Master of Data Science Online Programme
Course: Linear Algebra
SGA #2: Week 6

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Problem 2

Find a decomposition $A = U\Sigma V^T$ of the matrix

$$A = \begin{pmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix}, \tag{1}$$

where Σ is a rectangular diagonal matrix of size 4×3, U and V are orthogonal matrices, and the upper right element of V is equal to $1/\sqrt{2}$.

Solution

As per [3], we need to find the Singular Value Decomposition of (1) with the following algorithm.

1. Find matrix

$$B = A^T A = \begin{pmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix}^T \begin{pmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 28 & 14 & 28 \\ 14 & 7 & 14 \\ 28 & 14 & 28 \end{pmatrix}. \tag{2}$$

2. Find eigenvalues λ_i of (2). As per [1], eigenvalues of a matrix are the roots of the characteristic polynomial, i.e.

$$det(B - \lambda I) = 0, \tag{3}$$

where det is a function of columns of a square matrix. For a matrix $A_{3\times 3}$ the determinant is defined as $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$. So, our characteristic equation is

$$\begin{aligned} det\left[\begin{pmatrix} 28 & 14 & 28 \\ 14 & 7 & 14 \\ 28 & 14 & 28 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] &= 0, \\ det\left[\begin{pmatrix} 28 & 14 & 28 \\ 14 & 7 & 14 \\ 28 & 14 & 28 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right] &= 0, \\ det\left[\begin{pmatrix} 28-\lambda & 14 & 28 \\ 14 & 7-\lambda & 14 \\ 28 & 14 & 28-\lambda \end{pmatrix}\right] &= 0, \\ -\lambda^3 + 63\lambda^2 &= 0, \\ \lambda^2(63 - \lambda) &= 0. \end{aligned} \tag{4}$$

The equation (4) has three real roots: $\lambda_1 = 63, \lambda_2 = \lambda_3 = 0$. These are the eigenvalues of (2).

3. Find singular matrix Σ of size of (1) with the diagonal elements $\sigma_i = \sqrt{\lambda_i}$ such that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$. Since we only have three eigenvalues but four rows in (1), we need to append a row of all zeros at the bottom of Σ:

$$\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{63} & 0 & 0 \\ 0 & \sqrt{0} & 0 \\ 0 & 0 & \sqrt{0} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3\sqrt{7} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

4. Find eigenvectors of (2), As per [2], each eigenvalue has a corresponding eigenvector, which is a solution vector of the linear system $Bv_i = \lambda_i v_i \Rightarrow (B - \lambda_i I)v_i = 0$. Since we have three eigenvalues, there are three eigenvectors, each of can be obtained by solving its linear system. Let's find v_1 corresponding to λ_1:

$$\left[\begin{pmatrix} 28 & 14 & 28 \\ 14 & 7 & 14 \\ 28 & 14 & 28 \end{pmatrix} - \begin{pmatrix} 63 & 0 & 0 \\ 0 & 63 & 0 \\ 0 & 0 & 63 \end{pmatrix}\right] v_1 = \begin{pmatrix} -35 & 14 & 28 \\ 14 & -56 & 14 \\ 28 & 14 & -35 \end{pmatrix} v_1 = 0. \tag{6}$$

Since the matrix of (6) has rank 2, the solution has one free variable x_3, and thus the solution vector is $v_1 = x_3 \begin{pmatrix} 1 \\ 0.5 \\ 1 \end{pmatrix}$. By choosing $x_3 = 1, v_1 = \begin{pmatrix} 1 \\ 0.5 \\ 1 \end{pmatrix}$. For v_2 and v_3 we could construct and solve a similar system for $\lambda_2 = \lambda_3 = 0$. However since we are required to have a specific form for v_3, namely its first element must be $1/\sqrt{2}$, for v_3 we can choose any vector satisfying this condition without finding the eigenvector for v_3. This is valid, since the singular values for v_2 and v_3 are zeros, so in fact, it can be any vectors with the requirement of orthogonality to v_1. Looking at v_1 it is easy to obtain $v_3 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$. Obviously, v_1 and v_3 are orthogonal, since $\langle v_1, v_3 \rangle = 0$. To find $v_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ we need to require

$$\begin{cases} \langle v_1, v_2 \rangle = 0, \\ \langle v_2, v_3 \rangle = 0 \end{cases} \Rightarrow \begin{cases} 1 \cdot x + 0.5 \cdot y + 1 \cdot z = 0, \\ 1/\sqrt{2} \cdot x + 0 \cdot y - 1/\sqrt{2} z = 0 \end{cases} \tag{7}$$

Solving (7) yields $v_2 = z \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$. Let $z = 1$, then $x = 1$, and $y = -4$. So, $v_2 = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$.

