This paper is a slightly revised version of a paper that was presented in the workshop on Spatial and Temporal Reasoning, held at ECAI94, the 11th European Conference on Artificial Intelligence.

Some Observations and Puzzles about Composing Spatial and Temporal Relations

Brandon Bennett
Division of Artificial Intelligence
School of Computer Studies
University of Leeds, Leeds LS2 9JT, England
brandon@scs.leeds.ac.uk

September 24, 1994

1 Introduction

Since Allen's (1983) work on reasoning about temporal intervals, information about the *composition*[1] of relations has been exploited for automated reasoning in a variety of theories. In particular, such compositions have been used in reasoning about spatial relationships (Cui, Cohn and Randell 1992), (Egenhofer 1991).

Nevertheless, relational composition is by no means fully understood. Many questions regarding the effective generation and use of composition tables and the conditions under which they can be used to provide a *complete* inference mechanism remain unanswered (or even unasked). In this paper I assemble a number of observations and point out some questions which I believe to be important. This is an area of research that is potentially very significant for AI because it is possible that composition tables — perhaps in conjunction with other forms of *compiled* logical information — may provide the key to effective reasoning in seemingly intractable theories.

The composition of two binary relations can be defined as follows:

> Given a theory Θ in which a set $\mathcal{B}$ of mutually exhaustive and pairwise disjoint dyadic relations (a *basis* set) is defined, the composition, $Comp(R_1, R_2)$, of two relations R_1 and R_2 taken from $\mathcal{B}$ is defined to be: the disjunction of all relations R_3 in $\mathcal{B}$, such that, for arbitrary individual constants a, b, c, the formula $R_1(a,b) \wedge R_2(b,c) \wedge R_3(a,c)$ is consistent with Θ.[2]

This means that if $Comp(R_1, R_2) = R_3^1 \vee \ldots \vee R_3^n$ then

$$\forall x \forall y [R_1(x,y) \wedge R_2(y,z) \rightarrow (R_3^1(x,z) \vee \ldots \vee R_3^n(x,z)$$

and, furthermore, that $Comp(R_1, R_2)$ is the smallest subset of $\mathcal{B}$ for which such a formula is provable.

In other words $Comp(R_1, R_2)$ is the disjunction of all possible base relations which could hold between a and c. Thus, if one has a consistency checking algorithm for sets of instances of the relations in $\mathcal{B}$,

[1] Unfortunately, the terminology used to describe relational composition has not been consistent. Allen referred to compositions as "transitivity relations" and in Randell, Cohn and Cui (1992) they were called "transitive closures". However, these terms already have meanings different to what is intended, so the term "composition" (as used e.g. in Freksa (1992)) seems preferable.

[2] This is not the only definition found in the literature but it is equivalent to the definition given in Randell, Cohn and Cui (1992) and it also seems to correspond with what Egenhofer (1991) means by the composition of two relations. I shall later discuss another (stronger) definition.

R2(b,c) / R1(a,b)	DC	EC	PO	TPP	NTPP	TPP^{-1}	$NTPP^{-1}$	=
DC	no.info	DR,PO,PP	DR,PO,PP	DR,PO,PP	DR,PO,PP	DC	DC	DC
EC	DR,PO,PP^{-1}	DR,PO TPP,TP^{-1}	DR,PO,PP	EC,PO,PP	PO,PP	DR	DC	EC
PO	DR,PO,PP^{-1}	DR,PO,PP^{-1}	no.info	PO,PP	PO,PP	DR,PO,PP^{-1}	DR,PO PP^{-1}	PO
TPP	DC	DR	DR,PO,PP	PP	NTPP	DR,PO TPP,TP^{-1}	DR,PO PP^{-1}	TPP
NTPP	DC	DC	DR,PO,PP	NTPP	NTPP	DR,PO,PP	no.info	NTPP
TPP^{-1}	DR,PO,PP^{-1}	EC,PO,PP^{-1}	PO,PP^{-1}	PO,TPP,TP^{-1}	PO,PP	PP^{-1}	$NTPP^{-1}$	TPP^{-1}
$NTPP^{-1}$	DR,PO,PP^{-1}	PO,PP^{-1}	PO,PP^{-1}	PO,PP^{-1}	O	$NTPP^{-1}$	$NTPP^{-1}$	$NTPP^{-1}$
=	DC	EC	PO	TPP	NTPP	TPP^{-1}	$NTPP^{-1}$	=

