Numerical Analysis

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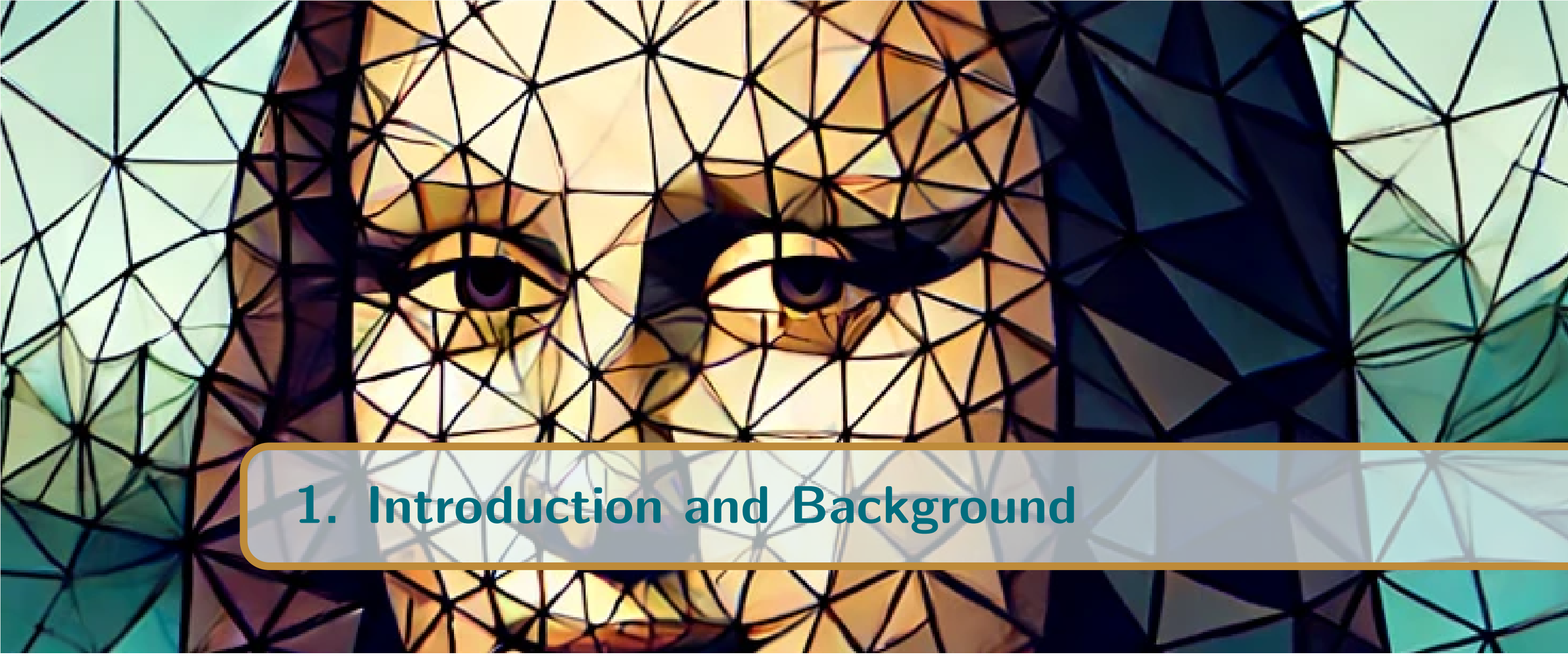

1. Introduction and Background

1.1 Summary

Summary 1.1 — Continuous functions on Ω and $\overline{\Omega}$. Let $\Omega \subset \mathbb{R}^n$ be an open set. $C(\Omega)$ denotes the set of all continuous functions defined on Ω. Similarly, $C(\overline{\Omega})$ denotes the set of all continuous functions defined on the closure of Ω.

For any $f \in C(\Omega)$ we have $f \in C(\overline{\Omega})$. However, for the converse, if $g \in C(\overline{\Omega})$, then if g is uniformally continuous and Ω is bounded, then g can continuously be extended to $\partial\Omega$. Note that $C(\Omega)$ functions can behave badly near $\partial\Omega$. For instance, consider the function $f : (0,1) \to \mathbb{R}$ given by $f(x) = \sin\big(1/x\big)$.

Summary 1.2 — The space of continuous 2π periodic functions. Consider the space of continouse functions defined on $\mathbb{R}$, i.e. $C(\mathbb{R})$. An important subset of this set is $C_p(2\pi)$ which is the set of all continuous 2π periodic functions where for $f \in C_p(2\pi)$ we have

$$f(x + 2\pi) = f(x), \qquad x \in \mathbb{R}.$$

This set, is in one-to-one correspondence with the set of all continuous function defined from the manifold S^1, or equivalently $\mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{R}$.

Summary 1.3 — Basis for the set of polynomials. Let $\mathbb{P}_n$ denote the set of all polynomials defined on $\mathbb{R}$ with degree less than or equal to n. Then a basis for this linear space is

$$\mathbb{B} = \{1, x, \cdots, x^n\}.$$

Thus the dimension of this space is $n + 1$.

