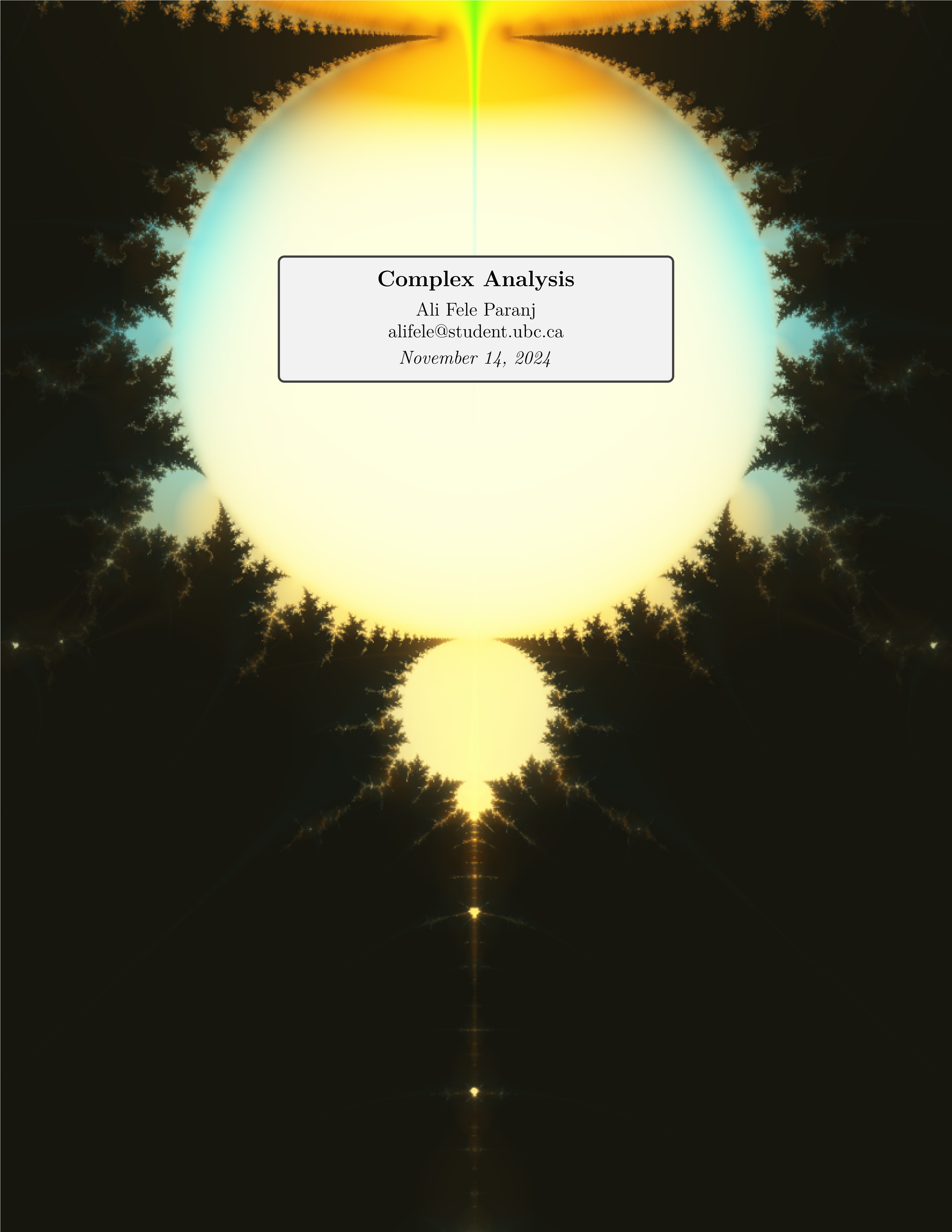

Complex Analysis

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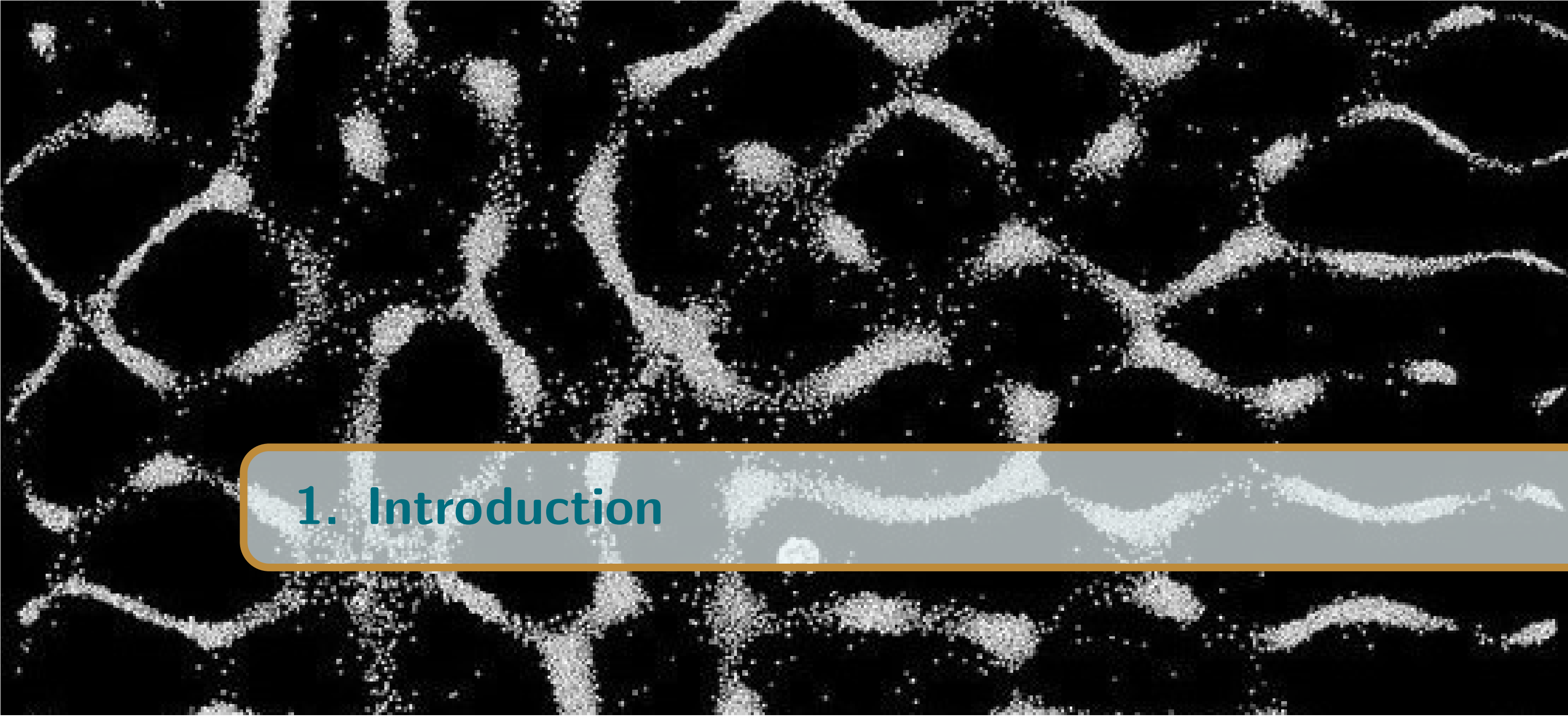
November 14, 2024

0. Contents



1. Introduction

1.1 Holomorphic Functions

We start with a definition of the holomorphic functions.

Definition 1.1 — Holomorphic functions. Let $f : \Omega \to \mathbb{C}$ be a functions defined on the open set $\Omega \subset \mathbb{C}$. Then f is holomorphic at $z_0 \in \Omega$ if

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = 0,$$

where $h \in \mathbb{C}$.

■ **Remark 1.1** Although the definition above resembles the definition of derivative for the real functions, but it is much more stronger. Later, we will see that the holomorphic functions are infinitely times differentiable, which is not necessarily true for the real differentiable functions. Also, we will see that every holomorphic function admits a power series, which is not again necessarily true in the case of real functions. I.e. there are real functions that are infinitely many times differentiable, but can not be expressed as a convergent power series. More on these later.

There are some very useful characterization of the holomorphic functions that comes in handy in making intuitions and proving some theorems in a more straight forward way. First, we will discuss the following observation.

Observation 1.1.1 — A bijection between complex numbers and 2×2 matrices. Observer the following multiplication between two complex variables:

$$(a + ib)(x + iy) = (ax - by) + (bx + ay).$$

This suggests that we can make the following one-to-one correspondence between the 2×2 matrices and complex numbers

$$a + ib \longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

The observation above is key to make some intuition about different notions in the complex analysis. For instance, for every complex map, we can associate it with a real map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

I.e. let $f = u + iv$. Then we can associate this with a real map $F(x, y) = (u(x, y), v(x, y))$. On the other hand, the differential of a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a 2×2 matrix. However, in the case of complex maps, the complex derivative of a map at a point is again a complex number. This suggests that for the map $F(x, y)$ where its Jacobian is give by

