



Manifolds

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1. Euclidean Spaces

1.1 Basic Notions and Definitions

1.1.1 A Review on the algebraic structures

Here in this chapter I will be covering the details of some notions that was challenging for me to digest in the first read.

Definition 1.1 — Axioms of Group. Group is a set A along with a binary operation $*$: $A \times A \rightarrow A$ that satisfies the following properties. Let $a, b, c \in A$, then

- **Associativity:** $a * (b * c) = (a * b) * c$.
- **Identity element:** $\exists 1 \in A$ such that

$$1 * a = a * 1 = a.$$

- **Inverse element:** $\forall a \in A \exists \hat{a} \in A$ such that

$$a * \hat{a} = \hat{a} * a = 1.$$

■ **Remark 1.1** A set along with a binary operation that does not satisfy any properties is called a **magma**. If the binary operation is only associative, then we are dealing with **semi-group**. If the binary operation has an identity element as well, then we call this algebraic structure as **monoid**.

Definition 1.2 — Axioms of Ring. A ring is a set R along with two operations $+$: $R \times R \rightarrow R$ and $*$: $R \times R \rightarrow R$, where

- $(R, +)$ is an Abelian group.
- $(R, *)$ is a monoid.
- The operator $(*)$ has distributive (left and right) law over $(+)$ i.e.

$$a * (b + c) = (a * b) + (a * c), \quad (b + c) * a = (b * a) + (c * a).$$

■ **Remark 1.2** **Field** is a ring where every non-zero element (i.e. inverse element in the $(R, +)$ group in the ring) has a multiplicative inverse.

Definition 1.3 — Axioms of Module. A **module** is a group M along with a ring R where the monoid of the ring acts on M (through scalar multiplication) (i.e. it satisfies the identity and compatibility properties) and satisfies the distributive property. I.e.

- **Compatibility of the monoid action:** $a, b \in R, u \in M$ then

$$a(bu) = (ab)u.$$

- **Identity of the monoid action:** Let 1 be the identity element of the ring R . Then $\forall u \in M$

$$1u = u1 = u.$$

- **Distribution law:** $a, b \in R$ and $u, v \in M$ then

$$- (a + b)u = au + bu.$$

$$- a(u + v) = au + av.$$

■ **Remark 1.3** A module (M, R) is called a **vector space**, if the **ring** R is a **field**.

Definition 1.4 — Axioms of Algebra. An Algebra over field F is a ring A that F acts on it (thus A has vector space structure as well), where the monoid operation of F (i.e. multiplication) satisfies the homogeneity property. I.e. for $r \in F$ and $u, v \in A$ we have

$$r(uv) = (ru)v = u(rv).$$

There are some important observations when combining different algebraic structures with each other to get a new one. The first is that when we combine two structures with different operators, then the operators need to satisfy the distributive laws. Also, note that when an algebraic structure (like group or monoid) acts on another algebraic structure, we need to have the identity and compatibility conditions satisfied.

The following diagram shows how different algebraic structures are combined with each other to produce another structure.

| |
|---|
| Group (G,*) |
| Ring |
| Group (G,+) |
| Monoid (G,*) |
| - Distribution (*) over (+) |
| Module |
| Group (M,+) |
| Ring (R,*,+) |
| - $R_{\text{mon}} @ M$ with "." |
| - Distribution of "." over (+) |
| Algebra |
| Ring (M,+,×) |
| Field (F, +, ×) |
| - (M_g, F) is a vector space |
| - $×$ in M_{mon} satisfies homogen cond. |

Note that in the figure above, I have used some non-standard notations to make the figure concise. For instance, the expression “ $R_{\text{mon}} @ M$ with \cdot ” means that the monoid structure in the field R acts on the group M with the (\cdot) symbol. Or the expression “ \times in M_{mon} satisfies homogen cond.” means the multiplication operation of the monoid structure inside the ring M satisfies the homogeneity condition (see the definition of the algebra in [Definition 1.4](#)). Finally, M_g means the group structure inside the ring M .

1.1.2 Directional Derivative

The notion of directional derivative is very central in generalization of the multi-variable calculus manifolds.

Definition 1.5 — Directional derivative. Let $f : U \rightarrow \mathbb{R}$ where $U \subset \mathbb{R}^n$ and $f \in C^\infty(U)$. Then we define a directional derivative at $p \in U$ and in the direction $v \in T_p(\mathbb{R}^n)$ as

$$D_v f \Big|_p = \lim_{t \rightarrow 0} \frac{f(p + tv) - f(p)}{t} = \frac{d}{dt} \Big|_{t=0} f(p + vt).$$

We denote the set of all directional derivatives at p by $\mathcal{D}(C_p^\infty(U))$.

■ **Remark 1.4** By the chain rule we have

$$D_v f \Big|_p = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(p).$$

Proposition 1.1 — Directional derivative is \mathbb{R} –linear map satisfying the Leibniz rule. A directional derivative D_v at point p is a \mathbb{R} –linear operator that maps

$$f \in C_p^\infty \mapsto D_v f \in \mathbb{R}$$

such that satisfies the Leibniz rule,

$$D_v(fg) = D_v(f)g + fD_v(g).$$

Proof. Let $D_v \in \mathcal{D}(C_p^\infty(U))$. Then by the remark above we can write it as

$$D_v f \Big|_p = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(p).$$

Since each of the partial derivatives above are \mathbb{R} -linear and satisfy the Leibniz rule, then D_v inherits those properties as well. \square

1.2 Tangent Spaces and Derivations

Definition 1.6 — Informal definition of tangent space. Let $p \in \mathbb{R}^n$. A tangent space at p denoted by $T_p(\mathbb{R}^n)$ is a linear space containing all of vectors emerging from p .

■ **Remark 1.5** Note that the definition above is an informal definition of the tangent space. For a more formal and technical definition, we can use the notion of curves, or the notion of manifolds. We won't touch this level of technicality in the early chapters.

To distinguish between points in \mathbb{R}^n and vectors in $T_p(\mathbb{R}^n)$ we denote a point $p \in \mathbb{R}^n$ by

$$p = (p^1, p^2, \dots, p^n),$$

while for a vector $v \in T_p(\mathbb{R}^n)$ we write

$$v = \langle v_1, v_2, \dots, v_n \rangle.$$

As we observed in [Definition 1.5](#), we have a natural one-to-one correspondence between each $v \in T_p(\mathbb{R}^n)$ and a $D_v \in \mathcal{D}(C_p^\infty(U))$, i.e. these linear spaces are isomorphic.

Proposition 1.2 Let $p \in \mathbb{R}^n$. The set of all directional derivatives at p , i.e. $\mathcal{D}(C_p^\infty(U))$ is isomorphic to the tangent space at p i.e. $T_p(\mathbb{R}^n)$.

Proof. Proof follows immediately from the following natural association in the definition of the directional derivative.

$$D_v f \Big|_p \longleftrightarrow \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(p)$$

\square

Definition 1.7 — Point derivations at a point. Let $p \in \mathbb{R}^n$, and U an open set containing p . A **derivation at p** or a **point-derivation of $C_p^\infty(U)$** is a *linear* operator

$$D : C_p^\infty(U) \rightarrow \mathbb{R}$$

such that satisfies the Leibniz property, i.e. for $f, g \in C_p^\infty(U)$ we have

$$D(fg) = D(f)g + fD(g).$$

We denote the set of all such maps as $\mathbb{D}(C_p^\infty(U))$.

■ **Remark 1.6** We know that a directional derivative at p satisfies the Leibniz rule. Thus $\mathcal{D}(C_p^\infty(U)) \subset \mathbb{D}(C_p^\infty(U))$. On the other hand, we know that both \mathcal{D} and \mathbb{D} are linear spaces. So $\mathcal{D}(C_p^\infty(U))$ is in fact a *linear subspace* of $\mathbb{D}(C_p^\infty(U))$. Thus the zero of these two linear spaces match.

The following Lemma follows from the algebraic property of the point derivation.

Lemma 1.1 Let D be a point derivation of $C_p^\infty(U)$. Then $D(c) = 0$ for any constant function c .

Proof. Let c be a constant function, i.e. a real number. Since D is R -linear, then we can write $D(c) = cD(1)$. On the other hand, from the Leibniz property we can write

$$D(1) = D(1 \cdot 1) = D(1) + D(1) = 2D(1)$$

Thus $D(1) = 0$, which implies $D(c) = 0$. \square

The following theorem is very important as it states that all of the point derivations are in fact the directional derivatives and vice-versa. This is a very interesting result, since we are in fact stating that an operator is a point derivative if and only if it satisfies an algebraic property. This means that we can abstract away all of the detailed limit processes in the definition of derivative, and replace that with an axiomatic requirement which is a purely algebraic property. We can see these kind of ideas, i.e. axiomatic generalization all over the mathematics. For instance, in the PDE theory, at some point we need to relax the definition of derivative and talk about the weak derivatives. To do so, we get an identity that derivatives satisfy and carefully use that identity to define the notion of weak derivatives.

Theorem 1.1 — The set of all point-derivations is isomorphic to the set of all directional derivatives.
Let $p \in U \subset \mathbb{R}^n$. Then $\mathbb{D}(C_p^\infty(U))$ is isomorphic to $T_p(\mathbb{R}^n)$.

Proof. Let $\phi : T_p(\mathbb{R}^n) \rightarrow \mathbb{D}(C_p^\infty(U))$ be a linear isomorphism between the linear spaces. We need to show that this map is surjective and bijective. To show the bijectivity, we use the fact that a linear map is bijective if and only if its kernel is a singleton. To find the kernel of the map, let need to find all points in $T_p(\mathbb{R}^n)$ that maps to the zero of \mathbb{D} . As we discussed in the remark of Definition 1.7, the zero \mathbb{D} is the same as the zero directional derivative, i.e. $D_v = 0$. We need to prove that v is the zero vector, i.e. the zero of $T_p(\mathbb{R}^n)$. To do this, we apply the D_v to the coordinate functions

$$0 = D_v(x^i) = \sum_{j=1}^n v^j \frac{\partial x^i}{\partial x^j} = v^j$$

Thus $v = 0 \in T_p(\mathbb{R}^n)$, thus ϕ is injective. In other words, the injectivity follows immediately from $T_p(\mathbb{R}^n)$ being isomorphic to \mathcal{D} , and \mathcal{D} being a linear subspace of \mathbb{D} .

To prove the surjectivity, let D be a point derivation at p , and let (f, V) be a representative of a germ in C_p^∞ . Marking V smaller if necessary, we may assume that V is an open ball, hence star shaped. By Taylor's approximation theorem we know there exists C^∞ functions $g_i(x)$ such that

$$f(x) = f(p) + \sum_{i=1}^n (x^i - p^i)g_i(x), \quad g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

From Lemma 1.1, we know that $D(f(p)) = 0$ as well as $D(p^i) = 0$. Thus we can write

$$D(f(x)) = \sum_{i=1}^n (D(x^i)g_i(x) + (p^i - p^i)D(g_i(x))) = \sum_{i=1}^n D(x^i)g_i(x) = \sum_{i=1}^n D(x^i) \frac{\partial f}{\partial x^i}(p).$$

This is in fact a directional derivative in the direction $v = \langle D(x^1), D(x^2), \dots, D(x^n) \rangle$. So for every D in \mathbb{D} we can find a vector in $T_p(\mathbb{R}^n)$. Thus ϕ is surjective. \square

Observation 1.2.1 Let D be a point derivation of $C_p^\infty(U)$. This corresponds to the directional derivative at p corresponding to the vector

$$v = \langle D(x^1), D(x^2), \dots, D(x^n) \rangle.$$

Observation 1.2.2 Let $p \in U \subset \mathbb{R}^n$. Then

$$T_p(\mathbb{R}^n) \equiv \mathcal{D}(C_p^\infty(U)) \equiv \mathbb{D}(C_p^\infty(U)),$$

i.e. they are all isomorphic linear spaces.

Because of the observation above, we identify the standard basis $\{e^1, e^2, \dots, e^n\}$ with the partial derivatives $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n}\}$. Thus we can write a vector $v \in T_p(\mathbb{R}^n)$ as

$$v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}.$$

1.3 Summary

Summary 🦋 1.1 — Don't confuse the naming conventions. Note the following different names that might be confusing in some cases!

- **k-linear function \equiv k-tensor \equiv multi-linear function:** All of these names are for functions $f : V^k \rightarrow \mathbb{R}$. We denote the set of all k-tensors defined on vector space V as $L_k(V)$. If $\alpha^1 \cdots \alpha^n$ are the dual basis for V , then a basis for $L_k(V)$ is

$$\mathcal{B}(L_k(V)) = \{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}_{\text{for all multi-indices } (i_1, \dots, i_k)}$$

Also we have

$$\dim L_k(V) = n^k$$

- **alternating k-tensors \equiv k-covectors \equiv multi-covectors of degree k:** All of these names are for function k-linear functions that are **alternating**. We denote the set of all k-covectors on V as $A_k(V)$. A basis for this set is

$$\mathcal{B}(A_k(V)) = \{\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}\}_{\text{for all multi-indices } (i_1, \dots, i_k) \text{ in ascending order}}$$

Also we have

$$\dim A_k(V) = \binom{n}{k}.$$

Summary 🦋 1.2 — Determining sign of permutation. Let $\sigma \in S_k$. There we can find the sign of the permutation by decomposing it into cycles, or counting its number of inversions. For instance, consider $\sigma \in S_9$ where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 9 & 3 & 7 & 8 & 5 & 6 & 1 \end{pmatrix}$$

(i) **Determining sign by cycle decomposition.** For σ we can write it as

$$\sigma = (1\ 2\ 4\ 3\ 9)(5\ 7)(6\ 8) = (1\ 9)(1\ 3)(1\ 2)(5\ 7)(6\ 8).$$

Since σ has 6 transpositions in it, then its sign is **even**.

(ii) **Determining sign by counting the number of inversions.** For σ the inversions are

$$(2, 1), (3, 1), (4, 3), (4, 1), (5, 1), (6, 1), (7, 5), (7, 6), (7, 1), \\ (8, 5), (8, 6), (8, 1), (9, 1), (9, 3), (9, 5), (9, 6), (9, 7), (9, 8).$$

Since σ has 18 inversions, then its sign is **even**.

Summary  **1.3** Let $f \in A_k(V)$ and $g \in A_l(V)$. Then

$$f \wedge g = (-1)^{kl}(g \wedge f).$$

In particular, if f is odd-linear function, then

$$f \wedge f = 0.$$

Summary  **1.4 — Some intuition about the alternating functions.** In this summary box, we will go through a concrete example to make some intuition about the alternating functions and their basis. Let $V = \mathbb{R}^3$, with standard basis $\{e_1, e_2, e_3\}$ and the dual basis $\{\alpha^1, \alpha^2, \alpha^3\}$. Let $f \in A_2(V)$. Then we know that we can decompose f in term of its basis

$$f = g_1 \alpha^1 \wedge \alpha^2 + g_2 \alpha^1 \wedge \alpha^3 + g_3 \alpha^2 \wedge \alpha^3.$$

The key step in making the intuition is to observe that for any l 1-covectors $\omega^1, \dots, \omega^l$ we have $(\omega^1 \wedge \dots \wedge \omega^l)(v_1, \dots, v_l) = \det [\omega^i(v_j)]$. Thus for the first term in the decomposition of $f(u, v)$ for $u, v \in \mathbb{R}^3$ we have

$$(\alpha^1 \wedge \alpha^2)(u, v) = \det \begin{bmatrix} \alpha^1(u) & \alpha^1(v) \\ \alpha^2(u) & \alpha^2(v) \end{bmatrix}.$$

This is the area covered by two vectors that we can get by projecting u, v onto the $x - y$ plane. With a similar argument, $\alpha^1 \wedge \alpha^3$ gives the area between the vectors when projected to the $x - z$ plane, and etc. The following figure summarizes these in a neat way.



In the figure above, we will get the red, green and orange volumes by applying $\alpha^1 \wedge \alpha^2$, $\alpha^2 \wedge \alpha^3$, $\alpha^1 \wedge \alpha^3$ on (u, v) respectively.

Summary 🦋 1.5 Consider the following $n \times n$ matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

The determinant is defined to be

$$\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

Summary 🦋 1.6 Let V be a vector space, and $\alpha^1, \dots, \alpha^k$ be 1-linear functions (i.e. 1-covectors) defined on V . Then we have

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \dots, v_k) = \det [\alpha^i(v_j)].$$

In particular, For two 1-covectors we have

$$(\alpha^1 \wedge \alpha^2)(v_1, v_2) = \det \begin{bmatrix} \alpha^1(v_1) & \alpha^1(v_2) \\ \alpha^2(v_1) & \alpha^2(v_2) \end{bmatrix}$$

See Problem [Problem 1.10](#) for the a detailed proof.

Summary 🦋 1.7 Let V be a vector space with dimension n , with $\{e_1, \dots, e_n\}$ as its basis and $\{\alpha^1, \alpha^2, \dots, \alpha^n\}$ as its dual basis. Let $A_k(V)$ be the set of all *alternating* k -linear functions defined on V . Then we have

$$\dim A_k = \binom{n}{k}.$$

In particular, let $V = \mathbb{R}^n$ with standard basis $\{e_1, e_2, e_3\}$, and the corresponding dual basis $\{\alpha^1, \alpha^2, \alpha^3\}$. Then we have

$$\begin{aligned} \mathbb{B}(A_0) &= \{1 \in \mathbb{R}\} \\ \mathbb{B}(A_1) &= \{\alpha^1, \alpha^2, \alpha^3\}, \\ \mathbb{B}(A_2) &= \{\alpha^1 \wedge \alpha^2, \alpha^1 \wedge \alpha^3, \alpha^2 \wedge \alpha^3\}, \\ \mathbb{B}(A_3) &= \{\alpha^1 \wedge \alpha^2 \wedge \alpha^3\}. \end{aligned}$$

It is clear from the basis sets above that the dimensions of A_0, A_1, A_2 and A_3 are 1, 3, 3, 1 respectively.

Also, observe that $A_k = 0$ for all $k > n$. This follows from the fact that $f \wedge f = 0$ for f a 1-covector.

Summary 🦋 1.8 Let V be n dimensional vector space with a basis $\{e_1, \dots, e_n\}$, and $\{\alpha^1, \dots, \alpha^n\}$ as its dual basis. Let $L_k(V)$ be the set of all k -linear functions defined on V . Then a basis for $L_k(V)$ is

$$\mathcal{B} = \{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$$

for all multi-indices (i_1, \dots, i_k) . This shows that

$$\dim L_k(V) = n^k.$$

For instance, Let $V = \mathbb{R}^2$. Then for $L_1(V)$ we have

$$\mathcal{B}(L_1(V)) = \{\alpha^1, \alpha^2, \alpha^3\}.$$

Similarly, for $L_2(V)$ we have

$$\mathcal{B}(L_2(V)) = \{\alpha^1 \otimes \alpha^2, \alpha^1 \otimes \alpha^3, \alpha^2 \otimes \alpha^3, \alpha^2 \otimes \alpha^1, \alpha^3 \otimes \alpha^1, \alpha^3 \otimes \alpha^2, \alpha^1 \otimes \alpha^1, \alpha^2 \otimes \alpha^2, \alpha^3 \otimes \alpha^3\}$$

For a side by side comparison, compare it with the basis for $A_2(V)$ that is

$$\mathcal{B}(A_2(V)) = \{\alpha^1 \wedge \alpha^2, \alpha^2 \wedge \alpha^3, \alpha^1 \wedge \alpha^3\}.$$

Summary 🦋 1.9 Looking at the dimension $L_1(V), L_2(V), \dots$, it reveals the fact that why most of the non-technical people are interested in interpreting tensors as multi-dimensional matrices! For instance, when $V = \mathbb{R}^2$ and $K = 3$, then $L_3(V)$ has dimension $2^3 = 8$ with the basis

$$\mathcal{B}(\mathbb{R}^2) = \{\alpha^1 \otimes \alpha^1 \otimes \alpha^1, \alpha^1 \otimes \alpha^1 \otimes \alpha^2, \alpha^1 \otimes \alpha^2 \otimes \alpha^1, \alpha^1 \otimes \alpha^2 \otimes \alpha^2, \alpha^2 \otimes \alpha^1 \otimes \alpha^1, \alpha^2 \otimes \alpha^1 \otimes \alpha^2, \alpha^2 \otimes \alpha^2 \otimes \alpha^1, \alpha^2 \otimes \alpha^2 \otimes \alpha^2\}$$

It is very tempting to think of elements of $L_3(V)$ as $2 \times 2 \times 2$ matrix like structures. In the case of $L_2(V)$ we can identify $L_2(V)$ with the set of all 2×2 matrices.

Summary 🦋 1.10 — **Determinant of a matrix is zero if its column vectors are linearly dependent.**

A k -tensor f is alternating if and only if $f(v_1, \dots, v_k) = 0$ whenever two of the vectors v_1, \dots, v_k are equal. With a simple generalization, we can state a more general statement that f is alternating k -tensor if and only if $f(v_1, \dots, v_k) = 0$ the vectors v_1, \dots, v_k are linearly dependent.

As an important (and simple!) Corollary for this statement, we can conclude that determinant of an $n \times n$ matrix (which is an alternating n -tensor) is zero when its column vectors are linearly dependent.

Summary 🦋 1.11 Let $U \subset \mathbb{R}^n$ open, and let $\{x_1, \dots, x_n\}$ be standard basis for \mathbb{R}^n , and $\{x^1, \dots, x^n\}$ the standard dual basis. Let $p \in U$. Then for $T_p(\mathbb{R}^n)$ and $T_p^*(\mathbb{R}^n)$ we have

$$\mathcal{B}(T_p(\mathbb{R}^n)) = \left\{ \frac{\partial}{\partial x_1} \Big|_p, \dots, \frac{\partial}{\partial x_n} \Big|_p \right\}, \quad \mathcal{B}(T_p^*(\mathbb{R}^n)) = \{dx^1 \Big|_p, \dots, dx^n \Big|_p\}.$$

1.4 Solved Problems

■ **Problem 1.1 — Algebra structure on C_p^∞ .** Define carefully addition, multiplication, and scalar multiplication in C_p^∞ . Prove that addition in C_p^∞ is commutative.

Solution First, note that the elements of C_p^∞ are actually the equivalence classes, where two functions are equivalent if they both define the same germ.

- For the definition of the addition, we can use the point-wise addition of the functions. However, we need to check to see if this definition is well-defined (i.e. the result of the addition of two functions does not depend on the choice of representative of the equivalence class). Let $f_1, f_2, g_1, g_2 \in C_p^\infty$ where f_1 and f_2 define the same germ, and similarly for g_1 and g_2 . Then, we claim that $f_1 + g_1$ define the same germ as $f_2 + g_2$. That is because for f_1, f_2 there is an open set U_1 containing p where $f_1(x) = f_2(x) \forall x \in U_1$. Similarly, there is an open set U_2 that contains p and for all $x \in U_2$ we have $g_1(x) = g_2(x)$. Let $W = U_1 \cap U_2$. Then on for all $x \in W$ we have $f_1(x) + g_1(x) = f_2(x) + g_2(x)$. Hence $f_1 + g_1$ defines the same germ as $f_2 + g_2$.
- For the scalar multiplication, we can use the notion of scalar multiplication in functions, and following an idea similar to the reasoning above, we can show that this definition is well-defined.
- For the multiplication on C_p^∞ we can use of the point-wise multiplication of functions as the definition, and with a similar reasoning to the one in item 1, we can show that this definition is well-defined.

For the commutativity of the addition on C_p^∞ , we need to emphasis that it follows immediately from the commutativity of the point-wise addition of functions.

■ **Problem 1.2 — Vector space structure on derivations at a point.** Prove that the set of all point derivatives is closed under addition and scalar multiplication.

Solution Let D and D' be derivations at $p \in \mathbb{R}^n$, and define $\hat{D} = D + D'$. Let $f, g \in C_p^\infty$. Then we can write

$$\hat{D}(fg) = (D + D')(fg)$$

On the other hand we have

$$D(fg) = D(f)g + fD(g), \quad D'(fg) = D'(f)g + fD'(g).$$

Adding two equations we will get

$$D(fg) + D'(fg) = (D(f) + D'(f))g + f(D(g) + D'(g))$$

Defining $\hat{D} = (D + D')(f) = D(f) + D'(f)$ we will get

$$\hat{D}(fg) = \hat{D}(f)g + f\hat{D}(g).$$

which shows that \hat{D} is also a point derivation at p . For the scalar multiplication, let $r \in \mathbb{R}$ and define $\tilde{D} = rD$. By defining $(rD)(f) = rD(f)$, we can write

$$(rD)(fg) = rD(fg) = rD(f)g + rfD(g) = rD(f)g + frD(g),$$

which shows that rD also satisfies the Leibniz property, this it is a point derivation.

■ **Problem 1.3** Let A be an algebra over a field K . If D_1 and D_2 are derivations of A , show that $D_1 \circ D_2$ is not necessarily a derivation (it is if D_1 or $D_2 = 0$), but $D_1 \circ D_2 - D_2 \circ D_1$ is always a derivation of A .

Solution TO BE ADDED.

■ **Problem 1.4** Find the inversions in the permutation $\tau = (1\ 2\ 3\ 4\ 5)$.

Solution This permutation can be written as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

Thus the number of inversions can be written as $(2, 1), (3, 1), (4, 1)$, and $(5, 1)$.

■ **Problem 1.5** Let $f : V^k \rightarrow \mathbb{R}$ be a k -linear function defined on vector space V . Show that the following function is alternating.

$$Af = \sum_{\sigma \in S_k} \text{sign}(\sigma) \sigma f.$$

Solution Let $\tau \in S_k$. Then

$$\tau(Af) = \sum_{\sigma \in S_k} \text{sign}(\sigma) (\tau\sigma)f = \sum_{\sigma \in S_k} \text{sign}(\sigma) \text{sign}(\tau) \text{sign}(\tau) (\tau\sigma)f = \text{sign}(\tau) \sum_{\sigma \in S_k} \text{sign}(\tau\sigma) (\tau\sigma)f$$

Note that since in the sum above σ runs through all permutations S_k , so does $\tau\sigma$. Thus we can write

$$\tau(Af) = \text{sign}(\tau)(Af).$$

This shows that Af is alternating.

■ **Problem 1.6** Let $f : V^k \rightarrow \mathbb{R}$ be a k -linear function. Show that Sf given below is symmetric.

$$Sf = \sum_{\sigma \in S_k} \sigma f.$$

Solution Let $\tau \in S_k$. Then we

$$\tau(Sf) = \sum_{\sigma \in S_k} \tau\sigma f = Sf.$$

Note that the last equality above holds, since σ runs through all permutations S_k and so does $\tau\sigma$.

■ **Problem 1.7** If f is a 3-linear function on a vector space V and $v_1, v_2, v_3 \in V$, what is $(Af)(v_1, v_2, v_3)$?

Solution

$$(Af)(v_1, v_2, v_3) = f(v_1, v_2, v_3) - f(v_2, v_1, v_3) + f(v_3, v_1, v_2) - f(v_1, v_3, v_2) + f(v_2, v_3, v_1) - f(v_3, v_2, v_1)$$

■ **Problem 1.8** Show that the tensor product of multi-linear functions is associative: If f, g , and h are multi-linear functions on V , then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h).$$

Solution We start with the left hand side. I.e.

$$\begin{aligned}
 ((f \otimes g) \otimes h)(v_1 \cdots, v_{k+l+m}) &= (f \otimes g)(v_1, \cdots, v_{k+l})h(v_{k+l+1}, \cdots, v_{k+l+m}) \\
 &= (f(v_1, \cdots, v_k)g(v_{k+1}, \cdots, v_{k+l}))h(v_{k+l+1}, \cdots, v_{k+l+m}) \\
 &= f(v_1, \cdots, v_k)(g(v_{k+1}, \cdots, v_{k+l})h(v_{k+l+1}, \cdots, v_{k+l+m})) \\
 &= (f \otimes (g \otimes h))(v_1, \cdots, v_{k+l+m}).
 \end{aligned}$$

■ **Problem 1.9** Consider following two ways that we can express the sum in the wedge product formula. Let $f \in A_k(V)$ and $g \in A_l(V)$. Then the wedge product $f \wedge g$ can be written as

$$(f \wedge g)(v_1, \cdots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sign}(\sigma) \sigma f(v_{\sigma(1)}, \cdots, v_{\sigma(k)})g(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)}).$$

Or, alternatively, we can write it as (k, l) -shuffle

$$(f \wedge g)(v_1, \cdots, v_{k+l}) = \sum_{(k,l)\text{-shuffles } \sigma} \text{sign}(\sigma) \sigma f(v_{\sigma(1)}, \cdots, v_{\sigma(k)})g(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)}).$$

Now, as a concrete example, let $f, g \in A_2(V)$. Write $f \wedge g$ in these two forms.

Solution First, we want to use the sum on the all permutations.

$$\begin{aligned}
 4(f \wedge g)(v_1, v_2, v_3, v_4) &= f(v_1, v_2)g(v_3, v_4) - f(v_2, v_1)g(v_3, v_4) + f(v_2, v_1)g(v_4, v_3) - f(v_1, v_2)g(v_4, v_3) \\
 &\quad - f(v_1, v_3)g(v_2, v_4) + f(v_3, v_1)g(v_2, v_4) - f(v_3, v_1)g(v_4, v_2) + f(v_1, v_3)g(v_4, v_2) \\
 &\quad + f(v_1, v_4)g(v_2, v_3) - f(v_4, v_1)g(v_2, v_3) + f(v_4, v_1)g(v_3, v_2) - f(v_1, v_4)g(v_3, v_2) \\
 &\quad + f(v_2, v_3)g(v_1, v_4) - f(v_3, v_2)g(v_1, v_4) + f(v_3, v_2)g(v_4, v_1) - f(v_2, v_3)g(v_4, v_1) \\
 &\quad - f(v_2, v_4)g(v_1, v_3) + f(v_4, v_2)g(v_1, v_3) - f(v_4, v_2)g(v_3, v_1) + f(v_2, v_4)g(v_3, v_1) \\
 &\quad + f(v_3, v_4)g(v_1, v_2) - f(v_4, v_3)g(v_1, v_2) + f(v_4, v_3)g(v_2, v_1) - f(v_3, v_4)g(v_2, v_1).
 \end{aligned}$$

Note that in every row, the functions are equal to each at (because f, g are alternating). Thus we have the factor of 4 not to count the redundant terms. However, we can do this sum using the $(2,2)$ -shuffles. This means that we only keep the very first column, as their argument permutation is the same as all $(2,2)$ -shuffles on 4 symbols. Thus we can write

$$\begin{aligned}
 (f \wedge g)(v_1, v_2, v_3, v_4) &= f(v_1, v_2)g(v_3, v_4) - f(v_1, v_3)g(v_2, v_4) + f(v_1, v_4)g(v_2, v_3) \\
 &\quad + f(v_2, v_3)g(v_1, v_4) - f(v_2, v_4)g(v_1, v_3) \\
 &\quad + f(v_3, v_4)g(v_1, v_2).
 \end{aligned}$$

Observation 1.4.1 How do we count the $(2,2)$ -shuffles of $\{v_1, v_2, v_3, v_4\}$? We start by a vertical line where in its left side we put v_1, v_2 and in its right side we write v_3, v_4 . Like the following table. This is already a shuffle (identity). To write the next shuffle, we keep v_1 in the first position, and write the next symbol whose its subscript is larger than 2 (i.e. v_3), and write the remaining symbols in the right hand side of the vertical line in increasing order. Then we continue this process, until there are no possible shuffles for its first element be v_1 . Then we put v_2 in the first place and right next to it we write the next symbol that its subscript is larger than 2 (i.e. v_3), and we continue.

| | |
|------------|------------|
| v_1, v_2 | v_3, v_4 |
| v_1, v_3 | v_2, v_4 |
| v_1, v_4 | v_2, v_3 |
| v_2, v_3 | v_1, v_4 |
| v_2, v_4 | v_1, v_3 |
| v_3, v_4 | v_1, v_2 |

In order to find the sign of each of these permutations, it is enough to extract the independent cycles. For instance, consider the following shuffle, corresponding to last row above.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

This permutation is the same as the permutation given by the cycle $\tau = (1\ 3)(2\ 4)$. To extract this we start with 1 and track where it goes, and we do this tracking until we get back to 1. Then we start this process with the remaining element until we get all of the cycles. This cycle decomposition of τ clearly shows that τ is an even permutation (can be written as 2 cycles). However, in the case of v_2, v_4, v_1, v_3 , the corresponding permutation is

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}.$$

We can write this as $\tau = (1\ 2\ 4\ 3)$. As with every cycle, we can decompose this into smaller cycles (not necessarily independent) as $\tau = (1\ 3)(1\ 4)(1\ 2)$. This shows that this particular permutation is odd.

■ **Problem 1.10 — Wedge product of covectors and determinant.** Let V be a vector space and let $\beta^1, \dots, \beta^l \in A_1(V)$. Prove that

$$(\beta^1 \wedge \dots \wedge \beta^l)(v_1, \dots, v_l) = \det [\beta^i(v_j)]$$

Solution This follows simply from the definition. From the definition of wedge product we have

$$(\beta^1 \wedge \dots \wedge \beta^l) = A [\beta^1(v_1) \dots \beta^l(v_l)] = \sum_{\sigma \in S_l} \text{sign}(\sigma) \beta^1(v_{\sigma(1)}) \dots \beta^l(v_{\sigma(l)}) = \det [\beta^i(v_j)]$$

where $[\beta^i(v_j)]$ is $l \times l$ square matrix.

■ **Problem 1.11** Verify

$$f \wedge g = (-1)^{kl}(g \wedge f)$$

for f and g being k and l linear maps correspondingly. Do this verification by considering $f \in A_2(V)$ and $g \in A_1(V)$.

Solution For $f \wedge g$ we can write (using $(2, 1)$ -shuffles)

$$(f \wedge g)(v_1, v_2, v_3) = f(v_1, v_2)g(v_3) - f(v_1, v_3)g(v_2) + f(v_2, v_3)g(v_1).$$

However, for $g \wedge f$, we can use $(1, 2)$ -shuffles to show

$$(g \wedge f)(v_1, v_2, v_3) = g(v_1)f(v_2, v_3) - g(v_2)f(v_1, v_3) + g(v_3)f(v_1, v_2).$$

Evaluating two expressions above reveals that $f \wedge g = g \wedge f$.

■ **Problem 1.12 — Tensor product of covectors (from W. Tu).** Let e_1, e_2, \dots, e_n be a basis for vector space V and let $\alpha^1, \dots, \alpha^n$ be its dual basis in V^\vee . Suppose $[g_{i,j}] \in \mathbb{R}^{n \times n}$ is an $n \times n$ matrix. Define a bilinear function $f : V \times V \rightarrow \mathbb{R}$ by

$$f(v, w) = \sum_{1 \leq i, j \leq n} g_{i,j} v^i w^j$$

for $v = \sum v^i e_i$ and $w = \sum w^j e_j$ in V . Describe f in terms of the tensor products of α^i and α^j , $1 \leq i, j \leq n$.

Solution We know that $v^i = \alpha^i(v)$, and similarly $w^j = \alpha^j(w)$. So we can write

$$v^i w^j = \alpha^i(v) \alpha^j(w) = (\alpha^i \otimes \alpha^j)(v, w).$$

Thus the bilinear function can be written as

$$f = \sum_{1 \leq i, j \leq n} g_{i,j} \alpha^i \otimes \alpha^j.$$

This bilinear function is very similar to the notion of weighted inner product.

■ **Problem 1.13 — Hyperplanes (from W. Tu).** (a) Let V be a vector space of dimension n , and $f : V \rightarrow \mathbb{R}$ a nonzero linear functional. Show that $\dim \ker f = n - 1$. A linear subspace of V of dimension $n - 1$ is called a hyperplane in V .

(b) Show that a nonzero linear functional on a vector space V is determined up to multiplicative constant by its kernel, a hyperplane in V . In other words, if f and $g : V \rightarrow \mathbb{R}$ are nonzero linear functionals and $\ker f = \ker g$, then $g = cf$ for some constant $c \in \mathbb{R}$.

Solution (a) We can use the Rank-Nullity theorem for the linear function f . This theorem states that

$$\dim \text{rank } f + \dim \ker f = \dim V.$$

We know that $\dim \text{rank } f = 1$, and $\dim V = n$. This implies $\dim \ker f = n - 1$.

(b) Let f, g be two linear functionals on V that has the same kernel, which we call it the set K . Since K is $n - 1$ dimensional linear subspace of K , then we can find a basis for it, call it $\mathcal{B}_1 = \{\hat{e}_1, \dots, \hat{e}_{n-1}\}$. Let \hat{e}_n be a vector normal to \mathcal{B}_1 . We can do all of these since V is n dimensional and we have just modified the basis vectors as described above. Let $z \in V$ where $z = \sum_{i=1}^n z_i \hat{e}_i$. Thus we have

$$f(z) = z_n f(\hat{e}_n), \quad g(z) = z_n g(\hat{e}_n).$$

This is true because $f(\hat{e}_i) = g(\hat{e}_i) = 0$ for all $\hat{e}_i \in \mathcal{B}$. Since f, g are non-zero functional, then we can write

$$\frac{f(z)}{g(z)} = \frac{f(\hat{e}_n)}{g(\hat{e}_n)} = c \quad \text{for some } c \in \mathbb{R}$$

Thus we can write

$$f = cg.$$

Note that c is determined by \hat{e}_n . \hat{e}_n itself is determined by the kernel of f, g , i.e. the hyperplane mentioned above.

Observation 1.4.2 I have this weird observation that I think it must be wrong, but I don't know how! Consider the set of all linear functionals defined on \mathbb{R}^2 . From duality, we know that this set (denoted by $L_1(\mathbb{R}^2)$) is isomorphic to \mathbb{R}^2 . But here comes the strange argument that leads to the non-real observation. As we saw in the question above, every hyperplane (in this case every line that passes through the origin) determines a linear functional up to a constant multiplication. On the other hand, we know that the set of all lines passing through the origin is the real projective plane (which is in fact $L_1(\mathbb{R}^2)/\sim$ where \sim identifies linear functionals that are constant multiple of each other), which definitely is not isomorphic to \mathbb{R}^2 .

■ **Problem 1.14 — A basis for k -tensors (from W. Tu).** Let V be a vector space of dimension n with basis e_1, \dots, e_n . Let $\alpha^1, \dots, \alpha^n$ be the dual basis for V^\vee . Show that a basis for the space $L_k(V)$ of k -linear functions on V is $\{\alpha^{i_1} \otimes \dots \otimes \alpha^{i_k}\}$ for all multi-indices (i_1, i_2, \dots, i_k) . In particular, this show that $\dim L_k(V) = n^k$.

Solution To show this, we need to show that the basis vectors are linearly independent, and spans all $L_k(V)$. To show the linear independence, consider the following linear combination of all basis vectors, where the total sum is set to zero.

$$\sum C_{i_1, \dots, i_k} \alpha^{i_1} \otimes \dots \otimes \alpha^{i_k} = 0,$$

where the sum is done on all of the multi-indices (i_1, \dots, i_k) . This sum implies that all of the coefficients are zero. We can see this by applying $(e_{j_1}, \dots, e_{j_k})$ to both sides. We have

$$(\alpha^{i_1} \otimes \dots \otimes \alpha^{i_k})(e_{j_1}, \dots, e_{j_k}) = 0$$

if there is any mismatch between the multi-indices I, J . This implies $C_{j_1, \dots, j_k} = 0$ for all multi-indices. Thus we can conclude that the basis vectors are linearly independent.

To show that the basis vectors span all the space $L_k(V)$, let $f \in L_k(V)$ be any k -tensor. Define

$$g = \sum f(e_{i_1}, \dots, e_{i_k}) \alpha^{i_1} \otimes \dots \otimes \alpha^{i_k}.$$

We can see that

$$g(e_{j_1}, \dots, e_{j_k}) = f(e_{j_1}, \dots, e_{j_k}) \quad \text{for all multi-indices}$$

From linearity we can deduce that $f = g$. Thus the basis vectors span the whole space.

■ **Problem 1.15 — A characterization of alternating k -vectors (from W. Tu).** Let f be a k -tensor on a vector space V . Prove that f is alternating if and only if f changes sign whenever two successive arguments are interchanged

$$f(\dots, v_{i+1}, v_i, \dots) = -f(\dots, v_i, v_{i+1}, \dots).$$

Solution The first direction follows immediately from the definition. I.e. assume f is alternating. Then sign of the permutation corresponding interchanging two successive arguments is a single transposition which has sign -1. Thus from definition we have $f(\dots, v_{i+1}, v_i, \dots) = -f(\dots, v_i, v_{i+1}, \dots)$.

For the converse, first observe that we can decompose arbitrary permutation to cycles, and then decompose cycles into transpositions. Eventually, we can decompose every transposition to transposition of adjacent elements. See the Lemma below

Lemma 1.2 — Decomposing arbitrary transposition to transposition of adjacent elements. Let $\tau = (k \ l)$ be an arbitrary transposition where $k < l$. Assume that k, l are m distance apart (i.e. $l - k = m$). Then we can write

$$(k \ l) = (k \ k+1) \cdot (k+1 \ k+2) \cdots (l-1 \ l) \cdot (l-2 \ l-1) \cdots (k \ k+1).$$

Thus $(k \ l)$ can be decomposed to $2m - 1$ adjacent transpositions.

Let $\sigma \in S_k$ be an arbitrary permutation. We can decompose it to the following permutations

$$\sigma = \tau_1 \tau_2 \cdots \tau_l,$$

where from the number of permutations we can see that $\text{sign}(\sigma) = (-1)^l$. Let τ_i be one of the transpositions above that interchanges two elements m distance apart. Then

$$\tau_i f = (-1)^{2m-1} f = -f.$$

This follows immediately from the Lemma above. Thus we can write

$$\sigma f = (-1)^l f = \text{sign}(\sigma) f.$$

This shows that f is an alternating k -tensor.

■ **Problem 1.16 — Another characterization of alternating k -tensors (from W. Tu).** Let f be a k -tensor on a vector space V . Prove that f is alternating if and only if

$$f(v_1, \dots, v_k) = 0$$

whenever two of the vectors v_1, \dots, v_k are equal.

