Some Remarks on Bergmann metrics

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1 Introduction

Let L be a holomorphic line bundle on a compact complex manifold M. A Kähler metric on M is *polarized* with respect to L if the Kähler form ω_g associated to g represents the Chern class $c_1(L)$ of L. Recall that if in a complex coordinate system $(z_1, \dots, z_n)$ of M the metric g is expressed by a tensor $(g_{j\bar{k}})_{1\leq j,\bar{k}\leq n}$ then ω_g is the d-closed $(1,1)$-form defined by $\frac{i}{2\pi}\sum_{j,\bar{k}=0}^{n} g_{j\bar{k}} dz_j \wedge d\bar{z}_k$.

The line bundle L is called a *polarization* of (M, g). In terms of cohomology classes, a Kähler manifold (M, g) admits a polarization if and only if ω_g is integral, i.e. its cohomology class $[\omega_g]_{dR}$ in the de Rham group, is in the image of the natural map $H^2(M, \mathbb{Z}) \hookrightarrow H^2(M, \mathbb{C})$. The integrality of ω_g implies, by a well-known theorem of Kodaira, that M is a projective algebraic manifold. This mean that M admits a holomorphic embedding into some complex projective space $\mathbb{C}P^N$. In this case a polarization L of (M, g) is given by the restriction to M of the hyperplane line bundle on $\mathbb{C}P^N$. Given a polarized Kähler metric g with respect to L, one can find a hermitian metric h on L with its Ricci curvature form $\mathrm{Ric}(h) = \omega_g$ (see Lemma 1.1 in [12]). Here $\mathrm{Ric}(h)$ is the 2-form on M defined by the equation:

$$\mathrm{Ric}(h) = -\frac{i}{2\pi}\partial\bar{\partial}\log h(\sigma(x), \sigma(x)), \tag{1}$$

for a trivializing holomorphic section $\sigma : U \subset M \to L \setminus \{0\}$ of L.

For each positive integer k, we denote by $L^{\otimes k}$ the k-th tensor power of L. It is a polarization of the Kähler metric kg and the hermitian metric h induces a natural hermitian metric h^k on $L^{\otimes k}$ such that $\mathrm{Ric}(h^k) = kg$.

Denote by $H^0(M, L^{\otimes k})$ the space of global holomorphic sections of $L^{\otimes k}$. It is in a natural way a complex Hilbert space with respect to the norm

$$\|s\|_{h^k} = \langle s, s\rangle_{h^k} = \int_M h^k(s(x), s(x))\frac{\omega_g^n(x)}{n!} < \infty,$$

for $s \in H^0(M, L^{\otimes k})$.

For sufficiently large k we can define a holomorphic embedding of M into a complex projective space as follows. Let $(s_0, \dots, s_{N_k})$, be a orthonormal basis for $(H^0(M, L^{\otimes k}), \langle \cdot , \cdot \rangle_{h^k})$ and let $\sigma : U \to L$ be a trivialising holomorphic section on the open set $U \subset M$. Define the map

$$\varphi_\sigma : U \to \mathbb{C}^{N_k+1} \setminus \{0\} : x \mapsto \left(\frac{s_0(x)}{\sigma(x)}, \dots, \frac{s_{N_k}(x)}{\sigma(x)}\right). \tag{2}$$

If $\tau : V \to L$ is another holomorphic trivialisation then there exists a non-vanishing holomorphic function f on $U \cap V$ such that $\sigma(x) = f(x)\tau(x)$. Therefore one can define a holomorphic map

$$\varphi_k : M \to \mathbb{C}P^{N_k}, \tag{3}$$

whose local expression in the open set U is given by (2). It follows by the above mentioned Theorem of Kodaira that, for k sufficiently large, the map φ_k is an embedding into $\mathbb{C}P^{N_k}$ (see, e.g. [6] for a proof).

Let $g_{FS}^{N_k}$ be the Fubini–Study metric on $\mathbb{C}P^{N_k}$, namely the metric whose associated Kähler form is given by

$$\omega_{FS}^{N_k} = \frac{i}{2\pi}\partial\bar{\partial}\log\sum_{j=0}^{N_k}|z_j|^2 \tag{4}$$

for a homogeneous coordinate system $[z_0, \dots, z_{N_k}]$ in $\mathbb{C}P^{N_k}$. This restricts to a Kähler metric $g_k = \varphi_k^* g_{FS}^{N_k}$ on M which is cohomologous to $k\omega_g$ and is polarized with respect to $L^{\otimes k}$. In [12] Tian christined the set of normalized metrics $\frac{1}{k}g_k$ as the *Bergmann* metrics on M with respect to L and he proves that the sequence $\frac{1}{k}g_k$ converges to the metric g in the C^2-topology (see Theorem A in [12]). This theorem was further generalizes by Ruan [10] who proved that the sequence $\frac{1}{k}g_k$ C^∞-converges to the metric g (see also [13]).

