

Lecture Notes For: Advanced Linear Algebra

Ali Fele Paranj
alifele@student.ubc.ca

February 5, 2023

Chapter 1

Matrices

1.1 What is a Matrix

A matrix is basically a notation convention that enables us to do some stuff more easily with a pencil and paper. A very similar concept to this is the long division algorithm for dividing two integers. For example consider the following long division (in French-European style) that we are all familiar with

$$\begin{array}{r|l} 198 & 12 \\ -12 & 16,5 \\ \hline 78 & \\ -72 & \\ \hline 60 & \\ -60 & \\ \hline 0 & \end{array}$$

So this notation and algorithms is to use some calculations more convenient when is done by hand with a pen and paper. So the matrix notation can also be thought as a computation convention. To make stuff more clear, consider the following example.

Example: Simple Pen and Paper Calculations

Consider V which is written as:

$$V = 2A + 3B + 4C$$

Given the following relation between A , B , and C , rewrite V in terms of x , y , and z .

$$A = x + 2y + 3z$$

$$B = 2x - y + z$$

$$C = -x - y + z$$

Solution 1.

To write V in terms of x , y , and z we write:

$$V = 2(x + 2y + 3z) + 3(2x - y + z) + 4(-x - y + z) \quad (1.1.1)$$

By arranging the terms using simple algebra we will have:

$$V = (2 + 6 - 4)x + (4 - 3 - 4)y + (6 + 3 + 4)z = 4x - 3y + 13z \quad (1.1.2)$$

Solution 2.

The calculations described in the first solution are not systematic. What I mean is that we started doing whatever we can do with you thinking about doing it in a more smart way that can also be systematically scaled to larger equations. This is where the matrices come into play. Matrices help us to do such calculations in a more algorithmic way (like the long division notation in which we do the calculations in an algorithmic way).

Let \mathbb{B} be the set of all *objects* that the V is expanded in terms of and call this set as the *basis* set. So for $\mathbf{V} = 2A + 3B + 4C$ we have the basis

$$\mathbb{B}_1 = \{A, B, C\}.$$

We can arrange the coefficients of V in basis \mathbb{B}_1 in the following way:

$$V_{\mathbb{B}_1} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}_{\mathbb{B}_1}$$

We call it the coordinates of V in the basis \mathbb{B}_1 . Since we want to write the vector V in terms of x, y , and z , we need to introduce the new basis \mathbb{B}_2 in the following way:

$$\mathbb{B}_2 = \{x, y, z\}$$

Since A, B , and C are expressed in terms of x, y , and z , we can arrange the coordinates of A, B , and C in the basis \mathbb{B}_2 in the following way:

$$L_{\mathbb{B}_1}^{\mathbb{B}_2} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2}$$

in which every column is the coefficients A, B , and C in the basis \mathbb{B}_2 respectively. Note the subscript and the superscripts of the matrix. This matrix means that its columns contains the coordinates of the basis \mathbb{B}_1 in the new basis \mathbb{B}_2 . So when it is applied to any vector that is described in basis \mathbb{B}_1 , we will get the components of that vector in the basis \mathbb{B}_2 . In other words:

$$V_{\mathbb{B}_2} = L_{\mathbb{B}_1}^{\mathbb{B}_2} V_{\mathbb{B}_1}$$

$$V_{\mathbb{B}_2} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}_{\mathbb{B}_1} \quad (1.1.3)$$

Considering the basic operations introduced with matrix notation, this matrix equation can be written in two ways as described below:

$$V_{\mathbb{B}_2} = 2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_{\mathbb{B}_2} + 3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}_{\mathbb{B}_2} + 4 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}_{\mathbb{B}_2} \quad (1.1.4)$$

The equation above is equivalent to equation 1.1.1 but described in other way! Also the other way to write the matrix equation 1.1.3 is the following way in which we have used the matrix multiplication conventions:

$$V_{\mathbb{B}_2} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix}_{\mathbb{B}_2} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}_{\mathbb{B}_1} = \begin{pmatrix} (1 * 2) + (2 * 3) + (-1 * 4) \\ (2 * 2) + (-1 * 3) + (-1 * 4) \\ (3 * 2) + (1 * 3) + (1 * 4) \end{pmatrix}_{\mathbb{B}_2} = \begin{pmatrix} 4 \\ -3 \\ 13 \end{pmatrix}_{\mathbb{B}_2} \quad (1.1.5)$$

which is essentially equivalent to the equation 1.1.2 but written in a different way.