Solution We first start with the forward direction. Assume f is alternating. Thus for $\sigma \in S_k$ we have $\sigma f = \text{sign}(\sigma) f$. Without loss of generality, we can assume $i < j$ and let $(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \in V^k$ where $v_i = v_j$. Thus we can write

$$f(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = f(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

Furthermore, let $\tau = (v_i, v_j)$, i.e. a transposition that interchanges v_i with v_j . Hence

$$\begin{aligned} f(v_1, \dots, v_i, \dots, v_j, \dots, v_k) &= \tau f(v_1, \dots, v_j, \dots, v_i, \dots, v_k) \\ &= -f(v_1, \dots, v_j, \dots, v_i, \dots, v_k). \end{aligned}$$

This implies

$$f(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = f(v_1, \dots, v_j, \dots, v_i, \dots, v_k) = 0$$

To show the converse, let $\sigma \in S_k$ be any permutation. We can decompose this permutation into transpositions, i.e. $\sigma = \tau_1 \tau_2 \cdots \tau_r$. Let τ be any of transpositions above that permutes $(i \ j)$. Let $(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \in V^k$. Since V^k is a vector space we can write

$$(v_1, \dots, v_i, \dots, v_j, \dots, v_k) - (v_1, \dots, v_i, \dots, v_i, \dots, v_k) - (v_1, \dots, v_j, \dots, v_j, \dots, v_k) = -(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

Applying f to both sides, and using the hypothesis and k -linearity of f we will get

$$f(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -f(v_1, \dots, v_j, \dots, v_i, \dots, v_k) = -\tau f(v_1, \dots, v_i, \dots, v_j, \dots, v_k).$$

Thus for each transposition in the decomposition $\sigma = \tau_1 \tau_2 \cdots \tau_r$ we will get a factor of -1. Thus

$$\sigma f = (-1)^r f = \text{sign}(\sigma) f$$

■ **Problem 1.17 — Wedge product and scalars (from W. Tu).** Let V be a vector space. For $a, b \in \mathbb{R}$, $f \in A_k(V)$, and $g \in A_\ell(V)$, show that $af \wedge bg = (ab)f \wedge g$.

Solution This follows immediately from the definition of wedge product.

$$af \wedge bg = \frac{1}{k!\ell!} A(af \otimes bg) = \frac{ab}{k!\ell!} A(f \otimes g) = (ab)f \wedge g.$$

■ **Problem 1.18 — Transformation rule for a wedge product of covectors (from W. Tu).** Suppose two sets of covectors on a vector space V , β^1, \dots, β^k and $\gamma^1, \dots, \gamma^k$ are related by

$$\beta^i = \sum_{j=1}^k a_j^i \gamma^j, \quad i = 1, 2, \dots, k,$$

for a $k \times k$ matrix $A = [a_j^i]$. Show that

$$\beta^1 \wedge \dots \wedge \beta^k = (\det A) \gamma^1 \wedge \dots \wedge \gamma^k.$$

Solution It will be helpful if we write the relation between β^i 's and γ^j 's as a matrix multiplication. To do so we can write

$$\begin{pmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^k \end{pmatrix} = \begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_k^1 \\ a_1^2 & a_2^2 & \cdots & a_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^k & a_2^k & \cdots & a_k^k \end{pmatrix} \begin{pmatrix} \gamma^1 \\ \gamma^2 \\ \vdots \\ \gamma^k \end{pmatrix}$$

On the other hand, we know

$$\begin{aligned} (\beta^1 \wedge \dots \wedge \beta^k)(v_1, \dots, v_k) &= \det \begin{pmatrix} \beta^1(v_1) & \beta^1(v_2) & \cdots & \beta^1(v_k) \\ \beta^2(v_1) & \beta^2(v_2) & \cdots & \beta^2(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \beta^k(v_1) & \beta^k(v_2) & \cdots & \beta^k(v_k) \end{pmatrix} \\ &= \det \left[\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_k^1 \\ a_1^2 & a_2^2 & \cdots & a_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^k & a_2^k & \cdots & a_k^k \end{pmatrix} \begin{pmatrix} \gamma^1(v_1) & \gamma^1(v_2) & \cdots & \gamma^1(v_k) \\ \gamma^2(v_1) & \gamma^2(v_2) & \cdots & \gamma^2(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^k(v_1) & \gamma^k(v_2) & \cdots & \gamma^k(v_k) \end{pmatrix} \right] \\ &= \det(A) \det \begin{pmatrix} \gamma^1(v_1) & \gamma^1(v_2) & \cdots & \gamma^1(v_k) \\ \gamma^2(v_1) & \gamma^2(v_2) & \cdots & \gamma^2(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^k(v_1) & \gamma^k(v_2) & \cdots & \gamma^k(v_k) \end{pmatrix} \end{aligned}$$

Thus we can write

$$(\beta^1 \wedge \dots \wedge \beta^k)(v_1, \dots, v_k) = \det(A) (\gamma^1 \wedge \dots \wedge \gamma^k)(v_1, \dots, v_k).$$

■ **Problem 1.19 — Transformation rule for k-covectors (from W. Tu).** Let f be a k -covector on a vector space V (of say, dimension n). Suppose two sets of vectors u_1, \dots, u_k and v_1, \dots, v_k in V are related by

$$u_j = \sum_{i=1}^k a_j^i v_i, \quad j = 1, \dots, k,$$

for a $k \times k$ matrix $A = [a_j^i]$. Show that

$$f(u_1, \dots, u_k) = (\det A) f(v_1, \dots, v_k).$$

By

Solution Since f is a k -vector, then we can expand it in its basis

$$f = \sum_{I \in AM} C_I \alpha^I,$$

where AM is the set of all ascending multi-indices $I = (i_1, \dots, i_k)$ for $1 \leq i_1 < i_2 < \dots < i_k \leq n$. Thus

$$f(u_1, \dots, u_k) = \sum_{I \in AM} C_I \alpha^I(u_1, \dots, u_k).$$

Let's focus on the term $I = (i_1, \dots, i_k)$. For this term we can write

$$\begin{aligned} (\alpha^{i_1} \wedge \dots \wedge \alpha^{i_k})(u_1, \dots, u_k) &= \det \begin{pmatrix} \alpha^{i_1}(u_1) & \alpha^{i_1}(u_2) & \dots & \alpha^{i_1}(u_k) \\ \alpha^{i_2}(u_1) & \alpha^{i_2}(u_2) & \dots & \alpha^{i_2}(u_k) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{i_k}(u_1) & \alpha^{i_k}(u_2) & \dots & \alpha^{i_k}(u_k) \end{pmatrix} \\ &= \det \left[\begin{pmatrix} \alpha^{i_1}(v_1) & \alpha^{i_1}(v_2) & \dots & \alpha^{i_1}(v_k) \\ \alpha^{i_2}(v_1) & \alpha^{i_2}(v_2) & \dots & \alpha^{i_2}(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{i_k}(v_1) & \alpha^{i_k}(v_2) & \dots & \alpha^{i_k}(v_k) \end{pmatrix} \begin{pmatrix} a_1^1 & a_2^1 & \dots & a_k^1 \\ a_1^2 & a_2^2 & \dots & a_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^k & a_2^k & \dots & a_k^k \end{pmatrix} \right] \\ &= \det(A) \det \begin{pmatrix} \alpha^{i_1}(v_1) & \alpha^{i_1}(v_2) & \dots & \alpha^{i_1}(v_k) \\ \alpha^{i_2}(v_1) & \alpha^{i_2}(v_2) & \dots & \alpha^{i_2}(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{i_k}(v_1) & \alpha^{i_k}(v_2) & \dots & \alpha^{i_k}(v_k) \end{pmatrix} \\ &= \det(A) (\alpha^{i_1} \wedge \dots \wedge \alpha^{i_k})(v_1, \dots, v_k) \end{aligned}$$

Thus we can write

$$f(u_1, \dots, u_k) = \sum_{I \in AM} C_I \alpha^I(u_1, \dots, u_k) = \det(A) \sum_{I \in AM} C_I \alpha^I(v_1, \dots, v_k)$$

or

$$\boxed{f(u_1, \dots, u_k) = \det(A) f(v_1, \dots, v_k).}$$

□

■ **Problem 1.20 — Vanishing of a covector of top degree (from W. Tu).** Let V be a vector space of dimension n . Prove that if an n -covector ω vanishes on a basis e_1, \dots, e_n for V , then ω is the zero covector on V .

Solution First, observe that for a vector space of dimension n we have

$$\dim(A_n) = \binom{n}{n} = 1, \quad \mathcal{B} = \{\alpha^1 \wedge \dots \wedge \alpha^n\}.$$

Let $f \in A_n(V)$. Thus we can write

$$f = g\alpha^1 \wedge \dots \wedge \alpha^n \quad \text{for some } g \in \mathbb{R}.$$

On the other hand, we know

$$(\alpha^1 \wedge \dots \wedge \alpha^n)(e_1, \dots, e_n) = \det [\alpha^i(e_j)] = 1.$$

This $f(e_1, \dots, e_n) = 0$ if and only if $g = 0$.

■ **Problem 1.21 — Linear independence of covectors (from W. Tu).** Let $\alpha^1, \dots, \alpha^k$ be 1-covectors on a vector space V . Show that $\alpha^1 \wedge \dots \wedge \alpha^k \neq 0$ if and only if $\alpha^1, \dots, \alpha^k$ are linearly independent in the dual space V^\vee .

Solution For the forward direction, we want to show

$$\alpha^1 \wedge \dots \wedge \alpha^k \neq 0 \implies \alpha^1, \dots, \alpha^k \text{ are linearly independent.}$$

To show this we will prove the contrapositive. Assume $\alpha^1, \dots, \alpha^k$ are linearly dependent. Thus we can write one of them as a linear combination of others. With out loss of generality, assume

$$\alpha^1 = c_2\alpha^2 + \dots + c_n\alpha^n,$$

where c_i s are not all zero. Consider

$$\alpha^1 \wedge \dots \wedge \alpha^k = c_2\alpha^2 \wedge \dots \wedge \alpha^k + \dots + c_k\alpha^k \wedge \dots \wedge \alpha^k = 0.$$

The last equality is true since $\alpha^I = 0$ if the multi-indices I as at least two repetitive indices.

For the converse, we want to show

$$\alpha^1, \dots, \alpha^k \text{ are linearly independent.} \implies \alpha^1 \wedge \dots \wedge \alpha^k \neq 0$$

Assume $\alpha^1, \dots, \alpha^k$ are linearly independent. Then it can be extended as a basis for V^\vee as $\alpha^1, \dots, \alpha^k, \dots, \alpha^n$. Let v_1, v_2, \dots, v_n be the dual basis for V . Then

$$(\alpha^1 \wedge \dots \wedge \alpha^k)(v_1, \dots, v_k) = \det [\alpha^i(v_j)] = 1 \neq 0.$$

Thus $\alpha^1 \wedge \dots \wedge \alpha^k$ can not be a zero k -covector.

■ **Problem 1.22 — Exterior multiplication (from W. Tu).** Let α be a non-zero 1-covector and γ a k -covector on a finite dimensional vector space V . Show that $\alpha \wedge \gamma = 0$ if and only if $\gamma = \alpha \wedge \beta$ for some $(k-1)$ -covector β on V .

Solution First, we prove the first direction. Let $\gamma \in A_k(V), \alpha \in A_1(V)$ such that $\gamma = \alpha \wedge \beta$ for some $\beta \in A_{k-1}(V)$. Then

$$\alpha \wedge \gamma = \alpha \wedge \alpha \wedge \beta = 0.$$

For the converse, let α and γ as before and we want to prove

$$\alpha \wedge \gamma = 0 \implies \gamma = \alpha \wedge \beta \text{ for some } \beta \in A_{k-1}(V).$$

We prove the contrapositive. Assume $\forall \beta \in A_{k-1}(V)$ we have $\gamma \neq \alpha \wedge \beta$. Then by wedge product of α to both sides we will get

$$\alpha \wedge \gamma \neq \alpha \wedge \alpha \wedge \beta = 0.$$

This completes the proof. □

■ **Problem 1.23 — A basis for 3-covectors (from W. Tu).** Let x^1, x^2, x^3, x^4 be the coordinates on \mathbb{R}^4 and p a point in \mathbb{R}^4 . Write down a basis for the vector space $A_3(T_p(\mathbb{R}^4))$.

Solution First, observe that a basis for $T_p(\mathbb{R}^4)$ is $\{\frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^4}|_p\}$, and for $T_p^*(\mathbb{R}^n)$ a standard basis is $\{dx_1|_p, \dots, dx_4|_p\}$. Thus a standard basis for $A_3(T_p(\mathbb{R}^4))$ would be

$$\mathcal{B} = \{dx^1 \wedge dx^2 \wedge dx^3|_p, dx^1 \wedge dx^2 \wedge dx^4|_p, dx^1 \wedge dx^3 \wedge dx^4|_p, dx^2 \wedge dx^3 \wedge dx^4|_p\}.$$

■ **Problem 1.24 — Wedge product of a 2-form with a 1-form (from W. Tu).** Let Ω be a 2-form and τ a 1-form on \mathbb{R}^3 . If X, Y, Z are vector fields on M find an explicit formula for $(\omega \wedge \tau)(X, Y, Z)$ in terms of the values of ω and τ on the vector fields X, Y, Z .

Solution From the definition of wedge product, and using the notion of $(2, 1)$ shuffles, that are $(1 < 2, 3), (1 < 3, 2), (2 < 3, 1)$ with respective signs $+, -, +$ we can write

$$(\omega \wedge \tau)(X, Y, Z) = (\omega \otimes \tau)(X, Y, Z) - (\omega \otimes \tau)(X, Z, Y) + (\omega \otimes \tau)(Y, Z, X),$$

or alternatively

$$(\omega \wedge \tau)(X, Y, Z) = \omega(X, Y)\tau(Z) - \omega(X, Z)\tau(Y) + \omega(Y, Z)\tau(X).$$

■ **Problem 1.25 — A closed 1-form on the punctured plane (from W. Tu).** Define a 1-form ω on $\mathbb{R}^2 - \{0\}$ by

$$\omega = \frac{1}{x^2 + y^2}(-ydx + xdy).$$

Show that ω is closed.

Solution define $f, g : \mathbb{R}^2 - \{0\} \rightarrow \mathbb{R}$ as

$$f(x, y) = \frac{-y}{x^2 + y^2}, \quad g(x, y) = \frac{x}{x^2 + y^2}.$$

Thus we can write ω as $\omega = fdx + gdy$. The exterior derivative of ω will be

$$d\omega = (g_x - f_y)dx \wedge dy.$$

By a simple calculation we can observe

$$f_x(x, y) = \frac{y^2 - x^2}{x^2 + y^2}, \quad g_y(x, y) = \frac{y^2 - x^2}{x^2 + y^2}.$$

Thus we will have

$$d\omega = 0.$$

■ **Problem 1.26 — A 1-form on \mathbb{R}^3 (from W. Tu).** Let ω be the 1-form $zdx - dz$, and let X be the vector field $y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on \mathbb{R}^3 . Compute $\omega(X)$ and $d\omega$.

Solution To calculate $\omega(X)$ we can write

$$\omega(X) = X\omega = (y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y})(zdx - dz) = yz - x.$$

And to calculate $d\omega$ we have

$$d\omega = dz \wedge dx$$

■ **Problem 1.27 — A 2-form on \mathbb{R}^3 (from W. Tu).** At each point $p \in \mathbb{R}^3$, define a bilinear function ω_p on $T_p(\mathbb{R}^3)$ by

$$\omega_p(a, b) = \omega_p \left(\begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix}, \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix} \right) = p^3 \det \begin{bmatrix} a^1 & b^1 \\ a^2 & b^2 \end{bmatrix},$$

For tangent vectors $a, b \in T_p(\mathbb{R}^3)$, where p^3 is the third component of $p = (p^1, p^2, p^3)$. Since ω_p is an alternating bilinear function on $T_p(\mathbb{R}^3)$, ω is a 2-form on \mathbb{R}^3 . Write ω in terms of the standard basis $dx^i \wedge dx^j$ at each point.

Solution In the definition of ω , we can replace determinant by the following wedge product

$$\omega_p = p^3 dx^1 \wedge dx^2.$$

■ **Problem 1.28 — Exterior calculus (from W. Tu).** Suppose the standard coordinates on \mathbb{R}^2 are called r and θ (this \mathbb{R}^2 is the (r, θ) plane not the (x, y) plane). If $x = r \cos \theta$ and $y = r \sin \theta$, calculate dx, dy , and $dx \wedge dy$ in terms of dr and $d\theta$.

Solution Using the relations $x = r \cos \theta$ and $y = r \sin \theta$, we can use the definition of exterior derivative to have

$$dx = \cos \theta \, dr - r \sin \theta \, d\theta, \quad dy = \sin \theta \, dr + r \cos \theta \, d\theta.$$

We can write this in a more clean way as

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} \begin{bmatrix} dr \\ d\theta \end{bmatrix}.$$

By using the fact proved in Problem [Problem 1.18](#) we will have

$$dx \wedge dy = \det(A) dr \wedge d\theta = r dr \wedge d\theta.$$

■ **Problem 1.29 — Exterior calculus (from W. Tu).** Suppose the standard coordinates on \mathbb{R}^3 are called ρ, ϕ and θ . If $x = \rho \cos \phi \sin \theta$, $y = \rho \sin \phi \sin \theta$, and $z = \rho \cos \theta$, calculate dx, dy, dz , and $dx \wedge dy \wedge dz$ in terms of $d\rho, d\phi$, and $d\theta$.

Solution Similar to the problem above, using the definition of the exterior derivative we have

$$\begin{aligned} dx &= \cos \phi \sin \theta \, d\rho - \rho \sin \phi \sin \theta \, d\phi + \rho \cos \phi \cos \theta \, d\theta, \\ dy &= \sin \phi \sin \theta \, d\rho + \rho \cos \phi \sin \theta \, d\phi + \rho \sin \phi \cos \theta \, d\theta, \\ dz &= \cos \theta \, d\rho - \rho \sin \theta \, d\theta. \end{aligned}$$

or in matrix notation we have

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{pmatrix} \cos \phi \sin \theta & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \theta & 0 & -\rho \sin \theta \end{pmatrix} \begin{bmatrix} d\rho \\ d\phi \\ d\theta \end{bmatrix}.$$

From the fact proved in Problem [Problem 1.18](#) we can write

$$dx \wedge dy \wedge dz = \det(A) \, d\rho \wedge d\theta \wedge d\phi.$$

We now need to calculate $\det(A)$

$$\det(A) = -\rho^2 \cos^2 \theta \sin \theta - \rho^2 \sin^3 \theta = -\rho^2 \sin \theta.$$

Thus we have

$$dx \wedge dy \wedge dz = (-\rho^2 \sin \theta) \, d\rho \wedge d\phi \wedge d\theta.$$

■ **Problem 1.30 — Wedge product (from W. Tu).** Let α be a 1-form and β a 2-form on \mathbb{R}^3 . Then

$$\begin{aligned} \alpha &= a_1 dx^1 + a_2 dx^2 + a_3 dx^3, \\ \beta &= b_1 dx^1 \wedge dx^3 + b_2 dx^3 \wedge dx^1 + b_3 dx^1 \wedge dx^2. \end{aligned}$$

Simplify the expression $\alpha \wedge \beta$ as much as possible.

Solution By following the rules for the wedge product we will have

$$\alpha \wedge \beta = (a_2 b_2 + a_3 b_3 - a_2 b_1) \, dx^1 \wedge dx^2 \wedge dx^3$$

■ **Problem 1.31 — Wedge product and cross product (from W. Tu).** The correspondence between differential forms and vector fields on an open subset of \mathbb{R}^3 also makes sense point wise. Let V be a vector space of dimension 3 with basis e_1, e_2, e_3 , and dual basis $\alpha^1, \alpha^2, \alpha^3$. To a 1-covector $\alpha = a_1 \alpha^1 + a_2 \alpha^2 + a_3 \alpha^3$ on V , we associate the vector $v_\alpha = \langle a_1, a_2, a_3 \rangle \in \mathbb{R}^3$. To the 2-covector

$$\gamma = c_1 \alpha^1 \wedge \alpha^3 + c_2 \alpha^3 \wedge \alpha^1 + c_3 \alpha^1 \wedge \alpha^2,$$

on V , we associate the vector $v_\gamma = \langle c_1, c_2, c_3 \rangle \in \mathbb{R}^3$. Show that under this correspondence, the wedge product of 1-covectors corresponds to the cross product of vectors in \mathbb{R}^3 : if $\alpha = a_1 \alpha^1 + a_2 \alpha^2 + a_3 \alpha^3$ and $\beta = b_1 \alpha^1 + b_2 \alpha^2 + b_3 \alpha^3$, then we have

$$v_{\alpha \wedge \beta} = v_\alpha \times v_\beta.$$

Solution Let $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$, and $\beta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$. Then for the wedge product we have

$$\alpha \wedge \beta = (a_1b_2 - a_2b_1)\alpha^1 \wedge \alpha^3 + (a_1b_3 - a_3b_1)\alpha^1 \wedge \alpha^2 + (a_2b_3 - a_3b_2)\alpha^2 \wedge \alpha^3.$$

We can associate this with vector $v_{\alpha \wedge \beta} = \langle a_1b_2 - a_2b_1, a_1b_3 - a_3b_1, a_2b_3 - a_3b_2 \rangle \in \mathbb{R}^3$. On the other hand, we can write $v_\alpha \times v_\beta = \langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle a_1b_2 - a_2b_1, a_1b_3 - a_3b_1, a_2b_3 - a_3b_2 \rangle$. This shows that

$$v_{\alpha \wedge \beta} = v_\alpha \times v_\beta.$$

■ **Problem 1.32 — Commutator of derivations and antiderivations (from W. Tu).** Let $A = \bigoplus_{k=-\infty}^{\infty} A^k$ be a graded algebra over a field K with $A^k = 0$ for $k < 0$. Let m be an integer. A superderivation of A of degree m is a K -linear map $D : A \rightarrow A$ such that for all k , $D(A^k) \in A^{k+m}$ and for all $a \in A^k$ and $b \in A^\ell$,

$$D(ab) = (Da)b + (-1)^{k\ell} aD(b).$$

If D_1 and D_2 are two superderivations of A of respective degrees m_1 and m_2 , define their commutator to be

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{m_1 m_2} D_2 \circ D_1.$$

Show that $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$. A superderivation is said to be even or odd depending on the parity of its degree. An even superderivation is a derivation; an odd superderivation is an antiderivation.

Solution To show that $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$ we need to show

$$[D_1, D_2](ab) \stackrel{?}{=} [D_1, D_2](a)b + (-1)^{k(m_1+m_2)} a[D_1, D_2](b).$$

Using the definition of $[D_1, D_2]$ we can expand this as

$$[D_1, D_2](ab) \stackrel{?}{=} [D_1(D_2(a)) - (-1)^{m_1 m_2} D_2(D_1(a))]b + (-1)^{k(m_1+m_2)} a[D_1(D_2(b)) - (-1)^{m_1 m_2} D_2(D_1(b))]. \quad (\text{Q.1})$$

To show this, we simply start with $[D_1, D_2]$ and apply it on $a \in A_k, b \in A_\ell$ and use the definition of $[D_1, D_2]$ to simplify it and turn it into the format of equation (Q.1).

$$\begin{aligned} [D_1, D_2](ab) &= (D_1 \circ D_2)(ab) - (-1)^{m_1 m_2} (D_2 \circ D_1)(ab) = D_1(D_2(ab)) - (-1)^{m_1 m_2} D_2(D_1(ab)) \\ &= D_1(D_2(a))b + (-1)^{m_1(k+m_2)} D_2(a)D_1(b) + (-1)^{m_2 k} [D_1(a)D_2(b) + (-1)^{m_1 k} aD_1(D_2(b))] \\ &\quad - (-1)^{m_1 m_2} [D_2(D_1(a))b + (-1)^{m_2(k+m_1)} D_1(a)D_2(b) + (-1)^{m_1 k} [D_2(a)D_1(b) + (-1)^{m_2 k} aD_2(D_1(b))]] \end{aligned}$$

Now by matching the terms we can write

$$\begin{aligned} [D_1, D_2](ab) &= [D_1(D_2(a)) - (-1)^{m_1 m_2} D_2(D_1(a))]b + (-1)^{k(m_1+m_2)} a[D_1(D_2(b)) - (-1)^{m_1 m_2} D_2(D_1(b))] \\ &\quad + \underbrace{D_1(a)D_2(b) [(-1)^{m_2 k} - (-1)^{m_2(k+2m_1)}]}_{=0}. \end{aligned}$$

The last term is equal to zero since if $m_2 k$ is even, then $m_2 k + 2m_1 m_2$ is also even, which leads to $(-1)^{m_2 k} - (-1)^{m_2(k+2m_1)} = -1 + 1 = 0$. As the other case, if $m_2 k$ is odd, then $m_2 k + 2m_1 m_2$ is also odd, where we will have $(-1)^{m_2 k} - (-1)^{m_2(k+2m_1)} = 1 - 1 = 0$. This proves that $[D_1, D_2]$ is indeed an superderivative of order $m_1 + m_2$.

2. Manifolds

Very important note: Throughout this text, to avoid repeating some words like *smooth*, and *etc*, we often omit them and we let the context to reflect these notions. So we emphasize that throughout this text, unless otherwise specified, by a **manifold** we always mean a C^∞ manifold. An atlas or a chart on a smooth manifold means an atlas or a chart contained in the differentiable structure of the smooth manifold.

2.1 Topological manifolds and Smooth manifolds

We start with the definition of topological manifolds. We will later study the smooth manifolds that are of our main interest in this lecture. But first, we need to review some basic definition.

Definition 2.1 — Hausdorff topological space. A set A , along with a collection of subsets of A called $\mathcal{T} \subset 2^A$, i.e. (A, \mathcal{T}) is called a topological space if we have

(I) $\emptyset \in \mathcal{T}$ and $A \in \mathcal{T}$.

(II) For any infinite collection $\{A_\alpha\}_\alpha \subset \mathcal{T}$ we have

$$\bigcup_{\alpha} A_\alpha \in \mathcal{T}.$$

(III) For a finite collection $\{A_i\}_{i \in I}$ where $I = \{1, 2, \dots, N\}$ for some $N \in \mathbb{N}$, we have

$$\bigcap_{i \in I} A_i \in \mathcal{T}.$$

(IV) $\forall x, y \in A$ there exists $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$ and we have $U \cap V = \emptyset$.

The following definition reviews the notion of second countable topological spaces.

Definition 2.2 — Second countable topological spaces. Let (X, \mathcal{T}) be a topological space. This space is second countable if there exists an at most countable collection of sets $\mathcal{B} \subset \mathcal{T}$ such that

$\forall U \in \mathcal{T}$ we have

$$U = \bigcup_{A \in \mathcal{B}} A.$$

I.e. any open set in \mathcal{T} can be written as a union of sets in \mathcal{B}

The last piece of definition that we need is the notion of locally Euclidean spaces.

Definition 2.3 — Locally Euclidean topological spaces. Let (X, \mathcal{T}) be a topological space. We say X is locally Euclidean if $\forall p \in X$, there exists an open neighborhood $p \in U \in \mathcal{T}$ such that is homeomorphic to an *open* subset of \mathbb{R}^n . I.e. there is a homeomorphism $\phi : U \rightarrow \mathbb{R}^n$. We call the pair $(U, \phi : U \rightarrow \mathbb{R}^n)$ a *chart*, U a *coordinate neighborhood* or a **coordinate open set**, and ϕ a *coordinate map* or *coordinate system* on U .

Definition 2.4 — Topological manifolds. A topological manifold is a Hausdorff, second countable topological space that is locally Euclidean space. It is said to be of dimension n , if it is locally Euclidean of dimension n .

In the following section, we will review the notion of compatible charts which will be central to our study of manifolds. Assume we have two charts (U, ϕ) and (V, ψ) . Since U, V are open in M (the manifold), then $U \cap V$ is open in U and V . Furthermore, since ϕ is a homeomorphism to an open subset of \mathbb{R}^n then $\phi(U \cap V)$ and $\psi(U \cap V)$ are also open. One way to show this, let $\phi(U \cap V) = B$, $\phi(U) = A$, and $A \setminus B = C$. We know that A is open, and $B \cup C = A$, $B \cap C = \emptyset$. Then A being open implies B and C is open. Now we can make the following definition for C^∞ -compatible maps.

Definition 2.5 — C^∞ compatible maps. Let M be a topological manifold, where (U, ϕ) and (V, ψ) are two charts. These charts are called C^∞ -compatible if the maps

$$\phi \circ \psi^{-1} : \psi(U \cap V) \rightarrow \phi(U \cap V) \quad \text{and} \quad \psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V),$$

are C^∞ . These two maps are called *transition functions* between charts.

■ **Remark 2.1** If two U, V in the definition above, i.e. $U \cap V = \emptyset$, then they are automatically compatible.

Definition 2.6 — C^∞ atlas. Let M be a topological manifold. The collection of charts $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}$ is called a C^∞ atlas, or simply an atlas if

- all of the charts are pairwise C^∞ compatible,
- the charts cover the whole manifold, i.e. $M = \bigcup_\alpha U_\alpha$.

Lemma 2.1 Let $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ be an atlas for the manifold M . Let (V, ψ) and (W, σ) be two charts where both of them are compatible with the atlas \mathcal{A} . Then these two charts are compatible with each other.

Proof. We start by showing that $\psi \circ \sigma^{-1}$ is C^∞ on $V \cap W$. Let $p \in V \cap W$. Then $\exists \alpha \in I$ such that $p \in U_\alpha$ for the chart (U_α, ϕ_α) . Then

$$(\psi \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \sigma^{-1}) : \sigma(W \cap U_\alpha \cap V) \rightarrow \psi(W \cap U_\alpha \cap V)$$

is C^∞ in $\sigma(W \cap U_\alpha \cap V)$, because it is the composition of C^∞ maps. Call $\mathbb{B}_\alpha = W \cap U_\alpha \cap V$. Then

what we have shows is simply $\forall p \in V \cap W$, there exists an open set $p \in \mathbb{B}_\alpha \subset V \cap W$ for $\alpha \in I$ and the map $\psi \circ \sigma^{-1}$ is C^∞ in \mathbb{B}_α . Since this holds for every $p \in V \cap W$ this proves that $\psi \circ \sigma^{-1}$ is C^∞ on $\sigma(V \cap W)$. Similarly, we can show $\sigma \circ \psi^{-1}$ is C^∞ on $\psi(V \cap W)$, and this completes the proof. \square

■ **Remark 2.2** Note that in an equality like $\psi \circ \sigma^{-1} = (\psi \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \sigma^{-1})$, two functions in the sides of the equality have different domains. So the equality sign here means that these two functions are equal on their common domain.

Definition 2.7 — Maximal atlas. Let \mathcal{U} be an atlas for the manifold M . Then \mathcal{U} is called maximal if it contains any other atlas of the manifold M . Or equivalently, if \mathcal{M} is any other atlas containing \mathcal{U} then $\mathcal{U} = \mathcal{M}$.

We can use the notion of the maximal atlas to define a smooth manifold.

Definition 2.8 — Smooth manifold. A topological manifold together with a maximal atlas is called a smooth manifold. The maximal atlas is also called a differentiable structure on M .

In practice, to show that a topological manifold is smooth, it is not necessary to exhibit a maximal atlas. Existence of any atlas will do so, as proposed by the following proposition.

Proposition 2.1 Any atlas of a locally euclidean space is contained in a *unique* maximal atlas.

Proof. Let $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}$ be any atlas for the locally Euclidean space M . Let $S = \{(V_i, \psi_i)\}$ denote the set of all charts compatible with \mathcal{U} . Construct the set $\mathfrak{M} = \mathcal{U} \cup S$. We claim that this set is a maximal atlas. Let (W, ψ) be a chart compatible with \mathfrak{M} . Then it should also be compatible with \mathcal{U} , thus it is contained in the set S , i.e. $(W, \psi) \in S$, thus in \mathfrak{M} . This shows that the atlas \mathfrak{M} is maximal.

To show the uniqueness, let \mathfrak{M}' be another maximal atlas. Since \mathfrak{M}' is compatible with \mathcal{U} , thus it is also compatible with \mathfrak{M} . But due to the construction it is contained in the new atlas, thus $\mathfrak{M}' \subset \mathfrak{M}$. Thus the maximal atlas is unique. \square

Considering the proposition above, we arrive at the following important observation.

Observation 2.1.1 — Showing a space is an smooth manifold. To show that a space is an smooth manifold, we just need to show that the space is

- a Hausdorff topological space that is also second countable,
- there exists any C^∞ atlas (not necessarily maximal).

2.2 Smooth maps on a manifold

In this section, we will study the notion of smooth maps between manifolds. We will use the notion of coordinate charts to transfer the notion of smooth maps from Euclidean spaces to manifolds. We start with the functions on manifolds.

Definition 2.9 — Smooth functions on manifolds. Let $f : M \rightarrow \mathbb{R}$ be a function on a manifold. The function f is smooth at point $p \in M$ if there exists a chart (U, ϕ) such that $p \in U$ and the map

$$f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$$

is smooth at p . The function f is said to be C^∞ on M if it is C^∞ at every point of M .

■ **Remark 2.3** Note that the smoothness of the function f in the definition above is independent of the local chart (U, ψ) that we choose. Let (V, ψ) be another local chart containing the point p . Then the map $f \circ \psi^{-1}$ is also smooth at p . Because

$$f \circ \psi^{-1} : (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}).$$

We know that the map $(\phi \circ \psi^{-1})$ is smooth at p (since the local charts in an atlas are compatible). Also $(f \circ \phi^{-1})$ is smooth (we have assumed so). Thus $f \circ \psi^{-1}$ is smooth.

Proposition 2.2 — Smoothness of real valued functions. Let M be a manifold of dimension n with atlas \mathfrak{U} , and let $f : M \rightarrow \mathbb{R}$ a real-valued function on M . The following are equivalent:

- (i) The function $f : M \rightarrow \mathbb{R}$ is C^∞ .
- (ii) The manifold M has an atlas such that for every chart (U, ϕ) in the atlas, $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$ is C^∞ .
- (iii) For every chart (V, ψ) on M , the function $f \circ \psi^{-1} : \psi(V) \rightarrow \mathbb{R}$ is C^∞ .

Proof. We prove a cyclic chain of implications $(ii) \implies (i) \implies (iii) \implies (ii)$.

$(ii) \implies (i)$: Let $p \in M$. From the assumption we know that there is an atlas with the desired property. Thus $\exists (U_\alpha, \phi_\alpha)$ that contains p and the function $f \circ \phi_\alpha^{-1} : \phi_\alpha(U) \rightarrow \mathbb{R}$ is smooth. Thus $f : M \rightarrow \mathbb{R}$ is smooth at p , and since the point p was arbitrary, then f is smooth on M .

$(i) \implies (iii)$: Let $p \in U$. From assumption we know that there is a coordinate open set U_α such that $f \circ \phi_\alpha^{-1} : \phi_\alpha(U) \rightarrow \mathbb{R}$ is smooth. Then we can write

$$f \circ \psi^{-1} = (f \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \psi^{-1})$$

which shows that $f \circ \psi^{-1}$ is smooth (look at the remark below).

$(iii) \implies (ii)$: This follows immediately from the definition. □

■ **Remark 2.4** Considering the “very important note” at the beginning of this chapter, we need to emphasize again that in part (iii) , when we say *for every chart* (V, ψ) *on* M , we mean a chart in the unique differentiable structure of the manifold that contains the atlas \mathfrak{U} .

Observation 2.2.1 — From W. Tu. The smoothness conditions from the proposition above will be a recurrent motif through out the book. To prove the smoothness of an object, it is sufficient that a smoothness criterion hold on the charts of some atlas. Once the object is shown to be smooth, it then follows that the same smoothness criterion holds on every chart on the manifold.

Definition 2.10 — Pull back of a function. Let $F : N \rightarrow M$ be a map between manifolds, and $\phi : M \rightarrow \mathbb{R}$. Then the pullback of the function ϕ by F is denoted by $F^* \phi$ and is defined as

$$F^* \phi = \phi \circ F.$$

■ **Remark 2.5** In this terminology, a function $f : M \rightarrow \mathbb{R}$ is smooth on a chart (U, ϕ) if and only if the pullback $(\phi^{-1})^* f$ is smooth on the subset $\phi(U)$ of Euclidean space.

2.2.1 Smooth maps between manifolds

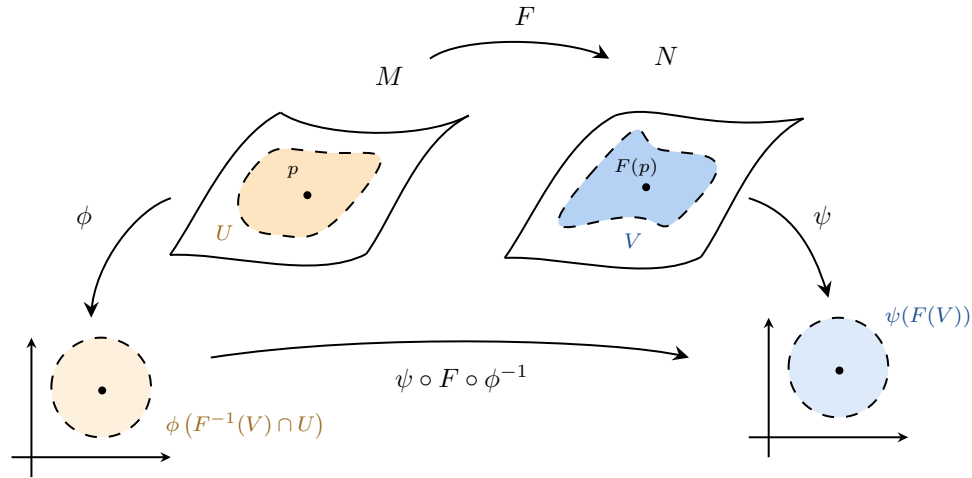
Here in this section we will discuss the smooth maps between manifolds. We will be able to recover the definition of the smooth functions on manifolds as a special case.

Definition 2.11 — Smooth maps between manifolds. Let M, N be manifolds with dimensions m, n respectively. A map $F : M \rightarrow N$ is said to be smooth at point $p \in M$, if there exists charts (U, ϕ) and (V, ψ) such that $p \in U$ and $F(p) \in V$, and the function

$$\psi \circ F \circ \phi^{-1} : \mathbb{R}^m \supset \phi(F^{-1}(V) \cap U) \rightarrow \mathbb{R}^n$$

is smooth at p . The map F is said to be smooth on manifold M if it is smooth at every point of the manifold.

■ **Remark 2.6** The following figure helps to digest the definition above.



■ **Remark 2.7** Since the Euclidean space is indeed a smooth manifold, then we can recover the definition of smooth functions on manifolds from the definition above. Let M be a manifold, and $N = \mathbb{R}^n$ with the atlas $\{(\mathbb{R}^n, \mathbb{1} : \mathbb{R}^n \rightarrow \mathbb{R}^n)\}$, then we will have the notion of vector valued functions on manifold. By setting $n = 1$ we will recover the definition of smooth function on manifold.

In the following proposition we will be showing that the smoothness of the map is independent of the charts chosen, thus the smoothness of the map is well-defined.

Proposition 2.3 — Smoothness of maps between manifolds is well-defined. Suppose $F : N \rightarrow M$ is C^∞ at $p \in N$. If (U, ϕ) is any chart in N that contains p and (V, ψ) is any chart in M that contains $F(p)$, then $\psi \circ F \circ \phi^{-1}$ is smooth at $\phi(p)$.

Proof. Since $F : N \rightarrow M$ is smooth at $p \in N$, then there are charts (G, γ) and (L, λ) (from the corresponding maximal atlases) such that $p \in G \subset N$ and $F(p) \in L \subset M$ and the function

$$(\lambda \circ F \circ \gamma^{-1}) : \gamma(F^{-1}(L) \cap G) \rightarrow \lambda(F(G))$$

is smooth at p . Consider the charts (U, ϕ) and (V, ψ) as above. Then the function

$$\psi \circ F \circ \phi^{-1} = (\psi \circ \lambda^{-1}) \circ (\lambda \circ F \circ \gamma^{-1}) \circ (\gamma \circ \phi^{-1}).$$

Note that the equality sign above merely means the functions are equal on their common domain, as the function in RHS and the function in LHS have different domains. We know that the coordinate maps ψ, λ , and γ, ϕ are compatible respectively. Thus the function $\psi \circ F \circ \phi^{-1}$ is smooth at $\phi(p)$. See the remark below for more details. \square

■ **Remark 2.8** In the proof above and in the equality

$$\psi \circ F \circ \phi^{-1} = (\psi \circ \lambda^{-1}) \circ (\lambda \circ F \circ \gamma^{-1}) \circ (\gamma \circ \phi^{-1})$$

thus equality sign does not indicate that these function are equal, but it just indicates that these two functions are equal in their common domain. To be more clear, for the function in the LHS we have

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \rightarrow \psi(F(U)).$$

But for the function in the LHS we have

$$(\psi \circ \lambda^{-1}) \circ (\lambda \circ F \circ \gamma^{-1}) \circ (\gamma \circ \phi^{-1}) : \phi(F^{-1}(L \cap V) \cap (G \cap U)) \rightarrow \psi(F(L \cap V))$$

Proposition 2.4 — Smoothness of maps in terms of charts. Let N and M be smooth manifolds, and $F : N \rightarrow M$ a continuous map. The following are equivalent:

- (i) The map $F : N \rightarrow M$ is C^∞ .
- (ii) There are atlases \mathfrak{U} for N and \mathfrak{V} for M such that for every chart (U, ϕ) in \mathfrak{U} and (V, ψ) in \mathfrak{V} the map

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \rightarrow \mathbb{R}^m$$

is C^∞ .

- (iii) For every chart (U, ϕ) on N and (V, ψ) on M , the map

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \rightarrow \mathbb{R}^m$$

is C^∞ .

Proof. We will prove this by showing a cyclic chain of implications $(ii) \implies (i) \implies (iii) \implies (ii)$.

- $(ii) \implies (i)$. Let $p \in N$. Then by hypothesis there are charts (U, ϕ) and (V, ψ) such that $p \in U$ and $F(p) \in V$ and the map

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \rightarrow \mathbb{R}^m$$

is smooth at $\phi(p)$, thus F is smooth at p . Since p is arbitrary, then F is smooth on N .