The aim of this paper is twofold. On one hand, in Section 2 we study, the polarized metrics g on M satisfying the equation

$$g_k = kg \tag{5}$$

(for some natural number k) which we call *self-Bergmann* metrics of degree k. If our Kähler manifold (M, g) is homogeneous and simply connected then the metric g is self-Bergmann of degree k for all sufficiently large k (for a proof see Theorem 2.1 below and cf. also [2]). In Theorem 2.4 and 2.6 we prove a sort of converse of Theorem 2.1 in the case of self-Bergmann metrics of degree 2 on $\mathbb{C}P^1$ induced by the Veronese map and in the case of self-Bergmann metrics of degree 1 on $\mathbb{C}P^1 \times \mathbb{C}P^1$ induced by the Segre map.

On the other hand, in Section 3, we consider the polarizations on non-compact Kähler manifolds (M, g). In particular we deal with the case of the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ equipped with the complete Kähler metric g^* whose associated Kähler form is given by

$$\omega^* = \frac{i}{2} \frac{dz \wedge d\bar{z}}{|z|^2}$$

and the polarization L given by the trivial bundle $L = \mathbb{C}^* \times \mathbb{C}$.

Our main results are contained in Theorem 3.5 where we describe all the hermitian metrics h^k on $L^{\otimes k} = L$ such that $\mathrm{Ric}(h) = \omega^*$ (in other words all the geometric quantizations on $(\mathbb{C}^*, \omega^*)$ (see Remark 2.3)). Moreover in Theorem 3.6 we calculate the set of Bergmann metrics $\frac{g_k}{k}$ and we prove that the sequence $\frac{g_k}{k}$ C^∞-converges to the metric g^* on every compact set $K \subset M$.

2 Self-Bergmann Metrics

As we pointed our in the introduction a large class of self-Bergmann metrics is given by the following:

Theorem 2.1 *(cfr. [2]) Let L be a polarization of a homogeneous and simply-connected compact Kähler manifold (M, g). Then g is self-Bergmann of degree k for every sufficiently large positive integer k.*

Proof: Recall that a Kähler manifold (M, g) is homogeneous if the group $\mathrm{Aut}(M) \cap \mathrm{Isom}(M, g)$ acts transitively on M, where $\mathrm{Aut}(M)$ denotes the group of holomorphic diffeomorphisms of M and $\mathrm{Isom}(M, g)$ the isometry group of M. Let k be large enough in such a way that the map $\varphi_k : M \rightarrow \mathbb{C}P^{N_k}$ given by (3) is an embedding. An easy calculation shows that

$$\omega_{g_k} = \varphi_k^*(\omega_{FS}^{N_k}) = k\omega_g + \frac{i}{2\pi}\partial\bar{\partial}\log\sum_{j=0}^{N_k} h^k(s_j, s_j) \tag{6}$$

where $\{s_0, \dots, s_{N_k}\}$ is the orthonormal basis for $(H^0(M, L^{\otimes k}, \langle \cdot, \cdot \rangle_{h^k})$, and where ω_{g_k}, in accordance with out notation, is the Kähler form associated to g_k. It turns out the if the manifold M is symply-connected then the holomorphic line bundle f^*L is isomorphic to L for any $f \in \mathrm{Aut}(M) \cap \mathrm{Isom}(M, g)$. Moreover the smooth function $\sum_{j=0}^{N_k} h^k(s_j, s_j)$ is invariant under the group $\mathrm{Aut}(M) \cap \mathrm{Isom}(M, g)$. Therefore, if (M, g) is assumed to be homogeneous then this function is constant which, by formula (6), implies that the metric g is self-Bergmann of degree k. $\square$

Remark 2.2 Note that the condition of simply-connectedness in Theorem 2.1 can not be relaxed. In fact the the n-dimensional complex torus M can be naturally endowed with a polarized flat metric g invariant by translation, making (M, g) into a homogeneous Kähler manifold. On the other hand the flat metric can not be the pull-back of the Fubini–Study metric via a holomorphic map (see Lemma 2.2 in [11] for a proof) and hence in particular condition (5) can not hold for any k (cf. also [8]).

Remark 2.3 In the terminology of quantization of a Kähler manifold (M, g) a pair (L, h) satisfying $\mathrm{Ric}(h) = \omega_g$ is called a *geometric quantization* of (M, g). In the work of Cahen–Gutt–Rawnsley the function $\sum_{j=0}^{N_k} h^k(s_j, s_j)$ is the central object of the theory (see [2], [3], [4], [5]). Infact it is one of the main ingredient needed to apply a procedure called *quantization by deformation* introduced by Berezin in his foundational paper [1]. Observe also that our definition of self-Bergmann metrics above is equivalent to the *regularity* of a quantization as defined in [2] and [3].

