From Points to Intervals

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Abstract

Representable relation algebras are characterised by a certain property of networks. Representations of relation algebras can be further classified as homogeneous or universal. These properties are also characterised by properties of networks properties (amalgamation and joint embedding properties). For the homogeneous, universal case a method of constructing interval algebras from point algebras is given. This is applied to give a metric algebra of intervals, a relation algebra capable of expressing Allen's interval relations as well as metric constraints.

1 Introduction and Background

A number of temporal reasoning devices can be classified as relation algebras (see [9]). These include the Allen Interval Algebra [1], a point-based version of Allen's Algebra and the metric point system of Dechter, Meiri and Pearl [4]. These have been used extensively in planning [2, 14, 5]. However, the point-based systems suffer from a lack of expressive power and Allen's Interval Algebra is intractable and does not represent metric information. In this paper all of these are shown to belong to a certain sub-class of the relation algebras - namely the relation algebras possessing universal, homogeneous representations. A characterisation of this class is given in terms of certain properties pertaining to networks - the joint embedding and amalgamation properties.

Further, it is shown that from any relation algebra of this type, it is possible to construct a relation algebra of intervals. In particular the Allen Interval Algebra can be built from the Point Algebra and starting with the metric system of Dechter, Meiri and Pearl, it is possible to construct an interval system which does represent metric information. This latter is preferable to other attempts to combine intervals and metrics [8, 13] because it is done in an entirely uniform way while the complexity of the propagation algorithm is the same as that of its competitors. Our system is likely to offer some efficiency advantage because of a simple rule for removing some disjuncts.

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1.1 Some temporal reasoners

In [1] Allen defines thirteen primitive relations which can hold between two intervals: before, meets, overlaps, starts, during, ends, equals and the converses of the first six. The composition of two primitive interval relations is given in a table e.g.

$$overlaps * during = \{during, starts, overlaps\}$$

An interval network consists of a complete directed graph on a set of nodes with each arc labelled by a disjunction of primitive interval relations. An interval network is *consistent* it it is modelled by a set of intervals lying on a linear flow. To determine whether an interval network is consistent has been shown to be NP-complete [15]. The point-based version P is similar but here there are only three primitive point relations: $<, =, >$. The composition table is much simpler, for example $< * <= \{<\}, < * >= \{<, =, >\}$. It has been shown that the consistency checking problem for the point algebra can be solved by a cubic-time algorithm [15]. The metric system M in [4] also uses networks, but here the arcs are labelled by an interval of rational numbers. The intended meaning of the arc

is that $a \leq (y - x) \leq b$[1] i.e. the directed distance from x to y lies inside the interval. Composition of two constraints is calculated by $[a_1, b_1] * [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$. Here again, provided there is only a single interval on each arc (no disjuncts) the problem of checking consistency can be done by the Floyd-Warshall algorithm in cubic time. Throughout this paper $\mathcal{I}$, will stand for the Allen Interval Algebra, $\mathcal{P}$ the point algebra and $\mathcal{M}$ the metric point system of [4].

The remainder of this paper is divided up as follows. In section 2 relation algebras are treated axiomatically as abstract structures. In section 3 networks are used to summarise a complete, but infinite, axiom schema. This network property is necessary and sufficient for a relation algebra to be representable. Network properties are also used to characterise RAs with homogeneous representations and those which have universal representations. In section 4 it is shown that for universal homogeneous relation algebras it is possible to build an algebra of intervals out of an algebra of points. In particular, the Allen interval algebra can be built out of the point algebra. Section 5 applies this result to the metric system of Dechter, Meiri and Pearl. An algebra of intervals in which metric constraints are also expressed is defined, thus combining the qualitative expressions of Allen's Interval Algebra with the quantitative expressive power of [4]. Finally, in a brief complexity sketch it is argued that this system is equivalent to competing systems, but with a uniform method of representation.

[1] Open, closed or semi-open intervals can be dealt with.

2 Relation algebras

All of the above are examples of relation algebras. A binary relation is just a set of pairs taken from some domain (example: '<'). Given two binary relations a and b over a domain D, it is possible to form the union $a \cup b$, intersection $a \cap b$ and the difference $a - b$. The composition of two binary relations $a * b$ is defined to be the set of all pairs (x, y) such that there is some z in D with $(x, z) \in a$ and $(z, y) \in b$; the converse $a^{\smile}$ of a is the set of all pairs (x, y) such that $(y, x) \in a$. The identity relation 1 is the set of all pairs (x, x) for any x in the domain D. A set of binary relations forms a concrete relation algebra if it contains the identity and is closed under the boolean operations, composition and converse.

2.1 Abstract Relation Algebras

Definition.

An (abstract) relation algebra $\mathcal{A}$ is a tuple $(A, \vee, -, 0, v, , *, ^{\smile}, 1)$ where $(A, \vee, -, 0, v)$ is a boolean algebra (v is the universal element), $*$ is an associative binary operator on $\mathcal{A}$ and for all $a, b, c \in \mathcal{A}$

1. $(a^{\smile})^{\smile} = a$
2. $1 * a = a * 1 = a$
3. $a * (b \vee c) = a * b \vee a * c$
4. $(a \vee b)^{\smile} = a^{\smile} \vee b^{\smile}$
5. $(a - b)^{\smile} = a^{\smile} - b^{\smile}$
6. $(a * b)^{\smile} = b^{\smile} * a^{\smile}$
7. $a * b \wedge c^{\smile} = 0 \Leftrightarrow b * c \wedge a^{\smile} = 0.$

It is easy to see that all these axioms are valid over the class of (concrete) algebras of binary relations. It is natural to ask whether these axioms, or indeed any other finite axiom set, are complete over that class. Lyndon [11] showed that the answer is negative [2]: there are (abstract) relation algebras i.e. structures satisfying axioms 1 to 7 above which are not isomorphic to any concrete algebra of binary relations. Lyndon gives an infinite set of conditions which are valid in any representable relation algebra and shows that they are complete for finite relation algebras. These conditions are rather complex and in section 3 we present an equivalent, though conceptually simpler, set of conditions in terms of networks and show that this is complete for the countable case too.