- $(i) \implies (iii)$. This follows immediately from [Proposition 2.3](#).
- $(iii) \implies (ii)$. Since N, M are smooth manifolds, then they have maximal atlases. Let \mathfrak{U} be the maximal atlas for N and \mathfrak{V} be the maximal atlas for M . This (ii) follows immediately from (iii) .

□

Proposition 2.5 — Composition of C^∞ maps. If $F : N \rightarrow M$ and $G : M \rightarrow P$ are C^∞ maps of manifolds, then the composite $G \circ F : N \rightarrow P$ is a smooth map.

Proof. We will demonstrate two proofs for this proposition to demonstrate what happens if we do not notice a possible level of abstraction. The first proof below is a very crude, hard-core, direct proof, that seems to be tough and easy to make mistakes, just because it is not encapsulating some of the details into another proposition. However, for the second proof, it will encapsulate some

of the details into the results of the [Proposition 2.4](#), which will allow us to do a more high level thinking.

proof 1. Let $p \in M$. Since F is smooth, then $\exists (U, \gamma)$ chart for M and $\exists (V, \phi)$ chart for M such that $p \in U$ and $F(p) \in V$ and the map

$$\phi \circ F \circ \gamma^{-1} : \gamma(F^{-1}(V) \cap U) \rightarrow \phi(V)$$

is smooth at $\gamma(p)$. Also, since G is smooth, then at $F(p)$, there exists the charts (W, λ) for M and (Y, ψ) for P such that $F(p) \in W$ and $G(F(p)) \in Y$ and the map

$$\psi \circ G \circ \lambda^{-1} : \lambda(G^{-1}(Y) \cap W) \rightarrow \psi(Y)$$

is smooth at $\lambda(F(p))$. Also note that since the charts (W, λ) and (V, ϕ) for M are compatible (they both are chosen from the maxima atlas), then the maps

$$\phi \circ \lambda^{-1} : \lambda(W \cap V) \rightarrow \phi(W \cap V) \quad \text{and} \quad \lambda \circ \phi^{-1} : \phi(W \cap V) \rightarrow \lambda(W \cap V)$$

are both smooth. Now it is time to glue all of these pieces together to infer the smoothness of the composite function $G \circ F$. Based on these pieces, the function

$$\psi \circ (G \circ F) \circ \gamma^{-1} = (\psi \circ G \circ \lambda^{-1}) \circ (\lambda \circ \phi^{-1}) \circ (\phi \circ F \circ \gamma^{-1}) : \gamma(F^{-1}(G^{-1}(Y) \cap V \cap W) \cap U) \rightarrow \psi(G(F(p)))$$

is smooth at $\gamma(p)$. Since the point $p \in N$ was arbitrary, then the map $G \circ F$ is smooth on M .

proof 2. In this proof, we will hide much of the details of the proof 1 in higher level of abstractions encapsulated in [Proposition 2.4](#). Since F is smooth, then $((i) \implies (iii))$ of [Proposition 2.4](#) for every chart (U, γ) for M and (V, ϕ) for M the map

$$\phi \circ F \circ \gamma^{-1} : \gamma(F^{-1}(V) \cap U) \rightarrow \phi(V)$$

is smooth.. Also, since G is smooth, then for every chart (Y, ψ) for P the map

$$\psi \circ G \circ \phi^{-1} : \phi(G^{-1}(Y) \cap V) \rightarrow \psi(Y)$$

is smooth. Since the composition of smooth real vector valued functions is smooth, then

$$\psi \circ (G \circ F) \circ \gamma^{-1} = (\psi \circ G \circ \phi^{-1}) \circ (\phi \circ F \circ \gamma^{-1})$$

is smooth. Since for every chart (U, γ) and (Y, ϕ) the function $\psi \circ (G \circ F) \circ \gamma^{-1}$ is smooth, then by $(iii) \implies (i)$ of [Proposition 2.4](#) we proved that $G \circ F$ is smooth. \square

2.3 Diffeomorphisms

We start with the definition of diffeomorphism.

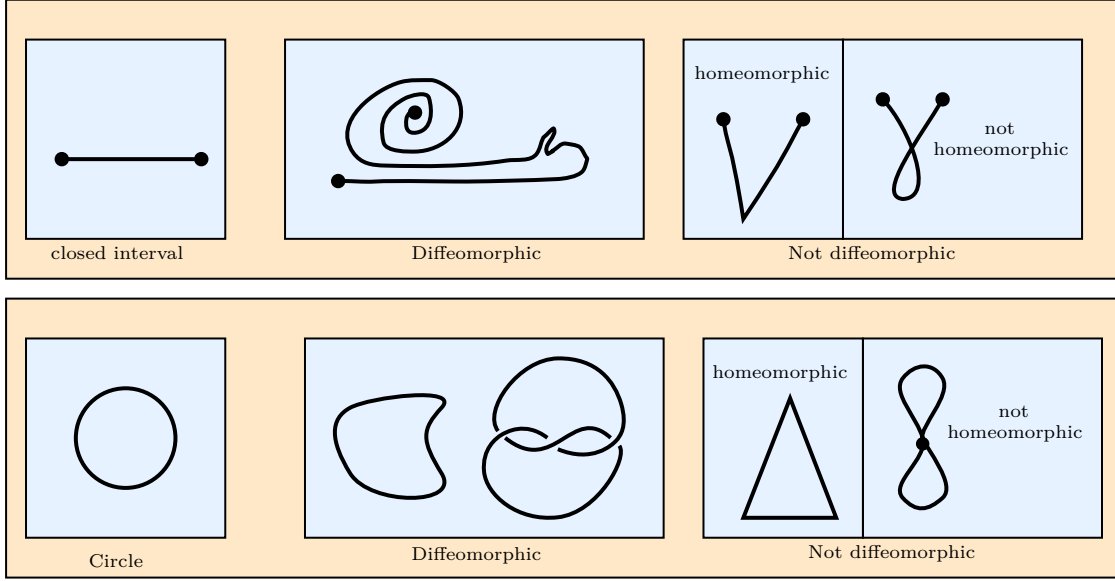
Proposition 2.6 — Diffeomorphism between manifolds. A map $F : M \rightarrow N$ between manifolds M and N is called a diffeomorphism if

- F is a bijection,
- F is C^∞ , and
- F^{-1} is also C^∞ .

The following observation box summarizes the differences between homeomorphisms and diffeomorphisms.

Observation 2.3.1 — Difference between homeomorphisms and diffeomorphisms. homeomorphisms are isomorphisms in the category of topological spaces. However, the diffeomorphisms are isomorphisms in the category of smooth manifolds, where the isomorphism preserves the differential structures. The following figure (original idea from^a) demonstrates some of these ideas.

^amathematical analysis lecture notes.



According to the two following propositions, in a *smooth* manifold, the coordinate maps are diffeomorphisms, and conversely, any diffeomorphism from an open subset of the manifold to an open subset of a Euclidean space can serve as a coordinate map.

Proposition 2.7 — Coordinate maps are diffeomorphisms. If (U, ϕ) is a chart on a manifold M of dimension n , then the coordinate map $\phi : U \rightarrow \phi(U) \in \mathbb{R}^n$ is a diffeomorphism.

Proof. For the proof, we will use the notion of smooth maps between manifolds in a smart way! First, observe that since ϕ is a coordinate map, thus it is automatically a homeomorphism. Thus we just need to show that ϕ and ϕ^{-1} are smooth. Let U be a manifold with an atlas $\{(U, \phi)\}$. Also, let $\phi(U) \subset \mathbb{R}^n$ be a manifold (this is indeed a manifold as it is an open subset of a Euclidean space), with an atlas $\{(\phi(U), \mathbb{1}_{\phi(U)})\}$. In this point of view, the coordinate map ϕ is in fact a map between manifolds. This is smooth, since

$$\mathbb{1}_{\phi(U)} \circ \phi \circ \phi^{-1} : \underbrace{\phi(U \cap \phi^{-1}(\phi(U)))}_{\phi(U)} \rightarrow \phi(U)$$

is smooth, then by Proposition 2.4 (ii) \implies (i), the map ϕ is smooth.

To show the smoothness of the map ϕ^{-1} , we observe that this is a map from the manifold with atlas $\{(\phi(U), \phi^{-1})\}$ to the manifold with atlas $\{(U, \phi)\}$. Since the following map

$$\phi \circ \phi^{-1} \circ \mathbb{1}_{\phi(U)} : \phi(U) \rightarrow \phi(U)$$

is smooth (since it is just the identity map), then by Proposition 2.4 (ii) \implies (i) the map ϕ^{-1} is smooth.

Putting these two results together, we have shown that the coordinate map ϕ is indeed a diffeomorphism from U to \mathbb{R}^n . \square

The following proposition is the converse of the proposition above.

Proposition 2.8 Let U be an open subset of a manifold M of dimension n . If $F : U \rightarrow F(U) \subset \mathbb{R}^n$ is a diffeomorphism onto an open subset of \mathbb{R}^n , then (F, U) is a chart in the differentiable structure of M .

Proof. Since F is a diffeomorphism, then it is certainly a homeomorphism. Thus we just need to show that (U, F) is a chart, i.e. compatible with the maximal atlas of the manifold (note that since M is a smooth manifold, then there exists a maximal atlas). Let $p \in U$ and let (V, ψ) be a chart that contains p . Then consider the function

$$F \circ \psi^{-1} : \psi(U \cap V) \rightarrow F(U \cap V) \quad \psi \circ F^{-1} : F(U \cap V) \rightarrow \psi(U \cap V).$$

By the proposition above, we know that since ϕ is a coordinate chart of a smooth manifold, then ϕ and ϕ^{-1} are both smooth. Then the functions above are a composition of smooth functions, thus they are smooth as well. This proves that (U, F) is compatible with the chart (V, ψ) . Since this chart was arbitrary, then it is compatible with the whole atlas, thus (U, F) is a chart. Also, by the maximality of the atlas, (U, F) is contained in the atlas. \square

Now using the notion of smooth maps between manifolds, we can define a Lie group.

Definition 2.12 — Lie group. A *Lie group* is a C^∞ manifold G having a group structure such that the multiplication map

$$\mu : G \times G \rightarrow G$$

and the inverse map

$$\iota : G \rightarrow G, \quad \iota(x) = x^{-1},$$

are both C^∞ .

■ **Remark 2.9** Similar to the definition above, a topological group is a topological space having a group structure such that the multiplication and the inverse maps are both continuous. Note that a topological group is required to be a topological space, but not a topological manifold.

The followings are some examples of Lie groups. We can easily check the group operation and the inverse map are both smooth by using the definitions that we have had above.

- (i) The Euclidean space \mathbb{R}^n is a Lie group under addition.
- (ii) The set \mathbb{C}^\times of nonzero complex numbers is a Lie group under multiplications.
- (iii) Subsequently, the unit circle in \mathbb{C}^\times is a Lie group under multiplication.

2.4 Partial Derivatives

In this section we will define the notion of partial derivatives of the maps between manifolds. We will use the notion of coordinate charts for this definition. Let N be a manifold of dimension n , and let (r^1, \dots, r^n) be the standard coordinate function on \mathbb{R}^n . For a chart (U, ϕ) on the manifold, ϕ is a vector valued function on manifold. We denote its coordinate functions as x^i where $x^i = r^i \circ \phi$. This is just a fancy (but very useful) way of writing the i -th coordinate of the vector valued function ϕ . Now we can define the directional derivatives for the functions on manifolds.

Definition 2.13 — Directional derivative. Let N be a manifold, and $f : N \rightarrow \mathbb{R}$ an smooth function. Let $p \in U$ and (U, ϕ) any chart containing p , where $\phi = (x^1, \dots, x^n)$. We define the direction derivative of f at p as

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial(f \circ \phi^{-1})}{\partial r^i}(\phi(p)) = \frac{\partial}{\partial r^i} \Big|_{\phi(p)} (f \circ \phi^{-1}).$$

■ **Remark 2.10** In the definition above, we can re-write it as

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial f}{\partial x^i}(\underbrace{\phi^{-1}(\phi(p))}_p) = \frac{\partial(f \circ \phi^{-1})}{\partial r^i}(\phi(p)).$$

We can now write

$$\frac{\partial(f \circ \phi^{-1})}{\partial r^i} = \frac{\partial f}{\partial x^i} \circ \phi^{-1} = [\phi^{-1}]^* \left[\frac{\partial f}{\partial x^i} \right],$$

where shows that $\frac{\partial(f \circ \phi^{-1})}{\partial r^i}$ is the pull back of the function $\frac{\partial f}{\partial x^i}$ when viewed as a function on the manifold. Since the pull back of $\frac{\partial f}{\partial x^i}$ is smooth (as $f \circ \phi^{-1}$ is smooth), then $\frac{\partial f}{\partial x^i}$ is smooth when viewed as a real valued function on the manifold.

Proposition 2.9 Suppose (U, x^1, \dots, x^n) is a chart on a manifold. Then we have

$$\frac{\partial x^i}{\partial x^j} = \delta_j^i.$$

Proof. We view x^i as a real valued function on the manifold. For a point p in the manifold, we can use the definition of partial derivative to write

$$\frac{\partial x^i}{\partial x^j}(p) = \frac{\partial(x^i \circ \phi^{-1})}{\partial r^j} = \frac{\partial(r^i \circ \phi \circ \phi^{-1})}{\partial r^j} = \frac{\partial r^i}{\partial r^j} = \delta_j^i.$$

Note that we have used the fact that $\phi = (x^1, \dots, x^n)$. □

Using the notion of partial derivatives, we can define the notion of Jacobian matrix for the maps between manifolds.

Definition 2.14 — Jacobian matrix of maps between manifolds. Let $F : N \rightarrow M$ be a smooth map, and let $(U, \phi) = (U, x^1, \dots, x^n)$ and $(V, \psi) = (V, y^1, \dots, y^m)$ be charts on N and M respectively such that $F(U) \subset V$. Denote by

$$F^i = y^i \circ F = r^i \circ \psi \circ F : U \rightarrow \mathbb{R}$$

the i -th component of F in the chart (V, ψ) . The following matrix

$$J = \left[\frac{\partial F^i}{\partial x^j} \right]$$

is called the Jacobian matrix of F relative to the charts given as above. In case M, N have the same dimension, the determinant of this matrix is called the Jacobian determinant of F

relative to the two charts. The Jacobian determinant is also written as

$$\det(J) = \frac{\partial(F^1, \dots, F^n)}{\partial(x^1, \dots, x^n)}.$$

2.5 Inverse function theorem

As we studied before, on a manifold N of dimension n , a diffeomorphism $f : U \rightarrow \mathbb{R}^n$ is indeed a coordinate map. In this section, we want to study the conditions under which a smooth map between manifolds can be upgraded to a local diffeomorphism. For this end, we will use a slight generalization of the inverse function theorem in the Euclidean spaces.

Definition 2.15 — Inverse function theorem in Euclidean spaces. Let $F : W \rightarrow \mathbb{R}^n$ where $W \subset \mathbb{R}^n$ be a smooth map. We say this function is a *local diffeomorphism* or locally invertible at point p , if there exists an open set $U \subset W$ containing p such that the Jacobian determinant is non-zero at p , i.e.

$$\det \left[\frac{\partial F^i}{\partial r^j} \right] (p) \neq 0.$$

We will skip proving this theorem here and we will just use it to prove the generalized version of this theorem for smooth maps between manifolds.

Theorem 2.1 — Inverse function theorem for manifolds. Let $F : N \rightarrow M$ be a smooth map between two manifolds of the same dimension, and $p \in N$. Suppose for some charts $(U, \phi) = (U, x^1, \dots, x^n)$ about p in N and $(V, \psi) = (V, y^1, \dots, y^n)$ about $F(p)$ in M , $F(U) \subset V$. Set $F^i = y^i \circ F$. Then F is locally invertible at p if and only if its Jacobian determinant $\det(\partial F^i / \partial x^j)(p)$ is nonzero.

Proof. First, observe the proof structure in the observation box below. We just need to show two following bi-directional implications.

$$F \text{ is locally invertible} \iff \psi \circ F \circ \phi^{-1} \text{ is locally invertible.}$$

and also

$$\det \left[\frac{\partial(\psi \circ F \circ \phi^{-1})^i}{\partial r^j} \right] (\phi(p)) \neq 0 \iff \det \left[\frac{\partial F^i}{\partial x^j} \right] (p) \neq 0.$$

For the first bi-directional implication since ϕ, ψ are coordinate maps, thus are invertible (they are bijective) and from the composition of functions it follows that the bi-directional implication is true.

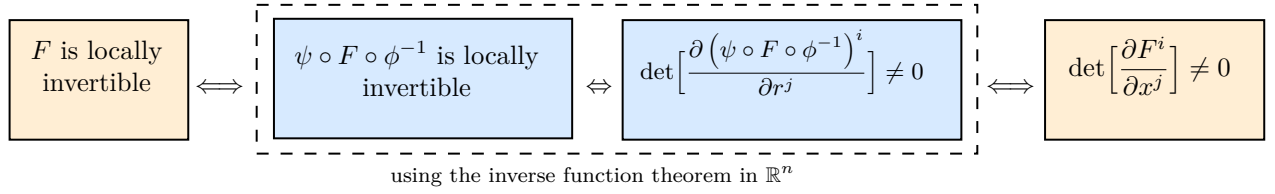
For the second bi-directional implication we start with the forward direction. We can write

$$\frac{\partial(\psi \circ F \circ \phi^{-1})^i}{\partial r^j} = \frac{\partial(r^i \circ \psi \circ F \circ \phi^{-1})}{\partial r^j} = \frac{\partial(y^i \circ F \circ \phi^{-1})}{\partial r^j} = \frac{\partial(F^i \circ \phi^{-1})}{\partial r^j} = \frac{\partial F^i}{\partial x^j}.$$

This proves the forward direction. We can do this starting from the last equation which proves the backward direction. \square

Observation 2.5.1 — Some hidden treasures! When proving the theorem above, we can figure out what should be the structure of the proof, even if we can not demonstrate the proof steps completely. For me, there is some sense of dimensional analysis if physics in what I am saying. By looking only at the ingredients of the problem/theorem/etc, we can figure out what is the possible way that we can combine those ingredients to achieve to a full proof. The following

figure reveals this structure for the proof.



Corollary 2.1 — Maps as local coordinate system. Let N be a manifold of dimension n . A set of n smooth functions F^1, \dots, F^n defined on a coordinate neighborhood (U, x^1, \dots, x^n) of a point $p \in N$ forms a coordinate system about p if and only if the Jacobian determinant

$$\left[\frac{\partial F^i}{\partial x^j} \right] (p) \neq 0.$$

Proof. define the vector valued function as $F = (F^1, \dots, F^n)$. Then we have

$$\det[\partial F^i / \partial x^j] \neq 0.$$

$$\iff F \text{ is locally invertible.}$$

$$\iff \text{There is a neighborhood } W \text{ containing } p \text{ such that } F : W \rightarrow F(W) \subset \mathbb{R}^n \text{ is a diffeomorphism.}$$

$$\iff (W, F) \text{ is a coordinate system in the maximal atlas of } N.$$

□

2.6 Quotients

The quotient construction is a process of simplification. Starting with an equivalence relation on a set, we identify each equivalence class to a point. If the original set is a topological space, then it is always possible to give the quotient space a topology so that the natural projection map (i.e. $\pi : S \rightarrow S/\sim, x \mapsto [x]$) becomes continuous. However, if the original set is a manifold, then it is not always true that the quotient set is also manifold, as the conditions like being Hausdorff or second countable might break.

Definition 2.16 — Quotient topology. Let (X, \mathcal{T}) be a topological space, and \sim an equivalence relation with π as the natural projection map. A set $U \in X/\sim$ is open, if and only if $\pi^{-1}(U) \in \mathcal{T}$, i.e. open.

■ **Remark 2.11** In the definition above, clearly the empty set \emptyset and the whole quotient space X/\sim are open. That is because $\pi^{-1}(\emptyset) = \emptyset \in \mathcal{T}$ (since the equivalence relation \sim partitions the set), and $\pi^{-1}(X/\sim) = X \in \mathcal{T}$. Also, for any map, in particular π we have

$$\pi^{-1}\left(\bigcup_{\alpha} U_{\alpha}\right) = \bigcup_{\alpha} \pi^{-1}(U_{\alpha}), \quad \pi^{-1}\left(\bigcap_i U_i\right) = \bigcap_i \pi^{-1}(U_i)$$

where $\pi^{-1}(U)$ means the pre-image of the set U under the map ϕ . This shows that an arbitrary union of the open sets in the quotient space is open, and also a finite intersection of open sets of the quotient space is also open. Thus the space $(X/\sim, \mathcal{T}')$ where $U \in \mathcal{T}'$ if and only if $\pi^{-1}(U) \in \mathcal{T}$

is a topological space.

Also Note that since π^{-1} maps open sets of X/\sim to the open sets of X , then π is automatically (i.e. by design) a continuous function.

2.6.1 Continuity of a map on a Quotient

Let S be a topological space and \sim an equivalence relation on this set. Give the quotient set S/\sim the quotient topology. Then a function $f : S \rightarrow Y$ from S to another topological space, where it is constant for all elements in the same equivalence class. This map then induces a map on S/\sim , i.e. $\bar{f} : S/\sim \rightarrow Y$ where

$$\bar{f}([p]) = f(p) \quad \text{for } p \in S.$$

or in other words

$$\bar{f}(\pi(p)) = f(p) \quad \text{or equivalently} \quad \bar{f} \circ \pi = f.$$

Proposition 2.10 The induced map $\bar{f} : S/\sim \rightarrow Y$ is continuous if and only if the map $f : S \rightarrow Y$ is continuous.

Solution First, we will show that continuity of \bar{f} implies the continuity of f . Let $U \in \mathcal{T}_Y$. Since \bar{f} is continuous, then $\bar{f}^{-1}(U) \in \mathcal{T}_{S/\sim}$. Furthermore, since π is continuous then $\pi^{-1}(\bar{f}^{-1}(U)) \in \mathcal{T}_S$, thus $(\bar{f} \circ \pi)$ is continuous. Since $f = \bar{f} \circ \pi$, then f is also continuous. As an alternative prove, we could start with the fact that $f = \bar{f} \circ \pi$, and since \bar{f} and π are both continuous, then so is f .

For the converse, we want to show that the continuity of f implies the continuity of \bar{f} . Let $V \in \mathcal{T}_Y$. Since f is continuous, then $f^{-1}(V) \in \mathcal{T}_S$. On the other hand, we have $f^{-1}(V) = (\pi^{-1} \circ \bar{f}^{-1})(V)$. Since π^{-1} sends open sets to open sets, then $\bar{f}^{-1}(V) \in \mathcal{T}_{S/\sim}$. Thus \bar{f} is continuous.

Definition 2.17 — Identification of a subset to a point. Let S be a topological space and $A \subset S$. Let \sim be a relation on S where

$$x \sim x \quad \forall x \in S,$$

and also

$$x \sim y \quad \forall x, y \in A.$$

This is an equivalence relation on S . We call the quotient space S/\sim is obtained from S by identifying the set A to a point.

2.7 Hausdorff and Second Countability Conditions

The quotient construction often does not preserve the second countability and Hausdorff property of a topological space. We start with the following proposition.

Proposition 2.11 Let X be a topological space, and \sim an equivalence relation on X . If X/\sim is Hausdorff then equivalent class $[p]$ of any $p \in X$ is closed in X .

Proof. In a Hausdorff space, every singleton set is closed. let $p \in X$. Then $\{\pi(p)\}$ is closed in X/\sim . Since π is continuous, then the pre image $\pi^{-1}(\{\pi(p)\}) = [p]$ is closed in X . This completes the proof. \square

■ **Remark 2.12** In the proof above, we claimed that in a Hausdorff space, every singleton set is closed. To see this, let $p \in S$ a Hausdorff space. Consider the singleton $\{p\}$ and its complement, i.e. $S - \{p\}$. We claim that set $S - \{p\}$ is open. To show this, let $U \in \mathcal{T}$ where $p \in U$. Since S is

Hausdorff, then there exists an open set $V \in \mathcal{T}$ such that contains any point $q \in S - \{p\}$ and do not overlap with U . This $q \in V \subset S - \{p\}$. Thus for every point $S - \{p\}$ we can find an open set containing q that is contained in the set $S - \{p\}$.

Definition 2.18 — Open equivalence relation. Let \sim be an equivalence relation defined on a set. This equivalence relation is said to be open, if $\pi(U)$ is open for some open set U

■ **Remark 2.13** We can give a useful characterization of open equivalence relations. Let U be an open set. Then $\pi(U)$ being open means that $\pi^{-1}(\pi(U))$ is also open. Thus an equivalence relation is open if and only if

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x]$$

is open.

Theorem 2.2 Let S be a topological space, and \sim an open equivalence relation defined on S . Then S/\sim is Hausdorff if and only if the graph R of the relation is closed in $S \times S$.

Proof. First, consider the following chain of bidirectional implications.

- R is closed in $S \times S$
- $\Leftrightarrow S \times S - R$ is closed in $S \times S$
- \Leftrightarrow For all $(x, y) \in (S \times S) - R$, there exist an open cell $U \times V \ni (x, y)$ such that $U \times V \cap R = \emptyset$.
- \Leftrightarrow For all pair $x, y \in S$ where $x \not\sim y$, there exists open sets $U \ni x$ and $V \ni y$ such that no element of U is equivalent to an element of V .
- \Leftrightarrow For all two points in $[x] \neq [y] \in S/\sim$ there exists open sets $U \ni x, V \ni y$ open in S such that $\pi(U) \cap \pi(V) = \emptyset$.

Now what remains to show is to show that the last statement (call it statement $*$) is equivalent to being Hausdorff. If $*$ is true, then since π is open, then S/\sim being Hausdorff follows immediately. For the converse, if S/\sim Hausdorff, then for $[x] \neq [y] \in S/\sim$ we can find open sets A, B containing $[x], [y]$ respectively where $A \cap B = \emptyset$. From the surjectivity of π we know that $\pi(\pi^{-1}(A)) = A$. Thus let $U = \pi^{-1}(A)$ and $V = \pi^{-1}(B)$, thus the statement $*$ follows. \square

A very simple use of the theorem above is that consider a case where the equivalence relation is simply the equality. Then we will have the following corollary.

Corollary 2.2 — Hausdorff characterization. Let S be a topological space. This space is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \in S \times S\}$ is closed in $S \times S$.

■ **Remark 2.14** Surprisingly, we can prove [Theorem 2.2](#) using the corollary above. See [Problem 2.21](#).

Similar to our procedure above, we can find some conditions under which a quotient space of a second countable space is still second countable. This criterion is given in the proposition below.

Proposition 2.12 — Second Countability of a Quotient Space . Let X be a second countable topological space, and let \sim be an equivalence relation. The quotient space X/\sim is second countable if \sim is an *open equivalence relation*, i.e. π is an open map.

Proof. We need to show that X/\sim possesses a countable basis for its open sets. We claim that if $\mathcal{B} = \{B_\alpha\}$ is countable basis for X , then $\{\pi(B_\alpha)\}$ is a countable basis for X/\sim . To show this let

$U \subset X/\sim$ be an open set. By continuity of π^{-1} the set $\pi^{-1}(U)$ is also open in X . Thus we can write

$$\pi^{-1}(U) \subset \bigcup_{\alpha \in I} B_\alpha.$$

Then we can write

$$U \subset \pi\left(\bigcup_{\alpha \in I} B_\alpha\right) = \bigcup_{\alpha \in I} \pi(B_\alpha).$$

Thus $\{\pi(B_\alpha)\}$ is indeed a countable basis for X/\sim . This completes the proof. \square

2.8 Real Projective Space

Here in this section we will review one of the very interesting manifolds that we can construct by quotient of another manifold. We start with the definition.

Definition 2.19 — Real Projective Space. Consider the set $\mathbb{R}^{n+1} - \{0\}$. Define the equivalence relation \sim as follows

$$x \sim y \iff x = ty \text{ for some } t \in \mathbb{R}$$

where $x, y \in \mathbb{R}^n - \{0\}$. The quotient set under this equivalence relation is called the real projective space or $\mathbb{R}P^n$.

■ **Remark 2.15** When I was writing this, it came to me what do we need to generalize this notion of projective plane to any vector space. Because the only ingredients that we use in this definition is the vector space properties of the underlying set.

Geometrically, the real projective plane can be thought of the set of all point passing through the origin. On the other hand, we know that every line passing through the origin hits the unit sphere S^n in exactly two antipodal points. The following proposition reveals the connections between the some quotient structure of unit spheres and the real projective space.

Proposition 2.13 Define an equivalence relation on S^n by identifying the antipodal points. I.e.

$$x \sim y \iff x = \pm y.$$

The quotient set S^n/\sim is homeomorphic to the real projective space.

Proof. To show this we need to find homeomorphisms between these two sets. Define $f : \mathbb{R}^{n+1} - \{0\} \rightarrow S^n$ be

$$f(x) = \frac{x}{\|x\|}$$

where $\|x\| = (\sum_i x_i^2)^{1/2}$ is the modulus of the point $x = (x_1, \dots, x_n)$. This function is continuous since the denominator is never zero and both nominator and denominator are continuous functions (note that $\|x\|$ is a composition of smooth functions). Consider the following diagram.

$$\begin{array}{ccc} \mathbb{R}^{n+1} - \{0\} & \xrightarrow{f} & S^n \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ \mathbb{R}P^n & \xrightarrow{\bar{f}} & S^n/\sim \end{array}$$

The function $\bar{f} : \mathbb{R}P^n \rightarrow S^n/\sim$ is a function induced by f given by

$$\bar{f}([x]) = \left[\frac{x}{\|x\|} \right] \in S^n/\sim.$$

This function is well defined as it does not depend on the representative of the equivalence class that we choose. Since for any other element in the equivalence class we will have

$$\bar{f}([tx]) = \left[\frac{tx}{\|tx\|} \right] = \left[\frac{tx}{|t|\|x\|} \right] = \left[\pm \frac{x}{\|x\|} \right] = \left[\frac{tx}{\|tx\|} \right] = \bar{f}([x]).$$

Since the function

$$\pi_2 \circ f$$

is continuous, then the function \bar{f} is also continuous. Next, define $g : S^n \rightarrow \mathbb{R}P^n - \{0\}$ given by $g(x) = [x]$. This map induces a map $\bar{g} : S^n/\sim \rightarrow \mathbb{R}P^n$ given by $\bar{g}([x]) = [x]$. By the same argument as above, \bar{g} is continuous and well defined. It just remains to show that these two maps are inverses of each other. To show this we have

$$(\bar{f} \circ \bar{g})([x]) = \bar{f}([x]) = \left[\frac{x}{\|x\|} \right] = [x],$$

and also

$$(\bar{g} \circ \bar{f})([x]) = \bar{g}\left(\left[\frac{x}{\|x\|} \right]\right) = \left[\frac{x}{\|x\|} \right] = [x].$$

Thus the spaces $\mathbb{R}P^n$ and S^n/\sim are homeomorphic to each other. \square

2.9 Real Projective Plane

Here in this section we will focus on the real projective plane, which is the set of all lines passing through origin in \mathbb{R}^3 . Although imagining an sphere with identified antipodal points are much easier than imagining the set of all line passing through the origin, but it is still hard to imagine the final geometrical and global properties of such a set. So we need more simplification.

Proposition 2.14 Let S^2/\sim be a unit sphere that its antipodal points are identified. This set is homeomorphic to the closed upper hemisphere where its antipodal points on the equator are identified.

Proof. The proof is given as the solution to [Problem 2.19](#). \square

Now, we can go further and simplify this space more.

Proposition 2.15 Let H^2/\sim be the quotient space adopted from the closed upper hemisphere by identifying the antipodal points on its equator. Also, let D^2/\sim be the quotient space adopted from the closed unit disk D^2 by identifying the antipodal points on its boundary circle. Then H^2/\sim and D^2/\sim are homeomorphic.

Proof. We need to find a homeomorphism between H^2/\sim and D^2/\sim . Consider the projection map

$$f : H^2 \rightarrow D^2 \quad (x, y, z) \mapsto (x, y).$$

Consider the following commutative diagram.

$$\begin{array}{ccc}
 H^2 & \xrightarrow{f} & D^2 \\
 \pi_1 \downarrow & \searrow \pi_2 \circ f & \downarrow \pi_2 \\
 H^2/\sim & \xrightarrow{\bar{f}} & D^2/\sim
 \end{array}$$

The map $\pi_2 \circ f$ is continuous and assumes a constant value for all of points in its domain that are in a same equivalence class. Thus this induces a continuous map $\bar{f} : H^2/\sim \rightarrow D^2/\sim$. To find the inverse for this function, consider

$$g : D^2 \rightarrow S^2 \quad (x, y) \mapsto (x, y, \sqrt{1 - (x^2 + y^2)}).$$

Consider the following commutative diagram.

$$\begin{array}{ccc}
 D^2 & \xrightarrow{g} & H^2 \\
 \pi_2 \downarrow & \searrow \pi_1 \circ g & \downarrow \pi_1 \\
 D^2/\sim & \xrightarrow{\bar{g}} & H^2/\sim
 \end{array}$$

The map $\pi_1 \circ g$ is continuous and assumes a constant value for all points in its domain that are in the same equivalence class. Thus it induces a map \bar{g} that is continuous.

We now need to show that \bar{f} and \bar{g} are inverses of each other. To see this consider the map

$$\bar{f} \circ \bar{g} : H^2/\sim \rightarrow H^2/\sim.$$

We can write

$$\bar{f}(\bar{g}([x, y])) = \bar{f}([x, y, \sqrt{1 - (x^2 + y^2)}]) = [x, y].$$

With a similar argument for $\bar{g} \circ \bar{f}$ we can show that these two functions are inverses of each other, thus we could find the explicit homeomorphism. \square

In summary, we have the following chain of homeomorphisms

$$\mathbb{R}P^2 \xrightarrow{\sim} S^2/\sim \xrightarrow{\sim} H^2/\sim \xrightarrow{\sim} D^2/\sim$$

2.10 Standard C^∞ Atlas on a Real Projective Space

2.11 Summary

Summary 🦋 **2.1 — A useful thing!**. Let N be a manifold of dimension n , and (U, ϕ) a local coordinate system containing $p \in N$. Then ϕ can be viewed as a vector valued real function $\phi : U \rightarrow \mathbb{R}^n$. Let r^i be the standard coordinate functions of \mathbb{R}^n . Then the i -th of ϕ can be written as.

$$x^i = r^i \circ \phi.$$

Summary 2.2 Let S be a manifold, and \sim an equivalence relation. Then S/\sim is

- Second countable if the equivalence relation is open.
- Hausdorff if the graph of the equivalence relation is *open* and its graph is closed in $S \times S$.

Summary 2.3 — Two ways to show that two quotient spaces are homeomorphic. In general, in order to show that two spaces are homeomorphic, we need to find an homeomorphism between the spaces (a continuous bijection with continuous inverse). In particular, if two spaces are quotient spaces, it is often easier to work with the original spaces (that are not necessarily homeomorphic). In this summary box I will talk about two important ways that we can do this.

1. **Direct method.** In this method we directly find the homeomorphism. Consider showing that the real projective space $\mathbb{R}P^n$ is homeomorphic to S^n/\sim where the equivalence relation \sim identifies the antipodal points in the unit sphere. The followings are the steps that we take

- Find $f : \mathbb{R}^n - \{0\} \rightarrow S^n$ where $\pi_2 \circ f$ assumes a constant value for all of the points in the same equivalence relation defined on $\mathbb{R}^{n+1} - \{0\}$. This induces a map $\bar{f} : \mathbb{R}P^n \rightarrow S^n/\sim$. From [Proposition 2.10](#) it follows that \bar{f} is continuous iff $\pi_2 \circ f$ is continuous (consider the following commutative diagram).

$$\begin{array}{ccc} \mathbb{R}^{n+1} - \{0\} & \xrightarrow{f} & S^n \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ \mathbb{R}P^n & \xrightarrow{\bar{f}} & S^n/\sim \end{array}$$

Also note that since the domain of \bar{f} is equivalence classes, we need to make sure that the value of the function does not depend on the specific representative that we choose for a particular equivalence class. I.e. we need to show that the function is well defined. (I think this is the same as the fact that this function assumes a constant value for all of the elements in the same equivalence class).

- We need to find $g : S^n \rightarrow \mathbb{R}^n - \{0\}$ that induces a map $\bar{g} : S^n/\sim \rightarrow \mathbb{R}^n - \{0\}/\sim$. Again, using [Proposition 2.10](#), \bar{g} is continuous if and only if $\pi_1 \circ g$ is continuous. Similar to what we did for \bar{f} we need to show that \bar{g} is indeed well defined.
- As the last step, we need to check if \bar{f} and \bar{g} are indeed inverses of each other, i.e. their composition leads to identity maps on the corresponding domains.

Also look at the proof of [Proposition 2.15](#) for a similar argument.

2. **Using compactness arguments.** Some times it is hard to find the homeomorphism directly. In this case we follow our approach as in [Problem 2.18](#) and [Problem 2.19](#), as well as the following important proposition

Proposition 2.16 A bijection from a compact space to a Hausdorff space is a homeomorphism.

Summary 🦋 2.4 There are some points in the proposition above that I will highlight by this summary box. First, note that the composition of two function, possibly one or both of them discontinuous, can be a continuous function (see [this](#)). Thus in the summary box above, neither of f or g need to be continuous. The function that need to be continuous are $\pi_2 \circ f$ and $\pi_1 \circ g$.

Summary 🦋 2.5 Let S be a topological space and G be a topological group. Consider the right action of the group G on S via the map

$$\alpha : S \times G \rightarrow S, \quad \alpha(s, g) = sg$$

If the right action is *continuous* (i.e. the map α is continuous) Then for $g \in G$ the map

$$\alpha_g : S \rightarrow S$$

is a *homeomorphism*. See [Problem 2.23](#) for the proof.

Summary 🦋 2.6 — **Showing a space is an smooth manifold.** To show that a space is an smooth manifold, we just need to show that the space is

- a Hausdorff topological space that is also second countable,
- there exists any C^∞ atlas (not necessarily maximal).

Summary 🦋 2.7 — **A confusing thing about the equivalence classes.** Define \sim on \mathbb{R} as

$$x \sim y \quad \text{iff} \quad x - y = 2n\pi \quad \text{for some } n \in \mathbb{Z}.$$

Then consider the following sets in \mathbb{R}/\sim .

$$V_1 = \{[t] \mid t \in (-\pi, \pi)\}, \quad V_2 = \{[t] \mid t \in (0, 2\pi)\}.$$

In words, what we really mean is that V_1 contains the equivalence classes whose representatives comes from the set $(-\pi, \pi)$. Similarly for V_2 , we say that it contains the equivalence classes whose representatives comes from the set $(0, 2\pi)$. We define the following coordinate maps

$$\begin{array}{ll} \phi_1 : V_1 \rightarrow \mathbb{R} & \phi_2 : V_2 \rightarrow \mathbb{R} \\ [t] \mapsto t & [t] \mapsto t \end{array}$$

Note that although the functions ϕ_1 and ϕ_2 look similar, but they are different functions as they have different domains. These two functions both return the representative of the equivalence classes in their argument. Note that we said *the representative* and not *a representative* because of the design of their domain. For instance, let's parse the following operation

$$\phi_1([23.5\pi]) = -0.5\pi.$$

Since the domain of ϕ_1 is the equivalence classes whose representatives comes from the set $(-\pi, \pi)$, then we need to find a representative for $[23.5\pi]$ that lives in $(-\pi, \pi)$. I.e. $[23.5\pi] = [1.5\pi] = [-0.5\pi]$.

Summary 🦋 2.8 — Strategy to show an equivalence relation is open. Let S be a topological space and \sim be an equivalence relation on S . This equivalence relation can also be given by a group action of G acting on S . I.e. we can write

$$x \sim y \quad \text{iff} \quad x = yg \quad \text{for some } g \in G.$$

Then to show that the equivalence relation \sim is open, or equivalently, to show that the projection map $\pi : S \rightarrow S/G$ is an open map, then we need to show for $U \subset S$ open we have

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} Ug$$

is open. By [Lemma 2.2](#), this set is open if and only if the group action is continuous. See [Problem 2.26](#) for a detailed example.

Summary 🦋 2.9 — Grassmannian. Grassmannian $G(k, n)$ is a generalization of the real projective plane. $G(k, n)$ defined to be the set all of k -planes passing through the origin as the linear subspaces of \mathbb{R}^n .

The Grassmannian $G(k, n)$ is in one-to-one correspondence with $F(k, n)/GL(k, \mathbb{R})$, where $F(k, n)$ is the set of all $n \times k$ matrices with rank k .

As a concrete example, consider $G(2, 4)$. The open coordinate sets are given as $U_{ij} = V_{ij}/GL(2, \mathbb{R})$, where

$$V_{ij} = \{A \in F(2, 4) \mid A_{ij} \text{ is nonsingular}\}.$$

The corresponding coordinate maps are induced by the maps

$$\tilde{\phi}_{ij} : V_{ij} \rightarrow \mathbb{R}^{2 \times 2}.$$

where for instance $\tilde{\phi}_{12}(A) = A_{34}A_{12}^{-1}$. See [Problem 2.26](#) for a detailed discussion.

2.12 Solved Problems

■ **Problem 2.1 — A C^∞ atlas on a circle (From W. Tu).** construct a C^∞ atlas for the unit circle S^1 .