1.2 Linear Operators and Matrices

Consider the vector spaces \mathcal{V} and \mathcal{W} by are spanned with $\mathbb{B}_{\mathcal{V}} = \{|a_i\rangle\}_{i=1}^N$ and $\mathbb{B}_{\mathcal{W}} = \{|b_j\rangle\}_{j=1}^M$ respectively. The act of a linear operator $A \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ on the vector $|v\rangle \in \mathcal{V}$ can be specified by its act on the basis of that vector space $\mathbb{B}_{\mathcal{V}}$. Suppose that A acts on the basis vectors $|a_i\rangle$ as the following:

$$|w_i\rangle = A|a_i\rangle = \sum_{j=1}^M \rho_{ji} |b_j\rangle \quad (1.2.1)$$

in which ρ_{ji} are the components of $|w_i\rangle$ in the basis $\mathbb{B}_{\mathcal{W}}$ and can be organized as column vectors:

$$\underline{w_i}_{\mathbb{B}_{\mathcal{W}}} = \begin{pmatrix} \rho_{1i} \\ \rho_{2i} \\ \vdots \\ \rho_{Mi} \end{pmatrix}_{\mathbb{B}_{\mathcal{W}}}$$

Now if A acts on the vector $|v\rangle = \sum_{i=1}^N \alpha_i |a_i\rangle$ can be written as:

$$|u\rangle = A|v\rangle = A \sum_{i=1}^N \alpha_i |a_i\rangle = \sum_{i=1}^N \alpha_i A|a_i\rangle = \sum_{i=1}^N \alpha_i \sum_{j=1}^M \rho_{ji} |b_j\rangle = \sum_{j=1}^M \underbrace{\sum_{i=1}^N \alpha_i \rho_{ji}}_{\eta_j} |b_j\rangle = \sum_{j=1}^M \eta_j |b_j\rangle$$

so the coordinates of the vector $|u\rangle = A|v\rangle$ will be:

$$\eta_j = \sum_{i=1}^N \alpha_i \rho_{ji} \quad (1.2.2)$$

which also can be written as:

$$\underline{u}_{\mathbb{B}_{\mathcal{W}}} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_j \end{pmatrix}_{\mathbb{B}_{\mathcal{W}}}$$

The above equation is very important and can be shown in matrix notation to make the idea more clear. To do this we need to build matrix $A_{\mathbb{B}_{\mathcal{V}}}^{\mathbb{B}_{\mathcal{W}}}$ by arranging the components of $\underline{u}_{\mathbb{B}_{\mathcal{W}}}$ as the columns of $A_{\mathbb{B}_{\mathcal{V}}}^{\mathbb{B}_{\mathcal{W}}}$:

$$A_{\mathbb{B}_{\mathcal{V}}}^{\mathbb{B}_{\mathcal{W}}} = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{M1} & \rho_{M2} & \cdots & \rho_{MN} \end{pmatrix}_{\mathbb{B}_{\mathcal{V}}}^{\mathbb{B}_{\mathcal{W}}} \quad (1.2.3)$$

So the equation 1.2.2 can be written as:

$$\boxed{\underline{u}_{\mathbb{B}_{\mathcal{W}}} = A_{\mathbb{B}_{\mathcal{V}}}^{\mathbb{B}_{\mathcal{W}}} \underline{v}_{\mathbb{B}_{\mathcal{V}}}} \quad (1.2.4)$$

Note that a matrix does not have any meaning by itself and is just a collection of numbers. To correspond any matrix with a linear operator $A \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, we need to fix the basis vectors $\mathbb{B}_{\mathcal{V}}$ and $\mathbb{B}_{\mathcal{W}}$.