[2] McKenzie [12] provides the simplest example of a relation algebra which does not possess a representation. But Gabbay and Hirsch are preparing a paper which gives a *non-standard* representation for any relation algebra, though here identity, converse and composition do not have their usual definitions

Notation

$a \leq b$ is an abbreviation for $a \vee b = b$. $a \wedge b =_{def} a - (a - b)$.

Definitions

An atom is a minimal non-zero element of $\mathcal{A}$. $\mathcal{A}$ is atomic if every non-zero element contains an atom. If $\mathcal{A}$ is atomic then every element is equal to the union of all its atoms. The set of all atoms of an RA $\mathcal{A}$ is denoted by $atoms_A$.

2.2 Relations as Linear Maps

Definitions

If $\mathcal{A}$ is atomic then $atoms_{\mathcal{A}}$ constitute a basis over the boolean ring $\{1, 0\}$ as every element of $\mathcal{A}$ is a sum of atoms. $\mathcal{A}$ becomes a module. Each relation r in $\mathcal{A}$ determines a linear map θ_r of the module where $\theta_r : a \rightarrow a * r$. r is said to be non-singular or invertible if the linear map θ_r is invertible. It follows that r is non-singular if and only if $r * (r^{\smile}) = 1_A$. The identity, 1, is always non-singular. If all the atoms of $\mathcal{A}$ are non-singular we say that $\mathcal{A}$ is non-singular and in this case $(atoms_{\mathcal{A}}, *)$ forms a group.

Examples

1. The metric point system $\mathcal{M}$ is a non-singular relation algebra. An atomic relation consists of a singleton interval $[d, d]$ and asserts that the directed distance between two points is exactly d. Since $[d, d] * [-d, -d] = [0, 0]$ (the identity relation) each atom must be invertible. $\mathcal{M}$ has infinitely many atoms.

2. The point algebra $\mathcal{P}$ with three atoms $\{=, <, >\}$ forms a three dimensional module. $\theta_=$ is the identity map with matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The matrix for $\theta_<$ is $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. and the matrix for $\theta_>$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Clearly $\theta_<$ and $\theta_>$ are singular.

3 Representations and Networks

3.1 Representations

Definition

A *representation* of a relation algebra $\mathcal{A}$ is a pair (X, I) where X is any set - the domain of the representation - and I is an isomorphism from $\mathcal{A}$ into the concrete algebra of relations on the set X.

3.2 Networks

Definitions

Let $\mathcal{A}$ be a relation algebra. A *network* N over $\mathcal{A}$ is a square anti-symmetric matrix whose entries are elements of $\mathcal{A}$. N is usually represented as a directed graph with each arc labelled by an element of $\mathcal{A}$. We write $n \in N$ if n is a node in the graph and $N(n, m)$ for the label on the arc from n to m. In this paper we always adopt the convention that a network is transitively closed (or locally consistent) [3] i.e. for any $l, n, m \in N$

$$N(n, m) * N(m, l) \geq N(n, m).$$

An $\mathcal{A}$-network is *consistent* (or satisfiable) if $\mathcal{A}$ has a representation (X, I) and there is a homomorphism h from N to X i.e. for all $n, m \in N(h(n), h(m)) \in N(n, m)$.

Notation

Let $\mathrm{Net}(\mathcal{A})$ be the class of all finite, atomic, transitively closed $\mathcal{A}$-networks.

Comments

1. In the planning literature [1, 15, 4] a network is considered as an example of a constraint satisfaction problem - given a (transitively closed) network, usually with non-atomic relations on the arcs, the problem is to discover whether the network is globally consistent. Global consistency has been taken to mean that the network can be tightened to an atomic network which is still transitively closed. The problem is one of choosing one atom from the relation on each arc such that the resulting atomic network is transitively closed. We shall see that an important assumption has been made, namely that a transitively closed, atomic network is globally consistent, that is it has a model. This assumption turns out to be valid only on a certain subclass of the class of all relation algebras - those with *universal* representations. See [9] for an example of an atomic, transitively closed network which has no model.

2. Another interpretation of networks will be developed in the following section. A network will be used as a kind of bridge between the purely syntactic properties of the relation algebra and the semantic properties of a representation.

3.3 Representations as Networks

Let X be a representation of a relation algebra $\mathcal{A}$. X can be viewed as a complete directed graph with each edge (x, y) labelled by all the relations which hold on that edge say $R(x, y)$. So $a \in R(x, y)$ iff $(x, y) \in a$. Now, $R(x, y)$ forms a filter since

$$a \in R(x, y) \& a \leq b \Rightarrow b \in R(x, y)$$
$$a, b \in R(x, y) \Rightarrow a \wedge b \in R(x, y)$$

[3] In some of the literature this property is referred to as 'path consistency'.

$R(x, y)$ is an ultrafilter since

$$a \in R(x,y) \Leftrightarrow -a \notin R(x,y)$$

If A is atomic then $R(x, y)$ must form a principal ultrafilter since

$$(x,y) \in \bigcap R(x,y).$$

So, if $\mathcal{A}$ is atomic we may view the representation X as a (usually infinite) atomic network over $\mathcal{A}$ with each arc labelled by the smallest relation which holds over that arc. Transitive closure is guaranteed by the fact that X is a representation. Now, given an atomic network N (always assumed to be transitively closed) we may ask whether N can be extended by adding new nodes and edges to an atomic network which forms a representation of $\mathcal{A}$.