$$JF = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix},$$

in order JF to represent a complex number, we need to have

$$\partial u/\partial x = \partial v/\partial y, \qquad \partial u/\partial y = -\partial v/\partial x.$$

This is known as the Cauchy-Riemann equations. Note that this is not a correct way to derive the Cauchy-Riemann equations, but its is just an intuitive way to see that C-R equations makes everything work smoothly.

Theorem 1.1 — Cauchy-Riemann Equations. Let $f : \Omega \to \mathbb{C}$ be complex differentiable. If $f = u + iv$ then

$$\partial u/\partial x = \partial v/\partial y, \qquad \partial u/\partial y = -\partial v/\partial x.$$

Proof. Since f is complex differentiable, then

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

In particular, it does so for $h = h_1 + ih_2$ going to zero by $h_1 \to 0, h_2 = 0$, as well as $h_1 = 0, h_2 \to 0$. Then we will have

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

This implies the Cauchy-Riemann equations. □

Definition 1.2 The following differentiation operations comes in handy for some applications.

$$\frac{\partial}{\partial z} = (\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}), \quad \frac{\partial}{\partial \bar{z}} = (\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y})$$

The following characterization of the holomorphic functions is very useful.

Proposition 1.1 — A useful characterization of C-R equations (i.e. Holomorphic functions). Let f be a complex map. Then C-R is equivalent to $\partial f/\partial \bar{z} = 0$.

Proof. Observe that

$$\frac{\partial f}{\partial \bar{z}} = (\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y})f.$$

Let $f = u + iv$. Then it is easy to check that $\partial f/\partial \bar{z} = 0$ if and only if we have

$$\partial u/\partial x = \partial v/\partial y, \qquad \partial u/\partial y = -\partial v/\partial x.$$

□

Corollary 1.1 A complex map f is holomorphic at z_0 in its domain if and only if we have

$$f(z) = f(z_0) + L(z - z_0) + |z - z_0|\Psi(z - z_0),$$

for some $L \in \mathbb{R}$ and $\Psi(z - z_0) \to 0$ as $z \to z_0$.

