The n-Pendulum Problem

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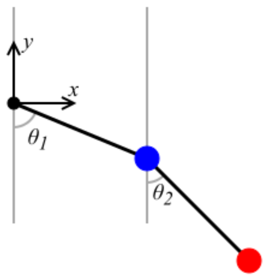
A. The double pendulum

I follow the derivation from here.



The two degrees of freedom are θ_1 and θ_2. Then

$$\begin{aligned}
x_1 &= l_1 \sin\theta_1 & \dot{x}_1 &= l_1\dot{\theta}_1 \cos\theta_1 \\
y_1 &= -l_1 \cos\theta_1 & \dot{y}_1 &= l_1\dot{\theta}_1 \sin\theta_1 \\
x_2 &= l_1 \sin\theta_1 + l_2 \sin\theta_2 & \dot{x}_2 &= l_1\dot{\theta}_1 \cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2 \\
y_2 &= -l_1 \cos\theta_1 - l_2\cos\theta_2 & \dot{y}_2 &= l_1\dot{\theta}_1 \sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2 .
\end{aligned}$$

Let's compute the Lagrangian, $\mathcal{L} = T - V$:

$$\begin{aligned}
V &= m_1 g y_1 + m_2 g y_2 \\
&= -\, m_1 g l_1 \cos\theta_1 - m_2 g (l_1\cos\theta_1 + l_2\cos\theta_2) \\
&= -\,(m_1+m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \\
T &= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 \\
&= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (l_2^2\dot{\theta}_1^2 + 2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1-\theta_2) + l_2^2\dot{\theta}_2^2) \\
&= \frac{1}{2}(m_1+m_2) l_1^2\dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1-\theta_2) \\
\mathcal{L} &= \frac{1}{2}(m_1+m_2) l_1^2\dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1-\theta_2) \\
&\quad + (m_1+m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2 .
\end{aligned}$$

We apply the Euler-Lagrange equations, which are:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \qquad \text{for } q_i = \theta_1, \theta_2 .$$

Therefore,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} &= (m_1+m_2)l_1^2\dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1-\theta_2)\\
\frac{\partial \mathcal{L}}{\partial \theta_1} &= -\, m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \sin(\theta_1-\theta_2) - (m_1+m_2) l_1 g \sin\theta_1
\end{aligned}$$

and

$$\begin{aligned}
\implies\ & (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2)\\
&\qquad = -m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \sin(\theta_1-\theta_2) - (m_1+m_2) l_1 g \sin\theta_1\\
\implies\ & (m_1+m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) + (m_1+m_2) g \sin\theta_1 = 0.
\end{aligned}$$

Again,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} &= m_2 l_2^2\dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1-\theta_2)\\
\frac{\partial \mathcal{L}}{\partial \theta_2} &= m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \sin(\theta_1-\theta_2) - m_2 g l_2 \sin\theta_2
\end{aligned}$$

and

$$\begin{aligned}
\implies\ & m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - m_2 l_1 \dot{\theta}_1 \sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2) = m_2 l_1 \dot{\theta}_1\dot{\theta}_2 \sin(\theta_1-\theta_2) - m_2 g \sin\theta_2\\
\implies\ & m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) + m_2 g \sin\theta_2 = 0.
\end{aligned}$$

Let's introduce $z_1 = \dot{\theta}_1, \dot{z}_1 = \ddot{\theta}_1, z_2 = \dot{\theta}_2, \dot{z}_2 = \ddot{\theta}_2$. This implies

$$\begin{cases}
(m_1+m_2)l_1\dot{z}_1 + m_2 l_2 \dot{z}_2 \cos(\theta_1-\theta_2) + m_2 l_2 z_2^2 \sin(\theta_1-\theta_2) + (m_1+m_2) g \sin\theta_1 = 0\\
m_2 l_2 \dot{z}_2 + m_2 l_1 \dot{z}_1 \cos(\theta_1-\theta_2) - m_2 l_1 z_1^2 \sin(\theta_1-\theta_2) + m_2 g \sin\theta_2 = 0
\end{cases}$$

$$\begin{cases}
(m_1+m_2)l_1\dot{z}_1 - m_2 l_1 \dot{z}_1 \cos^2(\theta_1-\theta_2) + m_2 l_2 z_2^2 \sin(\theta_1-\theta_2) + (m_1+m_2) g \sin\theta_1\\
\qquad + m_2 l_1 z_1^2 \sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2) - m_2 g \sin\theta_2 \cos(\theta_1-\theta_2) = 0\\
m_2 l_2 \dot{z}_2 (m_1+m_2) - m_2 l_1 z_1^2 \sin(\theta_1-\theta_2)(m_1+m_2) + m_2 g \sin\theta_2 (m_1+m_2) - m_2^2 l_2 \dot{z}_2 \cos^2(\theta_1-\theta_2)\\
\qquad - m_2^2 l_2 z_2^2 \sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2) - (m_1+m_2) m_2 g \sin\theta_1 \cos(\theta_1-\theta_2) = 0
\end{cases}$$

$$\begin{cases}
\dot{z}_1 = \frac{m_2 g \sin\theta_2 \cos(\theta_1-\theta_2) - m_2 \sin(\theta_1-\theta_2)[l_1 z_1^2 \cos(\theta_1-\theta_2) + l_2 z_2^2] - (m_1+m_2) g \sin\theta_1}{l_1[m_1+m_2\sin^2(\theta_1-\theta_2)]}\\
\dot{z}_2 = \frac{(m_1+m_2)[l_1 z_1^2 \sin(\theta_1-\theta_2) - g\sin\theta_2 + g\sin\theta_1\cos(\theta_1-\theta_2)] + m_2 l_2 z_2^2 \sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2)}{l_2[m_1+m_2\sin^2(\theta_1-\theta_2)]}
\end{cases}.$$

This forms a system of four first-order ODE in $\dot{z}_1, \dot{\theta}_1, \dot{z}_2, \dot{\theta}_2$ that we can solve numerically, using the Euler or Runge-Kutta method.

Содержание