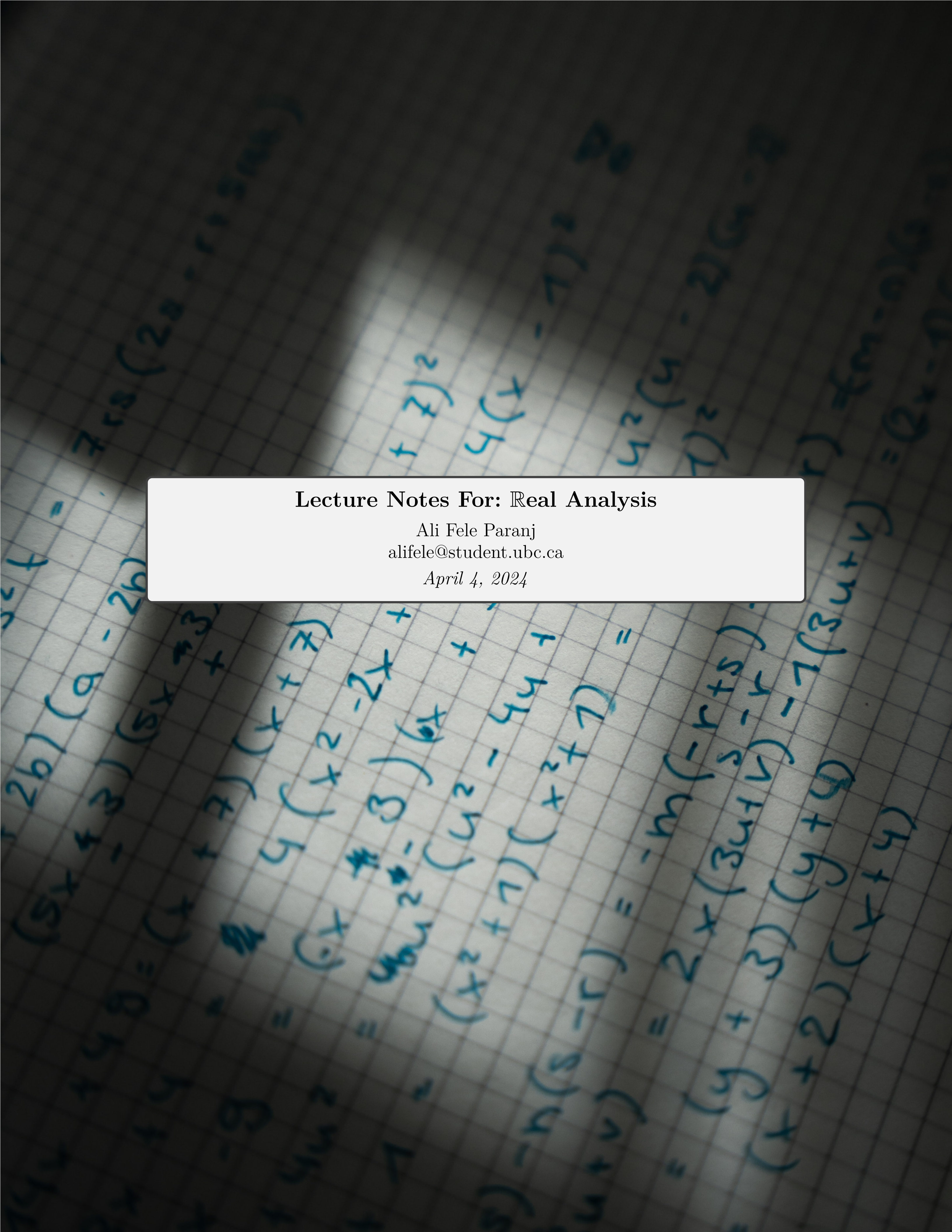

Lecture Notes For: ℝeal Analysis

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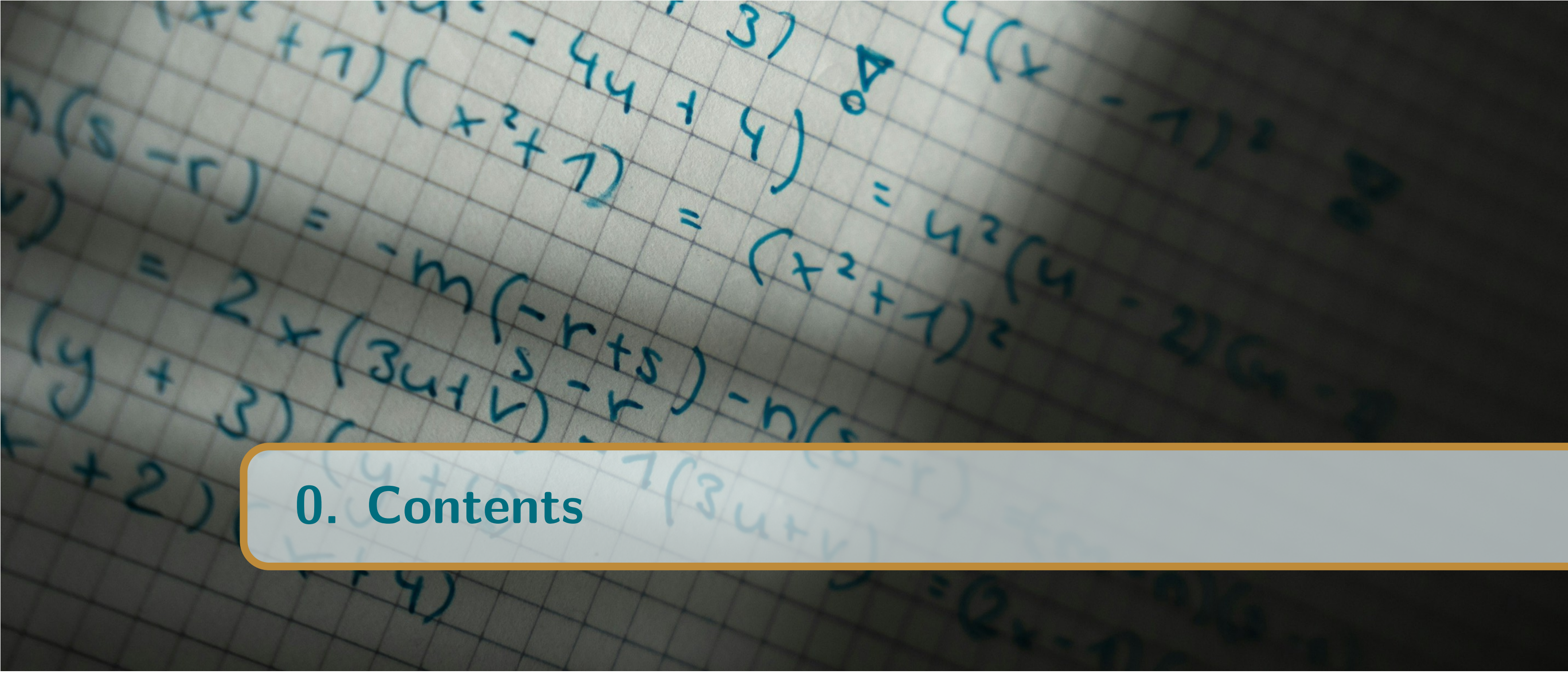
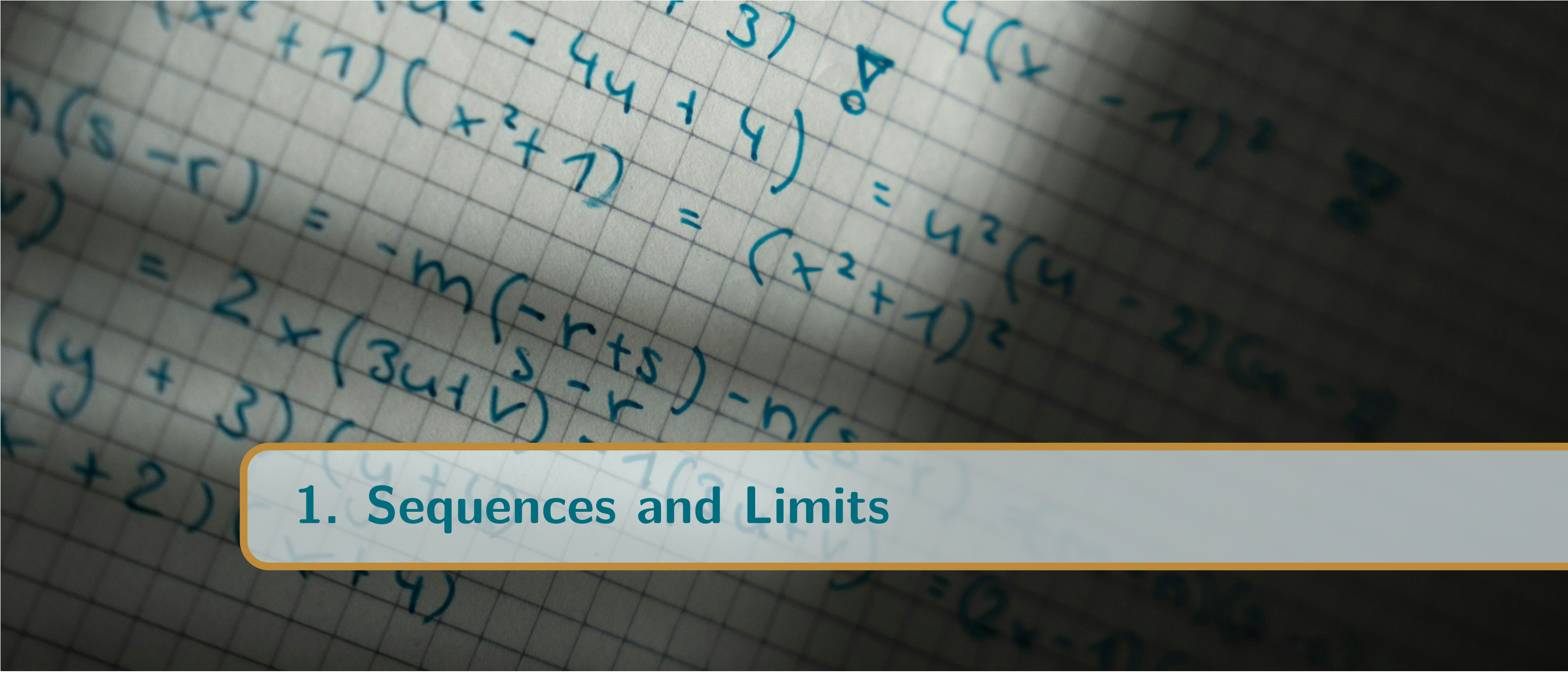


0. Contents



1. Sequences and Limits

> **Proposition 1.1 — Sequential characterization of countable sets.** Let X be a countable set. Then, X is a range of some sequence $\{x_n\}$. To put it on other words, there is an *onto* function $f : \mathbb{N} \to X$.

■ **Problem 1.1** Show that the following properties hold for arbitrary countable sets.

(a) All subsets of countable sets are countable.

(b) Any union of a pair of countable sets is countable.

Proof.

(a) Let A be a countable set, and $C \subseteq A$ be a subset. Since A is countable, then there is a numeration for its elements, i.e.
$$A = \{a_1, a_2, \cdots\}.$$
Then for $x \in C$, there exists $n \in \mathbb{N}$ such that $x = a_n$. Define $g : C \to \mathbb{N}$, and let $g(x) = n$. Since g is one-to-one, then C is at most countable.

(b) Let A, B be countable sets, and $X = A \cup B$. If A, B are finite, then the statement is trivially true. But if A, B are infinite, then since A and B are countable, then there exists bijection
$$g_A : A \to \mathbb{N},$$
$$g_B : B \to \mathbb{N}.$$
Let $x \in A \backslash B$. Then $g_A(x) = n$ for some $n \in \mathbb{N}$. Define $f : X \to \mathbb{N}$ and let $f(x) = 2n$. Also for $y \in B$, we have $g_B(y) = m$, then define $f(y) = 2m + 1$. Due to the construction, the function f is one-to-one, and since $\mathbb{N}$ is countable, then X is at most countable.

□

■ **Problem 1.2** Show that the following property holds for countable sets: If
$$S_1, S_2, S_3, \cdots$$
is a sequence os countable sets of real numbers,then the set S formed by taking all elements that belong to at least one of the sets S_i is also a countable set.

Proof. Let

$$X = \bigcup_{i \in \mathbb{N}} S_i.$$

Let $x \in X$. Then there exists $I \subseteq \mathbb{N}$ such that $\forall i \in I$ we have $x \in S_i$. Let $m = \min(I)$. Note that due to the construction of X, the set I is not empty, and since it is a subset of natural numbers, then there exists a unique minimum. So far, we have $x \in S_m$. Since S_m is countable, then there exists $f_m : S_m \to \mathbb{N}$ one-to-one (note that since there is a possibility that S_m might be finite, then requiring f_m be one-to-one, given that the co-domain is countable is enough for expressing the fact f_m is at most countable). Thus $f_m(x) = n$ for some $n \in \mathbb{N}$. Define $F : X \to \mathbb{N} \times \mathbb{N}$, and let $F(x) = (m, n)$. Do the the construction of F, it is one-to-one, and since $\mathbb{N} \times \mathbb{N}$ is countable, then X is at most countable. ☐

■ **Problem 1.3** Let X be a family of sets, and $\sim$ a relation on X defined as below,

$$A, B \in X,\ A \sim B \Leftrightarrow \exists f : A \to B,\ \text{a bijection.}$$

In words, when $A \sim B$ we can say A and B have the same cardinality. Show that $\sim$ is an equivalence relation.

A side note: There are lots of parallel notions for this in mathematics. For example in topology, a homeomorphism (bijective and continuous) define an equivalence relation on topological spaces, in differential geometry, a diffeomorphism does the job, in group theory, an isomorphism is the parallel notion, and in general, all of these notions are generalized with the notion of morphism in category theory.

Proof.

(a) **Reflexivity.** Let $A \in X$. Then $id_A : A \to A$, is an bijective map (identity map). Thus $A \sim A$.

(b) **Symmetry.** Let $A, B \in X$, and $A \sim B$. Thus we know $\exists f : A \to B$, a bijection. Let $g = f^{-1} : B \to A$. Due to the construction g is also bijective. Thus $B \sim A$.

(c) **Transitivity.** Let $A, B, C \in X$, $A \sim B$, and $B \sim A$. Then $\exists f : A \to B$ and $g : B \to C$ injective. Define $h : A \to B$, $h = g \circ f$. Due to the construction, h is also bijective. Thus $A \sim C$.

☐

■ **Problem 1.4** We define a real number to be *algebraic* is it is a solution of some polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

where all the coefficients are integers. A real number that is not algebraic, is called a transcendental number. Prove that the set of all algebraic numbers is countable.

Proof. The set of all polynomials of degree $n - 1$ is identified with $\mathbb{Z}^n$, which is countable. Thus the set all such polynomials is

$$X = \bigcup_{n \in \mathbb{N}} \mathbb{Z}^n.$$

As we proved in the Problem 1.2, X is countable. On the other hand, every polynomial has a countable number of zeros. Thus The set of all zeros of all polynomial with integer coefficients is countable. ☐

■ **Problem 1.5** Let $\{s_n\}$ be a sequence in $\mathbb{R}$ converging to $L \in \mathbb{R}$. Prove that the following definition is the equivalent to the original definition of converges.

$$\forall m \in \mathbb{N},\ \exists N \in \mathbb{N}:\ \forall n > N \text{ we have } |s_n - L| < 1/m.$$

Side note. This is a useful equivalent definition of convergence, because to check a sequence converges is enough to check the definition above for the integer m's only.

Proof. The fact that the original definition implies the definition above is trivial. It suffice to let $\epsilon = 1/m$. However, for the converse, we assume that the definition above is true and we want to infer the original definition of convergence. Let $\epsilon > 0$ is give. Then $\exists m \in \mathbb{N}$ such that $1/m < \epsilon$ (The Archimedes property for real numbers). On the other hand since we have assumed that the altered definition is true, them $\exists N \in \mathbb{N}$ such that $\forall n > N$ we have

$$|s_n - L| < 1/m < \epsilon.$$

This completes the proof. □

■ **Problem 1.6** If $\{s_n\}$ is a sequence of positive real numbers converging to a positive number L, show that $\{\sqrt{s_n}\}$ converges to $\sqrt{L}$.

Proof. Since $s_n \to L$, then $\exists N_1 > 0$ such that $\forall n > N_1$ we have $s_n < L + 1$. Thus $\sqrt{s_n} < \sqrt{L+1}$, which implies $\sqrt{s_n} + \sqrt{L} < \sqrt{L+1} + \sqrt{L} = M$, for $M \in \mathbb{R}$. For a given $\epsilon > 0$ let $\epsilon_0 = M\epsilon$. We know $\exists N_2 > 0$ such that $n > N_2$ we have $|s_n - L| = |\sqrt{s_n} - \sqrt{L}||\sqrt{s_n} + \sqrt{L}| < \epsilon_0$. Then $|\sqrt{s_n} - \sqrt{L}| < \epsilon_0/M = \epsilon$. This completes the proof. □

■ **Problem 1.7** Show that the sequence

$$a_n = n^p + \alpha_1 n^{p-1} + \alpha_2 n^{p-2} + \cdots + \alpha_p$$

diverges to ∞, where p is a positive integer and $\alpha_1, \alpha_2, \cdots, \alpha_p$ are real numbers (positive or negative).

Proof. We construct the sequence $\{b_n\}$ such that $b_n \leq a_n$ always hold.

$$b_n = n^p - |\alpha_1|n^{p-1} - |\alpha_2|n^{p-2} - \cdots - |\alpha_p|.$$

To demonstrate the proof in a more clear way, we assume $p = 4$. Now we can solve the following inequalities

$$\begin{aligned}
1/4n^4 - |\alpha_1|n^3 > 1/8n^4 &\implies n > |8\alpha_1|,\\
1/4n^4 - |\alpha_2|n^2 > 1/8n^4 &\implies n > |8\alpha_2|^{1/2},\\
1/4n^4 - |\alpha_3|n^1 > 1/8n^4 &\implies n > |8\alpha_3|^{1/3},\\
1/4n^4 - |\alpha_4|n > 1/8n^4 &\implies n > |8\alpha_4|^{1/4},
\end{aligned}$$

Let $N = \max\{|8\alpha_1|, |8\alpha_2|^{1/2}, |8\alpha_3|^{1/3}, |8\alpha_4|^{1/4}\}$. Then for $n > N$ all of the inequalities above holds. Thus we can write

$$n^4 + \alpha_1 n^3 + \alpha_2 n^2 + \alpha_3 n + \alpha_4 > n^4 - |\alpha_1|n^3 - |\alpha_2|n^2 - |\alpha_3|n - |\alpha_4| > 1/2n^4.$$

Since the very last term $(1/2n^4)$ diverges, then the sequence a_n diverges. □

■ **Problem 1.8** Prove that if $\{s_n\}$ bounded, then $\{s_n/n\}$ converges.

Proof. We propose that the $s_n/n \to 0$. Since s_n is bounded, then there is $M > 0$ such that $|s_n| < M$ for all $n \in \mathbb{N}$. For a given $\epsilon > 0$, let choose a natural number $N > M/\epsilon$. Then $\forall n > N$ we have

$$|s_n/n| < |s_n|/|n| < M/|n| < M\epsilon/M = \epsilon.$$

This implies that $\{s_n/n\}$ converges to zero. □

■ **Problem 1.9 — Order property in limits.** Let $\{t_n\}$ and $\{s_n\}$ be two real sequences that converge to T and S respectively, and also $\forall n \in \mathbb{N}$ we have $s_n \leq t_n$. Prove that $T \leq S$

Side note: By this proof you will show that the limits preserve the order property, and this fact is the main idea behind the squeeze theorem.

Proof. Since $t_n \geq s_n$ for all $n \in \mathbb{N}$, then $t_n - s_n \geq 0$. Thus $t_n - s_n \pm T \pm S < 0$, and we can re-arrange the terms as $(t_n - T) + (S - s_n) + (T - S) > 0$. On the other hand, since $s_n \to S$ and $t_n \to T$, then for $\epsilon > 0$ there exists $N_1 > 0, N_2 > 0$ such that $n > N_1$ implies $|s_n - S| < \epsilon/2$ and $n > N_2$ implies $|t_n - T| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$. So $\forall n > N$ we will have

$$\epsilon/2 + \epsilon/2 + (T - S) > (t_n - T) + (S - s_n) + (T - S) > 0.$$

This implies $T - S > -\epsilon$ for all $\epsilon > 0$. Thus we conclude $T - S \geq 0 \implies T > S$. □

■ **Problem 1.10** A careless student gives the following as a proof of the squeeze theorem. Find the flaw:

"If $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = L$, then take limits in the inequality $s_n \leq x_n \leq t_n$ to get $L \leq \lim_{n\to\infty} x_n \leq L$. This can only be true if $\lim_{n\to\infty} x_n = L$."

Proof. Formally, the student turns the inequality into two pieces, i.e. $s_n \leq x_n$ and $x_n \leq t_n$ and tries to apply the order property of limits to each of them. But the problem is that the squeeze theorem only states that s_n and t_n has limits and does not have any information about the existence of limit for x_n. Thus we can not simply apply the limit to the inequality and infer the squeeze theorem. □

■ **Problem 1.11** Let $\{s_n\}$ be a sequence of positive numbers. Show that the condition

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} < 1$$

implies $s_n \to 0$.

Proof. Let $\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \alpha < 1$. Also, since $\forall n \in \mathbb{N}$ we have $s_n > 0$, then $\alpha \geq 0$. So $0 \leq \alpha < 1$. Define $d = (1-\alpha)/2$, thus $1/2 \leq d < 1$ and $d > \alpha$ (in fact, d is sandwiched between α and 1). Since $s_{n+1}/s_n \to \alpha$, then $\exists N_1 > 0$ such that $\forall n > N_1$ we have

$$\frac{s_{n+1}}{s_n} < d \implies s_{n+1} < ds_n \implies s_{n+1} < ds_n < d^2 s_{n-1} < \cdots < d^n s_1.$$

Since all s_n are positive, then $s_1 = cd$, for some $c \in \mathbb{R}$. Thus we can write the inequality above as

$$s_{n+1} < d^{n+1}c \implies s_n < d^n c.$$

Now we can proceed with the $\epsilon - N$ proof directly, or we can use squeeze theorem. We know that

$$0 \leq s_n < d^n c.$$

Since both 0 and d^n converges to zero, then s_n also converges to zero, i.e. $s_n \to 0$. □

■ **Problem 1.12** Let $\{s_n\}$ be a sequence of positive numbers. Show that the condition

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} > 1$$

then $s_n \to \infty$.

Proof. Let $\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \alpha > 1$. Define $d = 1 + (\alpha - 1)/2$ (in fact d is sandwiched between 1 and α). Since $s_{n+1}/s_n \to \alpha$, then $\exists N > 0$ such that $\forall n > N$ we have $s_{n+1}/s_n > d$. As we seen in the solution of the problem above, this implies

$$s_n > d^n c,$$

where $c = s_1/d$. Now we can proceed with the $M - N$ proof to show that s_n actually diverges. ☐

■ **Problem 1.13** Let $\{s_n\}$ be a real sequence that is non-decreasing and bounded. Then prove that

$$s_n \to \sup_{n\in\mathbb{N}}(s_n).$$

Proof. Let $L = \sup_{n\to\infty} s_n$. Since $s_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, and is bounded, then the suprimum exists and $L \in \mathbb{R}$. Let $\epsilon > 0$ be given, and define $\beta = L - \epsilon$. Since L is the least upper bound for the sequence, then for $\exists N \in \mathbb{N}$ such that $s_N > \beta$, and since s_n is non-decreasing, $\forall n > N$ we have $s_n > \beta \implies |s_n - L| < \epsilon$. This implies that $s_n \to L$ as $n \to \infty$. ☐

■ **Problem 1.14** Decide on the convergence of the following sequence.

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \cdots$$

Proof. This sequence converges, since it is increasing and bounded. The fact that this sequence is increasing is trivial, but for being bounded we start with

$$\boxed{\sqrt{2} < 2} \implies 2+\sqrt{2} < 4 \implies \boxed{\sqrt{2+\sqrt{2}} < 2} \implies 2+\sqrt{2+\sqrt{2}} < 4 \implies \boxed{\sqrt{2+\sqrt{2+\sqrt{2}}} < 2} \implies \cdots.$$

Thus as we can see, all of the terms of the sequence is less than 2. Thus this sequence converges. ☐

■ **Problem 1.15** Prove that the monotone convergence theorem implies the following version of the nested interval theorem. Given a sequence of nested intervals $[a_n, b_n]$ such that $a_n < b_n$, and has the following arrangement

$$[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots,$$

and the $a_n - b_n \to 0$ (i.e. the length of the intervals shrinks to zero), then there is exactly one element at the intersection of the intervals.

Proof. Assume that we have the nested intervals with its length shrinking to zero, then we want to show that there is only one element at the intersection. Construct the sequence $\{a_n\}$ and $\{b_n\}$ as follows

$$a_n : a_1, a_2, a_3, \ldots, \quad b_n : b_1, b_2, b_3, \ldots.$$

Then it follows from the properties of nested intervals that $\{a_n\}$ is a non-decreasing sequence with b_1 as an upper bound, while $\{b_n\}$ is a non-increasing sequence with a_1 as a lower bound. Thus both sequences converge and we denote $a_n \to A$ and $b_n \to B$. Since $a_n < b_n$ for all $n \in \mathbb{N}$, thus $A \le B$. We claim that $A = B$. Assume other wise, i.e. $A < B$. Let $D = B - A$. Then for any given $n \in \mathbb{N}$ we have $a_n \le A < B \le b_n$, which implies $b_n - a_n > D$, which means $a_n - b_n$ can not converge to zero, which contradicts our assumption. Thus $A = B$. Also $A = B$ is the only common point in all the intervals. If there were other C contained in all of the intervals, then it would imply $C \le A$ and $C \ge B$, thus $A = B = C$. This completes the proof. ☐

■ **Problem 1.16** Prove that the nested interval theorem (stated as below) implies the monotone convergence theorem.