3.4 Characterisation of Representations by Networks

The finite relation algebras possessing representations can be characterised by an infinite axiom schema ([11]) but by no finite set of axioms. In this section a representation of a relation algebra is further classified as *universal* or *homogeneous*. Relation algebras with such representations are characterised by certain network properties. These properties are equivalent to infinite axiom schemas but they are conceptually clearer.

Definitions

Let $\mathcal{A}$ be an atomic relation algebra, let X be a representation of $\mathcal{A}$ and let $\mathcal{K}$ be the isomorphism type of any downward closed (i.e. if $N \in \mathcal{K}$ and M is a subnetwork of N then $M \in \mathcal{K}$) class of networks over $\mathcal{A}$. So K could be Net($\mathcal{A}$).

1. A local isomorphism θ of X is a finite partial map from X to itself say $\theta : \overline{x} \to \overline{y}$ ($\overline{x}, \overline{y}$ k-tuples from X) such that $R(x_i, x_j) = R(y_i, y_j)$ where R is considered as the function which identifies the relation which holds on a pair of elements of X. So θ preserves each edge relation in $\overline{x}$.

2. X is homogeneous if every local isomorphism of X extends to a full automorphism of X.

3. The *age* of a representation ($age(X)$) is defined to be the class of all finite atomic networks isomorphic to networks appearing in X. So $N \in age(X)$ iff there is an isomorphism from N into X.
   X is a *universal representation* of $\mathcal{A}$ if every finite, transitively closed, atomic $\mathcal{A}$-network appears somewhere in X. (More generally X is universal over $\mathcal{K}$ if every member of $\mathcal{K}$ appears somewhere in X).

4. $\mathcal{K}$ has the joint embedding property (JEP) if any two networks N and M in $\mathcal{K}$ can be jointly embedded in a network L of $\mathcal{K}$ so that N and M appear as sub-networks of L.

5. $\mathcal{K}$ has the Amalgamation Property (AP) if given any networks L, M_1 and M_2 from $\mathcal{K}$ and embeddings $f_i : L \to M_i$ $(i = 1, 2)$ there is a network $N \in \mathcal{K}$ and embeddings $g_i : M_i \to N$ such that $f_1 g_1 = f_2 g_2$. Less formally, given M_1 and M_2 with a common isomorphic network L, there is a way of gluing M_1 and M_2 together along L and forming a larger network N, still in $\mathcal{K}$

6. $\mathcal{K}$ has the triangle addition property (TAP) if given any network $N \in \mathcal{K}$ and any network of size three T (a triangle) such that N and T have one edge each labelled by the same relation then it is possible to embed N and T into an atomic network $L \in \mathcal{K}$ such that N and T share that same edge in L. TAP is a special case of AP.

Lemma 1 *A transitively closed atomic network N defines a representation of $\mathcal{A}$ iff for every edge (n, m) labelled by a and every pair of atoms b, c with $b * c \geq a$, there is a node $l \in N$ such that (n, l) is labelled b and (l, m) is labelled c.*

PROOF:

Assume that the edge property holds. Define $I : \mathcal{A} \to P(N \times N)$ by $I(a) = \{(n, m) : N(n, m) \leq a\}$. To show that I is an isomorphism it is easy to verify that the boolean operations and converse are preserved. The edge property guarantees that composition is preserved as well.

Theorem 2 *Let $\mathcal{A}$ by an atomic relation algebra with at most countably many atoms and let K be some non-empty, downward closed ($N \in \mathcal{K}, M \subseteq N \to N \in \mathcal{K}$) isomorphism class of finite, atomic, transitively closed A-networks.*

1. *$\mathcal{A}$ is a representable relation algebra [4] if and only if there is some $\mathcal{K}$ with the triangle addition property.*

2. *$\mathcal{A}$ has a representation X with $age(X) = \mathcal{K}$ iff $\mathcal{K}$ has TAP and JEP. $\mathcal{A}$ has a universal representation iff Net($\mathcal{A}$) has TAP and JEP.*

3. *$\mathcal{A}$ has a homogeneous representation, with $age(X) = \mathcal{K}$ iff $\mathcal{K}$ has JEP and AP. $\mathcal{A}$ has a homogeneous, universal representation iff Net($\mathcal{A}$) has JEP and TAP.*

PROOF:

1. $\Rightarrow$ If $\mathcal{A}$ has a representation X, then let $\mathcal{K}$ be the class of all (isomorphic copies of) finite networks which appear in X. Clearly $\mathcal{K}$ is downward closed. Also, let N be any network appearing in X with $N(n, m) = a$ say. If $a \leq b * c$ then it is necessary to show that the triangle T

[4] In this part the existence of a class $\mathcal{K}$ with TAP is equivalent to the infinite set of conditions given in [11]. However the condition on networks is conceptually more clear and this conditions is shown here to be necessary and sufficient for a countable set of atoms, as well as the finite case covered by Lyndon.

can be joined onto the edge (n, m) of N and form a network that still belongs to $\mathcal{K}$. But since X is a representation there must be some point l say with $X(n, l) = b$ and $X(l, m) = c$. So the complete network on $N \cup \{l\}$ is the required extension. Therefore $\mathcal{K}$ has TAP.

