Radial balanced metrics on the unit disk

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Abstract

Let Φ be a strictly plurisubharmonic and radial function on the unit disk $\mathcal{D} \subset \mathbb{C}$ and let g be the Kähler metric associated to the Kähler form $\omega = \frac{i}{2}\,\partial\bar{\partial}\Phi$. We prove that if g is g_{eucl}-balanced of height 3 (where g_{eucl} is the standard Euclidean metric on $\mathbb{C} = \mathbb{R}^2$), and the function $h(x) = e^{-\Phi(z)}$, $x = |z|^2$, extends to an entire analytic function on $\mathbb{R}$, then g equals the hyperbolic metric. The proof of our result is based on a interesting characterization of the function $f(x) = 1 - x$.

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1 Introduction and statement of the main results

Let $\Phi\colon M \to \mathbb{R}$ be a strictly plurisubharmonic function on an n-dimensional complex manifold M and let g_0 be a Kähler metric on M. Denote by $\mathcal{H} = L^2_{hol}(M,\, e^{-\Phi}\,\frac{\omega_0^n}{n!})$ the separable complex Hilbert space consisting of holomorphic functions φ on M such that

$$\langle \varphi, \varphi \rangle = \int_M e^{-\Phi}\, |\varphi|^2\, \frac{\omega_0^n}{n!} < \infty, \tag{1}$$

where ω_0 is the Kähler form associated to the Kähler metric g_0 (this means that $\omega_0(X, Y) = g_0(JX, Y)$, for all vector fields X, Y on M, where J is the complex structure of M). Assume for each point $x \in M$ there exists $\varphi \in \mathcal{H}$ non-vanishing at x. Then, one can consider the following holomorphic map into the N-dimensional ($N \leq \infty$) complex projective space:

$$\varphi_\Phi : M \to \mathbb{C}P^N : x \mapsto [\varphi_0(x), \ldots, \varphi_N(x)], \tag{2}$$

where $\varphi_j,\ j = 0, \ldots, N$, is a orthonormal basis for $\mathcal{H}$. In the case $N = \infty$, $\mathbb{C}P^\infty$ denotes the quotient space of $l^2(\mathbb{C}) \setminus \{0\}$ (the space of sequences z_j such that $\sum_{j=1}^\infty |z_j|^2 < \infty$), where two sequences z_j and w_j are equivalent iff there exists $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ such that $w_j = \lambda z_j$, $\forall j$.

Let g be the Kähler metric associated to the Kähler form $\omega = \frac{i}{2}\partial\bar{\partial}\Phi$ (and so Φ is a Kähler potential for g). We say that the metric g is g_0*-balanced of height* α, $\alpha > 0$, if $\varphi_\Phi^* g_{FS} = \alpha g$, or equivalently

$$\varphi_\Phi^* \, \omega_{FS} = \alpha\omega, \tag{3}$$

where g_{FS} is the Fubini–Study metric on $\mathbb{C}P^N$ and ω_{FS} its associated Kähler form, namely

$$\omega_{FS} = \frac{i}{2} \, \partial\bar{\partial} \log \sum_{j=0}^{N} |Z_j|^2,$$

for a homogeneous coordinate system $[Z_0, \ldots, Z_N]$ of $\mathbb{C}P^N$ (note that this definition is independent of the choice of the orthonormal basis). Therefore, if g is a g_0-balanced metric of height α, then αg is projectively induced via the map (2) (we refer the reader to the seminal paper [5] for more details on projectively induced metrics). In the case a metric g is g-balanced, i.e. $g = g_0$, one simply calls g a *balanced* metric.

The study of balanced metrics is a very fruitful area of research both from mathematical and physical point of view (see [2], [6], [7], [11], [12], [13], [15], [16], [17] and [18]). The map φ_Φ was introduced by J. Rawnsley [21] in the context of quantization of Kähler manifolds and it is often referred to as the *coherent states map*.

