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Quiz 2

I. Theory-based problems

Question 1: Revise the Slides

1.

$$\begin{aligned}
\text{Var}(aX+bY) =& \mathbb{E}[(aX+bY)^2] - (\mathbb{E}[aX+bY])^2 \\
=& \mathbb{E}[a^2X^2 + b^2Y^2 + 2abXY] - (a\mathbb{E}[X] + b\mathbb{E}[Y])^2 && (E.3) \\
=& a^2\mathbb{E}[X^2] + b^2\mathbb{E}[Y^2] + 2ab\mathbb{E}[XY] - a^2(\mathbb{E}[X])^2 - b^2(\mathbb{E}[Y])^2 - 2ab\mathbb{E}[X]\mathbb{E}[Y] && (E.3) \\
=& a^2(\mathbb{E}[X^2] - (\mathbb{E}[X])^2) + b^2(\mathbb{E}[Y^2] - (\mathbb{E}[Y])^2) + 2ab(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]) \\
=& a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]) \\
=& a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[X]\mathbb{E}[Y]) \\
=& a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab(\mathbb{E}[XY] - \mu_Y\mathbb{E}[X] - \mu_X\mathbb{E}[Y] + \mu_X\mu_Y) \\
=& a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab(\mathbb{E}[XY - \mu_Y X - \mu_X Y + \mu_X\mu_Y]) && (E.2 \text{ and } E.3) \\
=& a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab(\mathbb{E}[(X-\mu_X)(Y-\mu_Y)]) \\
=& a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(X,Y).
\end{aligned}$$

2. (a) If $P(A) = 0$, then clearly $P(A)P(B) = 0P(B) = 0$ and $P(A \cap B) = 0$. Therefore, $P(A \cap B) = P(A)P(B) = 0$.

   (b) Let A and B be independent and $P(B) > 0$. Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A).$$

   (c) Let A and B be independent. Then $P(A \cap B) = P(A)P(B)$. In addition, recall that $P(X) = P(X \cap Y) + P(X \cap Y^c)$. Then

$$\begin{aligned}
P(A^C)P(B) =& (1 - P(A))P(B) = P(B) - P(A)P(B) = P(B) - P(A \cap B) = P(A^C \cap B), \\
P(A^C)P(B^C) =& P(A^C)(1 - P(B)) \\
=& P(A^C) - P(A^C)P(B) = P(A^C) - P(A^C \cap B) = P(A^C \cap B^C), \\
P(A)P(B^C) =& P(A)(1 - P(B)) = P(A) - P(A)P(B) = P(A) - P(A \cap B) = P(A \cap B^C).
\end{aligned}$$

3.

$$\lim_{x \to -\infty} F_X(x) = \lim_{x \to -\infty} P(X \le x) = \lim_{x \to -\infty} P_X((-\infty, x]) = \lim_{x \to -\infty} P(\{\omega : X(\omega) \le x\}) = 0.,$$

where the last equality holds because there is clearly no ω for which $X(\omega) \le x$ for x approaching negative infinity.

$$\lim_{x \to \infty} F_X(x) = \lim_{x \to \infty} P(X \le x) = \lim_{x \to -\infty} P_X((-\infty, x]) = \lim_{x \to -\infty} P(\{\omega : X(\omega) \le x\}) = 1,$$

where the last equality holds because clearly for all $\omega, X(\omega) \le x$ for x approaching infinity.

4.

$$\mathbb{E}[X] = \int_0^1 x dF_X(x) = \int_0^1 x(1)dx = \Big[\frac{x^2}{2}\Big]_0^1 = \frac{1}{2}.$$

$$\mathbb{E}[X^2] = \int_0^1 x^2 dF_X(x) = \int_0^1 x^2(1)dx = \Big[\frac{x^3}{3}\Big]_0^1 = \frac{1}{3}.$$

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{1}{3} - (\frac{1}{2})^2 = \frac{1}{12}.$$

5. First, note that since the X's are independent, they are also uncorrelated. So $\text{Cov}(X_i, X_j) = 0$ for any $i, j = 1, 2, 3, 4$ such that $i \neq j$. Then

$$\begin{aligned}
\mathbb{E}[Y] =& \mathbb{E}\Big[\frac{X_1 + X_2 + X_3 + X_4}{4}\Big] \\
=& \frac{1}{4}(\mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[X_4]) && (E.3) \\
=& \frac{1}{4}(4)\Big(\frac{1}{2}\Big) \\
=& \frac{1}{2}. \\
\text{Var}[Y] =& \text{Var}\Big[\frac{X_1 + X_2 + X_3 + X_4}{4}\Big] \\
=& \Big(\frac{1}{4}\Big)^2(\text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) + \text{Var}(X_4)) && (Var.2) \\
=& \frac{1}{16}(4)\Big(\frac{1}{12}\Big) \\
=& \frac{1}{48}.
\end{aligned}$$