$\Leftarrow$

A countable nested sequence of finite networks (N_i) from $\mathcal{K}$ is constructed whose limit is a representation of $\mathcal{A}$. Let N_0 be any network in $\mathcal{K}$. Let the nodes of the N_i be taken from an enumerated sequence of nodes $(n_j)_{j<w}$. If (n_j, n_k) is an edge labelled by the atom a in the network N_i then since $\mathcal{A}$ has only countably many atoms there are only countably many pairs of atoms b, c with $b * c \geq a$. Enumerate the set of tuples (n_j, n_k, a, b, c) where $a \leq b * c$. Now to construct N_{i+1} from N_i pick the smallest tuple (n_j, n_k, a, b, c) such that $(n_j, n_k) \in N_i$ is labelled by a and remove it from the list. Since $\mathcal{K}$ has TAP it is possible to find some extension N_{i+1} of N_i which includes a node n in such a way that (n_j, n) is labelled b and (n, n_k) is labelled c. By lemma 1, the limit of this sequence is a representation of $\mathcal{A}$.

2. (a) $\Rightarrow$

   Suppose $age(X) = \mathcal{K}$. As in part 1, it follows that $\mathcal{K}$ has TAP. Also $\mathcal{K}$ has JEP since if $N, M \in \mathcal{K}$ they must appear in X, say as N_x and M_x. Let L be the network formed by the nodes of $N_x \cup M_x$ with arcs labelled by the atomic relation that holds in X. $L \in \mathcal{K}$.

   $\Leftarrow$

   Let $\mathcal{K}$ have TAP and JEP. As in part 1, a nested sequence of networks are chosen and this time the limit will be a universal representation of $\mathcal{A}$. In the enumeration of 'triangles to be completed' we must also include the list of all networks in $\mathcal{K}$ (each of which must be included in the structure). JEP guarantees that this can be done.

   (b) Let $K = Net(\mathcal{A})$.

3. (a) $\Rightarrow$

   Suppose $\mathcal{A}$ has a homogeneous representation X with $age(X) = \mathcal{K}$. Let $L, M_1, M_2 \in \mathcal{K}$ with embeddings $f_i : L \to M_i$. Let $M_1\theta_1$ and $M_2\theta_2$ be respective appearances of M_1 and M_2 in X (θ_i are isomorphisms from M_i into X). There is a local isomorphism from $L\theta_1$ to

$L\theta_2$ (namely ${\theta_1}^{-1} * \theta_2$). By homogeneity, this extends to a full automorphism α of X. To construct N embedding M_1 and M_2 consider the points $M_1\theta_1\alpha \cup M_2\theta_2$ in X. For each pair of nodes n, m from this set label the arc (n, m) by the atomic relation that actually holds on the two points in X. This labelling agrees with the arcs in M_1 and M_2 and makes them intersect correctly on L. This defines N which appears in X and therefore belongs to $\mathcal{K}$. The case where M_1 and M_2 do not intersect covers the JEP and the case where they do covers AP.

$\Leftarrow$

Suppose $\mathcal{K}$ has JEP and AP. A homogeneous representation X with $age(X) = \mathcal{K}$ is again constructed as the limit of a sequence. This time it is also necessary to ensure that every local isomorphism extends to an automorphism of the network. Suppose a local isomorphism $\theta : \bar{n} \to \bar{m}$ eventually appears in N. We wish to extend θ to an automorphism of the network. So take any $n \in N$, we must extend θ and N so that $n\theta$ and $n\theta^{-1}$ are defined and θ is still a local isomorphism. To define $n\theta$, take a copy N' of N and identify the tuple $\bar{n'}$ of N' with $\bar{m}$ of N (this can be done since θ preserves all edge relations from $\bar{n}$ to $\bar{m}$). The amalgamation property allows us to extend to a structure in $\mathcal{K}$ containing N and N'. Let n' be the element of N' which corresponds to n and set $n\theta = n'$. Similarly we can extend θ and N so that $n\theta^{-1}$ is defined. Now θ needs to be extended to only a countable number of nodes and there can only be a countable number of local isomorphisms which appear at any stage of the construction. Therefore it is possible to include all of these in the enumeration of 'tasks to be done'. In the limit, every local isomorphism extends to a full automorphism and therefore the limit is a homogeneous representation of $\mathcal{A}$.

(b) Again, let $\mathcal{K} = Net(\mathcal{A})$.

Question

The previous theorem exactly characterises the RAs with homogeneous, universal representations. Can it be proved that no finite set of axioms is capable of doing this?

3.5 Allen's Point Algebra and the Metric Point System have the Amalgamation and Joint Embedding Properties

It is quite straightforward to show that $\mathcal{P}$ and $\mathcal{M}$ have AP and JEP directly, but it will be useful for the section on metric interval algebra if instead we use the extension property (defined below) to show that both systems have a universal, homogeneous representation and then use theorem 2 to prove that they have AP and JEP.

Theorem 3 *Any non-singular RA $\mathcal{A}$ has a universal, homogeneous representation.*

PROOF:

This was first proved in [4] (theorem 3.3). Here, we use the so-called regular representation which is defined like this. The domain of the representation is the set of atoms of $\mathcal{A}$ itself. Each relation a of $\mathcal{A}$ is interpreted as a binary relation on $atoms_{\mathcal{A}}$ by

$$I(a) = \{(a_1, a_2) \in atoms_{\mathcal{A}} : a_1^{\smile} * a_2 \leq_{\mathcal{A}} a\}$$

(Since $\mathcal{A}$ is non-singular $a_1^{\smile} * a_2$ is itself an atom of $\mathcal{A}$, so every pair of elements of the domain are related by exactly one atomic element of $\mathcal{A}$.) It is a routine exercise to check that this does give a universal, homogeneous representation of $\mathcal{A}$.

