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Quiz 3

Reading in Hansen

Like I mentioned in my submission for Problem Set 1, I recognize that AP is, in general, easier to read because of the extra exposition it provides when compared to the Hansen lecture notes, but I still prefer the Hansen lecture notes because they are written in a manner to which I have become accustomed in textbooks over the years.

I prefer the fluidity, pacing, and notation of AP. Meanwhile, I prefer the succinct and direct approach that the Hansen lecture notes take. Although the notation in the Hansen lecture notes is not bad, there are times when I feel it can be a little overwhelming.

I think that the chapters 5-21 (every chapter after chapter 4) in Hansen's lecture notes are not covered at all in AP. This is just another reason why I personally prefer Hansen's lecture notes. However, AP clearly goes into extreme detail about topics discussed in the first 4 chapter of Hansen's lecture notes.

Theory

1. Let $\beta = \text{argmin}_b S(b_0, b_1)$ where

$$\begin{aligned} S(b_0,b_1) = \mathbb{E}[(Y-b_0-b_1X)^2] =& \mathbb{E}[(Y-b_0-b_1X)(Y-b_0-b_1X)] \\ =& \mathbb{E}[Y^2+b_0^2+b_1^2X^2-2b_0Y-2b_1XY+2b_0b_1X] \\ =& \mathbb{E}[Y^2]+b_0^2+b_1^2\mathbb{E}[X^2]-2b_0\mathbb{E}[Y]-2b_1\mathbb{E}[XY]+2b_0b_1\mathbb{E}[X] \end{aligned}$$

Then, we minimize in the following way:

$$\begin{aligned} 0 = \frac{\partial S(b_0,b_1)}{\partial b_0} =& 2b_0 - 2\mathbb{E}[Y] + 2b_1\mathbb{E}[X] \implies b_0 = \mathbb{E}[Y] - b_1\mathbb{E}[X] \\ 0 = \frac{\partial S(b_0,b_1)}{\partial b_1} =& 2b_1\mathbb{E}[X^2] - 2\mathbb{E}[XY] + 2b_0\mathbb{E}[X] \\ &\implies b_1 = (\mathbb{E}[X^2])^{-1}(\mathbb{E}[XY] - b_0\mathbb{E}[X]) \\ &\implies b_1 = (\mathbb{E}[X^2])^{-1}(\mathbb{E}[XY] - (\mathbb{E}[Y]-b_1\mathbb{E}[X])\mathbb{E}[X]) \\ &\implies (1-(\mathbb{E}[X^2])^{-1}(\mathbb{E}[X]\mathbb{E}[X]))b_1 = (\mathbb{E}[X^2])^{-1}(\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]) \\ &\implies b_1 = (\mathbb{E}[X^2]-\mathbb{E}[X]\mathbb{E}[X])^{-1}(\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]) \end{aligned}$$

Therefore,

$$\begin{aligned} \beta_1 =& (\mathbb{E}[X^2]-\mathbb{E}[X]\mathbb{E}[X])^{-1}(\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]) \\ \beta_0 =& \mathbb{E}[Y] - \beta_1\mathbb{E}[X] \end{aligned}$$

2. Let $f_{X_1,X_2}(x_1,x_2) = e^{(x_1+x_2)}1_{x_1\geq 0}1_{x_2\geq 0}$. Now,$f_{Y_1,Y_2}(y_1,y_2) = f_X(h^{-1}(y)) \cdot |J(y)|$ where

$$\begin{aligned} f_X(h^{-1}(y)) = f_X(X_1,X_2) =& e^{(x_1+x_2)}1_{x_1\geq 0}1_{x_2\geq 0} \\ =& e^{((y_1+y_2)/2+(y_1-y_2)/2)}1_{(y_1+y_2)/2\geq 0}1_{(y_1-y_2)/2\geq 0} \\ =& e^{y_1}1_{y_1=y_2}. \end{aligned}$$

and

$$J = \begin{vmatrix} \frac{\partial \frac{y_1+y_2}{2}}{\partial y_1} & \frac{\partial \frac{y_1+y_2}{2}}{\partial y_2} \\ \frac{\partial \frac{y_1-y_2}{2}}{\partial y_1} & \frac{\partial \frac{y_1-y_2}{2}}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

Therefore,

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) =& f_X(h^{-1}(y)) \cdot |J(y)| \\ =& e^{y_1} 1_{y_1=y_2} \cdot | -\frac{1}{2}| \\ =& \frac{1}{2} e^{y_1} 1_{y_1=y_2}. \end{aligned}$$

Programming

(in ECON_7022_Quiz_3.R file)

I have chosen to sample from Poisson, normal, and chi-squared distributions.

Each Poisson random variable had $\lambda = 5$, and thus, had an expected value and a variance of 5. I ended up getting a sample mean of 5.025 and a sample variance of 4.993374.

Each normally distributed random variable had a mean of 1 and a standard deviation of 3, and thus, had an expected value of 1 and a variance of 9. I ended up getting a sample mean of 0.9904346 and a sample variance of 9.486896.

Finally, each chi-squared random variable had 20 degrees of freedom, and thus, had an expected value of 20 and a variance of 40. I ended up getting a sample mean of 20.09426 and a sample variance of 40.45195.

All of these are very close to their theoretical values as expected since we are taking a large number of samples.