Lecture Notes For: Solved Problems in $\mathbb{R}$eal Analysis

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October 19, 2023

Introduction:
In this document, I have presented interesting questions in $\mathbb{R}$eal analysis. The questions are gathered from different books and websites (all of which are referenced). However, the solutions are written by myself, thus there is always a chance of mistake. Please let me know via email if you spot any.

Contents

1 Sets and Functions

1.1 Basic Set Theory

Question 1. If A, B and C are sets. prove the followings:

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(b) $C \backslash (A \cup B) = (C \backslash A) \cap (C \backslash B)$.

(c) $C \backslash (A \cap B) = (C \backslash A) \cup (C \backslash B)$.

Answer.

(a) *Proof.* Let P, Q, and R be logical statements. We can show (using the truth table) that the following biconditional implication is a tautology.

$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R).$$

Now let $x \in A \cap (B \cup C)$. Then $x \in A \wedge (x \in B \vee x \in C)$. Using the tautology above we can write $(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$, thus $x \in (A \cap B) \cup (A \cap C)$, which means $A \cap (B \cap C) \subset (A \cap B) \cup (A \cap C)$. Conversely, let $x \in (A \cap B) \cup (A \cap C)$. By definition $(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$. With the similar logic as above we can infer $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. Thus $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

(b) *Proof.* Let $x \in C \backslash (A \cup B)$. By definition $x \in C \wedge x \notin (A \cup B) \Leftrightarrow x \in C \wedge x \in \overline{A \cup B} \Leftrightarrow x \in C \wedge x \in \bar{A} \cap \bar{B} \Leftrightarrow x \in C \wedge (x \notin A \wedge x \notin B)$. Finally, using the following tautology

$$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge (P \wedge R),$$

we can write $(x \in C \wedge x \notin A) \wedge (x \in C \wedge x \notin B) \Leftrightarrow x \in (C \backslash A) \cap (C \backslash B)$, thus $C \backslash (A \cup B) subset (C \backslash A) \cap (C \backslash B)$. The converse can be shown is true following the similar logic as below, thus inferring $C \backslash (A \cup B) = (C \backslash A) \cap (C \backslash B)$. □

(c) *Proof.* Let $x \in C \backslash (A \cap B)$. By definition $x \in C \wedge x \notin (A \cap B) \Leftrightarrow x \in C \wedge x \in \overline{A \cap B} \Leftrightarrow x \in C \wedge x \in \bar{A} \cup \bar{B} \Leftrightarrow x \in C \wedge (x \notin A \vee x \notin B)$. Using the tautology in section (a), we can write $(x \in C \wedge x \notin A) \vee (x \in C \wedge x \notin B) \Leftrightarrow x \in (C \backslash A) \cup (C \backslash B)$, thus $C \backslash (A \cap B) \subset (C \backslash A) \cup (C \backslash B)$. The converse can be shown is true following the similar logic as below, thus inferring $C \backslash (A \cap B) = (C \backslash A) \cup (C \backslash B)$. □