Notice that one can easily give an alternative definition of balanced metrics (not involving projectively induced Kähler metrics) in terms of the reproducing kernel of the Hilbert space $\mathcal{H}$. Nevertheless the definition given here is motivated by the recent results on compact manifolds. In fact, it can be easily extended to the case when (M, ω) is a polarized compact Kähler manifold, with polarization L, i.e., L is a holomorphic line bundle L over M, such that $c_1(L) = [\omega]$ (see e.g. [1] and [3] for details). In the quantum mechanics terminology the bundle L is called the *quantum line bundle* and the pair (L, h) a *geometric quantization* of (M, ω). The problem of the existence and uniqueness of balanced metrics on a given Kähler class of a compact manifold M was solved by S. Donaldson [9] when the group of biholomorphisms of M which lifts to the quantum line bundle L modulo the $\mathbb{C}^*$ action is finite and by C. Arezzo and the second author in the general case (see also [19]).

Nevertheless, many basic and important questions on the existence and uniqueness of balanced metrics on noncompact manifolds are still open. For example, it is unknown if there exists a complete balanced metric on $\mathbb{C}^n$ different from the euclidean metric. The case of g_0-balanced metric on $\mathbb{C}^n$, where $g_0 = g_{eucl}$ is the Euclidean metric has been studied by the second author and F. Cuccu in [8]. There they proved the following.

Theorem A *Let g be a g_{eucl}-balanced metric (of height one) on $\mathbb{C}^n$. If Φ is rotation invariant then (up to holomorphic isometries) $g = g_{eucl}$.*

In this paper we are concerned with the g_{eucl}-balanced metrics g on the unit disk $\mathcal{D} = \{z \in \mathbb{C} \mid |z|^2 < 1\}$, where $g_{eucl} = dz \otimes d\bar{z}$ is the standard Euclidean metric on $\mathbb{C}$. In this case, the Hilbert space $\mathcal{H}$ consists of all holomorphic functions $\varphi\colon \mathcal{D} \to \mathbb{C}$ such that

$$\int_{\mathcal{D}} e^{-\Phi} |\varphi|^2 \frac{i}{2} dz \wedge d\bar{z} < \infty,$$

where Φ is a Kähler potential for g. Therefore $\mathcal{H}$ is the weighted Bergman space $L^2_{hol}(\mathcal{D}, e^{-\Phi}\omega_{eucl})$ on $\mathcal{D}$ with weight $e^{-\Phi}$. Notice that when $g = g_{hyp} = \frac{dz\otimes d\bar{z}}{(1-|z|^2)^2}$ is the hyperbolic metric on $\mathcal{D}$, then $\Phi(z) = -\log(1-|z|^2)$ is a Kähler potential for g_{hyp} and the Hilbert space $\mathcal{H} = L^2_{hol}(\mathcal{D}, e^{-\Phi}\omega_{eucl})$ consists of holomorphic functions f on $\mathcal{D}$ such that $\int_{\mathcal{D}}(1 - |z|^2)\, |f|^2 \frac{i}{2} dz \wedge d\bar{z} < \infty$. It is easily seen that $\sqrt{\frac{(j+1)(j+2)}{\pi}}\, z^j$, $j = 0, \ldots$ is an orthonormal basis of $\mathcal{H}$. The map (2), in this case, is given by:

$$\varphi_\Phi : \mathcal{D} \to \mathbb{C}P^\infty : z \mapsto [\ldots, \sqrt{\frac{(j+1)(j+2)}{\pi}}\, z^j, \ldots].$$

Thus,

$$\varphi_\Phi^* g_{FS} = \frac{i}{2}\partial\bar{\partial}\log\left[\frac{1}{\pi}\sum_{j=0}^{+\infty}(j+1)(j+2)\,|z|^{2j}\right] = \frac{i}{2}\partial\bar{\partial}\log\frac{1}{(1-|z|^2)^3} = 3\omega_{hyp}$$

and so g_{hyp} is a g_{eucl}-balanced metric of height $\alpha = 3$. Notice that the function $\Phi = -\log(1-|z|^2)$ is a radial function and $h(x) = e^{-\Phi(z)} = 1-|z|^2$, $x = |z|^2$, is an entire analytic function defined on all $\mathbb{R}$.

The following theorem, which is the main result of this paper, shows that the hyperbolic metric on the unit disk can be characterized by the previous data.