> For any sequence of intervals of real numbers $I_1 = [a_1, b_1], I_2 = [a_2, b_2]$ that are nested, i.e.
> $$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots,$$
> we have
> $$\bigcap_{n\in\mathbb{N}} I_n \neq \varnothing.$$

Proof. We will prove the monotone convergence theorem here only by considering the convergence of a non-decreasing sequence. The proof for other case (non-increasing sequence) will be very similar. Let $\{a_n\}$ be a non-deceasing and bounded sequence in $\mathbb{R}$. Let B denote the set of all upper bounds of the sequence a_n. Then define

$$b_n = \min\{\{\frac{k}{2^n} : k \in \mathbb{Z}\} \cap B\}.$$

With this choice of B, we will get the following nested sequences

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \cdots.$$

By the nested interval theorem, we know that the intersection of these intervals is not empty. Let $\alpha \in \bigcap_{n\in\mathbb{N}} I_n$, in which $I_n = [a_n, b_n]$. Thus $a_n \leq \alpha \leq b_n$. On the other hand, due to the construction of b_n, it follows that $\forall n \in \mathbb{N}$ we have $b_n - 1/2^n \notin B$, thus $\exists N_1 \in \mathbb{N}$ such that $a_N > b_n - 2^{-n}$, and since $\{a_n\}$ is non-decreasing, then $\forall n > N$ we have $a_n > b_n - 2^{-n}$. Thus for all $n > N$ we have

$$b_n - 2^{-n} \leq a_n \leq \alpha \leq b_n.$$

This implies $|a_n - \alpha| = \alpha - a_n < 2^{-n}$. So for a given $\epsilon > 0$ we can choose n large enough that $n > N$ and also $2^{-n} < \epsilon$. Thus $|a_n - \alpha| < \epsilon$. This shows $a_n \to \alpha$. Thus we proved that the nested interval property implies the convergence of the monotone and bounded sequences. □

■ **Problem 1.17** Let $\{x_n\}$ be a bounded sequence. Prove if every convergent subsequence of $\{x_n\}$ converges to zero, then $x_n \to 0$.

Proof. We use proof by contrapositive. I.e., we prove that if x_n does not converge to zero, then there exist a convergent subsequence of $\{x_n\}$ that does not go to zero. Assume x_n does not converge to zero. Then $\exists D > 0$ such that $\forall N \in \mathbb{N}$, there exists $n > N$ such that $|x_n| > D$. However, since the sequence is bounded, we can write the last expression as $D < |x_n| < M$ for some $M \in \mathbb{R}$. Now the subsequence $\{x_{n_k}\}$ where $\forall k \in \mathbb{N}$, $D < |x_{n_k}| < M$. This subsequence itself has a converging subsequence that does not converge to 0. (the existence of a converging subsequence is guaranteed since $\{x_n\}$ it is bounded and all its elements are real numbers). On the other hand, the subsequence of $\{x_{n_k}\}$ is also a subsequence of the original sequence. Thus we have found a subsequence that does not converge. This completes the proof. □

■ **Problem 1.18** Give an example of a sequence that has subsequences converging to every natural numbers (and not any other number).

Proof. First note that the sequence $\{1/n\}$ converges to 0 as $n \to \infty$. Now we can use a diagonalization argument similar to the Cantor's diagonalization argument to construct the desired sequence. The following organization does the job.

$$\begin{array}{ccccc}
1 & 1/2 & 1/3 & 1/4 & \cdots \\
1+1 & 1+1/2 & 1+1/3 & 1+1/4 & \cdots \\
2+1 & 2+1/2 & 2+1/3 & 2+1/4 & \cdots \\
3+1 & 3+1/2 & 3+1/3 & 3+1/4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

$\square$

■ **Problem 1.19** If a sequence $\{x_n\}$ has a property that

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = L,$$

prove that the sequence converges to L.

Proof. Let $\{x_{n_k}\}$ and $\{x_{n_l}\}$ be sub-sequences define by $n_k = 2k$ and $n_l = 2l + 1$. From the assumption we have $x_{n_k} \to L$ and $x_{n_l} \to L$. Let $\epsilon > 0$ be given. Then there exist integers K_1 and L_1 such that for all $k > K_1$ and all $l > L_1$ we have $|x_{n_k} - L| < \epsilon$ and $|x_{n_l} - L| < \epsilon$. Choose $Q = \max\{K_1, L_1\}$. Then $\forall l, k > Q$ we have

$$|x_{n_k} = L| < \epsilon, \quad |x_{n_l} - L| < \epsilon.$$

This implies that $\forall n > N$ where $N = 2Q$ we have

$$|x_n - L| < \epsilon.$$

This we conclude that the sequence $\{x_n\}$ converges to L. $\square$

■ **Problem 1.20** Prove that the Bolzano-Weierstrass theorem implies the convergence of the Cauchy sequences in $\mathbb{R}$.

Proof. Let $\{x_n\}$ be a sequence in $\mathbb{R}$ which is Cauchy. We know that every Cauchy sequence is bounded. Thus $\{x_n\}$ is bounded. On the other hand, due to the Bolzano-Weierstrass theorem, every bounded sequence has a convergent subsequence. Let $\{x_{n_l}\}$ be a convergent sub-sequence that $x_{n_l} \to L$ as $l \to \infty$ and $L \in \mathbb{R}$. For a given $\epsilon > 0$ we we can find $K > 0$ such that $\forall l > K$ we have $|x_{n_l} - L| < \epsilon/2$. Also we can fine $N_1 > 0$ such that $\forall n, m > N_1$ we have $|x_n - x_m| < \epsilon/2$. Let $N = \max\{N_1, n_K\}$. For all $n, n_l > N$ we have

$$|x_n - L| = |x_n - L \pm x_{n_l}| \le |x_n - x_{n_l}| + |x_{n_l} - L| < \epsilon.$$

This shows that $\{x_n\}$ converges to L as well. $\square$

■ **Problem 1.21** Prove that every bounded monotonic sequence is Cauchy (do not assume the monotone convergent theorem).

Proof. We need to use some notion of completeness of $\mathbb{R}$. We will use that the infimum or suprimum exists for the bounded sequences. Also, we will demonstrate the proof for non-decreasing sequences, but it is similar to extend the proof for non-increasing sequences as well. Let $\{x_n\}$ be a bounded non-decreasing sequences. Then the suprimum of the sequences exists. Let $L = \sup_{n\in\mathbb{N}}\{x_n\}$. Let $\epsilon > 0$ be given. Since L is the least upper bound, then $\exists N > 0$ such that $L - \epsilon < x_N \le L$. Furthermore, since the sequences is non-decreasing, in fact $\forall n > N$ we have $L - \epsilon < x_n < L$, thus the distance between any two elements passed N should be less than ϵ. I.e., $\forall n, m > N$ we have $|x_n - x_m| < \epsilon$. This completes the proof. $\square$

■ **Problem 1.22** Let $\{a_n\}$ be a sequence of positive real numbers. Proof the following identity.

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf \sqrt[n]{a_n} \leq \limsup \sqrt[n]{a_n} \leq \limsup \frac{a_{n+1}}{a_n}$$

Proof. Let $\beta_1 = \limsup \frac{a_{n+1}}{a_n}$ and $\beta_2 = \limsup \sqrt[n]{a_n}$. Let $B_1 > \beta_1$. Then we can find a natural number N such that $\forall n > N$ we have $a_{n+1}/a_n < B_1$. This implies $a_n < B_1^n C$ for some $C \in \mathbb{R}$. Then we have

$$\sqrt[n]{a_n} < B_1 \sqrt[n]{C} \implies \limsup \sqrt[n]{a_n} \leq B_1 \implies \boxed{\limsup \sqrt[n]{a_n} \leq \limsup \frac{a_{n+1}}{a_n}}.$$

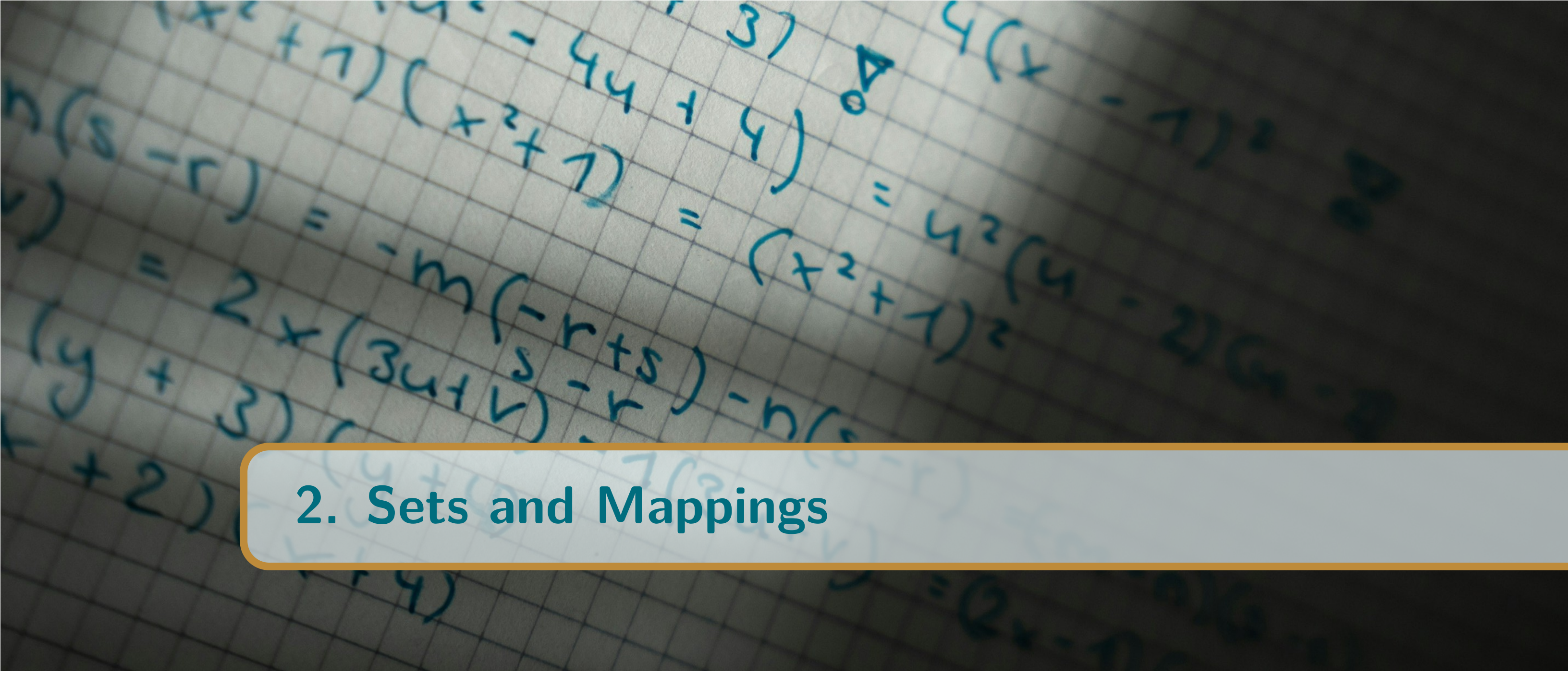
Similarly, let $\alpha_1 = \liminf \frac{a_{n+1}}{a_n}$ and $\alpha_2 = \liminf \sqrt[n]{a_n}$, and $A_1 < \alpha_1$. Then we can find an integer N such that $\forall n > N$, for some $C \in \mathbb{R}$ we have

$$a_{n+1}/a_n > A_1 \implies a_n > A_1^n C \implies \sqrt[n]{a_n} > A_1 \sqrt[n]{C} \implies \limsup \sqrt[n]{a_n} \geq A_1$$
$$\implies \boxed{\limsup \sqrt[n]{a_n} \geq \limsup \frac{a_{n+1}}{a_n}}.$$

Putting all together we have

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf \sqrt[n]{a_n} \leq \limsup \sqrt[n]{a_n} \leq \limsup \frac{a_{n+1}}{a_n}$$

□



2. Sets and Mappings

■ **Remark** Let $\mathbb{R}_+$ denote the real number greater than or equal to zero[1]. The we can view the association $x \mapsto x^2$ as a map from R to $\mathbb{R}_+$. When viewed so, the map is the surjective. Thus it is a reasonable convention not to identify this map with the map $f : \mathbb{R} \to \mathbb{R}$ defined by the same formula. To be completely accurate, we should therefore denote the set of arrival and the set of departure of the map into our notation, and for instance write

$$F_T^S : S \to T,$$

instead of our $f : S \to T$ notation. In practice, this notation is too clumsy, so that we omit the indices S, T. However, the reader should keep in mind the distinction between the maps

$$f_{\mathbb{R}_+}^{\mathbb{R}} : \mathbb{R} \to \mathbb{R}_+ \quad \text{and} \quad f_{\mathbb{R}}^{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$$

both defined by the association $x \mapsto x^2$. The first map is surjective, while the second one is not. Similarly the maps

$$f_{\mathbb{R}_+}^{R_+} : \mathbb{R}_+ \to \mathbb{R}_+ \quad \text{and} \quad f_{\mathbb{R}}^{\mathbb{R}_+} : \mathbb{R}_+ \to \mathbb{R}$$

defined by the same association $x \mapsto x^2$ are injective.

■ **Remark**

2.1 Problems

■ **Problem 2.1** Let X, Y be sets and $B \subseteq Y$. Prove

$$f^{-1}(B^c) = f^{-1}(B)^c.$$

Proof. First we show $f^{-1}(B^c) \subseteq f^{-1}(B)^c$. Let $x \in f^{-1}(B^c)$. Then $f(x) \in B^c \implies f(x) \notin B \implies x \notin f^{-1}(B) \implies x \in f^{-1}(B)^c$. For the other direction, assume $x \in f^{-1}(B)^c$. Then $x \notin f^{-1}(B) \implies f(x) \notin B \implies f(x) \in B^c \implies x \in f^{-1}(B^c)$. □

■ **Problem 2.2** Suppose both $f_1, f_2 : X \to Y$ are continuous, where X and Y are Hausdorff topological spaces. Then prove that for any $Q \subseteq X$ we have

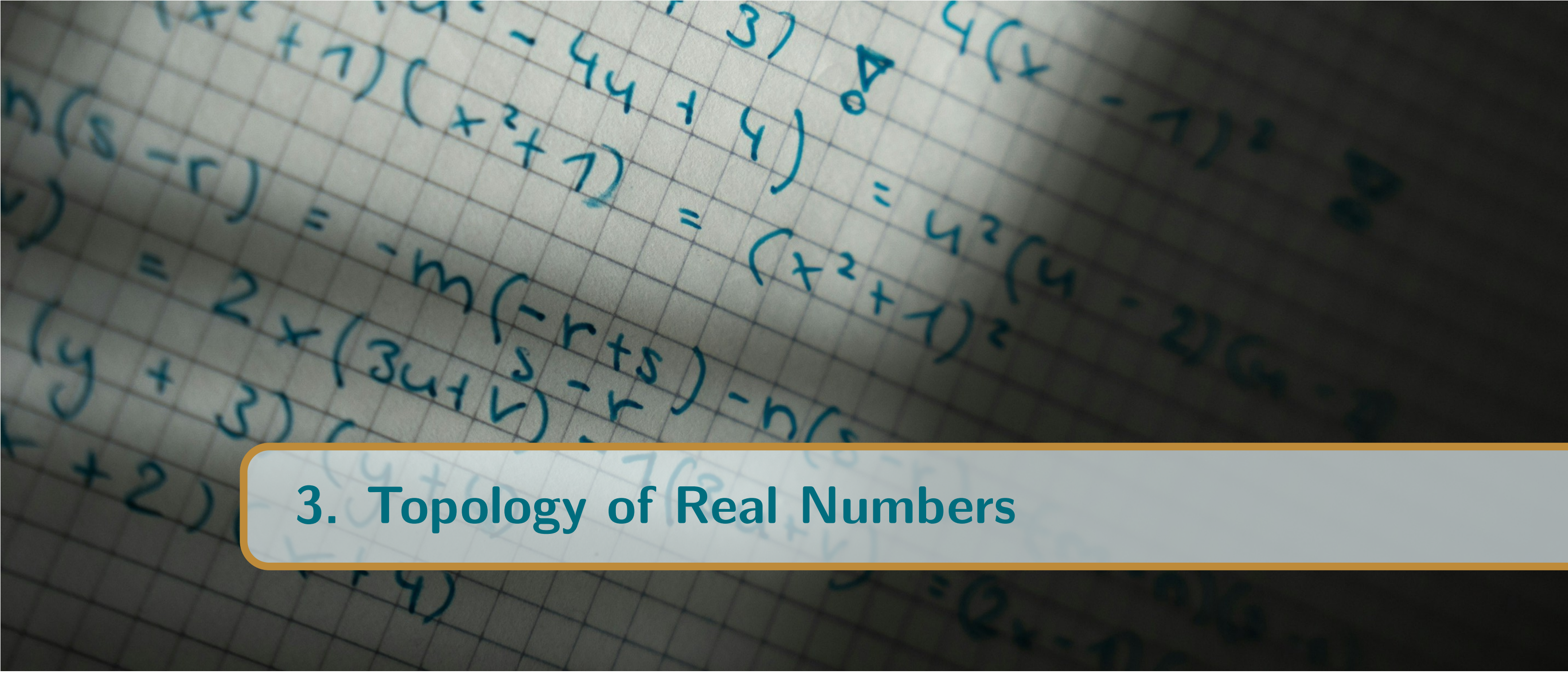
$$[f_1(q) = f_2(q) \quad \forall q \in Q] \implies [f_1(x) = f_2(x) \quad \forall x \in \overline{Q}].$$

[1] This note is from Segel, undergraduate analysis.

Proof. We will prove by contrapositive. The contrapositive will be

$$[\exists x \in \overline{Q} \text{ s.t. } f_1(x) \neq f_2(x)] \implies [\exists q \in Q \text{ s.t. } f_1(q) \neq f_2(q)].$$

Let $x \in \overline{Q}$ such that $f_1(x) \neq f_2(x)$. $x \in \overline{Q}$ implies $x \in Q$ or $x \in Q'$. In the former case, there is nothing to prove the consequent of the statement. Thus let $x \in Q'$. Also let $U, V \in \mathcal{T}_Y$ such that $f_1(x) \in U$ and $f_2(x) \in V$ and $U \cap V = \varnothing$. Since f_1 and f_2 are continuous, then $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are both open and contains x. Since $f_1^{-1}(U) \cap f_2^{-1}(V)$ is open, then $\exists G \in \mathcal{T}$ such that $x \in G \subseteq f_1^{-1}(U) \cap f_2^{-1}(V)$. Furthermore, since $x \in G'$, then $\exists q \in Q$ s.t. $q \in G$. Thus $f_1(q) \in U$ and $f_2(q) \in V$ and since $U \cap V = \varnothing$ then $f_1(q) \neq f_2(q)$ and this completes the proof. $\square$



3. Topology of Real Numbers

3.1 Introduction and Some Historical Notes

In this section we will construct the set of real numbers from the integers. We will assume that we know the integers and its basic arithmetic properties. However the fact is that the set of integers can be constructed by the set of positive integers (natural number) that can be constructed using the concept of the cardinality of a set and the set of all subsets of a set.

Ancient Greek scientists knew how to construct the rational and irrational numbers (like $\sqrt{2}$) with a compass and straightedge. But they did not know how to construct the number π with that setting. This problem was known for them as *the problem of squaring a circle.* In 1666, Newton showed that π can be constructed with an infinite sum. It was in late 1600's that Newton and Leibniz had vague notions of "limit" and "infinity". It was until early 1800's that there were no rigorous mathematical definition of these concepts. For example stuff by Fourier (like infinite Fourier series) made Laplace and Lagrange very uneasy! The infinite and limit concepts were more like a toolbox that were working very well on certain physical problems (for example in solving the PDE for hear equation). Finally In the early 1800s, a revolution happened in making these concepts precise. For example works done by Cauchy in 1820's and Weierstrass and Riemann (1850's and 1860's) had a significant contribution on these concepts.

3.1.1 A little note about Leopold Kronecker

In the lecture note by Francis Su in youtube, he talks about this famous saying from Kronecker:

> God created the integers. All else is the work of man!

And Su continues explaining that Kronecker was a finitist (following the finitism school of thoughts). When I heard this discussion his argue with Cantor came in my mind. In the Wikipedia page of Cantor we read that Kronecker was calling him as a "scientific charlatan", a "renegade" and a "corrupter of youth". So there is a connection with him being a finitist and having serious arguments with Cantor. It is also very interesting for me that one of his contributions which is Kronecker delta function kind of works with integers both in its index and its output!

Strangely, the quote that I have written above by Kronecker was his reply to the Lindemann when he proved that the number pi is a transcendental number. It is believed that he said "this is a beautiful but proves nothing. transcendental numbers do not exist!!"

3.2 Constructing Rational Numbers

To construct the rational numbers from integers, we need to use the concept of relations on a set. I will not talk about the concept of relations here as it is covered in other lecture notes. The relation that is of our interest is called a **equivalence relation**. Equivalence relation is a relation that has three properties called *reflexivity*, *symmetry*, and *transitivity*. We can define the rational numbers as:

$$\mathcal{Q} = \{\frac{a}{b} | a, b \in \mathcal{Z}, b \neq 0\}.$$

The notation $\frac{\cdot}{\cdot}$ as a equivalence relation: the $\frac{a}{b}$ is a representation of the ordered pair (a, b). We say $(a, b)\ (c, d)$ if and only if $ad = bc$. The relation is indeed a equivalence relation and this relation is in fact the equality relation for the rational numbers. For example $\frac{3}{5} = \frac{6}{10}$ because $3 * 10 = 6 * 5$.

It is very easy to show that this relation is an equivalence relation. However to check the transitivity property, we need to use the cancellation law. Keep in mind that we have not yet defied division for integers and the cancellation law is the next best thing to the division. The cancellation law for the integers is:

$$ab = ac, \quad a \neq 0 \quad \Rightarrow \quad b = c$$

So far we learned that we can construct the set of rational numbers like the following set:

$$\mathbb{Q} = \{\frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0\}.$$

However the question arise that what is the meaning of $\frac{a}{b}$. This is simply a representative of class of an equivalence defined on $\mathbb{Z} \times (\mathbb{Z} \backslash \{0\})$. The relation is defined as this:

> let $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0$. Then we write $(a, b) \sim (c, d)$ if and only if $ad = bc$. Then $\frac{a}{b}$ is an equivalence class such that:
>
> $$\frac{a}{b} = \{(c, d) \in \mathbb{Z} \times (\mathbb{Z} \backslash \{0\}) | (a, b) \sim (c, d)\}$$

As an example $\frac{1}{2} = \{(1, 2), (2, 4), (3, 6), \cdots\}$.

3.2.1 Defining addition for the rational numbers

So far we know how to add two integers but what does actually mean to add two rational numbers? We can throw any definitions that we want but we need to keep in mind that the definition should be well defined. In a sense that the definition does not depend on the representative of the class that we pick. For instance let's define the sum of rational numbers as:

> **A proposed definition for summation.** Let $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0$. Then let's define the summation of the rational numbers as the following:
>
> $$\frac{a}{b} + \frac{c}{d} = \frac{a + b}{c + d}.$$

The problem with the definition above is that it is not well defined, i.e. the result of the sum depends on the choice of representative for the class of interest. To illustrate that better let's do the following summation:

$$\frac{1}{2} + \frac{5}{3} = \frac{6}{5}$$

Now let's pick other representatives of the classes $\frac{1}{2}$ and $\frac{5}{3}$ which can be for instance $\frac{7}{14}$ and $\frac{10}{6}$. Now we expect to get a same result as before if we sum these two fractions:

$$\frac{7}{14}+\frac{10}{6}=\frac{17}{20}$$

It is clear that $\frac{17}{20}$ and $\frac{6}{5}$ are not equivalent. So if we define the summation in the specified way, then it is not well defined.

Also there is another problem. Defining the summation in this way will not extent the notion of sum for the integers. You can try summing $\frac{5}{1}+\frac{4}{1}$ and observe that the result is not the same as $5+4=9$.

Let's define that summation in the following way that is both well defined and also extends the notion of summation of the integers.

Definition 3.1 — Defining summation for the rational numbers. Let $\frac{a}{b}$ and $\frac{c}{d}$ be two rational numbers. Then we define the summation for rational numbers as:

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

To show that this definition is we ll defined, Let $\frac{a}{b}, \frac{c}{d}, \frac{a'}{b'}, \frac{c'}{d'}$ be rational numbers such that $(a,b)\sim(a',b')$ and $(c,d)\sim(c',d')$. We need to show that

$$\frac{ad+bc}{bd}=\frac{a'd'+b'c'}{b'd'}$$

Proof. Since $(a,b)\sim(a',b')$, then we can write $ab'=a'b$ and similarly since $(c,d)\sim(c',d')$ then $cd'=c'd$. Since $b,b',d,d'\neq 0$, then we can multiply bb' to the both sides of the second equation and dd' to both sides of the first equation. Then we will have:

$$\begin{aligned} ab'dd' &= a'bdd', \\ bb'cd' &= bb'c'd. \end{aligned}$$

By adding both sides of these equations then we will have:

$$\begin{aligned} ab'dd'+bb'cd' &= a'bdd'+bb'c'd, \\ (b'd')(ad+bc) &= (bd)(a'd'+b'c'). \end{aligned}$$

This clearly shows that $(ad+bc,bd)\sim(a'd'+b'c',b'd')$ hence

$$\frac{ad+bc}{bd}=\frac{a'd'+b'c'}{b'd'}$$

Now we can define the multiplication for the rational numbers.

Definition 3.2 — Multiplication of the rational numbers. Let $\frac{a}{b}$ and $\frac{c}{d}$ be rational numbers. Then we define the multiplication for the rational numbers as:

$$\frac{a}{b}\frac{c}{d}=\frac{ac}{bd}$$

Similar to the last part, we can show that this definition is well defined. Namely we can show that for rational numbers $\frac{a}{b}, \frac{a'}{b'}, \frac{c}{d}, \frac{c'}{d'}$ that $(a,b)\sim(a',b')$ and $(c,d)\sim(c',d')$, we have $(ac,bd)\sim(a'c',b'd')$.