1.3 Change of Basis

As we discussed earlier, the matrix representation of a linear operator $A \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ depends on the choice of basis $\mathbb{B}_{\mathcal{V}}$ and $\mathbb{B}_{\mathcal{W}}$ which are the basis of vector spaces \mathcal{V} and \mathcal{W} respectively. Now suppose that that in the vector spaces \mathcal{V} , the elements are described in basis $\mathbb{B}_1 = \{|e_i\rangle\}_{i=1}^N$ but we want to change it to the basis $\mathbb{B}_2 = \{|e'_j\rangle\}_{j=1}^N$. We need to know the relation between these two basis that is assumed to be the following:

$$|e_i\rangle = \sum_{j=1}^N \rho_{ji} |e'_j\rangle \quad (1.3.1)$$

Consider the vector $|v\rangle$ that is described in the basis \mathbb{B}_1 in the following way:

$$|v\rangle = \sum_{i=1}^N \alpha_i |e_i\rangle \quad (1.3.2)$$

$$|v\rangle = \sum_{i=1}^N \alpha_i |e_i\rangle$$

The coefficients of the expansion are called the *coordinates* of $|v\rangle$ in the basis \mathbb{B}_1 and can be shown like:

$$\underline{v}_{\mathbb{B}_1} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix}_{\mathbb{B}_1} \quad (1.3.3)$$

Now we can use the equation 1.3.1 to replace $|e_i\rangle$ in 1.3.2 with $|e'_j\rangle$:

$$|v\rangle = \sum_{i=1}^N \sum_{j=1}^N \alpha_i \rho_{ji} |e'_j\rangle = \sum_{j=1}^N \sum_{i=1}^N \rho_{ji} \alpha_i |e'_j\rangle = \sum_{j=1}^N \eta_j |e'_j\rangle \quad (1.3.4)$$

in which $\eta_j = \sum_{i=1}^N \rho_{ji} \alpha_i$ is the coordinates of $|v\rangle$ in the new basis \mathbb{B}_2 :

$$\underline{v}_{\mathbb{B}_2} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{pmatrix}_{\mathbb{B}_2}$$

The equation 1.3.4 can be written in the following matrix equation:

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{pmatrix}_{\mathbb{B}_2} = \underbrace{\begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{N1} & \rho_{N2} & \cdots & \rho_{NN} \end{pmatrix}}_{R} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix}_{\mathbb{B}_1} \quad (1.3.5)$$

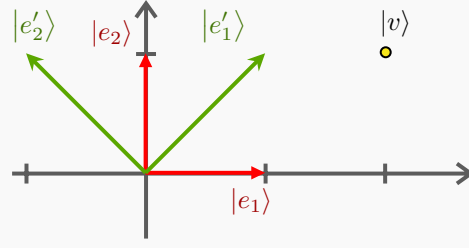
That can also be written in a more compact form:

$$\underline{v}_{\mathbb{B}_2} = R_{\mathbb{B}_1}^{\mathbb{B}_2} \underline{v}_{\mathbb{B}_1} \quad (1.3.6)$$

The matrix R is called the change of basis matrix.

Example: Change of Basis

Consider the following Cartesian plane:



The vectors of the plane can be expressed using any arbitrary basis two of which are provided here as an example:

$$\mathbb{B}_1 = \{|e_1\rangle, |e_2\rangle\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbb{B}_1}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathbb{B}_1} \right\}$$

$$\mathbb{B}_2 = \{|e'_1\rangle, |e'_2\rangle\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbb{B}_2}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathbb{B}_2} \right\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\mathbb{B}_1}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}_{\mathbb{B}_1} \right\}$$

It is clear that the vector $|v\rangle$ can be expanded like:

$$|v\rangle = 2|e_1\rangle + |e_2\rangle \quad (1.3.7)$$

So we can write:

$$\underline{v}_{\mathbb{B}_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}_{\mathbb{B}_1}$$

in which $\underline{v}_{\mathbb{B}_1}$ is the coordinates of $|v\rangle$ in the basis \mathbb{B}_1 . Now suppose that we want to find the coordinates of $|v\rangle$ in the basis \mathbb{B}_2 . To do that we need to write the $|e_1\rangle$ and $|e_2\rangle$ in terms of $|e'_1\rangle$ and $|e'_2\rangle$ (i.e. find the coordinates of elements of \mathbb{B}_1 in the basis \mathbb{B}_2):

$$\begin{aligned} |e_1\rangle &= |e'_1\rangle - |e'_2\rangle \\ |e_2\rangle &= |e'_1\rangle + |e'_2\rangle \end{aligned} \quad (1.3.8)$$

This can be written in the column vector format:

$$\underline{e_{1\mathbb{B}_2}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}_{\mathbb{B}_2}, \quad \underline{e_{2\mathbb{B}_2}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\mathbb{B}_2}$$