Theorem 1.1 *Let g be a Kähler metric on the unit disk $\mathcal{D}$. Assume that g admits a (globally) defined Kähler potential Φ which is radial and such that the function $h(x) = e^{-\Phi(z)}$, $x = |z|^2$, extends to a (real valued) entire analytic function on $\mathbb{R}$. If the metric g is g_{eucl}-balanced of height 3, then $g = g_{hyp}$.*

The proof of Theorem 1.1 is based on the following characterization of the function $f(x) = 1 - x$ very interesting on its own sake.

Lemma 1.2 *Let λ be a positive real number, and let $f\colon \mathbb{R} \to \mathbb{R}$ be an entire analytic function such that $f(x) > 0$ for all $x \in (0, 1)$. Define*

$$I_j = \int_0^1 f(t)\, t^j\, dt \quad \text{for } j \in \mathbb{N}. \tag{4}$$

If

$$\frac{2\,\lambda^2}{f^3(x)} = \sum_{j=0}^{+\infty} \frac{x^j}{I_j} \qquad \text{for all } x \in (0, 1), \tag{5}$$

then $f(x) = \lambda\,(1 - x)$ for all $x \in \mathbb{R}$.

Note that the radius of convergence of the series above is 1: indeed, since f is bounded and positive we have

$$\lim_{j \to +\infty} I_j^{1/j} = \lim_{j \to +\infty} \|t\|_{L^j((0,1),\, f(t)\, dt)} = \|t\|_{L^\infty((0,1),\, f(t)\, dt)} = \sup_{t \in (0,1)} |t| = 1.$$

Despite the very natural statement the proof of Lemma 1.2 is far from being trivial, and is based on a careful analysis of the behaviour of $f(x)$ and its derivatives as $x \to 1^-$.

In view of this lemma the authors believe the validity of the following conjecture which could be an important step towards the classification of g_{eucl}-balanced metrics of height α on the complex hyperbolic space $\mathbb{C}H^n$, namely the unit ball $B^n \subset \mathbb{C}^n$ equipped with the hyperbolic form $\omega_{hyp} = -\frac{i}{2}\,\partial\bar{\partial}\log(1 - \|z\|^2)$, $z \in B^n$.

Conjecture:
Fix a positive integer n and let

$$D_n = \{\, x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 < x_1 + \cdots + x_n < 1,\ x_j > 0 \,\}.$$

Suppose that there exists an integer $\alpha > n + 1$, a positive real number λ and an entire analytic function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) > 0$ for all $x \in D_n$ and

$$\frac{(\alpha - 1) \cdots (\alpha - n)\, \lambda^{n+1}}{f^\alpha(x)} = \sum_J \frac{x^J}{I_J(\alpha)},\ \forall J = (j_1, \ldots, j_n) \in \mathbb{N}^n,$$

where

$$I_J(\alpha) = \int_{D_n} f^{\alpha-(n+1)}(x)\, x^J\, dx_1 \cdots dx_n.$$

Then $f(x) = \lambda\,(1 - x_1 - \cdots - x_n)$.

Notice that Lemma 1.2 shows the validity of the previous conjecture for $n = 1$ and $\alpha = 3$.

Remark 1.3 The studies of balanced metrics on the unit ball $B^n \subset \mathbb{C}^n$ is far more complicated that one of studying the g_{eucl}-balanced metrics (we refer the reader to a recent paper of Miroslav Engliš [14] for the study of radial balanced metrics on B^n). The situation is similar in the compact case where there are no obstructions for the existence of g_0-balanced metrics (where g_0 is a fixed metric) on a given integral Kähler class of a compact complex manifold M while the existence of balanced metric on M is subordinated to the existence of a constant scalar curvature metric in that class (cf. [3] and [4]).

Remark 1.4 Lemma 1.2 should be compared with the following characterization of the exponential function due to Miles and Williamson [20] which is the main tool in [8] in order to prove Theorem A: *let* $f(x) = \sum_j b_j\, x^j$ *be an entire function on* $\mathbb{R}$ *such that* $b_0 = 1$, $b_j > 0$, $\forall j \in \mathbb{N}$, *and*

$$\int_{\mathbb{R}} \frac{b_j\, t^j}{f(t)}\, dt = 1, \;\; \forall j \in \mathbb{N},$$

then $f(x) = e^x$.

