Intervals, Points, and Branching Time[1]

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Abstract

This paper extends Allen's interval algebra to include points and either left or right-branching time. The branching time algebras each contain 24 relations: Allen's original 13 relations, 5 more relations resulting from the inclusion of points, and 6 relations because of the inclusion of branching time. In this paper, I also present: (1) A technique for automatically deriving algebras of relations; (2) a new way of representing temporal constraint networks — the constraint matrix; and (3) a new way of performing constraint propagation — via constraint matrix multiplication.

1 Introduction

It has been over a decade now since James F. Allen first introduced his algebra of time intervals in [Allen, 1983]. Since then, it has been the subject of many papers and the basis for representing time in many systems. A number of extensions and alternatives to Allen's algebra have been proposed over the years. At least one extension has integrated points with intervals, [Vilain, 1982], and another has added 11 interval relations to extend Allen's algebra to right-branching time, [Anger, et al., 1991][2]. This paper addresses both of these topics.

Branching time has been the subject of considerable study. In [McDermott, 1982] branching time plays an important role, but this work doesn't embrace Allen's interval algebra. In [Allen, et al. 1991] both the interval algebra and branching time appear together, but branching time and intervals are not truly combined. That is, no new interval relations pertaining to branches in time are added to Allen's algebra. By itself the algebra can only represent relations in a single, linear, non-branching timeline.

This paper extends Allen's algebra of time in two directions: (1) to include points, and in a way that is more parsimonious than the approach in [Vilain, 1982] — his approach required 13 new relations, mine only requires 5; and (2) to branching time, but in a different way from that of [Anger, et al., 1991]. They added 11 new interval-to-interval relations to Allen's 13 relations to achieve a single right-branching algebra. My approach only requires 6 new interval-to-interval relations, along with the 5 new point-based relations.[3]

In addition, the means by which my extensions have been obtained is of some interest itself. This is because the means were *mechanical* (i.e., the extensions have been computed). In the discussion to follow I will briefly review Allen's algebra and introduce my notation. I then will extend Allen's algebra to include points, discuss extensions and alternatives to Allen's interval relations, further extend my extension to include right-branching and left-branching time, introduce a new way to represent and propagate constraints, and then finish up with a description of the mechanical process by which I have accomplished this.

1. This work was supported by the U.S. Army Communications and Electronics Command contract DAAB07-93-C-B504.

2. I'd like to thank the anonymous reviewer who brought the work of Anger, Ladkin, and Rodriguez to my attention.

3. Interestingly, each of my branching algebras and the right branching algebra of [Anger, et al., 1991] consist of exactly 24 relations.

2 Allen's Algebra of Time

Allen's algebra of time is based on the 13 interval relations shown in Figure 1[1].

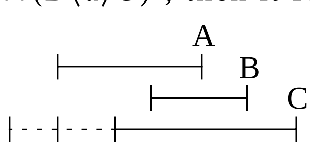
Relation	Depiction	Abbreviation
$X_{ip}\langle b\rangle Y_{ip}$ $Y_{ip}\langle bi\rangle X_{ip}$	X_{ip} Y_{ip}	b = before bi = after
$X_i\langle m\rangle Y_i$ $Y_i\langle mi\rangle X_i$	X_i Y_i	m = meets mi = met-by
$X_i\langle o\rangle Y_i$ $Y_i\langle oi\rangle X_i$	X_i Y_i	o = overlaps oi = overlapped-by
$X_i\langle s\rangle Y_i$ $Y_i\langle si\rangle X_i$	X_i Y_i	s = starts si = started-by
$X_{ip}\langle d\rangle Y_i$ $Y_i\langle di\rangle X_{ip}$	X_{ip} Y_i	d = during di = contains
$X_i\langle e\rangle Y_i$ $Y_i\langle e\rangle X_i$	X_i Y_i	e = equals
$X_i\langle f\rangle Y_i$ $Y_i\langle fi\rangle X_i$	X_i Y_i	f = finishes fi = finished-by

FIGURE 1. The elements of Allen's algebra of time.

If, for example, interval A *overlaps* interval B, we will write, $A\langle o\rangle B$. Disjunctions of relations are denoted by sets of relations. For example, if $\sigma = \{d, o, s\}$ then $A\langle\sigma\rangle C$ denotes the disjunction,

$$(A\langle d\rangle C) \vee (A\langle o\rangle C) \vee (A\langle s\rangle C)$$

The transitive property of relations can be used to define a *multiplication* operator for individual elements (i.e., interval relations) of Allen's algebra. For example, if $(A\langle o\rangle B) \wedge (B\langle d\rangle C)$, then it follows intu-



itively that $A\langle d, o, s\rangle C$ [2]. Thus we write

$$o \bullet d = \{d, o, s\} \qquad \textbf{[1]}$$

Let R_I be the set of all 13 relations:

$$R_I = \{b, bi, d, di, e, f, fi, m, mi, o, oi, s, si\} \qquad \textbf{[2]}$$

The operator, $\bullet$, together with R_I, doesn't really constitute an *algebra* in the mathematical sense, though, because it is not closed. However, if we define $\phi \bullet r = \phi$ for all $r \in R_I$, an algebra can be defined over the power set of R_I, $\Re_I = P(R_I)$.

To do this let $\rho, \sigma \in \Re_I$ and define their product as in [3]:

$$\rho \otimes \sigma = \bigcup_{(r \in \rho),\,(s \in \sigma)} r \bullet s \qquad \textbf{[3]}$$

It can be shown that $\otimes$ is closed and associative. Hence, $(\Re_I, \otimes)$ is a semi-group of order 2^{13} with unit element R_I. The operator $\otimes$ corresponds to the *constraints* function in [Allen, 1983]. In addition, it is well known that $(\Re_I, \cap)$ is a semi-group of the same order with the same unit element. The operators $\otimes$ and $\cap$ will be used later as forms of multiplication and addition, respectively.