Proof. Since $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$ so $ab' = a'b$ and $cd' = c'd$. By multiplying both sides of the equation we will have:

$$a'bc'd = ab'cd',$$

which clearly shows that $(ac, bd) \sim (a'c', b'd')$ hence

$$\frac{ac}{bd} = \frac{a'c'}{b'd'}.$$

3.2.2 Does $\mathbb{Q}$ extends $\mathbb{Z}$?

.

With the following correspondence (for $n \in \mathbb{Z}$)

$$\frac{n}{1} \leftrightarrow n,$$

we can show that the set $\{\frac{n}{1} | n \in \mathbb{Z}\}$ behaves exactly like the set of integers. In other words we say these two sets are isomorphic.

3.2.3 Orders in $\mathbb{Q}$

We know that the elements of $\mathbb{Z}$ are ordered (some elements are smaller or larger than the other ones). So the natural question that arise is that will this order be still valid for the points in $\mathbb{Q}$? To answer this question we need to rigorously define the order relation in $\mathbb{Z}$.

Definition 3.3 — Definition of Order. An order on a set S is a relation $<$ satisfying:

- Low of trichotomy: $\forall x, y \in S$, the only one of the following statements are true

$$x < y, \quad x = y, \quad y < x$$

- Transitivity: For $x, y, z \in S$, $x < y$ and $y < z$ implies $x < z$.

Note that this is a general definition of order on a set and is not restricted to our usual definition of order between real numbers. However, we can define the notion of the "usual" order in $\mathbb{Z}$ like the following:

Definition 3.4 — Order Relation on $\mathbb{Z}$. The order relation on $\mathbb{Z}$ denoted with the symbol $<$ is defined as the following. Let $a, b \in \mathbb{Z}$. We say $a < b$ if and only if $a - b$ a positive integer. The set of positive integers are defined as $\{1, 2, 3, 4, \cdots\}$.

■ **Example 3.1** Dictionary Oder on $\mathbb{Z}$ As stated earlier, we can extend the definition of order. A **dictionary order** on $\mathbb{Z}^2$ is the relation $\lessdot$ such that for $(a_1, a_2), (b_1, b_2) \in \mathbb{Z}^2$ we write $(a_1, a_2) \lessdot (b_1, b_2)$ if and only if $(a_1 < b_1)$ and if $a_1 = b_1$, then $a_2 < b_2$.
For example, given this relation we can write: $(3, 4) \lessdot (5, 1), (1, 0) \lessdot (1, 10), (3, 1) \lessdot (3, 5)$ ■

Definition 3.5 — Positive rational numbers. We say the rational number $\frac{a}{b}$ is positive if the integer ab is positive.

Definition 3.6 — Ordering of rational numbers. We say $\frac{a}{b} < \frac{c}{d}$ if $\frac{c}{d} - \frac{a}{b}$ is a positive rational number.

Given the ordering property of the rational numbers, we can look at the rational numbers with a new perspective.

3.2.4 $\mathbb{Q}$ Is a Field!

Field is one of many algebraic structures (like groups, rings, vector spaces, etc).

Definition 3.7 — Field. A field is a set F along with two operations $+, \times$ that holds the following properties:

- (A_1) : The set F is closed under $+$.
- (A_2) : $+$ is commutative.
- (A_3) : $+$ is associative.
- (A_4) : Every element in F has a additive inverse
- (A_5) : Every element in F has a additive identity (call it 0)
- (M_1) : The set F is closed under $\times$.
- (M_2) : $\times$ is commutative.
- (M_3) : $\times$ is associative.
- (M_4) : Every element in F (except for the additive inverse) has an multiplicative inverse.
- (M_5) : Even element in F has an multiplicative identity (call it 1).
- (D_1) : The operator $\times$ distributes over $+$.

■ **Example 3.2** $\mathbb{Q}$ is a field Question. Show that the set of rational numbers is a field.

Solution. We can start with finding the additive and multiplicative inverses and identities. It is obvious that:

- Additive identity: $\frac{0}{1}$.
- Additive inverse for $\frac{a}{b}$: $\frac{-a}{b}$.
- Multiplicative identity: $\frac{1}{1}$.
- Multiplicative inverse for $\frac{a}{b}$: $\frac{b}{a}$

Now we need to show that the conditions $A_1, A_2, A_3, M_1, M_2, M_3, D_1$ holds. Let $a, b \in \mathbb{Z}$. Then we know that $a+b$ and ab are also integers and are in $\mathbb{Z}$. So A_1, M_1 immediately follows from the definition of addition and multiplication for the rational numbers.

- A_2: We need to show that $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$ By following the addition defined for the rational numbers, we can write the expression for the LHS and RHS seperately and observe that those two are equal. So for LHS we have:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

- (A_3): We need to show $(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{a}{b} + (\frac{c}{d} + \frac{e}{f})$

  Following the definition of addition for the rational numbers, for the LHS we can write:

$$\frac{ad+bc}{bd} + \frac{e}{f} = \frac{fad+fbc+edb}{bdf}$$

And for the RHS we can write:

$$\frac{a}{b} + \frac{cf + de}{df} = \frac{adf + bcf + bde}{bdf}$$

Because of the associativity and commutativity properties of Z, we can conclude that RHS = LHS.

So we can observe that A_2, A_3, M_1, M_2 follows from the commutativity and associativity properties of the integers (which are considered as a ring). ∎

3.2.5 Constructing the Real Numbers from Rational Numbers

The rational numbers extends the set of integers in a very useful way. But it turns out that there are many holes in the set of rational numbers; i.e. there are some real numbers (real in the sense that we can construct some length equal to it on a paper) but that does not belong to the set of rational numbers. A very famous example is $\sqrt{2}$ that has been known from the ancient Greek. This number is the hypotenuse of a right triangle with sides equal to 1 (using the Pythagorean theorem). However we can prove that this number is not rational number.

Proof. Let x be number s.t. $x^2 = 2$. We claim that this number can not be a rational number. To show this, let's assume that x is rational. So we can write x as:

$$x = \{x = \frac{a}{b} | x^2 = 2, b \neq 0, (a, b) = 1\}$$

Note that we require $(a, b) = 1$ (i.e. relatively prime) as an extra condition since we know that in the class of equivalence with the representative $\frac{a}{b}$, there is an element $\frac{c}{d}$ such that (a, b) (c, d) (since (c, d) belongs to the $\frac{a}{b}$) and c, d are relatively prime. So we can write $x^2 = \frac{a^2}{b^2} = 2$. Then $a^2 = 2b^2$. We can easily show (by contrapositive) that if a^2 is even, then a is even as well. So for some integer k we can write $a = 2k$. Then $b^2 = 2k^2$, so b is also even. Hence for some integer l, $b = 2l$, and this is a contradiction because a, b are not relatively prime.

Now that we observed numbers like $\sqrt{x}$ are not rational, then we can say that the set of rational number $\mathbb{Q}$ does not have the **least upper bound property**.

Definition 3.8 — Least Upper Bound Property. A set S is said to have the least upper bound property every nonempty subset of S that is bounded above (thus has an upper bound), has a least upper bound (i.e. supremum) as well.

It is clear from the definition that the set $\mathbb{Q}$ does not have a least upper bound property since the set

$$A = \{x \in \mathbb{Q} | x^2 < 2\}$$

has an upper bound (like 2) but does not have a least upper bound. This indicates the wholes present in the set of rational numbers. However, we can extend the idea of rational numbers in a way that contains the set of rational numbers as a subset and also fills in the gaps. We can do that in many ways one of which is the concept of Dedekind cuts. Here is the definition of a Dedekind cut:

Definition 3.9 — Dedekind cut. A Dedekind cut α is a **subset of rational numbers** that has the following properties:

1. The set is not trivial (i.e. is not empty and does not contain all of the rationals),
2. is closed downwards. In other words $(x \in \alpha \wedge q \in \mathbb{Q}) \wedge q < x \Rightarrow q \in \alpha$, and

3. has no largest element. In other words: $\forall x \in \alpha, \exists r \in \alpha \quad s.t. \quad x < r$.

The set of real numbers can be defined using the idea of the Dedekind cuts in the following way:

Definition 3.10 — The Set of Real Number. The set of real numbers denoted by $\mathbb{R}$ is the set of all cuts:

$$\mathbb{R} = \{\alpha : \alpha \text{ is a cut}\}.$$

So when we refer to the real number $\sqrt{2}$, it is a set that:

$$\sqrt{2} = \{q \in \mathbb{Q} : x^2 < 2\}.$$

Remember that this set (which is also a cut) itself had not sup in the set of rational numbers. However the set of all such cuts (that we denoted that set as the set of real number), will have the least upper bound property. For an instance:

$$\sup\{x \in \mathbb{R} : x^2 < 2\} = \{x \in \mathbb{Q} : x^2 < 2\}$$

We can define the addition and multiplication operations for these sets (cuts) in a proper way that extends the idea of addition and multiplication for rationals (thus integers). Also, we can show that the set of real numbers also posses the order relations ($\alpha < \beta \quad iff \quad \alpha \subset \beta$). So we can show that the set of rational numbers form an **ordered field**. In fact we can show that the set of rational numbers is the only ordered field with the least upper bound property and for any other ordered field there is a one-to-one correspondence (bijective) between its elements and the set of real numbers. As an instance, we can define the addition and multiplication for the real number as:

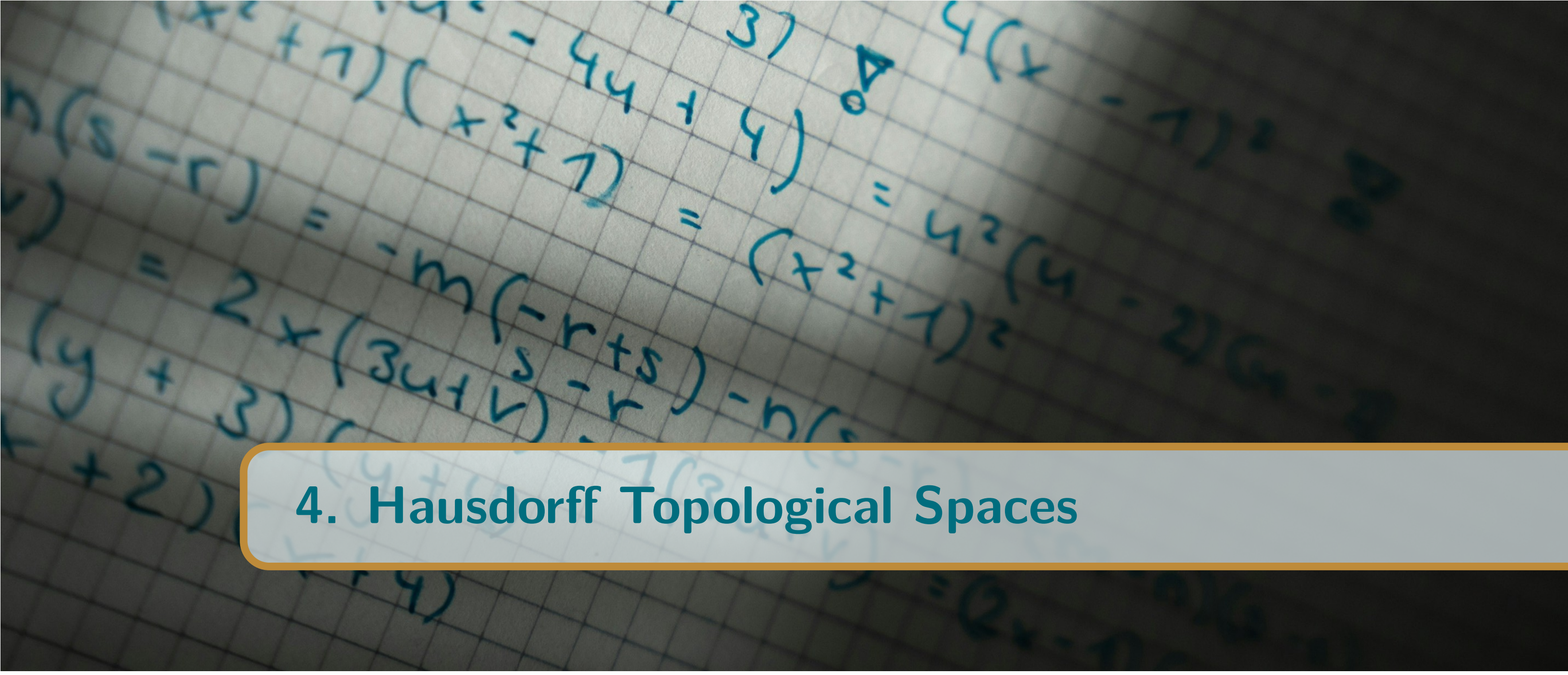
Definition 3.11 — Addition and Multiplication for Real Numbers. Let α and β be two Dedekind cuts. Then:

$$\alpha + \beta = \{r + s : r \in \alpha, s \in \beta\},$$
$$\alpha * \beta = \{rs : r \in \alpha, s \in \beta\}$$

One of the important things to check when defining a **binary operator** on a set ($O : S \to S$) is to check if the result of the operation still is in the set. So it is a good practice to show that $\alpha + \beta$ and $\alpha * \beta$ are still considered as cuts.

3.2.6 Note that I need to add them to the main text

- In a field, just one element (that is the additive identity) should have no inverse (no any other thing).



4. Hausdorff Topological Spaces

The following is a section of the great book “Mathematical Discovery” by Bruckner.

> Professional mathematicians must adhere to strict standards in their work. This entails providing precise definitions, even for seemingly familiar concepts. Such precision often requires the use of complex technical tools and methods. A mathematician must possess a clear understanding of fundamental concepts, such as the precise definition of a "curve," the mathematical interpretation of "traversing a curve with the inside to the left," the formal description of the number of "holes" in a pretzel, and the mathematical definition of area.
>
> **It’s important to note that this level of rigor and precision is not typically present when a mathematician initially approaches a problem and begins working on a solution.** In the early stages, ideas tend to be more abstract and intuitive. The refinement and meticulousness become evident only in the final drafts of mathematical work.

Thus, we first need to have a discussion that show that the ideas behind the abstractions and generalizations are achievable by careful studying the mathematical objects already around us. This this section focuses to motivate the reader towards the more abstract concepts.

Thus we will discuss that the $\mathbb{R}^n$ along with the Euclidean distance has some special properties (which later will be generalized to the concept of metric spaces), and then we will see that the notion of Euclidean distance give rise to special sets called open ball which will give rise to the notion of open sets. We will study the properties of these open sets and later we will study what if we define the notion of open sets on its own (without any need to any particular metric) which will lead to the notions and ides of topological spaces.

4.1 Motivation

Consider the set $\mathbb{R}^k$ which is a k fold Cartesian product of our favorite set $\mathbb{R}$!. We can also extend the notion of Euclidean distance in $\mathbb{R}$ (which was simply $|x-y|$ for $x, y \in \mathbb{R}$) to $\mathbb{R}^k$ as follows

$$|x-y| = \sqrt{\sum_{i=1}^{k}(y_i - x_i)^2}, \qquad x = (x_1, \cdots, x_k),\ y = (y_1, \cdots, y_k) \in \mathbb{R}^k.$$

We can easily observe that the Euclidean distance is a function $d : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ that satisfies the following properties

(i) $|x-y| \geq 0, \quad |x-y| = 0 \Leftrightarrow x = y,$

(ii) $|x-y| = |y-x|,$

(iii) $|x-y| \leq |x-z| + |z-y|$

Later, We will study the generalized idea of such functions defined on a set which will give rise to the concept of metric spaces.

Now, we intuitively define the notion of "open ball" centered at $x \in \mathbb{R}^k$ with radius $r \in \mathbb{R}$ as follows

$$\mathcal{B}_x(r) = \{y \in \mathbb{R}^k : \ |x-y| < r\}.$$

Then we define a set $A \subseteq \mathbb{R}^k$ to be a open set such that for every element $x \in$, we can have an open ball $\mathcal{B}_x(r)$ for some $r \in \mathbb{R}$ which is contained in A. More formally we can write

$$\forall x \in A, \ \exists r > 0 \text{ s.t. } x \in \mathcal{B}_x(r) \subseteq \mathbb{R}^k.$$

Since this notion is a very central one (as we will find out later), for $\mathbb{R}^k$, we have the notion of the set of all open sets of $\mathbb{R}^k$, for which we write $\mathcal{T}$.

Then we can go a little bit beyond the immediate intuition and define the notion of the set of all neighborhoods of x as

$$\mathcal{N}(x) = \{S \subseteq \mathbb{R}^k : \ \exists u \in \mathcal{T}, \ x \in u \subseteq S\}.$$

It immediately follows from the definition that all open balls containing x (not necessarily containing x) are in $\mathcal{N}(x)$, along with other sets which satisfies the required property. We are now in a good shape to study the properties of the open sets $u \in \mathcal{T}$. We claim the followings are some of such properties (which as it will turn out are the central properties in some sense)

(i)
$$\varnothing, \ \mathbb{R}^k \in \mathcal{T}.$$

(ii)
$$\forall \mathcal{G} \subseteq \mathcal{T} \text{ we have } \bigcup_{g \in \mathcal{G}} g \in \mathcal{T}.$$

(iii)
$$U_1, \cdots, U_n \in \mathcal{T}, \ n \in \mathbb{N} \implies \bigcap_{i=1}^{n} U_i \in \mathcal{T}.$$

(iv)
$$\forall x, y \in \mathbb{R}^k, \ \exists U, V \in \mathcal{T} \text{ s.t. } x \in U, \ y \in V, \quad U \cap V = \varnothing.$$

Proof. TO BE ADDED. $\square$

Since the notion of open sets is closely related with the distance function, thus there is no surprise that it can be used to express the ideas of convergence of a sequence in $\mathbb{R}^k$ (such a fundamental concept in analysis) with the new terminology. For instance, the following two statements are logically equivalent for $x_n \to \hat{x}$

(i) $\forall \epsilon > 0, \ \exists N > 0, \text{ s.t. } \forall n > N$ we have $|x_n - \hat{x}| < \epsilon.$

(ii) $\forall S \in \mathcal{N}(\hat{x}), \ \exists N > 0, \text{ s.t. } \forall n > N$ we have $x_n \in S.$

Proof. TO BE ADDED. $\square$

4.2 Metric Spaces

Although $\mathbb{R}^k$ is a very useful set synthesized in a special way to meet most of our requirements (for instance the completeness arguments in the sense of Cauchy sequence, least upper bound property, and etc.) but not every set we encounter is $\mathbb{R}^k$. We can have sets that are globally very different than the "*flat*" $\mathbb{R}^k$, for instance $\mathbb{S}^1$ (unit circle), $\mathbb{S}^1 \times \mathbb{S}^1$ (a tours), etc. One of the main approaches in dealing with such structures is to "locally" convert it (in a useful way) to a collection (or atlas) of subsets of $\mathbb{R}^k$ and then work with the original "alien" set in an indirect way by focusing on these local images in $\mathbb{R}^k$. Apart from this approach, it is also useful to generalize the notions of distance in a set, which will enable us working with other classes of abstract structures without relying on $\mathbb{R}^k$, as some of them are way larger than $\mathbb{R}^k$. For instance, consider the set of all bounded function $f : [0, 1] \to \mathbb{R}$. This set has a cardinality that is bigger than the cardinality of continuum. Also, might want to work with sets that are discrete in nature, like $\mathbb{N}$ which their cardinality is less than $\mathbb{R}^k$. So relying on $\mathbb{R}^k$ for all purposes is not feasible, thus it might be a good idea to have the notion of distance between elements in set.

Definition 4.1 — Metric Space. A metric space is simply (X, d) in which X is a set and $d : X \times X \to \mathbb{R}$ is a function called metric that satisfies the following properties

(i) $d(x, y) \geq 0, \quad d(x, y) = 0 \Leftrightarrow x = y.$

(ii) $d(x, y) = d(y, x).$

(iii) $d(x, y) \leq d(x, z) + d(z, y).$

In which $x, y, z \in X$.

Now we can easily see that all of the notions like open balls, open sets, and etc. which we defined for $\mathbb{R}^k$ can also be defined for a metric space. We can define different metrics on a particular set based on our demands. In fact there are infinitely many ways to come up with a metric function. **One of our main tasks in studying metric spaces is to show that there are some properties of a metric space that are independent of a particular defined metric**. As we will see later, this will give rise to more abstract construct called topological spaces.

Definition 4.2 — Open Ball in $\mathbb{R}^k$. Let (X, d) be a metric space. An open ball centered at $x \in X$ with radius r is the set

$$\mathcal{B}_r(x) = \{y \in X : d(y, x) < r\},$$

Definition 4.3 — Open Set in $\mathbb{R}^k$. Let (X, d) be a metric space and let $U \subseteq X$. U is open if

$$\forall x \in X,\ \exists \mathcal{B}_r(x) \text{ s.t. } \mathcal{B}_r(x) \subseteq U.$$

We denote the set of all open sets of X as $\mathcal{T}$.

A very useful intuition about open sets is that we can move around any points of the set (sufficiently small) and still be in the set. In other words, we can perturb the points of an open set (a sufficiently small perturbation) and still remain in the set.

Definition 4.4 — Neighborhood of x. Let (X, d) be a metric space. Then the set of all neighborhoods of x is

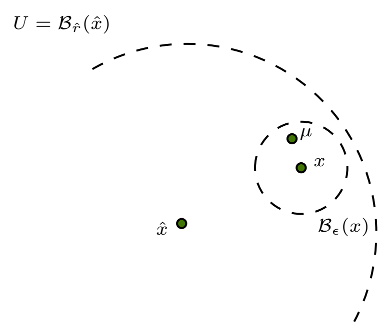
$$\mathcal{N}(x) = \{S \in \mathcal{P}(X) :\ \exists U \in \mathcal{T} \text{ s.t. } x \in U \subseteq S\}.$$

In other words, A neighborhood of $x \in X$, is the collection of all subsets of X, such that contains an open set containing x.

■ **Remark** An open ball is an open set. This is not a tautological statement. The word "open" in the notion of open ball, has nothing to do with the word "open" in the notion of open set. however, we can show that an open ball is indeed an open set, thus deserves the name "open". In more accurate language

Let $\hat{x} \in X,\ \hat{r} \in \mathbb{R},$ and define $U = \mathcal{B}_{\hat{r}}(\hat{x})$. Then U is an open set.

Proof. The proof of remark above can be facilitated by considering the following diagram.



Considering the visual idea, we can proceed with the proof. Let $x \in U$. Then let $r^* = r - d(x, \hat{x})$ and $\epsilon < r^*$. This implies that $d(x, \hat{x}) = \hat{r} - r^*$. We claim that $\mathcal{B}_\epsilon(x) \subseteq U$. Indeed, let $\mu \in \mathcal{B}_\epsilon(x)$. By definition $d(\mu, x) < \epsilon < r^*$. Thus

$$d(\hat{x}, \mu) \leq d(\hat{x}, x) + d(x, \mu) < (\hat{r} - r^*) + r^* = \hat{r}.$$

Thus we showed that $\mu \in \mathcal{B}_\epsilon(x)$ implies $\mu \in \mathcal{B}_{\hat{r}}(\hat{x})$. Thus we can conclude that $\mathcal{B}_\epsilon(x) \subseteq \mathcal{B}_{\hat{r}}(\hat{x})$, for any choice of x. Thus U is an open set. □

Following our arguments in the motivation section, we argued that the open sets of $\mathbb{R}^k$ satisfy some properties. If you read the proof closely, we used no facts very special about the Euclidean distance other than the properties described in the definition of a metric space. Thus it is not a surprise if we observe that those properties also hold for a general metric space.

Proposition 4.1 Let (X, d) be a metric space. Then the open sets determined by d satisfy the following properties

(i) $X, \varnothing \in \mathcal{T}$.

(ii) $\mathcal{G} \subseteq \mathcal{T} \implies \bigcup_{g \in \mathcal{G}} g \in \mathcal{T}$.

(iii) $U_1, \ldots, U_n \in \mathcal{T} \implies \bigcap_{i=1}^n U_i \in \mathcal{T}$.

(iv) $\forall x, y \in X,\ \exists U, V \in \mathcal{T}$ s.t. $x \in U,\ y \in\ V,\ U \cap V = \varnothing$.

Proof. TO BE ADDED. □

The metric spaces can be very abstract, like the space of all functions with a suitable metric indeed forms a metric space. However, this space is infinite dimensional and its cardinality is even larger than the cardinality of continuum. However, following the following definition we can decide if certain subsets of those abstract spaces are bounded or not.

Definition 4.5 — Bounded sets in a metric space. Let (X, d) be a metric space and $A \subseteq X$. Then A is bounded, if there exist $R \in \mathbb{R}$ and $x \in X$ such that $X \subseteq \mathcal{B}_R(x)$

Also, one interesting fact about the metric space, is that, no matter what is the nature of metric space, the union of all cocentric ball with natural numbers as their radius, centered at any point of the metric space covers the whole space. The following proposition makes this more formal.

Proposition 4.2 Let (X, d) be a metric space and $x \in X$. Then for

$$\mathcal{G} = \{\mathcal{B}_r(x) : r \in \mathbb{N}\},$$

we have

$$X = \bigcup \mathcal{G}.$$

Proof. Let $y \in X$. Then $d(x, y) = R$ for some $R \in \mathbb{R}$. Since $\exists N \in \mathbb{N}$ such that $R < N$, then $y \in \mathcal{B}_n(x)$ for all $n \geq N$. As for the converse, let $x \in \bigcup \mathcal{G}$. Then trivially $x \in X$. This completes the proof. □

Proposition 4.2 is a very interesting result. It is somehow fascinating that every metric space can be covered with countable number of cocentric open balls.