The paper contains another section where we prove Lemma 1.2 and Theorem 1.1.

2 Proof of the main results

In the proof of Lemma 1.2 we need the following elementary result.

Lemma 2.1 *Let* $r_0 \in \mathbb{N}$. *If a sequence* $\{c_j\}$ *satisfies* $c_j = O(j^{r_0})$ *as* $j \to +\infty$, *then the power series* $\sum_{j=0}^{+\infty} c_j\, x^j$ *converges in the interval* $(-1, 1)$ *to a function* $S(x)$ *such that* $S(x) = O((1-x)^{-r_0-1})$ *as* $x \to 1^-$.

Proof: If $r_0 = 0$ then the conclusion follows from the definition of the symbol O and the fact that $\sum_{j=0}^{+\infty} x^j = (1-x)^{-1}$. If, instead, $r_0 > 0$ then the conclusion follows similarly after the observation that $a_j = O((j+1) \cdot \ldots \cdot (j + r_0))$ and $\sum_{j=0}^{+\infty} (j+1) \cdot \ldots \cdot (j+r_0)\, x^j = r_0!\,(1-x)^{-r_0-1}$. $\square$

2.1 Proof of Lemma 1.2

By replacing $f(x)$ with $\lambda f(x)$ we may assume $\lambda = 1$. Unless otherwise stated, the variable x ranges in the interval $(0,1)$. The starting idea of the proof of Lemma 1.2 is the following. From the Taylor series of $f(x)$ at $x_0 = 1$,

$$f(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k f^{(k)}(1)}{k!} (1-x)^k, \tag{6}$$

we obtain an asymptotic estimate of the left-hand side of (5) (with $\lambda = 1$) as $x \to 1^-$. Moreover, by repeatedly integrating by parts we obtain, for every $j, k_0 \in \mathbb{N}$

$$I_j = \sum_{k=0}^{k_0} \frac{(-1)^k f^{(k)}(1)}{(j+1)\cdot\ldots\cdot(j+k+1)} + \frac{(-1)^{k_0+1}}{(j+1)\cdot\ldots\cdot(j+k_0+1)} \int_0^1 f^{(k_0+1)}(t)\, t^{j+k_0+1}\, dt. \tag{7}$$

Passing to the reciprocal $1/I_j$ and using Lemma 2.1 we obtain an asymptotic estimate of the right-hand side of (5) (with $\lambda = 1$). Since equality holds, we subsequently determine $f(1), f'(1), f''(1)$. Then, the proof is concluded by means of a more sophisticated argument.

Step 1: $f(1) = 0$ and $f'(1) = -1$. Denote by $k_0 \in \mathbb{N}$ the smallest natural number such that $f^{(k_0)}(1) \neq 0$. By (6) we get $f(x) = \frac{1}{k_0!} (-1)^{k_0} f^{(k_0)}(1) (1-x)^{k_0} (1 + O(1-x))$. In the sequel we will make often use of the following elementary expansion:

$$(1+t)^p = 1 + pt + O(t^2) \text{ as } t \to 0,\ p \in \mathbb{R}, \tag{8}$$

which implies, in particular, $(1 + O(1-x))^{-3} = 1 + O(1-x)$. Taking this into account, we deduce

$$\frac{2}{f^3(x)} = \frac{2\,(k_0!)^3\,(1 + O(1-x))}{(-1)^{k_0}\,(f^{(k_0)}(1))^3\,(1-x)^{3k_0}}. \tag{9}$$

Since we are assuming $f^{(k)}(1) = 0$ for $k < k_0$, and since the integral in (7) tends to zero at least as fast as $1/j$ as $j \to +\infty$, we may write

$$I_j = \frac{(-1)^{k_0} f^{(k_0)}(1)}{(j+1)\cdot\ldots\cdot(j+k_0+1)} (1 + O(1/j)),$$

which in turn, by (8), implies

$$\frac{1}{I_j} = \frac{(j+1)\cdot\ldots\cdot(j+k_0+1)}{(-1)^{k_0} f^{(k_0)}(1)} + O(j^{k_0}).$$