In view of Theorem 2.1 the following question naturally arises: *Let (M, g) be a homogenous and simply connected Kähler manifold (and hence g is self-Bergmann of degree k for k large) and let $\tilde{g}$ be a Kähler metric on M which is self-Bergmann of degree k. Can we conclude that also $\tilde{g}$ is homogeneous, namely there exists $f \in Aut(M)$ such that $\tilde{g} = f^*g$?*

When $M = \mathbb{C}P^N$, $g = g_{\omega_{FS}^N}$ and L is the hyperplane bundle, then the space $H^0(M, L)$ consisting of global holomorphic sections of L can be identified with the space of degree 1 homogeneous polynomials in the variables $\{z_0, \ldots, z_n\}$ (see, e.g. [6]). Let $\tilde{g}$ be a self-Bergmann metric of degree $k = 1$ then $N_k = \dim H^0(M, L) - 1 = N$ and the embedding φ_1 given by ((3) goes from $\mathbb{C}P^N$ to $\mathbb{C}P^N$. By the very definition of self-Bergmann metrics $\varphi_1^* g = \tilde{g}$ and since φ_1 belongs to the group $\mathrm{Aut}(\mathbb{C}P^N) = \mathrm{PGL}(N+1, \mathbb{C})$ we deduce that the previous question has a positive answer for $M = \mathbb{C}P^N$, $g = g_{\omega_{FS}^N}$ and $k = 1$.

The case of self–Bergmann metrics of any degree $k \geq 2$ on $\mathbb{C}P^N$ is much more complicated to handle even when $N = 1$. Nevertheless we prove the following:

Theorem 2.4 *Let $\tilde{g}$ be a self-Bergmann metric of degree 2 on $\mathbb{C}P^1$ induced by the Veronese map:*

$$\varphi : \mathbb{C}P^1 \to \mathbb{C}P^2 : [z_0, z_1] \mapsto [a z_0^2, b z_0 z_1, c z_1^2], \ a, b, c \in \mathbb{C}^*, \tag{7}$$

Then there exists $f \in PGL(2, \mathbb{C})$ such that $f^*(2g) = \tilde{g}$, where $g = g_{\omega_{FS}^1}$.

Proof: Under the action of $f \in PGL(2, \mathbb{C})$, we can suppose that the map (7) is given by

$$\varphi([z_0, z_1]) = [z_0^2, \alpha z_0 z_1, z_1^2],$$

for $\alpha \in \mathbb{C}^*$ (one simply defines $f([z_0, z_1]) = [\frac{1}{\sqrt{a}} z_0, \frac{1}{\sqrt{c}} z_1])$.

Observe that if $|\alpha|^2 = A = 2$ then $\varphi^* g_{FS}^2 = \varphi_2^* g_{FS}^2 = 2g$ which is self-Bergmann of degree k for large k by Theorem 2.1. Hence it is enough to show that if $\tilde{g}$ is self-Bergmann of degree 2 then $A = 2$. Let $\tilde{h}$ denote the hermitian structure on $H^0(M, L^{\otimes 2})$ such that $\mathrm{Ric}(\tilde{h}) = \omega_{\tilde{g}}$. Since $H^0(M, L^{\otimes 2})$ can be identified with the space homogeneous polynomials of degree 2 in z_0 and z_1, in order to prove our Theorem we need to show that if $\{z_0^2, \alpha z_0 z_1, z_1^2\}$ is a othonormal basis for $(H^0(M, L^{\otimes 2}), \langle \cdot , \cdot \rangle_{\tilde{h}})$ then $A = 2$.

In the chart $U_0 = \{z_0 \neq 0\}$, equipped with coordinate $z = \frac{z_1}{z_0}$, the Kähler form $\omega_{\tilde{g}}$ associated to $\tilde{g} = \varphi^* g_{FS}^2$ is given by:

$$\omega_{\tilde{g}} = \varphi^*(\omega_{FS}^2) = \frac{i}{2\pi} \partial \bar{\partial} \log(1 + A|z|^2 + |z|^4) = \frac{i}{2\pi} \frac{A + 4|z|^2 + A|z|^4}{(1 + A|z|^2 + |z|^4)^2} dz \wedge d\bar{z}.$$

Let $P(z_0, z_1)$ and $Q(z_0, z_1)$ be homogeneous polynomials of degre 2 in z_0 and z_1. We denote by small letter p and q their expression in U_0, namely

$p(z) = P(1, \frac{z_1}{z_0})$ and $q(z) = Q(1, \frac{z_1}{z_0})$. With the above notation the hermitian structure $\tilde{h}$ on U_0 is given by:

$$\tilde{h}(P, Q) = \frac{p(z)q(\bar{z})}{1 + A|z|^2 + |z|^4}.$$

Hence,

$$\langle P, Q\rangle_{\tilde{h}} = \int_{\mathbb{C}P^1} \tilde{h}(P, Q)\omega_{\tilde{g}} = \int_{\mathbb{C}} \frac{(A + 4|z|^2 + A|z|^4)p(z)q(\bar{z})}{(1 + A|z|^2 + |z|^4)^3} \frac{i}{2\pi} dz \wedge d\bar{z},$$

This can be written in polar coordinates $z = re^{i\theta}$ as

$$\langle P, Q\rangle_{\tilde{h}} = \frac{1}{\pi} \int_{r=0}^{+\infty} \frac{(A + 4r^2 + Ar^4)p(re^{i\theta})q(re^{-i\theta})}{(1 + Ar^2 + r^4)^3} r dr d\theta.$$

By the change of variable $r^2 = \rho$, one obtains:

$$\langle P, Q\rangle_{\tilde{h}} = \frac{1}{2\pi} \int_{\rho=0}^{+\infty} \frac{(A + 4\rho + A\rho^2)p(\sqrt{\rho}e^{i\theta})q(\sqrt{\rho}e^{-i\theta})}{(1 + A\rho + \rho^2)^3} d\rho. \tag{8}$$

