Geodesics on the equilibrium manifold*

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Abstract: We show the existence of a Riemannian metric on the equilibrium manifold such that a minimal geodesic connecting two (sufficiently close) regular equilibria intersects the set of critical equilibria in a finite number of points. This metric represents a solution to the following problem: given two (sufficiently close) regular equilibria, find the shortest path connecting them which encounters the set of critical equilibria in a finite number of points. Furthermore, this metric can be constructed in such a way to agree, outside an arbitrary small neighborhood of the set of critical equilibria, to any given metric with economic meaning.

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1 Introduction

We recall that the equilibrium manifold ($E(r)$ henceforth) is the set of pairs of price vectors p and endowments ω such that the aggregate excess demand function is equal to zero. One of its topological properties is the arc-connectedness [1, 4], i.e., it is always possible to link two arbitrarily chosen points (henceforth equilibria) in $E(r)$ with a continuous path. This property is important for policy making as it suggests that it is possible to design policies so that there is a continuous transition from one equilibrium to another more desirable one. Thus, there are some new insights for policy reform coming from the structure of the equilibrium manifold.

This point was noted by Balasko (see [4, p. 70]), where he raised the issue of choosing the path to follow on the equilibrium manifold to move from one equilibrium to another one. As Balasko observed: "...One of the most natural ideas would be to minimize length. Mathematically, this amounts to finding a geodesic[1] of the manifold E once it has been equipped with a suitable Riemannian structure (this means that a notion of distance extending the standard properties of the Euclidean distance has been defined on the manifold under consideration, here the equilibrium manifold E) conveying the major economic features involved in the planning problem. In this regard, formulating a reasonably relevant distance should be one of the first goals of planning theory. . . ".

It is natural that continuity of *price changes* is desirable for policy changes: as observed by [4, p.69], continuous paths should be preferred to discontinuous ones, with discontinuities being synonymous with catastrophes. However, the issue is delicate as the shortest path may involve intersections with critical equilibria. These equilibria are the very ones where there are discontinuous price changes or catastrophes. Then continuous paths which do not intersect the set of critical equilibria should be preferred to paths that are more "catastrophic". In fact, if a path should cross the set of critical equilibria, $E_c(r)$, properties such as local uniqueness of equilibria, continuity of the equilibrium price correspondence and the possibility of comparative static analysis (see [10], for the economic properties of the set of regular equilibria) would be lost giving rise to catastrophes [2].

A solution to the distance-catastrophes minimization problem mentioned above is equivalent to providing a positive answer to the following question: is it possible to construct a Riemannian metric on $E(r)$ such that a minimal geodesic connecting two regular equilibria intersects $E_c(r)$ in a finite number of points?

This problem is related to the work by [15] and [16]. In [16], under the assumption that the utility functions are real analytic, the equilibrium manifold and the set of critical equilibria become real analytic. This property allows to get a very

[1] A geodesic is a curve which minimizes distance between nearby points.

strong result: a minimal geodesic, joining any two regular equilibria, intersects transversally the whole set of critical equilibria. The assumption of real analyticity is a special case of smoothness. Under the standard smoothness assumption, the mathematical techniques and the results are different: in [15], in a smooth setting, the authors are only concerned with the codimension one stratum of the set of critical equilibria. In this case, it is necessary to modify the metric via a (smooth) partition of unity to make this stratum *totally geodesic* (see the following section for a definition). It is worth pointing out, the geodesic joining two sufficiently close regular equilibria should be perturbed in order not to intersect the strata of critical equilibria of codimension greater than one. As a consequence, the resulting perturbed path could not be a solution to the minimization problem because it could have lost the property of being a geodesic with respect to the given metric. Therefore the result obtained by [15] is a preliminary one.

This is clear if one reflects on the fact that a metric represents an algorithm (in this setting it could be identified with a redistribution policy) which, once given, allows us to construct without ambiguity the desired minimizing path (geodesic). Under this perspective there is a connection between our work and Garratt and Goenka's contribution [12]. In [12], the authors are concerned with the public finance issue of constructing a finite sequence of lump-sum taxes and transfers in such a way to avoid catastrophes (singularities) in the equilibrium manifold. Their policies stay within an open, connected set of regular economies in order to avoid counter-intuitive utility changes which may arise when a tax policy encounters singularities. The approach used in this paper allows us to consider the problem of income redistributions *with catastrophes* from the perspective of the equilibrium manifold. This means that instead of finding a policy on a subset of the regular economies as in [12], we look for a path on the equilibrium manifold. The policy can then be found through the projection of this path onto the space of economies (see Remark 3.4). We stress the importance of considering this kind of issues from the perspective of the equilibrium manifold. In fact this set is not simply a collection of prices and endowments such that the aggregate excess demand is zero. It conveys information not available in the space of economies which comes from the individualism and rationality assumptions (see [8]). In other words, its shape matters.[2]

