Catastrophes minimization on the equilibrium manifold

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Abstract: In a fixed total resources setting we show that there exists a Riemannian metric g on the equilibrium manifold, which coincides with any (fixed) Riemannian metric with economic meaning in an arbitrarily small neighborhood of the set of critical equilibria such that a minimal geodesic connecting two regular equilibria is arbitrarily close to a smooth path which minimizes catastrophes.

Keywords: Equilibrium manifold, regular economies, catastrophes, Riemannian metric.

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1 Introduction

It has been shown by [5] the existence of a Riemannian metric g on the equilibrium manifold $E(r)$, which coincides with any given metric g_{eco} with economic meaning outside an arbitrarily small neighborhood of the set of critical equilibria, such that a minimal geodesic connecting two regular equilibria intersects the set of critical equilibria $E_c(r)$ transversally in a finite number points (see the Introduction in [5] for an economic interpretation). The idea behind this construction is that discontinuities of prices (*catastrophes*), which can arise when a path crosses the set of critical equilibria, should be reflected by the Riemannian metric: hence catastrophic paths should be longer than regular paths. Since the metric by [5] does not provide a criterion to choose amongst paths which have a different but finite number of intersections with $E_c(r)$, it is an interesting economic problem to construct a metric which enables us to choose paths which minimize this number.

This issue is strictly related to the investigation on catastrophes minimization led by [6], where the authors have defined the length of a path γ, $l(\gamma)$, connecting two regular equilibria x and y of the equilibrium manifold $E(r)$, as the number of intersection points of γ with the set of critical equilibria, $E_c(r)$. Then the "distance" $d(x, y)$ between x and y can be defined as the infimum of $l(\gamma)$, for all γ connecting x and y. A *minimal* path γ, i.e., $l(\gamma) = d(x, y)$, is showed to exist. The crucial point is that in [6] there is not any Riemannian metric on the equilibrium manifold.

In this paper we show that there exists a Riemannian metric on $E(r)$ which enables to improve the results by [5, 6]. Our main result is Theorem 4.1, where we show that there exists a Riemannian metric g on $E(r)$ which coincides with g_{eco} in an arbitrarily small neighborhood of critical equilibria and which satisfies the following condition. A minimal geodesic connecting two regular equilibria x and y is arbitrarily close to a path still connecting x and y, which is minimal as in [6], i.e. $l(\gamma) = d(x, y)$.

This paper is organized as follows. Section 2 recalls the economic setting. In Section 3 we consider the catastrophes minimization problem relative to a one connected component of the codimension one stratum of the set of critical equilibria. Finally, in Section 4 we provide a solution to the minimization problem in the general case (Theorem 4.1).

2 Economic setting

We consider a pure exchange economy with l goods and m consumers. Let $S = \{p = (p_1, \dots p_l) \mid p_j > 0, j = 1, \dots l, p_l = 1\}$ be the set of normalized prices. Denote by $\Omega = (\mathbb{R}^l)^m$ the space of endowments $\omega = (\omega_1, \dots, \omega_m)$, $\omega_i \in \mathbb{R}^l$. We assume that the standard assumptions of smooth consumer's theory are satisfied (see [2] Chapter 2). The problem of maximizing the smooth utility function $u_i : \mathbb{R}^l \to \mathbb{R}$ subject to the budget constraint $p \cdot \omega_i = w_i$ gives the unique solution $f_i(p, w_i)$, i.e. consumer's i demand. Let

E be the closed set consisting of pairs $(p, \omega) \in S \times \Omega$ satisfying the following equations:

$$\sum_{i=1}^{m} f_i(p, p \cdot \omega_i) = \sum_{i=1}^{m} \omega_i.$$

The set E is a smooth submanifold of $S \times \Omega$ globally diffeomorphic to $\mathbb{R}^{lm}$ [2, p. 73]. Let $\pi : E \to \Omega$ be the *natural projection*, i.e. the smooth map defined by the restriction to E of $(p, \omega) \mapsto \omega$. Let E_c be the set of critical equilibria, namely the pairs $(p, \omega) \in E$ such that the derivative of π at (p, ω) is not onto. The set of *singular economies*, denoted by Σ, is the image via π of the set E_c. It is a closed and measure zero subset of Ω. The set of *regular economies* $\mathcal{R} = \Omega(r) \setminus \Sigma$ represent the regular values of the map π. The map $\pi_{|_{\pi^{-1}(R)}} : \pi^{-1}(R) \to R$ is a finite covering [2, p. 91].
If total resources are fixed, the equilibrium manifold is defined as