Now let $\text{Cov}(X_i, X_j) = 0.2$ for any $i, j = 1, 2, 3, 4$ such that $i \neq j$. Then

$$\begin{aligned}
\mathbb{E}[Y] =& \mathbb{E}\Big[\frac{X_1 + X_2 + X_3 + X_4}{4}\Big] \\
=& \frac{1}{4}(\mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[X_4]) && (E.3) \\
=& \frac{1}{4}(4)\Big(\frac{1}{2}\Big) \\
=& \frac{1}{2}. \\
\text{Var}[Y] =& \text{Var}\Big[\frac{X_1 + X_2 + X_3 + X_4}{4}\Big] \\
=& \Big(\frac{1}{4}\Big)^2\Big(\text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) + \text{Var}(X_4) + 2\text{Cov}(X_1, X_2) + 2\text{Cov}(X_1, X_3) \\
&+ 2\text{Cov}(X_1, X_4) + 2\text{Cov}(X_2, X_3) + 2\text{Cov}(X_2, X_4) + 2\text{Cov}(X_3, X_4)\Big) && (Var.3) \\
=& \frac{1}{16}\Big((4)\Big(\frac{1}{12}\Big) + 6(2)\Big(\frac{1}{5}\Big)\Big) \\
=& \frac{41}{240}.
\end{aligned}$$

The second random variable clearly has the higher variance since $\frac{41}{240} > \frac{1}{48}$.

Question 2: Matrix Algebra

Let $M = I_n - P$ where $P = X(X'X)^{-1}X'$.

(a) M is symmetric since we show that $M' = M$ with the following:

$$\begin{aligned} M' =& (I_n - P)' = (I_n - X(X'X)^{-1}X')' = I_n' - (X')'((X'X)')^{-1}(X)' \\ =& I_n - X(X'X)^{-1}X' = M. \end{aligned}$$

M is idempotent since we show that $MM = M$ with the following:

$$\begin{aligned} MM =& (I_n - P)(I_n - P) = (I_n - X(X'X)^{-1}X')(I_n - X(X'X)^{-1}X') \\ =& I_n I_n - 2I_n X(X'X)^{-1}X' + X(X'X)^{-1}X'X(X'X)^{-1}X' \\ =& I_n - 2X(X'X)^{-1}X' + X(X'X)^{-1}(X'X)(X'X)^{-1}X' \\ =& I_n - 2X(X'X)^{-1}X' + X(X'X)^{-1}X' \\ =& I_n - X(X'X)^{-1}X' = (I_n - P) = M. \end{aligned}$$

(b) Note that the trace operator is invariant under cyclic permutations, i.e. $tr(ABC) = tr(BCA) = tr(CAB)$ and that it is a linear operator, i.e. $tr(A + B) = tr(A) + tr(B)$. Also note that X is a $n \times k$ matrix. Then

$$\begin{aligned} tr(M) =& tr(I_n - P) = tr(I_n - X(X'X)^{-1}X') \\ =& tr(I_n) - tr(X(X'X)^{-1}X') \\ =& n - tr(X(X'X)^{-1}X') \\ =& n - tr((X'X)^{-1}X'X) \\ =& n - tr(I_k) \\ =& n - k \end{aligned}$$

(c)

$$\begin{aligned} X'\hat{u} &= X'(y - X\hat{\beta}) = X'(y - X(X'X)^{-1}X'y) \\ &= X'y - X'X(X'X)^{-1}X'y = X'y - X'y = 0. \end{aligned}$$

More Theory: (Bonus)

1. Suppose that P is a probability measure of the probability space $(\Omega, \mathscr{F}, P)$, and let

$$P'(A) = \frac{P(A \cap B)}{P(B)}$$

where A and B are events in $(\Omega, \mathscr{F}, P)$ with $P(B) > 0$. Then $P'(A) \geq 0$ for all events A since $P(B) > 0$ and $P(A \cap B) \in [0, 1]$. In addition,

$$P'(\Omega) = \sum_{A \in \mathscr{F}} P'(A) = \sum_{A \in \mathscr{F}} \frac{P(A \cap B)}{P(B)} = \frac{1}{P(B)} \sum_{A \in \mathscr{F}} P(B|A)P(A) = \frac{P(B)}{P(B)} = 1.$$

Finally, if $A_1, A_2, ... \in \mathscr{F}$ and $A_i \cap A_j = \varnothing$ for $i \neq j$, then

$$\begin{aligned}
P'\Big(\bigcup_{j=1}^{\infty} A_j\Big) =& P\Big(\bigcup_{j=1}^{\infty} A_j|B\Big) \\
=& \frac{1}{P(B)} P\Big(\bigcup_{j=1}^{\infty} A_j \cap B\Big) \\
=& \frac{1}{P(B)} \sum_{j=1}^{\infty} P(A_j \cap B) \qquad\qquad \Big(\text{since } P\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} P(A)\Big) \\
=& \sum_{j=1}^{\infty} \frac{P(A_j \cap B)}{P(B)} \\
=& \sum_{j=1}^{\infty} P'(A).
\end{aligned}$$

Thus, by definition P' is a probability measure.