Taking Lemma 2.1 into account, multiplication by x^j followed by summation over j yields

$$\sum_{j=0}^{+\infty} \frac{x^j}{I_j} = \frac{(k_0+1)!}{(-1)^{k_0} f^{(k_0)}(1)\,(1-x)^{k_0+2}} + O((1-x)^{-k_0-1}) \text{ as } x \to 1^-.$$

By comparing the last equality with (9) it follows that k_0 must satisfy $3k_0 = k_0 + 2$, and therefore $k_0 = 1$. This implies $f(1) = 0$ and $(f'(1))^3 = f'(1)$. Since $f(x) > 0$ for $x \in (0,1)$, $f'(1)$ must be negative and we conclude $f'(1) = -1$.

Step 2: $f''(1) = 0$. By Taylor expansion we have $f(x) = (1-x)\,[\,1 + \frac{1}{2} f''(1)\,(1-x) + O((1-x)^2)\,]$. Using (8) we get

$$\frac{2}{f^3(x)} = \frac{2}{(1-x)^3} - \frac{3\,f''(1)}{(1-x)^2} + O((1-x)^{-1}). \tag{10}$$

Choosing $k_0 = 2$ in (7) and arguing as before, we also find $1/I_j = (j+1)(j+2) - (j+1)\,f''(1) + O(1)$ and therefore by Lemma 2.1

$$\sum_{j=0}^{+\infty} \frac{x^j}{I_j} = \frac{2}{(1-x)^3} - \frac{f''(1)}{(1-x)^2} + O((1-x)^{-1}).$$

By comparing the last estimate with (10) for $x \to 1^-$ we deduce $f''(1) = 0$.

At this point one could try to obtain the higher order derivatives $f^{(k)}(1)$, $k \geq 3$, as in Steps 1 and 2. Unfortunately this does not work. Indeed one can easily verify that by iterating the previous procedure one gets $f^{(k)}(1)$, $k \geq 4$ in terms of $f^{(3)}(1)$ but the latter remains undetermined. In order to overcome this problem notice that the previous steps imply that the function

$$z(x) := \frac{2}{f^3(x)} - \frac{2}{(1-x)^3} - \frac{f'''(1)}{1-x} \tag{11}$$

is real analytic in a neighbourhood of $x = 1$. Indeed, we have $f(x) = (1-x)\,[1 - \frac{1}{6} f'''(1)\,(1-x)^2 + (1-x)^3\,\varphi(x)]$ for an entire analytic function $\varphi(x)$. Furthermore, $(1+t)^{-3} = 1 - 3t + t^2\,\psi(t)$, where $\psi(t)$ is analytic for $t \in (-1, +\infty)$ and the claim follows.

Further, by (5) (with $\lambda = 1$), $z(x)$ admits the following expansion around the origin $z(x) = \sum_{j=0}^{+\infty} a_j\, x^j$, where

$$a_j = 1/I_j - (j+1)(j+2) - f'''(1), \quad \text{for } j \in \mathbb{N}. \tag{12}$$

The proof of the lemma will be completed by showing that $z(x)$ vanishes identically. Indeed this is equivalent to

$$1/I_j = (j+1)(j+2) + f'''(1), \quad \text{for } j \in \mathbb{N}, \tag{13}$$

which plugged into (5) (with $\lambda = 1$) gives

$$\frac{2}{f^3(x)} = \frac{2}{(1-x)^3} + \frac{f'''(1)}{1-x}. \tag{14}$$

This shows that $f(1-t)$ is an odd function of t and therefore $f^{(4)}(1) = 0$. Taking this into account, and using (7) with $k_0 = 4$ we obtain

$$I_j = \frac{1}{(j+1)\,(j+2)} - \frac{f'''(1)}{(j+1)\cdot\ldots\cdot(j+4)} + O(j^{-6}),$$

which in turn implies $1/I_j = (j+1)\,(j+2) + f'''(1) - 4\,f'''(1)/j + O(j^{-2})$. By comparing the last expansion with (13) we deduce $f'''(1) = 0$. This and (14) imply $f(x) = 1-x$ and this concludes the proof of the lemma.