In Figure 1, inverses of individual relations have an "i" placed at the end of their abbreviation (e.g., $s^{-1} = si$). I will refer to elements of a power set, such as ρ, as **relation sets**. Inverses of relation sets are defined as follows:

$$\sigma^{-1} = \{r^{-1} \mid r \in \sigma \subseteq R_I\}.$$

The transitivity table for Allen's algebra is given in Table 1. Table entries that are larger than a single element have been abbreviated as shown in Table 3. Only the diagonal and upper-right half of the table is presented. The lower-

1. The subscripts on the Xs and Ys in Figure 1 will be explained in Section 3.

2. For convenience I have dropped the set brackets, "{ }", inside the angle brackets, "⟨ ⟩"

left half can be obtained from the well known identity,

$$(\rho \otimes \sigma)^{-1} = \sigma^{-1} \otimes \rho^{-1}, \text{ where } \rho, \sigma \in \Re_I. \quad \textbf{[4]}$$

3 Integrating Points and Intervals

In this section I will extend Allen's algebra to include points as well as intervals. The extension adds 5 new relations to those of Figure 1 — making for 18 basic relations (denoted by R_{IP}. The 5 relations are depicted in Figure 2. The subscripts on the Xs and Ys denote whether X or Y must be of type *interval* (i) or *point* (p), or can be of either type (ip).

Relation	Depiction	Abbreviation
$X_p \langle pf \rangle Y_i$ $Y_i \langle pfi \rangle X_p$	X_p at the end of Y_i	pf = point-finishes pfi = point-finished-by
$X_p \langle ps \rangle Y_i$ $Y_i \langle psi \rangle X_p$	X_p at the start of Y_i	ps = point-starts psi = point-started-by
$X_p \langle pe \rangle Y_p$ $Y_p \langle pe \rangle X_p$	X_p at the same point as Y_p	pe = point-equals

FIGURE 2. Point-to-point, point-to-interval, and interval-to-point relations.

This approach results in a transitivity table that doesn't permit certain multiplications to take place because of *interval/point type* restrictions. For example, the product, $f \bullet ps$, is undefined because f is an interval-to-interval relation (see the subscripts on X and Y in Figure 1) and ps is a point-to-interval relation. This condition manifests itself in the transitivity table as blank entries. Table 2 gives the lower-left half of the transitivity table.

In [Vilain, 1982] points were integrated with Allen's 13 interval-to-interval relations by adding 13 more relations that handled point-to-point, interval-to-point, and point-to-interval relationships. To understand the difference between my approach and that of [Vilain, 1982], consider Figure 3. It is based on a similar figure in [Rit, 1986] and depicts a mapping of intervals into the cartesian plane (i.e., $x = begin(A)$ and $y = end(A)$). A particular interval, Y, is shown in the figure and the remainder of the triangular region of intervals is divided and labeled according to the relationship each region's intervals have with Y (e.g., the dashed line labeled *meets*, has the property that every interval on it meets Y).

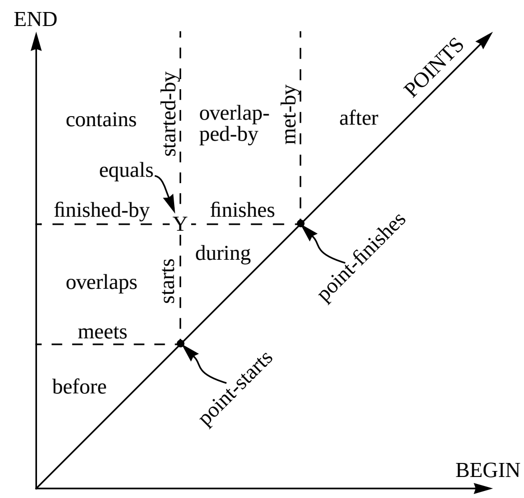


FIGURE 3. Interval & point relations mapped into a plane relative to interval Y.

Points are represented in Figure 3 by the diagonal line. Two points along this line are identified and labeled with their relationships to Y. The two points divide the diagonal into three *segments*. Adjacent to each segment is a triangular region *(before*, *during*, and *after*.). Simply stated, the approach in this paper *combines* the intervals on each of the three segments with the intervals of the adjacent regions as part of the relation to which the region corresponds; the approach in [Vilain, 1982] does not. Thus, in Figure 1, the relations *before*, *during*, and *after* allow both intervals and points to be compared.

4 Branching Time

To extend Allen's algebra to branching time, it is necessary first to consider what it means for two time points to exist on different branches. To do this we need to discuss temporal structures.

4.1 Temporal Structures

Figure 4 depicts the three structures of time considered in this paper. Left and right-branching time are viewed as separate structures. That is, either time branches to the right or to the left, but not both.

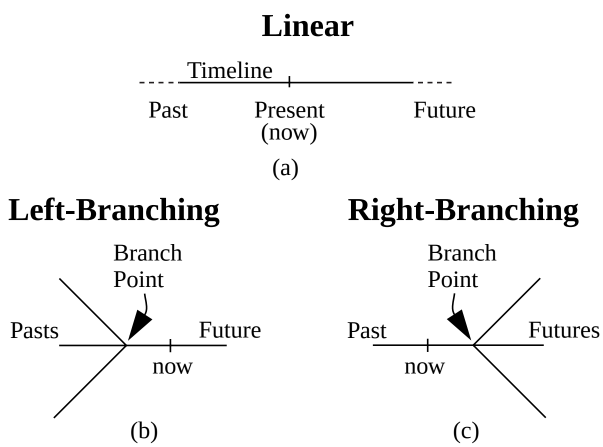


FIGURE 4. Three global structures of time.

As in [van Benthem, 1983], I'll define a **point structure** as an ordered pair, $(T, <)$, where T is a non-empty set and $<$ is a transitive binary relation on T. A **linear point structure** is a point structure such that, for any two time points x, y one and only one of the following three cases holds:

$$(x<y) \vee (x=y) \vee (x>y) \quad \textbf{[5]}$$

From this we have the transitivity table for a linear point structure (see Figure 5).

•	<	=	>
<	{<}	{<}	{<, =, >}
=	{<}	{=}	{>}
>	{<, =, >}	{>}	{>}

FIGURE 5. Transitivity table for linear time, point relations.

An interval in a point structure is a convex set with respect to $<$.