Proof. In general, for any complex map f we can write

$$f(z) = f(z_0) + \alpha(z - z_0) + \beta\overline{(z - z_0)} + \Psi(z - z_0)|z - z_0|.$$

(To see this why, first observe that we can write every complex map as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$, and then do the change of variable $x = (z + \bar{z})/2, y = (z - \bar{z})(2i)$). However, from the proposition above, we conclude that $\beta = 0$ for all z_0 in the domain. This completes the proof. □

Proposition 1.2 Let f be a holomorphic function on an open set. Then

$$f'(z_0) = \frac{\partial f}{\partial z} = 2\frac{\partial u}{\partial z}.$$

Since f is holomorphic at z_0, then the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

converges for any path on which $h = h_1 + ih_2$ approaches zero, in particular $(h_1 \to 0, h_2 = 0)$ and $(h_1 = 0, h_2 \to 0)$. Thus we conclude

$$f'(z) = \frac{1}{2}(\frac{\partial f}{\partial x}(z_0) + \frac{1}{i}\frac{\partial f}{\partial y}(z_0)).$$

(To see this modify the terms of the limit in the definition of the complex differentiation and use the paths above for appropriate limits and conclude the identity above). Thus we can write $f = u + iv$ and using the C-R equations (since f is holomorphic) we conclude that

$$f'(z) = 2\frac{\partial u}{\partial z}.$$

The following theorem is an important converse-like statement for the Cauchy-Riemann theorem.

Theorem 1.2 Let $f : \Omega \to \mathbb{C}$ be a function. Assume $f = u + iv$. If u, v are continuously differentiable on Ω, and satisfy the C-R equations, then f is holomorphic on Ω

Proof. Since u, v are continuously differentiable, then

$$u(x + h_1, y + h_2) - u(x, y) = \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + |h|\Psi_1(h),$$

and similarly

$$v(x + h_1, y + h_2) - v(x, y) = \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + |h|\Psi_2(h).$$

Since $f = u + iv$ then

$$f(z + h) - f(z) = \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + i\frac{\partial v}{\partial x}h_1 + i\frac{\partial v}{\partial y}h_2 + |h|\Psi(h)$$

. Using C-R we can write

$$f(z + h) - f(z) = (\frac{\partial u}{\partial x} + \frac{1}{i}\frac{\partial u}{\partial y})(h_1 + ih_2) + |h|\Psi(h).$$

Since

$$\frac{\partial u}{\partial x} + \frac{1}{i}\frac{\partial u}{\partial y} = 2\frac{d}{dz}u = \frac{d}{dz}f,$$

then we conclude that f is holomorphic (follows from the characterization of holomorphic functions). $\square$

1.2 Second Review Notes

These notes are from my second review of the chapter 2 of Stein Complex analysis.

A function $f : \Omega \to \mathbb{C}$ is holomorphic at $z_0 \in \Omega$ if

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists. This is equivalent to the existence of a complex number $a \in \mathbb{C}$ such that

$$f(z_0 + h) = f(z_0) + ha + h\psi(h)$$

where $\psi(h) \to 0$ as $h \to 0$. Alternatively, instead of $h\psi(h)$ we can write $o(h)$.

1.2.1 A Clear Discussion on the Holomorphic Functions

First, note the following association

$$\boxed{(a+ib)(h_1+ih_2) \quad \longleftrightarrow \quad \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}},$$

as well as

$$\boxed{f(x+iy) = u(x,y) + iv(x,y) \quad \longleftrightarrow \quad F(x,y) = (u(x,y), v(x,y))},$$

where f is a complex function whereas F is a map $\mathbb{R}^2 \to \mathbb{R}^2$. F is differentiable at $X \in \mathbb{R}^2$ if there exists a linear map (i.e. 2×2 matrix) J such that

$$F(X + \vec{h}) = F(X) + J\vec{h} + o(\left\|\vec{h}\right\|), \tag{1.2.1.I}$$

where $h = (h_1, h_2) \in \mathbb{R}^2$ and the linear map J is given by

$$J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}.$$

It is easy to check the following identity

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} a+d & b-c \\ c-b & a+d \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} a-d & -b-c \\ b+c & a-d \end{bmatrix}\begin{bmatrix} h_1 \\ -h_2 \end{bmatrix}.$$

Thus we can write the Jacobi matrix as

$$Jh = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} u_x+v_y & u_y-v_x \\ v_x-u_y & u_x+v_y \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} u_x-v_y & -u_y-v_x \\ u_y+v_x & u_x-v_y \end{bmatrix}\begin{bmatrix} h_1 \\ -h_2 \end{bmatrix}.$$

Using the associations we had before we can write

$$\frac{1}{2}\begin{bmatrix} u_x+v_y & u_x-v_x \\ v_x-u_x & u_x+v_y \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \quad \longleftrightarrow \quad \frac{1}{2}\left(\frac{\partial}{\partial x}(u+iv)-i(\frac{\partial}{\partial y}(u+iv))\right)(h_1+ih_2) = \frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right)(h_1+ih_2)$$
$$:= \frac{\partial f}{\partial z}h.$$

Similarly

$$\frac{1}{2}\begin{bmatrix} u_x-v_y & -u_y-v_x \\ u_y+v_x & u_x-v_y \end{bmatrix}\begin{bmatrix} h_1 \\ -h_2 \end{bmatrix} \quad \longleftrightarrow \quad \frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)(h_1-ih_2) := \frac{\partial f}{\partial \bar{z}}\bar{h}$$

Thus we can write

$$F(X+\vec{h}) = F(X) + J\vec{h} + o(\left\|\vec{h}\right\|)$$
$$\updownarrow$$
$$f(z+h) = f(z) + \frac{\partial f}{\partial x}h + \frac{\partial f}{\partial \bar{z}}\bar{h} + o(|h|).$$

Thus complex function $\Omega \to \mathbb{C}$ that is differentiable function is not necessarily holomorphic. However, it is holomorphic if and only if $\partial f/\partial \bar{z} = 0$.

