MATHEMATICS

When is the image of a proper map closed?

Asked 5 years ago Active 2 months ago Viewed 6k times

16

19

A map is called proper if the pre-image of a compact set is again compact.

In the Differential Forms in Algebraic Topology by Bott and Tu, they remark that the image of a proper map $f : \mathbb{R}^n \to \mathbb{R}^m$ is closed, adding the comment "(why?)".

I can think of a simple proof in this case for continuous f:

> If the image is not closed, there is a point p that does not belong to it and a sequence $p_n \in f(\mathbb{R}^n)$ with $p_n \to p$. Since f is proper $f^{-1}(\overline{B_\delta(p)})$ is compact for any δ. Let x_n be any point in $f^{-1}(p_n)$ and wlog $x_n \in f^{-1}(\overline{B_\delta(p)})$. Since in $\mathbb{R}^n$ compact and sequentially compact are equivalent, there exists a convergent subsequence x_{n_k} of x_n. From continuity of f: $f(x_{n_k}) \to f(x)$ for some x. But $f(x_{n_k}) = p_{n_k} \to p$ which is not supposed to be in the image and this gives a contradiction.

My problem is that this proof is too specific to $\mathbb{R}^n$ and uses arguments from basic analysis rather than general topology.

So the question is for what spaces does it hold that the image of a proper map is closed, how does the proof work, and is it necessary to pre-suppose continuity?

general-topology continuity compactness closed-map

Share Cite Improve this question Follow

edited Nov 6 '20 at 13:22

asked Jan 8 '16 at 10:57
s.harp
17.2k 2 26 57

5 Often map already implies continuity. I'd check the text for this. – Henno Brandsma Jan 8 '16 at 12:14

2 Answers

Active Oldest Votes

39

First of all the definition of a proper map assumes continuity by convention (I have not come across texts that say otherwise)

Secondly, here is a more general result -

Lemma : Let $f : X \to Y$ be a proper map between topological spaces X and Y and let Y be locally compact and Hausdorff. Then f is a closed map.

Proof : Let C be a closed subset of X. We need to show that $f(C)$ is closed in Y , or equivalently that $Y \setminus f(C)$ is open.

Let $y \in Y \setminus f(C)$. Then y has an open neighbourhood V with compact closure. Then $f^{-1}(\bar{V})$ is compact.

Let $E = C \cap f^{-1}(\bar{V})$. Then clearly E is compact and hence so is $f(E)$. Since Y is Hausdorff $f(E)$ is closed.

Let $U = V \setminus f(E)$. Then U is an open neighbourhood of y and is disjoint from $f(C)$.

Thus $Y \setminus f(C)$ is open. $\square$

I hope this helps.

EDIT: To clarify the statement U is disjoint from $f(C)$ -

Suppose $z \in U \cap f(C)$ Then there exists a $c \in C$ such that $z = f(c)$. This means $c \in f^{-1}(U) \subseteq f^{-1}(V) \subseteq f^{-1}(\bar{V})$. So $c \in C \cap f^{-1}(\bar{V}) = E$. So $z = f(c) \in f(E)$ which is a contradiction as $z \in U$.

Share Cite Improve this answer Follow

edited Jan 26 '19 at 3:33

answered Jan 9 '16 at 15:43
R_D
6,406 5 19 39

Why is U disjoint from $f(C)$? From your definition it is clear that $E \subseteq C$. So $f(E) \subseteq f(C)$. Hence

$V \setminus f(C) \subseteq V \setminus f(E) = U$. If the containment is proper then U may contain some element of $f(C)$. Who knows that? Isn't it so @R_D? – Dbchatto67 Jan 25 '19 at 10:26

Is it fine now? @Dbchatto67 – R_D Jan 26 '19 at 3:34

Yeah @R_D it's now absolutely fine. Thanks so much. – Dbchatto67 Jan 26 '19 at 6:04

Why is $f(E)$ compact? – Xiuyi Yang Mar 18 '20 at 3:26

@XiuyiYang continuous image of a compact set is compact. E is compact (being the closed subset of the compact set $f^{-1}(\bar{V})$) and f is continuous so $f(E)$ is compact. – R_D Mar 18 '20 at 14:58

12

One may generalize the result in R_D's answer even further:

> A proper map $f : X \to Y$ to a compactly generated Hausdorff space is a closed map (A space Y is called *compactly generated* if any subset A of Y is closed when $A \cap K$ is closed in K for each compact $K \subseteq Y$).
> **Proof:** Let $C \subseteq X$ be closed, and let K be a compact subspace of Y. Then $f^{-1}(K)$ is compact, and so is $f^{-1}(K) \cap C =: B$. Then $f(B) = K \cap f(C)$ is compact, and as Y is Hausdorff, $f(B)$ is closed. Since Y is compactly generated, $f(C)$ is closed in Y.

A locally compact space Y is compactly generated: If $A \subset Y$ intersects each compact set in a closed set, and if $y \notin A$, then A intersects the compact neighborhood K of y in a closed set C. Now $K \setminus C$ is a neighborhood of y disjoint from A, hence A is closed.

Share Cite Improve this answer Follow

edited Feb 28 '17 at 4:25
R_D
6,406 5 19 39

answered Jan 9 '16 at 16:29
Stefan Hamcke
24.6k 4 39 97