).$$

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■ **Remark** **Field** is a ring where every non-zero element (i.e. inverse element in the $(R, +)$ group in the ring) has a multiplicative inverse.

Definition 1.3 — Axioms of Module. A **module** is a group M along with a ring R where the monoid of the ring acts on M (through scalar multiplication) (i.e. it satisfies the idenity and compatibility properties) and satisfies the distributive property. I.e.

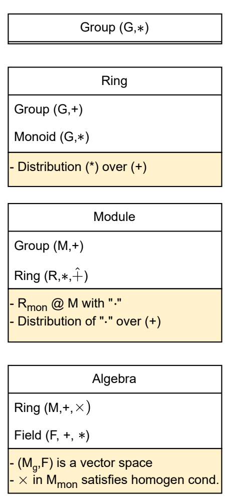
- **Compatibility of the monoid action**: $a, b \in R,\ u \in M$ then
$$a(bu) = (ab)u.$$
- **Identity of the monoid action**: Let 1 be the identity element of the ring R. Then $\forall u \in M$
$$1u = u1 = u.$$
- **Distribution law**: $a, b \in R$ and $u, v \in M$ then
  - $(a + b)u = au + bu$.
  - $a(u + v) = au + av$.

■ **Remark** A module (M, R) is called a **vector space**, if the **ring** R is a **field**.

Definition 1.4 — Axioms of Algebra. An Algebra over field F is a ring A that F acts on it (thus A has vector space structure as well), where the monoid operation of F (i.e. multiplication) satisfies the homogeneity property. I.e. for $r \in F$ and $u, v \in A$ we have
$$r(uv) = (ru)v = u(rv).$$

There are some important observations when combining different algebraic structures with each other to get a new one. The first is that when we combine two structures with different operators, then the operators need to satisfy the distributive laws. Also, note that when an algebraic structure (like group or monoid) acts on another algebraic structure, we need to have the identity and and compatibility conditions satisfied.

The following diagram shows how different algebraic structures are combined with each other to produce another structure.



Note that in the figure above, I have used some non-standard notations to make the figure concise. For instance, the expression "$R_{\mathbf{mon}}@M$ **with** $\cdot$" means that the monoid structure in the field R acts on the group M with the $(\cdot)$ symbol. Or the expression "$\times$ **in** $M_{\mathbf{mon}}$ **satisfies homogen cond.**" means the multiplication operation of the monoid structure inside the ring M satisfies the homogeneity condition (see the definition of the algebra in Definition 1.4). Finally, M_g means the group structure inside the ring M.