It follows immediately by (8) that $\{z_0^2, z_0 z_1, z_2^2\}$ (which on U_0 is given by $\{1, z, z^2\}$) is an orthogonal basis of $(H^0(M, L^{\otimes 2}), \langle \cdot , \cdot \rangle_{\tilde{h}})$. Furthermore,

$$\|z_0\|^2_{\tilde{h}} = \int_{\rho=0}^{+\infty} \frac{(A + 4\rho + A\rho^2)}{(1 + A\rho + \rho^2)^3} d\rho,$$
$$\|\alpha z_0 z_1\|^2_{\tilde{h}} = A \int_{\rho=0}^{+\infty} \frac{(A\rho + 4\rho^2 + A\rho^3)}{(1 + A\rho + \rho^2)^3} d\rho,$$
$$\|z_2^2\|^2_{\tilde{h}} = \int_{\rho=0}^{+\infty} \frac{(A\rho^2 + 4\rho^3 + A\rho^4)}{(1 + A\rho + \rho^2)^3} d\rho.$$

A direct calculation, using Lemma 2.5 below gives:

$$\|z_0\|^2_{\tilde{h}} = (\frac{A^3}{4} - A)I_3 + \frac{A}{4} I_2 + 1 - \frac{A^2}{8}, \tag{9}$$

$$\|\alpha z_0 z_1\|^2_{\tilde{h}} = (\frac{A^3}{2} - \frac{A^5}{8})I_3 + (A - \frac{3A^3}{8})I_2 + \frac{A^4}{16}, \tag{10}$$

$$\|z_2^2\|^2_{\tilde{h}} = (\frac{A^5}{16} - \frac{A^3}{4})I_3 + (\frac{3A^3}{8} - \frac{5A}{4})I_2 + \frac{3A}{8} I_1 + 1 - \frac{3A^2}{16} - \frac{A^4}{32}. \tag{11}$$

Hence it remains to show that if $A \neq 2$, then either $\|z_0\|^2_{\tilde{h}} \neq A\|z_0 z_1\|^2_{\tilde{h}}$, or $\|z_0\|^2_{\tilde{h}} \neq \|z_2^2\|^2_{\tilde{h}}$. Indeed we prove that $\|z_0\|^2_{\tilde{h}} \neq A\|z\|^2_{\tilde{h}}$. Suppose, by a contradiction that $\|z_0\|^2_{\tilde{h}} = A\|z_0 z_1\|^2_{\tilde{h}}$. By subtracting (9) from (10) one obtains:

$$-32 + 6A^2 + 3A^4 - 12AI_1 + (72A - 24A^3)I_2 + 6A^3(A^2 - 4)I_3 = 0. \quad (12)$$

We distinguish two cases: $0 < A < 2$ and $A > 2$.

For $0 < A < 2$, we easily obtain:

$$\begin{aligned}
I_1 &= \frac{\pi}{\sqrt{4 - A^2}} - \frac{2}{\sqrt{4 - A^2}} \arctan \frac{A}{\sqrt{4 - A^2}}, \\
I_2 &= \frac{2\pi}{(\sqrt{4 - A^2})^3} - \frac{A}{4 - A^2} - \frac{4}{(\sqrt{4 - A^2})^3} \arctan \frac{A}{\sqrt{4 - A^2}}, \\
I_3 &= \frac{6\pi}{(\sqrt{4 - A^2})^5} + \frac{A^3 - 10A}{2(4 - A^2)^2} - \frac{12}{(\sqrt{4 - A^2})^5} \arctan \frac{A}{\sqrt{4 - A^2}}.
\end{aligned}$$

By (12) one gets:

$$-(8 + A^2)\sqrt{4 - A^2} + 6A\pi - 12A \arctan \frac{A}{\sqrt{4 - A^2}} = 0,$$

which can be easily seen to be impossible for $0 < A < 2$. Indeed the function $F(A) = -(8+A^2)\sqrt{4 - A^2} + 6A\pi - 12A \arctan \frac{A}{\sqrt{4-A^2}}$ satisfies $F(0) = -16$, $\lim_{A \to 2^-} F(A) = 0, F'(0) = 6\pi$, $\lim_{A \to 2^-} F'(A) = 0$ and $F''(A) = -6\sqrt{4 - A^2}$ which implies that $F(A) < 0$ on the interval $(0, 2)$.

For $A > 2$, we get:

$$\begin{aligned}
I_1 &= -\frac{1}{\sqrt{A^2 - 4}} \log \frac{A - \sqrt{A^2 - 4}}{A + \sqrt{A^2 - 4}}, \\
I_2 &= \frac{A}{A^2 - 4} + \frac{2}{(\sqrt{A^2 - 4})^3} \log \frac{A - \sqrt{A^2 - 4}}{A + \sqrt{A^2 - 4}}, \\
I_3 &= \frac{A^3 - 10A}{2(A^2 - 4)^2} - \frac{6}{(\sqrt{A^2 - 4})^5} \log \frac{A - \sqrt{A^2 - 4}}{A + \sqrt{A^2 - 4}}.
\end{aligned}$$

By (12) one gets:

$$(8 + A^2)\sqrt{A^2 - 4} + 6A \log \frac{A - \sqrt{A^2 - 4}}{A + \sqrt{A^2 - 4}} = 0,$$

which can not hold for $A > 2$.

