Master of Data Science Online Programme
Course: Probability Theory
SGA #2: Space launches, insurance and lotteries

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March 10, 2023

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Problem 2

I can buy one lottery ticket out of two available. In the first lottery I can win \$100 with probability 0.1, and the price of ticket is \$10. In the second lottery, I can win \$50 with probability 0.1 and \$500 with probability 0.01. The price of ticket is \$20. To decide which ticket to buy I toss a fair coin once. I chose first ticket in case of head and second otherwise. Let X be random variable that denotes my net payout (taking into account price of a ticket). Find probability mass function of X (hint: use law of total probability). Show that expected value of X is an average of expected values of net payouts for each of two lotteries. Explain, why. Will it still hold if lotteries has different payouts or probabilities? Prove it.

Solution

Let's first obtain distributions and expected values for each lottery as two independent experiments. Let A be the random variable for payout in the first lottery less the ticket price of \$10 and B be the random variable for payout in the second lottery less the ticket price of \$20. Then the corresponding distributions of A and B can be shown as below:

A, \$	$100 - 10 = 90$	$0 - 10 = -10$
$P(A)$	0.1	$1 - 0.1 = 0.9$

Table 1: Distribution of payout for lottery A

B, \$	$50 - 20 = 30$	$500 - 20 = 480$	$0 - 20 = -20$
$P(B)$	0.1	0.01	$1 - 0.1 - 0.01 = 0.89$

Table 2: Distribution of payout for lottery B

The expected value of a discrete random variable is [1]

$$\mathbb{E}X = \sum_{i=1}^{n} x_i \cdot p_i, \tag{1}$$

where x_i is the i-th value of the random variable and p_i is the probability of the i-th value. Thus as per (1) and table 1, the expected value of A is

$$\mathbb{E}A = 90 \times 0.1 - 10 \times 0.9 = 0. \tag{2}$$

And as per (1) and table 2, the expected value of B is

$$\mathbb{E}B = 30 \times 0.1 + 480 \times 0.01 - 20 \times 0.89 = -10. \tag{3}$$

Interestingly, none of the two lotteries allows to win.

Now let's consider the random experiment of tossing the coin for choosing the lottery. For the fair coin we have equal probability of obtaining head and tail $P(Head) = P(Tail) = 1/2$. Since now we are considering new distribution for X as the result of dependent random experiments of coin tossing and buying a lottery ticket for either lottery A or B, we need to construct new distribution for X using the law of total probability. That is $P(X = a)$ or $P(X = b)$ now depends on $P(Head)$ or $P(Tail)$. The total probability [2]

$$P(X) = \sum_{i=1}^{n} P(\omega_i) \cdot P(X|\omega_i), \tag{4}$$

where in our case of a fair coin $P(\omega_i) = P(Head) = P(Tail) = 1/2$, and $P(X|\omega_i)$ are the corresponding conditional probabilities for each lottery. Re-writing for two lotteries and a fair coin, we get,

$$P(X) = \frac{1}{2} \cdot P(X|Head) + \frac{1}{2} \cdot P(X|Tail) = \frac{1}{2} \cdot [P(A) + P(B)], \tag{5}$$

Now we only need to calculate $P(X)$ for each $a \in A$ and $b \in B$:

- $P(X = -20) = 1/2 \times [P(A = -20) + P(B = -20)] = 1/2 \times (0 + 0.89) = 0.445$;

- $P(X=-10) = 1/2 \times [P(A=-10) + P(B=-10)] = 1/2 \times (0.9+0) = 0.45$;
- $P(X=30) = 1/2 \times [P(A=30) + P(B=30)] = 1/2 \times (0+0.1) = 0.05$;
- $P(X=90) = 1/2 \times [P(A=90) + P(B=90)] = 1/2 \times (0.1+0) = 0.05$;
- $P(X=480) = 1/2 \times [P(A=480) + P(B=480)] = 1/2 \times (0+0.01) = 0.005$.

So, finally the probability mass function for X with X sorted in increasing order can be shown as below

Coin toss	Tail	Head	Tail	Head	Tail
Lottery	B	A	B	A	B
X, \$	-20	-10	30	90	480
$P(X)$	0.445	0.45	0.05	0.05	0.005

Table 3: Probability mass function of X

Let's plot it.