Solution The unit circle C^1 can be described as a set of points $S^1 = \{e^{it} | t \in [0, 2\pi]\}$. Let U_1 and U_2 be two subsets of S^1 described as

$$U_1 = \{e^{it} | t \in (-\pi, \pi)\}, \quad U_2 = \{e^{it} | t \in (0, 2\pi)\}.$$

Consider the functions $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}$ for $\alpha = 1, 2$ given by

$$\begin{aligned} \phi_1(e^{it}) &= t, & -\pi < t < \pi, \\ \phi_2(e^{it}) &= t, & 0 < t < 2\pi. \end{aligned}$$

These functions are in fact different branches of the complex logarithm function $1/i \log(z)$, thus homeomorphisms onto their respective images. Thus $\{(U_1, \phi_1), (U_2, \phi_2)\}$ is an atlas for S^1 . To demonstrate the compatibility of these charts, we need to first calculate $U_1 \cap U_2$. This set has two connected components, i.e. $U_1 \cap U_2 = A \sqcup B$ where \sqcup is used to demonstrate the disjoint union of A, B . Explicitly, we can write

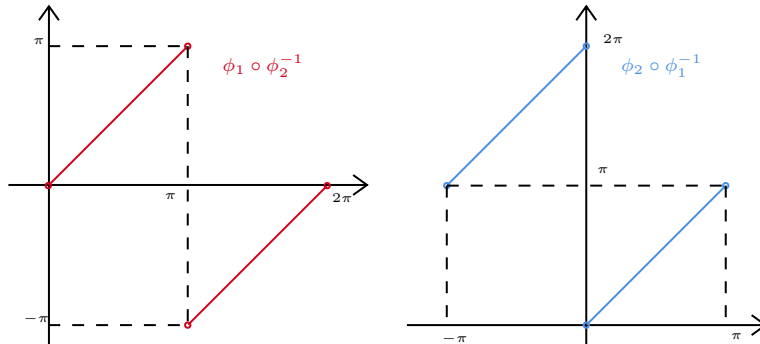
$$A = \{e^{it} | t \in (-\pi, 0)\}, \quad B = \{e^{it} | t \in (0, \pi)\}.$$

First, we start with the function $\phi_1 \circ \phi_2^{-1} : \underbrace{\phi_2(U_1 \cap U_2)}_{(0, \pi) \sqcup (\pi, 2\pi)} \rightarrow \underbrace{\phi_1(U_1 \cap U_2)}_{(-\pi, 0) \sqcup (0, \pi)}$. For this function we have

$$(\phi_1 \circ \phi_2^{-1})(t) = \begin{cases} t & t \in (0, \pi), \\ t - 2\pi & t \in (\pi, 2\pi). \end{cases}$$

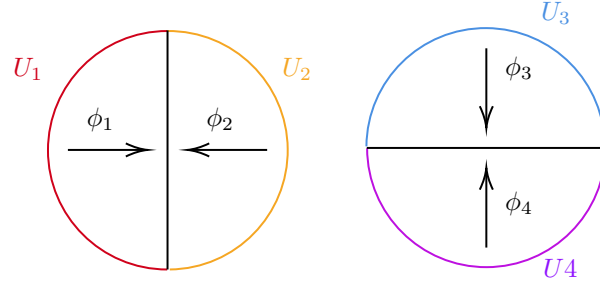
Similarly, for $\phi_2 \circ \phi_1^{-1} : \underbrace{\phi_1(U_1 \cap U_2)}_{(-\pi, 0) \sqcup (0, \pi)} \rightarrow \underbrace{\phi_2(U_1 \cap U_2)}_{(0, \pi) \sqcup (\pi, 2\pi)}$ we can write

$$(\phi_2 \circ \phi_1^{-1})(t) = \begin{cases} t + 2\pi & t \in (-\pi, 0), \\ t & t \in (0, \pi). \end{cases}$$



Observation 2.12.1 I was thinking about my solution to the problem above, and I thought it is wrong, as I was thinking that the function ϕ_1 is not homeomorphism as it is not continuous. But the point that I was missing is that this function is indeed continuous on its domain and the point of discontinuity (i.e. $x = \pi$) is not in the domain.

■ **Problem 2.2** Another C^∞ atlas on a circle In the previous problem, we constructed an atlas for a unit circle sitting in the complex plane. In this problem we are going to construct a different atlas for a unit circle sitting in the $x - y$ plane. The following diagram are the charts for this unit circle. Write these charts explicitly and check if they are pairwise compatible.



Solution The explicit formulas for the charts depicted above is as following

$$(U_1, \phi_1 : U_1 \rightarrow \mathbb{R}), (U_2, \phi_2 : U_2 \rightarrow \mathbb{R}), (U_3, \phi_3 : U_3 \rightarrow \mathbb{R}), (U_4, \phi_4 : U_4 \rightarrow \mathbb{R}),$$

where we have

$$\phi_1(x, y) = y, \quad \phi_2(x, y) = y, \quad \phi_3(x, y) = x, \quad \phi_4(x, y) = x.$$

Note that although some of the functions above might have a same formula, but they are different functions as they have different domains. To show that these functions are pairwise compatible, we start by noting that since $U_1 \cap U_2 = \emptyset$, thus (U_1, ϕ_1) and (U_2, ϕ_2) are compatible. With the same reasoning, the charts (U_3, ϕ_3) and (U_4, ϕ_4) are compatible. Now, we want to show that (U_1, ϕ_1) is compatible with (U_3, ϕ_3) . We need to show that

$$\phi_1 \circ \phi_3^{-1} : \underbrace{\phi_3(U_1 \cap U_3)}_{(-1,0)} \rightarrow \underbrace{\phi_1(U_1 \cap U_3)}_{(0,1)} \quad \text{and} \quad \phi_3 \circ \phi_1^{-1} : \underbrace{\phi_1(U_1 \cap U_3)}_{(0,1)} \rightarrow \underbrace{\phi_3(U_1 \cap U_3)}_{(-1,0)}$$

are C^∞ . To write them explicitly, we have

$$(\phi_1 \circ \phi_3^{-1})(x) = \phi_1(x, \sqrt{1-x^2}) = \sqrt{1-x^2}.$$

Also

$$(\phi_3 \circ \phi_1^{-1})(x) = \phi_3(-\sqrt{1-x^2}, x) = -\sqrt{1-x^2}.$$

We can see that both of these functions are C^∞ in their domain. Now, for the charts (U_1, ϕ_1) and (U_4, ϕ_4) need to show

$$\phi_1 \circ \phi_4^{-1} : \underbrace{\phi_4(U_1 \cap U_4)}_{(-1,0)} \rightarrow \underbrace{\phi_1(U_1 \cap U_4)}_{(-1,0)} \quad \text{and} \quad \phi_4 \circ \phi_1^{-1} : \underbrace{\phi_1(U_1 \cap U_4)}_{(-1,0)} \rightarrow \underbrace{\phi_4(U_1 \cap U_4)}_{(-1,0)}$$

are C^∞ in their domain. Explicitly, we have

$$(\phi_1 \circ \phi_4^{-1})(x) = -\sqrt{1-x^2}, \quad (\phi_4 \circ \phi_1^{-1})(x) = -\sqrt{1-x^2}.$$

With the same strategy, we can show that this collection of charts indeed makes a C^∞ atlas for the unit circle in $x - y$ plane.

■ **Problem 2.3 — The real line with two origins (from W. Tu).** Let A and B be two points not on the real line \mathbb{R} . Consider the set $S = (\mathbb{R} - \{0\}) \cup \{A, B\}$. For any two positive real numbers c, d , define

$$I_A(-c, d) = (-c, 0) \cup \{A\} \cap (0, d)$$

and similarly for $I_B(-c, d)$, with B instead of A . Define a topology on S as follows: On $(\mathbb{R} - \{0\})$, use the subspace topology inherited from \mathbb{R} , with open intervals as a basis. A basis of neighborhoods at A is the set $\{I_A(-c, d) \mid c, d > 0\}$; similarly, a basis of neighborhoods at B is $\{I_B(-c, d) \mid c, d > 0\}$.

- (a) Prove that the map $h : I_A(-c, d) \rightarrow (-c, d)$ defined by

$$\begin{aligned} h(x) &= x & \text{for } x \in (-c, 0) \cup (0, d), \\ h(A) &= 0 \end{aligned}$$

is a homeomorphism.

- (b) Show that S is locally Euclidean and second countable, but not Hausdorff.

Solution (a) We need to show that h is one-to-one, onto, and continuous, with continuous inverse. To show being one-to-one, let $x, y \in I_A(-c, d)$ such that $x \neq y$ and possibly one of them equal to A . If none is equal to A , then h is the identity map which is one-to-one. However, if one of them is equal to A , let's say $x = A$, then $h(x) = 0$ where $h(y) \in (-c, d) \cup (0, d)$, thus $h(y) \neq 0$. This proves that h is indeed one to one. To show that the function is onto, let $z \in (-c, d)$. Then if $z = 0$ we have $h(A) = z$, and if $z \neq 0$, we have $h(z) = z$. Thus the function h is a bijection.

As the second step, we need to show that this function is continuous with continuous inverse. From definition of continuity, we just need to show that both h and h^{-1} maps opens to opens (because for continuous function the pre-image of every open set is an open set; and to show that the inverse of the function is also continuous we need to show that the image of every open is also open). Let $U = (a, b) \subset I_A(-c, d)$ be an open set in the topology of S . If $A \notin (a, b)$, then the image of this set under h is (a, b) which is open in $(-c, d)$. But if $A \in (a, b)$, then the image of this set under the map h is $(a, 0) \cup (0, b) \cup \{0\} = (a, b)$, which is also open. Thus h^{-1} is continuous. To show the continuity of h^{-1} , let $(a, b) \subset (-c, d)$. If $0 \notin (a, b)$, then pre-image of this set under the map f is (a, b) that is open in S . However if $0 \in (a, b)$, then the pre-image of this set under f is the set $(-a, 0) \cup \{A\} \cup (0, b)$ which is indeed open in S as we can construct this with the basis if opens at A .

- (b) First, we show that S is locally Euclidean. To show this let $p \in S$. If $p \neq A$ and $p \neq B$, then we choose an open set $U = (a, b)$ containing p that does not contain neither of A and B . Then U is homeomorphic to (a, b) with the identity map. However if $p = A$, we choose any open $U = (a, b)$ containing p . Then $(U = (a, b), I_A(a, b))$ is a local chart. For the case where $p = B$, we can find a suitable chart with the same reasoning as for A . Thus we have shown that S is locally Euclidean.

To show that S is second countable, let \mathcal{B} be a basis for $\mathbb{R} - \{0\}$. Since \mathbb{R} is second countable, then \mathcal{B} is countable. Let $\mathbb{B} = \mathcal{B} \cup I_A(-c, d) \cup I_B(-c, d)$ is a countable basis for S for some $c, d > 0$. This shows that S is also second countable.

However, this space is not Hausdorff. To show this consider the points A, B . We can not find any two open U, V such that $A \in U$ and $B \in V$ and we have $U \cap V = \emptyset$. Or equivalently, for all open sets U, V such that $A \in U$ and $B \in V$ we have $U \cap V \neq \emptyset$. That is because from the basis of the open neighborhoods at A, B we have $U = I_A(-c, d)$ and $V = I_B(-e, f)$ for some $c, d, e, f > 0$. It is clear that $U \cap V \neq \emptyset$, thus S is not Hausdorff.

■ **Problem 2.4 — A sphere with a hair (from W. Tu)** . A fundamental theorem of topology, the theorem on invariance of dimension, states that if two nonempty open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are homeomorphic, then $n = m$. Prove that the sphere with a hair in \mathbb{R}^3 is not locally Euclidean at q (the point that the hair attaches to the sphere). Hence it cannot be a topological manifold.

Solution We will proceed with the proof by contradiction. Assume that the ball with hair at point q is homeomorphic to some open set in \mathbb{R}^n . Then q has an open neighborhood U homeomorphic to an open ball $B := \mathbb{B}(0, \epsilon) \in \mathbb{R}^n$, with q mapping to 0. We can restrict this homeomorphism to a homeomorphism $U - \{q\} \rightarrow B - \{0\}$. Now $B - \{0\}$ is either connected if $n \geq 2$ or has two connected components if $n = 1$, where $U - \{q\}$ has two connected components where one component is a 1 dimensional manifold (the hair) and the other component is a 2 dimensional manifold (the sphere). The only case where $B - \{0\}$ has two connected components is when $n = 1$. But because of the invariance of dimension principle, the sphere (2 dimensional manifold) can not be homeomorphic to a one dimensional manifold. Thus there is no such a homeomorphism between the hairy ball and \mathbb{R}^n and the hairy ball is not locally Euclidean in q .

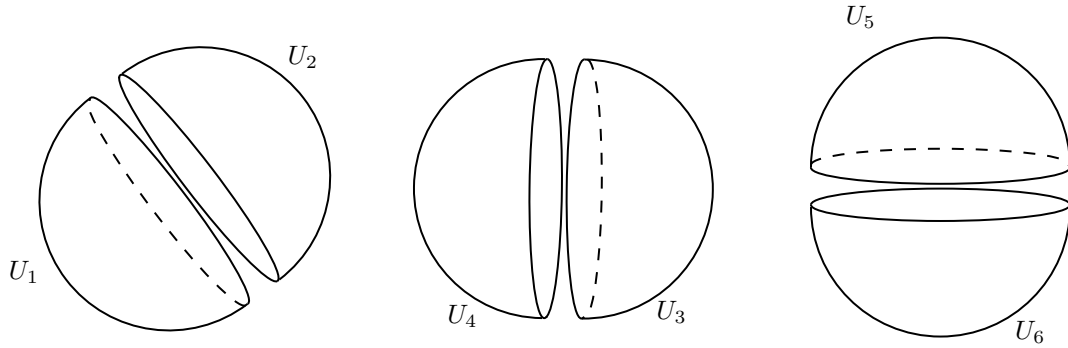
■ **Problem 2.5 — Charts on a sphere (from W. Tu)**. Let S^2 be the unit sphere, i.e.

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

in \mathbb{R}^3 . Define in S^2 the six charts corresponding to the six hemispheres (from the front, rear, right, left, upper, and lower hemispheres) as in the figure.

$$\begin{aligned} U_1 &= \{(x, y, z) \in S^2 \mid x > 0\}, & \phi_1(x, y, z) &= (y, z), \\ U_2 &= \{(x, y, z) \in S^2 \mid x < 0\}, & \phi_2(x, y, z) &= (y, z), \\ U_3 &= \{(x, y, z) \in S^2 \mid y > 0\}, & \phi_3(x, y, z) &= (x, z), \\ U_4 &= \{(x, y, z) \in S^2 \mid y < 0\}, & \phi_4(x, y, z) &= (x, z), \\ U_5 &= \{(x, y, z) \in S^2 \mid z > 0\}, & \phi_5(x, y, z) &= (x, y), \\ U_6 &= \{(x, y, z) \in S^2 \mid z < 0\}, & \phi_6(x, y, z) &= (x, y). \end{aligned}$$

Note that although some of the functions above might look similar (like ϕ_3 and ϕ_4) but they are in fact different functions as they have different domains. Show that $\phi_1 \circ \phi_4^{-1}, \phi_4 \circ \phi_1^{-1}$ is C^∞ on $\phi_4(U_1 \cap U_4), \phi_1(U_1 \cap U_4)$ respectively. Do the same same analysis for $\phi_6 \circ \phi_1^{-1}$.



Solution For the set $U_1 \cap U_4$ we have

$$U_1 \cap U_4 = \{(x, y, z) \in S^2 : x > 0 \text{ and } y < 0\}.$$

Thus we will have

$$\phi_4(U_1 \cap U_4) = \{(x, z) \in \mathbb{R}^2 : x^2 + z^2 \leq 1, x > 0\}.$$

Thus we can write

$$(\phi_1 \circ \phi_4^{-1})(\langle x, z \rangle) = \phi_1(\langle x, -\sqrt{1 - (x^2 + y^2)}, z \rangle) = \langle -\sqrt{1 - (x^2 + y^2)}, z \rangle$$

This is indeed a C^∞ vector valued function, since each component is a C^∞ function. Now to evaluate $\phi_4 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_4) \rightarrow \phi_4(U_1 \cap U_4)$ we need to first evaluate the set $\phi_1(U_1 \cap U_4)$. For this set we have

$$\phi_1(U_1 \cap U_4) = \{(z, y) \in \mathbb{R}^2 : z^2 + y^2 \leq 1, y < 0\}.$$

Then we can write

$$(\phi_4 \circ \phi_1^{-1})(\langle y, z \rangle) = \phi_4(\langle \sqrt{1 - (y^2 + z^2)}, y, z \rangle) = \langle \sqrt{1 - (y^2 + z^2)}, z \rangle.$$

This is indeed a C^∞ vector valued function.

To evaluate the function $\phi_6 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_6) \rightarrow \phi_6(U_1 \cap U_6)$, we first need to determine the domain of this function. First observe that

$$U_1 \cap U_6 = \{(x, y, z) \in S^2 : x > 0 \text{ and } y < 0\},$$

which is the same as $U_1 \cap U_4$. Then for the domain of the function of interest we can write

$$\phi_1(U_1 \cap U_6) = \{(y, z) \in \mathbb{R}^2 : z^2 + y^2 \leq 1 \text{ and } y < 0\},$$

$$(\phi_6 \circ \phi_1^{-1})(\langle y, z \rangle) = \phi_6(\langle \sqrt{1 - (y^2 + z^2)}, y, z \rangle) = \langle \sqrt{1 - (y^2 + z^2)}, y \rangle.$$

■ **Problem 2.6 — Existence of a coordinate neighborhood (from W. Tu).** Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ be the maximal atlas on manifold M . For any open set U in M and a point $p \in U$, prove the existence of a coordinate open set U_α such that $p \in U_\alpha \subset U$.

Solution Since $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ is an atlas, then for $p \in U$ given as above, we can find some $\alpha_1 \in I$ such that $p \in U_{\alpha_1}$. Consider the open set $W = U_{\alpha_1} \cap U$. The chart $(W, \phi_{\alpha_1}|_W)$ is in the atlas (since it is maximal), i.e. $\exists \alpha \in I$ such that $(U_\alpha, \phi_\alpha) = (W, \phi_{\alpha_1}|_W)$. This completes the proof.

■ **Problem 2.7 — An atlas for a product manifold (from W. Tu).** Prove the following proposition.

Proposition 2.17 If $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ are C^∞ atlases for the manifold M and N of dimensions m and n , respectively, then the collection

$$\{(U_\alpha \times V_i, \phi_\alpha \times \psi_i)\}$$

where

$$\phi_\alpha \times \psi_i : U_\alpha \times V_i \rightarrow \mathbb{R}^m \times \mathbb{R}^n$$

of charts is a C^∞ atlas on $M \times N$. Therefore, $M \times N$ is a C^∞ manifold of dimension $m + n$.

Solution In order to show that the collection $\mathfrak{U} = \{(U_\alpha \times V_i, \phi_\alpha \times \psi_i)\}$ is an atlas for $M \times N$, we need to show that charts are pairwise compatible as well as the set covering the whole space. To show that the chart covers the whole space $M \times N$, let $(p_1, p_2) \in M \times N$. Then $p_1 \in M$ and $p_2 \in N$. Then there are two coordinate open sets such that $p_1 \in U_{\alpha_1}$ and $p_2 \in V_{i_1}$. Thus the coordinate open set $U_{\alpha_1} \times V_{i_1}$ contains the point (p_1, p_2) .

To show that two any two charts in the collection \mathfrak{U} are compatible, let $(U_{\alpha_1} \times V_{i_1}, \phi_{\alpha_1} \times \psi_{i_1})$ and $(U_{\alpha_2} \times V_{i_2}, \phi_{\alpha_2} \times \psi_{i_2})$ be two charts. We claim that the corresponding coordinate maps are C^∞ . This follows directly from the fact the each component of the map these coordinate maps are C^∞ .

■ **Problem 2.8 — Smoothness of a projection map (from W. Tu).** Let M and N be manifolds and $\pi : M \times N \rightarrow M$, $\pi(p, q) = p$ the projection to the first factor. Prove that π is a C^∞ map.

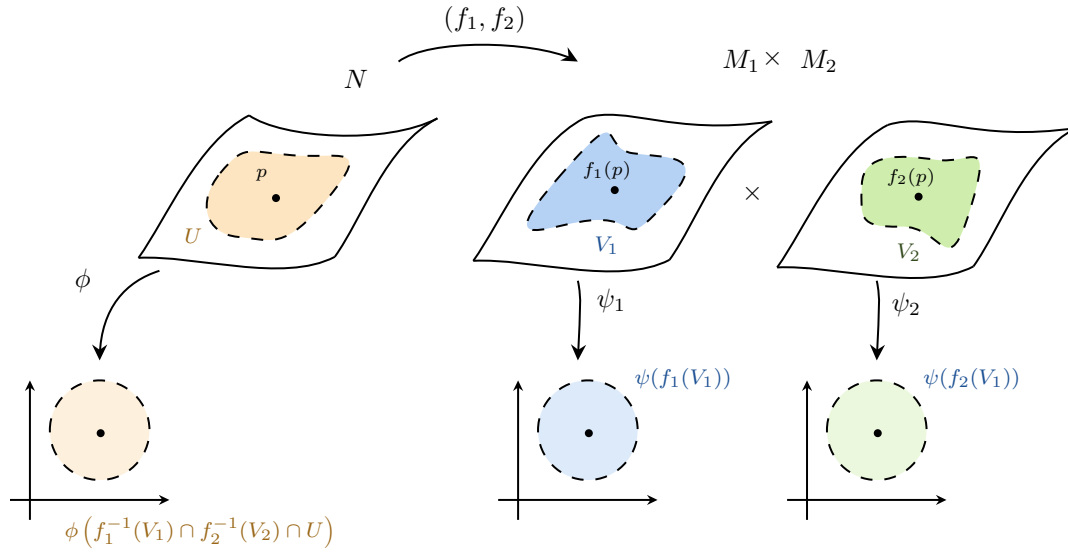
Solution Let $p \in M$ and $q \in N$, thus $(p, q) \in M \times N$. Let ϕ and ψ be two coordinate maps such that $\phi(p) = x$ and $\psi(q) = y$. Thus we can write $(\phi, \psi)(p, q) = (x, y)$. Consider the function

$$(\phi \circ \pi \circ (\phi \times \psi)^{-1})(x, y) = \phi(\pi(p, q)) = \phi(p) = x.$$

The function above is a C^∞ map from \mathbb{R}^{n+m} to \mathbb{R}^n (assuming M, N are m and n dimensional manifolds). This proves that the projection map π is smooth.

■ **Problem 2.9 — Smoothness of a map tp a Cartesian product (from W. Tu).** Let M_1, M_2 , and N be manifolds of dimensions m_1, m_2 and n respectively. Prove that a map $(f_1, f_2) : N \rightarrow M_1 \times M_2$ is smooth if and only if $f_i : N \rightarrow M_i$ for $i = 1, 2$ are both smooth.

Solution Consider the following picture.



For the first direction, we will show that smoothness of $(f_1, f_2) : N \rightarrow M_1 \times M_2$ implies the smoothness of $f_1 : N \rightarrow M_1$ and $f_2 : N \rightarrow M_2$. As depicted in the picture above, let ϕ be a coordinate map for N and ψ_1, ψ_2 be coordinate maps for M_1, M_2 respectively. Since (f_1, f_2) is smooth, then $(\psi_1 \times \psi_2) \circ (f_1, f_2)$ is smooth. Let $p \in N$, then

$$((\psi_1 \times \psi_2) \circ (f_1, f_2) \circ \phi^{-1})(\phi(p)) = ((\psi_1 \times \psi_2) \circ (f_1, f_2))(p) = (\psi_1 \times \psi_2)(f_1(p), f_2(p)) = (\psi_1(f_1(p)), \psi_2(f_2(p))).$$

This is a smooth map from \mathbb{R}^n to $\mathbb{R}^{m_1+m_2}$. Thus the components of the function are also smooth.

For the converse, we need to show that the smoothness of f_1 and f_2 implies the smoothness of (f_1, f_2) . Similar to the argument above, the smoothness of (f_1, f_2) follows immediately from the smoothness of the components.

■ **Problem 2.10 — Smooth functions on unit circle (W. Tu).** We have studied before that the unit circle S^1 defined by $x^2 + y^2 = 1$ in \mathbb{R}^2 is a C^∞ manifold. Prove that a smooth function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined on \mathbb{R}^2 restricts to a C^∞ function on S^1 .

Solution Consider the following inclusion map

$$i : S^1 \rightarrow \mathbb{R}^2$$

defined as $i(p) = (x(p), y(p))$, where x, y are the standard coordinate functions. The restriction of f to manifold will be

$$f|_{S^1} = f \circ i.$$

To show that $f|_{S^1}$ is smooth, we just need to show that the inclusion map is smooth. To show this we need to show that the components of this vector valued function is smooth. We start by showing that x is smooth. We use the same charts as in [Problem 2.2](#). Let $p \in S^1$. If $p \in U_3 \cap U_4$ then we have

$$\mathbb{1}_{(0,1)} \circ x \circ \phi_3^{-1} = \mathbb{1}_{(0,1)} : (-1, 1) \rightarrow \mathbb{R}^2, \quad \mathbb{1}_{(0,1)} \circ x \circ \phi_4^{-1} = \mathbb{1}_{(0,1)} : (-1, 1) \rightarrow \mathbb{R}^2,$$

which are both identity maps, thus smooth. So x is smooth on $U_3 \cap U_4$. For U_1 we have

$$\mathbb{1}_{(0,1)} \circ x \circ \phi_1^{-1}(x) = -\sqrt{1-x^2}, \quad \text{for } x \in \phi_1(U_1) = (-1, 1)$$

thus x is smooth on U_1 as well. For U_2 we have

$$\mathbb{1}_{(0,1)} \circ x \circ \phi_2^{-1}(x) = \sqrt{1-x^2}, \quad \text{for } x \in \phi_2(U_2) = (-1, 1)$$

thus x is smooth on U_2 as well. We can use a similar strategy to show that the coordinate function y is also smooth.

■ **Problem 2.11** The general linear group $\text{GL}(n, \mathbb{R})$ is the set of all real valued matrices with non-zero determinant under matrix multiplication. In other words

$$\text{GL}(n, \mathbb{R}) = \{A = [a_{ij}] \in \mathbb{R}^{n \times n} \mid \det(A) \neq 0\}.$$

We can see this as an open subset of $\mathbb{R}^{n \times n}$, thus it is a manifold. Show that this forms a Lie group.

Solution Let A, B be two matrices in the manifold. Then the i, j element of AB is

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

which is a polynomial in the coordinates of A and B , thus it is smooth. To show that the inverse is also smooth, for any function A in the manifold we have

$$A^{-1} = \frac{1}{\det(A)} \cdot (-1)^{i+j} ((i, j)\text{-minor of } A).$$

The (i, j) -minor of matrix A is the determinant of the sub matrix by deleting the i -th row and j -th column, which is again a polynomial in the coordinates of A and B , thus smooth (given that $\det(A) \neq 0$).

■ **Problem 2.12 — Jacobian matrix as a transition map (form W. Tu).** Let $(U, \phi) = (U, x^1, \dots, x^n)$ and $(V, \psi) = (V, y^1, \dots, y^n)$ be overlapping charts on a manifold M . The transition map $\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$ is a diffeomorphism of open subsets of \mathbb{R}^n . Show that its Jacobian matrix $J(\psi \circ \phi^{-1})$ at $\phi(p)$ is the matrix $[\partial y^i / \partial x^j]$ of partial derivatives at p .

Solution From the definition of the Jacobian matrix we can write

$$J(\psi \circ \phi^{-1}) = \frac{\partial(\psi \circ \phi^{-1})^i}{\partial r^j} = \frac{\partial(r^i \circ \psi \circ \phi^{-1})}{\partial r^j} = \frac{\partial(y^i \circ \phi^{-1})}{\partial r^j} = \frac{\partial y^i}{\partial x^j}.$$

Observation 2.12.2 — Some mnemonics for the Jacobian matrix. Here I introduce a symbolic mnemonic of remembering the form of the Jacobian matrix of a smooth map between manifolds. Let $F : N \rightarrow M$ a smooth map between manifolds. Since this function is from N to M , then the Jacobian matrix will be of the form

$$J(F) = [\partial F^i / \partial x^j]$$

where x^j a local coordinate in N and $F^i = y^i \circ F$ where y^i is a local coordinate in M . What we mean by local coordinates here is that for a point p on the manifold for which we want to calculate the Jacobian matrix, there are charts (U, x^1, \dots, x^n) and (V, y^1, \dots, y^n) on N and M respectively such that $p \in U$ and $F(U) \subset V$.

As another example, in the question above, since the function $\psi \circ \phi^{-1}$ is defined from

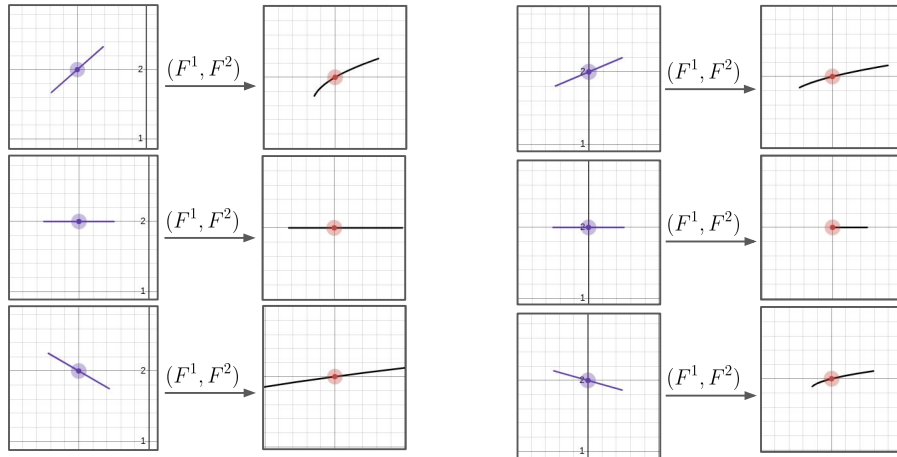
$\mathbb{R}^n \rightarrow \mathbb{R}^n$, then the Jacobian of this function starts with coordinates of \mathbb{R}^n , i.e. r^i .

■ **Problem 2.13 — From W. Tu.** Find all points in \mathbb{R}^2 in a neighborhood of which the functions given by $x^2 + y^2 - 1$ and y can serve as a local coordinate system.

Solution Define $F^1(x, y) = x^2 + y^2 - 1$ and $F^2(x, y) = y$. Then the pair (F^1, F^2) is locally invertible, thus can serve as a local diffeomorphism to \mathbb{R}^2 , thus a coordinate map if the Jacobian determinant $[\partial F^i / \partial x^j]$ is not zero. I.e.

$$\det \begin{bmatrix} 2x & 2y \\ 0 & 1 \end{bmatrix} = 2x \neq 0 \implies x \neq 0.$$

Thus the function $F = (F^1, F^2)$ can act as a local coordinate map everywhere except for on the points on the y axis. You can see why this happens in the plots below. As you can see, any path that is not crossing the y axis with zero vertical velocity is diffeomorphically mapped to a smooth curve. But when the curve passes through the y axis with a zero vertical velocity, then the curve folds on itself when mapped by F , thus F fails to be locally invertible and thus fails to be a coordinate map. Thus for any point that is not on the y axis, we can find an open set small enough that does not overlap with the y axis. But there is no such an open ball for the points on the y axis. You can try the online plotting tool that I have configured to generate the following plots [here](#).



■ **Problem 2.14 — Differentiable structure on \mathbb{R} .** Let \mathbb{R} be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \phi = \text{id} : \mathbb{R} \rightarrow \mathbb{R})$, and let \mathbb{R}' be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \psi : \mathbb{R} \rightarrow \mathbb{R})$, where $\psi(x) = x^{1/3}$.

- Show that these two differentiable structures are distinct.
- Show that there is a diffeomorphism between \mathbb{R} and \mathbb{R}' . (*Hint:* The identity map $\mathbb{R} \rightarrow \mathbb{R}$ is not the desired diffeomorphism; in fact, this map is not smooth).

Solution (a) Let M_1 be the atlas containing (\mathbb{R}, ϕ) and M_2 the atlas containing (\mathbb{R}, ψ) . Assume $M_1 = M_2$. Then (\mathbb{R}, ϕ) should be compatible with (\mathbb{R}, ψ) , i.e. the functions

$$\psi \circ \phi^{-1} : \mathbb{R} \rightarrow \mathbb{R}, \quad \phi \circ \psi^{-1} : \mathbb{R} \rightarrow \mathbb{R},$$

are smooth. This is not true since $(\psi \circ \phi^{-1})(x) = x^{1/3}$ that is not differentiable at $x = 0$. So $M_1 \neq M_2$ and these two atlases are distinct.

(b) For a more clear demonstration, the manifold with atlas generated by $(\mathbb{R}, \phi = \mathbb{1} : \mathbb{R} \rightarrow \mathbb{R})$ the manifold R , and call the other manifold the manifold R' . Define the map F between manifolds

$$F : R \rightarrow R'$$

as $F(p) = p^3$ for $p \in R$. The inverse of this map will be $F^{-1}(p) = p^{1/3}$. To show this map is a diffeomorphism, we need to show that the following

$$\psi \circ F \circ \phi^{-1} : \phi(U) \rightarrow \psi(F(U)), \quad \phi \circ F^{-1} \circ \psi^{-1} : \psi(V) \rightarrow \phi(F^{-1}(V)),$$

are smooth. In equations above, $(U, \phi : x \mapsto x)$ is some coordinate system (i.e. chart) on R whereas $(V, \psi : x \mapsto x^{1/3})$ is a coordinate system on R' . Let $\phi(x) = x \in \phi(U \cap V)$. Then

$$(\psi \circ F \circ \phi^{-1})(x) = \psi(F(x)) = \psi(x^3) = x,$$

which is the identity map and is smooth. Furthermore

$$(\phi \circ F^{-1} \circ \psi^{-1})(x) = \phi(F^{-1}(x^3)) = \psi(x) = x,$$

which is again the identity map and is smooth. Thus the map F is a diffeomorphism between the manifolds.

■ **Problem 2.15 — The smoothness of an inclusion map (From L. Tu).** Let M and N be manifolds and let q_0 be a point in N . Prove that the inclusion map $i_{q_0} : M \rightarrow M \times N$, $i_{q_0}(p) = (p, q_0)$, is C^∞ .

Solution Let $p \in M$ and let (U, ψ) , (V, ϕ) coordinate charts for M and N respectively where $x \in U$ and $q_0 \in V$. Then the chart $(U \times V, \psi \times \phi)$ will be a coordinate chart for $M \times N$ where $(p, q_0) \in U \times V$. To show the smoothness of i_{q_0} we need to show that the following

$$(\psi \times \phi) \circ i_{q_0} \circ \psi^{-1} : \psi(U) \rightarrow (\psi \times \phi)(i_{q_0}(U)).$$

Let $\psi(p) = x$. Then

$$((\psi \times \phi) \circ i_{q_0} \circ \psi^{-1})(x) = ((\psi \times \phi) \circ i_{q_0})(p) = (\psi \times \phi)(p, q_0) = (x, \phi(q_0)).$$

Thus $(\psi \times \phi) \circ i_{q_0} \circ \psi^{-1}$ maps $x \mapsto (x, \phi(q_0))$ where $\phi(q_0) \in \mathbb{R}^n$ a constant. This is smooth since each components is smooth.

■ **Problem 2.16 — Group of automorphisms of a vector space (from L. Tu).** Let V be a finite-dimensional vector space over \mathbb{R} , and $GL(V)$ the group of all linear automorphisms of V . Relative to an ordered basis $e = (e_1, \dots, e_n)$ for V , a linear automorphism $L \in GL(V)$ is represented by a matrix $[a_j^i]$ defined by

$$L(e_j) = \sum_i a_j^i e_i.$$

The map

$$\phi_e : GL(V) \rightarrow GL(n, \mathbb{R}), \quad L \mapsto [a_j^i],$$

is a bijection with an open subset of $\mathbb{R}^{n \times n}$ that makes $GL(V)$ into a smooth manifold, which we denote temporarily by $GL(V)_e$. If $GL(V)_u$ is the manifold structure induced from another ordered basis $u = (u_1, \dots, u_n)$ for V , show that $GL(V)_e$ is the same as $GL(V)_u$.

Solution To show that these manifolds are the same, we will show that the underlying set of these manifolds are the same. Thus we need to show $L \in GL(V)_e \implies L \in GL(V)_u$ and $L \in GL(V)_u \implies L \in GL(V)_e$. We start with the first implication. Let $L \in GL(V)_e$. Then we can write

$$\begin{bmatrix} L(e_1) \\ \vdots \\ L(e_n) \end{bmatrix} = A^T \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

For some matrix $A = [a_j^i]$. Thus $\phi_e(L) = A \in GL(n, \mathbb{R})$. Let $u = (u_1, \dots, u_n)$ be another basis for V . By the change of basis matrix (which is non-singular and invertible) we can write

$$\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = B^T \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

for some invertible matrix $B = [b_k^l]$. Since B is $n \times n$ invertible, thus $B \in GL(n, \mathbb{R})$. From the linearity of L we have

$$\begin{bmatrix} L(e_1) \\ \vdots \\ L(e_n) \end{bmatrix} = B^T \begin{bmatrix} L(u_1) \\ \vdots \\ L(u_n) \end{bmatrix}, \quad \begin{bmatrix} L(u_1) \\ \vdots \\ L(u_n) \end{bmatrix} = (B^T)^{-1} \begin{bmatrix} L(e_1) \\ \vdots \\ L(e_n) \end{bmatrix}.$$

Thus we can write

$$\begin{bmatrix} L(u_1) \\ \vdots \\ L(u_n) \end{bmatrix} = (B^T)^{-1} A^T B^T \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

Thus we can write

$$(L(u_1) \cdots L(u_n)) = BAB^{-1}(u_1 \cdots u_n)$$

Thus we can write

$$\phi_u(L) = BAB^{-1} = B\phi_e(L)B^{-1} \in GL(n, \mathbb{R}),$$

where the equality is true since $B \in GL(n, \mathbb{R})$ and $\phi_e(L) \in GL(n, \mathbb{R})$. Thus $\phi_u(L) \in GL(n, \mathbb{R})$. This implies $L \in GL(V)_u$. To show the second implication, we can use the similar idea as above.

■ **Problem 2.17 — Local coordinate systems (from L. Tu).** Find all points in \mathbb{R}^3 in a neighborhood of which the functions $x, x^2 + y^2 + z^2 - 1$, and z can serve as a local coordinate system.

Solution Define

$$F^1(x, y, z) = x, \quad F^2(x, y, z) = x^2 + y^2 + z^2 - 1, \quad F^3(x, y, z) = z.$$

Then the function $F = (F^1, F^2, F^3)$ can be a local coordinate map in a local coordinate system if its Jacobian determinant is nonzero. Thus we need to have

$$\det \begin{pmatrix} \partial F^1/\partial x & \partial F^1/\partial y & \partial F^1/\partial z \\ \partial F^2/\partial x & \partial F^2/\partial y & \partial F^2/\partial z \\ \partial F^3/\partial x & \partial F^3/\partial y & \partial F^3/\partial z \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 \\ 2x & 2y & 2z \\ 0 & 0 & 1 \end{pmatrix} = 2y \neq 0.$$

Thus $F = (F_1, F_2, F_3)$ is locally invertible, thus can be a local diffeomorphism to \mathbb{R}^3 , thus it can be a coordinate char for all points on the manifold where $y \neq 0$.

■ **Problem 2.18** Let I be the unit closed interval $I = [0, 1]$ and I/\sim the quotient space obtained from I by identifying the two points $\{0, 1\}$ to a point. Denote by S^1 the unit circle in the complex plane. Show that I is homeomorphic to S^1 .

Solution As we have discussed in the summary box [Summary 2.3](#), we will use the second method to show that an isomorphism exists between these two sets. Consider the function $f : I \rightarrow \mathbb{C}$ given by $f(t) = e^{2i\pi t}$. Thus function assumes the same value for the points $\{0, 1\}$ in its domain that are in the same equivalence class as well. Thus this induces a map $\bar{f} : I/\sim \rightarrow S^1$. Consider the following commutative diagram.

$$\begin{array}{ccc} I & \xrightarrow{f} & S^1 \\ \pi \downarrow & \nearrow \bar{f} & \\ I/\sim & & \end{array}$$

Since f is continuous, then by [Proposition 2.10](#) \bar{f} is also continuous. Because

- \bar{f} is continuous (from our reasoning above).
- \bar{f} is a bijection (note that f is not a bijection but \bar{f} is a bijection).
- I/\sim is compact (since it is a continuous image of a compact set I).
- S^1 is Hausdorff.

Then by the proposition embedded in [Summary 2.3](#) we conclude that \bar{f} is indeed a homeomorphism.

Observation 2.12.3 In the problem above, one might ask why we did not find the homeomorphism explicitly as we usually do in other problems. The answer is that we can still find the explicit homeomorphism but in doing so, we will face some difficulties (not in all of the problems though). For instance, let's try to find a continuous map $\bar{g} : S^1 \rightarrow I/\sim$ that is the inverse of \bar{f} . We might arrange the sets and maps in the following form to be able to use the theorems we have had in this chapter (like [Proposition 2.10](#)). Consider the following diagram.

$$\begin{array}{ccc} S^1 & \xrightarrow{g} & I \\ \downarrow id & \searrow \pi \circ g & \downarrow \pi \\ S^1 & \xrightarrow{\bar{g}} & I/\sim \end{array}$$

The appropriate function to choose is $g : S^1 \rightarrow I$ which is the complex logarithm function with its branch cut on the positive real axis. But this function is clearly discontinuous on the positive real axis. Although this does not mean that the map $\pi \circ g$ is not continuous, but to

show that this map is continuous we need to use the basic definitions to prove it and we can not use any higher level theorems (at least I am not aware of any). Then when we prove that $\pi \circ g$ is continuous, since it assumes a constant value on all of its points in the domain that are in the same equivalence class (the equivalence class here is the trivial equality relation, where each point has relation with only itself), then it induces a map \bar{g} where its continuity follows immediately from the continuity of $\pi \circ g$.

If you be sharp enough, you will understand that the last part of the reasoning is really redundant (i.e. concluding the continuity of \bar{g} from $\pi \circ g$). That is simply because \bar{g} is exactly the same as $\pi \circ g$ (can you see this from the diagram?)

■ **Problem 2.19** Let S^2 be the unit sphere and S^2/\sim the quotient set obtained from S^2 by identifying the antipodal points to a point. Also, let H^2 be the closed upper hemisphere and H^2/\sim be the quotient space obtained from H^2 by identifying the *antipodal points on its equator to a point*. Show that these two spaces are homeomorphisms.