A **branching point structure** is an ordered triple, $(T, <, \sim)$, where $(T, <)$ is a point structure and $\sim$ is an irreflexive, symmetric binary relation on T, called **incomparable**. I will use $\mathrm{l}\sim$ and $\mathrm{r}\sim$ for left and right-branching incomparability of points, respectively. This is not the same *incomparable* as that in [Anger, et al., 1991]. Theirs referred to intervals, not points. In the next section I will present my version of *incomparable* for intervals (which is the same as theirs).

4.2 Right-Branching Point Relations

Expression [5] ensures that a temporal structure is linear. In right branching time, for any two time points x and y one and only one of the follow four cases holds:

$$(x<y) \vee (x=y) \vee (x>y) \vee (x \langle \mathrm{r}\sim \rangle y) \quad \textbf{[6]}$$

However, [6] does not ensure that right-branching time actually branches to the right, as Figure 4(c) depicts. The following property, called *left linearity* in [van Benthem, 1983], ensures that it does:

$$(x < z) \wedge (y < z) \Rightarrow (x < y) \vee (x=y) \vee (x > y) \quad \textbf{[7]}$$

An example situation that satisfies [7] is depicted in Figure 6.

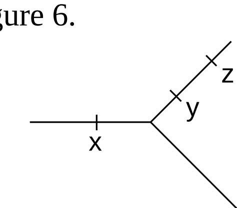


FIGURE 6. Left linearity in right-branching time.

Thus, the transitivity table for time points in right-branching time may be derived, and is shown in Figure 7. The quantity $\mathrm{r}\sim \bullet \mathrm{r}\sim$ depends on the number of branches that are allowed at a branch point. The table in Figure 7 assumes that two or more branches are allowed. If only binary branching is allowed, then $\mathrm{r}\sim \bullet \mathrm{r}\sim = \{<, =, >\}$. A similar condition holds for Figure 8 in the next section.

•	<	=	>	r~
<	{<}	{<}	{<, =, >}	{<, r~}
=	{<}	{=}	{>}	{r~}
>	{<, =, >, r~}	{>}	{>}	{r~}
r~	{r~}	{r~}	{>, r~}	{<, =, >, r~}

FIGURE 7. Transitivity table for time points in right-branching time.

4.3 Left-Branching Point Relations

For any two time points x and y in left-branching time, one and only one of the following four cases holds:

$$(x<y) \vee (x=y) \vee (x>y) \vee (x\langle l{\sim}\rangle y) \qquad \textbf{[8]}$$

To ensure that left-branching time actually branches to the left, as Figure 4(b) depicts. The following property, called *right linearity* in [van Benthem, 1983], is required:

$$(x>z) \wedge (y>z) \Rightarrow (x<y) \vee (x=y) \vee (x>y) \qquad \textbf{[9]}$$

Thus, the transitivity table for time points in left-branching time may be derived, and is given in Figure 8.

•	<	=	>	l~
<	{<}	{<}	{<, =, >,l~}	{l~}
=	{<}	{=}	{>}	{l~}
>	{<, =, >}	{>}	{>}	{>, l~}
l~	{<, l~}	{l~}	{l~}	{<, =, >, l~}

FIGURE 8. Transitivity table for time points in left-branching time.

4.4 Intervals in Branching Time

Using the transitivity table of Figure 7, the additional relations required to extend Allen's relations to right-branching time may be mechanically computed. There are 6 of them (see Figure 9).

The algebra of time that integrates intervals and points in right-branching time, then consists of the combined relations (denoted by R_{RB}) of Figures 1, 2, and 9 — altogether 24 relations. Table 5 gives the upper-right half of the transitivity table R_{RB}.

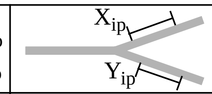
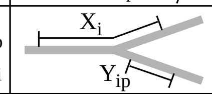
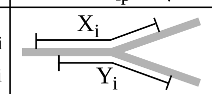
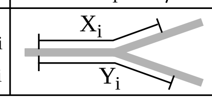
Relation	Depiction	Abbreviation
$X_{ip}\ \langle r{\sim}\rangle\ Y_{ip}$ $Y_{ip}\ \langle r{\sim}\rangle\ X_{ip}$		r~ = right-incomparable
$X_i\ \langle rb\rangle\ Y_{ip}$ $Y_{ip}\ \langle rbi\rangle\ X_i$		rb = right-before rbi = right-after
$X_i\ \langle ro\rangle\ Y_i$ $Y_i\ \langle roi\rangle\ X_i$		ro = right-overlaps roi = right-overlapped-by
$X_i\ \langle rs\rangle\ Y_i$ $Y_i\ \langle rs\rangle\ X_i$		rs = right-starts

FIGURE 9. Right-branching interval & point relations.

Consider the individual relations depicted in Figure 9. The distinctions made between pairs of intervals are based on the positions of the left end points relative to each other, and to the branch point. As long as the branch point is less than the right end of an interval, one may extend the right end to any desired degree without changing the relation.

With respect to left-branching time, we can use the transitivity table of Figure 8 to compute 6 new left-branching relations (see Figure 10).

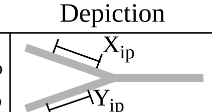
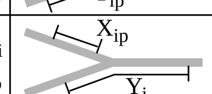
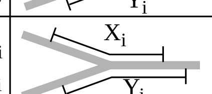
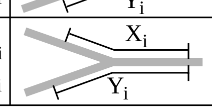
Relation	Depiction	Abbreviation
$X_{ip}\ \langle l{\sim}\rangle\ Y_{ip}$ $Y_{ip}\ \langle l{\sim}\rangle\ X_{ip}$		l~ = left-incomparable
$X_{ip}\ \langle lb\rangle\ Y_i$ $Y_i\ \langle lbi\rangle\ X_{ip}$		lb = left-before lbi = left-after
$X_i\ \langle lo\rangle\ Y_i$ $Y_i\ \langle loi\rangle\ X_i$		lo = left-overlaps loi = left-overlapped-by
$X_i\ \langle lf\rangle\ Y_i$ $Y_i\ \langle lf\rangle\ X_i$		lf = left-finishes

FIGURE 10. Left-branching interval & point relations.