$$E(r) = \{(p, \omega) \in S \times \Omega(r) | \sum_{i=1}^{m} f_i(p, p \cdot \omega_i) = r\},$$

where $r \in \mathbb{R}^l$ is the vector representing the total resources of the economy and $\Omega(r) = \{\omega \in \mathbb{R}^{lm} | \sum_{i=1}^{m} \omega_i = r\}$. Denote by $\pi : E(r) \to \Omega(r)$ the restriction of the natural projection to $E(r)$ and by $E_c(r)$ the set of critical points of π. The set $E_c(r)$ in the fixed total resource setting has a nice topological structure (see [1] and [2]):

Theorem 2.1 (Balasko) *$E(r)$ is a smooth manifold globally diffeomorphic to $\mathbb{R}^{l(m-1)}$ and $E_c(r)$ is a disjoint union of closed smooth submanifolds $\mathcal{S}_i,\ i = 1, \dots, inf(l-1, m-1)$ of $E(r)$. The manifold $\mathcal{S}_i$ has dimension $l(m-1) - i^2$ and $\mathcal{S}_i = \emptyset$ for $i > inf(l-1, m-1)$.*

Note that for a fixed i the manifold $\mathcal{S}_i$ could not be connected. Moreover, S_1 is the only stratum which disconnects $E(r)$.

3 Product metric on a connected component of S_1

In this section we analyze the catastrophes minimization issue by focusing our attention on a connected component of $\mathcal{S}_1$, the codimension one stratum of the set of critical equilibria $E_c(r)$. We denote this component by $\mathcal{S}$. Let g_{eco} be a fixed Riemannian metric on $E(r)$ with an *a priori* economic meaning. Our aim is to construct a Riemannian metric g on $E(r)$ which agrees with g_{eco} outside an arbitrarily small neighborhood of $\mathcal{S}$ and such that a minimal geodesic connecting two regular equilibria intersects $\mathcal{S}$ in the minimum number of points, namely 0 (resp. 1) if the regular equilibria belong (resp. do not belong) to the same connected component of $E(r) \setminus \mathcal{S}$ (see Corollary 3.3 below).
Let s_0 and s_1 be two critical equilibria belonging to $\mathcal{S}$. We start our analysis by constructing a neighborhood $\mathcal{S}^\delta$ of $\mathcal{S}$ ($\mathcal{S}^\delta$ depends on a real number $\delta > 0$ and $\mathcal{S}^\delta$ approaches $\mathcal{S}$ as $\delta \to 0$) and a Riemannian metric g on $E(r)$, which agrees with g_{eco} outside $\mathcal{S}^\delta$, satisfying the following property: *if $\gamma : I = [0, 1] \to E(r)$ is a minimal geodesic such that $\gamma(0) = s_0$ and $\gamma(1) = s_1$, then $\gamma(I) \subset \mathcal{S}$.*