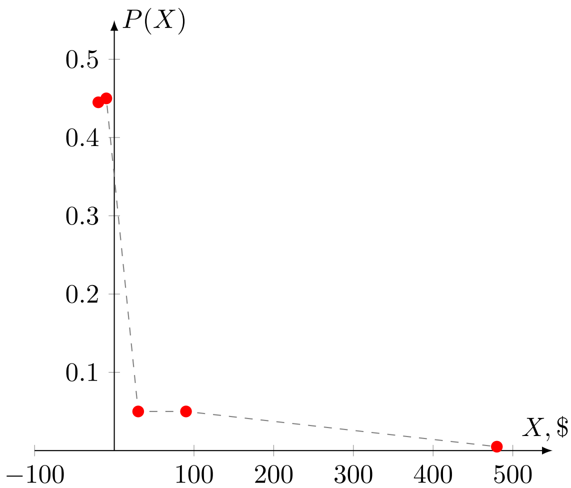

Figure 1: Plot of probability mass function of X

Let's find the expected value of X. As per (1),

$$\mathbb{E}X = -20 \times 0.445 - 10 \times 0.45 + 30 \times 0.05 + 90 \times 0.05 + 480 \times 0.005 = -5. \tag{6}$$

It is easily seen that the result of (6) is exactly the average of expected payout values for both lotteries: $(\mathbb{E}A + \mathbb{E}B)/2 = (0-10)/2 = -5$. This result is expected, since we used a fair coin to choose between the lotteries with equal probabilities of $1/2$. It is also expected, that the payout is still not positive, since both lotteries independently do not allow to win.

As per (5), $P(X)$ in fact is half the sum of the independent distributions of A and B, i.e. their average. So, even if parameters of A and B, such as the payout and/or probabilities change, $P(X)$ will still be their average, but only in the case of tossing a *fair* coin. Let's show that this condition of average holds only for a fair coin. Let $p \in [0,1]$ be the probability of obtaining a head, and $1-p$ be the probability of obtaining a tail. As per (4),

$$P(X) = p \cdot P(X|Head) + (1-p) \cdot P(X|Tail) = p \cdot P(A) + (1-p) \cdot P(B). \tag{7}$$

Equating (7) and (5), we get

$$\begin{aligned} p \cdot P(A) + (1-p) \cdot P(B) &= 1/2 \cdot [P(A) + P(B)], \\ p \cdot P(A) + P(B) - p \cdot P(B) &= 1/2 \cdot P(A) + 1/2 \cdot P(B), \\ p \cdot P(A) - p \cdot P(B) &= 1/2 \cdot P(A) + 1/2 \cdot P(B) - P(B), \\ p \cdot [P(A) - P(B)] &= 1/2 \cdot [P(A) - P(B)], \\ \text{since } A \text{ and } B \text{ are different, } P(A) - P(B) &\neq 0, \text{ thus} \\ p &= 1/2. \end{aligned} \tag{8}$$

This result means that $P(X)$ is the average of $P(A)$ and $P(B)$ only if $p = 1/2$, i.e. the coin is *fair*.

Answer

- The probability mass function can be expressed as per the table 3 and plot in Figure 1.
- The expected value of X is $\mathbb{E}X = -5 = (\mathbb{E}A + \mathbb{E}B)/2$, i.e. the average.
- This is only possible for a *fair* coin, since the probability of obtaining a head is the same as the probability of obtaining a tail which is $1/2$.
- This will hold even if distributions of A and B changes, since we equally choose one of the lotteries. However, it will not hold if we use an *unfair* coin. In this case, $\mathbb{E}X$ will not be the average of $\mathbb{E}A$ and $\mathbb{E}B$, since $P(X)$ is no longer equal to $1/2 \cdot [P(A) + P(B)]$.

References

[1] Ilya Schurov. *Expected value of random variable. Motivation and definition.* Faculty of Computer Science, Higher School of Economics. URL: https://smartedu.hse.ru/mod/page/0/756757.

[2] Ilya Schurov. *Law of total probability.* Faculty of Computer Science, Higher School of Economics. URL: https://smartedu.hse.ru/mod/page/0/756734.