Indeed the function $G(A) = (8+A^2)\sqrt{A^2-4} + 6A\log\frac{A-\sqrt{A^2-4}}{A+\sqrt{A^2-4}}$ satisfies $\lim_{A\to 2^+} F(A) = \lim_{A\to 2^+} F^{'}(A) = 0$, $\lim_{A\to+\infty} F(A) = \lim_{A\to+\infty} F^{'}(A) = +\infty$, and $F^{''}(A) = 6\sqrt{A^2-4}$ which implies that $F(A) > 0$ on $(2, +\infty)$.

$\square$

Lemma 2.5 *The following equalities hold:*

$$
\begin{aligned}
\int_{\rho=0}^{+\infty} \frac{\rho}{(1+A\rho+\rho^2)^3} d\rho &= \frac{1}{4} - \frac{A}{2} I_3;\\
\int_{\rho=0}^{+\infty} \frac{\rho^2}{(1+A\rho+\rho^2)^3} d\rho &= \frac{1}{4} I_2 + \frac{A^2}{4} I_3 - \frac{A}{8};\\
\int_{\rho=0}^{+\infty} \frac{\rho^3}{(1+A\rho+\rho^2)^3} d\rho &= \frac{1}{4} + \frac{A^2}{16} - \frac{3A}{8} I_2 - \frac{A^3}{8} I_3;\\
\int_{\rho=0}^{+\infty} \frac{\rho^4}{(1+A\rho+\rho^2)^3} d\rho &= \frac{3}{8} I_1 + \frac{3A^2}{8} I_2 + \frac{A^4}{16} I_3 - \frac{5A}{16} - \frac{A^3}{32},
\end{aligned}
$$

where

$$I_n = \int_{\rho=0}^{+\infty} \frac{1}{(1+A\rho+\rho^2)^n} d\rho, \quad n = 1, 2, 3.$$

Proof: Direct calculation integrating by parts. $\square$

Let consider now $M = \mathbb{C}P^1 \times \mathbb{C}P^1$ endowed with the metric $g = g_{FS}^1 + g_{FS}^1$ which we know to be self-Bergmann of degree k for all k (compare Theorem 2.1). In this case the map φ_1 (given by 3)) (which satisfies $\varphi_1^* g_{FS}^3 = g$) is given by:

$$\varphi_1 : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^3 : ([z_0, z_1], [w_0, w_1]) \mapsto [z_0w_0, z_0w_1, z_1w_0, z_1w_1].$$

The polarization L on M is the restriction to M of the hyperplane bundle on $\mathbb{C}P^3$ via the map φ_1 and a basis of $H^0(M, L)$ is $\{z_0w_0, z_0w_1, z_1w_0, z_1w_1\}$.

Theorem 2.6 *Let $\tilde{g}$ be a self-Bergmann metric of degree $k = 1$ on $M = \mathbb{C}P^1 \times \mathbb{C}P^1$ induced by the* Segree embedding *$\varphi : M \to \mathbb{C}P^3$ give by:*

$$\varphi([z_0, z_1], [w_0, w_1]) \mapsto [az_0w_0, bz_0w_1, cz_1w_0, dz_1w_1], a, b, c, d \in \mathbb{C}^*. \qquad (13)$$

Then there exists $f \in Aut(M) = PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$ such that $f^*g = \tilde{g}$.

Proof: The proof follows the same pattern of that of Theorem 2.4. First of all under the action of $f \in \mathrm{Aut}(M)$, we can suppose that the map (13) is given by

$$\varphi([z_0, z_1], [w_0, w_1]) = [\alpha z_0 w_0, z_0 w_1, z_1 w_0, z_1 w_1],$$

for $\alpha \in \mathbb{C}^*$. Indeed one takes $f([z_0, z_1], [w_0, w_1]) = [\frac{1}{b} z_0, \frac{1}{d} z_1], [\frac{d}{c} w_0, w_1]$. Hence it is enough to show that if $\tilde{g} = \varphi^* g_{FS}^3$ is a self-Bergmann metric of degree 1 then $A = |\alpha|^2 = 1$. Let $\tilde{h}$ be the hermitian structure on $H^0(M, L)$ such that $\mathrm{Ric}(\tilde{h}) = \omega_{\tilde{g}}$. In order to prove our Theorem it suffices to show that if $\{\alpha z_0 w_0, z_0 w_1, z_1 w_0, z_1 w_1\}$ is a othonormal basis for $(H^0(M, L), \langle \cdot , \cdot \rangle_{\tilde{h}})$ then $A = 1$. Let $U \cong \mathbb{C}^2$ be the chart on M defined by $(z_0, w_0) \neq (0, 0)$ equipped with coordinates $(z, w) = (\frac{z_1}{z_0}, \frac{w_1}{w_0})$. We can easily calculate the Kähler form $\omega_{\tilde{g}} = \varphi^*(\omega_{FS}^3)$ on U and obtain:

$$\omega_{\tilde{g}}^2 = \omega_g \wedge \omega_g = \frac{A(1 + |z|^2 + |w|^2) + |z|^2|w|^2}{(A + |z|^2 + |w|^2 + |z|^2|w|^2)^3} d\nu,$$

where $d\nu = (\frac{i}{2\pi})^2 dz \wedge d\bar{z} \wedge dw \wedge d\bar{w}$.