This is a very interesting and clear characterization of the holomorphic functions. The condition $\partial f/\partial \bar{z} = 0$ implies

$$\boxed{f \text{ holomorphic}} \iff \boxed{\frac{\partial f}{\partial \bar{z}} = 0} \iff \boxed{\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}} \quad \Longrightarrow \quad \boxed{u_x = v_y,\ u_y = -v_x},$$

which is the Cauchy-Riemann equations for a holomorphic function.

1.3 Summary

Summary 1.1 In a nutshell, every holomorphic function is a mapping from $\mathbb{R}^2 \supset \Omega \to \mathbb{R}^2$ such that its Jacobian is of the form

$$\begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix},$$

for some real numbers $\omega, \mu \in \mathbb{R}$. This property makes the holomorphic functions to be a special kind of maps from $\mathbb{R}^2$ to $\mathbb{R}^2$. In particular they preserve the angles, etc.

Summary 1.2 One interesting fact about the complex plane is that it brings together two fundamental behaviours "oscillation" and "growth" in an orthogonal way! This is a very lose statement, and what I mean is that "oscillation" and "growth" are two fundamental behaviours that a function does and for instance the function e^z has oscillator behaviour on the imaginary axis and growth behaviour on the real axis (and some combined behaviour on the quadrants.)

Summary 1.3 The zeros of a non-trivial holomorphic function are isolated. On other words if f is holomorphic on open Ω and $z_0 \in \Omega$ is a zero of f (i.e. $f(z_0) = 0$), then there is an open disk around z_0 such that it contains no other zeros of the function f, unless f is identically zero.

Theorem 1.3 — Local Characterization of a Holomorphic Function Near a Zero. Let f be holomorphic function on Ω with $z_0 \in \Omega$ such that $f(z_0) = 0$. Then there is an open neighborhood $z_0 \in U \subset \Omega$ such that there exist a non-vanishing holomorphic function g on U for which we can write

$$f(z) = (z - z_0)^n g(z),$$

where n is the order of the zero.

■ **Remark 1.2** Proof idea: Since f is holomorphic then it has a power series expansion near z_0 like $f(z) = a_n(z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \cdots$ for some $n \geq 1$ (note that the leading term is not constant as z_0 is a zero of f). By factoring $(z - z_0)^n$ we will have $f(z) = (z - z_0)^n(a_n + a_{n+1}(z - z_0) + \dots)$.

Using the definition of a pole and the theorem above we will have the following characterization of the poles of a holomorphic function.

Theorem 1.4 — Local Characterization of a Holomorphic Function Near a Pole. If f is holomorphic and has a pole at $z_0 \in \Omega$, then in a neighborhood of that point there is a non-vanishing holomorphic function $h(z)$ such that

$$f(z) = (z - z_0)^{-n} h(z),$$

for some $n \geq 1$. n is the order of the pole.

■ **Remark 1.3** Proof idea. By definition z_0 (i.e. the pole of f) is a zero of $1/f$. So by the local characterization of holomorphic functions near a zero we can write

$$1/f = (z - z_0)^n g(z)$$

where g is a non-vanishing holomorphic function one some open neighborhood of z_0. Define $h(z) = 1/g(z)$.

The theorem above will give us the following generalization of the power series for holomorphic functions.

Proposition 1.3 Let f be holomorphic on Ω except at z_0 that is a pole. Then on an open neighborhood near z_0 we can write

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + G(z)$$

where $G(z)$ is a holomorphic function on the same neighborhood.

■ **Remark 1.4** Proof idea: Using the characterization of holomorphic functions near a pole, we can write

$$f(z) = (z - z_0)^{-n} g(z)$$

where $g(z)$ is a non-vanishing holomorphic function on some open neighborhood containing z_0. Since $g(z)$ has a power series on that neighborhood we can write

$$g(z) = A_0 + A_1(z - z_0) + \cdots .$$

So we will have

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + G(z).$$

Summary 1.4 In the expansion $f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + G(z)$, the sum

$$\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + G(z)$$

is called the **principal part** of f and the coefficient a_{-1} is called the residue of f at z_0 and we write $\mathrm{res}_{z_0} f = a_{-1}$.

Theorem 1.5 — Generalization of Cauchy Integral Theorem. Let Ω be an open set that contain a toy contour γ and its interior. And let f be a holomorphic function in Ω except at poles $z_1, \cdots, z_n$ inside the contour. Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{i=1}^{n} \mathrm{res}_{z_i} f.$$

■ **Remark 1.5** Proof idea: By turning the contour into a keyhole like contour with narrow corridors, and letting the width of the corridors to go to zero, the integral of f on γ will be the same as the sum of integrals on small circles around each pole. Then by the characterization of holomorphic functions near poles, the only term from the principal part of function around each pole only the term with $1/(z - z_i)$ will be non-zero/

1.4 Solved Problems

■ **Problem 1.1 — Convergence of complex variables.** Let $\{z_n = a_n + ib_n\}_{n\in\mathbb{N}}$ for some $a_n \in \mathbb{R}, b_n \in \mathbb{R}$ be a sequence of complex numbers. Show that z_n converges to $w = \alpha + i\beta$ if and only if $a_n \to \alpha$ and $b_n \to \beta$ as $n \to \infty$.

