Master of Data Science Online Programme
Course: Calculus
SGA #5: Fundamental Theorem of Calculus

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Problem 1

Assume
$$\int_0^{2\pi} \frac{dx}{2-\sin(x)}. \tag{1}$$
Its antiderivative is
$$-\frac{2}{\sqrt{3}}\arctan\left(\frac{1-2\tan(\frac{x}{2})}{\sqrt{3}}\right)+C$$
(you can check directly by differentiation). Evidently, this function is 2π periodic, thus by the fundamental theorem of calculus the integral should be 0. In the same time the integrand positive on all the segment thus the integral should be positive. Where is the problem in this conundrum? State and prove the property of the function that breaks FTC.

Solution

First let's compute (1) directly:
$$\begin{aligned}\int_0^{2\pi} \frac{dx}{2-\sin(x)} &= -\frac{2}{\sqrt{3}}\arctan\left(\frac{1-2\tan(\frac{x}{2})}{\sqrt{3}}\right)\Bigg|_0^{2\pi} \\ &= -\frac{2}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right)+\frac{2}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right) \\ &= 0.\end{aligned} \tag{2}$$
Indeed, this is not what we expect when calculating the area under the curve.

Let's denote the integrand of (1) by $f(x)$:
$$f(x) = \frac{1}{2-\sin(x)}. \tag{3}$$
Let's check, if (3) is integrable over the whole segment $[0, 2\pi]$. The function is integrable over a segment $[a, b]$, if it is:

- monotonic over the whole segment, and
- continuous over the whole segment.

It is easy to show that the function is continuous over $[0, 2\pi]$. Indeed, the denominator $2-\sin(x)$ has no roots, since $\sin(x)$ is bounded by ± 1, i.e. the denominator is always positive. Thus there are no discontinuity points. At the same time, the function might have other critical points like extrema, at which it changes its direction, i.e. the function might be non-monotonic. Let's find the extrema of (3) by finding the roots of its first derivative:
$$\left(\frac{1}{2-\sin(x)}\right)' = \frac{\cos(x)}{(2-\sin(x))^2} = 0. \tag{4}$$
The roots of (4) are the roots of its nominator, i.e. $\cos(x) = 0 \Rightarrow x = \frac{\pi}{2} + 2\pi n, n \in \mathbb{Z}$. For the segment $[0, 2\pi]$ we have two roots: $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. The sign of the derivative is determined by the sign of its nominator, i.e. $\cos(x)$, since the denominator $(2-\sin(x))^2$ is always positive. The signs of the derivative and the function directions are the following:

1. $x \in \left[0, \frac{\pi}{2}\right)$: $\cos(x) > 0 \Rightarrow (4) > 0 \Rightarrow (3)$ is rising;
2. $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$: $\cos(x) < 0 \Rightarrow (4) < 0 \Rightarrow (3)$ is falling, $x = \frac{\pi}{2}$ is a local maximum;
3. $x \in \left(\frac{3\pi}{2}, 2\pi\right]$: $\cos(x) > 0 \Rightarrow (4) > 0 \Rightarrow (3)$ is rising again, $x = \frac{3\pi}{2}$ is a local minimum.

So, (3) is non-monotonic over the whole segment $[0, 2\pi]$, but it is piece-wise monotonic over the three segments: $\left[0, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, and $\left[\frac{3\pi}{2}, 2\pi\right]$. Thus we need to calculate (1) as the sum of the three definite integrals over each segment:

$$\begin{aligned}
\int_0^{2\pi} \frac{dx}{2-\sin(x)} &= \int_0^{\frac{\pi}{2}} \frac{dx}{2-\sin(x)} + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{dx}{2-\sin(x)} + \int_{\frac{3\pi}{2}}^{2\pi} \frac{dx}{2-\sin(x)} \\
&= -\frac{2}{\sqrt{3}} \left[\arctan\left(\frac{1-2\tan(\frac{x}{2})}{\sqrt{3}} \right)\Bigg|_0^{\frac{\pi}{2}} + \arctan\left(\frac{1-2\tan(\frac{x}{2})}{\sqrt{3}} \right)\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + \arctan\left(\frac{1-2\tan(\frac{x}{2})}{\sqrt{3}} \right)\Bigg|_{\frac{3\pi}{2}}^{2\pi} \right] \\
&= \frac{2\pi}{3\sqrt{3}} + \frac{\pi}{\sqrt{3}} + \frac{\pi}{3\sqrt{3}} \\
&= \frac{2\pi}{\sqrt{3}}.
\end{aligned} \tag{5}$$

Answer

We have shown that the function (3) is non-monotonic over the whole segment $[0, 2\pi]$, thus it is not integrable over this segment. This is why we got the result (2). However, the correct value of the definite integral over the segment $[0, 2\pi]$ is the sum of the definite integrals for each segment of monotonicity of (3) which is $2\pi/\sqrt{3} \approx 3.6276$.