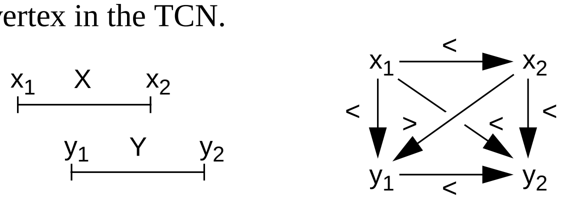
The algebra of time that integrates intervals and points in left-branching time consists of the combined relations (denoted by R_{LB}) of Figures 1, 2, and 10 — altogether 24 relations.

Table 6 gives the lower-left half of the transitivity table for the left-branching algebra.

5 Constraint Matrices

The prevailing method by which constraints between related intervals are represented *together* is by a temporal constraint network (TCN) — that is, a graph whose vertices represent temporal objects (e.g., points, intervals). Labels (i.e., relation sets) on the directed edges of the graph represent the relationship between the connected temporal objects. A graph, however, is a very amorphous structure when compared to, say, a matrix. In this section I will present an alternative to TCNs — the **constraint matrix**.

Consider Allen's *overlaps* relation, depicted in Figure 11(a). It can be represented in a point-based TCN, as shown in Figure 11(b). The corresponding constraint matrix is shown in Figure 11(c). Each row and column corresponds to a vertex in the TCN.



Depiction of X ⟨overlaps⟩ Y

(a)

Temporal Constraint Network

(b)

$$
\begin{array}{c|cccc}
 & x_1 & x_2 & y_1 & y_2 \\
\hline
x_1 & = & < & < & < \\
x_2 & > & = & > & < \\
y_1 & > & < & = & < \\
y_2 & > & > & > & =
\end{array}
$$

Constraint Matrix

(c)

FIGURE 11. An example constraint matrix.

The elements of the constraint matrix are the labels of edges in the TCN. A TCN is a completely connected graph, $G = (E, V)$. Given any two vertices, $v_1, v_2 \in V$, there is a corresponding edge, $(v_1, v_2) \in E$, and a label on that edge (e.g., $\sigma_{x_1, y_2} = \{<\}$). The TCN in Figure 11(b) depicts half of the edges. The other half are easily inferred by inverting relation sets. In the constraint matrix all of the edges' labels are explicit, including the fact that each temporal structure equals itself. A constraint matrix will always be square and have equality relations of some type on its diagonal.

The following definitions also hold. The element-wise **inverse** is denoted by $M^{-1} = (m_{i,j}^{-1})$, M^T denotes the **transpose**, and $M^S = (M^{-1})^T$ denotes the **skew**. Any constraint matrix M that corresponds to a TCN will be **skew-symmetric** (i.e., $M = M^S$).

6 Constraint Propagation

Consider what happens when we multiply an $n \times n$ constraint matrix M by itself (reusing the symbol $\otimes$). Letting $M = (m_{i,j})$, we define $M \otimes M = (\rho_{i,j})$ as follows:

$$\rho_{i,j} = \bigcap_{k=1}^{n} m_{i,k} \otimes m_{k,j} \qquad \textbf{[10]}$$

Consider a single term in the intersection in [10]. It represents a transitive relation between vertices v_i and v_j as derived through vertex v_k. The *summation* (i.e., intersection) from $k = 1$ through $k = n$ takes all n such transitive relation sets (i.e., through all n possible two-edge paths from v_i to v_j) and intersects them. Any relation that is not part of this intersection cannot be part of any **minimal labeling** of the network represented by M, as defined by [van Beek, 1989].

Furthermore, since equality relations (e.g., e and pe) act as multiplicative identity elements, the $k = i$ and $k = j$ terms in [10] consist of the relation set $m_{i,j}$ itself. Since [10] is an intersection, it follows that for $i, j = 1, \ldots, n$,

$$\rho_{i,j} \subseteq m_{i,j} \quad \textbf{[11]}$$

If $\rho_{i,j} = \phi$ (i.e., the empty set) for any element, then the constraint matrix is **inconsistent**. This leads to the following definition of constraint propagation as a fixed-point iteration.

6.1 Constraint Propagation Algorithm

Let C^{TCN} represent a temporal constraint network, in the form of a graph, and let C be its corresponding constraint matrix. If we use Allen's constraint propagation algorithm, $Propagate(C^{TCN})$, to propagate the constraints in C^{TCN}, then the constraint propagation operator, $\Pi(C)$, defined below, achieves the same effect in the constraint matrix, C.

To $\Pi(C)$:
- $C_{Temp1} \leftarrow C$
- $C_{Temp2} \leftarrow C \otimes C$
- While $C_{Temp1} \neq C_{Temp2}$
  - $C_{Temp1} \leftarrow C_{Temp2}$
  - $C_{Temp2} \leftarrow C_{Temp2} \otimes C_{Temp2}$
- Return C_{Temp2}

If at any time during the *while* statement an element of a matrix becomes ϕ, then C is inconsistent.

Note that [11] guarantees that the algorithm will terminate. In fact, $\Pi(C)$ expresses constraint propagation as a fixed-point iteration — a very common technique in numerical methods that exploits *contraction mappings*. According to [11], the operation of squaring a constraint matrix is a contraction mapping of sorts.

7 Derivation of Algebras

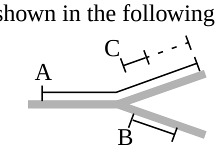
Consider the technique by which Allen came up with his original transitivity table. In the absence of evidence to the contrary, I assume that he did it by considering all 169 cases individually and I assume the same for [Anger, et al., 1991]. The latter required that 576 cases be considered! These are very tedious undertakings, but worse yet, they are prone to error[1]. In this section I will show how Allen's transitivity table can actually be computed. The value in being able to do this is that we can use the technique to extend Allen's algebra to include points and branching time, without all the tedium and potential for error. In fact, this was how the extensions presented in Figures 1, 2, 9, and 10 were derived.

The derivation technique is a two-step process: First, we derive the elements of the algebra, then the corresponding transitivity. I'll illustrate using Allen's algebra.