Proof. The proof is as follows

$\boxed{\Longrightarrow}$ Let otherwise. Without loss of generality, we can assume that a_n does not converge to α. Then $\exists\epsilon > 0$ such that $\forall N > 0$ we can find $n > N$ for which $|a_n - \alpha| > \epsilon$. This implies that

$$|(a_n - \alpha) + i(b_n - \beta)|^2 = |a_n - \alpha|^2 + |b_n - \beta|^2 > \epsilon^2$$

which implies

$$|z_n - \omega| = |(a_n - \alpha) + i(b_n - \beta)| > \epsilon$$

for some ϵ and for some $n > N$ for any choice of N. This is a contradiction, since implies z_n is not converging to w.

$\boxed{\Longleftarrow}$ Assume $\alpha_n \to a$ and $\beta_n \to b$ as $n \to \infty$. Fix $\epsilon > 0$. Let N be large enough such that

$$|a_n - \alpha| < \epsilon^2/2, \qquad |b_n - \beta| < \epsilon^2/2.$$

Then we can write

$$|(a_n - \alpha) + i(b_n - \beta)|^2 = |a_n - \alpha|^2 + |b_n - \beta|^2 < \epsilon^2.$$

This implies that z_n converges to w.

□

■ **Problem 1.2 — Completeness of $\mathbb{C}$.** Prove that the set of all complex numbers $\mathbb{C}$ is complete.

Proof. The convergence of a complex number is equivalent to the convergence of its real and imaginary parts. Since $\mathbb{R}$ is complete, it follows that $\mathbb{C}$ is also complete. □

2. Cauchy Theorem and Cauchy Integral Formula

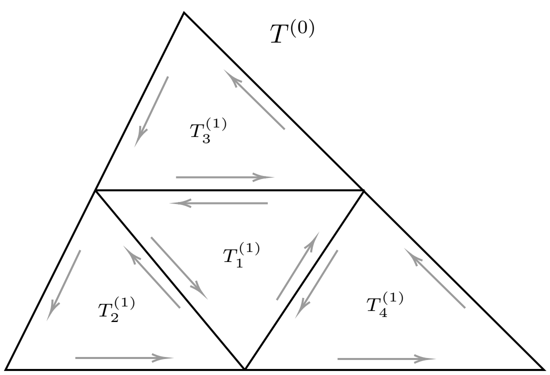
We start this chapter by the Goursat's theorem which in nutshell states that for a holomorphic function on an open region, it contour integral on a triangle contained in the open region is zero. We will use this idea later on to prove the Cauchy theorem.

Theorem 2.1 — Goursat's theorem. Let $\Omega \subset \mathbb{C}$ be an open set and $T \subset \Omega$ be a triangle that its interior is also contained in Ω. Then

$$\int_T f(z)dz = 0$$

whenever f is a holomorphic function on Ω.

Proof. To see the proof, we will use the following argument. Consider the triangle below. Starting



with the triangle $T^{(}0)$, we connect the mid-points of the edges and then we will get four *congruent* triangles. By a simple geometrical argument we can see that the radius and the diameter of the triangles $T^{(}1)_j$ for $j = 1, 2, 3, 4$ is half of the radius and perimeter of the original triangles. I.e. By doing the same process we will get smaller and smaller triangles at each step and at step n we have

$$d^{(}n) = 2^{-n}d^{(0)}, \qquad p^n = 2^{-n}p^{(0)}.$$

TODO: TO BE WRITTEN

□

2.1 Solved Problems

■ **Problem 2.1 — From Stein.** Suppose f is continuously complex differentiable on Ω, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω. Apply Green's theorem to show that

$$\int_T f(z)dz = 0.$$

This provides a proof of Goursat's theorem under the additional assumption that f' is continuous. *Hint: Green's theorem says that if (F, G) is a continuously differentiable vector field, then*

$$\int_T F dx + G dy = \int_{int(T)} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx dy.$$

For appropriate F and G, one can then use the Cauchy-Riemann equations.

Solution Let T be parameterized by piece-wise smooth $\gamma : [0, 1] \to \mathbb{R}$, given by $\gamma(t) = x(t) + iy(t)$. Let $f = u + iv$. By the definition of the Contour integral

$$\begin{aligned}\int_T f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt &= \int_a^b (u(x, y) + iv(x, y))(x'(t) + iy'(t))dt \\ = \int_a^b ((u + iv)x'(t) + (iu - v)y'(t))dt &= \int_T f dx + g dy,\end{aligned}$$

where

$$f(u, v) = u + iv, \qquad g(u, v) = iu - v.$$

Using Green's theorem

$$\int_T f dx + g dy = \int_{T^\circ} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy$$

where T° denotes the interior of the triangle. Observe that

$$\frac{\partial f}{\partial y} = u_y + iv_y, \qquad \frac{\partial g}{\partial x} = iu_x - v_x.$$

Thus

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = -(u_y + v_x) + (u_x - v_y) = 0,$$

where we have used the fact that the Cauchy-Riemann equations hold (as f is complex differentiable.) Thus