■ **Remark** One thing that the reader might note is the construct called “long line” which is a topological space which is longer than the real line in some sense. The long line consists of an uncountable number of copies of $[0, 1)$ “pasted together” end-to-end. This is a topological space and not a metric space. Thus out argument remains valid

4.2.1 Convergence in Metric Spaces

So far, we only hand the notion of sequence in sets like $\mathbb{R}, \mathbb{Z}, \mathbb{Q}$, etc. and we developed the notion of convergence of a sequence by $\epsilon - N$ business. Remember that a sequence is a fundamental concept which enables us to discover the word which could be reached via the infinitely long road of sequences! For a detailed discussion see my opinion piece titled by “What the Hell is Analysis?”. Now, the beauty of metric spaces is that we can analyze sequences in abstract metric spaces. This is automatically done via the structure of the metric function as for any pair $(x, y) \in X \times X$ it returns a real number which carries some intuitive information about the closeness of the points. So the if sequence $\{x_n\}$ whose elements live in X, converge to $\hat{x} \in X$, then this fact is basically translates to a bunch of real numbers approaching 0 which can be easily analyzed using numerous tools we have already developed (like the squeeze theorem, etc.)

For instance, in finding the solution of a particular ordinary differential equation, we can can come up with an iteration process (which is known as contraction mapping) which acts on the space of function. Then we can prove that the trajectory of points created by repeated application of the map on some initial point, will converge (in the sense that the distance between the output nth iteration of the map on the initial point and the point that it converges to is getting closed to zero). This leads to the well-known “Picard-Lindelöf” theorem which states that under some conditions, an initial value problem has unique solution.

Definition 4.6 Let (X, d) be a metric space. Then the sequence $\{x_n\}$ converges to $\hat{x} \in X$ if

$$\forall \epsilon > 0, \ \exists N > 0 : \ \forall n > N \text{ we have } d(x_n, \hat{x}) < \epsilon \ (\equiv x_n \in \mathcal{B}_\epsilon(\hat{x}))$$

Or alternatively, using the notion of Neighborhood

$$\forall S \in \mathcal{N}(\hat{x}),\ \exists N > 0 :\ \forall n > N \text{ we have } x_n \in S.$$

We can show that these two definitions are in fact equivalent by following argument

Proof. Since the statements (a) and (b) are equivalent, then we need to proof the both ways. Both proofs are straight and can be deduced by following the definitions.

- (a) $\implies$ (b): Given $S \in \mathcal{N}(\hat{x})$, $\exists \mathcal{B}_r(\bar{x})$ for $r > 0$ sufficiently small. Let $r = \epsilon$. Since (a) is true, then $\exists N > 0$ such that $\forall n > N$ we have $x_n \in \mathcal{B}_\epsilon(\hat{x}) \subseteq S$.

- (b) $\implies$ (a): Given $\epsilon > 0$, let $S = \mathcal{B}_\epsilon(\hat{x})$. Then since (b) is true, then $\exists N > 0$ suc that $\forall n > N$ we have $x_n \in S$, thus we conclude $x_n \in \mathcal{B}_\epsilon(\hat{x})$.

$\square$

Another interesting fact to consider about the metric spaces is the interplay between sequences and open sets. This interplay is important since the notion of sequences and converges is tightly bound with the notion of metric in a metric space. On the other hand, although the open sets are actually generated by the metric, but, as we will see later, they can have their own world, meaning that we can define then as sets that satisfy some basic axiom. Thus any useful interplay between sequences and converges between open sets can be useful in the future generalizations.

The following proposition defines the notion of open sets in a metric space (which we originally define by the notion of open balls) using the idea of sequences and their convergence.

Proposition 4.3 In a metric space (X, d), with subset A, the following are equivalent:

(a) A is an open set.

(b) For every $x \in A$, and every sequence $\{x_n\}$ obeying $x_n \to x$, one has $x_n \in A$ for all n sufficiently large. That is

$$\exists N \in \mathbb{N} :\ \forall n > N,\ x_n \in A.$$

Proof.

- $(a) \implies (b)$: We assume that A is open. Then for $x \in A$, we can pick a $\mathcal{B}_\epsilon(x) \subseteq A$ for some $\epsilon > 0$. Then from the definition of convergence, we know that if $x_n \to x$, then $\exists N > 0$ such that $\forall n > N$ we have $x_n \in \mathcal{B}_\epsilon(x)$ thus $x_n \in A$.

- $(b) \implies (a)$: We can prove this by both direct proof and also by contrapositive statement. For the direct proof, the idea is to consider the set of all sequences $x_n \to$ for $x \in A$. Then for each such sequence we can find $N > 0$, such that $\forall n > N$ we have $x_n \in A$. We pick x_{N+1} from all of such sequences and construct a set S such that $d(x, x_{N+1}) \in S$. We let r be the infimum of S. Infimum exists (since $S \subseteq \mathbb{R}$) and is nonzero (otherwise we could come up with a sequence that falsifies the assumption). We let $B = \mathcal{B}_r(x)$. Due to the construction $B \subseteq A$, thus implies A is an open set.

  However, the proof by contrapositive is much more straight forward. Assume a is not true. Then for a given $n \in \mathbb{N}$, let $\epsilon = 1/n$. Then since A is not open then $\exists x \in A$ such that $\mathcal{B}_\epsilon(x) \cap A \neq \varnothing$. Pick any $y \in \mathcal{B}_\epsilon(x) \cap A \neq$ and call it x_n. Due to the construction, $x_n \to x$, while $\forall n \in \mathbb{N},\ x_n \notin A$ which implies b is not true as well.

$\square$

4.3 Completeness of Metric Spaces

First we start with the notion of Cauchy sequences.

Definition 4.7 — Cauchy sequences. Let (X, d) be a metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is Cauchy if

$$\forall \epsilon > 0 : \ \exists N > 0 \text{ s.t. } \forall n, m > N \text{ we have } d(x_n, x_m) < \epsilon.$$

And then we have the following very important definition.

Definition 4.8 — Complete metric space. Let (X, d) be a metric space. This space is complete, if every Cauchy sequence has a limit in X.

The following proposition lists some of the important and basic properties of Cauchy sequences.

Proposition 4.4 Let (X, d) be a metric space. Then

(i) Every converging sequence is Cauchy.

(ii) Every Cauchy sequence is bounded.

(iii) Every closed subset of a complete metric space, is complete.

(iv) If a Cauchy sequence $\{x_n\}$ has a converging sub-sequence that converges to some point $\hat{x}$, then the full original Cauchy sequence must converge to $\hat{x}$.

(v) Every compact metric space is complete.

Proof. The proof for different items are as follows.

(i) Assume $\epsilon > 0$ is given and $\{x_n\}$ converges to $\hat{x} \in K$. From the properties of metric function we have
$$d(x_n, x_m) \leq d(x_n, \hat{x}) + d(x_m, \hat{x}).$$
Since $\{x_n\}$ converges, then $\exists N_1 > 0$ such that $\forall n > N_1$ we have $d(x_n, \hat{x}) < \epsilon/2$. So for $n, m > N$ we have
$$d(x_n, x_m) \leq d(x_n, \hat{x}) + d(x_m, \hat{x}) < \epsilon,$$
thus $\{x_n\}$ is Cauchy. □

(ii) Fix $\epsilon = 1$. Then since $\{x_n\}$ is Cauchy, then $\exists N > 0$ such that $d(x_N, x_m) < 1$ for all $m > N$. Let
$$\hat{R} = \{d(x_N, x_n) : \ n < N, n \in \mathbb{N}\}.$$
Let $R = \max\{1, \hat{R}\}$. Thus due the the construction we have
$$x_n \in \mathcal{B}_R(x_N), \quad \forall n \in \mathbb{N}.$$
This completes the proof. □

(iii) Let $\{x_n\}$ be a Cauchy sequence in F. Thus $A = \{x_n : \ n \in \mathbb{N}\} \subseteq F \subseteq X$. Since $\{x_n\}$ is also at X, then it converges to $\hat{x} \in X$. From definition of limit point (or derived set) we have $\hat{x} \in A'$. On the other hand $A \subseteq F \Leftrightarrow A' \subseteq F'$. Since F is closed, then $F' \subseteq F$. This implies $A' \subseteq F$. Thus $\hat{x} \in F$. This shows that F is complete. □

(iv) Let x_{n_k} be the converging sub-sequence, i.e. $x_{n_k} \to \hat{x}$ as $k \to \infty$. For a given ϵ, since $\{x_n\}$ is Cauchy, then $\exists N_1 > 0$ such that $\forall n, n_k > N_1$ we have $d(x_n, x_{n_k}) < \epsilon/2$. Also, since $\{x_{n_k}\}$ is converging to $\hat{x}$, then $\exists N_2 > 0$ such that $\forall n_k, n_l > N_2$ we have $d(x_{n_l}, x_{n_k}) < \epsilon/2$. Let $N = \max\{N_1, N_2\}$. Then $\forall n > N$ (also $n_k > N$) we have

$$d(x_n, \hat{x}) \leq d(x_n, x_{n_k}) + d(x_{n_k}, \hat{x}) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus the original sequence converges to $\hat{x}$. □

(v) Every compact metric space is sequentially compact (see Theorem 4.2), thus every sequence has a convergent sub-sequence that converges to a point in the metric space. Let $\{x_n\}$ be a Cauchy sequence. This will have a converging sub-sequence that converges to a point in $\hat{x} \in X$. But as we showed in (iv), this sequence itself converges to $\hat{x}$. Thus X is complete.

□

The following proposition is a very important one that has numerous use cases in the filed of functional analysis.

Proposition 4.5 For any non-empty set S, let $M(S)$ denote the set of all bounded real-valued functions on S. In symbols we have

$$M(S) = \{f : S \to \mathbb{R} \ : \ \sup_{x \in S} |f(x)| < \infty\}.$$

define $d : M(S) \times M(S) \to \mathbb{R}$ as

$$d(f, g) = \sup_{x \in S} |f(x) - g(x)|.$$

The $(M(S), d)$ is a **complete metric space.**

Proof. First we need to show that this is actually a metric space, i.e. the function d satisfies the metric properties. –> TOBE ADDED

For the second part, which is more important, is to show that this metric space is actually complete. To show this, let $\{f_n\}$ be a Cauchy sequence in $M(S)$. Thus from definition we have

$$\forall \epsilon > 0, \ \exists N > 0 \ \ \forall n, m > N \text{ we have } \sup_{x \in S} |f_n(x) - f_m(x)| < \epsilon.$$

From the definition of suprimum (least upper bound), we can conclude that

$$\forall x \in S \text{ we have } |f_n(x) - f_m(x)| < \epsilon.$$

This means that for every $x \in S$ the sequence $f_n(x)$ forms a Cauchy sequence in $\mathbb{R}$, thus it converges to some value, say $\hat{f}(x)$. We claim that the Cauchy sequence $\{f_n\}$ converges to $\hat{f}$. To show this we need to first show that $\hat{f} \in M(S)$ and the show that $f_n \to \hat{f}$.

To show $\hat{f} \in M(S)$ we seek help from an special function $\mathbf{0}$ which assigns the value $0 \in \mathbb{R}$ to every $x \in S$. Since $\{f_n\}$ is Cauchy, then it is bounded, thus $\exists R > 0$ such that $f_n \in \mathcal{B}_R(\mathbf{0})$ for all $n \in \mathbb{N}$. Using the definition of open ball we can write

$$\sup_{x \in S} |f_n(x) - \mathbf{0}(x)| = \sup_{x \in S} |f_n(x)| < R.$$

From the definition of sup we can conclude that

$$\forall x \in S, \ \forall n \in \mathbb{N} \text{ we have } f_n(x) < R.$$

By $n \to \infty$ we can write

$$\forall x \in S \text{ we have } \hat{f}(x) < R \implies \sup_{x \in S} |\hat{f}(x)| \leq R.$$

This shows that $\hat{f} \in M(S)$.

To show that $f_n \to \hat{f}$, let $\epsilon > 0$ be given. Then since for every $x \in A$ the sequence $f_n(x)$ converges to $\hat{f}(x)$ we can fine $N_x \in \mathbb{N}$ such that $\forall n > N_x$ we have $|f_n(x) - \hat{f}(x)| < \epsilon/2$. Let $N = \sup\{N_x\}$. We know that $N < +\infty$ since otherwise it means that there exists N_x for some $x \in A$ that is larger than any real number, which contradicts the fact that $f_n(x)$ converges to $\hat{f}(x)$. Now $\forall n > N_x$ we have

$$\forall x \in S: \ |f_n(x) - \hat{f}(x)| < \epsilon/2.$$

This implies

$$\sup_{x \in S} |f_n(x) - \hat{f}(x)| \leq \epsilon/2 < \epsilon.$$

Thus $f \to \hat{f}$, and this completes the proof. □

4.4 Hausdorff Topological Spaces

Metric spaces are natural extensions to the idea of R^k along with the Euclidean distance function. However, we can go even further, and show that some notions we encountered before are really independent of a particular metric. All such arguments are under the umbrella of topological spaces.

Definition 4.9 A topological space has two arguments: A set X, and a family $\mathcal{T}$ of subsets of X called "the open sets", that have the following properties.

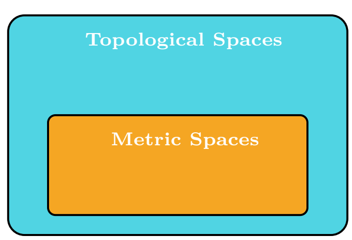
(i) Both $\varnothing$ and X are in $\mathcal{T}$.

(ii) Any union of open sets is open. That is for any subset $\mathcal{G} \subseteq \mathcal{T}$ one has $\bigcap \mathcal{G} \in \mathcal{T}$.

(iii) Any intersection of **finitely many** open sets is open. That is if $N \in \mathbb{N}$ and $U_1, \cdots, U_n \in \mathcal{T}$, then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

Given the definition above, we can now have the following important remark.

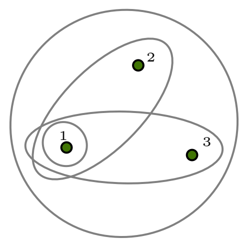
■ **Remark** In a nutshell, a metric space is a topological space whose opens sets are determined by the notion of open balls. The following diagram summarizes this fact.



So from now on, all of our definitions and proofs will be in a topological setting. The following example illustrates some topological spaces.

- Discrete topology: For any X let $\mathcal{T} = \mathcal{P}(X)$. Then any subset of X will be an open set.
- Trivial topology: For any X let $\mathcal{T} = \{\varnothing, X\}$.

- The usual topology on $\mathbb{R}^k$: Let $X = \mathbb{R}^k$. Let $U \in \mathcal{T}$ if $\forall x \in U$ there exists $\mathcal{B}_\epsilon(x) \subseteq U$ for some $\epsilon > 0$.
- Sorgenfrey line: Let $X = \mathbb{R}$. Then $G \subseteq \mathbb{R}$ belongs to $\mathcal{T}$ if and only if $\forall x \in G$, there exists $r > 0$ such that $[x, x + r) \subseteq G$.
- Let $X = \{1, 2, 3\}$ and $\mathcal{T} = \{\{1, 2\}, \{1, 3\}, \{1\}, \{1, 2, 3\}, \varnothing\}$. This is an example of non-Hausdorff topology, where there are not enough open sets available to isolate some points of the set via disjoint open sets. For instance, there are not disjoint open sets in $U, V \in \mathcal{T}$ such that $1 \in U$ and $2 \in V$. The following figure represents the configuration of the open sets.



4.4.1 Neighborhoods and Interior Points

Definition 4.10 Let $(X, \mathcal{T})$ be a topological space. Then for $x \in X$, we write $\mathcal{N}(x)$ to denote the set of all neighborhoods of x defined as

$$\mathcal{N}(x) = \{S \subset X : \exists u \in \mathcal{T} \text{ s.t. } x \in u \subseteq S\}.$$

Every open set containing $x \in X$ belongs to $\mathcal{N}(x)$. Often, some non-open sets do, too.

Lemma 4.1 Let $(X, \mathcal{T})$ be a topological space. Then the following are equivalent

(a) $A \in \mathcal{T}$.

(b) $\forall x \in A$ we have $A \in \mathcal{N}(x)$.

Proof. The proof is as follows:

- $(a) \implies (b)$: Since A is open, then trivially, for $x \in A$ we have $x \in A \subseteq A$. Thus following the definition of $\mathcal{N}(x)$ we conclude that $A \in \mathcal{N}(x)$.
- $(b) \implies (a)$: Let $x \in A$. Then from (b) we know that $A \in \mathcal{N}(x)$. Thus $\exists U_x \in \mathcal{T}$ such that $x \in U_x \subseteq A$. Then construct the set $\mathcal{G}$ as

$$\mathcal{G} = \bigcup_{x \in A} U_x.$$

From definition of open sets, $\mathcal{G}$ is open as it is union open sets. Also $x \in \mathcal{G}$ then $\exists U_x \subseteq A$, thus $\mathcal{G} \subseteq A$. On the other hand, $x \in A$ then $x \in U_x \subseteq \mathcal{G}$, thus $A \subseteq \mathcal{G}$, So we conclude $A = \mathcal{G}$.

□

The beautiful fact about topological spaces is that since they contain the metric spaces as a special case, then it means that we can generalize the ideas of "interior", "boundary", etc. using purely topological arguments.

Definition 4.11 Let A be any set in a topological space $(X, \mathcal{T})$. The set of interior points of A are defined as

$$A^\circ = \{x \in A : \exists U \in \mathcal{T} \text{ s.t. } x \in U \subseteq A\}.$$

Corollary 4.1 Let $(X, \mathcal{T})$ be a topological space and $A, B \subseteq X$. Then

$$A \subseteq B \implies A^\circ \subseteq B^\circ.$$

Proof. This corollary follows immediately from the definition. Let $x \in A^\circ$. Then $\exists U \in \mathcal{T}$ such that $x \in U \subseteq A \subseteq B$. Then $x \in B^\circ$ as well. □

The corollary above, somehow give the intuition, that A° is in some sense, the largest open set contained in A.

Proposition 4.6 Let $(X, \mathcal{T})$ be a topological space and A any set in X. Then

(a) A° is open, and $A^\circ \subseteq A$.

(b) If G is open and $G \subseteq A$, then $G \subseteq A^\circ$.

(c) A is open if and only if $A = A^\circ$.

Proof. The proof for different parts of the proposition are as follows

(a) Let $x \in A^\circ$. To show that A° is open it is enough to show that $A^\circ \in \mathcal{N}(x)$. From definition of an interior point, $\exists U \in \mathcal{T}$ such that $x \in U \subseteq A$. Now $\forall z \in U$ we have $z \in U \subseteq A$, thus $z \in A^\circ$, implying $U \subseteq A^\circ$, thus $A^\circ \in \mathcal{N}(x)$. So A° is an open set (following from the lemma we proved above). Furthermore, $A^\circ \subseteq A$ follows immediately from definition. Let $x \in A^\circ$. Then $\exists U \in \mathcal{T}$ such that $x \in U \subseteq A$ thus $A^\circ \subseteq A$.

(b) Since $G \in \mathcal{T}$, and $G \subseteq A$, then $\forall x \in G$ we have $x \in G \subseteq A$, thus $x \in A^\circ$. This implies $G \subseteq A^\circ$.

(c) This proof will have two parts. Part 1: A is open $\implies A = A^\circ$. Then we can write

$$x \in A \implies x \in A \subseteq A \implies x \in A^\circ \implies A \subseteq A^\circ.$$
$$x \in A^\circ \implies \exists U \in \mathcal{T} : x \in U \subseteq A \implies A^\circ \subseteq A.$$

Thus we can conclude that $A = A^\circ$. For the converse, we need to prove $A = A^\circ$ implies A is open. One of the important tools for this purpose is the lemma we proved before. To show that A is open we need to show that it is a Neighborhood of all of its elements (i.e. contains an open set which contains that point). More formally, let $x \in A$. Since $A = A^\circ$, then $x \in A^\circ$, Thus $\exists U \in \mathcal{T}$ such that $x \in U \subseteq A$. This implies that $A \in \mathcal{N}(x)$. thus we can conclude that A is an open set.

□

4.5 Closed sets and Closure

The interplay between true and false statements in logic, in which the latter is the negation of the former, leads to concepts like "De Morgan's" law. Because of this particular law, we have the dual concept of open sets, which we know as closed sets. There is nothing special about open sets that the notion of closed sets lack. They both are two sides of a single thing. So, we can actually build the whole concepts of topology out of closed sets.

Definition 4.12 Let $(X, \mathcal{T})$ be a topological space. Then $A \subseteq X$ is a closed set if and only if it complement A^c is an open set.

Note that the notion of closed set is **not** the negation of open sets. Thus we can have sets that are both open and closed, while we can have sets that are neither open not closed. For instance, in every topological space, the sets $\varnothing$ and X are always both open and closed (sometimes called clopen sets).

Lemma 4.2 Let $(X, \mathcal{T})$ be a topological space with $A \subseteq X$. Then the following two statements are equivalent.

(a) A is closed.

(b) For every $x \notin A$, $\exists U \in \mathcal{N}(x)$ such that $U \subseteq A^c$.

Proof. The proof will have two sections as follows:

- $(a) \implies (b)$: Since A is closed, then A^c is open, thus it is a neighborhood of all of its elements. So $\forall x \in A^c$, $\exists U \in \mathcal{T}$ such that $x \in U \subseteq A^c$.
- $(b) \implies (a)$: $\forall x \in A^c$, $\exists U \in \mathcal{N}(x)$ such that $x \in U \subseteq A^c$. Since $U \in \mathcal{N}(x)$, then $\exists V \in \mathcal{T}$ such that $x \in V \subseteq U \subseteq A^c$. Thus we conclude that $A^c \in \mathcal{N}(x)$. This implies that A^c is open, hence A is closed.

□

Proposition 4.7 Let $(X, \mathcal{T})$ be a topological space. Then

(a) Any intersection of closed sets is closed.

(b) Finite union of closed sets is closed.

Proof. The De Morgan's law play a central role in the proof.

(a) Let F be a set of closed sets. Then

$$(\bigcap F)^c = (\bigcup F^c)$$

is open as it is union of open sets (i.e. F^c). This implies that $\bigcap F$ is closed.

(b) Let $F = \{F_1, \cdots,_n\}$ be a collection of closed sets. Then

$$(\bigcup_{i=1}^{n} F_i)^c = \bigcap_{i=1}^{n} F_i^c$$

is open (as it is finite intersection of open sets. Thus $\bigcup_{i=1}^{n} F_i$ is closed.

□

Definition 4.13 Let $(X, \mathcal{T})$ be a topological space with $A \subseteq X$. Then the closure of A is defined as

$$\overline{A} = ((A^c)^\circ)^c$$

From the definition, we can conclude the following corollary

Corollary 4.2 Let $(X, \mathcal{T})$ be a topological space with $A, B \subseteq X$. Then we have

$$A \subseteq B \implies \overline{A} \subseteq \overline{B}$$

Proof. We can prove the statement following some basic set operations

$$A \subseteq B \implies A^c \supseteq B^c \implies (A^c)^\circ \supseteq (B^c)^\circ \implies ((A^c)^\circ)^c \subseteq ((B^c)^\circ)^c.$$

□

There is a very beautiful parallel between the notions of interior of a set and the closure of a set. For instance, the closure of a set, is the smallest closed set containing the original set. The parallel interior points and the closure is very similar to the parallel between infimum (biggest lower bound) and supremum (least upper bound).

Proposition 4.8 (a) $\overline{A}$ is closed and $A \subseteq \overline{A}$

(b) If F is closed and $A \subseteq F$, then $\overline{A} \subseteq F$.

(c) A is closed if and only if $A = \overline{A}$.

Proof. The proof for each statement is as follows:

(a) From definition, the closure of a set is a compliment of an open set (complement of $(A^c)^\circ$). Thus it is closed by definition. Further more, let $x \in A$. Then $x \notin A^c \implies x \notin (A^c)^\circ$. Thus $x \in ((A^c)^\circ)^c$, hence $A \subseteq \overline{A}$. Also, we can present this proof in a different way as follows

$$(A^c)^\circ \subseteq A^c \implies ((A^c)^\circ)^c \supseteq A.$$

(b) Since F is closed, then F° is open, and since it is open then $(F^c)^\circ = F^c$. Then we can write

$$A \subseteq F \implies A^c \supseteq F^c \implies (A^c)^\circ \supseteq F^c \implies ((A^c)^\circ)^c \subseteq F.$$

(c) This also follows immediately from the properties of open sets. For the first part, assume A is closed. Then A^c is open, and is the same as its interior, i.e. $(A^c)^\circ = A^c$. Computing the complements of both sides will result in $((A^c)^\circ)^c = A$, thus $\overline{A} = A$. As for the converse, Assume $A = \overline{A}$. Thus from definition $A = ((A^c)^\circ)^c$. Then $A^c = (A^c)^\circ$. This implies that A^c is open, hence A is closed.

□

4.6 Boundary Points

So far, we studied the notion of open sets that had some information about the interior of a set, and also we studied the notion of closed sets that has some information about the complement of a set. The notion of boundary of a set, kind of ties these two concepts to each other. In fact, as we will prove later, the boundary of a set, is the intersection of the closure of a set with the closure of its complement.

Definition 4.14 Let $(X, \mathcal{T})$ be a topological space with $A \subseteq X$. Then $x \in X$ is a *boundary point* of A if

$$\forall U \in \mathcal{T} \cap \mathcal{N}(x),\ U \cap A \neq \varnothing,\ U \cap A^c \neq \varnothing.$$

The set of all boundary points of A is denoted as ∂A.

Few points to easily digest the above definition. First, $U \in \mathcal{T} \cap \mathcal{N}(x)$ in words means U is an open set containing x. Also, note that the boundary point of A can be in A or in A^c. Also, equivalently, we can express the definition as x is a boundary point of A if $\forall U \in \mathcal{N}(x)$, $U \cap A \neq \varnothing$, $U \cap A^c \neq \varnothing$.

Corollary 4.3 Let $(X, \mathcal{T})$ be a topological space with $A \subseteq X$. Then

$$\partial(A) = \partial(A^c).$$

Proof. This follows immediately from the definition and the symmetrical appearance of A and A^c in the definition, such that interchanging their position does not matter. $\square$

Proposition 4.9 Let $(X, \mathcal{T})$ be a topological space. Then

(a) $\partial A = \overline{A} \cap \overline{A^c}$.

(b) A is closed if and only if $\partial A \subseteq A$; also, $\overline{A} = A \cup \partial A$.

(c) A is open if and only if $\partial A \subseteq A^c$; also, $A^\circ = A \backslash \partial A$.