Table 1: Composition table for 8 basic relations definable in the RCC calculus. If R_1(a,b) and R_2(b,c), it follows that R_3(a,c) where R_3 is looked up in the table. "no info." means that no base relation is excluded. Multiple entries in a cell are interpreted as disjunctions. Note that DR stands for DC and EC, PP for TPP and NTPP, PP^{-1} for TPP^{-1} and $NTPP^{-1}$, TP^{-1} for TPP^{-1} and =, and O for PO, TPP, NTPP, TPP^{-1}, $NTPP^{-1}$, and =.

the composition of any pair of relations taken from $\mathcal{B}$ can be straightforwardly computed. Given R_1 and R_2 ($\in\mathcal{B}$), one simply checks for all values of R_3 taken from $\mathcal{B}$, whether the situation described by $R_1(a,b)$, $R_2(b,c)$, $R_3(a,c)$ is possible. Thus if $\mathcal{B}$ consists of n relations then computing the of composition of a pair of relations in $\mathcal{B}$ will require n consistency checks.

Relational compositions provide for a very simple mode of inference: from $R_1(a,b)$ and $R_2(b,c)$ we can immediately deduce $Comp(R_1, R_2)(a,c)$. (The deductive power of this kind of inference will be discussed in section 5.) If such inferences are to be used frequently in some automated reasoning system, it is obviously a good idea to precompute all compositions in one or more sets of base relations that one wants to reason about. A *composition table* gives compositions of all pairs of relations in some basis; and is a central component of any composition-based reasoning system. Table 1 shows the composition table for eight basic topological relations defined in the calculus of Randell, Cui and Cohn (1992).

Relational composition (as defined above) can be regarded as a special case of the more general notion of the *locus of an unspecified relation.* If we have some theory and then specify some relations holding among a collection of objects, whenever the relation between some pair (or n-tuple) of objects is not given (or is only partially specified), we may enquire as to possible values of that relation in the context of the theory and the given facts. This set of values is the *locus* of that unspecified relation. I suggest that this mode of reasoning is very significant part of our assessment of partial information. However, it has not received much explicit attention form the AI community. Nevertheless, investigation of relational compositions (which may be called *loci of composition*) is a first step towards understanding more general forms of reasoning about unspecified relations.

2 Symmetry

Different types of reasoning will make use of different sets of relations depending upon what kind of information is relevant and what distinctions need to be made. Furthermore, for certain problems only very coarse distinctions need be made whereas in others fine discrimination is necessary. In view of these facts it is likely that a flexible architecture for reasoning based on composition tables will want to dynamically construct tables tailored to the requirements of a particular situation. This being so it is clearly useful to be able to construct these tables as efficiently as possible.

If a basis set contains n relations, then there will be n^2 table entries and if computing each entry requires making n consistency checks then the total number of consistency checks required to construct the table will be n^3.

However, a consideration of the structure of a composition table will reveal that it contains a large amount of redundant information. Hence much of the work done in consistency checking to compute such a table is also redundant. One sort of redundancy occurs because, if we compute each cell of a composition table separately, we end up checking the consistency of identical sets of relations several times. Further redundancy is introduced by the fact that any relation can be written in two ways: by inverting the relation and swapping the order of the arguments.