Solution We will use the same approach as the question above. Let $i : H^2 \rightarrow S^2$ be the inclusion map. This map is continuous. Consider the commuting diagram below

$$\begin{array}{ccc} H^2 & \xrightarrow{i} & S^2 \\ \pi_1 \downarrow & \searrow \pi_2 \circ i & \downarrow \pi_2 \\ H^2/\sim & \xrightarrow{\bar{i}} & S^2/\sim \end{array}$$

Since π_2 and i are both continuous maps, then $\pi_2 \circ i$ is also continuous. Also, note that this map assumes a constant value for all of the points in its domain that are in the same equivalence class. Thus it induces a map \bar{i} . Since $\pi_2 \circ i$ is continuous, then \bar{i} is also continuous. We can also easily show that \bar{i} is a bijection (although i was not a bijective map). Furthermore, since

- H^2/\sim is compact (since it is the continuous image of a compact set),
- S^2/\sim is Hausdorff (I will show below)

from the proposition in [Summary 2.3](#) we prove that \bar{i} is indeed an homeomorphism.

Showing that S^2/\sim is Hausdorff. It follows from [Theorem 2.2](#) that S^2/\sim is Hausdorff if and only if the equivalence relation is open and the graph of the equivalence relation is closed in $S^2 \times S^2$. From [Remark 2.13](#) it follows that \sim is open if and only if $\pi^{-1}(\pi(U))$ is open in S^2 for all open $U \subset S^2$. Let $U \subset S^2$ open. Then

$$\pi^{-1}(\pi(U)) = U \cap \{(x, y, z) \in \mathbb{R}^3 : -(x, y, z) \in U\}.$$

Since both of sets above are open, thus $\pi^{-1}(\pi(U))$ is open.

To show that the graph of the equivalence relation is closed in $S^2 \times S^2$ we have

$$\begin{aligned} R &= \{(x, y) \in S^2 \times S^2 : x \sim y\} = \{(x, y) \in S^2 \times S^2 : x = \pm y\} \\ &= \{(x, \pm x) : x \in S^2\} = \{(x, x) : x \in S^2\} \cup \{(x, -x) : x \in S^2\}. \end{aligned}$$

On the other hand, from [Corollary 2.2](#) we know that since S^2 is Hausdorff, then $\{(x, x) : x \in S^2\}$ is closed (as well as $\{(x, -x) : x \in S^2\}$). Thus R is closed in $S^2 \times S^2$.

■ **Problem 2.20 — Image of the inverse image of a map (from W. Tu).** let $f : X \rightarrow Y$ be a map of sets, and let $B \subset Y$. Prove that $f(f^{-1}(B)) = B \cap f(X)$. There fore if f is surjective, then $f(f^{-1}(B)) = B$.

Solution We basically want to show that the two following sets are equal

$$f(f^{-1}(B)) = B \cap f(X).$$

If $f(f^{-1}(B)) = \emptyset$, then it follows immediately that $f(f^{-1}(B)) = B \cap f(X) = \emptyset$. For that case that $f(f^{-1}(B)) \neq \emptyset$, let $y \in f(f^{-1}(B))$. Note that $f(f^{-1}(B)) \neq \emptyset \implies f^{-1}(B) \neq \emptyset$. This immediately implies that $y \in f(X)$. Also we can write

$$y \in f(f^{-1}(B)) \implies f^{-1}(y) \in f^{-1}(B) \implies y \in B.$$

Thus we get $y \in B \cap f(X)$. For the converse, let $y \in B \cap f(X)$. Then we can write

$$f^{-1}(y) \in f^{-1}(B \cap f(X)) = f^{-1}(B) \cap X = f^{-1}(B).$$

This implies

$$y \in f(f^{-1}(B)).$$

Now as a simple corollary, if f is surjective, then $f(X) = Y$, which implies

$$f(f^{-1}(B)) = B \cap Y = B.$$

■ **Problem 2.21 — Closedness of the diagonal of a Hausdorff space (from W. Tu).** Deduce [Theorem 2.2](#) from [Corollary 2.2](#).

Note. You might be quite confused as in the text we used the corollary mentioned above as an immediate result of the theorem above (which is the case in most of the cases). But in this case, the corollary has enough strength that can prove the original theorem.

Solution Let S be a topological space and \sim an open equivalence relation. We start with the forward direction. I.e. we want to show that S/\sim is Hausdorff implies that the graph of \sim is closed in $S \times S$. Consider the diagonal of S/\sim where

$$\Delta = \{(x, x) \in S/\sim \times S/\sim\}.$$

We know that Δ is closed in $S/\sim \times S/\sim$, as S/\sim is assumed to be Hausdorff and [Corollary 2.2](#) implies that Δ is closed in $S/\sim \times S/\sim$. Also, consider the graph of the equivalence relation

$$R = \{(x, y) \in S \times S \mid x \sim y\} = \{(x, y) \in S \times S \mid \pi(x) = \pi(y)\}.$$

This suggest that Δ is the image of R under the map $\Pi : S \times S \rightarrow S/\sim \times S/\sim$ that maps $(x, y) \in S \times S \mapsto (\pi(x), \pi(y))$. So we can write

$$\Delta = \Pi(R).$$

Or equivalently, we can state that R is the pre-image of Δ under that map Π , i.e. $R = \Pi^{-1}(\Delta)$. Since that components of the map Π are continuous, thus the pre-image of open sets are opens sets (or equivalently, the pre-image of closed sets are closed sets). Since Δ is closed in $S/\sim \times S/\sim$, then it implies that R is closed in $S \times S$.

To prove the converse, we need to show closedness of R in $S \times S$ implies S/\sim is Hausdorff. First note that Δ is the image of R under the map Π , i.e. $\Delta = \Pi(R)$. Since the equivalence relation is open, then π maps open sets in its domain to the open sets in its pre-domain (or equivalently, it maps closed sets to closed sets.). So does Π . Since R is closed in $S \times S$, then Δ is closed in $S/\sim \times S/\sim$. Then from [Corollary 2.2](#) it follows that S/\sim is Hausdorff.

■ **Problem 2.22 — Quotient of a sphere with antipodal points identified (from W. Tu).** Let S^n be the unit sphere centered at the origin in \mathbb{R}^{n+1} . Define an equivalence relation \sim on S^n by identifying antipodal points:

$$x \sim y \iff x = \pm y, \quad x, y \in S^n.$$

- (a) Show that \sim is an open equivalence relation.
- (b) Apply [Theorem 2.2](#) and [Corollary 2.2](#) to prove that the quotient space S^n/\sim is Hausdorff, without making use of the homeomorphism $\mathbb{R}P^n \simeq S^n/\sim$.

Solution In some parts of our solution for [Problem 2.19](#) we proved this for the case $n = 2$. To prove it for any $n \in \mathbb{N}$ we will use a pretty much same approach.

- (a) By [Remark 2.13](#) we need to show that for every $U \subset S^n$ the set $\pi^{-1}(\pi(U))$ is open in S^n . Let $U \subset S^n$ be an open set. Then

$$\pi^{-1}(\pi(U)) = U \cup \{x \in \mathbb{R}^{n+1} \mid -x \in U\}.$$

Since both sets in the RHS are open, then $\pi^{-1}(\pi(U))$ is open in S^n . Thus the equivalence relation is open.

- (b) We need to show that S^n/\sim is Hausdorff. Since the equivalence relation is open (proved in part (a) above), then by [Theorem 2.2](#) we just need to show that the graph of equivalence relation is closed in $S^n \times S^n$. For the graph of this equivalence relation we have

$$R = \{(x, y) \in S^n \times S^n \mid x \sim y\} = \{(x, y) \in S^n \times S^n \mid x = \pm y\} = \{(x, x) \mid x \in S^n\} \cup \{(x, -x) \mid x \in S^n\}.$$

Since S^n is Hausdorff (under the subspace topology), then by [Corollary 2.2](#) the two sets in the RHS above are both closed, thus making R closed, hence $S^n/\sim \times S^n/\sim$ is Hausdorff.

Be Careful Here! ⚠️ **2.12.1** In my proof above, Since S^n is Hausdorff, then by [Corollary 2.2](#) it implies that Δ is closed. In the proof above I also proved that $\{(x, -x) \mid x \in S^n\}$ is closed, but I have not proved it.

■ **Problem 2.23 — Orbit space of a continuous group action (from W. Tu).** Suppose a right action of a topological group G on a topological space S is continuous; this simply means that the map $\alpha : S \times G \rightarrow S$ describing the action is continuous. Define two points x, y of S to be equivalent if they are in the same orbit; i.e. there is an element $g \in G$ such that $y = xg$. Let S/G be the quotient space; it is called the orbit space of the action. Prove that the projection map $\pi : S \rightarrow S/G$ is an open map.

Observation 2.12.4 In the problem above, note and appreciate (!) that the group G is stated to be a topological group. If G was not a topological group, then we were not able to talk about the notion of continuity of the right group action. The continuity of the right group action implies that for any open set $U \subset S$ we have $\alpha^{-1}(U) \subset S \times G$ is open.

Solution First, we need to prove the following Lemma.

Lemma 2.2 Let S be a topological space and G be a topological group. Consider the right action of the group G on S via the map

$$\alpha : S \times G \rightarrow S, \quad \alpha(s, g) = sg$$

If the right action is *continuous* (i.e. the map α is continuous) Then for $g \in G$ the map

$$\alpha_g : S \rightarrow S$$

is a *homeomorphism*.

Proof. We need to do the following three steps

- We need to show that α_g is a bijection that is continuous with a continuous inverse. α_g is indeed inverse with the inverse given by $\alpha_{g^{-1}}$, where $g^{-1} \in G$ is the inverse element of $g \in G$ (note that since G is group then the inverse exists). Observe that

$$\alpha_g(\alpha_{g^{-1}}(x)) = xg^{-1}g = x, \quad \alpha_{g^{-1}}(\alpha_g(x)) = xgg^{-1} = x.$$

- We need to show that α_g is continuous. Since the map α is continuous, and α_g is just a restriction of the α for a fixed g , then α_g is also continuous.

Alternatively, we can show the continuity of α_g by showing that the pre-image of any open set $U \subset S$ is open under the subspace topology of $S \times \{g\}$ (where its open sets will be of the form $\mathcal{T}_S \times \{g\}$).

- We need to show that α_g^{-1} is continuous. Similar to our reasoning above, since α is continuous and α_g^{-1} is a restriction of α to the constant g^{-1} , α_g^{-1} is also continuous.

□

To show that π is an open map, by [Remark 2.13](#) we need to show that for any open $U \in S$ the set $\pi^{-1}(\pi(U))$ is open in S . Let $U \subset S$ be any open set. Then

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} Ug.$$

From the Lemma we prove above, for any fixed g , then map $\alpha_g : S \rightarrow S$ is a homeomorphism. Thus $\forall g \in G$ the set Ug is open in S . It follows immediately that $\pi^{-1}(\pi(U))$ is open as well.

■ **Problem 2.24 — Quotient of \mathbb{R} by $2\pi\mathbb{Z}$ (from L. Tu).** Let the additive group $2\pi\mathbb{Z}$ act on \mathbb{R} . on the right by $x \cdot 2\pi n = x + 2\pi n$, where n is an integer. Show that the orbit space $\mathbb{R}/2\pi\mathbb{Z}$ is a smooth manifold.

Solution To show that $X = \mathbb{R}/2\pi\mathbb{Z}$ is a smooth manifold, we need to show that X is a *Hausdorff* and *second countable topological space*. Furthermore, we need to exhibit a *maximal atlas*.

Showing that π is an open map. We first need to show that the projection map $\pi : \mathbb{R} \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ or equivalently $\pi : \mathbb{R} \rightarrow \mathbb{R}/\sim$ is an open map. Note that the equivalence relation \sim is defined as $x, y \in \mathbb{R}$ are equivalent, i.e. $x \sim y$ if $y = x \cdot g$ for some $g \in 2\pi\mathbb{Z}$ or equivalently, $y = x + 2\pi n$ for some $n \in \mathbb{Z}$. Since the group action map $\alpha(x, n) = x + 2\pi n$ is continuous, then by [Lemma 2.2](#) we conclude that the projection map π is open.

Showing that \mathbb{R}/\sim is second countable. Since π is an open map, then by [Proposition 2.12](#) we can conclude that \mathbb{R}/\sim is second countable.

Showing that \mathbb{R}/\sim is Hausdorff. Since π is an open map, to show that \mathbb{R}/\sim is Hausdorff, by [Theorem 2.2](#) we need to show that the graph of the equivalence relation R is closed in $\mathbb{R} \times \mathbb{R}$. The graph of this equivalence relation is


$$\begin{aligned} R &= \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \sim y\} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x \cdot g \text{ for some } g \in 2\pi\mathbb{Z}\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x + 2\pi n \text{ for some } n \in \mathbb{Z}\}. \end{aligned}$$

This R is the set of all lines in \mathbb{R}^2 with slope 1 with y-intercepts equal to $2\pi n$ for some $n \in \mathbb{Z}$. This set is indeed closed in \mathbb{R}^2 as its complement is open in \mathbb{R}^2 . We can prove the openness of R^c by first writing it as


$$R^c = \bigcup_{n \in \mathbb{Z}} S_n$$

where S_n is the open set between two lines with slope 1 and intercepts $2\pi(n)$ and $2\pi(n+1)$. Since R^c is union of open sets, thus it is open.

Exhibiting a maximal atlas See the box below.

Be Careful Here!  **2.12.2** For the last part of the question above, to show that $\mathbb{R}/2\pi\mathbb{Z}$ is a smooth manifold, I need to exhibit a maximal atlas (at least I think I need to do that!). To do this, I have a feeling that I need first to find any atlas, and then by using [Summary 2.6](#) we can show that the manifold is smooth. But, instead of designing a smooth atlas, I **think**^a I can show that this manifold is smooth by showing that it is homeomorphic to S^1 which is indeed a smooth manifold. Calling this homeomorphism as f , and considering the atlas $\{(U_\alpha, \phi_\alpha)\}$ as an atlas for S^1 , then the atlas for $\mathbb{R}/2\pi\mathbb{Z}$ will be $\{(f^{-1}(U_\alpha), \phi_\alpha \circ f)\}$.

^aSee the box below.

Be Careful Here!  **2.12.3 — Your thought in the box above is not actually true!** In the Be-Careful box above, I said I think I can show that the manifold is smooth by showing that it is homeomorphic to another manifold. This is not exactly true! Note that in the category of topological spaces the isomorphisms are the homeomorphisms, but in the category of smooth manifolds, the isomorphisms are the diffeomorphisms, not homeomorphisms. Thus in the Be-Careful box above we actually need to show that the manifold of interest is *diffeomorphic* to S^1 , not homeomorphic. The good news is that my proof below does not break apart. Surprisingly it is the same procedure in one of the exercises of L. Tu, which I have solved in [Problem 2.25](#).

(The old and vague solution) Showing that $\mathbb{R}/2\pi\mathbb{Z} \simeq S^1$. For this we need to find a homeomorphism. Consider the map f

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & S^1 \subset \mathbb{C} \\ \pi \downarrow & \nearrow \bar{f} & \\ \mathbb{R}/2\pi\mathbb{Z} & & \end{array}$$

where $x \mapsto e^{2\pi i x}$. This map assumes a constant value for all of the elements in the same equivalence class. But it induces the map \bar{f} , and the diagram above commutes. Continuity of f implies the continuity of \bar{f} . Although f has no inverse (since it is not one to one), but we can find an inverse for \bar{f} that is given by

$$\begin{aligned} \bar{f}^{-1} : S^1 &\rightarrow \mathbb{R} \\ e^{i\theta} &\mapsto \theta \pmod{2\pi}. \end{aligned}$$

Thus \bar{f} is a bijection. Since $\mathbb{R}/2\pi\mathbb{Z}$ is compact (continuous image of a compact set), and S^1 is Hausdorff (with the subspace topology of \mathbb{C}), by [Summary 2.3](#) we conclude that $\mathbb{R}/2\pi\mathbb{Z} \simeq S^1$.

The new and correct solution. For this, we present a C^∞ compatible chart for this manifold. First, consider the following open sets on the manifold

$$V_1 = \{[t] \mid t \in (-\pi, \pi)\}, \quad V_2 = \{[t] \mid t \in (0, 2\pi)\}.$$

In words, what we really mean is that V_1 contains the equivalence classes whose representatives comes from the set $(-\pi, \pi)$. Similarly for V_2 , we say that it contains the equivalence classes whose representatives comes from the set $(0, 2\pi)$. We define the following coordinate maps

$$\begin{aligned} \phi_1 : V_1 &\rightarrow \mathbb{R} & \phi_2 : V_2 &\rightarrow \mathbb{R} \\ [t] &\mapsto t & [t] &\mapsto t \end{aligned}$$

Note that although the functions ϕ_1 and ϕ_2 look similar, but they are different functions as they have different domains. These two functions both return the representative of the equivalence classes in their argument. Note that we said *the representative* and not *a representative*) because of the design of their domain. For instance, let's parse the following operation

$$\phi_1([23.5\pi]) = -0.5\pi.$$

Since the domain of ϕ_1 is the equivalence classes whose representatives comes from the set $(-\pi, \pi)$, then we need to find a representative for $[23.5\pi]$ that lives in $(-\pi, \pi)$. I.e. $[23.5\pi] = [1.5\pi] = [-0.5\pi]$.

We claim that the collection $\{(V_1, \phi_1), (V_2, \phi_2)\}$ is a C^∞ compatible atlas. To show this we simply need to show that the coordinate charts are compatible. We start with

$$\phi_1 \circ \phi_2^{-1} : \phi_2(V_1 \cap V_2) \rightarrow \phi_1(V_1 \cap V_2).$$

Note that

$$V_1 \cap V_2 = \{[t] \mid t \in (0, \pi) \cup (\pi, 2\pi)\}$$

or equivalently

$$V_1 \cap V_2 = \{[t] \mid t \in (-\pi, 0) \cup (0, \pi)\}.$$

Thus the function $\phi_1 \circ \phi_2^{-1}$ will be

$$\phi_1 \circ \phi_2^{-1} : (0, \pi) \cap (\pi, 2\pi) \rightarrow \mathbb{R}.$$

and is given by

$$(\phi_1 \circ \phi_2^{-1})(t) = \begin{cases} t & t \in (0, \pi), \\ t - 2\pi & t \in (\pi, 2\pi). \end{cases}$$

Similarly we can write

$$\phi_2 \circ \phi_1^{-1} : \underbrace{\phi_1(V_1 \cap V_2)}_{(-\pi, 0) \cap (0, \pi)} \rightarrow \phi_2(V_1 \cap V_2),$$

is $\phi_2 \circ \phi_1^{-1}$ is given by

$$(\phi_2 \circ \phi_1^{-1})(t) = \begin{cases} t + 2\pi & t \in (-\pi, 0), \\ t & t \in (0, \pi). \end{cases}$$

Since both of these functions are C^∞ in their corresponding domain, then it follows that the exhibited collection of coordinate charts is indeed a C^∞ atlas.

■ **Problem 2.25 — The circle as a quotient space.**

- (a) Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha=1}^2$ be the atlas of the circle S^1 in [Problem 2.1](#), and let $\bar{\phi}_\alpha$ be the map ϕ_α followed by the projection $\mathbb{R} \rightarrow \mathbb{R}/2\pi\mathbb{Z}$. On $U_1 \cap U_2 = A \sqcup B$, since ϕ_1 and ϕ_2 differ by an integer multiple of 2π , $\bar{\phi}_1 = \bar{\phi}_2$. Therefore, $\bar{\phi}_1$ and $\bar{\phi}_2$ piece together to give a well-defined map $\bar{\phi} : S^1 \rightarrow \mathbb{R}/2\pi\mathbb{Z}$. Prove that $\bar{\phi}$ is C^∞ .
- (b) The complex exponential $\mathbb{R} \rightarrow S^1, t \mapsto e^{it}$, is constant on each orbit of the action of $2\pi\mathbb{Z}$. Therefore, there is an induced map $F : \mathbb{R}/2\pi\mathbb{Z} \rightarrow S^1, F([t]) = e^{it}$. Prove that F is C^∞ .
- (c) Prove that $F : \mathbb{R}/2\pi\mathbb{Z} \rightarrow S^1$ is a diffeomorphism.

Solution

- (a) We define the map $\bar{\phi} : S^1 \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ as following

$$\bar{\phi}(x) = \begin{cases} (\pi \circ \phi_1)(x) & x \in U_1, \\ (\pi \circ \phi_2)(x) & x \in U_2. \end{cases}$$

Considering the atlas $\{(V_\alpha, \psi_\alpha)\}_{\alpha=1}^2$ for $\mathbb{R}/2\pi\mathbb{Z}$ that we constructed in [Problem 2.24](#), we can show that $\bar{\phi}$ is C^∞ by simply showing that it is a smooth map between manifolds (i.e. using [Definition 2.11](#)). Thus we need to show that the following maps (in their corresponding domains)

$$\begin{aligned} \psi_1 \circ \bar{\phi} \circ \phi_1^{-1} &: \phi_1(U_1 \cap \bar{\phi}^{-1}(V_1)) \rightarrow \psi_1(\bar{\phi}(U_1)), \\ \psi_1 \circ \bar{\phi} \circ \phi_2^{-1} &: \phi_2(U_2 \cap \bar{\phi}^{-1}(V_1)) \rightarrow \psi_1(\bar{\phi}(U_2)), \\ \psi_2 \circ \bar{\phi} \circ \phi_1^{-1} &: \phi_1(U_1 \cap \bar{\phi}^{-1}(V_2)) \rightarrow \psi_2(\bar{\phi}(U_1)), \\ \psi_2 \circ \bar{\phi} \circ \phi_2^{-1} &: \phi_2(U_2 \cap \bar{\phi}^{-1}(V_2)) \rightarrow \psi_2(\bar{\phi}(U_2)). \end{aligned}$$

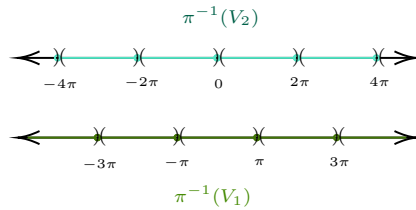
are smooth. To avoid multiple references to [Problem 2.1](#) we write the following definitions here.

$$U_1 = \{e^{it} | t \in (-\pi, \pi)\}, \quad U_2 = \{e^{it} | t \in (0, 2\pi)\}.$$

Now we can analyze all of these maps one by one. First, observe that

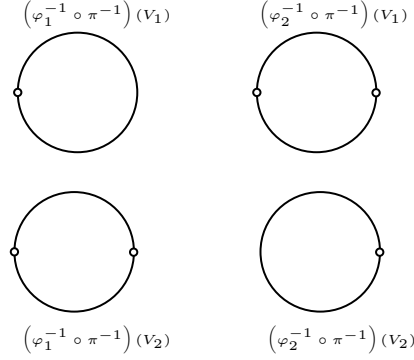
$$\pi^{-1}(V_1) = \bigcup_{n \in \mathbb{Z}} (-\pi + 2n\pi, \pi + 2n\pi), \quad \pi^{-1}(V_2) = \bigcup_{n \in \mathbb{Z}} (2n\pi, 2\pi + 2n\pi).$$

which are demonstrated in the following diagrams. On the other hand we have



$$\begin{aligned} U_1 \cap \bar{\phi}^{-1}(V_1) &= (\phi_1^{-1} \circ \pi^{-1})(V_1), & U_2 \cap \bar{\phi}^{-1}(V_1) &= (\phi_2^{-1} \circ \pi^{-1})(V_1), \\ U_1 \cap \bar{\phi}^{-1}(V_2) &= (\phi_1^{-1} \circ \pi^{-1})(V_2), & U_2 \cap \bar{\phi}^{-1}(V_2) &= (\phi_2^{-1} \circ \pi^{-1})(V_2). \end{aligned}$$

The image of these maps are shown in the figure below



Thus we will have

$$\begin{aligned}
 \phi_1(U_1 \cap \bar{\phi}^{-1}(V_1)) &= (-\pi, \pi), \\
 \phi_2(U_2 \cap \bar{\phi}^{-1}(V_1)) &= (0, \pi) \cup (\pi, 2\pi), \\
 \phi_1(U_1 \cap \bar{\phi}^{-1}(V_2)) &= (-\pi, 0) \cup (0, \pi), \\
 \phi_2(U_2 \cap \bar{\phi}^{-1}(V_2)) &= (0, 2\pi).
 \end{aligned}$$

Now we can re-write the maps of interest as

$$\begin{aligned}
 \psi_1 \circ \bar{\phi} \circ \phi_1^{-1} &: (-\pi, \pi) \rightarrow \psi_1(\bar{\phi}(U_1)), \\
 \psi_1 \circ \bar{\phi} \circ \phi_2^{-1} &: (0, \pi) \cup (\pi, 2\pi) \rightarrow \psi_1(\bar{\phi}(U_2)), \\
 \psi_2 \circ \bar{\phi} \circ \phi_1^{-1} &: (-\pi, 0) \cup (0, \pi) \rightarrow \psi_2(\bar{\phi}(U_1)), \\
 \psi_2 \circ \bar{\phi} \circ \phi_2^{-1} &: (0, 2\pi) \rightarrow \psi_2(\bar{\phi}(U_2)).
 \end{aligned}$$

which are given by

$$\begin{aligned}
 (\psi_1 \circ \bar{\phi} \circ \phi_1^{-1})(t) &= (\psi_1 \circ (\pi \circ \phi_1) \circ \phi_1^{-1})(t) = \psi_1(\pi(t)) = \psi_1([t]) = t, & t \in (-\pi, \pi). \\
 (\psi_1 \circ \bar{\phi} \circ \phi_2^{-1})(t) &= (\psi_1 \circ (\pi \circ \phi_2) \circ \phi_2^{-1})(t) = \psi_1(\pi(t)) = \phi_1([t]) = \begin{cases} t & t \in (0, \pi), \\ t - 2\pi & t \in (\pi, 2\pi). \end{cases} \\
 (\psi_2 \circ \bar{\phi} \circ \phi_1^{-1})(t) &= (\psi_2 \circ (\pi \circ \phi) \circ \phi_1^{-1})(t) = \psi_2(\pi(t)) = \psi_2([t]) = \begin{cases} t + 2\pi & t \in (-\pi, 0), \\ t & t \in (0, \pi). \end{cases} \\
 (\psi_2 \circ \bar{\phi} \circ \phi_2^{-1})(t) &= (\psi_2 \circ (\pi \circ \phi_2) \circ \phi_2^{-1})(t) = \psi_2(\pi(t)) = \psi_2(t) = t, & t \in (0, 2\pi).
 \end{aligned}$$

All of the maps above are C^∞ . This shows that $\bar{\psi}$ is indeed C^∞ .

- (b) With a similar approach as above, using [Definition 2.11](#) we need to show that the following maps are smooth.

$$\begin{aligned}
 \phi_1 \circ F \circ \psi_1^{-1} &: \psi_1(V_1 \cap F^{-1}(U_1)) \rightarrow \phi_1(F(U_1)) \\
 \phi_1 \circ F \circ \psi_2^{-1} &: \psi_2(V_2 \cap F^{-1}(U_1)) \rightarrow \phi_1(F(U_2)) \\
 \phi_2 \circ F \circ \psi_1^{-1} &: \psi_1(V_1 \cap F^{-1}(U_2)) \rightarrow \phi_2(F(U_1)) \\
 \phi_2 \circ F \circ \psi_2^{-1} &: \psi_2(V_2 \cap F^{-1}(U_2)) \rightarrow \phi_2(F(U_2))
 \end{aligned}$$

First, observe that

$$\begin{aligned} V_1 \cap F^{-1}(U_1) &= \{[t] \mid t \in (-\pi, \pi)\} \implies \psi_1(V_1 \cap F^{-1}(U_1)) = (-\pi, \pi), \\ V_2 \cap F^{-1}(U_1) &= \{[t] \mid t \in (0, \pi) \cup (\pi, 2\pi)\} \implies \psi_2(V_2 \cap F^{-1}(U_1)) = (0, \pi) \cup (\pi, 2\pi), \\ V_1 \cap F^{-1}(U_2) &= \{[t] \mid t \in (-\pi, 0) \cup (0, \pi)\} \implies \psi_1(V_1 \cap F^{-1}(U_2)) = (-\pi, 0) \cup (0, \pi), \\ V_2 \cap F^{-1}(U_2) &= \{[t] \mid t \in (0, 2\pi)\} \implies \psi_2(V_2 \cap F^{-1}(U_2)) = (0, 2\pi). \end{aligned}$$

Thus we can re-write the map definitions

$$\begin{aligned} \phi_1 \circ F \circ \psi_1^{-1} &: (-\pi, \pi) \rightarrow \phi_1(F(U_1)) \\ \phi_1 \circ F \circ \psi_2^{-1} &: (0, \pi) \cup (\pi, 2\pi) \rightarrow \phi_1(F(U_2)) \\ \phi_2 \circ F \circ \psi_1^{-1} &: (-\pi, 0) \cup (0, \pi) \rightarrow \phi_2(F(U_1)) \\ \phi_2 \circ F \circ \psi_2^{-1} &: (0, 2\pi) \rightarrow \phi_2(F(U_2)) \end{aligned}$$

These maps are given by

$$\begin{aligned} (\phi_1 \circ F \circ \psi_1^{-1})(t) &= \phi_1(e^{it}) = t, \quad t \in (-\pi, \pi). \\ (\phi_1 \circ F \circ \psi_2^{-1})(t) &= \begin{cases} t & t \in (0, \pi), \\ t - 2\pi & t \in (\pi, 2\pi). \end{cases} \\ (\phi_2 \circ F \circ \psi_1^{-1})(t) &= \begin{cases} t + 2\pi & t \in (-\pi, 0), \\ t & t \in (0, \pi). \end{cases} \\ (\phi_2 \circ F \circ \psi_2^{-1})(t) &= \phi_2(e^{it}) = t, \quad t \in (-\pi, \pi). \end{aligned}$$

All of the functions above are C^∞ . Thus F is C^∞ .

- (c) To show that $F : \mathbb{R}/2\pi\mathbb{Z} \rightarrow S^1$ is a diffeomorphism it is enough to show that $\bar{\phi} : S^1 \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ is inverse of it. Consider the following composition of maps

$$F \circ \bar{\phi} : S^1 \rightarrow S^1.$$

For the points $x \in U_1 \subset S^1$, since $U_1 = \{e^{it} \mid t \in (-\pi, \pi)\}$ we have

$$(F \circ \pi \circ \phi_1)(x) = (F \circ \pi \circ \phi_1)(\underbrace{e^{it}}_{t \in (-\pi, \pi)}) = (F(\pi(t))) = F([t]) = e^{it}.$$

Similarly, for the points $x \in U_2 \subset S^1$, since $U_2 = \{e^{it} \mid t \in (0, 2\pi)\}$ we have

$$(F \circ \pi \circ \phi_2)(x) = (F \circ \pi \circ \phi_2)(\underbrace{e^{it}}_{t \in (0, 2\pi)}) = F(\pi(t)) = F([t]) = e^{it}.$$


For the converse, consider the following map

$$\bar{\phi} \circ F : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}/2\pi\mathbb{Z}.$$

Let $[t] \in \mathbb{R}/2\pi\mathbb{Z}$. Then we have

$$(\bar{\phi} \circ F)([t]) = \bar{\phi}(e^{it}) = \pi(t + 2n\pi) = [t] \quad \text{for some } n \in \mathbb{Z}.$$

This shows that F is indeed a diffeomorphism between manifolds.

Summary  **2.10** In the question above, we proved (by finding an explicit diffeomorphism) that $\mathbb{R}/2\pi\mathbb{Z}$ is diffeomorphic to $S^1 \subset \mathbb{C}$.

■ **Problem 2.26 — The Grassmannian** $G(k, n)$. The Grassmannian $G(k, n)$ is the set of all k -planes through the origin in \mathbb{R}^n . Such a k -plane is a linear subspace of dimension k of \mathbb{R}^n and has a basis consisting of k linearly independent vectors a_1, \dots, a_k in \mathbb{R}^n . It is therefore completely specified by an $n \times k$ matrix $A = [a_1 \ \dots \ a_k]$ of rank k , where the rank of a matrix A , denoted by $\text{rank } A$ is defined to be the number of linearly independent columns of A . This matrix is called a matrix representative of the k -plane.

Two bases a_1, \dots, a_k and b_1, \dots, b_k determine the same k -planes if there is a change of basis matrix $g = [g_{ij}] \in GL(k, \mathbb{R})$ such that

$$\begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix} = g^T \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix},$$

or equivalently

$$b_j = \sum_i g_{ij} a_i,$$

or in matrix notation

$$B = Ag$$

where B, A are matrices whose columns are the vectors a_1, \dots, a_k and b_1, \dots, b_k respectively.

Let $F(k, n)$ denote the set of all $n \times k$ matrices of rank k , topologized as a subspace of $\mathbb{R}^{n \times k}$. Thus $F(k, n)$ is an open subset of $\mathbb{R}^{n \times k}$.

Let the group $GL(k, \mathbb{R})$ act on $F(k, n)$ on the right by $A \cdot g = Ag$ where the right hand side is the usual matrix multiplication. Consider the orbit space $F(k, n)/GL(k, \mathbb{R})$. Equivalently, we can show this space as the quotient space $F(k, n)/\sim$ where the equivalence relation \sim is given as

$$A \equiv B \quad \text{iff} \quad A = Bg \quad \text{for some } g \in GL(k, \mathbb{R}).$$

There is a bijection between the Grassmannian $G(k, n)$ and the orbit space $F(k, n)/GL(k, \mathbb{R})$. We give the Grassmannian $G(k, n)$ the quotient topology on $F(k, n)/GL$.

- Show that \sim is an open equivalence relation.
- Prove that the Grassmannian $G(k, n)$ is second countable.
- Let $S = F(k, n)$. Prove that the graph R in $S \times S$ of the equivalence relation \sim is closed. (*Hint:* Two matrices $A = [a_1 \ \dots \ a_k]$ and $B = [b_1 \ \dots \ b_k]$ are equivalent \Leftrightarrow every column of B is a linear combination of the columns of $A \Leftrightarrow \text{rank}[A \ B] \leq k \Leftrightarrow$ all $(k+1) \times (k+1)$ minors of $[A \ B]$ are zero).
- Prove that the Grassmannian $G(n, k)$ is Hausdorff.

Now we want to find a C^∞ atlas on the Grassmannian $G(k, n)$. For simplicity, we specialize to $G(2, 4)$. For any 4×2 matrix A , let A_{ij} be the 2×2 sub matrix consisting of its i th row and j th row. Define

$$V_{ij} = \{A \in F(2, 4) \mid A_{ij} \text{ is nonsingular}\}.$$

For any $A \in V_{ij}$, since $\det(A_{ij}) \neq 0$ (i.e. it is nonsingular), then we can perturb it small enough and by continuity of the determinant operator the nonsingularity of A_{ij} persists.

This shows that V_{ij} is indeed an open subset of $F(2, 4)$. Alternatively, we can show this by arguing that since the complement of V_{ij} is defined by vanishing of $\det(A_{ij})$, we conclude that V_{ij} is an open subset of $F(2, 4)$.

- (e) Prove that if $A \in V_{ij}$, then $Ag \in V_{ij}$ for any nonsingular matrix $g \in GL(2, \mathbb{R})$.

Define $U_{ij} = V_{ij}/\sim$. Since \sim is an open equivalence relation, U_{ij} is an open subset of $G(2, 4)$. For $A \in V_{12}$

$$A \sim AA_{12}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} I \\ A_{34}A_{12}^{-1} \end{bmatrix}.$$

This shows that the matrix representative of a 2-plane in U_{12} have a canonical form B where B_{12} is the identity matrix.

- (f) Show that the map $\tilde{\phi}_{12} : V_{12} \rightarrow \mathbb{R}^{2 \times 2}$,

$$\tilde{\phi}_{12}(A) = A_{34}A_{12}^{-1}$$

induces a homeomorphism $\phi_{12} : U_{12} \rightarrow \mathbb{R}^2 \times 2$.

- (g) Define similarly homeomorphism $\phi_{ij} : U_{ij} \rightarrow \mathbb{R}^2$. Compute $\phi_{12} \circ \phi_{23}^{-1}$, and show that it is C^∞ .
- (h) Show that $\{U_{ij} \mid 1 \leq i \leq j \leq 4\}$ is an open cover of $G(2, 4)$ and that $G(2, 4)$ is a smooth manifold.

Solution

- (a) According to [Lemma 2.2](#) the projection map $\pi : F(k, n) \rightarrow F(k, n)/GL$ is an open map if and only if the group action map $\alpha : F(k, n) \times G \rightarrow F(k, n)$ is continuous. Let $A \in F(k, n)$ and $g \in G$. Then $B = \alpha(A, g)$. For the element b_{kl} of B we have

$$b_{kl} = \sum_i A_{ki} g_{il}$$

which is a polynomial. Thus α is a continuous map. Now according to [Lemma 2.2](#) the equivalence relation \sim is open.

- (b) From [Proposition 2.12](#) it follows that since \sim is an open equivalence relation, then quotient space $G(k, n) = F(k, n)/\sim$ is second countable.
- (c) To show that R is closed by showing that its complement is open. Let $x \in R^c$. Then $x = [A \ B]$ for some matrices $A, B \in S$ where the matrix $[A \ B]$ has at least one $(k+1) \times (k+1)$ minor that is not zero. Then by continuity of determinant operation, for small enough $\epsilon > 0$ we can find an open ball \mathbb{B}_ϵ centered at $[A \ B] \in \mathbb{R}^{n \times 2k}$ such that for all $y \in \mathbb{B}_\epsilon$ the non-zero minor persists. This shows that R^c is open which implies the closedness of R .
- (d) Since \sim is an open equivalence relation and also the graph of the equivalence relation is closed in $S \times S$, then by [Theorem 2.2](#) it follows that $G(k, n)$ is Hausdorff.
- (e) First, observe that

$$(Ag)_{ij} = A_{ij}g.$$

Let $A \in V_{ij}$. Then from definition we know that $\det(A_{ij}) \neq 0$. Since $(Ag)_{ij} = A_{ij}g$ and $\det((Ag)_{ij}) = \det(A_{ij}) \det(g)$, then from nonsingularity of g it follows that $\det((Ag)_{ij}) \neq 0$. Hence $Ag \in V_{ij}$.

(f) Consider the following diagram.

$$\begin{array}{ccc} V_{12} & \xrightarrow{\tilde{\phi}_{12}} & \mathbb{R}^{2 \times 2} \\ \pi \downarrow & \nearrow \phi_{12} & \\ U_{12} & & \end{array}$$

First observe that $\tilde{\phi}_{12}$ assumes a constant value for all the points in its domain that are in the same equivalence class. That is because if for $A, B \in V_{12}$ we have $A \sim B$, then $\exists g \in GL(2, \mathbb{R})$ such that $A = Bg$. Thus

$$A_{34}A_{12}^{-1} = \tilde{\phi}_{12}(A) = \tilde{\phi}_{12}(Bg) = (Bg)_{34}(Bg)_{12}^{-1} = B_{34}gg^{-1}B_{12}^{-1} = B_{34}B_{12}^{-1}.$$

Also, since $\tilde{\phi}_{12}$ is continuous, then it follows that the induced map ϕ_{12} is also continuous.

Now, we need to find an inverse for this map. Consider the map

$$\psi_{12} : \mathbb{R}^{2 \times 2} \rightarrow U_{12}, \quad M \mapsto \begin{bmatrix} I \\ M \end{bmatrix}.$$

This inclusion map is indeed continuous. Now we need to show that ψ and ϕ are inverse maps. Consider

$$\psi \circ \phi : U_{12} \rightarrow U_{12}.$$

Let $A \in U_{12}$. Then we know that there exist a canonical representation of A as

$$A \sim \begin{bmatrix} I \\ A_{34}A_{12}^{-1} \end{bmatrix}.$$

Then

$$\psi(\phi(\begin{bmatrix} I \\ A_{34}A_{12}^{-1} \end{bmatrix})) = \psi(A_{34}A_{12}^{-1}) = \begin{bmatrix} I \\ A_{34}A_{12}^{-1} \end{bmatrix}$$

For the converse, consider the map

$$\phi \circ \psi : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}.$$

Let $M \in \mathbb{R}^{2 \times 2}$. Then

$$(\phi \circ \psi)(M) = \phi_{23}(\begin{bmatrix} I \\ M \end{bmatrix}) = M.$$

This shows that ψ and ϕ are inverse maps and this completes the proof.

(g) For this map we have

$$\phi_{12} \circ \phi_{23}^{-1} : \phi(U_{12} \cap U_{23}) \rightarrow \mathbb{R}^{2 \times 2}.$$

Note that the set $U_{12} \cap U_{23}$ is defined as the quotient space of $V_{12} \cap V_{23}$ where

$$V_{12} \cap V_{23} = \{A \in F(2, 4) \mid \det(A_{12}) \neq 0 \text{ and } \det(A_{23}) \neq 0\}.$$

For any matrix in $A \in U_{12}$ it will have a canonical form

$$A \sim \begin{bmatrix} I \\ A_{34}A_{23}^{-1} \end{bmatrix}$$

and similarly for $B \in U_{23}$

$$B \sim \begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1 \\ c & d \end{bmatrix}$$

where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = B_{14} B_{23}^{-1}$$

Let $M \in \phi(U_{12} \cap U_{23})$. Then

$$(\phi_{12} \circ \phi_{23}^{-1})(M) = \phi_1 \begin{bmatrix} M_{11} & M_{12} \\ 1 & 0 \\ 0 & 1 \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} M_{11} & M_{21} \\ 1 & 0 \end{bmatrix}^{-1}.$$

The result of the matrix multiplication above is a polynomial in terms of the components, thus it is C^∞ .

- (h) First observe that V_{ij} for $1 \leq i \leq j \leq 4$ are open subsets of $F(2, 4)$. For instance, if $A \in V_{12}$, then A_{12} is nonsingular. From the continuity of the determinant operator, the nonsingularity persists under small enough perturbation. Also, since the projection map π is an open map (shown in part (a)), then U_{ij} are open subsets of $G(2, 4)$. Now we need to show that all U_{ij} covers the whole space. Since there is a bijection between $G(2, 4)$ and $F(2, 4)/GL(2, \mathbb{R})$ (as we discussed in the statement of the problem), thus for any $x \in G(2, 4)$ there exists $A \in U_{ij}$ for some $1 \leq i \leq j \leq 4$. Thus the set $\{U_{ij} \mid 1 \leq i \leq j \leq 4\}$ is an open cover for $G(2, 4)$. Also from what we showed above, the atlas

$$\{(U_{ij}, \phi_{ij}) \mid 1 \leq i \leq j \leq 4\}$$

is C^∞ compatible. Thus $G(2, 4)$ is a smooth manifold.