Consider the normal bundle of $\mathcal{S}$ in $E(r)$, $N(S) = \{(x,v) \in \mathcal{S} \times \mathbb{R} \,|\, v \in N_x(S)\}$, where $N_x(\mathcal{S})$ is the g_{eco}-orthogonal complement of $\mathcal{S}$. By the Tubular Neighborhood Theorem [4, p. 76], there exists a diffeomorphism $T : U \to V$ from an open neighborhood U of S in $E(r)$ onto an open neighborhood V of $\mathcal{S}$ in $N(\mathcal{S})$, which maps each $s \in \mathcal{S}$ to the zero vector at s. Choose $\delta > 0$ sufficiently small in such a way that $\mathcal{S} \times (-\delta,\delta) \subset V \subset N(\mathcal{S})$ and let $\mathcal{S}^\delta \subset U = T^{-1}(\mathcal{S} \times (-\delta,\delta))$. Endow $\mathcal{S} \times (-\delta,\delta)$ with the *product metric* $g_p = g_s \oplus g_\delta$, where g_s is *any* Riemannian metric on $\mathcal{S}$ and g_δ is the standard Riemannian metric on $\mathbb{R}$. This means that given the two Riemannian manifolds $(\mathcal{S}, g_s)$ and $((-\delta,\delta), g_\delta)$, we consider the cartesian product $\mathcal{S} \times (-\delta,\delta)$ with the product structure [3, p. 42], i.e.,

$$(g_p)_{(x,y)}(v_1,v_2) = (g_s)_{(x)}(d\pi_s(v_1), d\pi_s(v_2)) + (g_\delta)_{(y)}(d\pi_\delta(v_1), d\pi_\delta(v_2)),$$

$$\forall (x,y) \in \mathcal{S} \times (-\delta,\delta),\ v_1, v_2 \in T_{(x,y)}(\mathcal{S} \times (-\delta,\delta))$$

where $\pi_s : \mathcal{S} \times (-\delta,\delta) \to \mathcal{S}$, and $\pi_\delta : \mathcal{S} \times (-\delta,\delta) \to (-\delta,\delta)$ are the natural projections. Finally, endow $\mathcal{S}^\delta$ with the *pull-back* metric

$$(g_p^*)_x(v,w) = (g_p)_{T(x)}(dT_x(v), dT_x(w)),\ \forall x \in \mathcal{S}^\delta, \forall v,w \in T_x\mathcal{S}^\delta.$$

Let C be a closed neighborhood of $\mathcal{S}$ such that $\mathcal{S}^\delta \subset C \subset U$. Consider the partition of unity $\lambda_\alpha : E(r) \to [0,1]$, $\alpha = 1,2$, subordinate to the open cover of $E(r)$ given by $U_1 = E(r) \setminus C$ and $U_2 = U$ such that

- $\lambda_1(x) + \lambda_2(x) = 1,\ \forall x \in E(r)$
- $\operatorname{supp} \lambda_\alpha \subset U_\alpha$, $\alpha = 1,2$.

Consider on $E(r)$ the Riemannian metric

$$g = \lambda_1 g_{eco} + \lambda_2 g_p^*.$$

It is equal to g_{eco} on $E(r) \setminus \operatorname{supp}(\lambda_2)$ and coincides with g_p on $\mathcal{S}^\delta$. Suppose now that γ is *any* smooth curve connecting s_0 and s_1, i.e. $\gamma(0) = s_0$ and $\gamma(1) = s_1$, such that $\gamma(I) \subset \mathcal{S}^\delta$ (see Figure 1).

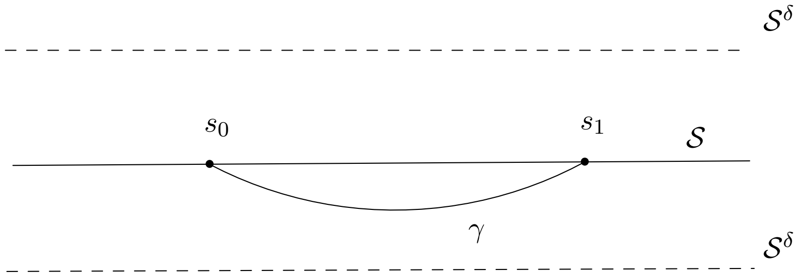

Figure 1: The path γ is not a minimal geodesic.