In order to prove that the the sequence $\{a_j\}$ vanishes identically we need the following steps.

Step 3. For every integer k_1 there exists a rational function $Q_{k_1}(j)$ such that

$$a_j = Q_{k_1}(j) + O(j^{-k_1}) \text{ as } j \to +\infty. \tag{15}$$

Observe, firstly, that if (15) holds for a particular $k_1 = \overline{k}_1$, then it also holds for every $k_1 < \overline{k}_1$ with $Q_{k_1} = Q_{\overline{k}_1}$. Hence, it suffices to prove (15) for $k_1 \geq 1$. Letting $k_0 = k_1 + 2$ in (7) we obtain:

$$I_j = [(j+1)\,(j+2)]^{-1}[1 + \tilde{Q}_{k_1}(j) + O(j^{-k_1-2})],$$

where

$$\tilde{Q}_{k_1}(j) = \sum_{k=3}^{k_1+2} \tfrac{(-1)^k\, f^{(k)}(1)}{(j+3)\cdot\ldots\cdot(j+k+1)} = \tfrac{-f'''(1)}{(j+3)(j+4)} + O(j^{-3}).$$

Therefore

$$1/I_j = (j+1)\,(j+2)\,[1 + \hat{Q}_{k_1}(j) + O(j^{-k_1-2})],$$

where $\hat{Q}_{k_1}(j) = -\frac{\tilde{Q}_{k_1}(j)}{1+\tilde{Q}_{k_1}(j)}$.

Letting $Q_{k_1}(j) = (j+1)\,(j+2)\,\hat{Q}_{k_1}(j) - f'''(1)$, the claim follows by the definition (12) of a_j. In the next step we will need the observation that

$$Q_{k_1}(j) = O(j^{-1}). \tag{16}$$

Step 4. The sequence $\{a_j\}$ defined before tends to zero faster than every rational function of j, namely $a_j = O(j^{-k_1})$ as $j \to +\infty$ for every integer k_1. This is proved by showing that for every $k_1 \geq 2$ and every rational function Q_{k_1} satisfying (15) we have $Q_{k_1}(j) = O(j^{-k_1})$. Suppose that this is not the case. Then, by (16), there exist positive integers $d < k_1$ and a rational function Q_{k_1} satisfying (15) such that the limit $\lim_{j\to+\infty} j^d\, Q_{k_1}(j)$ is a finite $c \neq 0$. This and (15) imply $a_j = c\, j^{-d} + O(j^{-d-1})$. Now recall that the sum of the series $\sum_{j=1}^{+\infty} x^j/j$ is the unbounded function $-\log(1-x)$, while the series $\sum_{j=1}^{+\infty} x^j/j^2$ converges to a bounded function in the interval $[-1,1]$. By comparison with these elementary series it follows that the $(d-1)$-th derivative of $\sum_{j=0}^{+\infty} a_j\, x^j$ is unbounded for x close to 1^-. But this is impossible because the last series converges to $z(x)$, which is analytic in a neighbourhood of $x=1$. This contradiction shows that $Q_{k_1}(j) = O(j^{-k_1})$ and the claim follows.

Step 5. The sequence $\{a_j\}$ is identically zero. Define $w_j = [(j+1)(j+2) + f'''(1)]\, I_j - 1$. Since $w_j = -I_j\, a_j$ and I_j is positive, it suffices to show that $w_j = 0$ for all $j \in \mathbb{N}$. This is achieved by representing w_j as the limit $\lim_{k_0\to+\infty} S_{k_0}(j)$ of the sum $S_{k_0}(j)$ defined below, and then by showing that $S_{k_0}(j)$ is infinitesimal as $k_0 \to +\infty$. Taking into account that $\int_0^1 (1-t)^k\, t^j\, dt = k!\, j!/(j+k+1)!$, multiplication of (6) by t^j followed by termwise integration over the interval $(0,1)$ yields

$$I_j = \sum_{k=0}^{+\infty} \tfrac{(-1)^k\, f^{(k)}(1)}{(j+1)\cdot\ldots\cdot(j+k+1)}.$$