Clearly a composition table can be constructed very easily once we know the set of consistent triangular configurations of relations drawn from the basis set under consideration. Furthermore, once we have determined whether a triangle is consistent, we have already determined the consistency of the essentially equivalent triangles obtained by rotating the original or inverting each of its relations. The exact number of triangles equivalent to a given triangle depends upon the distribution of symmetric and asymmetric relations and whether it contains duplicate relations.

The question which I now address is: How many essentially distinct triangles can be formed from s symmetric and a asymmetric relations?

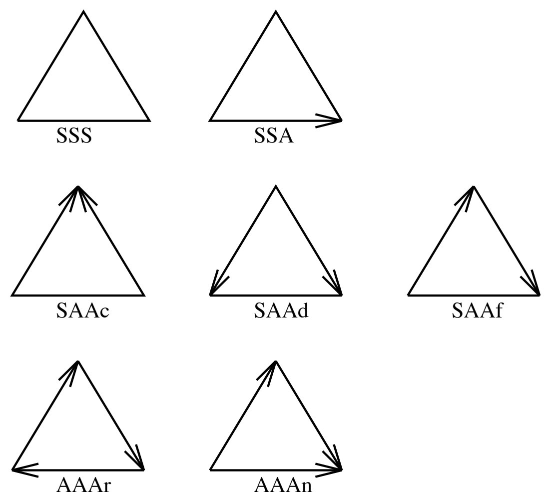

Figure 1: Possible Configurations of Symmetric and Asymmetric Relations

I consider an arbitrary set of relations consisting of s symmetric relations, a asymmetric relations and a further asymmetric relations which are their converses. Figure 1 shows all possible configurations of symmetric and asymmetric relations in a triangle, modulo rotation and flipping. The capital letters S and A stand for 'symmetric' and 'asymmetric' and indicate the numbers of each type of relation present in the triangle. The small letters 'c', 'd' and 'f', stand for 'converging', 'diverging' and 'following', which describe the different ways in which two asymmetric relations can be arranged. 'r' and 'n' denote rotating and not rotating configurations of three asymmetric relations.

To calculate the total number of essentially different triangles, the numbers of possible instantiations of each of these configurations were calculated case by case. After some manipulation, the following polynomial giving the total number, T of essentially distinct triangles in terms of s and a was arrived at:

$$T = \frac{1}{6}(s^3 + 3s^2 + 2s) + s^2a + s(2a^2 + a) + \frac{1}{3}(4a^3 + 2a)$$

We also know that the total number n of relations in a theory is equal to $s + 2a$, so $s = n - 2a$. By substituting $n - 2a$ for s in the polynomial we end up with a simpler equation primarily involving n:

$$T = \frac{1}{6}(n^3 + 3n^2 + 2n) - na$$

As the number of relations increases, the n^3 terms of the (second) equation will dominate. Thus for large n the number of distinct triangles will approach $n^3/6$.

The following table shows values of s, a, n^3, and T for a number of theories for which composition tables have been constructed. RCC-8 is the basis of eight topological relations defined in Randell, Cui and Cohn (1992). RCC-23 is a basis of spatial relations involving containment whose definition is discussed in Cohn, Randell, Cui and Bennett (1993) (the complete composition table is given in Bennett (1994)). IC-13 is Allen's (1983) temporal interval calculus; and LOS-14 is Galton's (1993) Line of Sight calculus. The final column gives T as a percentage of n^3.

Basis Set	s	a	n^3	T	%
RCC-8	4	2	512	104	20.3
RCC-23	7	8	21167	2116	17.4
IC-13	1	6	2197	377	16.8
LOS-14	2	6	2744	476	17.3

Hence, by looking at relational compositions as being charaterised by a set of consistent triangles rather than by a table and by taking advantage of rotational and mirror symmetry exhibited by these triangles, the computational work needed to determine the compositions of a set of relations can be reduced to approximately one sixth of what would be required using the naive, table-based approach.

