Manifolds

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0. Contents

1. Euclidean Spaces

1.1 Basic Notions and Definitions

Here in this chapter I will be covering the details of some notions that was challenging for me do digest in the first read.

> **Definition 1.1 — Axioms of Group.** Group is a set A along with a binary operation $* : A \times A \to A$ that satisfies the following properties. Let $a, b, c \in A$, then
>
> - **Associativity**: $a * (b * c) = (a * b) * c$.
>
> - **Identity element**: $\exists 1 \in A$ such that
>
> $$1 * a = a * 1 = a.$$
>
> - **Inverse element**: $\forall a \in A \ \exists \hat{a} \in A$ such that
>
> $$a * \hat{a} = \hat{a} * a = 1.$$

■ **Remark** A set along with a binary operation that does not satisfy any properties is called a **magma**. If the binary operation is only associative, then we are dealing with **semi-group**. If the binary operation has an identity element as well, then we call this algebraic structure as **monoid**.

> **Definition 1.2 — Axioms of Ring.** A ring is a set R along with two operations $+ : R \times R \to R$ and $* : R \times R \to R$, where
>
> - $(R, +)$ is an Abelian group.
>
> - $(R, *)$ is a monoid.
>
> - The operator $(*)$ has distributive (left and right) law over $(+)$ i.e.
>
> $$a * (b + c) = (a * b) + (a * c), \qquad (b + c) * a = (b * a) + (c * a).$$
>
> .

■ **Remark** **Field** is a ring where every non-zero element (i.e. inverse element in the $(R, +)$ group

in the ring) has a multiplicative inverse.

> **Definition 1.3 — Axioms of Module.** A **module** is a group M along with a ring R where the monoid of the ring acts on M (through scalar multiplication) (i.e. it satisfies the idenity and compatibility properties) and satisfies the distributive property. I.e.
>
> - **Compatibility of the monoid action**: $a, b \in R, \ u \in M$ then
> $$a(bu) = (ab)u.$$
>
> - **Identity of the monoid action**: Let 1 be the identity element of the ring R. Then $\forall u \in M$
> $$1u = u1 = u.$$
>
> - **Distribution law**: $a, b \in R$ and $u, v \in M$ then
>   - $(a + b)u = au + bu$.
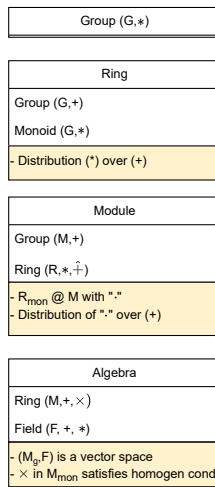>   - $a(u + v) = au + av$.

■ **Remark**   A module (M, R) is called a **vector space**, if the **ring** R is a **field**.

> **Definition 1.4 — Axioms of Algebra.** An Algebra over field F is a ring A that F acts on it (thus A has vector space structure as well), where the monoid operation of F (i.e. multiplication) satisfies the homogeneity property. I.e. for $r \in F$ and $u, v \in A$ we have
> $$r(uv) = (ru)v = u(rv).$$

There are some important observations when combining different algebraic structures with each other to get a new one. The first is that when we combine two structures with different operators, then the operators need to satisfy the distributive laws. Also, note that when an algebraic structure (like group or monoid) acts on another algebraic structure, we need to have the identity and and compatibility conditions satisfied.

The following diagram shows how different algebraic structures are combined with each other to produce another structure.

Group (G,$*$)

Ring
Group (G,+)
Monoid (G,$*$)
- Distribution ($*$) over (+)

Module
Group (M,+)
Ring (R,$*$,$\hat{+}$)
- R_{mon} @ M with "$\cdot$" - Distribution of "$\cdot$" over (+)

Algebra
Ring (M,+,$\times$)
Field (F, +, $*$)
- (M$_g$,F) is a vector space - $\times$ in M$_{mon}$ satisfies homogen cond.

Note that in the figure above, I have used some non-standard notations to make the figure concise. For instance, the expression "$R_{\mathbf{mon}}@M$ **with** $\cdot$" means that the monoid structure in the field R acts on the group M with the ($\cdot$) symbol. Or the expression "$\times$ **in** $M_{\mathbf{mon}}$ **satisfies homogen cond.**" means the multiplication operation of the monoid structure inside the ring M satisfies the homogeneity condition (see the definition of the algebra in Definition 1.4). Finally, M_g means the group structure inside the ring M.

1.2 Solved Problems

■ **Problem 1.1 — Algebra structure on** C_p^∞**.** Define carefully addition, multiplication, and scalar multiplication in C_p^∞. Prove that addition in C_p^∞ is commutative.

Solution First, note that the elements of C_p^∞ are actually the equivalence classes, where two functions are equivalent if they both define the same germ.

- For the definition of the addition, we can use the point-wise addition of the functions. However, we need to check to see if this definition is well-defined (i.e. the result of the addition of two functions does not depend on the choice of representative of the equivalence class). Let $f_1, f_2, g_1, g_2 \in C_p^\infty$ where f_1 and f_2 define the same germ, and similarly for g_1 and g_2. Then, we claim that $f_1 + g_1$ define the same germ as $f_2 + g_2$. That is because for f_1, f_2 there is an open set U_1 containing p where $f_1(x) = f_2(x) \ \forall x \in U$. Similarly, there is an open set U_2 that contains p and for all $x \in U_2$ we have $g_1(x) = g_2(x)$. Let $W = U_1 \cap U_2$. Then on for all $x \in W$ we have $f_1(x) + f_2(x) = g_1(x) + g_2(x)$. Hence $f_1 + g_1$ defines the same germ as $f_2 + g_2$.

- For the scalar multiplication, we can use the notion of scalar multiplication in functions, and following an idea similar to the reasoning above, we can show that this definition is well-defined.

- For the multiplication on C_p^∞ we can use of the point-wise multiplication of functions as the definition, and with a similar reasoning to the one in item 1, we can show that this definition is well-defined.

For the commutativity of the addition on C_p^∞, we need to emphasis that it follows immediately from the commutativity of the point-wise addition of functions.

■ **Problem 1.2 — Vector space structure on derivations at a point.** Prove that the set of all point derivatives is closed under addition and scalar multiplication.

Solution Let D and D' be derivations at $p \in \mathbb{R}^n$, and define $\hat{D} = D + D'$. Let $f, g \in C_p^\infty$. Then we can write

$$\hat{D}(fg) = (D + D')(fg)$$

On the other hand we have

$$D(fg) = D(f)g + fD(g), \qquad D'(fg) = D'(f)g + fD'(g).$$

Adding two equations we will get

$$D(fg) + D'(fg) = (D(f) + D'(f))g + f(D(g) + D'(g))$$

Defining $\hat{D} = (D + D')(f) = D(f) + D'(f)$ we will get

$$\hat{D}(fg) = \hat{D}(f)g + f\hat{D}(g).$$

which shows that $\hat{D}$ is also a point derivation at p. For the scalar multiplication, let $r \in \mathbb{R}$ and define $\tilde{D} = rD$. By defining $(rD)(f) = rD(f)$, we can write

$$(rD)(fg) = rD(fg) = rD(f)g + rfD(g) = rD(f)g + frD(g),$$

which shows that rD also satisfies the Leibniz property, this it is a point derivation.

■ **Problem 1.3** Let A be an algebra over a field K. If D_1 and D_2 are derivations of A, show that $D_1 \circ D_2$ is not necessarily a derivation (it is if D_1 or $D_2 = 0$), but $D_1 \circ D_2 - D_2 \circ D_1$ is always a derivation of A.