2. For slide 36,

$$P(A|C) = P(A \cap B|C) + P(A \cap B^C|C) = \frac{4}{9} + \frac{2}{9} = \frac{2}{3}.$$

$$P(B|C) = P(A \cap B|C) + P(A^C \cap B|C) = \frac{4}{9} + \frac{2}{9} = \frac{2}{3}.$$

$$P(A|C)P(B|C) = \frac{2}{3}\Big(\frac{2}{3}\Big) = \frac{4}{9}.$$

Thus,

$$P(A \cap B|C) = P(A|C)P(B|C) = \frac{4}{9},$$

so A and B are conditionally independent.

In addition,

$$\begin{aligned}
P(A) =& P(A|C)P(C) + P(A|C^C)P(C^C) \\
=& \frac{2}{3}\Big(\frac{1}{2}\Big) + [P(A \cap B|C^C) + P(A \cap B^C|C^C)]P(C^C) \\
=& \frac{1}{3} + \Big[\frac{1}{9} + \frac{2}{9}\Big]\Big(\frac{1}{2}\Big) \\
=& \frac{1}{3} + \frac{1}{6} \\
=& \frac{1}{2}.
\end{aligned}$$

Also,

$$\begin{aligned}
P(B) =& P(B|C)P(C) + P(B|C^C)P(C^C) \\
=& \frac{2}{3}\Big(\frac{1}{2}\Big) + [P(A \cap B|C^C) + P(A^C \cap B|C^C)]P(C^C) \\
=& \frac{1}{3} + \Big[\frac{1}{9} + \frac{2}{9}\Big]\Big(\frac{1}{2}\Big) \\
=& \frac{1}{3} + \frac{1}{6} \\
=& \frac{1}{2}.
\end{aligned}$$

Thus, we have

$$P(A)P(B) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}.$$

Finally,

$$P(A \cap B) = P(A \cap B|C)P(C) = \frac{4}{9}\left(\frac{1}{2}\right) = \frac{2}{9}.$$

So clearly, A and B are not independent because

$$P(A \cap B) = \frac{2}{9} \neq \frac{1}{4} = P(A)P(B).$$

Therefore, conditional independence does not imply independence.

For slide 37,

$$P(A|C) = P(A \cap B|C) + P(A \cap B^C|C) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$
$$P(B|C) = P(A \cap B|C) + P(A^C \cap B|C) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$
$$P(A|C)P(B|C) = \frac{2}{3}\left(\frac{2}{3}\right) = \frac{4}{9}.$$

Thus,

$$P(A \cap B|C) = \frac{1}{3} \neq \frac{4}{9} = P(A|C)P(B|C),$$

so A and B are not conditionally independent.

In addition,

$$\begin{aligned}
P(A) =& P(A|C)P(C) + P(A|C^C)P(C^C) \\
=& \frac{2}{3}\left(\frac{3}{4}\right) + [P(A \cap B|C^C) + P(A \cap B^C|C^C)]P(C^C) \\
=& \frac{1}{2} + [0 + 0]\left(\frac{1}{4}\right) \\
=& \frac{1}{2}.
\end{aligned}$$

Also,

$$\begin{aligned}
P(B) =& P(B|C)P(C) + P(B|C^C)P(C^C) \\
=& \frac{2}{3}\left(\frac{3}{4}\right) + [P(A \cap B|C^C) + P(A^C \cap B|C^C)]P(C^C) \\
=& \frac{1}{2} + [0 + 0]\left(\frac{1}{4}\right) \\
=& \frac{1}{2}.
\end{aligned}$$

Thus, we have

$$P(A)P(B) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}.$$

Finally,

$$P(A \cap B) = P(A \cap B|C)P(C) = \frac{1}{3}\left(\frac{3}{4}\right) = \frac{1}{4}.$$

So clearly, A and B are independent because

$$P(A \cap B) = P(A)P(B) = \frac{1}{4}.$$

Therefore, independence does not imply conditional independence.

In conclusion, it is clear that conditional independence and independence do not imply each other.

3. Let $x_2 > x_1$. Then

$$\begin{aligned}
F_x(x_2) - F_x(x_1) =& P(X < x_2) - P(X < x_1) \\
=& P_X((-\infty, x_2)) - P_X((-\infty, x_1)) \\
=& P(\{\omega : X(\omega) < x_2\}) - P(\{\omega : X(\omega) < x_1\}) \\
=& P(\{\omega : X(\omega) < x_2 \cap X(\omega) \geq x_1\}) \\
=& P(\{\omega : x_1 < X(\omega) < x_2\}) \\
=& P_X((x_1, x_2)) \\
=& P(x_1 < X < x_2).
\end{aligned}$$

4.

$$\begin{aligned}
f_X(x) =& P(X = x) \\
=& \sum_y P(X = x | Y = y) P(Y = y) \\
=& \sum_y P(X = x, Y = y) \\
=& \sum_y f_{X,Y}(x, y).
\end{aligned}$$

II. Empirical Problem

<u>Grades</u>

(a) (in ECON_7022_Quiz_2.R file)

(b) (in ECON_7022_Quiz_2.R file)