5. Find matrix V. Since, we have found all three vectors for V, now we only need to normalize each of them to get V:

$$V = \left[\frac{v_1}{||v_1||} \middle| \frac{v_2}{||v_2||} \middle| \frac{v_3}{||v_3||} \right] = \begin{pmatrix} \frac{2}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{3} & -\frac{4}{3\sqrt{2}} & 0 \\ \frac{2}{3} & \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{8}$$

6. Find matrix U by computing its columns as $u_i = \dfrac{Av_i}{\sigma_i}$:

$$u_1 = \frac{1}{3\sqrt{7}} \begin{pmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{7} \\ -1/\sqrt{7} \\ 2/\sqrt{7} \\ 1/\sqrt{7} \end{pmatrix}.$$

Now, since the rest two singular values are zeros and the fourth singular value is absent since (2) is the 3×3 matrix, we cannot compute u_2, u_3, and u_4 directly and need to obtain these vectors. Ideally u_2, u_3, u_4 are orthonormalized eigenvectors of matrix AA^T, and we could repeat the steps $1 - 5$ above to obtain such vectors. However, it is easily seen that since all other singular values are zeros, when composing matrix A back, the matrix product $U\Sigma$ will have only one non-zero column corresponding to the first singular value $3\sqrt{7}$, i.e. the first column. That means that u_2, u_3, and u_4 can be any orthonormal vectors. It easily seen that for u_2 we can take $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and for u_3 we can take $\begin{pmatrix} -1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$. To find u_4 we can apply the same technique used for finding v_2 above by constructing and solving a linear system that satisfies the condition of orthogonality (the equation for u_1 was multiplied by its denominator $\sqrt{7}$):

$$\begin{cases} \langle u_1, u_4 \rangle = 0, \\ \langle u_2, u_4 \rangle = 0, \\ \langle u_3, u_4 \rangle = 0 \end{cases} \Rightarrow \begin{cases} 1 \cdot x - 1 \cdot y + 2 \cdot w + 1 \cdot z = 0, \\ -1 \cdot x + 0 \cdot y + 0 \cdot w + 1 \cdot z = 0, \\ -1 \cdot x + 0 \cdot y + 1 \cdot w - 1 \cdot z = 0 \end{cases} \tag{9}$$

Solving (7) yields $u_4 = z \begin{pmatrix} 1 \\ 6 \\ 2 \\ 1 \end{pmatrix}$. Let $z = 1$, then $x = 1$, $y = 6$, and $w = 2$. So, $u_4 = \begin{pmatrix} 1 \\ 6 \\ 2 \\ 1 \end{pmatrix}$.

After normalizing u_2, u_3, and u_4 (u_1 is already normalized, since we obtained it from v_1),

$$U = \begin{pmatrix} 1/\sqrt{7} & -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{42} \\ -1/\sqrt{7} & 0 & 0 & 6/\sqrt{42} \\ 2/\sqrt{7} & 0 & 1/\sqrt{3} & 2/\sqrt{42} \\ 1/\sqrt{7} & 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{42} \end{pmatrix}. \tag{10}$$

Finally, let's check our decomposition by composing (1) back:

$$\begin{aligned} U\Sigma V^T &= \begin{pmatrix} \frac{1}{\sqrt{7}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{42}} \\ \frac{-1}{\sqrt{7}} & 0 & 0 & \frac{6}{\sqrt{42}} \\ \frac{2}{\sqrt{7}} & 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{42}} \\ \frac{1}{\sqrt{7}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{42}} \end{pmatrix} \begin{pmatrix} 3\sqrt{7} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{3} & -\frac{4}{3\sqrt{2}} & 0 \\ \frac{2}{3} & \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^T \\ &= \begin{pmatrix} 3 & 0 & 0 \\ -3 & 0 & 0 \\ 6 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & -\frac{4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix} \\ &= A. \end{aligned} \tag{11}$$

Answer

The required decomposition of (1) is:

$$A = U\Sigma V^T = \begin{pmatrix} \frac{1}{\sqrt{7}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{42}} \\ \frac{-1}{\sqrt{7}} & 0 & 0 & \frac{6}{\sqrt{42}} \\ \frac{2}{\sqrt{7}} & 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{42}} \\ \frac{1}{\sqrt{7}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{42}} \end{pmatrix} \begin{pmatrix} 3\sqrt{7} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{3} & -\frac{4}{3\sqrt{2}} & 0 \\ \frac{2}{3} & \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^T.$$

References

[1] Vsevolod Chernyshev. *Characteristic polynomial*. Faculty of Computer Science, Higher School of Economics. URL: https://smartedu.hse.ru/mod/page/0/701025.

[2] Vsevolod Chernyshev. *Eigenvectors and eigenvalues*. Faculty of Computer Science, Higher School of Economics. URL: https://smartedu.hse.ru/mod/page/0/701022.

[3] Dmitry Piontkovski. *Algorithm for finding SVD*. Faculty of Computer Science, Higher School of Economics. URL: https://smartedu.hse.ru/mod/page/0/701062.