Notice that it makes sense to integrate over the interval $(0,1)$ since, by assumption, f is entire (cf. Remark 2.2). Since $f(1) = f''(1) = 0$ and $f'(1) = -1$, the preceding formula leads to

$$w_j = \tfrac{f'''(1)}{(j+1)(j+2)} + \sum_{k=3}^{+\infty} \tfrac{(-1)^k\, [(j+1)(j+2)+f'''(1)]\, f^{(k)}(1)}{(j+1)\cdot\ldots\cdot(j+k+1)}. \tag{17}$$

For $k_0 \geq 3$ we may write $w_j = S_{k_0}(j) + R_{k_0}(j)$, where the partial sum $S_{k_0}(j)$ and the remainder $R_{k_0}(j)$ are given by

$$S_{k_0}(j) \quad = \quad \tfrac{f'''(1)}{(j+1)(j+2)} + \sum_{k=3}^{k_0} \tfrac{(-1)^k\, [(j+1)(j+2)+f'''(1)]\, f^{(k)}(1)}{(j+1)\cdot\ldots\cdot(j+k+1)}, \tag{18}$$

$$R_{k_0}(j) \quad = \quad \sum_{k=k_0+1}^{+\infty} \frac{(-1)^k\,[(j+1)(j+2)+f'''(1)]\,f^{(k)}(1)}{(j+1)\cdot\ldots\cdot(j+k+1)}.$$

With this notation, formula (17) is equivalent to saying that for every $j \in \mathbb{N}$ we have $S_{k_0}(j) \to w_j$ and $R_{k_0}(j) \to 0$ as $k_0 \to +\infty$. By (7), the remainder $R_{k_0}(j)$ also admits the following representation:

$$R_{k_0}(j) = \frac{(-1)^{k_0+1}\,[(j+1)(j+2)+f'''(1)]}{(j+1)\cdot\ldots\cdot(j+k_0+1)} \int_0^1 f^{(k_0+1)}(t)\,t^{j+k_0+1}\,dt,$$

which shows that $R_{k_0}(j) = O(j^{-k_0})$ as $j \to +\infty$. Furthermore, since $w_j = -I_j\,a_j$ and I_j is bounded, by Step 4 we have, in particular, $w_j = O(j^{-k_0})$ as $j \to +\infty$. It follows that $S_{k_0}(j) = O(j^{-k_0})$ as $j \to +\infty$ and therefore we may write

$$S_{k_0}(j) = \frac{P^1_{k_0}(j)}{(j+1)\cdot\ldots\cdot(j+k_0+1)}, \tag{19}$$

where $P^1_{k_0}(j) = m_{k_0}\,j + q_{k_0}$ is a convenient polynomial of degree $\deg P^1_{k_0} \le 1$ in the variable j. In order to show that $S_{k_0}(j)$ is infinitesimal as $k_0 \to +\infty$ we have to investigate the coefficients m_{k_0}, q_{k_0}. Observe, firstly, that from (17) we get

$$S_{k_0+1}(j) = S_{k_0}(j) + \frac{(-1)^{k_0+1}\,[(j+1)(j+2)+f'''(1)]\,f^{(k_0+1)}(1)}{(j+1)\cdot\ldots\cdot(j+k_0+2)}.$$

This and (19) yield

$$\frac{P^1_{k_0+1}(j)}{(j+1)\cdot\ldots\cdot(j+k_0+2)} = \frac{P^1_{k_0}(j)}{(j+1)\cdot\ldots\cdot(j+k_0+1)} + \frac{(-1)^{k_0+1}\,[(j+1)(j+2)+f'''(1)]\,f^{(k_0+1)}(1)}{(j+1)\cdot\ldots\cdot(j+k_0+2)}.$$

By summation of the two rational functions on the right-hand side of the last equality, and since the coefficients of j^2, j, j^0 in the numerator must equal the corresponding ones in the left-hand side, we deduce

$$\begin{aligned} 0 &= m_{k_0} + (-1)^{k_0+1}\,f^{(k_0+1)}(1), \\ m_{k_0+1} &= (k_0-1)\,m_{k_0} + q_{k_0}, \\ q_{k_0+1} &= (k_0+2)\,q_{k_0} - [2+f'''(1)]\,m_{k_0}. \end{aligned}$$