Now, consider a *linear subspace* of this space, the set of all polynomials that vanishes at 0 and 1 denoted by

$$\mathbb{P}_{n,0} = \{p \in \mathbb{P}_n \mid p(0) = p(1) = 0\}.$$

A basis for this linear subspace can be given as

$$\mathbb{B}_{n,0} = \{x(1-x), x^2(1-x), \cdots, x^{n-1}(1-x)\}.$$

Thus the dimension of this linear subspace is $\dim(\mathbb{P}_n) - 2$. The difference two in the dimension comes from the fact that polynomials in $\mathbb{P}_{n,0}$ vanished at two points of their domain. Thus the set of all polynomials of degree n that vanish at n points of their domain is a 1 dimensional linear subspace of $\mathbb{P}_n$.

1.2 Solved Problems

■ **Problem 1.1 — The space of solutions of an ODE (from Atkinson).** Show that the set of all continuous solutions of the differential equation $u''(x) + u(x) = 0$ is a finite-dimensional linear space. Is the set od all continuous solutions of $u''(x) + u(x) = 1$ is a linear space?

Solution Denote the set of all solutions for the ODE $u'' + u' = 0$ as

$$S = \{f \in C(\mathbb{R}) \mid f'' + f = 0\}.$$

We claim that S is a linear space. Because

- Closed under addition: Let $f, g \in S$. Then $f'' + f = 0$ and $g'' + g = 0$. From the linearity of derivative it follows that $(f+g)'' + (f+g) = 0$, hence $f + g \in S$.
- Existence of zero element: The function $g \equiv 0$ is in S.
- Existence of inverse element: Let $f \in S$. Then $f'' + f = 0$. Multiplying both sides by -1 we will get $(-f)'' + (-f) = 0$. Thus $-f \in S$.
- Closed under scalar multiplication: Let $f \in S$. Then $f'' + f = 0$. Multiplying both sides by $a \in \mathbb{R}$ we will get $(af)'' + (af) = 0$. Thus $af \in S$.
- Commutativity, associativity, distributivity, and follows from the same properties for the addition of functions.

To show that the dimension of this linear space is finite, consider two solutions $u_1, u_2 \in S$ such that their Wronskian is non-zero. I.e.

$$W(t) = \det \begin{pmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{pmatrix} \neq 0.$$

For the particular ODE given in this question, we can consider $u_1(t) = \cos(t)$ and $u_2(t) = \sin(t)$. Take any solution $v \in S$. Assume $v(0) = a$ and $v'(0) = b$. Consider $w(t) = pu_1(t) + qu_2(t)$ where $p, q \in \mathbb{R}$ chosen such that $v(0) = w(0)$ and $v'(0) = w'(0)$. Since both w, v are solutions of the ODE, then from the existence-uniqueness theorem, it follows that $v(t) = w(t)$. This shows that we can write every solution of the ODE in terms of u_1 and u_2. Thus S is a linear space of dimension 2.

The continuous solutions of the ODE $u'' + u = 1$ is not a linear space as it does not contain the zero element. However, we can show that this is an affine space.

■ **Problem 1.2 — Linear space (from Atkinson).** When is the set $\{v \in C[0,1] \mid v(0) = a\}$ a linear space?

Solution This set is a linear space only when $a = 0$. Otherwise, this set can not contain the zero function (to be served as the zero element of the vector space). Also, if $a \neq 0$, then this set will not be closed under addition and scalar multiplication.

■ **Problem 1.3 — Zero vector and linear independence (from Atkinson).** Show that in any linear space V, a set of vectors is always linearly dependent if one of the vectors is zero.

Solution Let $\{u_1, u_n, f\}$ be a collection of vectors where f is the zero vector. Let $\alpha_1 = \cdots = \alpha_n = 0$ and $\beta \neq 0$ and consider the sum

$$\alpha_1 u_1 + \cdots + \alpha_n u_n + \beta f = 0.$$

There is one non-zero coefficient β, thus the collection of vectors are linearly dependent.

■ **Problem 1.4 — Unique expansion in terms of basis vectors (from Atkinson).** Let $\{v_1, \cdots, v_n\}$ be a basis of an n-dimensional space V. Show that for any $v \in V$, there are scalars $\alpha_1, \cdots, \alpha_n$ such that

$$v = \sum_{i=1}^{n} \alpha_i v_i,$$

and the scalars $\alpha_1, \cdots, \alpha_n$ are uniquely determined by v.

Solution Let $\mathbb{B} = \{v_1, \cdots, v_n\}$ be a basis and let $v \in V$ be any vectors. Since $\mathbb{B}$ is a basis, then by definition the vectors $v_1, \cdots, v_n$ are

(I) linearly independent, and
(II) spans the whole space.

(II) implies the existence of the scalars $\alpha_1 \cdots \alpha_n$ such that

$$v = \sum_{i}^{n} \alpha_i v_i.$$

Furthermore, (I) implies the uniqueness of these scalars. To see this we will use the proof by contradiction. Consider the $\beta_1, \cdots, \beta_n$ where we have $\beta_i \neq \alpha_i$ at least for one $1 \leq i \leq n$. Then

$$v = \sum_{i=1}^{n} \alpha_i v_i, \qquad v = \sum_{i=1}^{n} \beta_i v_i.$$

Subtracting these two expressions we will get

$$0 = \sum_{i=1}^{n} (\alpha_i - \beta_i) v_i.$$

Since $\alpha_i \neq \beta_i$ for at least one index i. From the definition of linear dependence, this implies that the collection of vectors in $\mathbb{B}$ is linearly dependent that contradicts (I) which is a contradiction.