3 Existential Import

As discussed above, relational compositions have been regarded as of importance principally because of their potential role in consistency checking (and hence automated reasoning in general). In this section I want to draw attention to another aspect of relational composition which is not so widely recognised by the AI community. I shall remark upon what may be called the *existential import* of relational compositions. Although this aspect of composing relations has been considered by some researches in temporal reasoning (in particular Ladkin (1987) and Ladkin and Maddux (1988)), as far as I know it has not been considered at all in spatial reasoning.

First, consider a 'rival' definition of the composition of two relations which is standard in set theory (and is central to Ladkin's (1987) (1988) analysis of Allen's (1983) interval calculus):

> Let R_1 be a relation from A to B and R_2 be a relation from B to C (i.e. A, B and C are sets, $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$), Then the *composition* of R_1 with R_2, $R_1 \circ R_2$ is the set of all ordered pairs, $\langle a, c\rangle \in A \times C$ such that, for some $b \in B$, $\langle a, b\rangle \in R_1$ and $\langle b, c\rangle \in R_2$.

This purely *extensional* definition seems rather different form the one given above. It is in fact strictly stronger that the previous definition because not only does it ensure that whenever we have $R_1(a, b)$ and $R_2(b, c)$ we must also have $Comp(R_1, R_2)(a, c)$ but it also requires that whenever $Comp(R_1, R_2)(a, c)$ (i.e. a and b are related by any one of the base relations making up the (generally) disjunctive relation $Comp(R_1, R_2)(a, c)$ then there mush exists some regions, say b, such that $R_1(a, b)$ and $R_2(b, c)$.

But consider, for example, the composition table entry given in Table 1 for the relations EC and TPP. This tells us that for any two regions, a and b, if there is some third region c such that $\mathsf{EC}(a, c)$ and $\mathsf{TPP}(c, b)$, then the relation between a and c must be either EC,PO or PP (PP means TPP or NTPP). If we generalise and formalise this fact in 1st-order logic we get the following formula:

$$\forall x \forall y[\ \exists z[\mathsf{EC}(x, z) \wedge \mathsf{TPP}(z, x)] \rightarrow (\mathsf{EC}(x, y) \vee \mathsf{PO}(x, y) \vee \mathsf{TPP}(x, y) \vee \mathsf{NTPP}(x, y))]$$

Although I have not done an exhaustive analysis it seems that such conditionals, arising from Table 1 can always be strengthened to bi-conditionals. That is, in this case, if we know that

$$(\mathsf{EC}(x, y) \vee \mathsf{PO}(x, y) \vee \mathsf{TPP}(x, y) \vee \mathsf{NTPP}(x, y))$$

we can infer that

$$\exists z[\mathsf{EC}(x,z) \wedge \mathsf{TPP}(z,x)].$$

This is illustrated in Figure 2 in which we see that if a and b are related in any one of these four ways, then there must exist a region c such that $\mathsf{EC}(a,c)$ and $\mathsf{TPP}(c,b)$.[3]

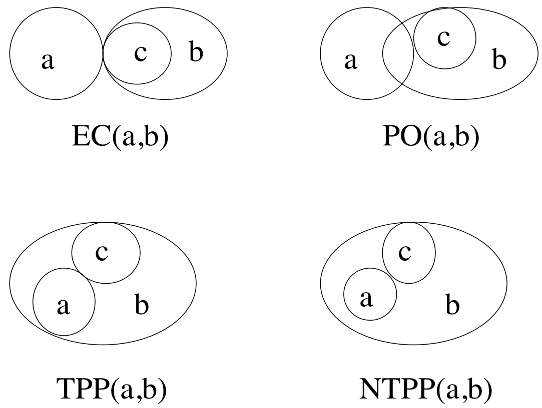

Figure 2: Relations between a and b where there is a region c such that $\mathsf{EC}(a,c)$ and $\mathsf{TPP}(c,b)$

If the strengthening to biconditionals which I have described holds for the composition table for some set of base relations (in some theory), then relational composition as defined in section 1 will, for that set of relations, coincide with the extensional definition given above. But more importantly, it will mean that composition tables can be employed to facilitate a new mode of inference: to deduce from a relation between two objects the existence of a third object related to the original two in a specific way.