Corollary 4 *The metric point system $\mathcal{M}$ has such a representation. This representation turns out to be $\mathbb{Q}$.*

Definition

Let $\mathcal{A}$ be a representable relation algebra and let N be a (not necessarily atomic) $\mathcal{A}$-network. A model of N is a homomorphism I from N into some representation X of $\mathcal{A}$. So for all $n, m \in N$ the relation $N(n, m)$ must hold between the pair $I(n), I(m)$. A (not necessarily atomic) finite network N has the *extension property* [5] if for any subnetwork M of N every model of M extends to a model of N.

Definition

A *simple* M-network is a transitively closed M-network with a single interval on each arc. (In the terminology of [4] this is called an STP.)

Theorem 5 *Any (transitively closed) simple M-network has the extension property.*

PROOF:

Let N be such a network and let M be any subnetwork. Let I map the nodes of M to Q so that $I(m_2) - I(m_1) \in M(m_1, m_2)$. We must show that I can be extended to all the nodes of N. So, let $n \in N - M$. In order to be consistent with the relation $N(m, n)$, n must be assigned a value from $S_m = I(m) + N(m, n)$. So the problem reduces to finding a value which belongs to S_m for all $m \in M$, in other words we must show that $\bigcap_{m \in M} S_m$ is non-empty. Now, each pair S_m, S_{m*} intersect, otherwise the triangle $(m, m*, n)$ would not be transitively closed. So $\{S_m : m \in M\}$ is a finite set of convex sets of rationals which

[5] In [4] this property is called *decomposability.*

intersect pairwise. It is not hard to show by induction on $|M|$ (using convexity and linearity) that such a set must have a common intersection, and this allows us to assign a value to n and in a similar way, all the remaining nodes in N.

Corollary 6 *The point algebra $\mathcal{P}$ has a universal, homogeneous representation.*

PROOF:

The domain of the representation will be the regular representation of M, namely $\mathbb{Q}$.

Universality. Let N be an atomic $\mathcal{P}$-network. Translate N into a simple M-network by replacing '$=$' by $[0,0]$, '$<$' by $(0,\infty)$ and '$>$' by $(-\infty,0)$. The previous theorem allows us to find a model of this network in $\mathbb{Q}$.

Homogeneity. Let $\bar{x}$ and $\bar{y}$ be locally isomorphic tuples in $\mathbb{Q}$. A full automorphism is constructed by a back and forth argument, each step using the extension property.

Corollary 7 *Both $Net(\mathcal{P})$ and $Net(\mathcal{M})$ have AP and JEP.*

Corollary 8 *Any (transitively closed, not necessarily atomic) $\mathcal{P}$-network is nearly minimal i.e. for each arc and each relation other than equality on that arc there is a model of the network which realises that relation. (Note: this is weaker than the extension property, which does not hold for general $\mathcal{P}$-networks). Hence, any transitively closed $\mathcal{P}$-network is satisfiable.*

PROOF:

The following sketch is an alternative to the original proof in [3]. First note that if any of the arcs has only the atomic relation '$=$' on it we may reduce the network by 'collapsing' the nodes together. So assume every arc has at least one of '$<$' and '$>$' on it. The proof is similar to the proof of theorem 5. Here we assign values to a node n of N, but ensuring that no two nodes are assigned the same value. As before it is necessary to show that $\bigcap_{m\in M} S_m$ is non-empty. But here the S_m may not be convex (consider $\{<,>\}$). Nevertheless, each S_m can be turned into a convex set S^*_m by adding at most a single point. Therefore $\bigcap_{m\in M} S^*_m$ is non-empty and it is not hard to see that it is, in fact, an infinite intersection. So removing the finite number of points added to the S_m will still leave an infinite intersection. Therefore it is possible to assign n a value, moreover this can be chosen to be different to all the previous values.

4 Intervals from Points

In this section we devise a construction which builds an 'interval' algebra from a 'point' algebra. The 'point' algebras which are suitable are those with homogeneous, universal representations - far more general than the usual three atomic relations on a linearly ordered set. Let $\mathcal{A}$ be an atomic relation algebra with a universal homogeneous representation X. Fix one atomic relation r from $\mathcal{A}$ (principal example: $<$ in $\mathcal{P}$).

Definitions

1. The set of r-intervals in X, denoted X_r^2 is the set of ordered pairs (x, y) such that $(x, y) \in r$. Let α and β be r-intervals in X, respectively (α_1, α_2) and (β_1, β_2).

2. Define the atomic interval relation $I(\alpha, \beta)$ between a and b as the 2 by 2 matrix $\begin{bmatrix} R(\alpha_1, \beta_1) & R(\alpha_1, \beta_2) \\ R(\alpha_2, \beta_1) & R(\alpha_2, \beta_2) \end{bmatrix}$ where $R(a_i, b_j)$ is the atomic relation which holds in X between a_i and b_j $(i, j = 1, 2)$.

3. The set of atomic interval relations over $\mathcal{A}$ is the set of all atomic 2 by 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$\begin{aligned} a * c^{\smile} &\geq r \\ b * d^{\smile} &\geq r \\ a^{\smile} * b &\geq r \\ c^{\smile} * d &\geq r. \end{aligned}$$

   In other words the network below must be transitively closed.

4. We may use the same matrix notation as a shorthand for non-atomic interval relations so $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be $\bigcup_{atoms \alpha \leq a, \beta \leq b, \gamma \leq c, \delta \leq d} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. The set of all finite unions of atomic interval relations is denoted by $\mathcal{A}_r^2$. Now given any two r-intervals there is exactly one atomic interval relation which holds between them. The boolean operations pose no problem but it is not automatic that the composition of two atomic interval relations will be a union of atomic interval relations.