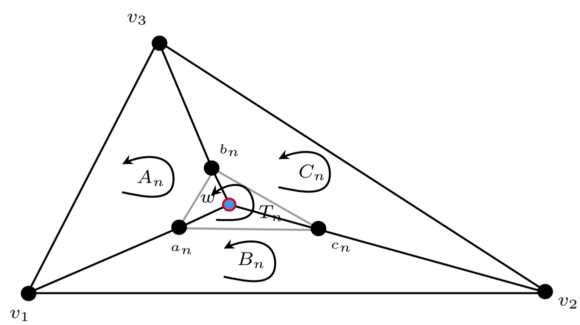
$$\int_T f(z)dz = 0.$$

Observation 2.1.1 In the question above, using the fact that f is complex differentiable, *plus* the extra regularity condition, that f' is continuous, we can simply use the Green's theorem to prove a variant of the Goursat's theorem. The reason that we need the extra regularity is that for the Green's theorem, we need the partial derivatives of the components of the vector field to be continuous.

■ **Problem 2.2 — From Stein.** Let Ω be an open subset of C and let $T \subset \Omega$ be a triangle whose interior is also contained in Ω. Suppose that f is a function holomorphic in Ω except possibly at a point $\omega \in T^\circ$, where T° is the interior of the triangle. Prove that if f is bounded near w, then

$$\int_T f(z)dz = 0.$$

Solution Consider the following figure where we connect the point $w \in T^\circ$ to the edges. On



each of the line segments (v_i, w) we choose the points a_n, b_n, c_n respectively for $i = 1, 2, 3$ such that $a_n, b_n, c_n \to w$ as $n \to \infty$. By the cancellation of the integral on the paths that have been traversed twice on opposite directions we can write

$$\int_T f(z)dz = \int_{A_n} f(z)dz + \int_{B_n} f(z)dz + \int_{C_n} f(z)dz + \int_{T_n} f(z)dz,$$

were A_n, B_n, and C_n are the tetragons as shown above, and T_n is the central triangle. Observe that the integration over each tetragon can further be broken down into two integral on two triangles, where by applying the Goursat's theorem it evaluates to zero. Thus the only remaining term will be

$$\int_T f(z)dz = \int_{T_n} f(z)dz.$$

We can write

$$|\int_T f(z)dz| = |\int_{T_n} f(z)dz| \leq \max_{z\in T_n} |f(z)|\ell(T_n).$$

Since the function f is bounded around w then $\max_{z\in T_n} |f(z)| \leq C$ for some $C \in \mathbb{R}$ and $\ell(T_n) \to 0$ as $n \to \infty$, we will have

$$\int_T f(z)dz = 0.$$

■ **Problem 2.3 — From Stein.** Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$ of the image of f satisfies

$$2|f'(0)| \leq d.$$

Moreover, it can be show that the equality holds precisely when f is linear, $f(z) = a_0 + a_1 z$. *Hint: You can use the fact that* $2f'(0) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi)-f(-\xi)}{\xi^2} d\xi$ *whenever* $0 < r < 1$.

Solution Using the Cauchy integral formula we can write

$$f'(0) = \frac{1}{2\pi i}\int_{|\xi|=r}\frac{f(\xi)}{\xi^2}d\xi = -\int_{|\xi|=r}\frac{f(-\xi)}{\xi^2}d\xi,$$

where the second equality holds by a simple change of variable. Then we can write

$$2f'(0) = \frac{1}{2\pi i}\int_{|\xi|=r}\frac{f(\xi)-f(-\xi)}{\xi^2}d\xi.$$

Then

$$\begin{aligned} 2|f'(0)| &= \frac{1}{2\pi i}|\int_{|\xi|=r}\frac{f(\xi)-f(-\xi)}{\xi^2}d\xi| \\ &\leq \frac{1}{2\pi i}\int_{|\xi|=r}\frac{|f(\xi)-f(-\xi)|}{|\xi^2|}d\xi \\ &\leq \frac{d}{2\pi}\int_{|\xi|=r}\frac{1}{|\xi|^2}d\xi \\ &\leq \frac{d}{2\pi}\frac{2\pi r}{r^2} = d/r. \end{aligned}$$

This for all $r \in (0,1)$ we have $2|f'(0)| \leq d/r$. This implies

$$2|f'(0)| \leq \inf_{r\in(0,1)}\frac{d}{r} = d.$$

■ **Problem 2.4 — From Stein.** If f is holomorphic function on the strip $-1 < y < 1$ and $x \in \mathbb{R}$ with

$$|f(z)| \leq A(1+|z|)^\eta, \qquad \eta \text{ a fixed real number}$$

for all z in that strip, show that for each integer $n \geq 0$ there exists $A_n \geq 0$ so that

