# Master of Data Science Online Programme
# Course: Calculus
# SGA #4: Chain Rule

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# Problem 1

Let

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2}, & \text{if } x^2+y^2>0 \\ 0, & \text{if } x=y=0, \end{cases} \tag{1}$$

where $x(t) = y(t) = t$. In the same time

$$\frac{d}{dt}(f(x(t),y(t))|_{t=0} = \frac{1}{2}, \tag{2}$$

while

$$f'_x(0,0) = f'_y(0,0) = 0, \tag{3}$$

thus the chain rule does not coincide with the direct computation.

Carefully examine all involved entities. State the property of the initial function or its derivative that breaks the chain rule (please, show that this function has indeed this property, not just name it) and thus should be necessary for chain rule.

## Solution

### Approach 1

The function is called differentiable at point $(a,b)$, if it is close to its tangent plane, i.e.

$$f(x,y) = f(a,b) + f'_x(a,b)(x-a) + f'_y(a,b)(y-b) + R(x,y), \tag{4}$$

where $f'_x$ and $f'_x$ are partial derivatives of $f(x,y)$ towards $x$ and $y$ respectively, and $R(x,y)$ is an error of approximation of $f(x,y)$ by its tangent plane. This error is an infinitesimal function (little-o) towards the change in variables and can be treated as the distance between $(x,y)$ and $(a,b)$. In our case it is the Euclidean distance:

$$R(x,y) = o(\rho) = o(\sqrt{(x-a)^2+(y-b)^2}). \tag{5}$$

Let's check if (1) is an infinitesimal function (little-o) towards the error (5) at $(0,0)$ by finding the limit of their ratio:

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{f(x,y)}{\sqrt{x^2+y^2}} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^2}{(x^2+y^2)^{3/2}} = \begin{Bmatrix} x=t \\ y=t \\ t\to 0 \end{Bmatrix} = \lim_{t\to 0} \frac{t\cdot t^2}{(t^2+t^2)^{3/2}} = \lim_{t\to 0} \frac{t^3}{\sqrt{8}t^3} = \sqrt{\frac{1}{8}} \approx \frac{1}{3}. \tag{6}$$

The result of (6) is obviously not infinitesimal, which means that (4) does not hold and (1) is not differentiable at $(0,0)$. In this case the chain rule is not applicable.

### Approach 2

Sufficient condition of differentiability of $f(x,y)$ at point $(a,b)$: if both partial derivatives $f'_x$ and $f'_y$ are continuous at $(a,b)$, then $f(x,y)$ is differentiable at $(a,b)$. Let's check if both partial derivatives are continuous at our critical point $(0,0)$: their limits must be finite and be equal to: $f'_x(0,0) = 0$ and $f'_y(0,0) = 0$ respectively. First let's do the calculation for the partial derivative $f'_x$:

$$f'_x = \frac{\partial f}{\partial x} = \left(\frac{xy^2}{x^2+y^2}\right)'_x = \frac{y^2(x^2+y^2) - xy^2\cdot 2x}{(x^2+y^2)^2} = \frac{y^2(y^2-x^2)}{(x^2+y^2)^2}. \tag{7}$$

It easily seen that $f'_x$ is an indeterminate form 0/0 at (0,0), if calculated directly, so we need to find its limit at different approaches:

1. Let $x=t$ and $y=0$ (horizontal approach). Then (7) is

$$f'_x(x=t, y=0) = \lim_{t\to 0} \frac{0^2(0^2-t^2)}{(t^2+0^2)^2} = \lim_{t\to 0} \frac{0}{t^4} = 0.$$

2. Let $x=0$ and $y=t$ (vertical approach). Then (7) is

$$f'_x(x=0, y=t) = \lim_{t\to 0} \frac{t^2(t^2-0^2)}{(0^2+t^2)^2} = \lim_{t\to 0} \frac{t^4}{t^4} = 1.$$

3. Finally, let $x = t$ and $y = t$ (linear approach). Then (7) is

$$f'_x(x = t, y = t) = \lim_{t \to 0} \frac{t^2(t^2 - t^2)}{(t^2 + t^2)^2} = \lim_{t \to 0} \frac{0}{t^4} = 0.$$

We can easily see that the results are different, thus the limit of (7) when $(x, y) \to (0, 0)$ does not exist. This means that the partial derivative $f'_x$ is discontinuous at (0,0).

Now let's repeat the calculation for $f'_y$:

$$f'_y = \frac{\partial f}{\partial y} = \left( \frac{xy^2}{x^2 + y^2} \right)'_y = \frac{2xy(x^2 + y^2) - xy^2 \cdot 2y}{(x^2 + y^2)^2} = \frac{2x^3 y}{(x^2 + y^2)^2}. \tag{8}$$

It easily seen that $f'_y$ is an indeterminate form 0/0 at (0,0), if calculated directly, so we need to find its limit at different approaches:

1. Let $x = t$ and $y = 0$ (horizontal approach). Then (8) is

$$f'_y(x = t, y = 0) = \lim_{t \to 0} \frac{2t^3 \cdot 0}{(t^2 + 0^2)^2} = \lim_{t \to 0} \frac{0}{t^4} = 0.$$

2. Let $x = 0$ and $y = t$ (vertical approach). Then (8) is

$$f'_y(x = 0, y = t) = \lim_{t \to 0} \frac{2 \cdot 0^3 \cdot t}{(0^2 + t^2)^2} = \lim_{t \to 0} \frac{0}{t^4} = 0.$$

3. Finally, let $x = t$ and $y = t$ (linear approach). Then (8) is

$$f'_y(x = t, y = t) = \lim_{t \to 0} \frac{2t^3 t}{(t^2 + t^2)^2} = \lim_{t \to 0} \frac{2t^4}{4t^4} = \frac{1}{2}.$$

We can easily see that the results are different, thus the limit of (8) when $(x, y) \to (0, 0)$ does not exist. This means that the partial derivative $f'_y$ is discontinuous at (0,0). Since both $f'_x$ and $f'_y$ are discontinuous at $(0, 0)$, (1) is not differentiable at $(0, 0)$, Thus the chain rule is not applicable.

## Answer

While the function (1) is defined at $(0, 0)$, its both partial derivatives $f'_x$ and $f'_y$ are not continuous at this critical point, which means that the function is not differentiable at $(0, 0)$. Also, we have shown that (1) cannot be approximated by its tangent plane at $(0, 0)$ which again means that the function is not differentiable at this critical point. Thus the chain rule breaks in this case.