With a slight abuse of notation, identifying $\mathcal{S}^\delta$ and $\mathcal{S}\times(-\delta,\delta)$ via the diffeomorphism T, we can write $\gamma(t) = (\gamma_s(t), \gamma_\delta(t))$, where the first component belongs to $\mathcal{S}$ and the second one belongs to $(-\delta,\delta)$. Then

$$l_g(\gamma) = \int_0^1 \|\gamma'(t)\|_{g_p} dt = \int_0^1 (\|\gamma_s'(t)\|_{g_s} + \|\gamma_\delta'(t)\|_{g_\delta}) dt \geq \int_0^1 \|\gamma_s'(t)\|_{g_s} dt = l_g(\gamma_s).$$

(Here $l_g(\gamma)$ (resp. $l_g(\gamma_s)$) denotes the length of γ (resp. γ_s) with respect to g, $\|\gamma'(t)\|_{g_p} = ((g_p)_{\gamma(t)}(\gamma'(t), \gamma'(t)))^{1/2}$ and similarly for the other terms).

Assume now that γ is a minimal geodesic, i.e., $l_g(\gamma) \leq l_g(\sigma)$ for all path $\sigma : I \to E(r)$ connecting s_0 and s_1, Then the previous inequality is indeed an equality, i.e. $\|\gamma_\delta'(t)\|_{g_\delta} = 0$ and this forces the geodesic to lie on $\mathcal{S}$, i.e. $\gamma(I) \subset \mathcal{S}$.

What if a subset of γ does not belong to $\mathcal{S}^\delta$ as in Figure 2?

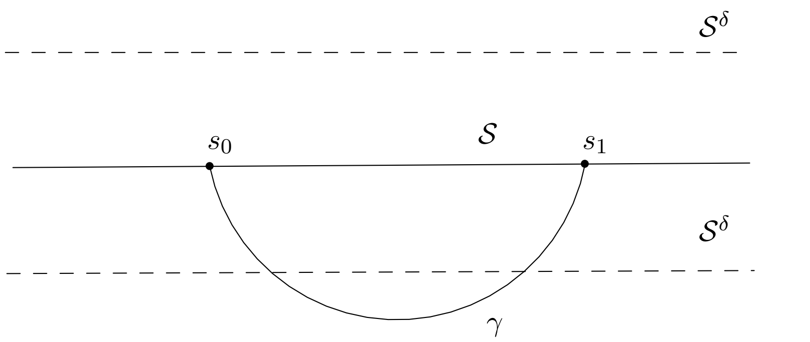

Figure 2: The path γ is not a minimal geodesic.

In this case we need to change the metric g in a suitable way. Observe as the

Riemannian metric g_s on $\mathcal{S}$ has not played any essential role so far. Let $\mathrm{diam}_{g_s}(\mathcal{S})$ be the diameter of $\mathcal{S}$ with respect to the metric g_s, namely

$$\mathrm{diam}_{g_s}(\mathcal{S}) = \max_{x,y\in\mathcal{S}} d_{g_s}(x,y),$$

where $d_{g_s} : \mathcal{S}\times\mathcal{S} \to \mathbb{R}_+$ is the distance on $\mathcal{S}$ induced by the metric g_s , i.e.

$$d_{g_s}(x,y) = \inf_{\sigma} \int_0^1 ||\sigma'(t)||_{g_\mathcal{S}} dt$$

and σ is varying amongst all the paths $\sigma : I \to \mathcal{S}$ such that $\sigma(0) = x$ and $\sigma(1) = y$. By choosing g_s in such a way that $\mathrm{diam}_{g_s}(\mathcal{S}) \leq 2\delta$ [1], where δ is the radius of $\mathcal{S}^\delta$, one can change the metric g accordingly. Then, for any curve γ as in Figure 2, it is easily seen that $l_g(\gamma) > d_g(s_0, s_1)$ and, therefore, γ cannot be a minimal geodesic.

We summarize what we have just showed in the following proposition.