7.1 Deriving an Algebra's Elements

To derive the elements of Allen's algebra, begin with an algebra of time points based on the set of linear-time, point relations, $R_{Pt} = \{<, =, >\}$. The transitively table for individual relations in R_{Pt} was given in Figure 5. Let $\Re_{Pt}$ denote the power set of R_{Pt}. In the same way that the operator in [3] was defined, we can define an operator, $\otimes_{\mathrm{Pt}}$. Then the semi-group, $(\Re_{Pt}, \otimes_{\mathrm{Pt}})$ corresponds to the point algebra of [Vilain, et al., 1990]. A constraint matrix, C, of elements of $\Re_{Pt}$ can be propagated using the constraint propagation operator, Π, defined in Section 6.1.

1. I only examined a few of the entries in detail in the transitivity table of [Anger, et al., 1991]. Still, I found an error: The table entry (in their notation) for, *pP;U*, should include the elements *di* and *fi* (as shown in the following figure) but doesn't.



The closest thing in this paper to their result is the entry for *rb⊗r~={b, di, fi, m, o, pfi, rb, ro, r~}* in Table 5. Recall, though, that their algebra does not include points.

Now, look back at the constraint matrix example in Figure 11(c). Dotted lines partition the matrix into four 2x2 submatrices. The two partitions along the diagonal are identical, and denote the relationships between the beginning and ending points of any non-degenerate interval. The off-diagonal partition on the upper-right denotes the unique relationship that exists between the beginning and ending points of two intervals that overlap. The partition on the lower-left of the diagonal is the skew of the one on the upper-right.

The main idea behind the mechanical derivation of the elements of Allen's algebra is to set up a matrix that is similar in structure to the one shown in Figure 11(c). This is done in [12].

$$W(r_{11}, r_{12}, r_{21}, r_{22}) = \begin{bmatrix} = & < & r_{11} & r_{12} \\ > & = & r_{21} & r_{22} \\ r_{11}^{-1} & r_{21}^{-1} & = & < \\ r_{12}^{-1} & r_{22}^{-1} & > & = \end{bmatrix} \quad \textbf{[12]}$$

The elements, r_{11}, r_{12}, r_{21}, and r_{22}, denote individual relations chosen from R_{Pt}. There are $3^4 = 81$ ways to choose these elements. Thus, we may form 81 matrices of the form given by [12]. If we propagate each one using Π, all but 13 of them will be inconsistent. Examining the 13 consistent matrices reveals that each one corresponds to one of Allen's relations. The matrix in Figure 11(c) is one of these. In all of the consistent matrices, the two 2x2 partitions on the diagonal are identical to those of Figure 11(c). The matrices are uniquely identified by their upper-right, off-diagonal partition; we will refer to it as the **identifying submatrix**. Seven of the identifying submatrices are shown in [13]. Each is named according to the relation of Allen's to which it corresponds. The other 6 identifying submatrices are the skews of these seven (note that $E = E^S$).

$$B = \begin{bmatrix} < & < \\ < & < \end{bmatrix}, D = \begin{bmatrix} > & < \\ > & < \end{bmatrix}, E = \begin{bmatrix} = & < \\ > & = \end{bmatrix},$$
$$F = \begin{bmatrix} > & < \\ > & = \end{bmatrix}, M = \begin{bmatrix} < & < \\ = & < \end{bmatrix}, O = \begin{bmatrix} < & < \\ > & < \end{bmatrix},$$
$$S = \begin{bmatrix} = & < \\ > & < \end{bmatrix} \quad \textbf{[13]}$$

This identifies the elements of the algebra. Obviously, interpreting the *meaning* of the resulting consistent matrices and giving them names is not done mechanically.

The integration of points with Allen's algebra requires only some very simple modifications to the procedure given above. First, the four instances of the symbols "<" and ">" in [12] need to be replaced with "≤" and "≥", respectively. Note that these two new relations actually correspond to the relation sets {<, =} and {>, =}, respectively. This modification permits the intervals to be degenerate (i.e., points).

Then the four elements, r_{11}, r_{12}, r_{21}, & r_{22}, need to be chosen, not from R_{Pt}, but rather from {≤, =, ≥}. This results once again in 81 constraint matrices. Propagating each one of them results in 18 consistent matrices. Examining them reveals that 13 of them correspond to Allen's algebra and 5 correspond to the relations given in Figure 2.

However, the 13 matrices that correspond to Allen's relations do so in a slightly different manner from the 13 considered previously. The difference is in the 2x2 partitions along the diagonal. Previously, these were always the same. Now there are some differences. In particular, they appear in one of two forms:

$$I_i = \begin{bmatrix} = & < \\ > & = \end{bmatrix} \text{ and } I_{ip} = \begin{bmatrix} = & \leq \\ \geq & = \end{bmatrix} \quad \textbf{[14]}$$

The former corresponds to a non-degenerate interval, the latter to an interval or a point.

In the 5 other matrices a third form of 2x2 partition is observed along the diagonal:

$$I_p = \begin{bmatrix} = & = \\ = & = \end{bmatrix} \quad \textbf{[15]}$$

It corresponds to a degenerate interval (i.e., a point).

This is how the subscripts in Figures 1, 2, 9, and 10 were determined.

7.2 Deriving the Transitivity Table

The second stage in the process of deriving an algebra is to derive the transitivity table. Consider the 18 elements in R_{IP}. Let $\omega_1, \omega_2 \in R_{IP}$. Our goal is to determine $\omega_1 \otimes \omega_2$.