Proof. The proof of each section is as follows:

(a) This proof has two sections. First we show that $\partial A \subseteq \overline{A} \cap \overline{A^c}$. We use the proof by contrapositive. Let $x \notin \overline{A} \cap \overline{A^c}$. Thus $x \notin ((A^c)^\circ)^c \cap (A^\circ)^c$. Then by De Morgan's law we have $x \in (A^c)^\circ \cap A^\circ$. This says that x should be in the interior of A or A^c, each of which implies that $x \notin \partial A$. That is because

$$\begin{aligned} &x \in (A^c)^\circ \implies \exists U \in \mathcal{T}: \ x \in U \subseteq A^c \implies A \notin \partial A \\ &x \in A^\circ \implies \exists V \in \mathcal{T}: \ x \in V \subseteq A \implies A \notin \partial A. \end{aligned}$$

However for the converse, we show $x \notin \partial A$ leads to $x \notin \overline{A} \cap \overline{A^c}$ (i.e. contrapositive of the actual statement that we need to prove). Let $x \notin \partial A$. Then $\exists U \in \mathcal{N}(x)$ such that $U \cap A = \varnothing$ or $U \cap A^c = \varnothing$, each of which implies $x \notin \overline{A} \cap \overline{A^c}$. Indeed $U \cap A = \varnothing$ implies $x \notin \overline{A}$, thus $x \notin \overline{A} \cap \overline{A^c}$. Similarly $U \cap A^c = \varnothing$ implies $x \notin \overline{A^c}$, thus $x \notin \overline{A} \cap \overline{A^c}$.

(b) This proof has two sections. THIS PROOF TO BE COMPLETED.

- For the first part, we want to show that A is closed implies $\partial A \subseteq A$. For this purpose we can take very different ways, i.e. to use the the tools already developed (like the one we proved in section (a)), or to use the basic definitions. For instance to assume that we want to use the tool developed in part (a). Let $x \in \partial A$. So, from (a) we can write $x \in \overline{A} \cap \overline{A^c} \Leftrightarrow x \in ((A^c)^\circ)^c \cap (A^\circ)^c$. Since A is closed, then A^c is open. This implies that $A^c = (A^c)^\circ$. Thus $x \notin (A^c)^\circ \cup A^\circ \Leftrightarrow x \notin A^c \cap A^\circ \Leftrightarrow x \in A \cap (A^\circ)^c$, thus $x \in A$. This concludes that $x \in A$, hence $\partial A \subseteq A$.

  However, we can take slightly different approaches as well. Since A is open, then A^c is closed. To show $\partial A \subseteq A$ we can show by contrapositive i.e. $x \notin A \implies x \notin \partial A$. Since $x \notin A$, then $x \in A^c$, and since A^c is open then $x \in (A^c)^\circ$. This implies $x \notin ((A^c)^\circ)^c$, thus $x \notin \overline{A}$. From (a), again, we can conclude that $x \notin \partial A$.

$\square$

4.7 Limit Points and Isolated Points

The notion of a limit point is very important in a topological space, as it has a very intuitive sequential characterization in a metric space, and has a very interesting connection with other topological notions such as closed and open sets.

Definition 4.15 Let $(X, \mathcal{T})$ be a topological space. A point $x \in X$ is a limit point of $A \subseteq X$ if

$$\forall U \in \mathcal{N}(x), \text{ we have } (U \backslash \{x\}) \cap A \neq \varnothing.$$

4.8 Sequential Characterization

Naturally, we have the concept of sequences and limits of sequences in metric spaces. On the other hand, we studied that the topological spaces are generalized form of metric spaces. This means that all of topological concepts we covered so far (i.e. open sets, closed sets, interior points, closure, limit points, etc) can be characterized with the notion of a sequence and convergence in metric spaces. This is very important since sequences are some tools that are easier to conceive intuitively.

The concept of limit point in the topological sense is easier to characterize sequentially, and then characterize other notions using the interplay between the limit point and those concepts. The following proposition reveals a very important characterization.

Proposition 4.10 In a metric space (X, d) with $A \subseteq X$. the followings are equivalent

(a) x is a limit point of A

(b) There exists a sequence x_n with distinct elements such that $x_n \to x$.

Proof. This proof will have two parts as follows.

- $(a) \implies (b)$. We assume that x is a limit point of A and we need to come up with a smartly designed sequence with distinct elements, all of which lies in A such that $x_n \to x$. Since $x \in A'$, then $\forall \mathbb{B}(x; r)$ for $r \in \mathbb{R}$ we have $\mathbb{B}(x; r) \cap A \neq \varnothing$. Let $r_1 = 1$. Choose $x_1 \in \mathbb{B}(x; r_1) \cap A$. Let $r_2 = d(x_1, x)/2$ and choose $x_2 \in \mathbb{B}(x; r_2) \cap A$. Similarly, let $r_3 = d(x_2, x)/2$ and choose $x_3 \in \mathbb{B}(x; r_3) \cap A$, and we continue the construction. Then, due to the construction, all of the elements of $\{x_n\}$ has distinct elements. Also since $d(x_n, x) \leq 2^{-n}$, then $x_n \to x$. This complete the proof.

- $(b) \implies (a)$. We assume that there is a sequence $\{x_n\}$ with distinct elements in A that approaches x. Let $S \in \mathcal{N}(x)$. The from definition there is a ball $\mathbb{B}[x; r)$ for some $r \in \mathbb{R}$ such that $\mathbb{B}[x; r) \subseteq S$. On the other hand, from the definition of convergence we know $\exists N > 0$ such that $\forall n > N$ we have $x_n \in \mathbb{B}[x; r)$. Since x_n has all distinct elements, then $\mathbb{B}[x; r)$ still contains x_n excluding at most one element. Hence $S \backslash \{x\} \cap A \neq \varnothing$. Since S was chosen arbitrary, then this is true for all $S \in \mathcal{N}(x)$. This completes the proof.

□

We can have a similar approach and characterize the notion of isolated points with the notion of sequences in a metric space. Since the set of isolated points of A is $A \backslash A'$, then we can have the following proposition.

Proposition 4.11 Let (X, d) be a metric space with $A \subseteq X$. Then $x \in A$ is an isolated point of A is there are no sequence with distinct elements that approach x.

Proof. TO BE ADDED. □

One of the useful interplay between the notion of limit point and the notion of open sets is the following proposition.

Proposition 4.12 Let $(X, \mathcal{T})$ be a topological space. Then

$$G \in \mathcal{T} \Longleftrightarrow G \cap (G^c)' = \varnothing,$$

in which $(\cdot)'$ denotes the set of all limit points.

Using this interplay we can have a sequential characterization of the notion of open set.

Proposition 4.13 — Sequential characterization of open sets. Let (X, d) be a metric space. Then $A \subseteq X$ is open if and only there are no sequences approaching x with distinct elements all of which lies in A^c.

As we saw before, the notion of boundary points is very crucial, since the notion of interior points and also closure of a set can be define using that (i.e. $A^\circ = A \backslash \partial A$ and $\overline{A} = A \cup \partial A$). The following proposition makes a connection between the notion of a boundary point in topological space and the notion of sequence and its limit in a metric space.

Proposition 4.14 Let (X, d) be a metric space. Then $x \in \partial A$ if and only if there exist two sequence x_n and y_n such that $x_n \to x$ and $y_n \to x$, and $x_n \in A,\ y_n \in A^c$ for all n sufficiently large.

Proof. Let $x \in \partial A$. Since (X, d) is a topological space, then $\forall \epsilon > 0$, $\exists \mathcal{B}_\epsilon(x)$ such that $\mathcal{B}_\epsilon(x) \cap A \neq \varnothing$ and $\mathcal{B}_\epsilon(x) \cap A^c \neq \varnothing$. We construct the sequences $\{x_n\}$ and $\{y_n\}$ with the following construction. For a given $n \in \mathbb{N}$, let $\epsilon = 1/n$. Then pick $x_n \in \mathcal{B}_\epsilon(x) \cap A$ and $y_n \in \mathcal{B}_\epsilon(x) \cap A^c$. Due to the construction, we have $x_n \to x$ also $y_n \to x$ with the required property that $x_n \in A$ and $y_n \in A^c$ for all $n \in \mathbb{N}$. □

4.8.1 Base of Topology

The notion of the base of topology is a very useful and practical tool to analyze relatively complex topological spaces. The following is from the Wikipedia article about the notion of bases of a topology

> Bases are ubiquitous throughout topology. The sets in a base for a topology, which are called *basic open sets*, are often easier to describe and use than arbitrary open sets. Many important topological definitions such as *continuity* and *convergence* can be checked using only basic open sets instead of arbitrary open sets. Some topologies have a base of open sets with specific useful properties that may make checking such topological definitions easier.
>
> Not all families of subsets of a set X form a base for a topology on X. Under some conditions detailed below, a family of subsets will form a base for a (unique) topology on X, obtained by taking all possible unions of subfamilies. Such families of sets are very frequently used to define topologies. A weaker notion related to bases is that of a *subbase* for a topology. Bases for topologies are also closely related to *neighborhood bases*.

There are many ways to interpret the idea of topological bases. Here, we will cover two of them.

Definition 4.16 Let $(X, \mathcal{T})$ be a topological space. Then $\mathcal{B} \subseteq \mathcal{T}$ is a basis for topology if and only if $\forall U \in \mathcal{T}$ there exists $B \subseteq \mathcal{B}$ such that $\bigcup B = U$.

The definition above, focuses on an existing topological space with given $\mathcal{T}$. However, we can have a quite opposite point of view, with some flavors of reverse engineering. Consider the following proposition.

Proposition 4.15 Let X be a non-empty set. Take a collection of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$. Try using the sets in $\mathcal{B}$ to define the notion of neighborhood of $x \in X$ as follows

$$\mathcal{N}(x) = \{S \subseteq x : \ \exists U \in \mathcal{B} \text{ s.t. } x \in U \subseteq S\}.$$

Then declare a set $G \subseteq X$ to be "open" if and only of $G \in \mathcal{N}(x)$ holds for all $x \in G$.

The construction above, defines a Hausdorff topological space if $\mathcal{B}$ satisfies the following properties.

(a) $\bigcap \mathcal{B} = X$ [i.e. every point in X belongs to at least one set $B \in \mathcal{B}$].

(b) Whenever $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

(c) $\forall x, y \in X$ we have $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1$ and $x \in B_2$ while $B_1 \cap B_2 = \varnothing$.

Such a set $\mathcal{B}$ is called a base for $\mathcal{T}$.

Proof. TO BE ADDED. □

■ **Example 4.1** Let $(\mathbb{R}, d)$ be a metric space in which d is the Euclidean metric. This metric induces the notion of open balls in $(\mathbb{R}, d)$. Consider the set

$$\mathcal{B} = \{\mathcal{B}_\epsilon(x) : \ \epsilon, x \in \mathbb{Q}\}.$$

There are in fact balls which are centered at rational numbers and has rational radius. We can prove that this set is a basis for $(\mathbb{R}, \mathcal{T})$ in which $\mathcal{T}$ is the usual topology. We can show that via definition of basis (i.e. to show that every $U \in \mathcal{T}$) can be written as a union of subsets of $\mathcal{B}$, or, alternatively, we can show $\mathcal{B}$ is basis via showing that we can construct $\mathcal{T}$ via the construction discussed in the proposition above (which boils down to showing that $\mathcal{B}$ satisfies the required conditions).

The basis $\mathcal{B}$ is interesting since it is countable. The topological spaces that admit a countable basis are called *second-countable* spaces. ■

4.8.2 Summary

Here in this section we are going to summarize all of the properties of topological spaces.

Summary 4.1 — Open sets and the interior points. Let $(X, \mathcal{T})$ be a topological spaces. Then $A \in \mathcal{T}$ is called an open set and has the following properties

- A is open if and only if $A = A^\circ$.
- A is open if and only if $\forall x \in A$ we have $A \in \mathcal{N}(x)$.

- A is open if and only if $A \cap (A^c)' = \varnothing$.
- $A^\circ = A \backslash \partial A$

Summary 4.2 — Closed sets and the closure. Let $(X, \mathcal{T})$ be a topological space. Then

- F is closed if and only if F^c is open.
- F is closed if and only if $F = \overline{F}$.
- F is closed if and only if $F' \subseteq F$.
- $\overline{F} = F \cup \partial F$.

Summary 4.3 — Subset preserving operations!. Let $(X, \mathcal{T})$ be a topological space, and $A, B \subseteq X$. If $A \subseteq B$, then

- $A^\circ \subseteq B^\circ$.
- $\overline{A} \subseteq \overline{B}$.
- $A' \subseteq B'$.

4.9 Compactness

We start with the definition of compactness.

Definition 4.17 — Compactness. Let $(X, \mathcal{T})$ be a topological space. Then $K \subseteq X$ is compact, if for every collection of open sets $\mathcal{G} \subseteq \mathcal{T}$ satisfying $K \subseteq \bigcup_{G \in \mathcal{G}} G$ (which is called an open cover), we have a finite sub collection $\{G_1, G_2, \cdots, G_N\}$ for some $N \in \mathbb{N}$ such that $K \subseteq \bigcup_{i=1}^{N} G_i$ (which is called a finite sub-cover). In words, a set is compact, if every open cover admits a finite sub-cover.

■ **Remark** Compactness is the next best thing to finiteness. It's so valuable that when it is absent, we sometimes switch to a new topology in which compactness is present.

Furthermore, the notion of compactness is expressing the fact that if $K \subseteq X$ is compact, then we can find an open set G that can be constructed via union of only finite ingredients in $\mathcal{T}$ and $K \subseteq G$.

Lemma 4.3 Let $(X, \mathcal{T})$ be a topological space. Then $S \subseteq X$ finite, is a compact set.

Proof. Let $\mathcal{G} \subseteq \mathcal{T}$ be an open cover for $S = \{x_1, \cdots, x_N\}$ for some $N \in \mathbb{N}$. Then since $S \subseteq \bigcap \mathcal{G}$, then each $x_i \in S$ belongs to some G_i in $\mathcal{G}$. Then $\{G_1, \cdots, G_N\}$ is a sub-cover that is finite. □

Lemma 4.4 Let $(\mathbb{R}, |\cdot|)$ be a metric space. Then $\mathbb{Z} \subseteq \mathbb{R}$ is not compact.

Proof. To show $\mathbb{Z} \subseteq \mathbb{R}$ is not compact, we need to find an open cover that fails to have a finite sub-cover. Let $\mathcal{G} = \{\mathcal{B}_{1/4}(x) : x \in \mathbb{Z}\}$. $\mathcal{G}$ is an open cover, but it fails to have any sub-cover. Since any $G \in \mathcal{G}$ covers only one integers, and for any choice of $N \in \mathbb{N}$ the sub-cover $H = \{G_1, \cdots, G_N\}$ won't cover all of $\mathbb{Z}$. Thus $\mathbb{Z}$ is not compact. □

The following is a very important proposition which helps in making some intuition about the compact sets.

Proposition 4.16 Let (X, d) be a metric space, and $K \subseteq X$ compact. Then K is bounded.

Proof. Let $x \in X$. Then from Proposition 4.2 we define

$$\mathcal{G} = \{B_r(x) : r \in \mathbb{N}\},$$

and we get $X = \bigcup \mathcal{G}$. Then $K \subseteq \mathcal{G}$. Thus $\mathcal{G}$ is an open cover. Since $\mathcal{K}$ is compact, then there is a finite sub-cover $\mathcal{H} = \{B_{r_1}(x), \cdots, B_{r_N}(x)\}$ for some $N \in \mathbb{N}$, such that $K \subseteq \bigcup \mathcal{H}$. Let $r = \max\{r_1, \cdots, r_N\}$ and $z \in K$. Then $\exists B_{r_i}(x) \in \mathcal{H}$ such that $z \in B_{r_i}(x) \subseteq B_r(x)$. Thus $K \subseteq B_r(x)$. This shows that K is bounded. □

Lemma 4.5 In $(\mathbb{R}, |\cdot|)$, the set $S = \{1/n : n \in \mathbb{N}\}$ is not compact, but the set $\overline{S} = S \cup \{0\}$ is compact.

Proof. To show that S is not compact, we can easily show that there is an open cover that fails to have a finite sub-cover. A great candidate for that is a family of open sets each of which contains only one $x \in S$. Consider

$$\mathcal{G} = \{\mathcal{B}_r(n) : n \in \mathbb{N}, r = 1/(n+1)^2\}.$$

This is an open cover all if its elements are open sets and $S \subseteq \mathcal{G}$. Since each element of $\mathcal{G}$ contains only one element of S, then $\mathcal{G}$ fails to have finite subset that covers S. Thus S is not compact.

The second part of proof is to show that $S \cup \overline{S}$ is compact. Let $\mathcal{G}$ be an open cover. Then there is an open set $G \in \mathcal{G}$ such that $0 \in G$. Since G is open and contains 0, and also the sequence $\{1/n\}$ goes to 0, then $\exists N \in \mathbb{N}$ such that $\forall n > N$ we have $1/n \in G$. Thus G contains all but finitely many elements of S. However for each $1/n$ with $1 \leq n \leq N$ we have $G_n \in \mathcal{G}$ such that contains $1/n$. Thus $\{G, G_1, \cdots, G_N\}$ is a finite sub-cover, thus the set $S \cup \overline{S}$ is compact. □

■ **Remark** The proposition above can be extended to show that for any convergent sequence in any metric space, the closure of the range is compact.

The following proposition is an important one, as it is true only in Hausdorff topological spaces, and not in general topological spaces.

Proposition 4.17 Let $(X, \mathcal{T})$ be a **Hausdorff** topological space and $K \subseteq X$ compact. Then K is closed. In other words, in any Hausdorff topological space, a compact set contains all of its limit points.

Proof. Since this proposition is only true for Hausdorff topological spaces, then we have the hint that we should use some properties specific to the Hausdorff topological spaces. We want to show that K^c is open. Since an open set is a neighborhood of all of its elements, thus for $y \in K^c$ we need to find an open set S such that $y \in S \subseteq K^c$.

Let $y \in K^c$. Then since $\forall x \in K$, there exists $U_x, V_x \in \mathcal{T}$ such that $x \in U_x$ and $y \in V_x$ and $U_x \cap V_x = \varnothing$. This is the same as writing $V_x \subseteq U_x^c$. Clearly, $\mathcal{G} = \{U_x : x \in K\}$ is an open cover for K. But since K is compact, then $\mathcal{G}$ admits an open sub-cover. Thus $\exists x_1, \cdots, x_N \in K$ for some $N \in \mathbb{N}$ such that $K \subseteq \bigcup_{i=1}^N U_{x_i}$. Then we can write

$$K^c \supseteq \bigcap_{i=1}^N U_{x_i}^c \supseteq \bigcap_{i=1}^N V_{x_i}.$$

Note that $S = \bigcap_{i=1}^{N} V_{x_i}$ is open as it is finite intersection of open sets, and due to the construction $y \in S$. Thus $y \in S \subseteq K^c$. This implies that the set K^c is open, which implies the set K is closed. This completes the proof. ☐

■ **Remark** It is very important to note that Proposition 4.17 is **only** true for the Hausdorff topological spaces. For instance let $X = \{1, 2, 3\}$, and $\mathcal{T} = \{\varnothing, X, \{1\}\}$. Then $(X, \mathcal{T})$ is a topological spaces that is not Hausdorff (as there are no disjoint open sets separating $1, 2 \in X$). The set $\{2\}$ is compact (as it is finite), but it is not closed (since its complement is not open).

The following proposition highlights how the property of compactness of a set gets inherited by certain type of its subsets.

Proposition 4.18 Let $(X, \mathcal{T})$ be a topological space and $K \subseteq X$ is compact. Then

$$F \subseteq K \text{ closed} \implies F \text{ is compact.}$$

Proof. There are two ways to proof this. Beginner's way and Pro's way!

- **Beginner's proof.** Since F is closed, then F^c is open, hence it is neighborhood of all of its elements. Thus $\forall x \in F^c$, $\exists u_x \in \mathcal{T}$ such that $x \in u \subseteq F^c$, which also implies $u_x \cap F = \varnothing$. Let $\mathcal{U} = \{u_x : x \in F^c\}$. Now let $\mathcal{G}$ be an open cover for F. Then $\mathcal{U} \cup \mathcal{G}$ is an open cover for K as every $z \in K$ belongs to at lest one open set in $\mathcal{U}$ or in $\mathcal{G}$. Since K is compact, thus it admits finite sub-cover $H = \{H_1, \cdots H_N\} \subseteq \mathcal{U} \cup \mathcal{G}$ for some $N \in \mathbb{N}$ such that $\forall z \in K$ we have some $H_i \in H$ such that $z \in H_i$. Let $y \in F$. Then $\exists H_i \in H$ such that $y \in H_i$. But note that y does not belong to any $u_x \in \mathcal{U}$. Thus $H_i \subseteq \mathcal{G}$. So we can conclude that H has a subset $\mathcal{H}$ that all if its elements belongs to $\mathcal{G}$, which implies $\mathcal{H}$ is a finite sub-cover for F. Thus we conclude that F is compact.
- **Pro's Proof.** Let $\mathcal{G} = \{G_\alpha, \alpha \in A\}$ be an open cover for F, in which A is an index set. Since F is closed, then F^c is open, and $\mathcal{G} \cup \{F^c\}$ is an open cover for K. On the other hand, since K is compact, thus there is a sub-cover consisting of $G_{\alpha_1}, \cdots, G_{\alpha_N}$ for some $N \in \mathbb{N}$ and possibly F^c. Then
$$K \subseteq (F^c) \cup (\bigcup_{i=1}^{N} G_{\alpha_n}).$$
Let $y \in F \subseteq K$. Then $\exists G_{\alpha_n}$ for some $n \leq N$ such that $y \in G_{\alpha_n}$. Thus $\{G_{\alpha_1}, \cdots, G_{\alpha_n}\}$ is a finite sub-cover and this concludes the proof.

☐

Using Proposition 4.18 we can have some useful corollaries for special topological spaces, like Hausdorff topological space.

Corollary 4.4 Let $(X, \mathcal{T})$ be a HTS, and $K \subseteq X$ compact, and $F \subseteq X$ closed. Then

$$F \cap K \text{ is compact.}$$

Proof. This immediately follows from Proposition 4.18 and Proposition 4.17. Since $(X, \mathcal{T})$ is an HTS, and K is compact, then K is closed, which implies $K \cap F$ is closed (since F is closed and any intersection of closed sets is closed). Thus $K \cap F \subseteq K$ is compact. ☐

Also, the following is a very important corollary of the proposition above, which will be used later to prove the Heine-Borel theorem.

Corollary 4.5 Let $(X, \mathcal{T})$ be a topological space and $K \subseteq X$ is compact. If $A \subseteq K$ is an infinite set, then $A' \neq \varnothing$.

Proof. We proceed with the proof by contrapositive. Since $A' = \varnothing$, then $\overline{A} = A \cup A' = A$, hence A is closed. Since $A \subseteq K$, then A is also compact. Then since $A' = \varnothing$, then $\forall x \in A$, we can fine open $S_x \in \mathcal{N}(x)$ such that $A \cap S_x \backslash \{x\} = \varnothing$, i.e. S_x contains no other elements of A. Let $\mathcal{S} = \{S_x : \ x \in A\}$. Due to the construction $\mathcal{S}$ is an open cover of A. But since A is compact, then there exists $\{S_{x_1}, \cdots, S_{x_N}\}$ for some $N \in \mathbb{N}$ such that covers A. However, S_{x_i}s are disjoint (because of the construction), then each S_{x_i} contains only one element x_i, thus the set A is finite. This completes the proof via showing that the contrapositive is true. □

4.9.1 Characterization of Compact Using Closed Sets

Since the notions of closedness and openess of sets are quite dual, then we expect for every notion characterized using open sets, have a dual characterization using the notions of closed sets, and the notion of compactness is no difference. We start with the following definition.

Definition 4.18 — Finite intersection property. A family of sets $\mathcal{F}$ has the **finite intersection property** if whenever $N \in \mathbb{N}$ and $F_1, \cdots, F_N$ are sets in $\mathcal{F}$, one has $\bigcap_{n=1}^{N} F_n \neq$.

■ **Example 4.2** The following sets has the finite intersection properties.

- $\mathcal{F} = \{[-1, 1], [-0.5, 0.5], [-0.25, 0.25]\}$.
- $\mathcal{G} = \{\{1, 2, 3\}, \{2, 3\}, \{3\}\}$.
- $\mathcal{H} = \{[-1/n, 1/n] : \ n \in \mathbb{N}\}$.

■

There is a beautiful parallel between the notion of finite intersection property and having a finite sub-cover. The following theorem makes this parallel more clear.

Theorem 4.1 — Characterization of compact sets with closed sets. Given a HTS $(X, \mathcal{T})$ and a **closed** set $K \subseteq X$, then the following are equivalent.

(a) K is **compact**.

(b) Every collection of **closed** subsets of K with **finite intersection property**, has a **non-empty intersection**.

Proof. Here, I will proved two proves for this theorem.

- **First Proof.** The proof will have two parts.
  - ◐ $(a) \implies (b)$ Let $K \subseteq X$ be compact, and also let $\{F_\alpha\}_{\alpha \in A}$ be a collection of closed subsets of K with finite intersection property. We claim that $\bigcap_{\alpha \in A} F_\alpha$ is non-empty. Suppose otherwise, i.e. $\bigcap_{\alpha \in A} F_\alpha = \varnothing$. Then since $X = \varnothing^c$, we have
  $$K \subseteq (\bigcap_{\alpha \in A} F_\alpha)^c = \bigcup_{\alpha \in A} F_\alpha^c.$$
  Since K is compact, then there is $J \subseteq A$ finite such that $K \subseteq \bigcup_{\alpha \in J} F_\alpha^c$. Using the De Morgan's law we can write
  $$\bigcap_{\alpha \in J} F_\alpha \subseteq K^c.$$

However, since $F_\alpha \subseteq K$ for all $\alpha \in A$, then $\bigcap_{\alpha\in J} F_\alpha = \varnothing$. This contradicts the fact that $\mathcal{F}$ has finite intersection property.