Proposition 3.1 *Let g_{eco} be a metric on $E(r)$ with an economic meaning. Then there exists a Riemannian metric g on $E(r)$ which coincides with g_{eco} outside an arbitrarily small neighborhood of $\mathcal{S}$ and such that any minimal geodesic connecting two critical equilibria $s_0, s_1 \in \mathcal{S}$ must belong to $\mathcal{S}$.*

Remark 3.2 Notice that the submanifold $S \subset E(r)$ is *totally geodesic* with respect to the metric g constructed in Proposition 3.1 (this property has also played a crucial role to show the main result in [5]). Indeed, the condition that any minimal geodesic connecting two critical equilibria of $\mathcal{S}$ must belong to $\mathcal{S}$ implies that any geodesic $\gamma : I \to \mathcal{S}$ starting at $s = \gamma(0) \in \mathcal{S}$ and such that $\gamma'(0) \in T_s\mathcal{S}$ must belong to $\mathcal{S}$, i.e. $\mathcal{S}$ is totally geodesic. On the other hand there exist totally geodesic submanifolds where this condition is not fulfilled. Take, for example, the unit sphere $S^2 \subset \mathbb{R}^3$ of equation $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ (with the spherical metric induced by the Euclidean metric of $\mathbb{R}^3$), its totally geodesic submanifold (the equator) $S^1 = S^2 \cap \{z = 0\}$ and the points $x = (1,0,0)$ and $y = (-1,0,0)$. Then the smooth curve $\gamma : I \to S^2$, $\gamma(t) = (\cos \pi t, 0, \sin \pi t)$, namely the meridian of S^2 connecting x and y and passing through the north pole $(0,0,1) = \gamma(\frac{1}{2})$, is a minimal geodesic of S^2 which does not belong to S^1.

We finally state and prove the main result of this section.

Corollary 3.3 *Let g_{eco} be a metric on $E(r)$ with an economic meaning. Then there exists a Riemannian metric g on $E(r)$ which coincides with g_{eco} outside an arbitrarily small neighborhood of $\mathcal{S}$ and such that any minimal geodesic $\gamma : [0,1] \to E(r)$ connecting*

[1] A metric g_s with this property can be constructed as follows. Consider, for example, the Riemannian metric $\hat{g}$ on $E(r)$ obtained by the restriction of the spherical metric on $S^{l(m-1)} = E(r) \cup \{\infty\}$, the *Alexandroff one-point compactification* of $E(r) \simeq \mathbb{R}^{l(m-1)}$. Then the restriction of $\hat{g}$ on S, still denoted by $\hat{g}$, satisfies $\mathrm{diam}_{\hat{g}} < \infty$. Therefore the metric $g_s = \frac{2\delta}{\mathrm{diam}_{\hat{g}}(S)}\hat{g}$ satisfies $\mathrm{diam}_{g_s}(S) \leq 2\delta$.

two regular equilibria x and y intersects $\mathcal{S}$ in at most one point. Moreover γ intersects $\mathcal{S}$ exactly in one point if and only if x and y belong to different connected components of $E(r) \setminus \mathcal{S}$.

Proof: It follows by Theorem 2.1 that $\mathcal{S}$ is a connected codimension one submanifold of $E(r)$ and hence $E(r) \setminus \mathcal{S}$ consists of two disjoint connected open sets, say D_1 and D_2, which have $\mathcal{S}$ as common boundary (see [6] for details) . Since $\mathcal{S}$ is a totally geodesic submanifold of the Riemannian manifold $(E(r), g)$ the geodesic joining x and y intersects $\mathcal{S}$ in a finite number of points (cfr. Remark 3.2 or [5]). If $x, y \in D_1$ (resp. $x, y \in D_2$) then $\sharp\{\mathrm{Im}\,\gamma \cap \mathcal{S}\}$ is even. Hence, by Proposition 3.1, γ cannot cross $\mathcal{S}$. Similarly, if $x \in D_1$ and $y \in D_2$ (or viceversa) $\sharp\{\mathrm{Im}\,\gamma \cap \mathcal{S}\}$ is odd. Hence, again by Proposition 3.1, γ is forced to intersect $\mathcal{S}$ in exactly one point. $\square$

4 Main result

In this section our aim is to minimize catastrophes in the general case, where $E_c(r)$ is composed by a disjoint union of closed smooth submanifolds S_i, i denoting the codimension of the *stratum* of the set of critical equilibria with respect to $E(r)$. In [6] we have considered a similar minimization problem. We have defined the length of a path γ, $l(\gamma)$, connecting any two regular equilibria x and y, as the number of its intersection points with $E_c(r)$ and the "distance" $d(x, y)$ as the infimum of $l(\gamma)$, where γ is varying amongst all the smooth curves joining x and y. We have shown the existence of a minimal path γ, i.e., such that $l(\gamma) = d(x, y)$.[2]