Let $\widehat{R_{IP}}$ denote the 18 identifying submatrices of R_{IP}. They consist of the matrices in [13] and their inverses, along with the matrices in [16] and their inverses:

$$PF = \begin{bmatrix} > & = \\ > & = \end{bmatrix}, PS = \begin{bmatrix} = & < \\ = & < \end{bmatrix}, PE = \begin{bmatrix} = & = \\ = & = \end{bmatrix}. \textbf{[16]}$$

Also let $I_k', I_k'' \in \{I_i, I_{ip}, I_p\}$, where $k = 1, 2$, and let $W_k \in \widehat{R_{IP}}$ be the identifying submatrix for ω_k. Now, to each element ω_k, there corresponds a 4x4 matrix like that of [12], which can be represented using 2x2 matrices as follows:

$$\Omega_k = \begin{bmatrix} I_k' & W_k \\ W_k^S & I_k'' \end{bmatrix} \quad \textbf{[17]}$$

Next, define a 2x2 matrix that expresses complete uncertainty with respect to 4 points:

$$U = \begin{bmatrix} R_{Pt} & R_{Pt} \\ R_{Pt} & R_{Pt} \end{bmatrix} \quad \textbf{[18]}$$

Then define a 6x6 matrix that combines the Ω_k matrices of each ω_k:

$$Q = \begin{bmatrix} I_1' & W_1 & U \\ W_1^S & I_1'' & W_2 \\ U^S & W_2^S & I_2'' \end{bmatrix} \quad \textbf{[19]}$$

Note that Q is equivalent to a TCN of three intervals, X, Y, and Z, where $X\langle\omega_1\rangle Y$, $Y\langle\omega_2\rangle Z$, and the relationship between X and Z is unknown. Note also that I_1'' was put into Q, not I_2'. Although it doesn't matter which is chosen, if one is I_i and the other is I_p then Q will be inconsistent. For example, *finishes* relates intervals to intervals and *point-finishes* relates points to intervals, hence $f \otimes pf$ is undefined.

The central result then depends on what happens to U when we compute $\Pi(Q)$. But, before we continue, one more definition is needed. Given two constraint matrices $M = (m_{i,j})$ and $N = (n_{i,j})$ both having the same dimensions, we say that M is **subequivalent** to N (written $M \ll N$) if $\forall(i, j)(m_{i,j} \subseteq n_{i,j})$.

Now, suppose that $\Pi(Q)$ is consistent and let $\widehat{W}$ denote the upper-right 2x2 partition of $\Pi(Q)$ (i.e., the portion that corresponds to U in Q).

Finally, let $\Theta = \left\{ W \in \widehat{R_{IP}} \middle| W \ll \widehat{W} \right\}$, and let θ be the set of elements in R_{IP} that correspond to Θ. Which finally brings us to the desired product, $\omega_1 \otimes \omega_2 = \theta$.

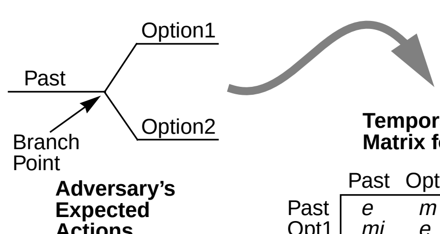
For example, letting ω_1 and ω_2 be the Table 2 relations *during (d)* and *overlaps (o)*, respectively, then $\widehat{W} = \begin{bmatrix} R_{Pt} < \\ R_{Pt} < \end{bmatrix}$. From [13] and [16] it then follows that $\Theta = \{B, D, M, O, PS, S\}$ which corresponds to $d \otimes o$ in Table 2.

This technique was used to generate all four interval-based transitivity tables contained in this paper, and may even be used to generate constraint algebras for non-temporal systems (e.g., a spatial algebra of rectangles).

8 Conclusions

I have extended Allen's interval algebra to both points and left and right-branching time. I have also presented a formalism for automatically deriving algebras of relations, along with a new way of representing temporal constraint networks as matrices and of performing constraint propagation based on constraint matrix multiplication.

All of the work described in this paper has been implemented in a system known as the *Temporal Constraint Manager* and is being used as a key element in a research program that is investigating automated, adversarial planning. The program's approach to planning involves first representing the expected actions of an adversary using a temporal constraint network. Right-branching time is used to represent situations where the adversary may take one of several possible courses of action. Then planning is done against the "backdrop" of the adversary's expected actions.



Temporal Constraint Matrix for Planning

	Past	Opt1	Opt2	Br Pt
Past	*e*	*m*	*m*	*pfi*
Opt1	*mi*	*e*	*rs*	*psi*
Opt2	*mi*	*rs*	*e*	*psi*
Br Pt	*pf*	*ps*	*ps*	*pe*

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Table 1: Allen's Interval Transitivity Table (also see Table 3)

B	**BI**	**D**	**DI**	**E**	**F**	**FI**	**M**	**MI**	**O**	**OI**	**S**	**SI**
B	R_I	Z_1	B	B	Z_1	B	B	Z_1	B	Z_1	B	B
	BI	Z_2	BI	BI	BI	BI	Z_2	BI	Z_2	BI	Z_2	BI
		D	R_I	D	D	Z_1	B	BI	Z_1	Z_2	D	Z_2
			DI	DI	Z_4	DI	Z_5	Z_4	Z_5	Z_4	Z_5	DI
				E	F	FI	M	MI	O	OI	S	SI
					F	Z_6	M	BI	Z_4^{-1}	Z_7	D	Z_7
						FI	M	Z_4	O	Z_4	O	DI
							B	Z_6	B	Z_4^{-1}	M	M
								BI	Z_5^{-1}	BI	Z_5^{-1}	BI
									Z_7^{-1}	Z_3	O	Z_5
										Z_7	Z_5^{-1}	Z_7
											S	Z_8
												SI

B	B																	
BI	R_{IP}	BI																
D	B	BI	D															
DI	Z_{10}^{-1}	Z_9^{-1}	Z_3	DI														
E	B	BI	D	DI	E													
F	B	BI	D	Z_9^{-1}	F	F												
FI	B	Z_9^{-1}	Z_4^{-1}	DI	FI	Z_6	FI											
M	B	Z_9^{-1}	Z_4^{-1}	B	M	Z_4^{-1}	B	B										
MI	Z_{10}^{-1}	BI	Z_5^{-1}	BI	MI	MI	MI	Z_8	BI									
O	B	Z_9^{-1}	Z_4^{-1}	Z_{10}^{-1}	O	Z_4^{-1}	Z_7^{-1}	B	Z_4	Z_7^{-1}								
OI	Z_{10}^{-1}	BI	Z_5^{-1}	Z_9^{-1}	OI	OI	Z_4	Z_5	BI	Z_3	Z_7							
S	B	BI	D	Z_{10}^{-1}	S	D	Z_7^{-1}	B	MI	Z_7^{-1}	Z_5^{-1}	S						
SI	Z_{10}^{-1}	BI	Z_5^{-1}	DI	SI	OI	DI	Z_5	MI	Z_5	OI	Z_8	SI					
PE	B	BI	D	—	—	—	—	—	—	—	—	—	—	PE				
PF	B	BI	D	BI	PF	PF	PF	PS	BI	D	BI	D	BI	—	—			
PFI	B	Z_9^{-1}	Z_4^{-1}	—	—	—	—	—	—	—	—	—	—	PFI	Z_6	—		
PS	B	BI	D	B	PS	D	B	B	PF	B	D	PS	PS	—	—	B	—	
PSI	Z_{10}^{-1}	BI	Z_5^{-1}	—	—	—	—	—	—	—	—	—	—	PSI	MI	—	Z_8	—