3. Tangent Spaces

3.1 Tangent Spaces

For several reasons, the notion of tangent space is more easily developed using algebraic view point rather than the geometric one. Then we can easily construct our geometric intuition from using the developed theory. We start with the notion of derivation.

Definition 3.1 — Derivation at a point. Let M be a manifold and p a point of manifold. Let $C_p^\infty(M)$ denote the germ of C^∞ functions at p . Derivation at p is a linear map

$$D : C_p^\infty(M) \rightarrow \mathbb{R}$$

such that for $f, g \in C_p^\infty(M)$ we have

$$D(fg) = D(f)g + fD(g).$$

The definition above is a purely algebraic one.

■ **Remark 3.1** The set of all point derivations at p forms a vector space.

Definition 3.2 — Tangent space. Let M be a manifold and $p \in M$. The tangent space at p denoted by $T_p(M)$ is the vector space of all point derivations at p .

In the following proposition we will see a very familiar point derivation.

Proposition 3.1 — Partial derivative $\partial/\partial x^i|_p$ is a tangent vector. Let M be a manifold and $p \in M$. Let (U, ϕ) be a coordinate chart containing p and $f : M \rightarrow \mathbb{R}$ a function defined at a neighborhood of p . By definition we have

$$\frac{\partial}{\partial x^i} \Big|_p f = \frac{\partial}{\partial r^i} \Big|_{\phi(p)} (f \circ \phi^{-1})$$

where $x^i = r^i \circ f$ with r^i as the standard coordinate function of \mathbb{R}^n . The map $\frac{\partial}{\partial x^i} \Big|_p$ is a point derivation at p , thus a tangent vector.

Proof. First observe that $\partial/\partial x_i|_p$ is indeed a linear map from $C_p^\infty(M)$ to \mathbb{R} . Let $f, g \in C_p^\infty(U)$

and (U, x^1, \dots, x^n) a coordinate chart. Then we have

$$\begin{aligned} \frac{\partial}{\partial x^i} \Big|_p (fg) &= \frac{\partial}{\partial r^i} \Big|_{\phi(p)} ((fg) \circ \phi^{-1}) = \frac{\partial}{\partial r^i} \Big|_{\phi(p)} (f \circ \phi^{-1} \cdot g \circ \phi^{-1}) \\ &= \frac{\partial}{\partial r^i} \Big|_{\phi(p)} (f \circ \phi^{-1}) \cdot g(p) + f(p) \cdot \frac{\partial}{\partial r^i} \Big|_{\phi(p)} (g \circ \phi^{-1}) \\ &= \left(\frac{\partial}{\partial x^i} \Big|_p f \right) g + f \left(\frac{\partial}{\partial x^i} \Big|_p g \right). \end{aligned}$$

□

Definition 3.3 — Differential of a smooth map. Let $F : N \rightarrow M$ be a smooth map between manifolds. This map induces a *linear* map called the *differential of F*

$$F_* : T_p N \rightarrow T_{F(p)} M$$

where for $X_p \in T_p N$ we have

$$(F_*(X_p))f = X_p(f \circ F)$$

where f is a germ at p .

Proposition 3.2 Let $F : N \rightarrow M$. Its differential F_* is a derivation at $F(p)$ for $p \in N$.

Proof. See [Problem 3.1](#)

□

Differential of a map between Euclidean spaces

To see what kind of an object is the differential of a map, we calculate it explicitly on Euclidean spaces. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a map and let x^1, \dots, x^n be the standard coordinate functions of \mathbb{R}^n while y^1, \dots, y^m be the standard coordinate functions of \mathbb{R}^m . Let $p \in \mathbb{R}^n$. since F_* is a linear map between two vector spaces, then we can write it as a matrix once we fix the basis for the spaces. The basis for $T_p(\mathbb{R}^n)$ is the set $\{\partial/\partial x^i|_p\}_i$, while for $T_p(\mathbb{R}^m)$ is $\{\partial/\partial y^j|_p\}_j$. To get the matrix representation of F_* we need to find its effect on the basis vectors

$$F_*\left(\frac{\partial}{\partial x^j} \Big|_p\right) = \sum_k a_j^k \frac{\partial}{\partial y^k} \Big|_{F(p)}.$$

By applying it on y_i we will get

$$a_j^i = F_*\left(\frac{\partial}{\partial x^j} \Big|_p\right)y^i = \frac{\partial}{\partial x^j} \Big|_p (y^i \circ F) = \frac{\partial}{\partial x^j} \Big|_p F^i.$$

This is precisely the Jacobian matrix of F evaluated at p .

The Chain Rule

A nice thing behind this algebraic point of view is that we can now write the notion of chain rule in a much simpler form. The following propositions captures this idea.

Proposition 3.3 — The chain rule. Let $F : N \rightarrow M$ and $G : M \rightarrow P$ be maps between manifolds, and $p \in N$. Then the differential of F at p and G at $F(p)$ are linear maps

$$T_p N \xrightarrow{F_*, p} T_p M \xrightarrow{G_*, F(p)} T_p P.$$

Then we have

$$(G \circ F)_{*,p} = G_{*,F(p)} F_{*,p}.$$

Proof. Let $X_p \in T_p(N)$. From definition we have for some $f : G \rightarrow \mathbb{R}$ we have

$$((G \circ F)_{*,p} X_p) f = X_p(f \circ G \circ F) = X_p((f \circ G) \circ F) = (F_{*,p} X_p)(f \circ G).$$

Let $Y_q = F_{*,p} X_p$ where $q = F(p)$. Then

$$Y_q(f \circ G) = (G_{*,q} Y_q) f.$$

This implies that

$$(G \circ F)_{*,p} = G_{*,q} F_{*,p}$$

and this completes the proof. \square

■ **Example 3.1** Let $F : \mathbb{R} \rightarrow \mathbb{R}^3$ and $G : \mathbb{R}^3 \rightarrow \mathbb{R}$. Then the differential of the composite function $G \circ F$ is the matrix multiplication of the differential of each map. I.e.

$$(G \circ F)_* = G_* F_* = (G_x \ G_y \ G_z) \cdot \begin{pmatrix} F'_1 \\ F'_2 \\ F'_3 \end{pmatrix} = G_x F'_1 + G_y F'_2 + G_z F'_3 = \frac{\partial G}{\partial x} \frac{dF_1}{dt} + \frac{\partial G}{\partial y} \frac{dF_2}{dt} + \frac{\partial G}{\partial z} \frac{dF_3}{dt},$$

which is the same chain rule in calculus. ■

Writing the chain rule in this form not only is visually appealing, but also we can deduce two important Corollaries from it as follows.

Corollary 3.1 — Diffeomorphism of Manifolds vs. Isomorphism of Tangent Spaces. Let $F : N \rightarrow M$ be a diffeomorphism of manifolds and $p \in N$. Its differential F_* is an isomorphism of tangent spaces $T_p N$ and $T_{F(p)} M$.

Proof. F being a diffeomorphism of manifolds, it implies that we have $G : M \rightarrow N$ such that $G \circ F = \mathbb{1}_N$ and $F \circ G = \mathbb{1}_M$. Now using [Proposition 3.3](#) we can write

$$\begin{aligned} (G \circ F)_* &= G_* \circ F_* = (\mathbb{1}_N)_* = \mathbb{1}_{T_p N} \\ (F \circ G)_* &= F_* \circ G_* = (\mathbb{1}_M)_* = \mathbb{1}_{T_{F(p)} M}. \end{aligned}$$

This shows that F_* and G_* are the right and left inverses of each other, hence an isomorphism of vector spaces. \square

Corollary 3.2 — Invariance of Dimension. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open subsets that are diffeomorphic. Then $m = n$.

Proof. Let $F : U \rightarrow V$ be a diffeomorphism and $p \in U$. Then F_* is an isomorphism of tangent spaces $T_p U$ and $T_{F(p)} V$, thus of the same dimension. Since in a Euclidean space the tangent space has the same dimension as the space itself (by definition and the way that we construct a basis for tangent space using the standard coordinate vectors) then this implies that $n = m$. \square

Corollary 3.3 — Tangent Space Has the Same Dimension as Manifold. Let M be a manifold of dimension n and $p \in M$. Then $T_p(M)$ is isomorphic to \mathbb{R}^n and has dimension n .

Proof. Let (U, ϕ) be a coordinate chart containing the point p . Since M is assumed to be a smooth manifold, then the coordinate map ϕ automatically upgrades to a diffeomorphism between U and \mathbb{R}^n . Thus the tangent spaces $T_p(U)$ and $T_{\phi(p)}(\mathbb{R}^n)$ are isomorphic. This implies that $T_p(U)$ is also of dimension n . \square

3.2 Basis for the Tangent Space at a Point

Proposition 3.4 — Differential of a Coordinate Map. Let M be a manifold, $p \in M$, and $(U, x^1 \dots, x^n)$ a coordinate chart containing p . Then

$$\phi_*\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial}{\partial r^i}\Big|_{\phi(p)}$$

Proof. This follows immediately from the definition. Namely for a function f defined on U we have

$$\left(\phi_*\left(\frac{\partial}{\partial x^i}\Big|_p\right)\right)f = \frac{\partial}{\partial x^i}\Big|_p(f \circ \phi) = \frac{\partial}{\partial r^i}\Big|_{\phi(p)}(f \circ \phi \circ \phi^{-1}) = \frac{\partial}{\partial r^i}\Big|_{\phi(p)}f.$$

This implies

$$\phi_*\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial}{\partial r^i}\Big|_{\phi(p)}.$$

□

■ **Remark 3.2** Remember that in [Definition 2.13](#) we defined

$$\frac{\partial}{\partial x^i}\Big|_p f = \frac{\partial}{\partial r^i}\Big|_{\phi(p)}(f \circ \phi^{-1}). \quad (\clubsuit)$$

where (U, ϕ) is a coordinate chart containing p and $x^i = r^i \circ \phi$. The proposition above justifies this definition. This proposition states that $T_p(M) \simeq T_{\phi(p)}(\mathbb{R}^n)$ with ϕ_* as an isomorphism. This basically means that $\partial/\partial x^i$ maps to $\partial/\partial r^i$ with map ϕ_* , or equivalently, $\partial/\partial r^i$ maps to $\partial/\partial x^i$ with the map ϕ_*^{-1} . I.e.

$$\frac{\partial}{\partial x^i}\Big|_p \mapsto \frac{\partial}{\partial r^i}\Big|_{\phi(p)} \quad \text{via } \phi_*$$

and

$$\frac{\partial}{\partial r^i}\Big|_p \mapsto \frac{\partial}{\partial x^i}\Big|_{\phi(p)} \quad \text{via } \phi_*^{-1}$$

Thus starting with the LHS of (\clubsuit) we can write

$$\frac{\partial}{\partial x^i}\Big|_p f = (\phi_*^{-1}\left(\frac{\partial}{\partial r^i}\Big|_{\phi(p)}\right))f = \frac{\partial}{\partial r^i}\Big|_{\phi(p)}(f \circ \phi^{-1}).$$

Proposition 3.5 — Basis for Tangent Space. Let M be a manifold of dimension n and $p \in M$. Then the vectors $\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p$ is a basis for the tangent space $T_p(M)$.

Proof. Let (U, ϕ) be a coordinate chart containing p . By [Corollary 3.3](#) we see that $T_p M$ is isomorphic to \mathbb{R}^n with ϕ_* as a possible isomorphism. Since an isomorphism of vector spaces carries basis to a basis, and by [Proposition 3.4](#) ϕ_* maps the standard basis $\frac{\partial}{\partial r^1}\Big|_{\phi(p)}, \dots, \frac{\partial}{\partial r^n}\Big|_{\phi(p)}$ to $\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p$ respectively, then this forms a basis for $T_p(M)$. □

■ **Remark 3.3** For a point $p \in M$, once we fix a coordinate chart containing p we have a basis for $T_p(M)$

Proposition 3.6 — Transition Matrix for Coordinate Vectors. Let M be a manifold and $p \in M$. Let $(U, \phi = (x^1, \dots, x^n))$ and $(V, \psi = (y^1, \dots, y^n))$ be two coordinate charts containing p . By Proposition 3.5 the following two collection of tangent vectors form two possible bases for $T_p M$

$$\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}, \quad \left\{ \frac{\partial}{\partial y^1} \Big|_p, \dots, \frac{\partial}{\partial y^n} \Big|_p \right\}.$$

The transition rule between these two basis is given by

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i},$$

on $U \cap V$.

Proof. We can think of this as ψ is a map between manifolds, i.e. $\psi : U \cap V \subset M \rightarrow \mathbb{R}^n$. Considering the chart $(W, \mathbb{1}_{\mathbb{R}^n})$ for the target manifold of ψ , what we are looking after is to see how does ψ_* maps the tangent vector $\partial/\partial x^j$. For ψ_* it is a map from $T_p N$ to $T_{\psi(p)} \mathbb{R}^n$ where the basis for $T_{\psi(p)} \mathbb{R}^n$ is the collection $\{\partial/\partial r^i\}_i$. Thus we can write

$$\psi_*(\partial/\partial x^j|_p) = \sum_k a_j^k \partial/\partial r^k|_{\psi(p)}.$$

Applying r^i to both sides will result in

$$a_j^i = \phi_*(\partial/\partial x^j|_p) r^i = \partial/\partial x^j|_p (r^i \circ \phi) = \partial/\partial x^j|_p y^i.$$

So far we have shown that

$$\psi_*\left(\frac{\partial}{\partial x^j}\right) = \sum_i \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial r^i}$$

Now by applying ψ_*^{-1} to both sides and using the observation in Remark 3.2 we can write

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}$$

where we have used the fact that ψ_*^{-1} is linear. □

■ **Remark 3.4** The relation above can be written symbolically as matrix multiplication

$$\begin{bmatrix} \partial/\partial x^1 \\ \partial/\partial x^2 \\ \vdots \\ \partial/\partial x^n \end{bmatrix} = \begin{pmatrix} \partial y^1/\partial x^1 & \partial y^1/\partial x^2 & \dots & \partial y^1/\partial x^n \\ \partial y^2/\partial x^1 & \partial y^2/\partial x^2 & \dots & \partial y^2/\partial x^n \\ \vdots & \vdots & \ddots & \vdots \\ \partial y^n/\partial x^1 & \partial y^n/\partial x^2 & \dots & \partial y^n/\partial x^n \end{pmatrix}^T \begin{bmatrix} \partial/\partial y^1 \\ \partial/\partial y^2 \\ \vdots \\ \partial/\partial y^n \end{bmatrix}$$

A Local Expression for the Differential

We start with the following proposition.

Proposition 3.7 — Local Expression for the Differential. Let $F : N \rightarrow M$ be a map between manifolds, and $p \in N$. Let (U, x^1, \dots, x^n) be a coordinate chart in N containing p and (V, y^1, \dots, y^n) containing $F(p)$ be a coordinate chart in M . Relative to the bases $\{\partial/\partial x^j|_p\}$ for $T_p N$ and $\{\partial/\partial y^i|_{F(p)}\}$ for $T_p M$, the differential $F_* : T_p N \rightarrow T_p M$ is represented by the matrix

$$[\partial F^i / \partial x^j(p)],$$

where $F^i = y^i \circ F$.

■ **Remark 3.5** We can state the inverse function theorem for manifolds [Theorem 2.1](#) in a coordinate free description: A C^∞ function $F : N \rightarrow M$ is locally invertible at point $p \in N$ if and only if $F_* : T_p N \rightarrow T_p M$ is an isomorphism of vector spaces.

3.3 Curves in a Manifold

The notion of a curve in a manifold is a very important one as it will allow us to define the tangent space in an alternative way. We start by the definition of a curve in a manifold.

Definition 3.4 — Curves in a Manifold. Let M be a manifold and $p \in M$. a *curve* in manifold is a map

$$c : (a, b) \rightarrow M.$$

Usually $0 \in (a, b)$ and we say c is a *curve starting at p* if $c(0) = p$. The *velocity vector* $c'(t_0)$ of the curve at time $t_0 \in (a, b)$ is defined to be

$$c'(t_0) = c_* \left(\frac{d}{dt} \Big|_{t_0} \right) \in T_{c(t_0)} M.$$

■ **Remark 3.6** Note that the velocity vector of the curve c at time t_0 denoted by $c'(t_0)$ is a tangent vector at $T_{c(t_0)} M$. This in particular can be a source of confusion when the target space of the curve is \mathbb{R} . That is because in the calculus the notation $c'(t)$ is the derivative of a real-valued function and is therefore a scalar. If it is necessary to distinguish between these two meanings of $c'(t)$ when c maps into \mathbb{R} , we will write $\dot{c}(t)$ for the calculus derivative. For a clarification on this point visit [Problem 3.2](#).

■ **Remark 3.7** Alternative notations for $c'(t_0)$ are

$$\frac{d}{dt} \Big|_{t_0} c, \quad \frac{dc}{dt}(t_0).$$

However, I prefer the notation introduced in the definition box above as it has more emphasis on the fact that $c'(t)$ is a vector in $T_p M$.

■ **Example 3.2** Velocity vector of a curve is consistent with the results in calculus In this example we will demonstrate that the notion of the velocity vector of a curve is consistent with the similar notion in calculus. Define $c : \mathbb{R} \rightarrow \mathbb{R}^2$.

$$c(t) = (t^2, t^3).$$

The velocity vector $c'(t)$ is in tangent space $T_{c(t)} \mathbb{R}^2$. Let $(\mathbb{R}^2, \mathbb{1}_{\mathbb{R}^2})$ be the atlas for manifold \mathbb{R}^2 . Then we can write $c'(t)$ in terms of the basis for tangent space.

$$c'(t) = a \frac{\partial}{\partial x} \Big|_{c(t)} + b \frac{\partial}{\partial y} \Big|_{c(t)}.$$

To determine the coefficients we apply the standard coordinate function x, y to both sides and we will get

$$c'(t)x = a, \quad c'(t)y = b.$$

From definition we have

$$c'(t)x = c_* \left(\frac{d}{dt} \Big|_t \right) x = \frac{d}{dt} \Big|_t (x \circ c) = \frac{d}{dt} \Big|_t (t^2) = 2t.$$

Similarly we can get $c'(t)y = 3t^2$. Thus

$$c'(t) = (2t) \frac{\partial}{\partial x} \Big|_{c(t)} + (3t^2) \frac{\partial}{\partial y} \Big|_{c(t)}.$$

■

The steps that we took in the example above motivates us for the following proposition.

Proposition 3.8 — Velocity Vector in a Local Coordinate. Let $c : (a, b) \rightarrow M$ be a smooth curve, and let (U, x^1, \dots, x^n) be a coordinate chart about $c(t)$. Write $c^i = x^i \circ c$ for the i -th component of c in the chart. Then $c'(t) \in T_{c(t)}M$ is given by

$$c'(t) = \sum_i \dot{c}_i \frac{\partial}{\partial x^i} \Big|_{c(t)}.$$

Thus relative to the bases $\{\partial/\partial x^i|_p\}$ for $T_{c(t)}M$ the velocity vector $c'(t)$ can be written as

$$\begin{bmatrix} \dot{c}_1(t) \\ \dot{c}_2(t) \\ \vdots \\ \dot{c}_n(t) \end{bmatrix}$$

Proof. First, observe that by definition $c'(t) \in T_{c(t)}M$ thus can be written in the local coordinate of $T_{c(t)}M$ as

$$c'(t) = \sum_i a^i \frac{\partial}{\partial x^i} \Big|_{c(t)}.$$

Now we apply both sides to x_j , and simply use the definition of the differential of a map and write

$$x^j = c_* \left(\frac{d}{dt} \Big|_t \right) x^j = \frac{d}{dt} \Big|_t (x^j \circ c) = \dot{c}^j(t).$$

This completes the proof. □

So far, we observed that every smooth curve c at p in a manifold M gives rise to a tangent vector in T_pM . The following proposition states the converse of this, i.e. for every tangent vector in tangent space T_pM there exists a curve passing through p .

Proposition 3.9 — Existence of a Curve with a Given Initial Vector. For any point p in a manifold M and any tangent vector $X_p \in T_pM$, there are $\epsilon > 0$ and a smooth curve $c : (-\epsilon, \epsilon) \rightarrow M$ such that $c(0) = p$ and $c'(0) = X_p$

Proof. Let $p \in M$ and $X_p \in T_pM$. Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart centered at p . The differential of ϕ given by ϕ_* maps X_p to $Y_0 \in T_0\mathbb{R}^n$. For instance, assume

$$X_p = \sum_i a^i \frac{\partial}{\partial x^i} \Big|_p.$$

Applying ϕ_* to both sides and using [Proposition 3.4](#) we will get

$$Y_0 = \sum_i a^i \frac{\partial}{\partial r^i} \Big|_0$$

where r^i is the standard coordinate function of \mathbb{R}^n . Define the smooth curve $\gamma : (-\epsilon, \epsilon) \rightarrow \mathbb{R}^n$ given by

$$\gamma(t) = (a^1 t, \dots, a^n t).$$

For small enough ϵ this curve is in $\phi(U)$. We claim that the curve $c : (-\epsilon, \epsilon) \rightarrow M$ given by

$$c = \phi^{-1} \circ \gamma$$

has the desired property. First, observe by definition we will have $c(0) = p$

$$c(0) = \phi^{-1}(\gamma(0)) = p.$$

For $c'(0)$ we use the definition

$$c'(0) = c_*(d/dt|_0) = (\phi^{-1} \circ \gamma)_*(d/dt|_0) = (\phi_*^{-1} \gamma_*)(d/dt|_0)$$

It now remains to calculate the $(\phi_*^{-1} \gamma_*)$. To calculate this, first observe that γ_* maps $d/dt|_0$ to Y_0 . And also observe that ϕ_*^{-1} maps X_p to Y_0 . I.e.

$$Y_0 = \phi_* X_p, \quad Y_0 = \gamma_* \frac{d}{dt} \Big|_0.$$

Then we will get

$$\phi_* X_p = \gamma_* \frac{d}{dt} \Big|_0 \implies X_p = \phi_*^{-1} \gamma_* \frac{d}{dt} \Big|_0.$$

Thus we have

$$c'(0) = \phi_*^{-1} \gamma_* \frac{d}{dt} \Big|_0 = X_p$$

and this completes the proof. \square

Now using the notion of curves, we can calculate the differential of a map between manifolds via curves. The following proposition discusses this statement.

Proposition 3.10 — Computing Differentials Using Curves. Let $F : N \rightarrow M$ be a smooth map between manifolds, and $p \in N$ and $X_p \in T_p N$. Let c be a curve on N such that $c(0) = p$, and $c'(0) = X_p$. Then

$$F_{*,p} X_p = (F \circ c)'(0).$$

Proof. We simply follow the definition

$$(F \circ c)'(0) = (F \circ c)_* \left(\frac{d}{dt} \Big|_0 \right) = F_* c_* \left(\frac{d}{dt} \Big|_0 \right) = F_* \left(c_* \left(\frac{d}{dt} \Big|_0 \right) \right) = F_* X_p$$

and this completes the proof. \square

3.4 Immersion and Submersion

We start with the definitions of Immersion and Submersion.

Definition 3.5 — Immersion and Submersion. Let $F : N \rightarrow M$ be a smooth map, and let $F_{*,p}$ denote its differential at a point $p \in N$. Then

- F is an Immersion if and only if $F_{*,p}$ is injective.
- F is a Submersion if and only if $F_{*,p}$ is surjective.

Now we can define the notion of critical point and regular points of a mapping.

Definition 3.6 — Critical and Regular points of a smooth map between manifolds. Let $F : N \rightarrow M$ be a smooth map between manifolds. A point $p \in N$ is

- a *critical point* if $F_{*,p}$ fails to be surjective, and
- a *regular point* if $F_{*,p}$ is surjective.

A point $q \in M$ is critical value of the map F if it is the image of a critical point.

The following proposition gives a more intuitive characterization of the critical points of function defined from a manifold to the real number line.

Proposition 3.11 — Critical points of a map to real number line. Let $f : N \rightarrow \mathbb{R}$ be a smooth map from manifold N to the real number line. A point $p \in N$ is a critical point of the map f in and only if all of its partial derivatives relative to some chart $(U, \phi = (x^1, \dots, x^n))$ are zero at p , i.e.

$$\left. \frac{\partial}{\partial x^i} \right|_p f = 0 \quad \text{for all } i = 1, \dots, n.$$

Proof. The differential of f in the local coordinates can be written as the matrix

$$f_{*,p} = \left[\left. \frac{\partial}{\partial x^1} \right|_p f, \dots, \left. \frac{\partial}{\partial x^n} \right|_p f \right].$$

The point p is a critical point if and only if $f_{*,p}$ is not surjective. Since the image of $f_{*,p}$ is \mathbb{R} , then its column space is either one dimensional or zero dimensional. Thus $f_{*,p}$ is not surjective if and only if all of its entries are zero, i.e.

$$\left. \frac{\partial}{\partial x^i} \right|_p f = 0 \quad \text{for all } i = 1, \dots, n.$$

□

3.5 Vector Fields on Manifolds

Intuitively speaking, vector fields assign to every point p in manifold M a tangent vector from $T_p M$. Or, formally speaking, A vector field on a manifold M is a sections of the tangent bundles TM of manifold M .

As we will see later, every smooth vector field can be seen locally as the velocity vector field of a fluid flow. The path traced out by a point under this flow is called an *integral curve* of the vector field. In other words, integral curves are curves whose velocity vector field is the restriction of the given vector field to the curve. Finding the integral curve is equivalent to solving a system of first order differential equations.

The set of all vector fields on a manifold, denoted by $\mathfrak{X}(M)$ has a vector space structure. Introducing the notion of multiplication by Lie bracket $[\cdot, \cdot]$ makes it into a Lie algebra. More details are covered throughout this section.

3.5.1 Smoothness of vector fields

The smoothness of the vector fields on a manifold, formally seen as sections of tangent bundle over a manifold, can be traced back to the smoothness of the section. However, to keep stuff simple, we will give some other characterization of smooth vector fields.

Proposition 3.12 — Smoothness of a Vector Field on a Chart. Let M be a manifold with $p \in M$. And let (U, x^1, \dots, x^n) be a local chart on the manifold. A vector field given as

$$X = \sum_i a^i \frac{\partial}{\partial x^i} \Big|_p$$

is smooth on U if and only if the coefficient functions a^i are smooth on U .

Proposition 3.13 — Smoothness of a Vector Field in Terms of Functions. A vector field X defined on manifold M is smooth, if and only if for any smooth function f defined on M , the function Xf is also smooth.

Observation 3.5.1 — Vector fields as derivations of algebra $C^\infty(M)$. By the characterization of smooth vector fields over a manifold given by the proposition above, we can see a vector field X on a manifold M as a linear map on the algebra of C^∞ functions on manifold.

$$X : C^\infty(M) \rightarrow C^\infty(M).$$

It is also straight forward to show that X is a derivation of the algebra $C^\infty(M)$, i.e. for $f, g \in C^\infty(M)$ we have

$$X(fg) = X(f)g + fX(g).$$

3.6 Integral Curves

Intuitively speaking, an integral curve of a smooth vector field is the curve traced out by a point under the given vector field. The following is a formal definition of an integral curve of a vector field.

Definition 3.7 — Integral Curve of a Vector Field. Let X be a smooth vector field defined on a manifold M . An integral curve of X is a curve $c : (a, b) \rightarrow M$ such that

$$c'(t) = X|_{c(t)},$$

where $X|_{c(t)}$ means restricting vector field X to the curve $c(t)$. An integral curve is called *maximal* if its domain can not be extended to a larger interval.

■ **Remark 3.8** We call point $p \in M$ the initial point of the curve c and say c starts in p if $c(0) = p$. Also we write $c_t(p)$ instead of $c(t)$. With this notation the flow structure of c will be more evident, namely $c_s(c_t(p)) = c_{s+t}(p)$.

As we can observe in the following example, computing the integral curve passing through a point $p \in M$ is the same as solving an initial value problem.

■ **Example 3.3** Consider the Euclidean space $M = \mathbb{R}^2$ as a manifold. Define the vector field X to be

$$X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$

We want to find an integral curve passing $p = (1, 0)$. For $c : (a, b) \rightarrow \mathbb{R}^2$ given as $c(t) = (x(t), y(t))$ to be an integral curve, we need its differential be the same as $X|_{c(t)}$ for $t \in (a, b)$. By [Proposi-](#)

tion 3.8 we can write

$$c'(t) = \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y}.$$

And for $c'(t)$ to be an integral curve we need to have

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}, \quad x(0) = 1, y(0) = 0.$$

Solving the initial value problem above we will get

$$x(t) = \cos(t), \quad y(t) = \sin(t).$$

■

Given a vector field X on manifold M , finding the integral curve of X starting at a point $p \in M$ reduces to solving an initial value problem as described below. By Definition 3.7 a curve $c : (a, b) \rightarrow M$ is an integral curve starting at p if we have

$$X|_{c(t)} = c'(t) \quad \text{for } t \in (a, b), \quad c(0) = p.$$

Choosing a local coordinate chart (U, x^1, \dots, x^n) that contains p , we can write the restricted vector field $X|_{c(t)}$ as

$$X|_{c(t)} = \sum_i a^i(c(t)) \frac{\partial}{\partial x^i} \Big|_p.$$

On the other hand, by Proposition 3.8 we can write $c'(t)$ in the local coordinates as

$$c'(t) = \sum_i \dot{c}^i(t) \frac{\partial}{\partial x^i} \Big|_p,$$

where $\dot{c}^i = x^i \circ c$. Thus we will have

$$\dot{c}^i(t) = a^i(c(t)) \quad \text{for } i = 1, \dots, n, \quad \text{and} \quad c(0) = p.$$

Since the coefficient functions a^i are all smooth, then by the existence and uniqueness theorem we conclude the existence of the desired integral curve.

Theorem 3.1 — Existence-Uniqueness Theorem of ODE. Let $V \subset \mathbb{R}^n$ be an open set, and $f : V \rightarrow \mathbb{R}^n$ be a *smooth* vector field defined on V . For any $p \in V$, the integral curve of f starting at p , or equivalently, the solution of the ODE system given by

$$\frac{dy}{dt}(t) = f(y(t)), \quad y(0) = p,$$

has a unique *smooth* solution $y : (a(p), b(p)) \rightarrow \mathbb{R}^n$ where $(a(p), b(p))$ is the maximal open interval containing 0 on which y is defined.

A smooth vector field, gives us more than what we have stated above. The following theorem states such an statement.

Theorem 3.2 — Smooth dependence of starting point of an integral curve. Let $V \subset \mathbb{R}^n$ be an open set, and $f : V \rightarrow \mathbb{R}^n$ be a smooth vector field defined on V . Then for every $p \in \mathbb{R}^n$, there exist an open neighborhood W containing p and a number $\epsilon > 0$, and a smooth map *smooth map* y

$$y : (-\epsilon, \epsilon) \times W \rightarrow \mathbb{R}^n$$

such that for all $q \in W$ the map $y(t, q)$ as a function of t is an integral curve of f starting at q .

Or equivalently, it is the unique solution of the ODE system given as

$$\frac{\partial y}{\partial t}(q, t) = f(y(q, t)), \quad y(0, q) = q.$$

We can use the theorems above to prove a similar statement for the case of manifold as given in the following proposition.

Proposition 3.14 — Existence of Local Smooth Flow. Let M be a manifold and X be a *smooth* vector field defined on a U . Then for any point $p \in U$, there are a neighborhood W containing p , an $\epsilon > 0$, and a *smooth* map

$$F : (-\epsilon, \epsilon) \times W \rightarrow U$$

such that for all $q \in U$ the map $F(t, q)$ as a function of t is an integral curve of X starting at q . The map F is called a *local flow* generated by the vector field X .

■ **Remark 3.9** We write $F_t(q)$ instead of $F(t, q)$. Let $t, s \in (-\epsilon, \epsilon)$ such that $F_t(F_s(q))$ and $F_{t+s}(q)$ are both defined. These two maps as a function of t are integral curves of X starting at $F_s(q)$ that corresponds to $t = 0$. By the existence and uniqueness theorem it follows that

$$F_t(F_s(q)) = F_{t+s}(q).$$

■ **Remark 3.10** The map $F(t, q)$ is called a *local flow*, and for each $q \in W$ the map $F(t, q)$ is an integral curve of X starting at q and is called a *flow line*. Thus local flow is a map from $(-\epsilon, \epsilon) \times W$ to U while a flow line is a particular integral curve.

■ **Remark 3.11** If the local flow F is defined on $\mathbb{R} \times M$ it is called a *global flow*. Thus for all $t \in \mathbb{R}$ and $q \in M$

$$F_t(q) \circ F_{-t}(q) = F_{-t}(q) \circ F_t(q) = F_0(q) \mathbb{1}_M.$$

This shows that $F_t : M \rightarrow M$ is a diffeomorphism (is smooth while its inverse is also smooth). Thus we can summarize that a global flow gives rise to a *one-parameter group of diffeomorphisms* of M .

We can summarize the discussion above in the following definition.

Definition 3.8 — Local Flow on a Manifold. Let M be a manifold and $p \in M$, and U be a coordinate open set containing p . A *local flow* defined on U is a smooth map

$$F : (-\epsilon, \epsilon) \times W \rightarrow U$$

where $\epsilon > 0$ and W is an open neighborhood of p , writing $F(t, q) = F_t(q)$, such that

- $F_0(q) = q$,
- $F_t(F_s(q)) = F_{t+s}(q)$ whenever both sides are defined.

3.7 The Lie Bracket

As we say before, a vector field X on a manifold M is \mathbb{R} -linear map from the algebra $C^\infty(M)$ to $C^\infty(M)$ that satisfies the Leibniz rule, i.e. for $f, g \in C^\infty(M)$ we have

$$X(fg) = X(f)g + fX(g).$$

And we also saw that the set of all vector fields on M denoted by $\mathfrak{X}(M)$ has a vector space structure. However, for $Y \in \mathfrak{X}(M)$, the vector fields $X \circ Y = XY$ and YX are

- \mathbb{R} -linear, and
- smooth (because by [Proposition 3.13](#) $X(Y(f))$ is smooth given $Y(f)$ is smooth, and similarly for $Y(X(f))$).

However, XY fails to be a derivation of the algebra $C^\infty(M)$. To see this, let $f, g \in C^\infty(M)$. Then

$$XY(fg) = X(Y(fg)) = X(Y(f)g + fY(g)) = XY(f)g + Y(f)X(g) + X(f)Y(g) + fXY(g).$$

Because of the two middle terms, XY fails to be a derivation. However, the symmetries of the expression suggests that $XY - YX$ will not have the two middle terms and thus will be a derivation. By explicit calculation as above we will have

$$(XY - YX)(fg) = (XY - YX)(f)g + f(XY - YX)(g).$$

Writing $[X, Y] = XY - YX$ motivates us for the definition of a Lie bracket.

Definition 3.9 — Lie bracket of two vector fields. Let X, Y be two vector fields defined on manifold M . Their Lie Bracket $[X, Y]$ is a vector field on M that at p has the value

$$[X, Y]_p f = (X_p Y - Y_p X)f,$$

for any germ of a smooth function f at p .

■ **Remark 3.12** The Lie bracket $[X, Y]_p$ is a derivation of $C_p^\infty(M)$, hence is a tangent vector.

Proposition 3.15 — Properties of Lie Bracket. Let X, Y, Z be vector fields over a manifold M . Then

- (i) **Bilinearity:** $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ and $[Z, aX + bY] = a[Z, X] + b[Z, Y]$.
- (ii) **Anticommutativity:** $[X, Y] = -[Y, X]$,
- (iii) **Jacobi identity:** $\sum_{\text{cyclic}} [X, [Y, Z]] = 0$.

Proof. The property (i) follows immediately from the definition of Lie bracket. For (ii), first we expand the sum

$$\sum_{\text{cyclic}} [X, [Y, Z]] = [X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]].$$

Now we can expand each term from the definition of Lie bracket.

$$\begin{aligned} [X, [Y, Z]] &= X[Y, Z] - [Y, Z]X = XYZ - XZY - YZX + ZYX, \\ [Z, [X, Y]] &= Z[X, Y] - [X, Y]Z = ZXY - ZYX - XYZ + YXZ, \\ [Y, [Z, X]] &= Y[Z, X] - [Z, X]Y = YZX - YXZ - ZXY + XZY. \end{aligned}$$


Thus in the cyclic sum all of terms with opposite sign cancel each other, hence the cyclic sum is zero. \square

The following definition abstracts the notion of Lie bracket to be a product operation that satisfies some axioms.

Definition 3.10 — Lie Algebra. A Lie algebra over a field K is a vector space V over K along with a product $[\cdot, \cdot] : V \times V \rightarrow V$, called the *bracket*, such that for $X, Y, Z \in V$ and $a, b \in K$ we have

- (i) **Bilinearity:** $[aX + bY, Z] = a[X, Y] + b[Y, Z]$ and $[Z, aX + bY] = a[Z, X] + b[Z, Y]$.
- (ii) **Anticommutativity:** $[X, Y] = -[Y, X]$,
- (iii) **Jacobi identity:** $\sum_{\text{cyclic}} [X, [Y, Z]] = 0$.

■ **Example 3.4** For any vector space V define $[X, Y] = 0$ for all $X, Y \in V$. With this bracket, V becomes a Lie algebra called *abelian Lie algebra*. ■

Be Careful Here!  **3.7.1 — Lie algebra might not even be an algebra.** From the definition of algebra, as an algebraic structure, the product operation has to be associative. In terms of bracket, for it to be associative we need to have

$$[X, [Y, Z]] = [[X, Y], Z] \quad \text{for all } X, Y, Z \in V.$$

An abelian algebra is automatically associative, thus it is algebra. However, a bracket in a Lie algebra, might not always be associative. Thus a Lie algebra might not even be an algebra. In Lie algebra, the product operation satisfies the Jacobi identity instead of associativity property. However, note the following important remark.

■ **Remark 3.13** If A is any algebra over a field K , then the product

$$[x, y] = xy - yx,$$

for $x, y \in A$ turns A into a Lie algebra over K .

■ **Example 3.5 — Lie bracket as the bracket.** Let M be a manifold and $\mathfrak{X}(M)$ be the set of all smooth vector fields defined on M . Then $\mathfrak{X}(M)$ is a Lie algebra with the Lie bracket as the bracket. ■

Proposition 3.16 Let V be a Lie algebra over a field K . Define

$$\text{ad}_X : V \rightarrow V, \quad \text{ad}_X(Y) = [X, Y].$$

Prove that ad_X is a derivation V

Proof. From the Jacobi identity, for all $X, Y, Z \in V$ we have

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

By using bilinearity and anticommutativity properties we can re-write this expression as

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]].$$

We can re-write this as

$$\text{ad}_X([Y, Z]) = [\text{ad}_X(Y), Z] + [Y, \text{ad}_X(Z)].$$

This completes the proof. □

3.8 Pushforward of a Vector Field

A smooth map $F : N \rightarrow M$ between manifolds N and M , pushes forward a tangent vector $X_p \in T_p(N)$ where $p \in N$ by its differential. I.e. $F_*(X_p)$ is the pushforward of X_p . However, this notion does not exist in general for vector fields. That is because if F is not injective and for

$p, q \in N$ two distinct points such that $z = F(p) = F(q)$, then the pushforward of X_p and X_q by F_* are two tangent vectors at $T_z(M)$, and there is no reason that $F_*(X_p)$ should be equal to $F_*(X_q)$.

However, if F is a *diffeomorphism*, then it makes sense to talk about the pushforward of a vector field by F . That is because F is injective, then there is no ambiguity of the type discussed above, and also since F is surjective, the pushed forward vector field is defined everywhere on M .

Definition 3.11 — F -related Vector Fields. Let $F : N \rightarrow M$ be a smooth map between manifolds and X, \bar{X} be two vector fields defined in N and M respectively. Then we say X is F -related to \bar{X} if for all $p \in N$ we have

$$F_*X_p = \bar{X}_{F(p)}.$$

The following proposition gives a useful characterization of the F -related vector fields.

Proposition 3.17 — A characterization of related vector fields. Let $F : N \rightarrow M$ be a smooth map between manifolds, and X and \bar{X} be two vector fields defined on N and M respectively. Then X and \bar{X} are F -related if and only if for all $g \in C^\infty(M)$ we have

$$X(g \circ F) = (\bar{X}g) \circ F.$$

Proof. **TODO: To be added.** □

Proposition 3.18 Let $F : N \rightarrow M$ be a smooth map between manifolds. Let X, Y be smooth vector fields on N that are F -related to the smooth vector fields \bar{X}, \bar{Y} on M respectively. The the Lie bracket $[X, Y]$ defined on N is F -related to Lie bracket $[\bar{X}, \bar{Y}]$ defined on M .

Proof. **TODO: To be added.** □

3.9 Submanifolds

So far, despite our thorough analysis of manifolds, we have little tools to see if a given topological space is a manifold. Of course, we can use the definition of a smooth manifold and directly show that a given topological space is second countable and Hausdorff, and also exhibit a smooth atlas, hence a manifold. While this is a possibility, but it is seldom done in mathematics. That is why we tend to develop more tools that helps us in identifying if a space is a manifold. For instance, the quotient space construction and the discussion of the criteria under which the quotient space is second countable and Hausdorff was an effort in this direction. Here in this subsection we will develop much more powerful tools to show a particular space is manifold. These tools will be used extensively in the next chapters.

3.9.1 Submanifolds

We will introduce the notion of submanifolds. As a very simple example, the xy plane in \mathbb{R}^3 is a *regular submanifold* of the manifold \mathbb{R}^3 where it is defined as the set on which the z coordinate function vanishes.