The crucial difference with respect to the present paper is that in [6] the equilibrium manifold is not endowed with a Riemannian structure. Moreover, according to this definition of distance, only S_1 matters in their analysis, since it disconnects. On the contrary, in this paper we have the more powerful result that there is a Riemannian metric which realizes the distance. In [5] we have also considered the catastrophes-minimization problem in the general set up. We have showed that there exists a Riemannian metric on $E(r)$ such that a geodesic joining any two regular equilibria intersects $E_c(r)$ in a finite number of points. This construction does not rule out that the geodesic may intersect S_1 even when x and y belong to the same connected component. Both the results by [5, 6] are improved by the following theorem, where we construct a Riemannian metric on $E(r)$ such that a minimal geodesic realizes the distance in terms of catastrophes.

Theorem 4.1 *Let g_{eco} be any Riemannian metric on the equilibrium manifold $E(r)$. Then there exists a Riemannian metric g on $E(r)$ which coincides with g_{eco} in an arbitrarily small neighborhood of $E_c(r)$ and which satisfies the following condition. Let $\sigma : [0, 1] \to E(r)$ be a minimal geodesic connecting two regular equilibria. Then there*

[2]It is worth pointing out that the the map $d : (E(r) \setminus E_c(r)) \times (E(r) \setminus E_c(r)) \to \mathbb{R}$, defined by $d(x, y)$ is not a *distance* in the standard metric spaces terminology. For example, if x and y are distinct points belonging to the same connected components $E(r) \setminus E_c(r)$ then $d(x, y) = 0$. Nevertheless, it defines a distance on the quotient space $(E(r) \setminus E_c(r))/\sim$, where $x \sim y$ if and only if they belong to the same connected component of $E(r) \setminus E_c(r)$ (see [6] for details).

exists a smooth curve $\gamma : [0,1] \to E(r)$, joining x and y, arbitrarily close to σ and intersecting $E_c(r)$ transversally in a finite number of points equals to the distance $d(x,y)$.

Proof: By Theorem 2.1 we can write $\mathcal{S}_1 = \cup_j \mathcal{S}^j$ where each $\mathcal{S}^j$ is a connected codimension one submanifold of $E(r)$ and $\mathcal{S}^j \cap \mathcal{S}^k = \emptyset$, for $j \neq k$. It is not hard to extend the construction used to prove Corollary 3.1 (each $\mathcal{S}^j$ plays the role of $\mathcal{S}$) to get a Riemannian metric g on $E(r)$ such that any minimal geodesic $\sigma : I \to E(r)$ connecting two regular equilibria $x, y \in E(r) \setminus E_c(r)$ intersects each $\mathcal{S}^j$ in at most one point and exactly in one point if x and y belong to different connected components of $E(r) \setminus \mathcal{S}^j$. Note that σ does not generally realize the distance $d(x,y)$ between x and y since the set $A = \operatorname{Im} \sigma \cap (E_c(r) \setminus S_1)$ can be non-empty. If $A = \emptyset$ then $\gamma = \sigma$ is the desired path. Otherwise, by the Transversality Theorem [4, p. 68-69], σ can be perturbed to a smooth path γ joining x and y arbitrarly close to σ and transversal to $E_c(r) \setminus S_1$. Since the codimension of $E_c(r) \setminus S_1$ is greater than one, γ will be transversal to $E_c(r) \setminus S_1$ if it does not intersect it and we are done. □

Remark 4.2 In the case of two agents and two commodities, i.e. $l = m = 2$, it follows by Theorem 2.1 that there are not strata of critical equilibria of codimension greater than one: then the minimal geodesic connecting two regular equilibria needs not to be perturbed. In the general case, the probability that this perturbation will actually occur is zero since transversality is a generic property.

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