Table 2: Interval & Point Transitivity Table (also see Tables 3 & 4)

Table 3: Relation Set Abbreviations (Part 1 of 4)

R_I	{B, BI, D, DI, E, F, FI, M, MI, O, OI, S, SI}
Z_1	{B, D, M, O, S}
Z_2	{BI, D, F, MI, OI}
Z_3	{D, DI, E, F, FI, O, OI, S, SI}
Z_4	{DI, OI, SI}
Z_5	{DI, FI, O}
Z_6	{E, F, FI}
Z_7	{BI, MI, OI}
Z_8	{E, S, SI}

Table 4: Relation Set Abbreviations (Part 2 of 4)

R_{IP}	{B, BI, D, DI, E, F, FI, M, MI, O, OI, S, SI, PE, PF, PFI, PS, PSI}
Z_9	{B, D, M, O, PS, S}
Z_{10}	{BI, D, F, MI, OI, PF}

Table 3: Right Branching Interval & Point Transitivity Table (also see Tables 3, 4, & 7)

B	**BI**	**D**	**DI**	**E**	**F**	**FI**	**M**	**MI**	**O**	**OI**	**S**	**SI**	**PE**	**PF**	**PFI**	**PS**	**PSI**	**RB**	**RBI**	**RO**	**ROI**	**RS**	**R~**
B	R_{ip}	Z_9	B	B	Z_9	B	B	Z_9	B	Z_9	B	B	B	Z_9	B	B	B	B	Z_{11}	B	Z_9	B	Z_{12}
	BI	Z_{13}	BI	BI	BI	BI	Z_{13}	BI	Z_{13}	BI	Z_{13}	BI	BI	BI	BI	Z_{13}	BI	R~	RBI	RBI	RBI	RBI	R~
		D	R_{IP}	D	D	Z_9	B	BI	Z_9	Z_{10}	D	Z_{10}	—	—	B	—	BI	Z_{12}	RBI	Z_{11}	Z_{14}	Z_{14}	R~
			DI	DI	Z_4	DI	Z_{16}	Z_4	Z_{16}	Z_4	Z_{16}	DI	DI	Z_4	DI	Z_{16}	DI	RB	Z_{17}	RO	Z_{18}	RO	Z_{19}
				E	F	FI	M	MI	O	OI	S	SI	—	—	PFI	—	PSI	RB	RBI	RO	ROI	RS	R~
					F	Z_6	M	BI	Z_4^{-1}	Z_7	D	Z_7	—	—	PFI	—	BI	Z_{19}	RBI	Z_{17}	Z_{20}	Z_{20}	R~
						FI	M	Z_4	O	Z_4	O	DI	—	—	PFI	—	DI	RB	Z_{17}	RO	Z_{18}	RO	Z_{19}
							B	Z_6	B	Z_4^{-1}	M	M	—	—	B	—	PFI	B	Z_{17}	B	Z_4^{-1}	M	Z_{19}
								BI	Z_{16}^{-1}	BI	Z_{16}^{-1}	BI	—	—	PSI	—	BI	R~	RBI	RBI	RBI	RBI	R~
									Z_7^{-1}	Z_3	O	Z_5	—	—	B	—	DI	Z_{22}	Z_{17}	Z_{23}	Z_{24}	Z_{25}	Z_{19}
										Z_7	Z_{16}^{-1}	Z_7	—	—	DI	—	BI	Z_{19}	RBI	Z_{17}	Z_{20}	Z_{20}	R~
											S	Z_8	—	—	B	—	PSI	Z_{22}	RBI	Z_{23}	Z_{26}	Z_{27}	R~
												SI	—	—	DI	—	PSI	RB	RBI	RO	ROI	RS	R~
													PE	PF	—	PS	—	—	RBI	—	—	—	R~
														—	PE	—	BI	R~	RBI	RBI	RBI	RBI	R~
															—	M	—	—	Z_{17}	—	—	—	Z_{19}
																—	PE	B	RBI	B	D	PS	R~
																	—	—	RBI	—	—	—	R~
																		RB	Z_{28}	RB	Z_{17}^{-1}	RB	Z_{29}
																			RBI	Z_{29}^{-1}	Z_{13}	Z_{13}	R~
																				Z_{30}	Z_{15}	Z_{16}	Z_{19}
																					Z_{30}^{-1}	Z_{30}^{-1}	R~
																						Z_{21}	R~
																							R_{RB}