Theorem 9 *Let $\mathcal{A}$ be a universal, homogeneous, atomic relation algebra and let X be any universal, homogeneous representation of A.*

1. *$\mathcal{A}_r^2$ forms a relation algebra. The identity relation is $\begin{bmatrix} 1_{\mathcal{A}} & r \\ r^{\smile} & 1_{\mathcal{A}} \end{bmatrix}$ and the inverse*

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{\smile} = \begin{bmatrix} a_{11}{}^{\smile} & a_{12}{}^{\smile} \\ a_{21}{}^{\smile} & a_{22}{}^{\smile} \end{bmatrix}.$$

The composition of any two atomic interval relations is calculated thus:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} * \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} * b_{11} \cap a_{12} * b_{21} & a_{11} * b_{12} \cap a_{12} * b_{22} \\ a_{21} * b_{11} \cap a_{22} * b_{21} & a_{21} * b_{12} \cap a_{22} * b_{22} \end{bmatrix}.$$

in other words, ordinary matrix multiplication with $\cap, *$ in place of $+, \times$ respectively.

2. *X_r^2 forms a universal, homogeneous representation of $\mathcal{A}_r^2$.*

PROOF:

1. There is a potential problem because there is no distribution law for intersection ($a * (b \cap c) = a * b \cap a * c$ is not generally true in relation algebras) and consequently the matrix multiplication may not be associative. But for the restricted case of r-interval relations over a universal, homogeneous RA the matrix multiplication turns out to be associative. We show this by proving that the matrix product $R * S$ calculates the composition of the interval relations R and S correctly. This will also prove that the compositions of two interval relations is a union of atomic interval relations. To do this we need a lemma.

Lemma 10 *Let $\mathcal{A}$ be a homogeneous, universal, atomic relation algebra, let R and S be the matrices of atomic interval relations and let α and β be any r-intervals. Then $(\alpha, \beta) \in R * S$ (the matrix product) iff there is some r-interval γ such that $(\alpha, \gamma) \in R$ and $(\gamma, \beta) \in S$.*

Proof of Lemma

$\Leftarrow$

Suppose $(\alpha, \gamma) \in R$ and $(\gamma, \beta) \in S$. $(\alpha, \gamma) \in R$ means that $(\alpha_i, \gamma_j) \in R_{ij}(i, j = 1, 2)$. Similarly $(\gamma_j, \beta_k) \in S_{jk}(j, k = 1, 2)$. Considering the triangle $(\alpha_i, \gamma_j, \beta_k)$ in the representation X, it must be the case that $(\alpha_i, \beta_k) \in R_{ij} * S_{jk}(i, j, k = 1, 2)$.

Therefore $(\alpha_i, \beta_k) \in R_{ij} * S_{1k} \cap R_{i2} * S_{2k}$ which is the entry calculated by the matrix product. Hence $(\alpha, \beta) \in R * S$.

$\Rightarrow$

Suppose; $(\alpha, \beta) \in R * S$. Let the atomic interval relation between a and b be $T (T \subseteq R * S)$. Since R is an interval relation, the network is consistent. By considering each triangle, the following network is transitively closed.

Universality guarantees that this network occurs in the representation and homogeneity ensures that the local isomorphism from this occurrence to

extends to a full automorphism of the representation. Therefore there is an r-interval $\gamma = (\gamma_1, \gamma_2)$ such that $(\alpha, \gamma) \in R$ and $(\gamma, \beta) \in S$ as required.

Proof of Theorem (continued).

Composition of binary relations is always associative, so by the lemma, matrix multiplication in $\mathcal{A}_r^2$ is associative. The other axioms are straightforward to check.

2. We have already seen that X_r^2 is a representation of $\mathcal{A}_r^2$. It is easy to show that it is universal by

(a) taking any interval network
(b) converting to a network of points
(c) using the universality of X to find the network in X and
(d) convert back to X_r^2 thus showing that the original network appears in X_r^2.

Homogeneity is handled in exactly the same way.

Corollary 11 *1. The Allen Interval Algebra can be constructed from the point algebra $\mathcal{P}$ and has a universal, homogeneous representation as ordered pairs of rationals. It is isomorphic to the relation algebra with atomic relations the set of two by two matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with elements $<, >$ and $=$ such that $a * c^{\smile} \geq$ '$<$' ; $b * d^{\smile} \geq$ '$<$' etc. The atomic interval relations and their corresponding matrices are*

$$
\begin{array}{ll}
\text{'}equals\text{'} & \begin{bmatrix} = & < \\ > & = \end{bmatrix}; \\
\text{'}before\text{'} & \begin{bmatrix} < & < \\ < & < \end{bmatrix}; \\
\text{'}meets\text{'} & \begin{bmatrix} < & < \\ = & < \end{bmatrix}; \\
\text{'}overlaps\text{'} & \begin{bmatrix} < & < \\ > & < \end{bmatrix}; \\
\text{'}starts\text{'} & \begin{bmatrix} = & < \\ > & < \end{bmatrix}; \\
\text{'}during\text{'} & \begin{bmatrix} > & < \\ > & < \end{bmatrix}; \\
\text{'}ends\text{'} & \begin{bmatrix} > & < \\ > & = \end{bmatrix}
\end{array}
$$

plus the converses of the last six.

2. It is possible to take the interval algebra, fix any one atomic relation say 'overlaps' and then define a relation algebra of 'intervals of intervals'. Here an interval will be any pair of intervals i, j such that i overlaps j.

5 Intervals with metrics

Since the metric point system $\mathcal{M}$ has a homogeneous, universal representation (corollary 4) the construction of theorem 9 produces a relation algebra of intervals with metrics.