4 Neighbourhoods

An idea that has been brought to the fore by Freksa (1992) is that of *conceptual neighbourhood.* It was originally applied to temporal intervals as described by Allen (1983). Freksa (1992) tells us that:

"Two relations between pairs of events are (conceptual) neighbours *if they can be transformed into one another by continuously deforming (i.e. shortening, lengthening, moving) the events (in a topological sense)."*

and

"A set of relations between pairs of events forms a (conceptual) neighbourhood *if its elements are path-connected through 'conceptual' neighbour relations."*

Clearly, these definitions assume an interpretation of *events* as line segments — i.e. as one dimensional regions — and very similar definitions could be given for spatial regions of 2, 3 or more dimensions.

Given a basis set of spatial relations, we can consider the possibilities for direct transition between one relation and another. That is if two bodies occupy regions between which a relation R_1 holds is it possible that, by continuously moving or deforming the bodies, they can come to occupy regions related by R_2, without passing through a state in which they occupy regions related by another relation different from R_1 and R_2. If this can be done we can say that R_1 and R_2 are (conceptual) neighbours. Such neighbourhoods of spatial relations have been investigated in a number of works (e.g. (Galton 1993)).

A feature of the composition table for (temporal) intervals given by Allen (1983) pointed out by Freksa (1992) is that all entries in the table are disjuncts over some conceptual neighbourhood. Fur-

[3]It is an open question whether this could be proved directly from the axioms of the calculus specified by Randell, Cui and Cohn (1992)

thermore, in the calculus of Randell, Cui and Cohn (1992), provided we specify that bodies can only occupy connected regions of space, it turns out that all composition table entries are again conceptual neighbourhoods.

Freksa's interprets his findings as showing some deep connection between indefinite situations arising from partial information (e.g. the disjunctive entries of composition tables) and coarse description of a situation in which *neighbouring* relations are assimilated to a more general relation (which includes all relations in some conceptual neighbourhood).

However, the generality of the connection between these two kinds of indefinite information is not clear. To illustrate this I now give two examples of compositions of spatial relations, in which the resulting disjunctive relation is not a conceptual neighbourhood.

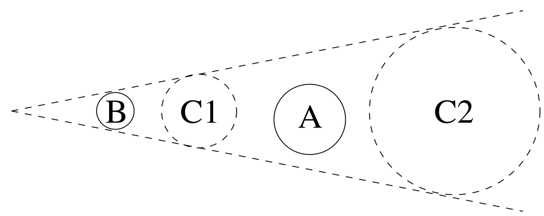

Figure 3: A *Line of Sight* composition

Figure 3 is given by Galton (1994) to show that in his *Line of Sight* calculus composition table entries are not necessarily conceptual neighbourhoods. The figure represents the situation where, from the point of view of an observer (situated at the l.h.s.), the body A is completely hidden behind a body B. We then learn that a body C is exactly hidden by B. According to Galton's calculus there are then just two possible relations between A and C: either C hides A or A is in front of C.

But there is a feature of Galton's calculus which may suggest that this is not a *bona fide* counter-example. It is that bodies are not allowed to overlap. This means that we do not have complete freedom to deform them within the space. If we allowed overlapping then there would be intermediate possibilities for the position of C which would form a neighbourhood linking $C1$ and $C2$.

Figure 4 shows a discontinuous composition in a relation set which is not subject to this criticism.

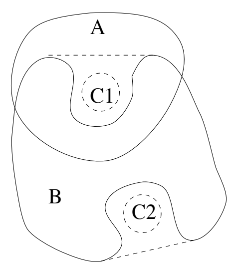

Figure 4: Another discontinous composition

Here we have region A overlapping B in such a way that the region occupied by the concavities in B overlaps A but does not intersect the boundary of A. This relation is somewhat complex but it can be defined fairly easily in the calculus of Randell, Cui and Cohn (1992). We also know that B contains another region C completely within its concavities. Possible relations between A and C are then either that C is completely inside A or that C is completely outside A. Of course if C were a 2 piece region then it could have a part inside A and a part outside A; but, even if we allow this, the 3 possibilities do not form a conceptual neighbourhood because a body cannot, by continuous topological deformation, go from being completely inside a region to acquiring an extra part which is completely outside that region.