By arranging these columns into the columns of matrix we will get the change of basis matrix:

$$\mathbf{R} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2}$$

By inserting 1.3.8 in 1.3.7 we can get the expanded form of $|v\rangle$ in the new basis \mathbb{B}_2 .

$$|v\rangle = 2(|e'_1\rangle - |e'_2\rangle) + (|e'_1\rangle + |e'_2\rangle) = 3|e'_1\rangle - |e'_2\rangle$$

The above calculations can also be done by applying the change of basis matrix R on the coordinates of $|v\rangle$ in \mathbb{B}_1 , i.e. $\underline{v}_{\mathbb{B}_1}$:

$$\underline{v}_{\mathbb{B}_2} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2} \begin{pmatrix} 2 \\ 1 \end{pmatrix}_{\mathbb{B}_1} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}_{\mathbb{B}_2}$$

The idea behind the change of basis matrix and the matrices corresponding to linear operator are very similar and this similarity can lead to confusions. For example the equations 1.2 and 1.3 are very similar but are very different at the same time. In the former case the vector $|v\rangle$ changes to another vector $|w\rangle$ but in the latter case the vector $|v\rangle$ remains unchanged and only gets represented with new coordinates in the new basis. So when we can know that a matrix changes the basis only and keeps the vector unchanged or maps the vector into another vector? The following example would clarify the differences.

Example: Linear Operators Matrices vs. Change of Basis Matrices

Question.

Consider the matrix A

$$A = \begin{pmatrix} -1 & -1 \\ -3 & 3 \end{pmatrix}$$

What is the effect of this matrix on a arbitrary vector $|v\rangle = \alpha_1 |a_1\rangle + \alpha_2 |a_2\rangle$?

Answer.

Case 1.

You might answer "it depends on the vector $|v\rangle$ ". But the truth is that this question is not a correct question. As emphasized before, a matrix by itself is just a collection of numbers and does not have any meaning. By fixing the vector spaces and the corresponding basis vectors, then the matrix A gets its meaning and corresponds to the a linear operator $A \in \mathcal{L}(\mathcal{V}, \mathcal{V})$.

Let's assume:

$$\begin{aligned} \mathbb{B}_1 &= \{|a_1\rangle, |a_2\rangle\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbb{B}_1}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathbb{B}_1} \right\} \\ \mathbb{B}_2 &= \{|b_1\rangle, |b_2\rangle\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbb{B}_2}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathbb{B}_2} \right\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\mathbb{B}_1}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}_{\mathbb{B}_1} \right\} \end{aligned}$$

So the vector $|v\rangle$ can be written as:

$$|v\rangle = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbb{B}_1} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathbb{B}_1}$$

Then the matrix

$$A_{\mathbb{B}_1}^{\mathbb{B}_2} = \begin{pmatrix} -1 & -1 \\ -3 & 3 \end{pmatrix}_{\mathbb{B}_1}^{\mathbb{B}_2}$$

has the following interpretation:

The operator A maps the basis vectors in the following way

$$\begin{aligned} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbb{B}_1} &\xrightarrow{A} \begin{pmatrix} -1 \\ -3 \end{pmatrix}_{\mathbb{B}_2} = -1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\mathbb{B}_1} - 3 \begin{pmatrix} -1 \\ 1 \end{pmatrix}_{\mathbb{B}_1} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathbb{B}_1} &\xrightarrow{A} \begin{pmatrix} -1 \\ 3 \end{pmatrix}_{\mathbb{B}_2} \end{aligned}$$

So $A|v\rangle$ will be:

$$A|v\rangle = \alpha_1 \begin{pmatrix} -1 \\ -3 \end{pmatrix}_{\mathbb{B}_2} + \alpha_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix}_{\mathbb{B}_2}$$

Case 2.

Let's consider the following sets as the basis

$$\begin{aligned} \mathbb{B}_1 &= \{|a_1\rangle, |a_2\rangle\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbb{B}_1}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathbb{B}_1} \right\} \\ \mathbb{B}_2 &= \{|b_1\rangle, |b_2\rangle\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbb{B}_2}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathbb{B}_2} \right\} = \left\{ \begin{pmatrix} -1/6 \\ 1/6 \end{pmatrix}_{\mathbb{B}_1}, \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix}_{\mathbb{B}_1} \right\} \end{aligned}$$