But this is a rather uninteresting algebra of intervals as an interval here is defined by a single, fixed atomic relation. That means that all intervals have to be of the same size, an over-restricted definition. An interval is more usually considered as any pair of points with the first one less than the second. In order to deal with these it is necessary to consider non-atomic networks. Recall that the regular representation of M is $\mathbb{Q}$.

5.1 Definition of $\mathcal{M}^2$

1. An *interval* is a pair of rationals (p, q) such that $p < q$.

2. An *elementary metric interval relation* R is a two by two matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C and D are intervals of rationals such that the $\mathcal{M}$-network

   is transitively closed. $(p^-, p^+)R(q^-, q^+)$ asserts that $q^- - p^- \in A, q^+ - p^- \in B$ etc.

3. Such relations are composed according to the rule

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} * \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} (A+E) \cap (B+G) & (A+F) \cap (B+H) \\ (C+E) \cap (D+G) & (C+F) \cap (D+H) \end{bmatrix}$$

   in other words ordinary matrix multiplication with addition of intervals and intersection instead of multiplication and addition respectively.

4. The identity is $\begin{bmatrix} [0,0] & (0,\infty) \\ (-\infty,0) & [0,0] \end{bmatrix}$ (note that this is not atomic) and the converse of a relation $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is $\begin{bmatrix} -A & -C \\ -B & -D \end{bmatrix}$.

5. More general metric interval relations can be formed as disjuncts of elementary ones. The compliment of an elementary relation will typically be a disjunct. Non-elementary matrices can be introduced e.g. $R \vee S$. The product

$$(R \vee S).(T \vee U) = RT \vee RU \vee ST \vee SU.$$

Thus the disjunct $i\{<,>\}j$ can be expressed as

$$i\begin{bmatrix} (-\infty,0) & (-\infty,0) \\ (-\infty,0) & (-\infty,0) \end{bmatrix} \vee \begin{bmatrix} (0,\infty) & (0,\infty) \\ (0,\infty) & (0,\infty) \end{bmatrix} j.$$

Note this could not be expressed as a network in the point algebras $\mathcal{P}$ or $\mathcal{M}$.

6. The efficiency will be enhanced if a disjunction is reduced by the rule: $M \vee N \Rightarrow N$ if $M \subseteq N$ (where $M \subseteq N$ if each of the four entries of M is a subset of the corresponding entry of N). Also for matrices with three of the four entries equal

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \vee \begin{bmatrix} A & B \\ C & D^* \end{bmatrix} \Rightarrow \begin{bmatrix} A & B \\ C & D \vee D^* \end{bmatrix}$$

provided $D \vee D^*$ is an interval.

As before, it is necessary to check that composition is associative and this is done by showing that matrix product is isomorphic to composition of relations. The critical section of the proof takes two intervals α and β related by the matrix product $R * S$. It is required to show that there exists a third interval γ such that α and γ are related by R and γ and β are related by S. But this follows from the fact that $\mathcal{M}$ has the extension property (theorem 5).

Expressive Power

This system is capable of expressing all of Allen's interval relations e.g. 'overlaps' is written as $\begin{bmatrix} (0,\infty) & (0,\infty) \\ (-\infty,0) & (0,\infty) \end{bmatrix}$. A constraint on the duration of i can be expressed using this format as $i\begin{bmatrix} [0,0] & [d,e] \\ [-e,-d] & [0,0] \end{bmatrix} i$ where d and e are respectively lower and upper bounds on the duration of i. Thus, the qualitative expressive power of Allen's system is combined with the quantitative power of the metric system $\mathcal{M}$ of Dechter, Meiri and Pearl.

5.2 Complexity

It is possible to use any of the algorithms from the literature in conjunction with this language, but for the present suppose we use a fixpoint algorithm to calculate the transitive closure of a network

$$Propagate(R(i,j),N) :-$$

$$Makequeue(R(i,j),Q),$$

$$Process-queue(Q) \text{ till empty.}$$

and define

$$Process-queue(R(i,j) : Rest-of-queue) :-$$

$$\forall k \in N[\text{ let } N'(i,k) := N(i,k) \cap R(i,j) * N(j,k)$$

$$\text{and if } N'(i,k) \neq N(i,k) \text{ then add it to the queue}$$

$$N(i,k) := N'(i,k)],$$

$$Process - queue(Rest - of - queue).$$

together with a similar calculation for $N(k,j)$. This is equivalent to the Allen propagation algorithm.

$\mathcal{M}^2$ is a highly expressive language and the worst case complexity of checking the consistency of a network will be at least as bad as its two sublanguages $\mathcal{M}$ and $\mathcal{A}$, i.e. it is NP-hard. In fact, a non-deterministic Turing machine could solve the problem in polynomial time since the non-disjunctive case can be solved in cubic time (below) so consistency checking for $\mathcal{M}^2$ is NP-complete.

But if we restrict to certain fragments of the full language we obtain the following results.

- A network with only elementary metric interval relations on the arcs (no disjuncts) can be checked in cubic time. This follows from the proof in [4] that computing the transitive closure of an $\mathcal{M}$-network with only one interval on each arc (in their terminology an STP), can be done in cubic time, and computes the minimal network.

- If all the relations on the arcs of the network are *pure Allen relations* i.e. equivalent to a union of some of the thirteen primitive interval relations, then the matrix product (which is calculated in constant time) will produce the same result as the Allen transitivity table. Therefore the same complexity and completeness results will hold i.e. consistency checking is NP-complete, but the Allen propagation algorithm provides a useful approximation in cubic time.