$$|f^{(n)}(x)| \leq A_n(1+|x|)^\eta, \qquad \text{for all } x \in \mathbb{R}.$$

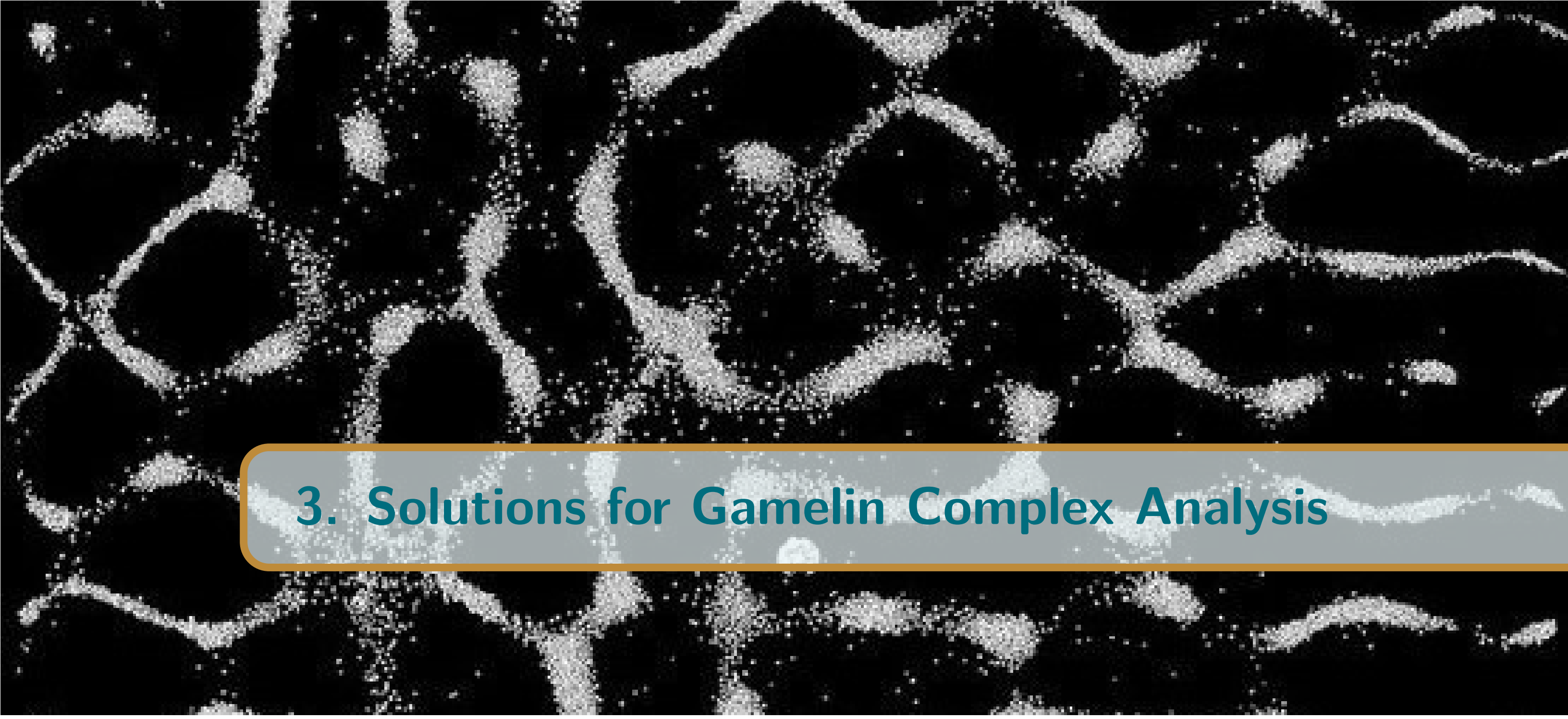
Hint: Use the Cauchy inequalities.

Solution Let $x \in \mathbb{R}$. This point is in the strip where f is holomorphic. Consider the set $|z-x| = R$ that is a circle with radius R centered at x. Then So using the Cauchy inequalities

$$|f^{(n)}(x)| \leq \frac{n!}{R^n}\sup_{|z-x|=r}|f(z)| \leq \frac{An!}{R^n}\sup_{|z-x|=r}(1+|z|)^\eta \leq \frac{An!}{R^n}(1+|x+R|)^\eta \leq \frac{An!}{R^n}(1+|x|+R)^\eta.$$

By letting $R = 1/2$ and using the fact that $3/2 + |x| \leq 3/2(1+|x|)$, we will get

$$|f^{(n)}(x)| \leq \frac{An!}{(1/2)^n}\cdot(\frac{3}{2})^\eta(1+|x|)^\eta = A_n(1+|x|)^\eta$$



3. Solutions for Gamelin Complex Analysis

Theorem 3.1 — Runge's Approximation Theorem. Let K be a compact subset of $\mathbb{C}$ contained in a an open set D. If f is analytic (holomorphic) on D, then it can be approximated *on* K uniformally by *rational* functions with poles *off* K.

There is some flexibility regarding the location of the poles of the approximating rational functions. This is captured by the following Lemma.

Lemma 3.1 Let K be a compact subset of $\mathbb{C}$, and U be a *connected open* subset of $\mathbb{C}^*\backslash K$, and let $z_0 \in U$. Every *rational* function f with poles at U can be approximated uniformally *on* K with *rational* with poles at z_0.

3.1 Approximation Theorems

■ **Problem 3.1** Show that any analytic function $f(z)$ on a domain D can be approximated normally on D by a sequence of rational functions that are analytic on D.

Solution Let $K \subset D$ be any compact subset of D. By Theorem 3.1 f can be approximated uniformly on K by rational functions with poles off K. Let $U = \mathbb{C}^*\backslash K$, and ;et $z_0 \in U \cap (\overline{D})^c$ (where $\overline{D}$ is the closure of D). By Lemma 3.1 each of the approximating functions above can be approximated by rational functions with poles at z_0. Since $z_0 \in (\overline{D})^c$, then all of these approximating rational functions are holomorphic (analytic on D). So f can be approximated normally (on every compact subset of D) with rational functions that are holomorphic on D.

