Topology

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1. Topological Spaces and Continuous Function

Definition 1.1 — Topology on a set. Let X be a set. A topology on X is a collection of subsets of X, called *open sets* and denoted by $\mathcal{T}$, that satisfies

- $X, \varnothing \in \mathcal{T}$,
- For an *arbitrary* collection of open sets $\{U_\alpha\}_{\alpha \in J}$ we have
$$\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}.$$
- For a *finite* collection of open sets $\{U_1, \cdots, U_n\}$ for some $n \in \mathbb{N}$ we have
$$\bigcap_{i=1}^{n} U_i \in \mathcal{T}$$

Definition 1.2 — Finer and Coarse Topologies. Let X be a set and $\mathcal{T}_1$ and $\mathcal{T}_2$ two topologies. The we say the topology $\mathcal{T}_1$ is finer if we have $\mathcal{T}_\in \subset \mathcal{T}_\infty$. Or alternatively, we can call the topology $\mathcal{T}_2$ to be coarser.

Definition 1.3 — Bases for a topology. Let X be a set. A collection of subsets of X, denoted by $\mathbb{B}$, is called a basis for the topology on X if we have

- $\forall x \in X$ there exists $B \in \mathbb{B}$ such that $x \in B$.
- If for sets $B_1, B_2 \in \mathbb{B}$ we have $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathbb{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Definition 1.4 — Topology Generated by a Basis. Let X be a set and $\mathbb{B}$ a basis for a topology. We *define the topology $\mathcal{T}$ generated by* $\mathbb{B}$ as follow: For any $U \in \mathcal{T}$ and $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

■ **Remark 1.1** We can now check that if $\mathcal{T}$ generated by $\mathbb{B}$ is really a topology. We need to show that $\mathcal{T}$ has the topology properties.

- The statement $\varnothing \in \mathcal{T}$ is vacuously true. To see $X \in \mathcal{T}$, Consider $x \in X$. Then from the definition of a basis $\exists B \in \mathbb{B}$ such that $x \in B$ and since X is the whole space, thus $x \in B \subset X$.

- Let $\{U_\alpha\}_{\alpha \in J}$ be an arbitrary collection of sets in $\mathcal{T}$. Consider $x \in \bigcup_\alpha U_\alpha$. Then there exists an index α such that $x \in U_\alpha$. Since $U_\alpha \in \mathcal{T}$ thus there exists $B \in \mathbb{B}$ such that $x \in B \subset U_\alpha$ which implies $x \in B \subset \bigcup_\alpha U_\alpha$, hence $\bigcup_\alpha U_\alpha \in \mathcal{T}$.
- To see that the intersection of a finite collection of sets in $\mathcal{T}$ is also in $\mathcal{T}$, first if $U_1 \in \mathcal{T}$ and $U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$. That is because for $x \in U_1 \cap U_2$ we know that $\exists B_1, B_2 \in \mathbb{B}$ such that $x \in B_1 \cap B_2$. Thus from the definition of a basis $\exists B_3 \in \mathbb{B}$ such that $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$, hence $U_1 \cap U_2 \in \mathcal{T}$. Then by induction we can conclude that for any finite collection of sets in $\mathcal{T}$ their intersection is also in $\mathcal{T}$.

Proposition 1.1 — Characterization of Open Sets in Terms of Basis. Let X be a set and $\mathbb{B}$ a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the collection of all unions of $\mathbb{B}$.

Proof. Let $\mathcal{A}$ be the collection of all unions of $\mathbb{B}$. We need to show the equality of the sets

$$\mathcal{A} = \mathcal{T}.$$

Let $A \in \mathcal{A}$. Thus $A = \bigcup_\alpha B_\alpha$ for $B_\alpha \in \mathbb{B}$. Let $x \in A$. Then we can choose any B_α and we will have $x \in B_\alpha \subset A$. Thus $A \in \mathcal{T}$. To show the converse let $A \in \mathcal{T}$. Thus for any $x \in A$ we have $B_x \in \mathbb{B}$ such that $x \in B_x \subset A$. We can write

$$A = \bigcup_{x \in A} B_x.$$

Thus $A \in \mathcal{A}$. This completes the proof. □

Be Careful Here! 1.0.1 — Be Aware of the Terminology. From the proposition above, we can say that a subset $U \subset X$ is open (i.e. $U \in \mathcal{T}$) if it can be written as union of sets in $\mathbb{B}$. This union, however, need not be unique. This is a very important difference with the notion of basis in linear algebra, where given a basis for a space, we can then write any vector in the space uniquely by the basis vectors.

1.1 Solved Problems

■ **Problem 1.1 — Finite Complement Topology.** Let X be a set and let $\mathcal{T}_f$ be the collection of all subsets U of X such that $X - U$ is either finite ir is all of X. Show that $\mathcal{T}_f$ is a topology on X.

Solution First, observe that since X is the whole space, for any subset $U \subset X$ we have $X - U = U^c$. So we can write $\mathcal{T}_f$ as

$$\mathcal{T}_f = \{U \subset X \mid U^c = X \text{ or } U^c \text{ is finite}\}.$$

We need to check if $\mathcal{T}_f$ has the properties of topology.

- It is immediate that $\varnothing \in \mathcal{T}_f$ and $X \in \mathcal{T}_f$.
- Let $\{A_\alpha\}_{\alpha \in I}$ be an arbitrary collection where $A_\alpha \in \mathcal{T}_f$ for all $\alpha \in I$. Let $C = \bigcup_\alpha A_\alpha$. For C to be in $\mathcal{T}_f$, C^c needs to be X or finite.

$$C^c = \bigcap_\alpha A_\alpha^c.$$

  Since all A_α are in $\mathcal{T}_f$, then A_α^c are X or finite. Thus C^c is also X or finite, hence $C \in \mathcal{T}_f$

- Let $\{A_1, \cdots, A_n\}$ be finite collection of open sets. Consider $D = \bigcap_i A_i$. Since

$$D^c = \bigcup_i A_i^c,$$

and A_i^c are finite or X, then D^c is also finite or X, hence $D \in \mathcal{T}_f$.

■ **Problem 1.2** Construct a set and give it a finite complement topology.

Solution Consider the set

$$X = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}.$$

The collection of sets $\mathcal{T} = \{A_n\}_{n \in \mathbb{N}}$, where

$$A_n = X - \{\frac{1}{1}, \frac{1}{2}, \cdots, \frac{1}{n}\},$$

is a finite complement topology.