◑ $(b) \implies (a)$ Assume $\mathcal{G} = \{G_\alpha\}_{\alpha\in A}$ an open cover for K. We are safe to assume $G_\alpha \cap K \neq \varnothing$ for all $\alpha \in A$. We want to show that K has a finite sub-cover. In other words $\exists J \subseteq A$ and finite such that $\{G_\alpha\}_{\alpha\in J}$ is a finite open cover for K. We use the idea of proof by contradiction. So assume K fails to have a finite sub-cover. Thus $\forall J \subseteq A$ and finite, one has $K \not\subseteq \bigcup_{\alpha\in J} G_\alpha$. Thus $K\backslash(\bigcup_{\alpha\in J} G_\alpha) \neq \varnothing$. By a careful design of a collection of closed subsets of K that has finite intersection property but fails to have a non-empty intersection, we can finish the proof. Let $F_\alpha = K\backslash G_\alpha$. Clearly F_α is closed and $F_\alpha \subseteq K$. We claim that $\mathcal{F} = \{F_\alpha : \ \alpha \in A\}$ has finite intersection property. That is because for any finite $J \subseteq A$ we have

$$\bigcap_{\alpha\in J} F_\alpha = \bigcap_{\alpha\in J}(K \cap G_\alpha^c) = K \cap (\bigcap_{\alpha\in J} G_\alpha^c) = K \cap (\bigcup_{\alpha\in J} G_\alpha)^c = K\backslash(\bigcup_{\alpha\in J} G_\alpha) \neq \varnothing.$$

However, since $\mathcal{G}$ is an open cover for K and $K \subseteq \bigcup \mathcal{G}$, then

$$\bigcap_{\alpha\in A} F_\alpha = \bigcap_{\alpha\in A}(K \cap G_\alpha^c) = K \cap (\bigcap_{\alpha\in A} G_\alpha^c) = K \cap (\bigcup_{\alpha\in A} G_\alpha)^c = K\backslash(\bigcup_{\alpha\in A} G_\alpha) = \varnothing.$$

This contradicts (b), thus we conclude that there is a finite open cover.

- **Second Proof.** This proof uses the idea of contrapositive. Thus, instead of showing $(a) \Leftrightarrow (b)$ we show $\neg(a) \implies \neg(b)$.

♪ $\neg(b) \implies \neg(a)$. Since we assume (b) is not true, then there exists a collection of closed subsets of K, i.e. $\mathcal{F} = \{F_\alpha\}_{\alpha\in A}$ with finite intersection property that fails to have a non-empty intersection. We want to construct an open cover for K that fails to have a sub-cover. Let $G_\alpha = F_\alpha^c$. Clearly G_α is open. We claim that $\mathcal{G} = \{G_\alpha\}_{\alpha\in A}$ is an open cover for K that fails to have any sub-cover. $\mathcal{G}$ is an open cover for K. Because

$$\bigcup_{\alpha\in A} G_\alpha = \bigcup_{\alpha\in A} F_\alpha^c = [\bigcap_{\alpha\in A} F_\alpha]^c = [\varnothing]^c = X,$$

and since $K \subseteq X$ then $K \subseteq \bigcup_{\alpha\in A} G_\alpha$, hence $\mathcal{G}$ is an open cover for K. Also, $\mathcal{G}$ fails to admit a sub-set as sub-cover for K. That is because, for any finite $J \subseteq A$ we have

$$K\backslash(\bigcup_{\alpha\in J} G_\alpha) = K \cap (\bigcup_{\alpha\in J} F_\alpha^c)^c = K \cap (\bigcap_{\alpha\in J} F_\alpha) \neq \varnothing$$

The last term in the expression above is not empty due to the finite intersection property of $\mathcal{F}$. Thus K fails to have any finite sub-cover.

♫ $\neg(a) \implies \neg(b)$. Since (a) is not true, then it has an open cover $\{G_\alpha\}_{\alpha\in A}$ that fails to have any finite sub-cover. We assume $G_\alpha \cap K \neq \varnothing$ for all $\alpha \in A$. In other words $K \subseteq \bigcup_{\alpha\in A} G_\alpha$ but for any finite subset $J \subseteq A$ we have $K \not\subseteq \bigcup_{\alpha\in J} G_\alpha$. Let $F_\alpha = K\backslash G_\alpha$. Since K and G_α^c are closed, then F_α is also closed and due to the construction $F_\alpha \subseteq K$. We claim that the collection $\mathcal{F} = \{F_\alpha\}_{\alpha\in A}$ has finite intersection property, but fails to have a non-empty intersection. $\mathcal{F}$ has finite intersection property, since for any finite$J \subseteq A$ we have

$$\bigcap_{\alpha\in J} F_\alpha = K \cap (\bigcap_{\alpha\in J} G_\alpha^c) = K \cap (\bigcup_{\alpha\in J} G_\alpha)^c = K\backslash(\bigcup_{\alpha\in J} G_\alpha) \neq \varnothing.$$

The last term in the expression above is not empty since K fails to have any finite sub-cover. However,

$$\bigcap_{\alpha\in A} F_\alpha = K \cap (\bigcap_{\alpha\in A} G_\alpha^c) = K \cap (\bigcup_{\alpha\in A} G_\alpha)^c = K \backslash (\bigcup_{\alpha\in A} G_\alpha) = K \backslash \mathcal{G} = \varnothing,$$

since $\mathcal{G}$ is an open cover for K.

□

4.9.2 Sequential Characterization of Compact Sets

Similar to the other notions we have encountered so far, in a metric space, a compact sets has a sequential characterization. The following theorem is a very central and important theorem for metric spaces. The Heine-Borel theorem follows as a corollary of the theorem below.

Theorem 4.2 Let (X, d) be a metric space with $K \subseteq X$. Then The following are equivalent.

(a) The set K is compact.

(b) Every sequence $\{x_n\}$ in K has a converging sub-sequence, whose limit lies in K.

Proof. This theorem has two parts as follows.

(a) $(a) \implies (b)$ Let $\{x_n\}$ be a sequence in K, and $A = \{x_n : n \in \mathbb{N}\}$. If A is finite, then $\{x_n\}$ has a constant sub-sequence, thus (b) follows trivially. However, if A is not finite, then by Corollary 4.5 we know that $A' \neq \varnothing$. Also since $A \subseteq K$ implies $A' \subseteq K'$ but since K is a compact set in an HTS, then K is closed, hence $A' \subseteq K$. Let $z \in A'$, $m \in \mathbb{N}$, and $\epsilon = 1/m$. Then $\mathbb{B}(z; \epsilon) \cap A \neq \varnothing$. Thus $\exists n_m \in \mathbb{N}$ such that $x_{n_m} \neq z$ and $x_{n_m} \in A$. The sub-sequence $\{x_{n_k}\}$ approaches the point z. This completes the proof.

(b) $(b) \implies (a)$ TOBEADDED

□

4.10 Problems

■ **Problem 4.1** Give an example of each of the following or prove that no such a set exists.

(i) A non-empty set with no accumulation points and no isolated points.

(ii) A non-empty set with no interior points and no isolated points.

(iii) A non-empty set with no boundary points and not isolated points.

Proof.

(i) No such a set exists. Because $A^{iso} = A \backslash A'$. Thus if $x \notin A'$, then $x \in A^{iso}$.

(ii) $(\mathbb{R}^2, \mathcal{T})$, with the usual topology. Then any closed curve in the plane will have this property.

(iii) $\mathbb{R}$ in $(\mathbb{R}, \mathcal{T})$ with usual topology.

□

■ **Problem 4.2** Must every boundary point of set be also an accumulation point of that set?

Proof. We want to check if $\partial A \subseteq A'$. While this intuitively might look correct, but an isolated point of a set is indeed a boundary point. However, it is not an accumulation point. □

■ **Problem 4.3** Let E be a set and $\{x_n\}$ a sequence of distinct points, not necessarily elements of E. Suppose that $\lim_{n\to\infty} x_n = x$ and that $x_{2n} \in E$ and $x_{2n+1} \notin E$ for all $n \in \mathbb{N}$. Show that x is a boundary point of E.

Proof. Since $x_n \to x$, then all of its subsequences converge to x. Thus $\forall U \in \mathcal{N}(x)$, we can find $N > 0$ such that $\forall n > N$ we have $x_{2n} \in U$ and $x_{2n+1} \in U$. Thus $U \cap E \neq \varnothing$ and $U \cap E^c \neq \varnothing$. This proves that x indeed a boundary point. □

■ **Problem 4.4** Show that a set, all of whose points are isolated, must be closed.

Proof. This statement is vacuously true (as a set with all isolated points has not limit points, thus it vacuously contains all of its limit points.) However, here, we will provide a detailed proof using the first principles. Let F be a set that all of its points are isolated points. Thus for every $x \in F$ we can find an open set $U_x \in \mathcal{T}$ such that $U_x \cap F = \{x\}$. Consider F^c. Then the set $G_x = U_x \backslash \{x\}$ is open, since $\{x\}$ is closed. Thus $\forall y \in G_x$ we can fine an open set $V \in \mathcal{T}$ such that $V \subseteq G_x \subseteq F^c$. This implies that F^c is indeed open, thus F is closed. □

■ **Problem 4.5** Show that the closure operation has the following properties.

(i) $E_1 \subseteq E_2 \implies \overline{E_1} \subseteq \overline{E_2}$.

(ii) $\overline{E_1 \cup E_2} = \overline{E_1} \cup \overline{E_2}$.

(iii) $\overline{E_1 \cap E_2} \subset \overline{E_1} \cap \overline{E_2}$.

(iv) Give an example of two sets E_1 and E_2 such that

$$\overline{E_1 \cap E_2} \neq \overline{E_1} \cap \overline{E_2}.$$

Proof.

(i) (a) **First method.** We will use the definition that $\overline{E} = ((E^c)^\circ)^c$, and also the fact that $E_1 \subseteq E_2 \implies E_1^\circ \subseteq E_2^\circ$. Then we can write

$$E_1 \subseteq E_2 \implies E_1^c \supseteq E_2^c \implies (E_1^c)^\circ \supseteq (E_2^c)^\circ \implies ((E_1^c)^\circ)^c \subseteq ((E_2^c)^\circ)^c \implies \boxed{\overline{E_1} \subseteq \overline{E_2}}.$$

(b) **Second method.** Let $E_1 \subseteq E_2$. Then it follows that $E_1' \subseteq E_2'$. That is because for $x \in E_1$, for all $U \in \mathcal{N}(x)$ we have $U \backslash \{x\} \cap E_1 \neq \varnothing$. However since $E_1 \subseteq E_2$ then $U \backslash \{x\} \cap E_2 \neq \varnothing$. Then it follows that $E_1 \cup E_1' \subseteq E_2 \cup E_2'$, hence $\overline{E_1} \subseteq \overline{E_2}$.

(ii) From definition, it follows that

$$\overline{E_1 \cup E_2} = (E_1 \cup E_2) \cup (E_1 \cup E_2) = (E_1 \cup E_1') \cup (E_2 \cup E_2') = \overline{E_1} \cup \overline{E_2}.$$

(iii) First we need to prove $(E_1 \cap E_2)' \subseteq E_1' \cap E_2'$ (note that the equality might also hold, however, we are only interested in the $\subseteq$ case). Let $x \in (E_1 \cap E_2)'$. Then $\forall U \in \mathcal{N}(x)$ we have $U \backslash \{x\} \cap (E_1 \cap E_2) \neq \varnothing$. Thus $U \backslash \{x\} \cap E_1 \neq \varnothing$ and $U \backslash \{x\} \cap E_2 \neq \varnothing$, which implies $x \in E_1' \cap E_2'$.
Now let $y \in \overline{E_1 \cap E_2}$. Then from definition we can write

$$y \in (E_1 \cap E_2) \cup (E_1 \cap E_2)'.$$

Then it means $y \in E_1 \cap E_2$ or $y \in (E_1 \cap E_2)'$. Since the a set is always a subset of its closure, then the former implies $y \in \overline{E_1} \cap \overline{E_2}$. Also, from the latter we have $y \in (E_1 \cap E_2)'$ we implies $y \in E_1' \cap E_2'$. Since the derived set is a subset of the closure of a set, then $y \in \overline{E_1} \cap \overline{E_2}$. This completes the proof.

(iv) We basically want to show case where the following statements is true

$$(E_1 \cup E_1') \cap (E_2 \cup E_2') \nsubseteq (E_1 \cap E_2) \cup (E_1' \cap E_2').$$

Let E_1 be an open set, and E_2 be a set with $x \in E_2$, an isolated point, such that $x \in \partial E_1$. Then $\overline{E_1 \cap E_2} = \varnothing$ while $\overline{E_1} \cap \overline{E_2} = \{x\}$.

□

■ **Problem 4.6** Show that for any sets E_1, E_2 in a topological space $(X, \mathcal{T})$ we have

$$(E_1 \cup E_2)^\circ \supseteq E_1^\circ \cup E_2^\circ.$$

Also, give an example of two sets E_1 and E_2 such that

$$(E_1 \cup E_2)^\circ \neq E_1^\circ \cup E_2^\circ.$$

Proof. Let $x \in E_1^\circ \cup E_2^\circ$. Then $x \in E_1^\circ$ or $x \in E_2^\circ$. In the former case, since E_1° is an open set, so $E_1^\circ \in \mathcal{N}(x)$. Thus $\exists U \in \mathcal{T}$ such that $x \in U \subseteq E_1$. This implies that $x \in U \subseteq E_1 \cup E_2$. Thus $x \in (E_1 \cup E_2)^\circ$. The proof for the latter case is also very similar. Thus we can conclude that $(E_1 \cup E_2)^\circ \supseteq E_1^\circ \cup E_2^\circ$.
As for the example, Let E_1 and E_2 be two closed sets that part of their boundary is the same, i.e. $\partial E_1 \cap \partial E_2 = G \neq \varnothing$. Then G is not boundary of $E_1 \cup E_2$ anymore, thus $G \subseteq (E_1 \cup E_2)^\circ$. However, $G \nsubseteq E_1^\circ E_2^\circ$, since $G \nsubseteq E_1^\circ$ and $G \nsubseteq E_2^\circ$, as it was a boundary set for both E_1 and E_2. □

■ **Problem 4.7** Let $(X, \mathcal{T})$ be a topological space, and $E \subseteq X$. Prove that E' is closed.

Proof.

(i) **First method.** Let $x \in (E')^c$. Then $x \notin E'$. This implies that we can fine an open set $U \in \mathcal{T} \cap \mathcal{N}(x)$ such that $U \backslash \{x\} \cap E = \varnothing$. We claim that $U \backslash \{x\} \cap E' = \varnothing$. Because otherwise, there is $y \in U \backslash \{x\}$ and $y \in E'$. From the former, we conclude that $\exists V \in \mathcal{T}$ such that $y \in V \subseteq U \backslash \{x\}$ (because $U \backslash \{x\}$ is an open set, thus belongs to $\mathcal{N}(y)$ for all $y \in U \backslash \{x\}$). On the other hand, the latter implies $V \backslash \{y\} \cap E \neq \varnothing$. These two implies $U \backslash \{x\} \cap E \neq \varnothing$ which is a contradiction to the original assumptions. We then conclude $U \backslash \{x\} \subseteq (E')^c$. Thus $x \in U \subseteq (E')^c$, which implies $(E')^c$ is open, hence E' is closed.

(ii) **Second method.** We can use the fact that a set is closed if and only if it is equal to its closure. So we can write

$$\overline{E'} = E' \cup E''.$$

Then what remains to be proved is to prove that $E'' \subseteq E'$. We use the contrapositive arguments. I.e., we want to prove for $x \in X$, we have $x \notin E'$ implies $x \notin E''$. Lets assume $x \notin E'$. Thus we can find an open set $U \in \mathcal{T} \cap \mathcal{N}(x)$ such that $U \backslash \{x\} \cap E = \varnothing$. We claim $U \backslash \{x\} \cap E' = \varnothing$ as well. Because otherwise y such that $y \in U \backslash \{x\}$ and $y \in E'$. The former implies existence of open set $V \in \mathcal{T}$ such that $x \in V \subseteq U$. However, the latter, implies $V \backslash \{y\} \cap E \neq \varnothing$. Thus we conclude $U \backslash \{x\} \cap E \neq \varnothing$, which is contradiction. Thus $U \backslash \{x\} \cap E' = \varnothing$. Thus by contrapositive we can conclude $E'' \subseteq E'$. So

$$\overline{E'} = E',$$

which implies the closeness of E'.

□

■ **Problem 4.8** Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$. Prove that A° is open.

Proof. Let $x \in A^\circ$. Then $\exists U \in \mathcal{T}$ such that $x \in U \subseteq A$. Then $\forall z \in U$ we can find an open set $V \in \mathcal{T}$ such that $z \in V \subseteq U$ (because every open set belongs to $\mathcal{N}$ all of its elements). Thus $z \in A^\circ$. This implies $x \in U \subseteq A^\circ$. Thus $A^\circ \in \mathcal{N}(x)$ for all $x \in A^\circ$. We can now conclude A° is open. □

■ **Problem 4.9** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then prove the followings.

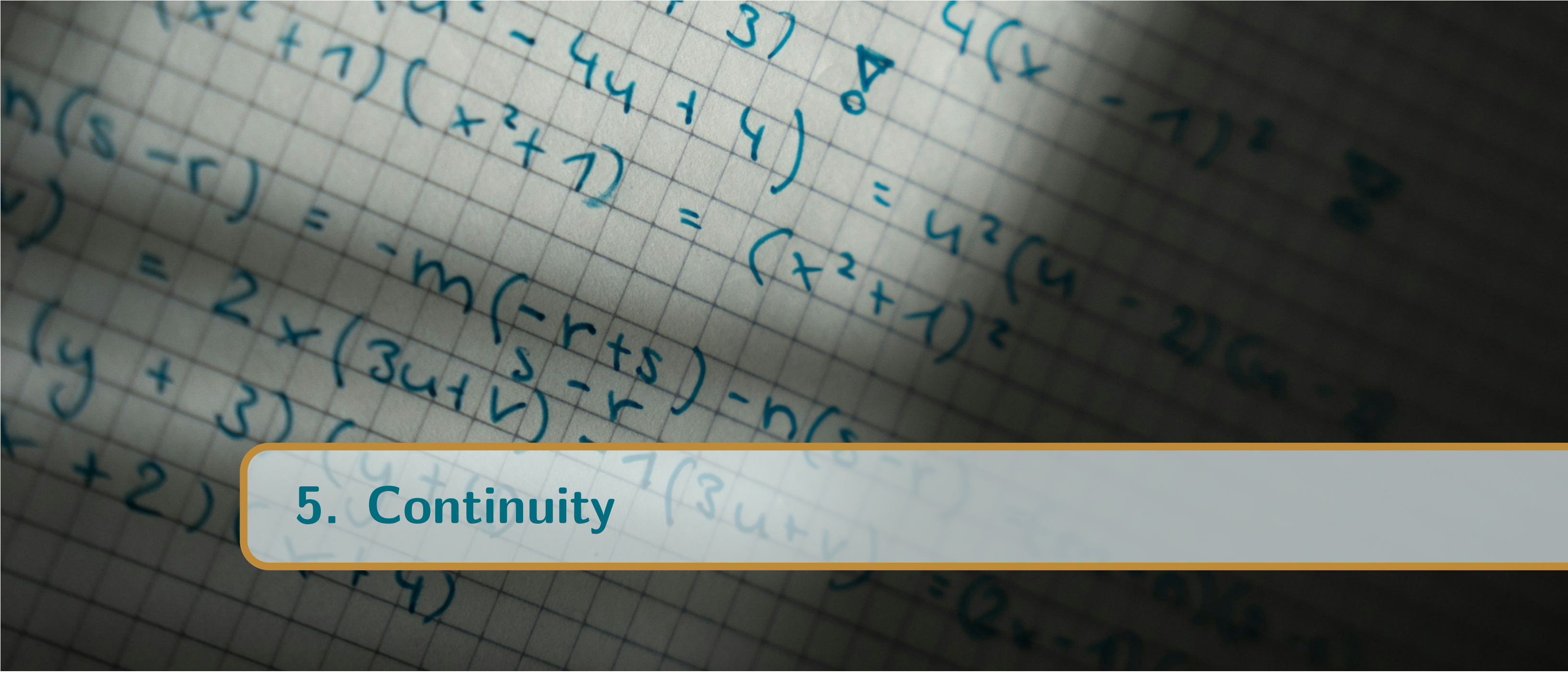
(a) A is open if and only if $A^\circ = A$.

(b) A is closed if and only if $\overline{A} = A$

Proof. (a)

(b) (i) **Method 1.** We use the identity $\overline{A} = A \cup A'$. Thus we just need to prove A is closed if and only if $A' \subseteq A$. Assume A is closed, then A^c is open. Now we want to prove $A' \subseteq A$. By the contrapositive argument, we can instead prove $x \notin A$ implies $x \notin A'$. Let $x \in A^c$ (which we have assumed is open). Then $\exists U \in \mathcal{T}$ such that $x \in U \subseteq A^c$. Thus $U \subseteq A = \varnothing$, in particular $U \backslash \{x\} \subseteq A = \varnothing$. Thus $x \notin A'$. So we proved that closeness of A implies $A' \subseteq A$. For the converse, we have

□



5. Continuity

In this section we will cover the topics related to the continuity of functions from one topological space to another topological space. Also, we will put much emphasis on the properties of continouse functions from one metric space to anther metric space (in particular functions from $\mathbb{R} \to \mathbb{R}$). The notion of continuity is one of the central concepts in analysis which shows up anywhere there is a trace of analysis!

5.1 Basic Definitions and Characterizations

We start with the most general definition of a continouse function in a topological setting.

Definition 5.1 Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. We say a function $f : X \to Y$ is continouse [on $(X, \mathcal{T}_X]$ iff

$$\forall G \in \mathcal{T}_Y \text{ we have } f^{-1}(G) \in \mathcal{T}_X.$$

f Note that $f^{-1}(G)$ means the pre-image of the set G, and does not mean the inverse of the function f.

In words, a function from X to Y is continouse if the pre-image of every open set in Y is also an open set in X.

Lemma 5.1 $f : X \to Y$ is continouse if and only if $f^{-1}(C)$ is closed for every closed set C in Y.

Proof. First, we need to prove the following identity

$$f^{-1}(C^c) = (f^{-1}(C))^c.$$

To show this we first show $f^{-1}(C^c) \subseteq (f^{-1}(C))^c$. Let $x \in f^{-1}(C^c)$. Then

$$x \in f^{-1}(C^c) \implies f(x) \in C^c \implies f(x) \notin C \implies x \notin f^{-1}(C) \implies x \in (f^{-1}(C))^c.$$

Then we need to show $(f^{-1}(C))^c \subseteq f^{-1}(C^c)$. Let $x \in (f^{-1}(C))^c$. Then

$$x \in (f^{-1}(C))^c \implies x \notin f^{-1}(C) \implies f(x) \notin C \implies f(x) \in C^c \implies x \in f^{-1}(C^c).$$

Thus we proved that $f^{-1}(C^c) = (f^{-1}(C))^c$. Now to prove the lemma, for the $\implies$ direction, we assume that f is continouse. Let $C \subseteq Y$ be a closed set. Then $C^c \in \mathcal{T}_Y$. Since f is continouse, then $f^{-1}(C^c) \in \mathcal{T}_X$. From the identity we proved above, it immediately follows that $(f^{-1}(C))^c \in T_x$, thus $f^{-1}(C)$ is closed. Now for the other direction, we assume that $f^{-1}(C)$ is closed for every closed set C in Y. Let $G \in \mathcal{T}_Y$. Then G^c is closed. From our assumption, we know that $f^{-1}(G^c) = (f^{-1}(G))^c$ is closed. Thus $f^{-1}(G) \in \mathcal{T}_X$. So the pre-image of G is open. Then we conclude that f is continouse. □

■ **Example 5.1** In any metric space (X, d), for any point $p \in X$, the function $f : X \to \mathbb{R}$ defined as

$$f(x) = f(x, p), \qquad x \in \mathbb{R},$$

is continouse.

Proof. To show this, we need to show that $f^{-1}(G)$, where G is open in $\mathbb{R}$, is open in X. Let $U = (a, b)$ be an interval in $\mathbb{R}$ for some $a, b \in \mathbb{R}$. We will have three cases of such interval, where $a > 0, b > 0$, $a < 0, b > 0$, and $a < 0, b < 0$. In the third case, $f^{-1}(U) = \varnothing$ which is indeed open. For the first case, we have

$$f^{-1}(U) = \mathbb{B}[p, b) \backslash \mathbb{B}[p, a] = \mathbb{B}[p, b) \cup (\mathbb{B}[p, a])^c.$$

The right hand side is the intersection of two open sets. Thus $f^{-1}(U)$ is open. For the second case, where $a < 0, b > 0$, the calculations becomes even simpler since

$$f^{-1}(U) = \mathbb{B}[p, b)$$

which is clearly an open set in the metric space X. To complete the proof, we need to also consider more complicated open sets in $\mathbb{R}$ which is the union of intervals. Let G be an open set in $\mathbb{R}$ with usual topology. Then G is a union of intervals $G = \bigcup_i G_i$, where G_i are intervals. □

■

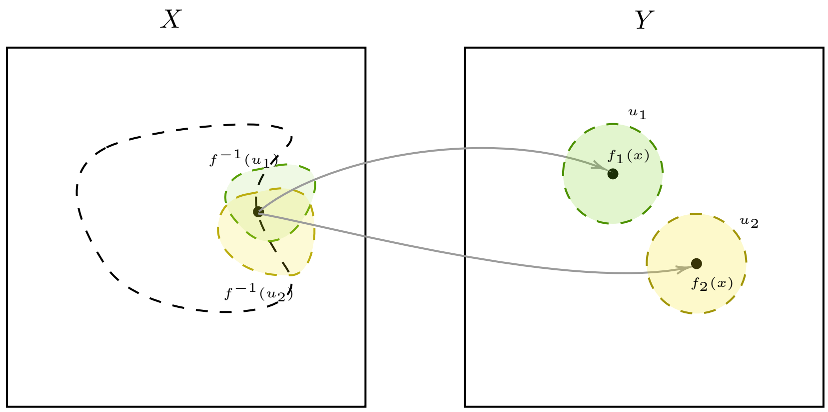
The following theorem, characterizes the behaviour of continouse functions on dense sets.