■ **Problem 3.2** Show that there is a sequence of polynomials $\{p_n(z)\}$ such that $p_n(z) \to 1$ if $\text{Re}(z) > 0$, $p_n(z) \to 0$ if $\text{Re}(z) = 0$, and $p_n(z) \to -1$ if $\text{Re}(z) < 0$.

Solution TODO: TO BE ADDED.

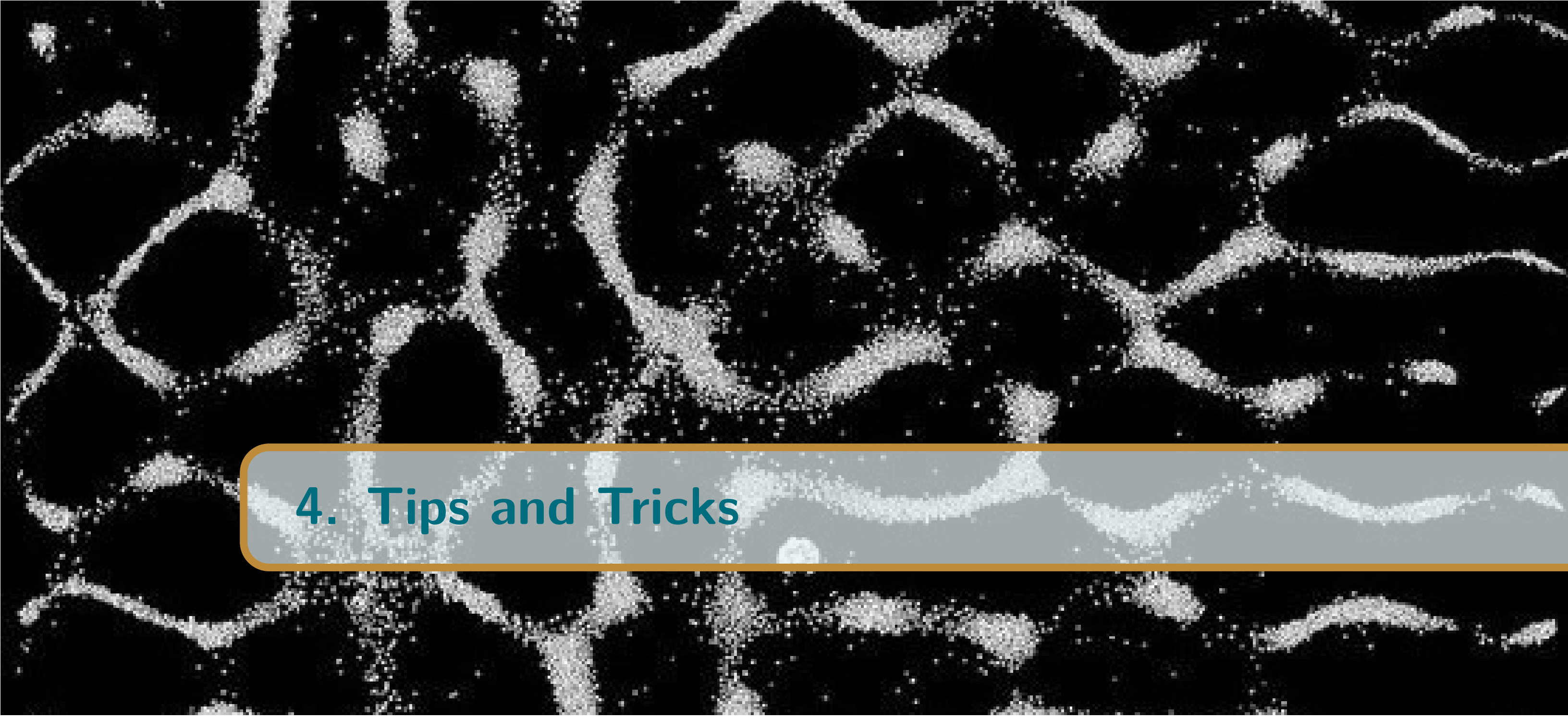
■ **Remark 3.1** I have not been able to solve the question above yet. The followings are my attempts

- I was thinking what if we use some conformal map to map the Left-Right half planes to the Interior-Exterior of the unit disk where the imaginary axis maps to the boundary of the disk.

This is easily possible by simply rotating the Riemann sphere by 90 degrees around the y axis. I am not sure if this mapping will help.

- The function $\tanh(z) = (e^z - e^{-z})/(e^z + e^{-z})$ has a very interesting behaviour. Define $f_n(z) = \tanh(nz)$. Then for the Left half plane $f_n(z) \to -1$ while for the Right half plane $f_n(z) \to 1$. But on the imaginary axis the function oscillates. Not sure if some modifications of this function will help.

■ **Problem 3.3** Let $\{z_j\}$ be a sequence of distinct points in a domain D that accumulates on ∂D, and let $\{w_j\}$ be a sequence of complex numbers. Show that there is an analytic function $f(z)$ on D such that $f(z_j) = w_j$ for all j. The sequence $\{z_j\}$ is called an interpolating sequence for analytic functions on D.



4. Tips and Tricks

Observation 4.0.1 The transformation $1/z$ corresponds to the rotation of the Riemann sphere by 180 degrees around the x axis.

■ **Remark 4.1** These remarks are temporarily.

- normal convergence is the uniform convergence in compact subsets.
- When a compact set K is contained in an open set D, then the boundary of D, i.e. ∂D is uniformly away from the boundary of K (i.e. ∂K).