Let $P \in H^0(M, L) = span\{z_0 w_0, z_0 w_1, z_1 w_0, z_1 w_1\}$ We denote by small letter p its expression in the chart U, namely $p(z, w) = P(1, \frac{w_1}{w_0}, \frac{z_1}{z_0}, \frac{z_1}{z_0}\frac{w_1}{w_0})$. With the above notation the hermitian structure $\tilde{h}$ on U is given by:

$$\tilde{h}(P, Q) = \frac{p(z, w) q(\bar{z}, \bar{w})}{A + |z|^2 + |w|^2 + |z|^2|w|^2}.$$

Hence,

$$\langle P, Q \rangle_{\tilde{h}} = \int_M \tilde{h}(P, Q) \frac{\omega_{\tilde{g}}^2}{2!} = \frac{1}{2} \int_{\mathbb{C}^2} \frac{(A(1 + |z|^2 + |w|^2) + |z|^2|w|^2) p\bar{q}}{(A + |z|^2 + |w|^2 + |z|^2|w|^2)^4} d\nu,$$

for $P, Q \in H^0(M, L)$.

It follows that $\{\alpha z_0 w_0, z_0 w_1, z_1 w_0, z_1 w_1\}$ (which on U is given by $\{\alpha, w, z, zw\}$) is a othogonal basis of $(H^0(M, L), \langle \cdot, \cdot \rangle_{\tilde{h}})$. By passing in polar coordinates, a straightforward calculation gives:

$$\|\alpha z_0 w_0\|_{\tilde{h}}^2 = \|z_1 w_1\|_{\tilde{h}}^2 = \frac{1 - 3A + 2A^2 - A \log A}{48(A - 1)^2} \tag{14}$$

and

$$\|z_0 w_1\|_{\tilde{h}}^2 = \|z_1 w_0\|_{\tilde{h}}^2 = \frac{2 - 3A + A^2 + A \log A}{48(A - 1)^2}. \tag{15}$$

It is now easy to see that (14) and (15) are equal if and only if $A = 1$ which concludes the proof of our theorem. □

3 Quantizations and Bergmann metrics of $(\mathbb{C}^*, g^*)$

In this section we consider the case of a complete Kähler manifold (M, g). Let L be a holomorphic line bundle on M endowed with an hermitian structure h. Following Tian (Sect. 4 in [12]) we denote by $H^0_{(2)}(M, L^{\otimes k})$ the Hilbert space consisting of all L^2 integrable global holomorphic sections of $L^{\otimes k}$, namely

$$s \in H^0_{(2)}(M, L^{\otimes k}) \Leftrightarrow \langle s, s\rangle_{h^k} = \int_M h^k(s(x), s(x))\frac{\omega_g^n(x)}{n!} < \infty.$$

Let $\{s_j\}_{j\geq 0}$ be an orthonormal basis of $(H^0_{(2)}(M, L^{\otimes k}), \langle\ \cdot\ ,\cdot\ \rangle_{h^k})$. One if his main result , which generalizes the above mentioned Theorem A, is summarized in the following:

Theorem 3.1 *(Tian) Let M be a complete Kähler manifold with a polarized Kähler metric g and let L be a holomorphic line bundle with hermitian metric h such that its Ricci curvature form satisfies: Ric$(h) = \omega_g$. Then for any compact set $K \subset M$ and k sufficiently large*

$$\omega_k = \frac{i}{2\pi}\partial\bar{\partial}\log\sum_{j=0}^{+\infty}|s_j|^2 \tag{16}$$

defines a Kähler form on K. Moreover if g_k denotes the Kähler metric on K associated to ω_k (i.e. $\omega_{g_k} = \omega_k$) then the sequence of metrics $\frac{g_k}{k}$ C^2-converges to the Kähler metric g on K.

As in the compact case, a geometric quantization of a complete Kähler manifold (M, g) is given by a pair (L, h), where L is a holomorphic line bundle on M equipped with a hermitian metric h such that $\mathrm{Ric}(h) = \omega_g$ (see Remark 2.3)). The metrics $\frac{g_k}{k}$ (defined only on compact sets $K \subset M$) are called the *Bergmann metrics* on (M, g).

Remark 3.2 In analogy with the compact case, we say that a Kähler metric on a complete manifold is *self-Bergmann* of degree k if $g_k = kg$. Observe that this implies that g_k is globally defined on M and not only in a compact set $K \subset M$. A slight modification of Theorem 2.1 shows that in a homogeneous and simply-connected Kähler manifold (M, g) then the metric g is self-Bergmann of degree k for all k. Therefore, for example, the flat metric on the complex Euclidean space $\mathbb{C}^n$ is self-Bergmann of degree k.