Theorem 5.1 Let $f_1, f_2 : X \to Y$, where $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are Hausdorff topological spaces, and f_1 and f_2 are continouse. Then for any $Q \subseteq X$ we have

$$\left[f_1(x) = f_2(x)\ \forall x \in Q\right] \implies \left[f_1(x) = f_2(x)\ \forall x \in \overline{Q}\right].$$

Proof. In the proof of this theorem, we will utilize the fact that the spaces are Hausdorff spaces (i.e. every two point can be distinguished by two disjoint open sets). The following figure will make the proof idea more tangible.

Let $x \in \overline{Q}$. Then $x \in Q \cup \partial Q$. If $x \in Q$, then there is nothing to prove. When $x \in \partial Q, then we proceed with the proof by contradiction. Assume f_1(x) \neq f_2(x)$. Then since Y is a Hausdorff topological spaces, we can fine open sets $u_1, u_2 \in \mathcal{T}_Y$ such that $f_1(x) \in u_1$ and $f_2(x) \in u_2$ but $u_1 \cap u_2 = \varnothing$. Then consider the pre-image of these open sets. Since f is continouse, then $f_1^{-1}(u_1)$ and $f_2^{-1}(u_2)$ are open sets in X that contains x. Let $v = f_1^{-1}(u_1) \cap f_2^{-1}(u_2)$. v is open in X that contains x. Since x is a boundary point, then $v \cap Q \neq \varnothing$. Let $y \in v \cap Q$. Then $f_1(y) \in u_1$ and $f_2(y) \in u_2$ but since $f_1(y) = f_2(y)$, then $u_1 \cap u_2 \neq \varnothing$. This contradicts our initial assumption that u_1 and u_2 are disjoint. This completes the proof.



□

■ **Remark** The theorem above indicates that if two continouse function from $\mathbb{R} \to \mathbb{R}$, agree on all values of the form $m/2^n$, for all $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$, then they should agree on the whole real line.

5.2 Continuity and Compactness

The compactness of the domain of a continouse function has many interesting implications.

Theorem 5.2 — Continouse image of a compact set. Let $f : X \to Y$ be a continouse function, and X compact. Then $f(X)$ is compact in Y.

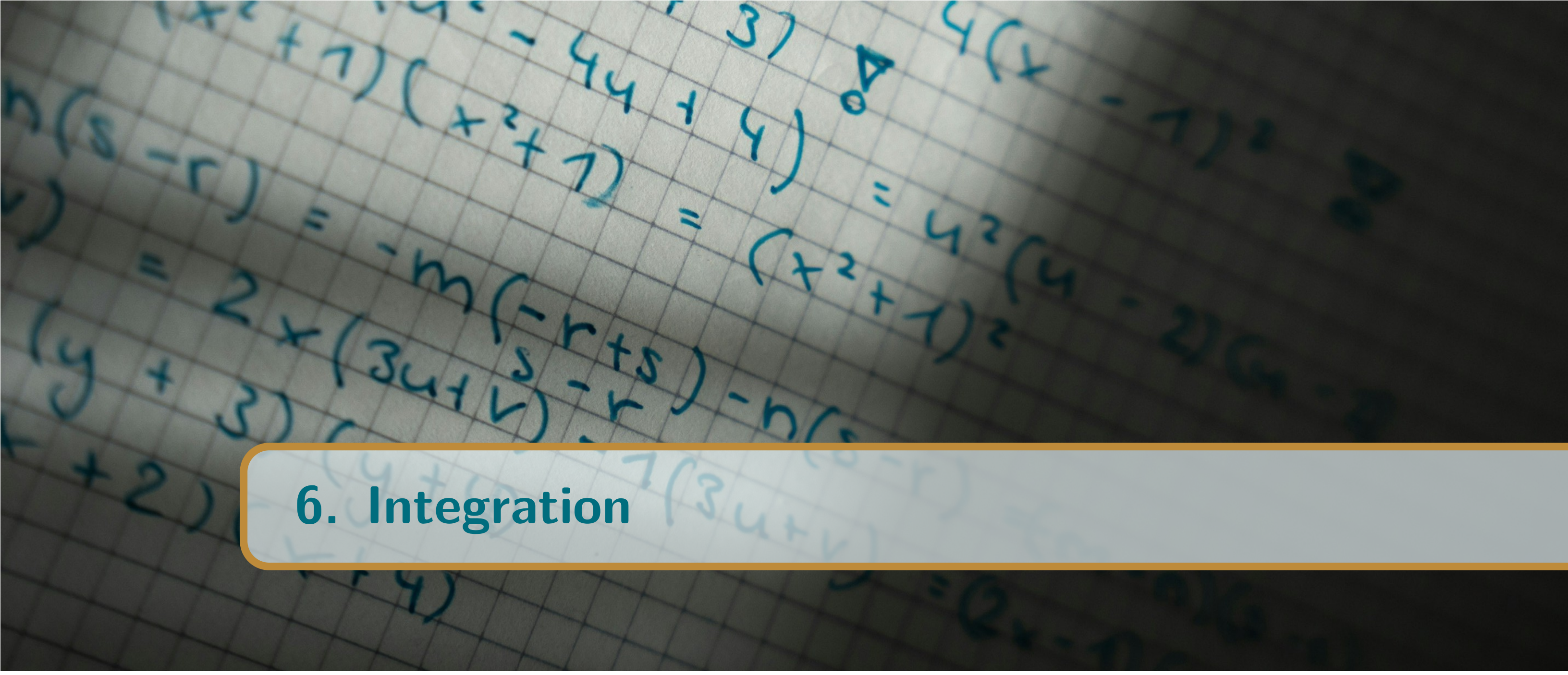
Proof. Let $\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$ be an open cover for $f(X)$. Then we claim that $\tilde{\mathcal{G}} = \{f^{-1}(G_\alpha)\}_{\alpha \in I}$ is also an open cover for X. That is because

$$\forall x \in X, \; \exists \alpha \in I \text{ s.t. } f(x) \in G_\alpha \implies x \in f^{-1}(G_\alpha),$$

thus the $\tilde{\mathcal{G}}$ covers every element of the set X. Since X is compact, then there exists a finite sub-cover. I.e. there exists $I' \subseteq I$ finite such that $\tilde{\mathcal{G}}' = \{f^{-1}(G_\alpha)\}_{\alpha \in I'}$ is a finite cover for X. Now we claim that $\mathcal{G}' = \{G_\alpha\}_{\alpha \in I'}$ is also a finite sub-cover for $f(X)$. This is true since for every $y \in f(X)$ we have

$$f^{-1}(y) \in X \implies \exists \alpha \in I' \text{ s.t. } f^{-1}(y) \in f^{-1}(G_\alpha) \implies y \in G_\alpha.$$

Thus $\mathcal{G}'$ covers every element of the set $f(X)$, thus it is a valid finite sub-cover for $\mathcal{G}$. This completes the proof. □



6. Integration

Integrals, are one of the very central notions in the world of analysis that has numerous rules both in developing new foundational theories as well as lots of uses in the word of applications. Integrals act as a very useful norm in different spaces of functions, they help us in generalizing the notion of derivative (weak derivative), they act like linear operators between function spaces, they appear in some of the most important formulations of the natural sciences (like calculus of variations), and many many more applications. Here in this chapter we will cover the basics of the notion of integration of *real* functions, and later we will see other variations of the integrals. To avoid unnecessary abstraction, we will mainly deal with functions from reals to reals, but it will be very straight forward to generalize these concepts to the complex valued functions, or function defined on $\mathbb{R}^n$ to $\mathbb{R}^m$. We start with the notion of Riemann integrable functions.

6.1 Riemann Integrable Functions

Definition 6.1 — Riemann Integrable Functions. Let f be a bounded real function define on $[a, b] \subset \mathbb{R}$. Let partition P be the set of points $\{x_i \in [a, b] \;:\; a = x_0 \leq x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n = b\}$ for some $n \in \mathbb{N}$. define $\Delta x_i = x_i - x_{i-1}$, and

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x), \qquad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x).$$

Then define the following sums

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i, \qquad L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i.$$

We define the upper and lower Riemann sums as

$$\overline{\int_a^b} f \; dx = \inf U(P, f), \qquad \underline{\int_a^b} f \; dx = \sup L(P, f),$$

which are called upper and lower Riemann integrals respectively. Note that the suprimum and the infimum are take on all possible partitions.

We say that the function f is Riemann integrable and write it as $R \in \mathcal{R}$ if

$$\overline{\int_a^b} f\ dx = \underline{\int_a^b} f\ dx.$$

And we denote this common value as

$$\int_a^b f\ dx.$$

■ **Remark** Note that the upper and lower Riemann integrals exists, as in the definition we assumed that the function f is bounded. If f is not bounded, then M_i or m_i might not exist for some interval $[x_{i-1}, x_i]$, leading that the upper or lower Riemann sum might not exist.

At this point, this definition might seem useless as we are talking about things like the suprimum or infimum on all possible partitions. The set of all partitions of $[a,b]$ seems to be a set that is hard to characterize. However, as we will see, this "hard to use" definition will take us to somewhere that is very interesting. However, at this stage, the following example is aimed to emphasis the subtleties of the definition above.

■ **Example 6.1 — Attempting to integrate $f(x) = x$.** Let $f : [0,1] \to \mathbb{R}$ where $f(x) = x$. We want to use the definition above to calculate the integral of $f(x)$ in $[0,1]$. To start with something, first, we want to consider a equidistant partition of the interval $[0,1]$ to n intervals that all has the same length. Then the partition will be $P = \{\frac{0}{n}, \frac{1}{n}, \cdots, \frac{n-1}{n}, \frac{n}{n}\}$. Then we will have

$$M_i = \sup_{x \in I_n} f(x) = \frac{i}{n}, \qquad m_i = \inf_{x \in I_n} f(x) = \frac{i-1}{n}, \qquad \Delta x_i = \frac{1}{n}.$$

So the upper and lower Riemann sums will be

$$U(P,f) = \sum_{i=1}^n M_i \Delta x_i = \frac{1}{n^2}\sum_{i=1}^n i = \frac{n(n+1)}{2n^2} = \frac{1}{2} + \frac{1}{2n^2},$$
$$L(P,f) = \sum_{i=1}^n m_i \Delta x_i = \frac{1}{n^2}\sum_{i=1}^n (i-1) = \frac{n(n-1)}{2n^2} = \frac{1}{2} - \frac{1}{2n^2}$$

Now, it is quite clear that out of all possible equidistence partitions, the infimum of $U(P,f)$ will be $1/2$, and the suprimum of $L(P,f)$ will be $1/2$ as well. **But**, I need to emphasis that this does not imply anything about the upper and lower Riemann integrals. That is because, the upper (lower) Riemann sum is the infimum (suprimum) considered on all partitions. However, here, we have only considered the partitions with equal distance intervals. However, one might come up with an elegantly designed partition that can change the game. This at this stage, we can not even integrate the function $f(x) = x$. We will make our original definition of Riemann integrablity by a very useful generalization to the Riemann-Stieltjes integrals. ■

Proposition 6.1 — Properties of the Riemann sums. Let $f : [a,b] \to \mathbb{R}$ bounded, and let P be any partition. Then we have

(a) $L(P,f) \le U(P,f)$.

(b) $\exists m, M \in \mathbb{R}$ such that $m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$.

Proof. (a) This follows immediately from the properties of suprimum and infimum.

(b) Since f is bounded, then $\exists m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M \ \forall x \in [a, b]$. Then

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i \leq \sum_{i=1}^{n} M \Delta x_i = M \sum_{i=1}^{n} \Delta x_i = M(b-a),$$
$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i \geq \sum_{i=1}^{n} m \Delta x_i = m \sum_{i=1}^{n} \Delta x_i = m(b-a).$$

Note that we use the telescoping property of the sums above. Combining the results from above with the result of part (a) we can write

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a).$$

□

6.2 Riemann-Stieltjes Integrals

At this point we will generalize the notion of Riemann integration to Riemann-Stieltjes integration. The idea behind this generalization will be more clear later.

Definition 6.2 — Riemann-Stieltjes Integrals. TO BE ADDED

The following definition of the refinement of a partition, will help up to prove some statements that will help us in making useful tools out of the definitions above.

Definition 6.3 — Refinement of a partition. We say that the partition P^* is a refinement of the partition P if $P \subseteq P^*$. Given two partitions P_1, P_2, their **common refinement** is the partition $P = P_1 \cup P_2$.

The following theorem will show the significance of the notion of the refinements.

Lemma 6.1 If P^* is a refinement of P, then we have

(i) $L(P, f, \alpha) \leq L(P^*, f, \alpha)$

(ii) $U(P^*, f, \alpha) \leq L(P, f, \alpha)$

Proof. We will prove the first statement, and the proof for the second statement will be analogous. First, assume that the refinement P^* has only one extra point, say x^* that falls in the i-th interval i.e. $I_i = [x_{i-1}, x_i]$. Thus this interval will turn into two intervals $I_i^{(1)} = [x_{i-1}, x^*]$ and $I_i^{(2)} = [x^*, x_i]$. Let

$$M_i = \sup_{x \in I_i f(x)} f(x), \qquad w_1 = \sup_{x \in I_i^{(1)}} f(x), \qquad w_2 = \sup_{x \in I_i^{(2)}} f(x).$$

Then, from the properties of suprimum it follows that $w_2 \leq M_i$ and also $w_1 \leq M_i$. Then

$$U(P, f, \alpha) - U(P^*, f, \alpha) = M_i(\alpha_i - \alpha_{i-1}) - \big(w_1(\alpha(x^*) - \alpha_{i-1}) + w_2(\alpha_i - \alpha(x^*))\big)$$

Since $w_1 \leq M_i$, and $w_2 \leq M_i$, and α is non-decreasing, then

$$w_1(\alpha(x^*) - \alpha_{i-1}) \leq M_i(\alpha(x^*) - \alpha_{i-1}), \quad w_2(\alpha(x^*) - \alpha_{i-1}) \leq M_i(\alpha(x^*) - \alpha_{i-1})$$

Then we can write

$$U(P, f, \alpha) - U(P^*, f, \alpha) \geq M_i(\alpha_i - \alpha_{i-1}) - M_i(\alpha_i - \alpha_{i-1}) = 0.$$

then it immediately follows that

$$U(P^*, f, \alpha) \leq U(P, f, \alpha)$$

□

The following important proposition will take advantage of the lemma above.

Proposition 6.2 Let $f : [a, b] \to \mathbb{R}$ bounded, and $\alpha : [a, b] \to \mathbb{R}$ non-decreasing. Then

$$\underline{\int_a^b} f \, d\alpha \leq \overline{\int_a^b} f \, d\alpha.$$

Proof. Let set $\mathbb{P}$ denote the set of all partitions on the interval $[a, b]$. Then for $P_1, P_2 \in \mathbb{P}$, let their common refinement be $P^* = P_1 \cup P_2$. Then from the properties of the common refinement we can write

$$L(P_1, f, \alpha) \leq L(P^*, f, \alpha), \qquad U(P^*, f, \alpha) \leq U(P_1, f, \alpha).$$

However, it follows from the properties of the upper and lower Riemann sums for partition P^* that

$$L(P^*, f, \alpha) \leq U(P^*, f, \alpha).$$

Combining these two results will lead to

$$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha) \qquad \forall P_1, P_2 \in \mathcal{P}$$

If P_2 is fixed, then taking sup on all $P_1 \in \mathcal{P}$ we will have

$$\underline{\int_a^b} f \, d\alpha \leq U(P_2, f, \alpha).$$

Now by taking inf on all $P_2 \in \mathcal{P}$ we will have

$$\underline{\int_a^b} f \, d\alpha \leq \overline{\int_a^b} f \, d\alpha.$$

□

The following theorem comes very handy in the applications.

Theorem 6.1 $f \in \mathcal{R}_\alpha[a, b]$ if and only if $\forall \epsilon > 0$ there exists a partition such that

$$U(P, f, f\alpha) - L(P, f, \alpha) < \epsilon.$$

Proof.
$\boxed{\Longrightarrow}$ We assume $f \in \mathcal{R}_\alpha[a, b]$. Then

$$\underline{\int_a^b} f \, d\alpha = \overline{\int_a^b} f \, d\alpha = I$$

Since $I = \inf_{p \in \mathcal{P}} U(P, f, \alpha)$, then there exists $P_1 \in \mathcal{P}$ such that

$$U(P_1, f, \alpha) < I + \epsilon/2.$$

Similarly, $\exists P_2 \in \mathcal{P}_\in$ such that

$$L(P_2, f, \alpha) > I - \epsilon/2.$$

Let P^* be the common refinement of P_1, P_2. I.e. $P^* = P_1 \cup P_2$. Then

$$I - \epsilon/2 < L(P_2, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_1, f, \alpha) < I + \epsilon/2.$$

Then the maximum difference between $U(P^*, f, \alpha)$ and $L(P^*, f, \alpha)$ can be ϵ. I.e.

$$U(P^*, f, \alpha) - L(P^*, f, \alpha) < \epsilon.$$

$\boxed{\Longleftarrow}$ We can do this in two different ways. For the first method, I will use the proof by contradiction. But to do this, first observe that $\inf_{P \in \mathbb{P}} U(P, f, \alpha)$ and $\sup_{P \in \mathbb{P}} L(U, f, \alpha)$ exists. That is because if $\inf_{P \in \mathbb{P}} U(P, f, \alpha) = -\infty$, then $\sup_{P \in \mathbb{P}} L(P, f, \alpha) = -\infty$ as well. Similarly, if $\sup_{P \in \mathbb{P}} L(P, f, \alpha) = \infty$, then $\inf_{P \in \mathbb{P}} L(P, f, \alpha) = \infty$ as well. In either case, the hypothesis fails to be true. Thus $\exists \gamma_1, \gamma_2 \in \mathbb{R}$ such that $\gamma_1 = \inf_{P \in \mathbb{P}} U(P, f, \alpha)$ and $\gamma_2 = \sup_{P \in \mathbb{P}} L(P, f, \alpha)$. From Proposition 6.2 it follows that $\gamma_2 \leq \gamma_1$. We claim that $\gamma_2 = \gamma_1$. Because other wise, we let $\epsilon = \gamma_1 - \gamma_2$. Then by hypothesis we can find $P \in \mathbb{P}$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

On the other hand, from the properties of sup and inf we know that

$$U(P, f, \alpha) \geq \gamma_1, \qquad L(P, f, \alpha) \leq \gamma_2$$

Thus it follows

$$U(P, f, \alpha) - L(P, f, \alpha) \geq \epsilon,$$

which is a contradiction.
There is also a much more higher level proof that utilizes the previous results. Let $\epsilon > 0$ given. Then $\exists P \in \mathbb{P}$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

We know that for all $P \in \mathbb{P}$ we have

$$L(P, f, \alpha) \leq \underline{\int} f \, d\alpha \leq \overline{\int} f \, d\alpha \leq U(P, f, \alpha).$$

Thus

$$0 \leq \overline{\int} f \, d\alpha - \underline{\int} f \, d\alpha < \epsilon.$$

However, since this is true for any $\epsilon > 0$, then we can conclude that the upper and lower integrals are equal, thus $f \in \mathcal{R}_\alpha[a, b]$. $\square$

Corollary 6.1 Let $f : [a, b] \to \mathbb{R}$ be bounded. Assume for some $\epsilon > 0$ and some partition $P \in \mathcal{P}$ we have

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

Then this holds for every refinement of P (With the same ϵ).

Proof. Let P^* be any refinement of P. Then

$$L(P, f, \alpha) \leq L(P^*, f, \alpha), \qquad U(P^*, f, \alpha) \leq U(P, f, \alpha).$$

We can rearrange this as

$$L(P, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P, f, \alpha).$$

The it immediately follows that

$$U(P^*, f, \alpha) - L(P^*, f, \alpha) < \epsilon.$$

□

The following theorem is one of our major results so far. Thus theorem highlights the fact that how we can arrive at useful results from abstract definitions.

Theorem 6.2 — Continuous functions are Riemann-Stieltjes integrable.. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then it is Riemann-Stieltjes integrable.

Proof. Choose η small enough such that

$$(\alpha(b) - \alpha(a))\eta < \epsilon.$$

Then since the function f is continuous on a compact set $[a, b]$, it is uniformally continuous. So for η chosen as above, we can find $\delta > 0$ such that for every $t, s \in [a, b]$ we have

$$|t - s| < \delta \implies |f(t) - f(s)| < \eta.$$

Then if P is a partition that $\Delta x_i < \delta$ for all i, then we have

$$M_i - m_i < \eta.$$

Now consider the following difference between the upper and lower Riemann sums

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i \leq \sum_{i=1}^{n} \eta\Delta\alpha_i = \eta(\alpha(b) - \alpha(a)) < \epsilon.$$

So the function f is Riemann-Stieltjes integrable. □

The following theorem is also useful as it relaxes some of the requirements on the function f to be Riemann-Stieltjes integrable, and puts more constraints on the integrator α. The proof will be very similar to the proof above.

Proposition 6.3 Let $f : [a, b] \to \mathbb{R}$ monotone function, and α continuous on $[a, b]$. Then $f \in \mathcal{R}_\alpha[a, b]$ (note that we still require α to be monotone).

Proof. Assume that the function f is non-decreasing (the proof for the other case will be analogous). For a given $\epsilon > 0$, choose $\eta > 0$ small enough that

$$(f(b) - f(a))\eta < \epsilon.$$

Since α is continuous on the compact set $[a, b]$, then it is uniformally continuous. Thus for chosen η as above, we can find $\delta > 0$ such that for all $t, s \in [a, b]$ we have

$$|t - s| < \delta \implies |\alpha(t) - \alpha(s)| < \eta.$$

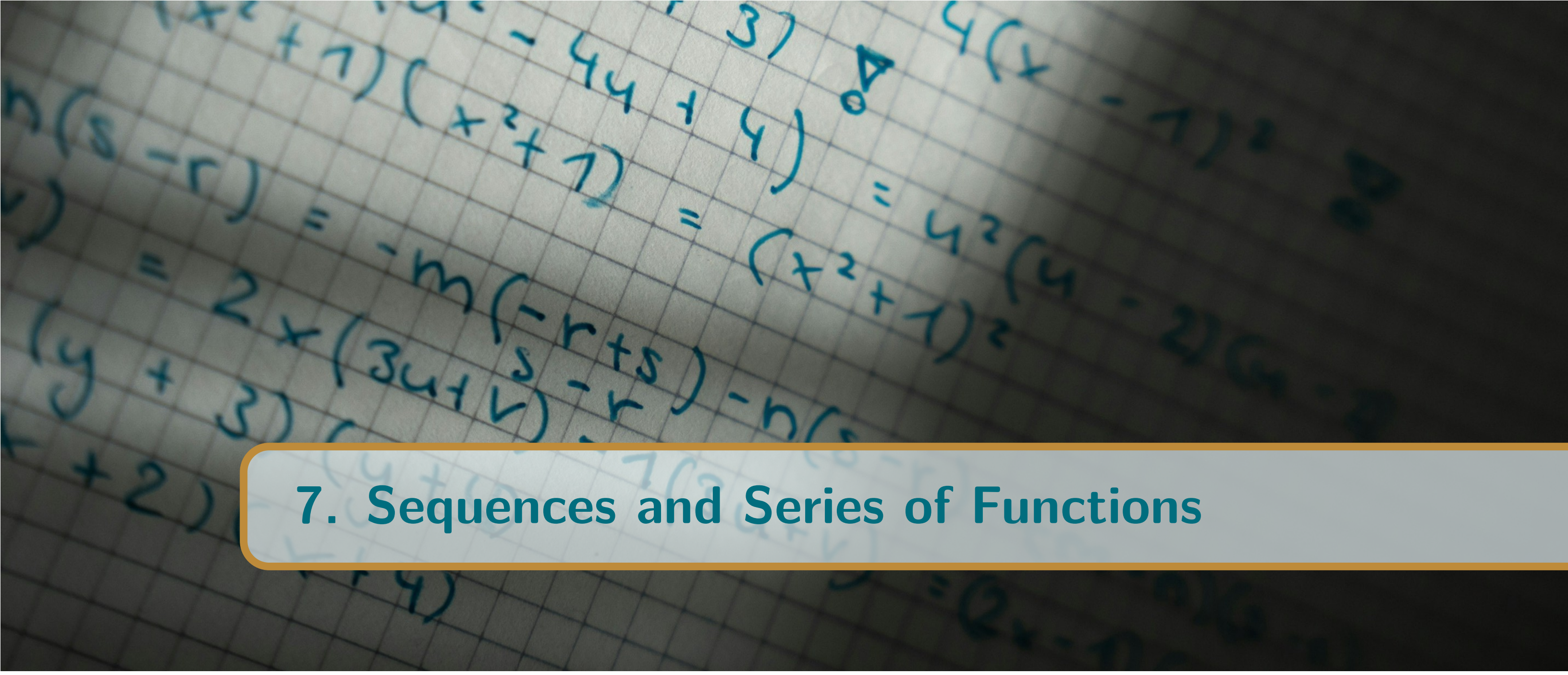
Let P be any partition that $\Delta x_i < \delta$ for all i. Then $\Delta\alpha_i < \eta$ for all i. So we can write

$$\sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i < \eta\sum_{i=1}^{n}(M_i - m_i).$$

Since f is non-decreasing, then for all i we have $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Then the sum above telescopes and we will have

$$U(P, f, \alpha) - L(P, f, \alpha) < \eta(f(b) - f(a)) < \epsilon.$$

□



7. Sequences and Series of Functions

7.1 Basic Notions and Definitions

In this chapter we will study the sequence and series of functions. We will study different notions of convergence, namely point-wise convergence, uniform convergence, etc. Note that in this chapter, we will try to be as concrete as possible, avoiding unnecessary abstraction.

Definition 7.1 — Point-wise convergence. Let $\{f_n\}$ be a sequence of functions $f : [0,1] \to \mathbb{R}$. Then we say that this sequence converges *point-wise* to the function $f : [0,1] \to \mathbb{R}$, and show it as $f_n(x) \to f(x)$ if

$$\forall \epsilon > 0, x \in [0,1],\ \exists N > 0 \text{ s.t. } \forall n > N \text{ we have } |f_n(x) - f(x)| < \epsilon.$$

We write this as $\lim_{n\to\infty} f_n(x) = f(x)$.