Since the series (6) converges together with all its derivatives at $x = 0$, it follows that for every $h \in \mathbb{N}$ we have $m_{k_0} = o(k_0!\,k_0^{-h})$ as $k_0 \to +\infty$. The same holds for q_{k_0} because $q_{k_0+1} = (k_0+2)\,[m_{k_0+1} - (k_0-1)\,m_{k_0}] - [2 + f'''(1)]\,m_{k_0}$. Hence, by (19), it follows that $S_{k_0}(j) \to 0$ as $k_0 \to +\infty$. Since we have already observed that $S_{k_0}(j) \to w_j$ as $k_0 \to +\infty$, we finally conclude that $w_j = 0$ for all j, as claimed. $\square$

Remark 2.2 The assumption in Lemma 1.2 that f is an entire analytic function can be relaxed. If equality (6) holds in the interval $(-\varepsilon, 2+\varepsilon)$ for some $\varepsilon > 0$, then the same proof shows that $f(x) = \lambda\,(1-x)$ in that interval.

2.2 Proof of Theorem 1.1

Since the function $h(x) = e^{-\Phi(z)}$, $x = |z|^2$, extends to all real numbers it follows that $e^{-\Phi(z)}$ does not blow up at the boundary of $\mathcal{D}$. This implies that the monomials z^j, $j = 0, 1\dots$ are an orthogonal basis of $\mathcal{H} = L^2_{hol}(\mathcal{D},\, e^{-\Phi}\omega_{eucl})$. Hence the sequence $\sqrt{b_j}\, z^j$, $j = 0, \dots$, with

$$b_j = \left(\int_{\mathcal{D}} e^{-\Phi}\, |z|^{2j}\, \frac{i}{2}\, dz \wedge d\bar{z}\right)^{-1},$$

is an orthonormal basis of $\mathcal{H}$ and the Kähler metric g is g_{eucl}-balanced of height 3 iff

$$\frac{i}{2}\, \partial\bar{\partial} \log\Big(\sum_{j=0}^{\infty} b_j\, |z|^{2j}\Big) = 3\omega = 3\, \frac{i}{2}\, \partial\bar{\partial}\Phi.$$

This implies that the function $\Phi(z) - \log(\sum_{j=0}^{\infty} b_j\, |z|^{2j})^{\frac{1}{3}}$ is a radial harmonic function on $\mathcal{D}$ and hence equals a constant, say Φ_0. By setting $f(x) = (\sum_{j=0}^{\infty} b_j\, x^j)^{-\frac{1}{3}}$, $x \in [0, 1)$, and by the definition of the b_j's one then gets

$$e^{-\Phi_0}\, b_j \int_{\mathcal{D}} f(|z|^2)\, |z|^{2j}\, \frac{i}{2}\, dz \wedge d\bar{z} = 1,\ \forall j \in \mathbb{N}. \tag{20}$$

Observe that again the assumption that $h(x) = e^{-\Phi(z)}$, $x = |z|^2$, extends to an entire analytic function on $\mathbb{R}$ implies the same property for $f(x) = h(x)\, e^{\Phi_0}$. By passing to polar coordinates $z = \rho e^{i\theta}$, $\rho \in [0, +\infty)$, $\theta \in [0, 2\pi)$ and by the change of variables $t = \rho^2$ one obtains:

$$\pi e^{-\Phi_0}\, b_j \int_0^1 f(t)\, t^j\, dt = 1,\ \forall j \in \mathbb{N}.$$

By setting $\lambda^2 = \frac{\pi e^{-\Phi_0}}{2}$, $I_j = \frac{1}{2\lambda^2 b_j}$ and by the definition of $f(x)$ one gets (4) and (5). Therefore, by Lemma 1.2, $f(x) = \lambda\, (1 - x)$, i.e. $\Phi(z) = \Phi_0 - \log\lambda - \log(1 - |z|^2)$, which implies $\omega = \frac{i}{2}\, \partial\bar{\partial}\Phi = -\frac{i}{2}\, \partial\bar{\partial}\log(1 - |z|^2) = \omega_{hyp}$ and this concludes the proof of the theorem. □

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