In order to describe all the geometric quantizations of a Kähler manifold (M, g) one gives the following (cf. e.g. [9]):

Definition 3.3 *Two holomorphic hermitian line bundles* (L_1, h_1) *and* (L_2, h_2) *on a Kähler manifold* (M, g) *are called equivalent if there exists an isomorphism of holomorphic line bundles* $\psi : L_1 \to L_2$ *such that* $\psi^* h_2 = h_1$.

Let us denote by $[L, h]$ the equivalence class of (L, h) and by $\mathcal{L}(M, g)$ the set of equivalence classes. We refer the reader to [2] for the proof of the following:

Theorem 3.4 *The group* $Hom(\pi_1(M), S^1)$ *acts transitively on the set of equivalence classes* $\mathcal{L}(M, g)$.

In Theorem 3.5 below we describe this action in the case of $(\mathbb{C}^*, g^*)$. We first observe that any holomorphic line bundle L on $\mathbb{C}^*$ is holomorphically trivial. Let h be the hermitian metric on L given by:

$$h(f(z), f(z)) = e^{\frac{-\pi}{2}\log^2|z|^2}|f(z)|^2.$$

for a holomorphic function f on $\mathbb{C}^*$. It is easily seen that $\mathrm{Ric}(h_0) = \omega^*$ and hence L is a quantization of $(\mathbb{C}^*, g^*)$. We can prove now the first result of this section:

Theorem 3.5 *The group*

$$Hom(\pi_1(\mathbb{C}^*), S^1) = Hom(\mathbb{Z}, S^1) \cong S^1 \cong \frac{\mathbb{R}}{\mathbb{Z}}$$

acts on the set of equivalence classes $\mathcal{L}(\mathbb{C}^*, g^*)$ *by defining:*

$$[\lambda] \cdot (L, h) = (L, h_\lambda), \tag{17}$$

where $[\lambda]$ *denotes the equivalence class of* λ *in* $S^1 \cong \frac{\mathbb{R}}{\mathbb{Z}}$ *and* h_λ *is the hermitian metric on* L *defined by:*

$$h_\lambda(f(z), f(z)) = |z|^{2\lambda} h(f(z), f(z)), \tag{18}$$

for a holomorphic function f *on* $\mathbb{C}^*$.

Proof: Let λ and μ be real numbers such that $\lambda - \mu \in \mathbb{Z}$. It is easy to see that the map

$$\psi : (L, h_\mu) \to (L, h_\lambda) : (z, t) \mapsto (z, z^{\nu - \lambda} t)$$

is a holomorphic automorphism of the trivial bundle and $\psi^*(h_\lambda) = h_\nu$, namely $[L_0, h_\mu] = [L_0, h_\lambda]$. Furthermore, if $\lambda - \mu \notin \mathbb{Z}$ then $[L, h_\lambda] \neq [L, h_\mu]$. Indeed, suppose that $\psi : L \to L$ is a holomorphic automorphism of the trivial bundle, such that $\psi^* h_\lambda = h_\mu$. It follows that $\psi(z,t) = (z, f(z)t)$, where f is a holomorphic function on $\mathbb{C}^*$, satisfying $|f(z)|^2 = |z|^{2(\mu-\lambda)}$. This is impossible unless $\lambda - \mu$ is an integer. $\square$

Given a natural number k it follows immediately that the trivial bundle L endowed with the hermitian structure

$$h^k(f(z), f(z)) = e^{\frac{-k\pi}{2}\log^2|z|^2}|f(z)|^2$$

defines a quantization of $(\mathbb{C}^*, kg^*)$. By Theorem 3.5 we know that every class in $\mathcal{L}(\mathbb{C}^*, kg^*)$ can be represented by a pair (L, h_λ^k), where

$$h_\lambda^k(f(z), f(z)) := e^{\frac{-k\pi}{2}\log^2|z|^2}|z|^{2\lambda}|f(z)|^2. \tag{19}$$

and two such pairs (L, h_λ^k) and (L, h_μ^k) are equivalent iff $[\lambda] = [\mu]$. In what follows, to simplify the notation, we consider the class corresponding to $\lambda = 0$, namely the trivial bundle L on $\mathbb{C}^*$ endowed with the hermitian metric

$$h^k(f(z), f(z)) := e^{\frac{-k\pi}{2}\log^2|z|^2}|f(z)|^2.$$

It follows that the space $((H^0_{(2)}(\mathbb{C}^*, L), \langle\ \cdot\ ,\ \cdot\ \rangle_{h^k})$, which we will denote by $\mathcal{H}_k$, equals the space of holomorphic functions f in $\mathbb{C}^*$ such that

$$\|f\|^2_{h^k} = \langle f, f\rangle_{h^k} = \int_{\mathbb{C}^*} e^{\frac{-k\pi}{2}\log^2|z|^2}|f(z)|^2 k\frac{i}{2}\frac{dz\wedge d\bar{z}}{|z|^2} < +\infty.$$