B	B																							
BI	R_{IP}	BI																						
D	B	BI	D																					
DI	Z_{10}^{-1}	Z_{31}^{-1}	Z_{35}	DI																				
E	B	BI	D	DI	E																			
F	B	BI	D	Z_{9}^{-1}	F	F																		
FI	B	Z_{31}^{-1}	Z_{36}^{-1}	DI	FI	Z_{44}	FI																	
M	B	Z_{31}^{-1}	Z_{36}^{-1}	B	M	Z_{36}^{-1}	B	B																
MI	Z_{10}^{-1}	BI	Z_{5}^{-1}	BI	MI	MI	MI	Z_{8}	BI															
O	B	Z_{31}^{-1}	Z_{36}^{-1}	Z_{10}^{-1}	O	Z_{36}^{-1}	Z_{7}^{-1}	B	Z_{36}	Z_{7}^{-1}														
OI	Z_{10}^{-1}	BI	Z_{5}^{-1}	Z_{9}^{-1}	OI	OI	Z_{4}	Z_{5}	BI	Z_{3}	Z_{7}													
S	B	BI	D	Z_{10}^{-1}	S	D	Z_{7}^{-1}	B	MI	Z_{7}^{-1}	Z_{5}^{-1}	S												
SI	Z_{10}^{-1}	BI	Z_{5}^{-1}	DI	SI	OI	DI	Z_{5}	MI	Z_{5}	OI	Z_{8}	SI											
PE	B	BI	D	—	—	—	—	—	—	—	—	—	—	PE										
PF	B	BI	D	BI	PF	PF	PF	PS	BI	D	BI	D	BI	—	—									
PFI	B	Z_{31}^{-1}	Z_{36}^{-1}	—	—	—	—	—	—	—	—	—	—	PFI	Z_{44}	—								
PS	B	BI	D	B	PS	D	B	B	PF	B	D	PS	PS	—	—	B	—							
PSI	Z_{10}^{-1}	BI	Z_{5}^{-1}	—	—	—	—	—	—	—	—	—	—	PSI	MI	—	Z_{8}	—						
LB	B	L~	LB	Z_{33}^{-1}	LB	LB	Z_{40}^{-1}	B	L~	Z_{40}^{-1}	Z_{39}^{-1}	LB	Z_{39}^{-1}	—	—	B	—	L~	LB					
LBI	Z_{32}^{-1}	LBI	Z_{37}^{-1}	LBI	LBI	LBI	LBI	Z_{37}^{-1}	LBI	Z_{37}^{-1}	LBI	Z_{37}^{-1}	LBI	LBI	LBI	LBI	Z_{37}^{-1}	LBI	Z_{49}	LBI				
LO	B	LBI	LO	Z_{32}^{-1}	LO	LO	Z_{42}^{-1}	B	LBI	Z_{42}^{-1}	Z_{37}^{-1}	LO	Z_{37}^{-1}	—	—	B	—	LBI	LB	Z_{48}^{-1}	Z_{50}			
LOI	Z_{10}^{-1}	LBI	Z_{38}	Z_{34}^{-1}	LOI	LOI	Z_{41}^{-1}	Z_{5}	LBI	Z_{46}^{-1}	Z_{45}^{-1}	Z_{38}	Z_{45}^{-1}	—	—	DI	—	LBI	Z_{37}	Z_{31}^{-1}	Z_{35}	Z_{50}^{-1}		
LF	B	LBI	LO	Z_{34}^{-1}	LF	LF	Z_{43}^{-1}	M	LBI	Z_{47}^{-1}	Z_{45}^{-1}	LO	Z_{45}^{-1}	—	—	PFI	—	LBI	LB	Z_{31}^{-1}	Z_{36}^{-1}	Z_{50}^{-1}	Z_{44}	
L~	Z_{33}^{-1}	L~	Z_{39}^{-1}	L~	L~	L~	L~	Z_{39}^{-1}	L~	Z_{39}^{-1}	L~	Z_{39}^{-1}	L~	L~	L~	L~	Z_{39}^{-1}	L~	Z_{48}	L~	Z_{39}^{-1}	L~	L~	R_{LB}

Table 4: Left Branching Interval & Point Transitivity Table (also see Tables 3, 4, & 8)

Table 5: Relation Set Abbreviations (Part 3 of 4)

R_{RB}	{B, BI, D, DI, E, F, FI, M, MI, O, OI, S, SI, PE, PF, PFI, PS, PSI, RB, RBI, RO, ROI, RS, R~}
Z_{11}	{B, D, M, O, PS, RBI, RO, ROI, RS, S}
Z_{12}	{B, RB, R~}
Z_{13}	{BI, D, F, MI, OI, PF, RBI, ROI}
Z_{14}	{D, RBI, ROI}
Z_{15}	{D, DI, E, F, FI, O, OI, RO, ROI, RS, S, SI}
Z_{16}	{DI, FI, O, RO}
Z_{17}	{RBI, RO, ROI, RS}
Z_{18}	{RO, ROI, RS}
Z_{19}	{RB, R~}
Z_{20}	{RBI, ROI}
Z_{21}	{E, RS, S, SI}
Z_{22}	{B, RB}
Z_{23}	{B, M, O, RO}
Z_{24}	{D, O, RO, ROI, RS, S}
Z_{25}	{O, RO}
Z_{26}	{D, ROI}
Z_{27}	{RS, S}
Z_{28}	{D, DI, E, F, FI, O, OI, RB, RBI, RO, ROI, RS, S, SI}
Z_{29}	{B, DI, FI, M, O, PFI, RB, RO, R~}
Z_{30}	{DI, FI, O, RB, RO}

Table 6: Relation Set Abbreviations (Part 4 of 4)

R_{LB}	{B, BI, D, DI, E, F, FI, M, MI, O, OI, S, SI, PE, PF, PFI, PS, PSI, LB, LBI, LF, LO, LOI, L~}
Z_{31}	{B, D, LB, LO, M, O, PS, S}
Z_{32}	{BI, D, F, LB, LF, LO, LOI, MI, OI, PF}
Z_{33}	{BI, LBI, L~}
Z_{34}	{D, LB, LO}
Z_{35}	{D, DI, E, F, FI, LF, LO, LOI, O, OI, S, SI}
Z_{36}	{DI, LOI, OI, SI}
Z_{37}	{LB, LF, LO, LOI}
Z_{38}	{LF, LO, LOI}
Z_{39}	{LBI, L~}
Z_{40}	{BI, LBI}
Z_{41}	{D, LO}
Z_{42}	{BI, LOI, MI, OI}
Z_{43}	{F, LF}
Z_{44}	{E, F, FI, LF}
Z_{45}	{LB, LO}
Z_{46}	{D, F, LF, LO, LOI, OI}
Z_{47}	{LOI, OI}
Z_{48}	{B, D, LB, LO, L~, M, O, PS, S}
Z_{49}	{D, DI, E, F, FI, LB, LBI, LF, LO, LOI, O, OI, S, SI}
Z_{50}	{D, LB, LO, O, S}