■ **Remark** Note the order of the quantifiers in the definition above. What we are basically saying is that for any choice of ϵ and for any x, we can find N (that depends on ϵ and x) such that for all $n > N$ we have $|f_n(x) - f(x)| < \epsilon$. There is **NO** guarantee that by using this N, for any other point x' we have $|f_n(x') - f(x)| < \epsilon$.

Thinking more carefully, we can see that this is nothing other than a collection of sequences in $\mathbb{R}$ that are indexed by index the set $[0,1]$. For instance, for $x = 0.5$, $\{f_n(0.5)\}_n$ is a sequence in $\mathbb{R}$ like any other value of $x \in [0,1]$. That is why a sequence of functions can be thought of as an indexed sequence. And the point-wise notion of convergence to $f(x)$ is nothing other than indexing the converged value by x. Back to the previous example, for $x = 0.5$, if we know that $\{f_n(0.5)\}_n$ convergence, the symbol $f(0.5)$ is the most appropriate symbol to represent the limit. Then for $x = 0.6$, if $\{f_n(x)_n\}$ converges, we can say it converges to $f(0.6)$. So the point-wise convergence is simply saying that all of the indexed sequences converge to a set whose values are indexed by the same index as the index of the corresponding sequence.

Because of our discussion above, it is not surprising if we say that there is no guarantee that some good properties of f_n carries over to f (like continuity, differentiability, integrability, etc). That is because the point-wise notion of convergence is only expressing the convergence of a bunch of sequences (indexed with real numbers in [0,1]) in a neat way.

As we will discover through this chapter, it turns out that the question about if the properties of the function f_n carries over the properties of the f through the point-wise convergence, is essentially the same type of question that if in a multivariate limit, exchange of limit order is allowed.

■ **Example 7.1 — Point-wise convergence does not guarantee differentiability.** Let $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \frac{1}{n}\sin(2^n x).$$

For this function we have

$$\lim_{n\to\infty} f_n(x) = 0, \quad \forall x \in \mathbb{R}.$$

However, by calculating the derivative, we will have

$$f'(x) = \frac{2^n}{n}\cos(2^n x),$$

that does not converge point wise. ■

■ **Example 7.2 — Point-wise convergence does not guarantee continuity.** Let $\{f_n\}$ be a sequence of functions $f : [0,1] \to \mathbb{R}$ where $f_n(x) = x^n$. Then it is immediate that $f_n(x) \to 0$ for all $x \in [0,1)$ and $f_n(1) \to 1$. Thus $f_n(x) \to f(x)$ where

$$f(x) = \begin{cases} 0 & x \in [0,1), \\ 1 & x \in 1. \end{cases}$$

It is clear that all of f_n are continuous but the limiting function f is not. ■

■ **Example 7.3 — Point-wise convergence does not guarantee integrablity.** Let $f : (0,1] \to \mathbb{R}$ defined as

$$f_n(x) = \begin{cases} n & 0 < x < \frac{1}{n}, \\ 0 & \frac{1}{n} \leq x \leq 1 \end{cases}$$

This function converges to $f(x) \equiv 0$ point wise on $(0,1]$. However,

$$\lim_{n\to\infty} \int_0^1 f_n(x) = 1$$

whereas

$$\int_0^1 f(x) = 0.$$

Thus the limiting function (point-wise) do not have the same integral as the limit of the integrals of the functions in the sequence. z1 ■

Definition 7.2 — Uniform convergence. let $\{f_n\}$ be a sequence of functions $f : [0,1] \to \mathbb{R}$. Then we say that the sequence converges to $f : [0,1] \to \mathbb{R}$ if

$$\forall \epsilon > 0, \ \exists N > 0 \text{ s.t. } \forall n > N \text{ we have} \qquad |f_n(x) - f(x)| < \epsilon \quad \forall x \in [0,1].$$

We show this by $f_n \to f$.

■ **Remark** Note the order of the quantifiers and compare that closely with the definition of the point-wise convergence. In the uniform convergence, for any choice of ϵ, we can find N that for all $n > N$ we have $|f_n(x) - f(x)| < \epsilon$ that holds true for any $x \in [0,1]$. Thus in some sense, the function f_n converges to f as a whole, and not in a point by point sense. This is also evident in the notation that we use to demonstrate the uniform convergence $f_n \to f$, that gives the feeling that the functions f_n converges to f as a whole object.

Theorem 7.1 — Cauchy criteria of convergence. Let $\{f_n\}_n$ be a sequence of functions $f_n : E \to \mathbb{R}$. This sequence converges to f uniformally, if and only if $\forall \epsilon > 0$, $\exists N > 0$ s.t. $\forall n, m > N$ we have $|f_n(x) - f_m(x)| < \epsilon \; \forall x \in E$.

Proof. $\boxed{\Longrightarrow}$ For this direction, we need to show that uniform convergence imply the sequence to be Cauchy. Let $\epsilon > 0$ given. Then since $f_n \to f$, then we can find integer N such that for all $n, m > N$ and $x \in E$

$$|f_n(x) - f(x)| < \epsilon/2, \qquad |f_m(x) - f(x)| < \epsilon/2$$

Thus we can write

$$|f_n(x) - f_m(x)| \le |f_n(x) - f(x)| + |f_m(x) + f(x)| < \epsilon$$

$\boxed{\Longleftarrow}$ For this direction, we need to prove that the Cauchy criteria satisfied implies the uniform convergence. First, observe that for any fixed x, the sequence $\{f_n(x)\}$ is Cauchy in $\mathbb{R}$. Thus it converges to some real number $f(x)$. Given $\epsilon > 0$ choose N large enough that $\forall n, m > N$ we get

$$|f_n(x) - f_m(x)| < \epsilon/2 \qquad \forall x \in E.$$

Then, for all $x \in E$ we can write

$$|f_n(x) - f(x)| \le |f_n(x) - f_m(x)| + |f_m(x) - f(x)|.$$

The first term in the right hand side is less than $\epsilon/2$. And since $f_m(x) \to f(x)$, we can choose m large enough that $|f_m(x) - f(x)| < \epsilon/2$. Thus we will get

$$|f_n(x) - f(x)| < \epsilon.$$

This completes our proof. □

The following criterion often comes very handy and useful.

Proposition 7.1 Let $\{f_n\}$ be a sequence of functions defined on E that converges point-wise to $f(x)$. Define

$$M_n = \sup_{x \in E} |f_n(x) - f(x)|.$$

Then $f_n \to f$ if and only if $M_n \to 0$.

.

Proof. $\boxed{\Longleftarrow}$ For this direction, as assume $M_n \to 0$ and we prove that $f_n \to f$. For given $\epsilon > 0$ we can find $N > 0$ such that for all $n > N$ we have

$$M_n < \epsilon.$$

By the properties of the suprimum, we know that $\forall x \in E$ we have $|f_n(x) - f(x)| < M_n$. This implies

$$|f_n(x) - f(x)| < \epsilon \qquad \forall x \in E.$$

Thus we conclude $f_n \to f$.
$\boxed{\Longrightarrow}$ For this direction we assume $f_n \to f$ and we prove that $M_n \to 0$. For a given $\epsilon > 0$ we can find $N > 0$ such that for all $n > N$ we have

$$|f_n(x) - f(x)| < \epsilon \qquad \forall x \in E.$$

Then it follows from the definition of suprimum that

$$M_n = \sup_{x \in E} |f_n(x) - f(x)| \le \epsilon.$$

This implies that $M_n \to 0$ as $n \to \infty$. □

7.2 Uniform Convergence and Continuity

As we say in Example 7.2 it is very easy to show that by point-wise convergence, the continuity of the functions does not carry over to the limiting function. This example is a very simple and good example where the continuity of a sequence of functions does not carry over to the limiting function. The following remark also shows the fact that asking about the continuity of the limiting function is in fact a question about the possibility of the exchange of the orders of the limit.

■ **Remark** Let $\{f_n\}$ be a set of functions defined on E. Assume that all of these functions are continuous on E, i.e. $\forall x \in E$ we have

$$\lim_{t\to x} f_n(t) = f_n(x),$$

and by using the fact that $f_n(x) \to f$ we can write

$$\lim_{n\to\infty} \lim_{x\to t} f_n(t) = \lim_{n\to\infty} f_n(x) = f(x).$$

However, if $f(x)$ is continuous, then we can write

$$\lim_{t\to x} f(t) = f(x).$$

but since $f_n(t) \to f(t)$, then we can re-write the expression above as

$$\lim_{t\to x} \lim_{n\to\infty} f_n(t) = f(x).$$

Combining this with the result above we can get

$$\boxed{\lim_{n\to\infty} \lim_{x\to t} f_n(t) \stackrel{?}{=} \lim_{x\to t} \lim_{n\to\infty} f_n(t)}.$$

The following theorem shows the relation between uniform convergence and continuity.

Theorem 7.2 — Uniform convergence allows the exchange of the order of the limits. Let $\{f_n\}$ be a sequence of functions defined on E a subset of a metric space (M, d). Assume $f_n \to f$ uniformally where f is also defined on E. Let x be a limit point of E and suppose that

$$\lim_{t\to x} f_n(t) = A_n \qquad (n = 1, 2, 3, \cdots).$$

Then $\{A_n\}$ converges and

$$\lim_{n\to\infty} A_n = \lim_{t\to x} f(t),$$

which is equivalent to

$$\lim_{n\to\infty} \lim_{t\to x} f_n(t) = \lim_{t\to x} \lim_{n\to\infty} f_n(t).$$

Proof. First part:
First, we need to prove that the sequence $\{A_n\}$ converges. We will show $\{A_n\}$ is Cauchy by using the fact that $f_n \to f$ uniformally. We can do this in two way, where the first one is much more high level (in the sense that it is close to everyday conversation) and the second way is much more verbose revealing the underlying gears. I have seen that Rudin sometimes uses the first way of reasoning in his proofs which is very fast and easy to follow.
Let $\epsilon > 0$ given. Then $\exists N > 0$ such that $\forall n, m > N$ we have

$$|f_n(t) - f_m(t)| < \epsilon \qquad \forall t \in E.$$

Now taking limit $t \to x$ we will get

$$|A_n - A_m| < \epsilon.$$

This shows that $\{A_n\}$ is Cauchy, this converges.
To reveal what is happening behind the scene of the reasoning above (when we simply say taking limit $t \to x$), we provide the following reasoning. Given $\epsilon > 0$, choose N_1 large enough that $\forall n, m > N_1$ we have

$$|f_n(t) - f_m(t)| < \epsilon/3 \qquad \forall t \in E.$$

We can always find such N_1 because $f_n \to f$ by hypothesis.
Now, let x be a limit point of E. Then by choosing t close enough to x we will get

$$|f_n(t) - A_n| < \epsilon/3, \qquad |f_m(t) - A_m| < \epsilon/3.$$

For all $t \in E$ we can write

$$|A_n - A_m| = |A_n - A_m \pm f_n(t) \pm f_m(t)| \leq |f_n(t) - A_n| + |f_m(t) - A_m| + |f_n(t) - f_m(t)|.$$

Now by choosing n, m large enough, and t close enough to x we will get

$$|A_n - A_m| < \epsilon.$$

This shows that the sequence $\{A_n\}$ is a real Cauchy sequence, thus converges.
Second part:
Now, we want to show that $\lim_{t\to x} f(t) \to A$ as $n \to \infty$, where x is a limit point of E, and A is the limit of A_n as $n \to \infty$. I.e. $\lim_{n\to} A_n = A$. For given $\epsilon > 0$ and all $t \in E$, we can write

$$|f(t) - A| \leq \underbrace{|A - f_n(t)|}_{\leq |A_n - A| + |A_n - f_n(t)|} + |f(t) - f_n(t)| \leq |A_n - A| + |A_n - f_n(t)| + |f(t) - f_n(t)|.$$

We are in a good shape now, since each of the terms above can be controlled. To be more specific, since $f_n \to f$ uniformally, and $\lim_{n\to\infty} A_n = A$, then we can choose n large enough such that

$$|f(t) - f_n(t)| < \frac{\epsilon}{3}, \quad |A_n - A| < \frac{\epsilon}{3}, \qquad \forall t \in E.$$

And for this choice of n, since $\lim_{t\to x} f_n(t) = A_n$, we can choose t close enough to x such that

$$|A_n - f_n(t)| < \frac{\epsilon}{3}.$$

Putting all the pieces together, we shown that for given $\epsilon > 0$, we can choose t close enough to x such that

$$|f(t) - A| < \epsilon.$$

Thus $f(t) \to A$ as $t \to x$. Thus in a nutshell

$$\lim_{n\to\infty} \lim_{t\to x} f_n(t) = \lim_{t\to x} \lim_{n\to\infty} f_n(t).$$

□

The theorem that we proved above is a very important theorem, as many important conclusions will follow as a kind of immediate corollary from that. The following important result is one of those.

Corollary 7.1 Let $\{f_n\}$ be a sequence of continuous functions that converges uniformally to f on E. Then f is continuous.

Proof. This is an immediate corollary of the Theorem 7.2. To be more explicit, since $f_n \to f$ uniformally on E, then we can do the following exchange of limits.

$$\lim_{n\to\infty} \underbrace{\lim_{t\to x} f_n(t)}_{f_n(x)} = \lim_{t\to x} \underbrace{\lim_{n\to\infty} f_n(t)}_{f(t)}.$$

where we have used the fact that f_n are all continuous. Thus we have

$$\lim_{t\to x} f(t) = f(x).$$

This implies the continuoity of f and completes the proof. ☐

■ **Remark** Note that the converse of the corollary above is not true. I.e. we can have a set of continuous functions $\{f_n\}$ that converges point-wise to a continuous function f, but the converges fails to be uniform. The case we studied in Example 7.3 is of this type. You can easily see this fails to be a uniform convergence by applying Proposition 7.1.

In the remark above we say that the converse of the corollary above is not always true. However, the following proposition provides the necessary conditions under which the converse of the corollary above is also true.

Proposition 7.2 — Dini's Theorem. Let K be a **compact** set, and

(i) $\{f_n\}$ is a sequence of **continuous** functions.
(ii) $f_n(x)$ converges to $f(x)$ **point-wise** on K, where f is also **continuous**.
(iii) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$ and $n = 1, 2, \ldots$.

Then f_n is converging uniformally to f on K.

Proof. Define an auxiliary function $g_n(x) = f_n(x) - f$. Because $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, and since $f_n(x) \to f(x)$ for all $x \in K$ then we conclude that $g_n(x) \geq g_{n+1}(x)$. Also, observe that because f_n and f are both continuous on K, then g_n is also continuous on K for all $n \in \mathbb{N}$. To prove the theorem, it is equivalent to prove that $g_n \to 0$ uniformally.
Let $\epsilon > 0$ given. Define

$$K_n = \{x \in K : \ g_n(x) \geq \epsilon\}.$$

Since g_n is continuous, then K_n is closed. To show this, consider K_n^c, which is the set of point $x \in K$ such that $g_n(x) < \epsilon$. Since g_n is continouse, we can make small enough perturbation to x and still the value of function g_n less than ϵ. Thus K_n^c is open, implying the set K_n is closed. Since $K_n \subseteq K$ and K_n is closed, thus K_n is also compact. On the other hand, since $g_n(x) \geq g_{n+1}(x)$, then we have $K_n \supset K_{n+1}$.
Since $g_n(x) \to 0$ point wise, then for any $x \in K$, we have $x \notin K_n$ if n is large enough. Thus $x \notin \bigcap K_n$. This implies $\bigcap K_n = \varnothing$. From the nested intervals theorem for real intervals, then we conclude that $K_N = \varnothing$ for all N large enough (otherwise, their infinite intersection will contain at least one element). This proves that for all n large enough, we have

$$0 \leq g_n(x) < \epsilon.$$

This implies the uniform convergence. ☐

■ **Remark** Note that the third condition in the Dini's theorem can be also $f_n(x) \leq f_{n+1}(x)$. The proof will be similar to the prove provided above.

The following example emphasizes that fact that the compactness is really needed for the proposition above.

■ **Example 7.4** The function $f_n(x) = \frac{1}{nx+1}$ converges point wise and monotonically to 0 on (0,1), but the convergence fails to be uniform. ■

■ **Example 7.5 — Non-example for monotonicity (the third condition).** Consider the following function

$$f_n(x) = \begin{cases} 0 & x \in [0, \frac{1}{2n}), \\ 4nx - 2 & x \in [\frac{1}{2n}, \frac{3}{4n}), \\ 4 - 4nx & x \in [\frac{3}{4n}, \frac{1}{n}), \\ 0 & x \in [\frac{1}{n}, 1]. \end{cases}$$

This sequence if functions converge point wise to the zero function ($f \equiv 0$) on $[0,1]$ which is a compact set. Note that f_n and f are both continuous. But the convergence fails to be uniform since $\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1$ for all $n \in \mathbb{N}$, which implies the convergence is not uniform[1]. ■

7.2.1 Exploiting some of the algebraic structures

At this stage, we can exploit some of the algebraic structures of the objects we are working with. To do this, we first start with the following definition.

Definition 7.3 Let X be a metric space. We denote with $C(X)$ the following set of functions.

$$C(X) = \{f : X \to \mathbb{C} : \ f \text{ is } \textbf{continuous} \text{ and } \textbf{bounded}\}.$$

■ **Remark** Note that if X is compact, then boundedness of f is for free.

The following proposition reveals the underlying algebraic structure of this set.

Proposition 7.3 The set $C(X)$ is a vector space for which we can define the following norm

$$\|f\| = \sup_{x\in X} |f(x)|.$$

Proof. Part 1: Showing $C(X)$ is a vector space. It is very straight forward to show this, and we just need to check the properties of the vector spaces to see if they are satisfied. Here, I will point out the important properties. First, note that we have a zero element, which is the zero function. Also, note that if a function f is continouse and bounded, then the function $-f$ defined as $-f(x) = -1 * f(x)$. The rest is very easy to verify.
Showing $\|f\|$ satisfies the norm properties. First, we need to show that the norm defined as above is positive definite, i.e. $\|f\| = 0$ if and only if $f \equiv 0$. $f \equiv 0$ implying $\|f\| = 0$ is trivial, and $\|f\| = 0$ implying $f \equiv 0$ follows from the definition of the suprimum. Also, from definition of suprimum is follows that $\|kf\| = |k|\|f\|$ for $k \in \mathbb{C}$. Lastly, we need to show the triangle inequality, which follows from the fact that for $h(x) = f(x) + g(x)$ we have $|h(x)| \leq |f(x)| + |g(x)|$. Then it follows from the properties of the suprimum that $\|h\| \leq \|f\| + \|g\|$. □

Thus we saw that $C(X)$ is a normed linear space. The following Lemma shows that we can easily define $C(X)$ to be also a metric space.

[1] This example is taken from Math CounterExample.

Lemma 7.1 Let V be a vector space with a norm $\|\cdot\|$. Then

$$d(x,y) = \|x-y\|,$$

satisfies the properties of a metric.

Proof. To be added later. $\square$

With the Lemma above in hand, we can have the following definitions.

Definition 7.4 Let $C(X)$ be the set of all complex valued, bounded, and continuous functions defined on a metric space X. $C(X)$ is a metric space with a metric

$$d(f,g) = \sup_{x\in X} |f(x)-g(x)|.$$

With the definition above in hand, we can put Proposition 7.1 in a different way

> A sequence $\{f\}$ converges to f with respect to the metric of $C(X)$ if and only if $f_n \to f$ uniformally on X.

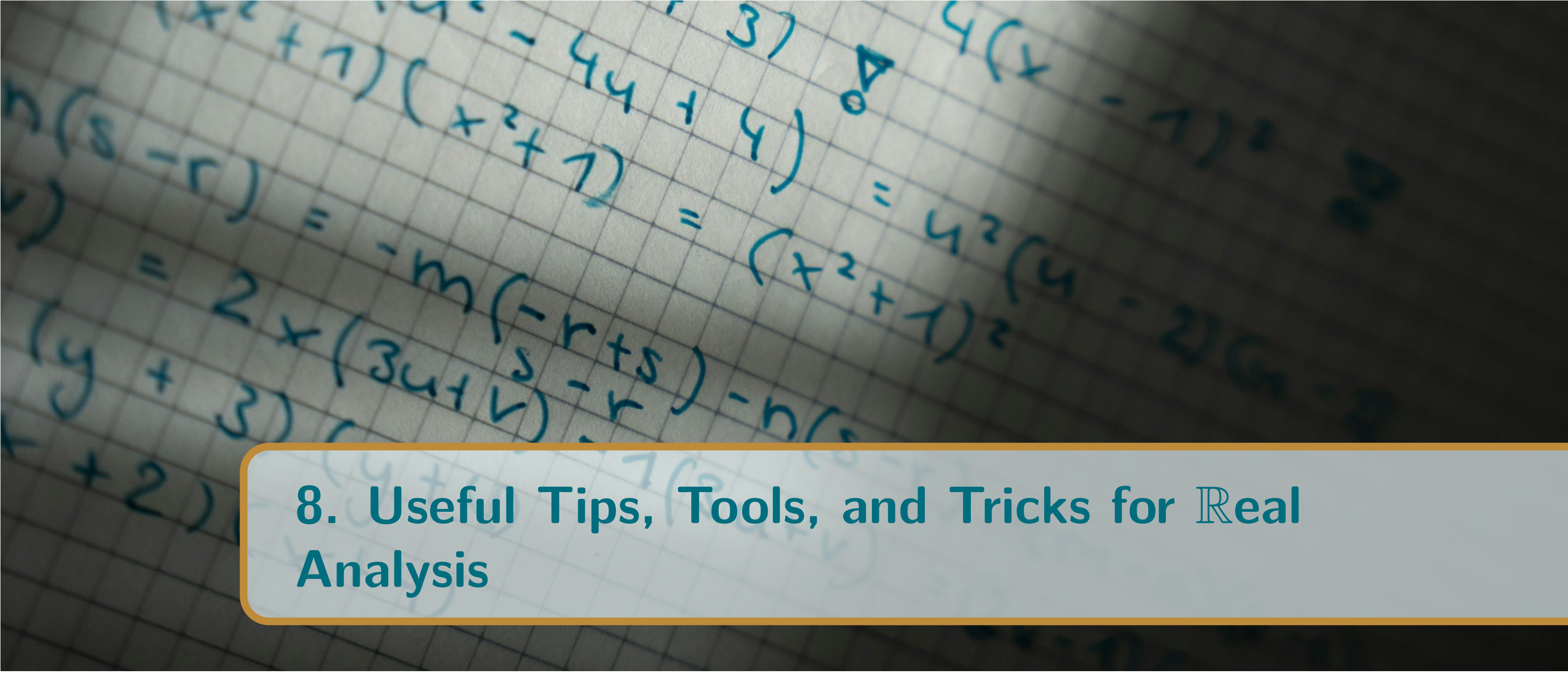
The following theorem is very central.

Theorem 7.3 The metric space $C(X)$ with metric

$$d(f,g) = \|f-g\|$$

where $\|\cdot\|$ is the sup-metric, is **complete**.

Proof. The proof of this theorem is nothing more than gluing together the pieces of theorems and propositions that we have had before. Let $\{f_n\}$ be a Cauchy sequence in $C(X)$. Thus by Theorem 7.1, we can conclude that converges uniformally to some f. We now need to show that $f \in C(X)$. First, since f_n is continuous for all $n = 1, 2, 3, \cdots$, by Corollary 7.1 we conclude that f is also continuous. Secondly, since $f_n \to f$ uniformally, then for all n large enough we have $|f_n(x) - f(x)| < 1$ for all $x \in X$. Thus f is bounded, i.e. $\sup_{x\in X} |f(x)| < M$. This implies $d(f,0) < M$ which implies f is also bounded (i.e. there exists some open ball with finite radius that contains f). This shows that the limiting object f is both bounded and continuous. This $f \in C(X)$. This implies that the metric space $C(X)$ is complete. $\square$



8. Useful Tips, Tools, and Tricks for $\mathbb{R}$eal Analysis

In this chapter we will cover some useful tips and tricks in solving the problem. This is not a complete and comprehensive list of tricks, and the list will grow larger as I encounter more of these in different resources.

Trick 8.1 There is a way to produce multiplication from squaring and addition/difference. Suppose that a set S is closed under squaring, as well as addition and subtraction. Thus if $a, b \in S$ then $a + b \in S$, $a - b \in S$, thus $(a-b)^2 \in s$ as well as $(a+b)^2 \in S$. Thus $(a+b)^2 - (a-b)^2 \in S$ as well. Thus

$$(a+b)^2 - (a-b)^2 = 4ab \in S.$$

Thus we can conclude that being closed under addition, subtraction, and squaring, leads to being closed under multiplication as well.

One useful use of this trick is in when we try to prove if f and g are RiemannStieltjes integrable, then we want prove that fg is also RiemannStieltjes integrable. We do so by observing that f, g are RS integrable, then so is $f + g$ and $f - g$ (which are proved by simpler theorem) and also f^2 is RS integrable. Thus from the trick above we can conclude that fg is also RS integrable.

8.1 Construction Zone

Some tricks for real analysis. I will expand this list and also explain each one with its particular use case throughout this chapter

1. archimedes
2. well ordering principle
3. unpack defn
4. IVT
5. MVT
6. three equiv. defn for cts.
7. sequential compactness

8. heine borel
9. bolzano weierstrass
10. lim $x_n = L$ iff limsup $x_n = \liminf x_n = L$
11. sequential characterization of open, closed sets, and others
12. union of countable set is countable
13. add subtrace and multiply divide
14. define new set or function and play with it
15. construct sequence
16. triangle inequality
17. X compact then f:X->R has max/min in X and f(X) compact
18. work backwards and apply previous results
19. contrapositives
20. denseness of Q in R
21. finite intersection prop
22. cantor intersection thm
23. squeeze thm
24. limit laws
25. monotone convergence thm
26. show seq cauchy (by completeness)
27. diff between squares
28. exploit the defn of convergence
29. consider all cases
30. maybe try proof by contradiction
31. ep ball defn of x in cl(A)
32. if A is subset of B and B closed then cl(A) is subset of B
33. if A is subset of B and A open then A subset of int(A)
34. HTS1-HTS4