One can check that the functions z^j, with $j \in \mathbb{Z}$, form an orthogonal system for $\mathcal{H}_k$. Since every holomorphic function in $\mathbb{C}^*$ can be expanded in Laurent series, it follows that z^j are in fact a complete orthogonal system. Their norms are given by

$$\begin{aligned}\|z^j\|^2_{h_0^k} &= k\int_{\mathbb{C}^*} e^{\frac{-k\pi}{2}\log^2|z|^2}|z|^{2j}\frac{i}{2}\frac{dz\wedge d\bar{z}}{|z|^2}\\ &= k\pi\int_0^{+\infty} e^{\frac{-k\pi}{2}\log^2 r^2} r^{2j}\frac{2r}{r^2}dr.\end{aligned}$$

By the change of variable $e^\rho = r^2$ one gets

$$\begin{aligned}\|z^j\|^2_{h^k} &= k\pi\int_{-\infty}^{+\infty} e^{\frac{-k\pi}{2}\rho^2}e^{j\rho}d\rho = k\pi e^{\frac{j^2}{2k\pi}}\int_{-\infty}^{+\infty} e^{-\left(\sqrt{\frac{k\pi}{2}}\rho-\sqrt{\frac{1}{2k\pi}}j\right)^2}\\ &= k\pi e^{\frac{j^2}{2k\pi}}\sqrt{\frac{2}{k\pi}}\int_{-\infty}^{+\infty} e^{-t^2}dt = \sqrt{2k}\pi e^{\frac{j^2}{2k\pi}},\end{aligned}$$

Then a orthonormal basis for $\mathcal{H}_k$ is given by

$$s_j = (\frac{1}{\sqrt{2k\pi}} e^{-\frac{j^2}{2k\pi}})^{\frac{1}{2}} z^j$$

and by formula (16) we get:

$$\omega_k = \frac{i}{2\pi} \partial\bar{\partial} \log \sum_{j\in\mathbb{Z}} e^{-\frac{j^2}{2k\pi}} |z|^{2j}. \tag{20}$$

Let $\frac{g_k}{k}$ be the corresponding sequence of Bergmann metrics (which are defined, by Theorem 3.1, on every compact set $K \subset \mathbb{C}^*$ for k sufficiently large). The following Theorem extends Tian's theorem 3.1 in the case of the punctured plane endowed with the metric g^*.

Theorem 3.6 *Let $\mathbb{C}^*$ be endowed with the complete metric g^*. Then the sequence of Bergmann metrics $\frac{g_k}{k}$ C^∞-converges to the metric g^* on every compact set $K \subset \mathbb{C}^*$.*

Proof: By formula (20) it is enough to show that the sequence of functions

$$f_k(x) = \frac{1}{k} \log(\sum_{j\in\mathbb{Z}} e^{\frac{-j^2}{2k\pi}} x^j) \tag{21}$$

(defined on $\mathbb{R}^+$) C^∞-converges to the function $f(x) = \frac{\pi}{2}\log^2 x$ on every compact set $C \subset \mathbb{R}^+$. In order to prove it we apply the Poisson summation formula (see p. 347, Theorem 24 in [7]) to the function $f(j) = e^{\frac{-j^2}{2k\pi}} x^j = e^{\frac{-j^2}{2k\pi} + j\log x}$. Namely, one has: $\sum_{j\in\mathbb{Z}} f(j) = \sum_{j\in\mathbb{Z}} \hat{f}(j)$, where $\hat{f}(j) = \int_{-\infty}^{+\infty} e^{-2\pi i j \nu} f(\nu)$. By a straightforward calculation one gets:

$$\begin{aligned} \hat{f}(j) &= e^{k\frac{\pi}{2}(2\pi i j - \log x)^2} \int_{-\infty}^{+\infty} e^{-\frac{1}{2\pi k}(\nu + 2\pi^2 i j k - \pi k \log x)^2} \\ &= 2\pi\sqrt{k} e^{k\frac{\pi}{2}\log^2 x} e^{-2k\pi^2 j(\pi j - i\log x)}. \end{aligned}$$

Thus

$$\begin{aligned} \lim_{k\to\infty} \frac{1}{k} \log \sum_{j\in\mathbb{Z}} f(j) &= \lim_{k\to\infty} \frac{1}{k} \log \sum_{j\in\mathbb{Z}} \hat{f}(j) \\ &= \frac{\pi}{2}\log^2 x + \lim_{k\to\infty} \frac{1}{k} \log \sum_{j\in\mathbb{Z}} e^{-2k\pi^2 j(\pi j - i\log x)}. \end{aligned}$$

It is now immediate to see that the sequence $\sum_{j\in\mathbb{Z}} e^{-2k\pi^2 j(\pi j - i\log x)}$ C^∞-converges to the constant function 1 on every compact set $C \subset \mathbb{R}^+$, which concludes the proof of our Theorem. Indeed,

$$|\sum_{j\in\mathbb{Z}} e^{-2k\pi^2 j(\pi j - i\log x)}| \leq 1 + \sum_{j\in\mathbb{Z}\setminus\{0\}} e^{-2k\pi^3 j^2} < 1 + \int_{-\infty}^{+\infty} e^{-2k\pi^3 t^2} dt = 1 + \frac{1}{\sqrt{2k}\pi}.$$

□

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