Given these counter-examples it seems that the *locus* of an unspecified relation — i.e. the set of its possible values given certain other information — is not in general a conceptual neighbourhood.

Nevertheless, this does not necessarily mean that there is no reason to think of conceptual neighbourhoods as more significant than arbitrary disjunctions of relations. From the cognitive point of view, it is is probably true that a set of alternatives which form a conceptual neighbourhood is more easy to imagine and reason with than an arbitrary disjunction of possible relations. Moreover, in cases where disjunctive information and conceptual neighbourhood do coincide it may also be possible to exploit this in automatic reasoning.

5 Triangle Consistency

I conclude with a brief outline of another open question (raised by Cui, Cohn and Randell (1993)) which is of great significance in determining the usefulness of composition based reasoning.

Although composition tables provide for efficient implementation of certain inference steps, in general they cannot provide all the inferences we need for complete reasoning. What is not clear is under what conditions (i.e. for what types of theory and classes of additional facts) can composition tables provide a complete test of consistency.

In describing a spatial or spatio-temporal situation, amongst the most common facts will be statements of relations holding between objects. A fairly strong consistency criterion for such sets of facts is *triangle consistency*:

> A set of basic relations is triangle consistent iff, for every set of 3 objects, the three relations between the objects are consistent (if for some triangle one or more relations is unspecified, that triangle is also consistent). [4]

This can be generalised to the case where the facts are not exclusively basic relations but may also be disjunctions of relations taken from some basis set: we simply require that each disjunctive relation can be instantiated with one of its disjuncts in such a way that the resulting set of base relations is triangle-consistent.

Unfortunately, triangle-consistency does not guarantee genuine (global) consistency. For example, if a theory defines its objects to have the properties of circles of equal size, then any given circle can be externally connected to a maximum of six other circles; but this restriction could not be enforced by triangle checking.

Composition tables can be regarded as a *compilation* of all information in a logical theory that is relevant to determining triangle-consistency. Thus, to get the most out of composition-based reasoning, it is essential to know under what conditions triangle consistency is equivalent to global consistency.

[4] The notion of triangle consistency which I define here may superficially appear to be very similar to that of *3-consistency*, which is applied to constraint networks representing (binary) constraint satisfaction problems (CSPs) (see e.g. Tsang (1989) (1993)) and indeed there are analogies between the two concepts. However, 3-consistency (and related notions such as *arc consistency* and *path consistency*) have been defined and studied within the framework of CSPs which is somewhat different than the current setting.

In a CSP we have a set of variables and a set of constraints on possible values of these variables. These constraints can be regarded as a set of tuples of possible assignments (perhaps not explicitly given but checked on demand by some procedure) or as specified by a particular type of 1st-order theory (see Mackworth (1977)). However, the natural 1st-order specification of a CSP will not be akin to our 1st-order formalism for spatial reasoning. This is because the variables in a CSP are naturally represented by variables in its 1st-order representation and the predicates correspond to constraints; but in our case it is the values of *relations* which we wish to check for consistency and the dependencies between these relations are encoded in axioms and definitions involving quite complex quantifications. It may in some cases be possible to *reify* relations and thus turn a problem which is initially cast as one of determining whether consistent values can be assigned to relations into a problem of assigning values to ordinary (nominal) variables; but in general this would be far from straightforward.

6 Conclusion

I have discussed a number of loosely related issues surrounding the construction and use of composition tables for automated reasoning. I hope that some of the questions that I have raised will lead to further research in this potentially fruitful area.

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