- For general metric constraints with disjunctions, the problem is NP-complete. Dechter, Meiri and Pearl left open the problem of whether the fixpoint algorithm must terminate at all. If the metric values are comensurate (the ratios are rational) then without loss it may be assumed that all the metric constraints have integer bounds. In this case the fixpoint algorithm will certainly terminate and if the number of integers lying in any constraint has a fixed bound then the algorithm will terminate in cubic time. The arguement is exactly the same when analysing $\mathcal{M}^2$ except the bound must apply to the number of atomic matrices with integer entries, within the constraint.

  The remaining problem is that calculating the fixpoint (transitive closure) is not a *complete* deductive mechanism. It is easy to devise inconsistent but transitively closed networks in $\mathcal{M}$ and hence in $\mathcal{M}^2$. However, computing the transitive closure may give a useful first approximation for a consistency checker which then proceeds by brute force to test each choice of disjunctions for consistency. (The constraint technology will clearly be useful here.)

Examples

1. From $i\,\{d\}\,j$ and $j\,\{o\}\,k$ we should deduce that $i\,\{<, m, o, s, d\}\,k$.
$i \begin{bmatrix} (-\infty,0) & (0,\infty) \\ (-\infty,0) & ((0,\infty) \end{bmatrix} j$ and $j \begin{bmatrix} (0,\infty) & (0,\infty) \\ (-\infty,0) & (0,\infty) \end{bmatrix} k$ gives
$i \begin{bmatrix} (-\infty,0) & (0,\infty) \\ (-\infty,0) & ((0,\infty) \end{bmatrix} \begin{bmatrix} (0,\infty) & (0,\infty) \\ (-\infty,0) & (0,\infty) \end{bmatrix} k$ or
$i \begin{bmatrix} (-\infty,\infty) \cap (-\infty,\infty) & (-\infty,\infty) \cap (0,\infty) \\ (-\infty,\infty) \cap (-\infty,\infty) & (-\infty,\infty) \cap (0,\infty) \end{bmatrix} k$ i.e.
$i \begin{bmatrix} (-\infty,\infty) & (0,\infty) \\ (-\infty,\infty) & (0,\infty) \end{bmatrix} k$ which says that both endpoints of i must lie before the end of k, as required.

2. Let j start at least 5 second after i finishes and let j finish less than 10 seconds after i starts.

It should be possible to deduce that the duration of i is less than 5 seconds. Well, $i \begin{bmatrix} (0,\infty) & (0,10) \\ (5,\infty) & (5,\infty) \end{bmatrix} j$ and $j \begin{bmatrix} (-\infty,0) & (-\infty,-5) \\ (-10,0) & (-\infty,-5) \end{bmatrix} i$ so $i \begin{bmatrix} (-10,10) & (-\infty,5) \\ (-5,\infty) & (-\infty,\infty) \end{bmatrix} i$.
Intersecting this with the initial constraint on i, gives $i \begin{bmatrix} [0,0] & (0,5) \\ (-5,0) & (0,0) \end{bmatrix} i$ as required.

5.3 Comparisons

A number of other attempts have been made to combine qualitative and quantitative reasoning ([7], [8], [10], [13]). The language $\mathcal{M}^2$ of this paper has two main advantages. Firstly, it uses the same uniform representation for all relations. There is no need to refer to a special table when dealing with an interval constraint and a seperate table for metrics. All constraints are represented as matrices and compositions are calculated by matrix multiplication. By contrast, [7], [8] and [13] are all essentially hybrid systems which handle metric and interval information seperately and translate from one to the other.

The other advantage of $\mathcal{M}^2$ is its expressive power. When disjunctive relations are allowed it is possible to express constraints which are neither point-based metric nor qualitative interval relations. For example to assert that interval i either starts more than

5 seconds after interval j ends or ends more than 10 seconds before j ends, we use the disjunction

$$\begin{bmatrix} (-\infty,5) & (-\infty,5) \\ (-\infty,5) & (-\infty,5) \end{bmatrix} \vee \begin{bmatrix} (-\infty,\infty) & (10,\infty) \\ (-\infty,10) & (-\infty,10) \end{bmatrix}.$$

Note that this could not be represented directly in any of the competing systems. It would be necessary to construct additional intervals and put constraints on these. This expressive power is achieved without additional complexity cost (the complexity of checking the consistency of a network in a sub-language of $\mathcal{M}^2$ is the same as that in competing systems).

More tentatively, there is one further advantage. When dealing with the disjunctive case, the simple algorithm for combining matching disjuncts (see section 5, Definition of $\mathcal{M}^2$, 6) is very straightforward and will improve the efficiency considerbly. Disjuncts like $before, meets$ in Allen's language translate to

$$\begin{bmatrix} (0,\infty) & (0,\infty) \\ (0,\infty) & (0,\infty) \end{bmatrix} \vee \begin{bmatrix} (0,\infty) & (0,\infty) \\ `[0,0]' & (0,\infty) \end{bmatrix}.$$

which gets rewritten as $\begin{bmatrix} (0,\infty) & (0,\infty) \\ `[0,\infty)' & (0,\infty) \end{bmatrix}$ thus eliminating a disjunct which could improve the efficiency. Theoretical results about average case performance are hard to provide in this area so this is most likely to be judged, eventually, by empirical results.

6 Conclusion

An interval relation algebra can be constructed from a point relation algebra provided it is universal and homogeneous. This allows us to construct a metric interval algebra from the metric system $\mathcal{M}$. This representation permits the expressing of Allen type disjuncts like $i\{<,>\}j$. It is thus more expressive than other systems that allow quantitative, metric information. As with these systems [4] the propagation algorithm will be complete and of cubic complexity if there are no disjuncts buts its performance in general is intractable.

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