Definition 3.12 — Regular Submanifold. Let M be a manifold of dimension n . A subset $S \subset M$ is called a *regular sub-manifold* of dimension k , if for any $p \in S$, there exist a chart $(U, \phi = x^1, \dots, x^n)$ such that the set $U \cap S$ is defined by vanishing $n - k$ coordinate functions. I.e. we need to have

$$U \cap S = \{p \in U \mid x^{k+1}(p) = \dots = x^n(p) = 0\},$$

where we assumed that by renumbering the coordinate functions, we may assume that these vanishing coordinates are x^{k+1}, \dots, x^n . We call such chart an *adapted chart* of S relative to M . Furthermore, we call $n - k$ the *co-dimension* of S .

■ **Remark 3.14** In different wording, we could say that if (U, ϕ) is an adapted chart of S relative to M , then it means that $S \cap U$ is precisely the zero level set of $n - k$ coordinate functions of ϕ .

■ **Remark 3.15** As a topological space, the submanifold S is required to have the subspace topology.


■ **Example 3.6 — A trivial submanifold.** Let S be an open subset of manifold M . Then S is defined by vanishing of no coordinates of any chart overlapping by S . Thus every open subset of a manifold, is trivially a regular submanifold. ■

■ **Example 3.7 — A submanifold of \mathbb{R}^2 .** The set $S = (-1, 1) \times \{0\}$ (as a subset of x -axis) is a regular submanifold of \mathbb{R}^2 . That is because considering the chart $\{U = (-1, 1) \times (-1, 1), \phi = (x, y)\}$ for any $p \in S$ we have $y = 0$. Thus S is a regular submanifold of \mathbb{R}^2 , and (U, ϕ) is an adapted chart. However, note that the chart $(V = \mathbb{R}^2, \psi = (x, y))$ is not an adapted chart since $V \cap S = (-1, 1) \times \{0\}$ but the point on V where one of the coordinate functions vanish, say x or y , is the set $\mathbb{R} \times 0$ or $\{0\} \times \mathbb{R}$ none of which are equal to S . For the same reason the chart $(W = (-2, 2), \omega = (x, y))$ is also not an adapted chart. ■

The following important proposition states that the regular submanifold is also a manifold.

Proposition 3.19 — Regular Submanifold is a Manifold. Let S be a regular submanifold of the manifold M . Let $\mathfrak{U} = \{(U, \phi)\}$ be a collection of *compatible* and *adapted* charts that *covers* S . Then the collection $\{(U \cap S, \phi_S)\}$ is a compatible chart for S , hence S is a manifold. Furthermore, if S is defined by the vanishing of $n - k$ local coordinate maps, then S has coordinate k .

Proof. To show that the collection $\{(U \cap S, \phi_S)\}$ is an atlas, we first need to show that it covers the whole space S . This follows immediately from the fact that the collection \mathfrak{U} covers the set S . To show the compatibility of the charts in the collection $\{(U \cap S, \phi_S)\}$, Let $(U, \phi = (\phi^1, \dots, \phi^n))$ and $(V, \psi = (\psi^1, \dots, \psi^n))$ be two coordinate charts where both of them contains the point $p \in S$. By re-numbering the coordinate component function we assume that ϕ^k, \dots, ϕ^n and ψ^k, \dots, ψ^n vanish at p . Since $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are both smooth, then it follows immediately that the corresponding restricted charts are also compatible. □

Be Careful Here!  **3.9.1** What about the other properties of manifold? In the proposition above we only exhibited a compatible chart, but we did not checked for other properties of manifolds, like being Hausdorff or second countable. These two properties follows from the fact that S is a subset of M and has the subspace topology. Thus given that M is second countable and Hausdorff, then S is also second countable and Hausdorff.

Definition 3.13 — Level Set of a Map. Let $F : N \rightarrow M$ be a map between manifolds. For $c \in M$ the level set of F with level c is the set

$$F^{-1}(\{c\}) = \{x \in N \mid F(x) = c\}.$$

■ **Remark 3.16** In the case $F : N \rightarrow \mathbb{R}^m$ the set $F^{-1}(\{0\})$ is called the *zero set* of the function F .

■ **Remark 3.17** Recall that a point $c \in M$ is a regular value of the map F if either it is not in the image of F (i.e. $F^{-1}(\{c\}) = \emptyset$), or for all $p \in F^{-1}(\{c\})$ the differential $F_{*,p} : T_p N \rightarrow T_p M$ is surjective.

Definition 3.14 — Regular Level-Set. The inverse image of a regular value of a map is called a regular level set. I.e. Let $F : N \rightarrow M$ be a manifold, and $c \in M$ be a *regular value* of the map F . Then level set $F^{-1}(c)$ is a *regular level set* of F .

■ **Remark 3.18** In different words, a level set of a map is a *regular level set* if for every point on the level set, the differential of the map is *surjective*. This implies that $\dim N \geq \dim M$. To see this observe that for the linear map $F_* : T_p N \rightarrow T_{F(p)} M$, the maximum number of linearly independent rows or columns is $\min\{\dim N, \dim M\}$. In order for F_* to be full rank, the rank of F_* should be equal to $\dim M$. This means that we need to have

$$\min\{\dim N, \dim M\} = \dim M \implies \boxed{\dim N \geq \dim M}.$$

■ **Example 3.8 — S^2 in \mathbb{R}^3 .** We can express the manifold S^2 as a regular level set of a carefully designed map and we will make an adopted chart for it. To do this, consider the map

$$g : \mathbb{R}^3 \rightarrow \mathbb{R}$$

given by

$$g(x, y, z) = x^2 + y^2 + z^2.$$

Its level set with level 1 is given as

$$S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\},$$

which is precisely S^2 as a subset of \mathbb{R}^3 . We can equivalently express this as a zero set of $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ given as

$$f(x, y, z) = x^2 + y^2 + z^2 - 1.$$

To see if this level set is a regular level set or not, we need to determine if the Jacobian matrix of f is full rank on S^2 . The Jacobian matrix of this map is given by

$$f_*(x, y, z) = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}.$$

The rank of this matrix is 1 except when $x = y = z = 0$. Since $(0, 0, 0) \notin S^2$, then all of the points on the 2-sphere are regular level sets. To create an adopted chart, we will use a trick which will be more formalized in the next sections. Let p be a point on S^2 . Since f_* is surjective at p , then not all of its elements are zero. This translates to the fact that there is at least one coordinate of f_* that does not vanish. Without loss of generality we can assume at p we have $x(p) \neq 0$. Then consider the coordinate map given by $\phi = (f, y, z)$. Its Jacobian matrix will be

$$J[\phi] = \begin{bmatrix} 2x & 2y & 2z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and its determinant will be $\det[J[\phi]] = 2x$. We can observe that at p the determinant is not zero. So by [Corollary 2.1](#) we can see that there exists an open neighborhood containing p , i.e. U_p on which ϕ is locally invertible, thus it is a local coordinate chart. Thus for the point p the chart (U_p, ϕ) is an adopted chart (since $U_p \cap S^2$ is defined by the vanishing of the first element of ϕ), hence $(U_p \cap S^2, \phi_S = (y, z))$ will be a local chart for S^2 . This shows that S^2 , as a regular level set of f is a submanifold. ■

The idea of the example above is very important, as we can use its strategy almost verbatim to prove the following proposition.

Proposition 3.20 — Regular Level Set of a Function is a Sub-Manifold. Let $f : N \rightarrow \mathbb{R}$ be a function defined on the manifold. Let $c \in \mathbb{R}$. If c is a *regular* value of the function f , then its level curve $f^{-1}(\{c\})$ is a regular level curve, and is a sub-manifold.

Proof. To start, we need to following Lemma.

Lemma 3.1 — Zero Level Curve. Let $f, g : N \rightarrow \mathbb{R}$ be real-valued functions on a manifold N . Let $c \in \mathbb{R}$. Then the level set of f with level c is the same as the level set of g with level 0. Furthermore, $f^{-1}(\{c\})$ is a regular level curve if and only if $g^{-1}(\{0\})$ is a regular level set.

Proof of the lemma. Let S, S' be defined as

$$S = \{p \in N \mid f(p) = c\}, \quad S' = \{p' \in N \mid g(p) = 0\}.$$

We will need to show $S = S'$. We can see this by

$$p \in S \Leftrightarrow f(p) = c \Leftrightarrow f(p) - c = 0 \Leftrightarrow g(p) = 0 \Leftrightarrow p \in S'.$$

To see proof for the second part, note that $f_* = g_*$, thus f_* will be surjective if and only if g_* is surjective, thus $f^{-1}(\{c\})$ will be a regular level set if and only if $g^{-1}(\{0\})$ is a regular level set. \square

By using the Lemma above, let $g = f - c$. So the level set $f^{-1}(\{c\})$ will be the same as $g^{-1}(\{0\})$, and will be a regular level set if and only if $g^{-1}(\{0\})$ is a regular level set. So we will prove that the zero set of g is a submanifold. Let $p \in S = g^{-1}(\{0\})$, and let $(U, \phi = (x^1, \dots, x^n))$ be a local chart containing p . Then the differential of g at p will be given by

$$g_{p,*} = \left[\frac{\partial}{\partial x^1} \Big|_p g \quad \cdots \quad \frac{\partial}{\partial x^n} \Big|_p g \right].$$

Since the level set $g^{-1}(\{0\})$ is a regular level set, $g_{p,*}$ is surjective. This implies that not all of the components of $g_{p,*}$ be zero at the same time (or alternatively, there exists at least one element in $g_{p,*}$ that is not zero. In a nutshell, for any choice of p , we can find at least one element of $g_{p,*}$ that is not zero. Fix $p \in g^{-1}(\{0\})$ and without loss of generality assume that $\partial g / \partial x^1 \neq 0$. Then consider the map $\phi(g, x^2, \dots, x^n)$ from some open neighborhood of p to \mathbb{R}^n . Its Jacobian matrix is given by

$$J[\phi] = \begin{bmatrix} \frac{\partial}{\partial x^1} \Big|_p g & \frac{\partial}{\partial x^2} \Big|_p g & \cdots & \frac{\partial}{\partial x^n} \Big|_p g \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The determinant of the Jacobian matrix above is given by $\det(J[\phi]) = \frac{\partial}{\partial x^1} \Big|_p g \neq 0$. Thus by [Corollary 2.1](#), there exists an open neighborhood U_p containing p such that ϕ is locally invertible, thus (U_p, ϕ) is a local chart. Observe that $S \cap U_p$ is given by vanishing the first component of ϕ , i.e. $g = 0$. Thus (U_p, ϕ) is an adopted chart. Since the point p was arbitrary, then we proved that for all $p \in S$ there exist an adopted chart, hence S is a submanifold. \square

Now we can extend this theorem more generally form maps between manifolds. This theorem is often called the regular level set theorem, implicit function theorem, pre-image theorem, etc. We will use the name “regular level set theorem”.

Theorem 3.3 — Regular Level Set Theorem. Let $F : N \rightarrow M$ be map between manifolds, and let $c \in M$. The level set $F^{-1}(\{c\})$ is a submanifold of dimension $n - m$ if the level set is regular.

Proof. Assume $F^{-1}(\{c\})$ is non-empty (otherwise it is trivially a submanifold). Let $(V, \psi = (y^1, \dots, y^m))$ be a local chart on M centered at c , i.e. $\psi(c) = 0$. Then we will have $F^{-1}(\{c\}) = F^{-1}(\psi^{-1}(\{0\})) = (\psi \circ F)^{-1}(\{0\})$. If we call $F^i = y^i \circ F$ then the level set $S = F^{-1}(\{c\})$ is the common zero set of the functions F^1, \dots, F^m . Fix $p \in N$ and choose a local chart (U, ψ) that contains p and $U \subset F^{-1}(V)$. Since S is a regular level set, then $F_{*,p}$ is surjective which also implies that $F_{*,p}$ has at least m linearly-independent vectors. Without loss of generality we can assume that the first m columns in $F_{*,p}$ are linearly-independent, hence

$$\det \underbrace{\left[\frac{\partial F^i}{\partial x^j} \right]_{1 \leq i, j \leq m}}_D \neq 0.$$

Replace first m coordinates of ϕ with F^1, \dots, F^m . I.e. let $\phi = (F^1, \dots, F^m, x^{m+1}, \dots, x^n)$. The Jacobian matrix of this map is given by

$$J[\phi] = \begin{bmatrix} D & * \\ 0 & I \end{bmatrix}.$$

and its determinant is given by

$$\det J[\phi] = \det S \neq 0.$$

Thus the map ϕ is locally invertible in a neighborhood of p possibly smaller than U , call it U_p . We can also now see that $S \cap U_p$ is defined by the common zero set of F^1, \dots, F^m . Thus (U_p, ϕ) is an adopted chart for S , hence it is a submanifold. \square

Regular level set theorem, is very powerful tool to prove that some spaces are manifolds. For instance, the following proposition reflects an example of this elegant use.

Proposition 3.21 — Special Linear Group is a Manifold. The special linear group $SL(n, \mathbb{R})$ is a subset of $GL(n, \mathbb{R})$ that contains matrices with determinant 1. $SL(n, \mathbb{R})$ is a manifold.

Proof. Let $\det : GL(n, \mathbb{R}) \rightarrow \mathbb{R}$ be a function which is the determinant function we know from linear algebra. Then $SL(n, \mathbb{R})$ can be defined by the level set $\det^{-1}(\{1\})$. To show that this level set is regular, we need to show that the determinant function is surjective on the level set. To see this, we first need to calculate the differential of \det function. Let $m_{i,j}$ denote the (i, j) -minor of a matrix. Given a matrix $A \in GL(n, \mathbb{R})$ we can expand the determinant function as

$$\det(A) = \sum_j (-1)^{i+j} a_{ij} m_{ij}.$$

This implies that

$$\left. \frac{\partial}{\partial a^{ij}} \right|_A \det = (-1)^{i+j} m_{ij}$$

By [Proposition 3.11](#) we have

$$\det \text{ is not surjective} \implies \left. \frac{\partial}{\partial a^{ij}} \right|_A \det = 0 \forall i, j \implies m_{ij} = 0 \forall i, j \implies \det(A) = 0.$$

So we can conclude

$$\det(A) \neq 0 \implies \det \text{ is surjective.}$$

Since $SL(n, \mathbb{R})$ is a subset of $GL(n, \mathbb{R})$, then it follows that the matrices in this set are not singular, hence \det is surjective which implies that the level set $SL(n, \mathbb{R})$ is a regular level set, hence, is a regular manifold. \square

3.9.2 The Rank of a Smooth Map

Recall that the rank of a smooth map between manifolds is defined by the rank of its differentials. Consider the following important observation.

Observation 3.9.1 — Surjectivity implies maximal rank, but the converse is not true! Let $F : V \rightarrow W$ be a linear map between vector spaces. If F is surjective, then its rank is *maximal*. However, the converse is not true. If F has full rank, then this does not imply that F is surjective. The converse is only true when we have $\dim V \geq \dim W$.

Also, as a general fact from linear algebra, recall that the rank of F is at most $\min\{\dim V, \dim W\}$. I.e. $\text{rank } F \leq \min\{\dim V, \dim W\}$.

So for a smooth map between manifolds, say $F : N \rightarrow M$, assuming that it has maximal rank at $p \in N$, we will have the following three cases that are not mutually exclusive:

- (i) if $\dim N = \dim M$, then since F_* has full rank, thus its determinant is not singular, hence it is locally invertible. So F is a local diffeomorphism at p .
- (ii) If $\dim N \leq \dim M$, then by the observation box above, the maximal rank does not necessarily imply being surjective. However the maximality of the rank implies that $\text{rank } F = \dim N$, hence F_* is injective, thus an immersion.
- (iii) If $\dim N \geq \dim M$, then the map F is an submersion.

In this section, we will basically apply the idea of the constant rank theorem for Euclidean spaces to the manifolds. To make sure we understand every step, we are giving a proof of the constant rank theorem in the case of Euclidean spaces.

Theorem 3.4 — Constant Rank Theorem for Euclidean Spaces. Let $f : \mathbb{R}^n \supset U \rightarrow \mathbb{R}^m$ be a map between Euclidean spaces. Assume that f has a constant rank k in a neighborhood of $p \in U$. Then there exists diffeomorphisms of G and F (as a local coordinate changes) such that

$$(F \circ f \circ G^{-1})(x^1, \dots, x^n) = (x^1, \dots, x^k, 0, \dots, 0).$$

Proof. We will do the proof for the case where $n = m = 2$ and $k = 1$ to better highlight the ideas behind the proof. Consider the Jacobian matrix of f at p given as

$$Jf = \begin{bmatrix} \partial_x f^1 & \partial_y f^1 \\ \partial_x f^2 & \partial_y f^2 \end{bmatrix}.$$

□

Since f has constant rank 1, then we should be able to find a non-zero column vector in the Jacobian matrix. Without loss of generality we can assume that $\partial_x f^1 \neq 0$. Let $G(x, y) = (f^1(x, y), y)$. Its Jacobian matrix is given by

$$JG = \begin{bmatrix} \partial_x f^1 & \partial_y f^1 \\ 0 & 1 \end{bmatrix}$$

and its determinant is $\partial_x f^1$ which is non-zero from the assumption. Thus it is locally invertible, i.e. $\exists V \ni f(p)$ and $U_1 \ni p$ such that $G : U_1 \rightarrow V$ is a local diffeomorphism. We can make U_1 smaller such that $U_1 \subset U$. Thus on U_1 the function f has constant rank, and G is locally invertible. Observe that $f^1 G^{-1} = u$ since $(u, v) = (G \circ G^{-1})(u, v) = (f^1 \circ G^{-1}, y \circ G^{-1})(u, v)$, thus we can write

$$f \circ G^{-1} = (u, h(u, v))$$

where $h = y \circ G^{-1}$. Note that since G is a local diffeomorphism, then $f \circ G^{-1}$ has constant rank 1 on U_1 as well (similar to f). Thus in its Jacobian matrix

$$J(f \circ G^{-1}) = \begin{bmatrix} 1 & 0 \\ \partial_x h & \partial_y h \end{bmatrix}.$$

In order for $f \circ G^{-1}$ have the same rank as f on U_1 we need to have $\partial_y h = 0$ everywhere on U_1 . This means that h is only a function of u . Define $F(x, y) = (x, y - h(y))$ which is a diffeomorphism. This implies

$$(F \circ f \circ G^{-1})(u, v) = (u, 0).$$

■ **Example 3.9 — A useful characterization of constant rank for a map.** Let $f : \mathbb{R}^n \supset U \rightarrow \mathbb{R}^m$ be a map that has a non-zero Jacobian at $p \in U$. Then by continuity, it has non-zero Jacobian at a neighborhood of p , hence it has a constant rank. Thus non-zero Jacobian for a map at a point, implies constant rank of map at a neighborhood of that point. ■

This idea of the constant rank theorem can easily be extended to the manifolds with almost no cost. The following proposition captures this idea.

Proposition 3.22 — Constant Rank Theorem for Manifolds. Let $F : N \rightarrow M$ be a map between manifold that has constant rank k in a neighborhood of $p \in N$. Then there exists charts (U, ϕ) centered at $p \in N$ and (V, ψ) centered at $F(p) \in M$ such that

$$(\psi \circ F \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

Proof. Let $(U, \bar{\phi})$ and $(V, \bar{\psi})$ be two charts centered at $p \in N$ and $F(p) \in M$ respectively. Observe that since $\bar{\phi}$ and $\bar{\psi}$ are diffeomorphisms, then the map $H = \bar{\psi} \circ F \circ \bar{\phi}$ has a constant rank k (another way to see this is to consider the differential $H_* = \bar{\psi}_* \circ F_* \circ \bar{\phi}_*$, and since $\bar{\psi}_*$ and $\bar{\phi}_*$ are isomorphisms, then H_* has the same rank as F_*). Now by the constant rank theorem for the vector spaces, there exists local diffeomorphisms G in a neighborhood of $\phi(p)$ and F in a neighborhood of $\psi(F(p))$ such that

$$(F \circ H \circ G^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

Let $\phi = G \circ \bar{\phi}$ and $\psi = F \circ \bar{\psi}$. This completes the proof. □

By the regular level set theorem, we know that the pre-image of a regular value of map between manifolds is a regular submanifold. However, we can use constant rank theorem to show it suffices for the map to have a constant map in a neighborhood of the pre-image, and this guarantees that the level set is a regular submanifold.

Proposition 3.23 — Constant Rank Level set Theorem. Let $F : N \rightarrow M$ be a map between manifolds, and $c \in M$. If F has a constant rank in a neighborhood of $F^{-1}(\{c\})$, then the level set $F^{-1}(\{c\})$ is a regular submanifold.

Proof. Let $p \in F^{-1}(c)$. In order for $F^{-1}(c)$ to be a manifold, then we need to find a chart containing p such that the intersection of the local open coordinate neighborhood with $F^{-1}(c)$ is defined by vanishing of k coordinate functions of ϕ . The key in finding such a local chart in the constant rank theorem for the manifolds. For $p \in F^{-1}(c)$ chosen as above, by the constant rank theorem we can find local charts (U, ϕ) containing p , and (V, ψ) containing $F(p)$ such that we can write

$$(\psi \circ F \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

Thus the level set $(\psi \circ F \circ \phi^{-1})^{-1}(0) \subset \mathbb{R}^n$ is defined by vanishing of the coordinate vectors r^1, \dots, r^k . On the other hand, this zero set, is the image of $F^{-1}(c) \cap U$ under the map ϕ . Because

$$\phi(F^{-1}(c)) = \phi(F(\psi(0)^{-1})^{-1}) = (\phi \circ F^{-1} \circ \psi^{-1})(0) = (\psi \circ F \phi^{-1})^{-1}(0).$$

Thus $F^{-1}(c) \cap U$ is defined by the vanishing of the coordinate functions r^1, \dots, r^k . Since the point p was arbitrary, then this proves that $F^{-1}(c)$ is a regular submanifold. \square

The constant rank level set theorem is a very powerful theorem that helps us to show some very important level sets are in fact manifolds. The orthogonal group $O(n)$ is an example.

Proposition 3.24 — The orthogonal group $O(n)$ is a manifold.. The orthogonal group $O(n)$ as a subset of $GL(n, \mathbb{R})$ is the set of all matrices in $GL(n, \mathbb{R})$ such that $A^T A = I$. $O(n)$ is a regular submanifold of $GL(n, \mathbb{R})$, thus is a manifold.

Proof. Define the map $f : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$ given as $f(A^T A) = I$. So the orthogonal group defined above can also be seen as the level set $f^{-1}(I)$. To show that $O(n)$ is a regular submanifold, we will show f has a constant rank in a neighborhood of $f^{-1}(I)$, and in fact on the entire $GL(n, \mathbb{R})$. To see this, first observe that for any $A, B \in GL(n, \mathbb{R})$ there exists $C \in GL(n, \mathbb{R})$ such that $B = AC$, or $B = r_C(A)$. Consider

$$f(AC) = (AC)^T(AC) = C^T(A^T A)C = C^T f(A)C,$$

or in a more useful notation

$$(f \circ r_C)(A) = (l_{C^T} \circ r_C \circ f)(A).$$

Since A was arbitrary, then $f \circ r_C = l_{C^T} \circ r_C \circ f$. By the chain rule we have

$$f_{*,AC} \circ (r_C)_{*,A} = (l_{C^T})_{*,*} \circ (r_C)_{*,*} \circ f_{*,A}.$$

Since the right and the left multiplication are diffeomorphisms, thus we have

$$\text{rank } f_{*,AC} = \text{rank } f_{*,A}.$$

Thus f has constant rank on $GL(n, \mathbb{R})$. From the constant rank level set theorem, we conclude that $O(n)$ is a regular submanifold of $GL(n, \mathbb{R})$. \square

Immersion and Submersion theorem

The constant rank level set theorem helps us to write a local canonical form for any submersion and immersion. The development of this idea is based on following two observations.

Observation 3.9.2 — Submersion and Immersion Implies Maximal Rank. Let $F : N \rightarrow M$ be a map between manifolds of dimension n and m respectively.

- F is an immersion $\implies n \leq m$, $\text{rank } F = n$.
- F is a submersion $\implies n \geq m$, $\text{rank } F = m$.

We can see that in both cases, the map being a submersion or immersion implies the maximality of its rank.

Observation 3.9.3 — Rank Maximality is an open Condition. Let $F : N \rightarrow M$ be a map between manifolds where it has a maximal rank at $p \in N$. Then the maximality of rank persists in an open neighborhood of p . To see why this is true, observe that F_* has k (the maximal rank of F at p) linearly independent columns. By rearranging the columns, assume that the first k columns are linearly independent. This implies that the determinant of the first $k \times k$ block of F_* is non zero. Since determinant is continuous, then this non-zero determinant persists in an open neighborhood of $p \in N$.

In conclusion, we see that being an immersion or submersion at a point, implies the rank is constant on an open neighborhood of that point. Thus we can write the following normal forms for immersion and submersion.

Proposition 3.25 — Normal Forms for Immersion and Submersion. Let $F : N \rightarrow M$ be a map between manifolds, and let $p \in N$. Then

- If F is immersion at p , then there are charts (U, ϕ) containing $p \in N$ and (V, ψ) containing $F(p) \in M$ such that

$$(\psi \circ F \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^n, 0, \dots, 0).$$

- If F is submersion at p , then there are charts (U, ϕ) containing $p \in N$ and (V, ψ) containing $F(p) \in M$ such that

$$(\psi \circ F \circ \phi^{-1})(r^1, \dots, r^m, r^{m+1}, \dots, r^n) = (r^1, \dots, r^m).$$

Proof. The proof of this proposition is based on the fact that being an immersion or submersion at a point implies the maximality of the rank, and also the fact that the maximality of the rank is an open condition, and finally applying the constant rank theorem level set theorem. \square

Observation 3.9.4 The important observation of the proposition above is that every immersion locally looks like an inclusion map, and every submersion, locally looks like a projection map.

3.10 Summary

Summary 🦋 3.1 — Coordinate Map and Its Differential. Let M be a manifold and $p \in M$. Let (U, ϕ) be a coordinate chart containing p . We saw in [Proposition 2.7](#) that ϕ is a diffeomorphism between U and \mathbb{R}^n . In [Corollary 3.3](#) we also saw that $T_p(U)$ is isomorphic to $T_{\phi(p)}(\mathbb{R}^n)$. In a nutshell

Under a coordinate chart (U, ϕ) , U is diffeomorphic to \mathbb{R}^n and at a point $p \in U$ the tangent space at p , i.e. $T_p(U)$, is isomorphic to the tangent space at $\phi(p)$, i.e. $T_{\phi(p)}\mathbb{R}^n$.

Summary 🦋 3.2 Let $F : V \rightarrow W$ be a linear map between vector spaces. If F is surjective then $\dim V \geq \dim W$. See [Remark 3.18](#) for more details and reasoning.

Summary 🦋 3.3 — Differential of a Map Between Euclidean Spaces. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Its differential at point $p \in \mathbb{R}^n$ is its Jacobian matrix evaluated at point p .

$$F_* = \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^n} \\ \frac{\partial F^2}{\partial x^1} & \cdots & \frac{\partial F^2}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1} & \cdots & \frac{\partial F^m}{\partial x^n} \end{pmatrix} (p)$$

Summary 🦋 3.4 Differential of a coordinate map acting on a tangent vector Let M be a manifold and $(U, \phi = (x^1, \dots, x^n))$ a coordinate chart containing $p \in M$. Then we have

$$\phi_*\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial}{\partial r^i}\Big|_{\phi(p)},$$

or equivalently

$$\phi_*^{-1}\left(\frac{\partial}{\partial r^i}\Big|_{\phi(p)}\right) = \frac{\partial}{\partial x^i}\Big|_p.$$

To show these equations we use the definition of directional derivatives on a manifold ([Definition 2.13](#)). However, we can see that how these equations can reproduce the definition of the directional derivative on a manifold. Let $f : M \rightarrow \mathbb{R}$

$$\frac{\partial}{\partial x^i}\Big|_p f = \phi_*^{-1}\left(\frac{\partial}{\partial r^i}\Big|_{\phi(p)}\right) f = \frac{\partial}{\partial r^i}\Big|_{\phi(p)} (f \circ \phi^{-1}).$$

where we have used the definition of the differential of a map.

Summary 🦋 3.5 — Bases for Tangent Space at a Point on a Manifold. Let M be a manifold with $p \in M$. $T_p M$ is the the tangent space at point p that contains all of the point derivatives $D : C_p^\infty \rightarrow \mathbb{R}$. For each coordinate chart containing p we can write a natural bases for $T_p M$. For instance, let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart containing p . Then the collection

of tangent vectors

$$\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$$

is a basis for $T_p M$. While choosing a different coordinate chart containing p , e.x. $(V, \psi = (y^1, \dots, y^n))$ we can write another natural basis for $T_p M$

$$\left\{ \frac{\partial}{\partial y^1} \Big|_p, \dots, \frac{\partial}{\partial y^n} \Big|_p \right\}.$$

The change of basis matrix R maps these two basis to each other. This map is given by

$$\begin{bmatrix} \frac{\partial}{\partial x^1} \\ \frac{\partial}{\partial x^2} \\ \vdots \\ \frac{\partial}{\partial x^n} \end{bmatrix} = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} & \dots & \frac{\partial y^1}{\partial x^n} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} & \dots & \frac{\partial y^2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y^n}{\partial x^1} & \frac{\partial y^n}{\partial x^2} & \dots & \frac{\partial y^n}{\partial x^n} \end{pmatrix}^T \begin{bmatrix} \frac{\partial}{\partial y^1} \\ \frac{\partial}{\partial y^2} \\ \vdots \\ \frac{\partial}{\partial y^n} \end{bmatrix}$$

Summary 🦋 3.6 — Velocity vector of a curve into \mathbb{R}^n . Let $c : (a, b) \rightarrow \mathbb{R}^n$. Then its velocity vector at $t_0 \in (a, b)$ is given by

$$c'(t) = \begin{bmatrix} \dot{c}_1(t_0) \\ \dot{c}_2(t_0) \\ \vdots \\ \dot{c}_n(t_0) \end{bmatrix}$$

where c_1, \dots, c_n are the components of the curve c . See [Example 3.2](#) for a worked example.

Summary 🦋 3.7 — Finding Local Integral Curves. Given a vector field X on manifold M , finding the integral curve of X starting at a point $p \in M$ reduces to solving an initial value problem as described below. By [Definition 3.7](#) a curve $c : (a, b) \rightarrow M$ is an integral curve starting at p if we have

$$X|_{c(t)} = c'(t) \quad \text{for } t \in (a, b), \quad c(0) = p.$$

Choosing a local coordinate chart (U, x^1, \dots, x^n) that contains p , we can write the restricted vector field $X|_{c(t)}$ as

$$X|_{c(t)} = \sum_i a^i(c(t)) \frac{\partial}{\partial x^i} \Big|_p.$$

On the other hand, by [Proposition 3.8](#) we can write $c'(t)$ in the local coordinates as

$$c'(t) = \sum_i \dot{c}^i(t) \frac{\partial}{\partial x^i} \Big|_p,$$

where $c^i = x^i \circ c$. Thus we will have

$$\dot{c}^i(t) = a^i(c(t)) \quad \text{for } i = 1, \dots, n, \quad \text{and} \quad c(0) = p.$$

Since the coefficient functions a^i are all smooth, then by the existence and uniqueness theorem we conclude the existence of the desired integral curve.

Summary 🦋 3.8 — **Lie bracket and the bracket.** *Lie bracket* is defined as

$$[X, Y] = XY - YX,$$

for $X, Y \in \mathfrak{X}(M)$ as vector fields defined on the manifold M . However, *the bracket*, is the product operator defined on Lie algebra that satisfies three properties in [Definition 3.10](#). The set $\mathfrak{X}(M)$ along with the Lie bracket as the bracket is a Lie algebra.

Summary 🦋 3.9 — **Regular Submanifold Characterized by an Adapted Chart.** In different wording, we could say that if (U, ϕ) is an adapted chart of S relative to M , then it means that $S \cap U$ is precisely the zero level set of $n - k$ coordinate functions of ϕ .

3.11 Solved Problems

■ **Problem 3.1 — The differential of a map.** Check that $F_*(X_p)$ is a derivation at $F(p)$ and that $F_* : T_p N \rightarrow T_{F(p)} M$ is a linear map.

Solution We start with showing that $F_*(X_p)$ is a derivation at $F(p)$. By definition we have

$$\begin{aligned} [(F_*(X_p))(f \cdot g)](F(p)) &= [X_p((f \cdot g) \circ F)](p) = [X_p((f \circ F) \cdot (g \circ F))](p) \\ &= [X_p(f \circ F)](p) \cdot g(F(p)) + f(F(p)) \cdot [X_p(g \circ F)](p) \\ &= [(F_*(X_p))f](F(p)) \cdot g(F(p)) + f(F(p)) \cdot [(F_*(X_p))g]. \end{aligned}$$

This shows that $F_*(X_p)$ is indeed a derivation at $F(p)$. Furthermore, to prove the linearity, let $X_p, Y_p \in T_p(N)$. Then

$$(F_*(X_p + Y_p))f = (X_p + Y_p)(f \circ F) = X_p(f \circ F) + Y_p(f \circ F) = (F_*(X_p))f + (F_*(Y_p))f.$$

■ **Problem 3.2 — Velocity vector versus the calculus derivative (from L. Tu).** Let $c : (a, b) \rightarrow \mathbb{R}$ be a curve with target space \mathbb{R} . Verify that $c'(t) = \dot{c}(t)d/dx|_{c(t)}$.

Solution Let $f : \mathbb{R} \rightarrow \mathbb{R}$. From definition

$$c'(t_0)f = c_*\left(\frac{d}{dt}\Big|_{t_0}\right)f = \frac{d}{dt}\Big|_{t_0}(f \circ c) = \dot{c}(t_0)\frac{df}{dx},$$

where we have used the chain rule. Thus

$$c'(t_0) = \dot{c}(t)\frac{d}{dx}\Big|_{c(t_0)}.$$

■ **Problem 3.3 — Differential of left multiplication (from L. Tu).** If g is a matrix in the general linear group $GL(n, \mathbb{R})$, let $\ell_g : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$ be left multiplication by g . Thus $\ell_g(B) = gB$ for any $B \in GL(n, \mathbb{R})$. Since $GL(n, \mathbb{R})$ is an open subset of the vector space $\mathbb{R}^{n \times n}$, the tangent space $T_g(GL(n, \mathbb{R}))$ can be identified with $\mathbb{R}^{n \times n}$. Show that with this identification the differential $(\ell_g)_{*,I} : T_I(GL(n, \mathbb{R})) \rightarrow T_g(GL(n, \mathbb{R}))$ is also left multiplication by g .

Solution We will use the idea of [Proposition 3.10](#) to compute the differential using a curve. Let $c : (-\epsilon, \epsilon) \rightarrow GL(n, \mathbb{R})$ be a curve such that $c(0) = I$ and $c'(0) = X_p$. Then

$$(\ell_g)_{*,I}X_p = (\ell_g \circ c)'(0) = (gc)'(0) = gc'(0) = gX_p.$$

■ **Problem 3.4 — Differential of a map (from L. Tu.).** Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the map

$$(u, v, w) = F(x, y) = (x, y, xy).$$

Let $p = (x, y) \in \mathbb{R}^2$. Compute $F_*(\partial/\partial x|_p)$ as a linear combination of $\partial/\partial u, \partial/\partial v$, and $\partial/\partial w$ at $F(p)$

Solution We know that $F_*(\partial/\partial x|_p) \in T_{F(p)}\mathbb{R}^3$. Letting $(\mathbb{R}^3, \mathbb{1}_{\mathbb{R}^3} = (u, v, w))$ be an atlas for \mathbb{R}^3 , we will have

$$F_*(\partial/\partial x|_p) = a\partial/\partial u|_{F(p)} + b\partial/\partial v|_{F(p)} + c\partial/\partial w|_{F(p)}.$$

Applying u, v, w to both sides we will get

$$a = F_*(\partial/\partial x|_p)u, \quad b = F_*(\partial/\partial x|_p)v, \quad c = F_*(\partial/\partial x|_p)w.$$

Thus

$$a = 1, \quad b = 0, \quad c = 1.$$

■ **Problem 3.5 — Differential of a Linear Map.** Let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. For any $p \in \mathbb{R}^n$, there is a canonical identification of $T_p(\mathbb{R}^n)$ to \mathbb{R}^n give by

$$\sum a^i \frac{\partial}{\partial x^i} \Big|_p \mapsto a = \langle a^1, \dots, a^n \rangle.$$

Show that the differential $L_{*,p} : T_p(\mathbb{R}^n) \rightarrow T_{L(p)}\mathbb{R}^m$ is the map $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ itself with the identification of the tangent spaces as above.

Solution TODO: Answer to be added.

■ **Problem 3.6 — Differential of a Map (from W. Tu).** Fix a real number α and define $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\begin{bmatrix} u \\ v \end{bmatrix} = (u, v) = F(x, y) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Let $X = -y\partial/\partial x + x\partial/\partial y$ be a vector field on \mathbb{R}^2 . If $p = (x, y) \in \mathbb{R}^2$ and $F_*(X_p) = (a\partial/\partial u + b\partial/\partial v)|_{F(p)}$, find a and b in terms of x, y , and α .

Solution We start with

$$F_*(X_p) = (a\partial/\partial u + b\partial/\partial v)|_{F(p)}.$$

Applying u and v to both sides we will get

$$a = F_*(X_p)u, \quad b = F_*(X_p)v.$$

To compute a we will have

$$a = F_*(X_p)u = X_p(u \circ F) = \left(-y \frac{\partial}{\partial x} \Big|_p + x \frac{\partial}{\partial x} \Big|_p\right)(x \cos \alpha - y \sin \alpha) = -y \cos \alpha - x \sin \alpha.$$

Similarly for b we can write

$$b = F_*(X_p)v = X_p(v \circ F) = \left(-y \frac{\partial}{\partial x} \Big|_p + x \frac{\partial}{\partial x} \Big|_p\right)(x \sin \alpha + y \cos \alpha) = -y \sin \alpha + x \cos \alpha.$$

■ **Problem 3.7 — Transition matrix for coordinate vectors (from L. Tu).** Let x, y be the standard coordinates on \mathbb{R}^2 , and let U be the open set

$$U = \mathbb{R}^2 - \{(x, 0) | x \geq 0\}.$$

On U the polar coordinates r, θ are uniquely defined by

$$x = r \cos \theta, \quad y = r \sin \theta$$

where $r > 0$ and $0 < \theta < 2\pi$. Find $\partial/\partial r$ and $\partial/\partial \theta$ in terms of $\partial/\partial x$ and $\partial/\partial y$.

Solution We need to find the change of basis matrix. We can either use the idea of [Summary 3.5](#) or do the computations manually. We will do the latter. We expand $\partial/\partial r$

$$\partial/\partial r = a\partial/\partial x + b\partial/\partial y.$$

Applying both sides to x and y respectively we will get

$$a = \partial x/\partial r = \cos \theta, \quad b = \partial y/\partial r = \sin \theta.$$

Similarly for $\partial/\partial \theta$ we can write

$$\partial/\partial \theta = c\partial/\partial x + d\partial/\partial y.$$

Applying both sides to x and y respectively we will get

$$c = \partial x/\partial \theta = -r \sin \theta, \quad d = \partial y/\partial \theta = r \cos \theta.$$

■ **Problem 3.8 — Velocity vector (from L. Tu).** Let $p = (x, y)$ be a point in \mathbb{R}^2 . Then

$$c_p(t) = \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad t \in \mathbb{R},$$

is a curve with initial point $p \in \mathbb{R}^2$. Compute the velocity vector $c'_p(0)$.

Solution We can either use the result of the [Proposition 3.8](#) or we can do a manual calculation, which we will do the latter. Since $c'_p(t) \in T_p(\mathbb{R}^2)$, then we can expand it in the bases induced by the local coordinates.

$$c'_p(t) = c_{*,p}(\frac{\partial}{\partial t}\Big|_0) = a \frac{\partial}{\partial x}\Big|_p + b \frac{\partial}{\partial y}\Big|_p.$$

Now applying x to both sides we will get

$$a = c_{*,p}(\frac{\partial}{\partial t}\Big|_0)x = \frac{\partial}{\partial t}\Big|_0(x \circ c_p(t)) = -2x \sin 2t - 2y \cos 2t.$$

Similarly, by applying y to both sides we will get

$$b = c_{*,p}(\frac{\partial}{\partial t}\Big|_0)y = \frac{\partial}{\partial t}\Big|_0(y \circ c_p(t)) = 2x \cos 2t - 2y \sin 2t.$$

■ **Problem 3.9 — Tangent Space to a Product.** If M and N are manifolds, let $\pi_1 : M \times N \rightarrow M$ and $\pi_2 : M \times N \rightarrow N$ be the two projections. Prove that for $(p, q) \in M \times N$,


$$(\pi_{1*}, \pi_{2*}) : T_{(p,q)}(M \times N) \rightarrow T_p M \times T_q N$$

is an isomorphism.

Solution Consider the map

$$(\pi_1, \pi_2) : M \times N \rightarrow M \times N.$$

This is indeed a diffeomorphism, and in fact $(\pi_1, \pi_2) = \mathbb{1}_{M \times N}$. This implies that (π_{1*}, π_{2*}) is an isomorphism of vector spaces.

Be Careful Here!  **3.11.1** I am not sure if my question above is a correct one. However, I have provided a more direct proof below.

Second solution: We will show this by explicitly constructing a basis for $T_{(p,q)}(M \times N)$ as well as $T_p M \times T_q N$ and we will show that the map (π_{1*}, π_{2*}) carries the bases to bases, hence is an isomorphism of vector spaces.

Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart containing $p \in M$ and $(V, \psi = (y^1, \dots, y^m))$ be a coordinate chart containing $q \in N$. Then a coordinate chart for $M \times N$ that contains (p, q) would be $(U \times V, (\phi \circ \pi_1, \psi \circ \pi_2) = (\bar{x}^1, \dots, \bar{x}^n, \bar{y}^1, \dots, \bar{y}^m))$. Observe that

$$\pi_{1*} \left(\frac{\partial}{\partial \bar{x}^i} \Big|_{(p,q)} \right) = a_i^1 \frac{\partial}{\partial x^1} \Big|_p + \dots + a_i^n \frac{\partial}{\partial x^n} \Big|_p.$$

By applying x^j to both sides we get

$$a_i^j = \pi_{1*} \left(\frac{\partial}{\partial \bar{x}^i} \Big|_{(p,q)} \right) x^j = \frac{\partial}{\partial \bar{x}^i} \Big|_{(p,q)} (x^j \circ \pi_1) = \delta_{ij}.$$

Thus we can write

$$\pi_{1*} \left(\frac{\partial}{\partial \bar{x}^i} \Big|_{(p,q)} \right) = \frac{\partial}{\partial x^i} \Big|_p,$$

and by a similar calculation for π_{2*} we get

$$\pi_{2*} \left(\frac{\partial}{\partial \bar{y}^i} \Big|_{(p,q)} \right) = \frac{\partial}{\partial y^i} \Big|_q.$$

On the other hand, observe that a basis for $T_{(p,q)}(M \times N)$ is

$$\left\{ \frac{\partial}{\partial \bar{x}^1} \Big|_{(p,q)}, \dots, \frac{\partial}{\partial \bar{x}^n} \Big|_{(p,q)}, \frac{\partial}{\partial \bar{y}^1} \Big|_{(p,q)}, \dots, \frac{\partial}{\partial \bar{y}^m} \Big|_{(p,q)} \right\}$$

and a basis for $T_p(M) \times T_q(N)$ is

$$\left\{ \left(\frac{\partial}{\partial x^1} \Big|_p, 0 \right), \dots, \left(0, \frac{\partial}{\partial x^n} \Big|_p \right), \left(0, \frac{\partial}{\partial y^1} \Big|_q \right), \dots, \left(0, \frac{\partial}{\partial y^m} \Big|_q \right) \right\}$$

We can now see that the map (π_{1*}, π_{2*}) maps these two bases to each other, thus it is an isomorphism of vector spaces.

■ **Problem 3.10 — Differentials of multiplication and inverse (from L. Tu).** Let G be a Lie group with multiplication map $\mu : G \times G \rightarrow G$, inverse map $\iota : G \rightarrow G$, and identity element e .

(a) Show that the differential at the identity of the multiplication map μ is addition:

$$\mu_{*,(e,e)} : T_e G \times T_e G \rightarrow T_e G, \quad \mu_{*,(e,e)}(X_e, Y_e) = X_e + Y_e$$

(b) Show that the differential at the identity of ι is the negative

$$\iota_{*,e} : T_e G \rightarrow T_e G, \quad \iota_{*,e}(X_e) = -X_e.$$

Solution

- (a) We will use the idea of [Proposition 3.10](#) to compute the differential. Let $c : (-\epsilon, \epsilon) \rightarrow G \times G$ for some $\epsilon > 0$, where $c(t) = (g_1(t), g_2(t))$ such that we have $x(0) = (e, e)$ and $c'(0) = (X_e, Y_e)$. Then we can write

$$\mu_{*,(e,e)}(X_e, Y_e) = (\mu \circ c)'(0) = (g_1 g_2)'(0) = g_1'(0)g_2(0) + g_1(0)g_2'(0) = X_e + Y_e,$$

where we have used the fact that $(\cdot)'$ is a derivation.

- (b) We again start with the idea of [Proposition 3.10](#). Let $c : (-\epsilon, \epsilon) \rightarrow G$ for some $\epsilon > 0$ such that $c(0) = e$ and $c'(0) = X_e$. Thus we can write

$$\iota_{*,e}X_e = (\iota \circ c)'(0).$$

Now we aim at finding $(\iota \circ c)'(0)$ by using the identity

$$\mu(c(t), (\iota \circ c)(t)) = e, \quad \text{for } t \in (-\epsilon, \epsilon).$$

Since the function above (which is from \mathbb{R} to G) is constant, thus its derivation will be zero. I.e.

$$(c \cdot (\iota \circ c))(0) = c'(0) \underbrace{\iota(c(0))}_e + \underbrace{c(0)}_e (\iota \circ c)'(0) = 0$$

where we have used the fact that $\iota(e) = e$ and $(\cdot)'$ is a derivation. This implies that

$$(\iota \circ c)'(0) = -X_p.$$

■ **Problem 3.11 — Transforming vectors to coordinate vectors.** Let X_1, \dots, X_n be n vector fields on an open subset U of a manifold of dimension n . Suppose that at $p \in U$, the vectors $(X_1)_p, \dots, (X_n)_p$ are linearly independent. Show that there is a chart (V, x^1, \dots, x^n) about p such that $(X_i)_p = (\partial/\partial x^i)_p$ for $i = 1, \dots, n$.

Solution Let (V, y^1, \dots, y^n) be a coordinate chart containing p . We expand the tangent vectors $(X_1)_p, \dots, (X_n)_p$ in terms of the bases $\partial/\partial y^1|_p, \dots, \partial/\partial y^n|_p$ and arrange the coordinates as the columns of a matrix P . Symbolically we can write

$$[(X_1)_p \ (X_2)_p \ \cdots \ (X_n)_p] = [\partial/\partial y^1|_p \ \partial/\partial y^2|_p \ \cdots \ \partial/\partial y^n|_p] P.$$

Since $(X_i)_p$ are linearly independent, so P is non singular and is invertible. Thus we can write

$$[\partial/\partial y^1|_p \ \partial/\partial y^2|_p \ \cdots \ \partial/\partial y^n|_p] = [(X_1)_p \ (X_2)_p \ \cdots \ (X_n)_p] P^{-1}.$$

Define

$$x^j = \sum_i [P^{-1}]_{ji} y^i.$$

Since P^{-1} is non singular and (y^1, \dots, y^n) is a coordinate maps, this implies that (x^1, \dots, x^n) is also a coordinate map, thus $(V, (x^1, \dots, x^n))$ is also a chart. From the change of bases rule for the differential in charts we know that

$$\partial/\partial y_i = \sum_j \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial x_j}$$

where from the definition x^j

$$\partial/\partial y^i = \sum_j [P^{-1}]_{ji} \frac{\partial}{\partial x^j}.$$

Symbolically we can write this as

$$[\partial/\partial y^1 \ \partial/\partial y^2 \ \cdots \ \partial/\partial y^n] = [\partial/\partial x^1 \ \partial/\partial x^2 \ \cdots \ \partial/\partial x^n] P^{-1}.$$

By multiplying both sides as P we will get

$$[\partial/\partial x^1 \ \partial/\partial x^2 \ \cdots \ \partial/\partial x^n] = [\partial/\partial y^1 \ \partial/\partial y^2 \ \cdots \ \partial/\partial y^n] P$$

Thus we can conclude that

$$[\partial/\partial x^1 \ \partial/\partial x^2 \ \cdots \ \partial/\partial x^n] = [(X_1)_p \ (X_2)_p \ \cdots \ (X_n)_p].$$

■ **Problem 3.12 — Local Maxima (form L. Tu).** A real valued function $f : M \rightarrow \mathbb{R}$ on a manifold is said to have a local maximum at $p \in M$ if there is a neighborhood U of p such that $f(p) \geq f(q)$ for all $q \in U$.

- (a) Prove that if a differentiable function $f : I \rightarrow \mathbb{R}$ defined on an open interval I has a local maximum at $p \in I$ then $f'(p) = 0$.
- (b) Prove that a local maximum of a C^∞ function $f : M \rightarrow \mathbb{R}$ is a critical point of f .

Solution

- (a) Let f has a local maximum at $p \in I$. Thus $f(p) \geq f(x)$ for all x sufficiently close to p from left. Thus

$$f'(p) = \lim_{x \rightarrow p^-} \frac{f(x) - f(p)}{x - p} \leq 0.$$

On the other hand for all x sufficiently close to p from left we have $f(p) \geq f(x)$. This implies that

$$f'(p) = \lim_{x \rightarrow p^+} \frac{f(x) - f(p)}{x - p} \geq 0.$$

Combining the two inequalities above we will get

$$f'(p) = 0.$$


- (b) Let $p \in M$ be a local maximum of a function f . Let $X_p \in T_p M$ be any tangent vectors, and c be a smooth curve with velocity vector X_p that starts from p . Then

$$X_p f = (f \circ c)'(0).$$

Note that $f \circ c$ is a real function from $\mathbb{R} \rightarrow \mathbb{R}$ that has a local maxima at $t = 0$, thus $(f \circ c)'(0) = 0$. Let $(U, \phi = (x^1, \dots, x^n))$ be any coordinate chart. From the fact above we know that

$$\left. \frac{\partial}{\partial x^i} \right|_p f = 0 \quad \text{for } i = 1, 2, \dots, n.$$

So by [Proposition 3.11](#) we conclude that p is a critical point.

Be Careful Here!  **3.11.2 — The converses is not true.** Note that the converse of the theorem above is not true. A critical point of a real-valued function is not necessarily an local maxima. It can be a so called saddle point. For instance, the function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $x^3 - 6xy + y^2$ has a saddle point at $(0, 0)$.

■ **Problem 3.13 — Maxima Integral Curve (from L. Tu).** Let $X = x^2 d/dx$ be a vector field on the real number line \mathbb{R} . Find maximal integral curve of X starting at $x = 2$.

Solution Let $c : (a, b) \rightarrow \mathbb{R}$ given as $c(t) = x(t)$ such curve such that $c(0) = 2$. By [Proposition 3.10](#) we can write

$$c'(t) = \dot{c}(t) \frac{d}{dt} = \dot{x} \frac{d}{dx}.$$

For c to an integral curve we need to have

$$c'(t) = X|_{c(t)} \quad \text{for } t \in (a, b).$$

Thus we will have

$$\dot{x} = x^2, \quad x(0) = 2.$$

By solving this initial value problem with separation of variables method, we will get

$$x(t) = \frac{-2}{2t + 1}.$$

The maximal domain for this curve is $t \in (-\infty, -1/2)$.

■ **Problem 3.14 — Recovering vector field from a global flow (from L. Tu).** Let $F : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a global flow on \mathbb{R}^2 give by

$$F\left(t, \begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Recover the vector field that generates this global flow.

Solution Since F is a global flow, then for any given $p = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, $F(t, p)$ will be an integral curve starting at p . Thus its velocity vector at p will give the vector field at p . The velocity vector of F at p is given by

$$X_p = \frac{\partial F}{\partial t} \Big|_0(0, p) = \begin{bmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{bmatrix} \Big|_0 \begin{bmatrix} x \\ y \end{bmatrix} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$

Thus the generating vector field is given as

$$X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$

■ **Problem 3.15 — Equality of vector fields (from L. Tu).** Show that two C^∞ vector fields X and Y on a manifold M are equal if and only if for every C^∞ function f on M we have $Xf = Yf$.

Solution

\Rightarrow This direction is obvious.

\Leftarrow For this direction we will use the proof by contrapositive. Let X, Y be two vector fields that do not match at some $p \in M$, i.e. $X_p \neq Y_p$. With reference to a local coordinate chart we can expand X_p, Y_p in terms of the local bases vectors. We will have

$$X_p = \sum_i a^i \frac{\partial}{\partial x^i} \Big|_p, \quad Y_p = \sum_i b^i \frac{\partial}{\partial x^i} \Big|_p.$$

Since $X_p \neq Y_p$ then there exists at least one i such that $a^i \neq b^i$. Define $f : M \rightarrow \mathbb{R}$ to be $f = x^i$. Then

$$X_p f = a^i, \quad Y_p f = b^i.$$

Since $a^i \neq b^i$, then we conclude that $X_p f \neq Y_p f$. This implies $Xf \neq Yf$.

■ **Problem 3.16 — Vector field on an odd sphere (from L. Tu).** Let $x^1, y^1, \dots, x^n, y^n$ be the standard coordinates on \mathbb{R}^{2n} . The unit sphere S^{2n-1} in \mathbb{R}^{2n} is defined by the equation $\sum_{i=1}^n (x^i)^2 + (y^i)^2 = 1$. Show that

$$X = \sum_i^n \underbrace{-y^i}_{a^i} \frac{\partial}{\partial x^i} + \underbrace{x^i}_{b^i} \frac{\partial}{\partial y^i}$$

is a nowhere-vanishing smooth vector field on S^{2n-1} . Since all spheres of the same dimension are diffeomorphic, this proves that on every odd-dimensional sphere there is a nowhere-vanishing smooth vector field. It is a classical theorem of differential and algebraic topology that on an even-dimensional sphere every continuous vector field must vanish somewhere.

Solution For this, we first use the following result that is proved elsewhere in the problems.

Lemma 3.2 The unit sphere S^n in \mathbb{R}^{n+1} is defined by the equation $\sum_{i=1}^{n+1} (x^i)^2 = 1$. For $p = (p^1, \dots, p^{n+1}) \in S^n$, a necessary and sufficient condition for

$$X_p = \sum a^i \frac{\partial}{\partial x^i} \Big|_p \in T_p(\mathbb{R}^{n+1})$$

to be tangent to S^n at p is $\sum a^i p^i = 0$.

Let $p = (p^1, q^1, \dots, p^n, q^n)$ be a point on the unit sphere S^{2n-1} . Thus from definition we have

$$\sum_{i=1}^n (p^i)^2 + (q^i)^2 = 1.$$

At this point the vector field X is

$$X_p = \sum_{i=1}^n -q^i \frac{\partial}{\partial x^i} + p^i \frac{\partial}{\partial y^i}.$$

Thus it follows immediately that

$$\sum_{i=1}^n a^i p^i + b^i q^i = \sum_{i=1}^n -q^i p^i + p^i q^i = 0.$$

This shows that X_p is a tangent vector to $p \in S^{2n-1}$. On the other hand, for the vector field X to vanish, all of its coefficient functions must vanish, which occurs only at the origin, where it is not located on the unit sphere S^{2n-1} .

We summarize the result of this problem in the theorem below.

Theorem 3.5 — Hairy ball theorem. On an even-dimensional unit sphere, every continuous tangent vector field should vanish somewhere.

■ **Problem 3.17 — Maximal integral curve on a punctured line (from L. Tu).** Let M be $\mathbb{R} - \{0\}$ and let X be the vector field d/dx on M . Find the maximal integral curve of X starting at $x = 1$.

Solution Let $c : (a, b) \rightarrow \mathbb{R}$ be the integral curve of $X = d/dx$ starting at $p = 1$. Since $c'(t) = \dot{c}d/dx$, then we need to have

$$\dot{c} = 1, \quad c(0) = 1.$$

Solving the initial value problem above we will get

$$c(t) = t + 1.$$

The maximal domain for this curve is $(0, \infty)$.

■ **Problem 3.18 — Integral curves in the plane (from L. Tu).** Find the integral curves of the vector field

$$X_{(x,y)} = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} = \begin{bmatrix} x \\ -y \end{bmatrix} \quad \text{on } \mathbb{R}^2.$$

Solution Let $c : (a, b) \rightarrow \mathbb{R}^2$ given by $c(t) = (x(t), y(t))$ to be an integral curve. Then we need to have

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

Thus the integral curves are given as

$$c(t) = (x(t), y(t)) = (Ae^t, Be^{-t}),$$

where A and B are determined by the starting point of the curve.

■ **Problem 3.19 — Hypersurface (from L. Tu).** Show that the solution set of the equation $x^3 + y^3 + z^3 = 1$ in \mathbb{R}^3 is a manifold of dimension 2.

Solution Define $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ as $f(x, y, z) = x^3 + y^3 + z^3$. The solution set of $x^3 + y^3 + z^3 = 1$ is the same as the level curve of f with level 1. Since $f_* = [3x^2 \ 3y^2 \ 3z^2]$ is not surjective only at $(0, 0, 0)$, and this point is not on the level set, then f_* is surjective for all points of the level set, hence a regular level set. By [Theorem 3.3](#) the level set is a submanifold of dimension $3 - 1 = 2$, and by [Proposition 3.19](#) it is a manifold of dimension 2.

■ **Problem 3.20 — Regular values (from L. Tu).** Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = x^3 - 6xy + y^2.$$

Find all values $c \in \mathbb{R}$ for which the level set $f^{-1}(c)$ is a regular submanifold of \mathbb{R}^2 .

Solution A regular level set of a function is the pre-image of a regular value of that function. We need to find the critical points of the map. The differential of the map is given by

$$F_* = \begin{bmatrix} 3x^2 - 6y & -6x + 2y \end{bmatrix}$$

Thus F is not surjective at the points $(0, 0)$ and $(6, 18)$ in its domain. So for all $c \in \mathbb{R} - \{0, -108\}$ the level curve $f^{-1}(c)$ will be a regular level set.

■ **Problem 3.21 — Solution set of one equation (from L. Tu).** Let x, y, z, w be the standard coordinates on \mathbb{R}^4 . Is the solution set of $x^5 + y^5 + z^5 + w^5 = 1$ in \mathbb{R}^4 a smooth manifold? Explain why or why not. (Assume that the subset is given the subspace topology).

Solution Define $F : \mathbb{R}^4 \rightarrow \mathbb{R}$ as $F(x, y, z, w) = x^5 + y^5 + z^5 + w^5$. The solution set of the equation above is the same as the level set $F^{-1}(\{1\})$. We need to show that this level set is a regular level set. To see this consider the differential of F give as

$$F_* = \begin{bmatrix} 5x^4 & 5y^4 & 5z^4 & 5w^4 \end{bmatrix}$$

By [Proposition 3.11](#) the only critical point of this map is the origin $(0, 0, 0, 0)$ which is not in the solution set, i.e. $F(0, 0, 0, 0) \neq 1$. This implies that the level set is a regular level set, hence a regular submanifold, hence a smooth manifold of codimension 1, or equivalently of dimension $4 - 1 = 3$.

■ **Problem 3.22 — Solution set of two equations (from L. Tu).** Is the solution set of the system of equations

$$x^3 + y^3 + z^3 = 1, \quad z = xy,$$

in \mathbb{R}^3 a smooth manifold? Prove your answer.

Solution Define the map $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ as $F = (f_1, f_2) = (x^3 + y^3 + z^3 - 1, xy - z)$. The solution set of the system of equations above is the same as the zero set of the function F . In order for this zero set to be regular, we demand that the map F be surjective on this level set. The differential of the maps is given by

$$F_* = \begin{bmatrix} 3x^2 & 3y^2 & 3z^2 \\ y & x & -1 \end{bmatrix}$$

In order for F not to be surjective, all of (2×2) -minors of the F_* should vanish. I.e. we need to have

$$x^3 = y^3, \quad x^2 + yz^2 = 0, \quad y^2 + xz^2 = 0.$$

The simultaneous solution of these equations is

$$\{(0, 0, z) \mid z \in \mathbb{R}\} \cup \{(x, x, \pm\sqrt{-x}) \mid x < 0\}.$$

Non of these points is on the level set, (i.e. do not satisfy the set of two equations). So we conclude that the solution set is a regular level set, hence a regular submanifold, hence a manifold of codimension 2, or alternatively a manifold of dimension $3 - 2 = 1$.

■ **Problem 3.23 — Solution set of two polynomial equations (from L. Tu).** Decide whether the subset S of \mathbb{R}^3 defined by the two equations

$$x^3 + y^3 + z^3 = 1, \quad x + y + z = 0,$$

is a regular submanifold of \mathbb{R}^3 .

Solution Define $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ as $(f_1, f_2) = (x^3 + y^3 + z^3, x + y + z)$. The solution set of the two polynomial equations above is the same as the level set $F^{-1}(0, 1)$. The Jacobian of this map at $p \in \mathbb{R}^3$ is given by

$$F_* = \begin{bmatrix} 3x^2 & 3y^2 & 3z^2 \\ 1 & 1 & 1 \end{bmatrix}$$

For F_* not to be surjective on the level set, then all of its 2×2 minors should vanish. I.e. in order for F_* not to be surjective we need to have

$$3x^2 - 3y^2 = 0, \quad 3x^2 - 3z^2 = 0, \quad 3y^2 - 3z^2 = 0.$$

The solution set of these equations is $A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$ where

$$\begin{aligned} A_1 &= \{(t, t, t) : t \in \mathbb{R} - \{0\}\}, & A_2 &= \{(-t, t, t) : t \in \mathbb{R} - \{0\}\}, \\ A_3 &= \{(t, -t, t) : t \in \mathbb{R} - \{0\}\}, & A_4 &= \{(t, t, -t) : t \in \mathbb{R} - \{0\}\}, & A_5 &= \{(0, 0, 0)\}. \end{aligned}$$

Non of these solution sets satisfy the two polynomial equations (or equivalently are not on the level set of F with level $(1, 0)$). So the level set is a regular level set, hence a manifold of degree $3-2=1$.

■ **Problem 3.24 — Maximal integral curve in the plane (from L. Tu).** Find the maximum integral curve $c(t)$ starting at the point $(a, b \in \mathbb{R}^2)$ of the vector field $X_{(x,y)} = \partial/\partial x + x\partial/\partial y$ on \mathbb{R}^2 .

Solution Let $c : (\alpha, \beta) \rightarrow \mathbb{R}^2$ be an integral curve given as $c(t) = (x(t), y(t))$ of X starting at (a, b) . Since

$$c'(t) = \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y}.$$

Thus we need to have

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ x \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Solving the ODE system above we will get

$$x(t) = t + A, \quad y(t) = \frac{1}{2}t^2 + At + B.$$

Applying the initial conditions we will get

$$A = a, \quad B = b.$$

So the integral curve is given as

$$c(t) = (t + a, \frac{1}{2}t^2 + at + b).$$

The maximal domain for this integral curve is $(-\infty, \infty)$.

■ **Problem 3.25 — Integral curve starting at a zero of a vector field (from L. Tu).**

- Suppose the smooth vector field X on a manifold M vanishes at a point $p \in M$. Show that the integral curve of X with initial point p is the constant curve $c(t) \equiv p$.
- Show that if X is the zero vector field on a manifold M , and $c_t(p)$ is the maximal integral curve of X starting at p , then the one parameter group of diffeomorphisms $c : \mathbb{R} \rightarrow \text{Diff}(M)$ is the constant map $c(t) \equiv \mathbb{1}_M$.

Solution

- (a) Let $c : (a, b) \rightarrow M$ be the integral curve of X starting at p . Thus we have the following initial value problem

$$\dot{c}^i(t) = 0, \quad c^i(0) = p^i.$$

This implies that the integral curve is the constant curve $c(t) \equiv p$.

- (b) For this we need to find the global flow generated by the vector field. Let the global flow be $F : \mathbb{R} \times M \rightarrow M$. From part (a) above we can see that

$$F(t, p) \equiv p \quad \forall t \in \mathbb{R}, \quad p \in M.$$

Thus for a fixed $p \in M$ the function $F_t : M \rightarrow M$ is the identity map $\mathbb{1}_M$.

■ **Problem 3.26 — Maxima integral curve (from L. Tu).** Let X be the vector field xd/dx on \mathbb{R} . For each $p \in \mathbb{R}$, find the maximal integral curve $c(t)$ of X starting at p .

Solution Let $c : (a, b) \rightarrow \mathbb{R}$ given as $c(t) = x(t)$ be the maximal integral curve of interest. Then we will have the following initial value problem.

$$\dot{x} = x, \quad x(0) = p.$$

The solution of the initial value problem above is given as

$$x(t) = pe^t.$$

The maximal domain of this curve is given as $(-\infty, \infty)$.

■ **Problem 3.27 — Maximal integral curve (from L. Tu).** Let X be the vector field x^2d/dx on the real line \mathbb{R} . For each $p > 0$ in \mathbb{R} , find the maximal integral curve of X with initial point p .

Solution Let $c : (a, b) \rightarrow \mathbb{R}$ given as $c(t) = x(t)$ be the maximal integral curve of interest. Then we will have the following initial value problem.

$$\dot{x} = x^2, \quad x(0) = p.$$

The solution of the initial value problem above is given as

$$x(t) = \frac{-1}{t - 1/p} = \frac{p}{1 - pt}.$$

If $p > 0$, the maximal domain for this curve is $(-\infty, 1/p)$. However, if $p < 0$, then the maximal domain for this curve is given as $(1/p, \infty)$.

■ **Problem 3.28 — Reparameterization of an integral curve.** Suppose $c : (a, b) \rightarrow M$ is an integral curve of the smooth vector field X on M . Show that for any real number s , the map

$$c_s : (a + s, b + s) \rightarrow M, \quad c_s(t) = c(t - s),$$

is also an integral curve of X .

Solution Without loss of generality, let $0 \in (a, b)$. Let $c(0) = p$, and $x'(0) = X_p$. Then $s \in (a + s, b + s)$. So for $c_s(s)$ we have

$$c_s(s) = c(s - s) = c(0) = p.$$

Furthermore, since from the chain rule we have $c'_s(t) = c'(t - s)$ then

$$c'_s(s) = c'(0) = X_p.$$

Thus $c_s(t)$ is an integral curve of X that passes from the point p at $t = s$.

Observation 3.11.1 — Uniqueness of integral curve. As we discussed in [Theorem 3.1](#), the solution of an initial value problem is unique. But based on the observation above, a more accurate term to state the statement above is to say that the solution to an initial value problem is unique up to a reparameterization.

■ **Problem 3.29 — Lie bracket of vector fields (from L. Tu).** If f and g are C^∞ functions and X and Y are C^∞ vector fields on a manifold M , show that

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$

Solution To see this, first note that X, Y are both derivations. Thus expanding the Lie bracket we will get

$$[fX, gY] = (fX)(gY) - (gY)(fX)$$

For the first term in RHS we have

$$(fX)(gY) = fX(gY) = f[X(g)Y + gXY] = fX(g)Y + fgXY,$$

while for the second term we have

$$(gY)(fX) = gY(fX) = g[Y(f)X + fYX] = gY(f)X + gfYX.$$

Combining these two terms while noting that $fg = gf$

$$[fX, gY] = fg(XY - YX) + fX(g)Y - gY(f)X = fg[X, Y] + fX(g)Y - gY(f)X.$$

■ **Problem 3.30 — Lie bracket of vector fields on \mathbb{R}^2 (from L. Tu).** Compute the Lie bracket

$$[-y\partial/\partial x + x\partial/\partial y, \partial/\partial x]$$

on \mathbb{R}^2 .

Solution Using the definition we can write

$$\begin{aligned} [-y\partial/\partial x + x\partial/\partial y, \partial/\partial x] &= (-y\partial/\partial x + x\partial/\partial y)(\partial/\partial x) - (\partial/\partial x)(-y\partial/\partial x + x\partial/\partial y) \\ &= -y\partial^2/\partial x^2 + x\partial^2/\partial y\partial x + y\partial^2/\partial x^2 - x\partial^2/\partial x\partial y \\ &= 0. \end{aligned}$$

■ **Problem 3.31 — Lie bracket in local coordinates.** Consider two C^∞ vector field X, Y on \mathbb{R}^n :

$$X = \sum_i a^i \frac{\partial}{\partial x^i}, \quad Y = \sum_j b^j \frac{\partial}{\partial x^j},$$

where a^i, b^j are C^∞ functions on \mathbb{R}^n . Since $[X, Y]$ is also a C^∞ vector field \mathbb{R}^n ,

$$[X, Y] = \sum_k c^k \frac{\partial}{\partial x^k}$$

for some C^∞ function c^k . Find the formula for c^k in terms of a^i and b^j .

Solution By the bilinearity of the Lie bracket, we can write

$$[X, Y] = \sum_i [a^i \frac{\partial}{\partial x^i}, \sum_j b^j \frac{\partial}{\partial x^j}] = \sum_{i,j} [a^i \frac{\partial}{\partial x^i}, b^j \frac{\partial}{\partial x^j}] = \sum_{i,j} (a^i b^j [\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] + a^i \frac{\partial b^j}{\partial x^i} \frac{\partial}{\partial x^j} - b^j \frac{\partial a^i}{\partial x^j} \frac{\partial}{\partial x^i})$$

By using the fact that $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$ for all i, j , and by re-arranging the similar terms we can get

$$[X, Y] = \sum_k \det \begin{bmatrix} \sum_i a^i \frac{\partial}{\partial x^i} & \sum_j b^j \frac{\partial}{\partial x^j} \\ a^k & b^k \end{bmatrix} \frac{\partial}{\partial x^k}.$$

Thus we have

$$c^k = \det \begin{bmatrix} \sum_i a^i \frac{\partial}{\partial x^i} & \sum_j b^j \frac{\partial}{\partial x^j} \\ a^k & b^k \end{bmatrix}.$$

■ **Problem 3.32 — Regular Submanifolds.** Suppose that a subset S of \mathbb{R}^2 has the property that locally on S one of the coordinates is a C^∞ function of the other coordinate. Show that S is a regular submanifold of \mathbb{R}^2 .

Solution We can show this in two ways. First by finding an explicit adopted chart, and second, by showing that this graph is a regular level set of a smooth map. For the first method, let $y = f(x)$, and $p \in S$, thus $y(p) = f(x(p))$. Let U be any open set containing p , then define the coordinate map $\phi = (x, f(x) - y)$. Note that the Jacobian of this map is given by

$$J[\phi] = \begin{pmatrix} 1 & 0 \\ f_x & -1 \end{pmatrix}$$

hence it is non-singular. Thus ϕ is a local diffeomorphism, so it is a local coordinate map. With this choice of the local chart, the set $S \cap U$ is defined by vanishing the second coordinate of ϕ . Thus $(S \cap U, \phi)$ is an adopted chart, and since $p \in S$ was chosen arbitrary, the set S is a regular submanifold, hence a manifold of dimension 1. For the second method, we need show that S , as the graph of a smooth map $y = f(x)$ is a regular zero set of the map $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ given as $F(x, y) = f(x) - y$. The differential of this map is

$$F_* = [f_x \quad -1]$$

and by [Proposition 3.11](#) we conclude that F is always surjective, thus its zero level curve is regular, hence the set S is a regular submanifold.

■ **Problem 3.33 — Graph of a smooth function.** Show that the graph $\Gamma(f)$ of a smooth function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$\Gamma(f) = \{(x, y, f(x, y)) \in \mathbb{R}^3\}$$

is a regular submanifold of \mathbb{R}^3 .

Solution We need to show that the graph $\Gamma(f)$ is a regular level set of a smooth map. Define $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ as $F(x, y, z) = z - f(x, y)$. Then the graph $\Gamma(f)$ is the zero set of this map. The differential of this map is given by

$$F_* = [-f_x, -f_y, 1]$$

where by [Proposition 3.11](#) is surjective. Thus the zero set is a regular level set, hence $\Gamma(f)$ is a smooth manifold.

■ **Problem 3.34 — Euler's Formula (from L. Tu).** Let F be a homogeneous polynomial of degree k . Thus for any $t \in \mathbb{R}$ we will have

$$F(tx_0, \dots, tx_n) = t^k F(x_0, \dots, x_n).$$

Show that

$$\sum_{i=0}^n x_i \frac{\partial F}{\partial x_i} = kF.$$

Solution To see this we differentiate both sides of the identity $F(tx_0, \dots, tx_n) = t^k F(x_0, \dots, x_n)$ with respect to t and utilize the chain rule to get

$$\sum_i x_i \frac{\partial F}{\partial x_i}(tx_0, \dots, tx_n) = kt^{k-1} F(x_0, \dots, x_n).$$

Let $t \rightarrow 0$. Then we will get

$$\sum_i x_i \frac{\partial F}{\partial x_i}(x_0, \dots, x_n) = kF(x_0, \dots, x_n).$$

■ **Problem 3.35 — Product of regular submanifolds (from L. Tu).** If S_i is a regular submanifold of the manifold M_i for $i = 1, 2$, prove that $S_1 \times S_2$ is a regular submanifold of $M_1 \times M_2$.

Solution Let M_1 and M_2 be manifolds of dimension m_1, m_2 . Since S_1 is a regular submanifold of M_1 , then for $p \in S_1$, there exists a local chart $(U_p, \phi = (\phi^1, \dots, \phi^{m_1}))$ such that $S_1 \cap U_p$ is the zero set of (ϕ^1, \dots, ϕ^k) for some $k \leq m_1$. Similarly for S_2 , for any $q \in S_2$, there exists a local chart $(V_q, \psi = (\psi^1, \dots, \psi^{m_2}))$ such that $S_2 \cap V_q$ is the zero set of (ψ^1, \dots, ψ^l) for some $l \leq m_2$. Let $(p, q) \in S_1 \times S_2$. Then $(S_1 \times S_2) \cap (M_1 \times M_2)$ is defined as the zero set of $(\phi^1, \dots, \phi^k, \psi^1, \dots, \psi^l)$, hence is a regular submanifold of codimension $(m_1 + m_2) - (k + l)$.

■ **Problem 3.36 — Complex Special Linear Group (from L. Tu).** The complex special linear group $SL(n, \mathbb{C})$ is the subgroup of $GL(n, \mathbb{C})$ consisting of $n \times n$ complex matrices of determinant 1. Show that $SL(n, \mathbb{C})$ is a regular submanifold of $GL(n, \mathbb{C})$ and determine its dimension. (This problem requires a rudimentary knowledge of complex analysis).

Solution The proof is exactly the same as the proof of [Proposition 3.21](#) with the only difference that in order to show that the determinant function $\det : GL(n, \mathbb{C}) \rightarrow \mathbb{C}$ is differentiable, we need to highlight that \det function is a holomorphic function as it is a polynomial of its components, hence it is differentiable everywhere on its domain.

■ **Problem 3.37 — The Transversality Theorem.** newEnv

4. Lie Groups and Lie Algebra

We start with the definition of a Lie group.

Definition 4.1 — Lie Group. A Lie group G is a smooth manifold that also has a group structure, and the multiplication and the inverse map are both smooth. I.e. the two maps

$$\mu : G \times G \rightarrow G, \quad \iota : G \rightarrow G$$

are smooth maps. These maps are give as

$$\mu(a, b) = ab, \quad \iota(a) = a^{-1},$$

for $a, b \in G$.

Similarly to the notion of a group acting on a set, a group can also act on itself which has its own terminology.

Definition 4.2 — Translation. Let G be a group. Every $a \in G$ induces two maps

$$l_a : G \rightarrow G, \quad r_a : G \rightarrow G,$$

given as

$$l_a(g) = ag, \quad r_a(g) = ga,$$

for $g \in G$. The maps l_a and r_a are called left / right multiplication by a or left/right translation by a .

Proposition 4.1 Let G be a Lie group and $a \in G$. Then the left translation $l_a : G \rightarrow G$ is a diffeomorphism.

Proof. First, we show that l_a has an inverse, hence is a bijection. We claim that $l_{a^{-1}}$ where $a^{-1} = \iota(a)$ is the inverse of l_a . To see these we use the fact that $l_a \circ l_b = l_{ab}$. Thus

$$l_a \circ l_a^{-1} = l_{aa^{-1}} = l_e = \mathbb{1}_G.$$

Similarly

$$l_a^{-1} \circ l_a = l_{a^{-1}a} = l_e = \mathbb{1}_G.$$

Now it remains to show that l_a and $l_{a^{-1}}$ are both smooth. This follows immediately from the fact that l_a and l_a^{-1} are restrictions of μ to $\{a\} \times G$ and $\{a^{-1}\} \times G$ respectively. \square

Since Lie groups has extra structures (compared to a group), its homomorphisms should also preserve that structure. This is reflected in the definition below.

Definition 4.3 — Lie Group Homomorphism. A map between two Lie groups $F : H \rightarrow G$ is a Lie group homomorphism if it is a group homomorphism, and a C^∞ map.

■ **Remark 4.1** A group homomorphism is a map $F : H \rightarrow G$ such that

$$F(ab) = F(a)F(b)$$

for $a, b \in H$. In a functional notation, this can be written as

$$F \circ l_x = l_{F(x)} \circ F.$$

■ **Remark 4.2** Let $a, b = e_H$ and denote the identity of G as e_G . Since F is a homomorphism we have

$$F(e_H e_H) = F(e_H)F(e_H) \implies F(e_H) = F(e_H)F(e_H).$$

This implies that we must have

$$F(e_H) = e_G.$$

Thus the homomorphism maps the identity element to the identity element.

Now we start by introducing a very important Lie group, called the general linear group $GL(n, \mathbb{R})$.

Proposition 4.2 The general linear group, denoted by $GL(n, \mathbb{R})$, topologized as a subset of $\mathbb{R}^{n \times n}$ is a Lie group.

Proof. The general linear group with the given topology is an open subset of $\mathbb{R}^{n \times n}$, thus it is a manifold. The group structure of this set is also obvious. Now what remains to show is to show that the multiplication and the inverse maps are both smooth. Let $A, B \in GL(n, \mathbb{R})$. Then from the definition of matrix multiplication we know

$$[AB]_{i,j} = \sum_k A_{i,k} B_{k,j}.$$

So the (i, j) element of AB is a polynomial in terms of the coordinates of A and B , this is a smooth function. For the inverse, we know that

$$[A^{-1}]_{i,j} = \frac{1}{\det(A)} (-1)^{i+j} ((j, i)\text{-minor } A).$$

Again, since (j, i) -minor of a matrix is a polynomial in terms of its elements, and since $\det(A) \neq 0$, we conclude that ι is also continuous. \square

4.1 Summary

Summary 🦋 4.1 — Some Vocabulary. Don't confuse the following names!

- General Linear Group $GL(n, \mathbb{R})$: $n \times n$ real matrices with a non-zero determinant.
- Special Linear Group $SL(n, \mathbb{R})$: $n \times n$ real matrices whose determinant is one.
- Orthogonal Group $O(n)$: All matrices on $GL(n, \mathbb{R})$ that are symmetric. I.e. $O(n) = \{A \in GL(n, \mathbb{R}) : A^T A = I\}$.

5. Appendix

5.1 Change of Basis

Let V be a vector space of dimension n , and $\mathbb{B}_1 = \{e_i\}_{i=1}^n$ be a basis. Then for $\alpha \in V$ we can write it uniquely as

$$\alpha = \sum_i \alpha^i e_i.$$

Once we know a particular basis, we can do the following association

$$e_i \longleftrightarrow \begin{bmatrix} 0 \\ \vdots \\ \underbrace{1}_{i\text{-th element}} \\ \vdots \\ 0 \end{bmatrix} \quad \alpha \longleftrightarrow \vec{a} = \begin{bmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ \alpha^{n-1} \\ \alpha^n \end{bmatrix}$$

And we define the transpose operator to be $\vec{a}^T = (\alpha^1 \cdots \alpha^n)$. Then we can write the unique decomposition of α to the basis vectors as

$$\alpha = (\alpha_1 \cdots \alpha_n) \begin{bmatrix} e^1 \\ \vdots \\ e^n \end{bmatrix}$$

where these symbols act with the usual multiplication of vectors. Since these representations of elements of the vector space by column vectors depends on the particular basis that we choose, then the natural question is that how these symbols will change if we change the basis. Assume that the old basis is changed to the new $\mathbb{B}_2 = \{e'_i\}_{i=1}^n$ where

$$\begin{bmatrix} e'_1 \\ e'_2 \\ \vdots \\ e'_n \end{bmatrix} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1N} \\ r_{21} & r_{22} & \cdots & r_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ r_{N1} & r_{N2} & \cdots & r_{NN} \end{pmatrix}^T \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = R^T \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

To find the coordinates of α in the new basis we can write

$$\alpha = \underbrace{(\alpha_1 \cdots \alpha_n)}_{(\beta_1 \cdots \beta_n)} (R^T)^{-1} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

Thus the relation between the new coordinate of α and the old coordinate is

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = R \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

To observe how does linear map representation in new coordinates change consider the equation

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = M \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

where the representation of column vectors and the matrix M is in the basis \mathbb{B}_1 . If we change it to a new basis by the change of basis matrix R , then for the new coordinates we will have

$$R \begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix} = MR \begin{bmatrix} y'_1 \\ \vdots \\ y'_n \end{bmatrix}.$$

This implies

$$\begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix} = \underbrace{R^{-1}MR}_{M'} \begin{bmatrix} y'_1 \\ \vdots \\ y'_n \end{bmatrix}.$$

Summary 5.1 — Change of basis. In a nutshell, Let V is a vector space of dimension N . Then after specifying a basis $\mathbb{B}_1 = \{e_i\}$, we can represent the elements of vector space as column vectors. If we want to change the basis to a new basis $\mathbb{B}_2 = \{e'_i\}$ with the change of basis matrix R (whose j -th column contains the coordinates of e'_i in terms of \mathbb{B}_1 basis vectors). Then the relation between new coordinates and the old coordinates is

$$\underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\text{represented according to } \mathbb{B}_1} = R \underbrace{\begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix}}_{\text{represented according to } \mathbb{B}_2}.$$

Also if

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}_{\mathbb{B}_1} = M \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{\mathbb{B}_2}$$

then

$$\begin{bmatrix} y'_1 \\ \vdots \\ y'_n \end{bmatrix}_{\mathbb{B}_2} = R^{-1} M R \begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix}.$$

5.2 Change of Basis (A second go!)

This is my 89053291-th time that I am thinking about the change of basis and what is an appropriate way to express this. I always forget how to write the change of basis matrix appropriately. Should it appear as R^{-1} , or as R^T ? The right hand side should be the new basis vectors, or they should be on the right hand side, and all these kind of questions. Different approaches by different people (for example Sadri Hassani book) make it even more challenging to remember a working principle for the future references. Here is the way that I did last time and it worked very well. I first started with a first initial guess

$$\begin{bmatrix} e'_1 \\ \vdots \\ e'_n \end{bmatrix} = R \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}.$$

where $\{e_1, e_2, e_3\}$ is the old basis vectors and the ones with ' appearing on them are the new ones. Let v be any vector in your vector space. We can write it as

$$v = \sum_i \nu_i e_i \quad \text{or} \quad v = [\nu_1 \quad \cdots \quad \nu_n] \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}.$$

From the change of basis formula we can it as

$$v = [\nu_1 \quad \cdots \quad \nu_n] R^{-1} \begin{bmatrix} e'_1 \\ \vdots \\ e'_n \end{bmatrix}.$$

This implies that the new coordinates are

$$[\nu'_1 \quad \cdots \quad \nu'_n] = [\nu_1 \quad \cdots \quad \nu_n] R^{-1}$$

Thus the new coordinates in terms of the old coordinates will be written as

$$\begin{bmatrix} \nu'_1 \\ \vdots \\ \nu'_n \end{bmatrix} = (R^T)^{-1} \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_n \end{bmatrix}.$$

We don't want the matrix above look that ugly! So do the right thing from the beginning and start with

$$\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = R^T \begin{bmatrix} e'_1 \\ \vdots \\ e'_n \end{bmatrix}.$$