This is where differential geometry plays its role: the Riemannian metric allows us to consider the intrinsic geometry of the equilibrium manifold. In this paper we construct a Riemannian metric on $E(r)$ such that a minimal geodesic connecting two sufficiently close regular equilibria intersects the *whole set* (all the strata) of

[2]For recent literature which explores the structure of the equilibrium manifold and its importance in aggregation theory and testability and identification issues in general equilibrium see [6, 7, 8].

critical equilibria in a finite number of points (Theorem 3.1). The endpoints of the geodesic are assumed to be "sufficiently close" regular equilibria because we are concerned with a fixed total resources setting. Our analysis is local but the Riemannian metric is defined globally, i.e., the algorithm can be applied for any two sufficiently close regular equilibria. Moreover, our local results (see points 2. and 3. in Theorem 3.1) are strong enough to rule out the possibility of intersecting the set of critical equilibria in more than one point.

In this paper we are not concerned with the issue of minimizing the number of the intersections of the path with the set of critical equilibria. This leads to an entirely different problem related to the work by [14], where the authors have defined the length between any two regular equilibria as the number of intersection points of the evolution path connecting them with the set of critical equilibria. They show that there exists a minimal path according to this definition of length. But in [14] no Riemannian metric is constructed: it has no connection with [15], [16] and the current paper.

It is worth noting that it is not our aim to suggest here a criterion according to which one can prefer a metric with respect to its economic meaning. We agree with Balasko [4, p. 70] that the connection between metric and economics still has to be investigated and we do not want to tackle this ambitious issue here. Nevertheless our construction allows us to obtain a further interesting result. Suppose we are given a metric with an *a priori* economic meaning, g_{eco}. Then one can construct a metric g, namely a solution to the distance-catastrophes minimization problem, which agrees with g_{eco} outside an arbitrarily small neighborhood of the set of critical equilibria. The metric g_{eco}, which can be *any metric*, plays here the role played by g_Φ in [15], where g_Φ was the *pull-back metric* (see [15]) of the Euclidean metric g_{can}, $E(r)$ being globally diffeomorphic to $\mathbb{R}^{l(m-1)}$ via the diffeomorphism $\Phi : E(r) \to \mathbb{R}^{l(m-1)}$ (see [4]).

Some features that characterize our analysis deserve a few comments. First, we do not consider the issue of the *Walras tatonnement*. We assume that the "fast dynamics" (see [4, p. 24]) of prices originated by the difference between demand and supply is zero. This assumption is consistent with the fact that the path we want to choose entirely belongs to the manifold $E(r)$. We realize that time and adjustment process are one of the main issues with the classical Arrow-Debreu model. The ideas developed in this paper can potentially be extended to consider out-of-equilibrium price dynamics. Second, in our construction we heavily rely on the very nice topological properties enjoyed by the set of critical equilibria $E_c(r)$, this set being a finite and disjoint union of closed smooth submanifolds $\mathcal{S}_j$ of $E(r)$ (see [3, 4]). An analogous result could not have been obtained working on the space of endowments $\Omega(r)$, since the set of singular economies could behave very badly (cf. Remarks 3.2 and 3.4).

This paper is organized as follows. Section 2 recalls the economic model and some mathematical results. Section 3 is devoted to the proof of our main result, Theorem 3.1.

2 Mathematical Preliminaries

In this section we collect some results that will be important for analysis (see [11], [13], and [15] for further details). A *Riemannian metric* g on a smooth k-dimensional manifold $X \subset \mathbb{R}^n$ is a family of inner products g_x on T_xX, $x \in X$, varying smoothly with x. A smooth manifold with a Riemannian metric is called a *Riemannian* manifold.

Example 2.1 We denote by g_{can} the Euclidean metric on $\mathbb{R}^n$, i.e., the metric defined by $g_{can}(v_1, v_2) = v_1 \cdot v_2$, where $v_1 \cdot v_2$ denotes the standard inner product in $\mathbb{R}^n$.

Example 2.2 If X is a smooth submanifold of a Riemannian manifold (Y, h), we can define a Riemannian metric g on X by restricting h to TX, the tangent bundle of X. Therefore every manifold $X \subset \mathbb{R}^n$ admits a Riemannian metric, obtained by restricting g_{can} to TX.

Example 2.3 Let $\Phi : X \to Y$ be a diffeomorphism. A Riemannian metric h on Y induces a Riemannian metric on X, called the *pull-back metric*, which we denote by $\Phi^*(h)$. This is defined by:

$$(\Phi^*(h))_x(v, w) = h_{\Phi(x)}(d\Phi_x(v), d\Phi_x(w)), \forall x \in X, \forall v, w \in T_xX.$$

Two Riemannian manifolds (X, g) and (Y, h) are *isometric* if there exists a diffeomorphism $\Phi : X \to Y$ such that $g = \Phi^*(h)$. The map Φ is then called an *isometry* between (X, g) and (Y, h).

We can use the metric g of a Riemannian manifold (X, g) to calculate the length of its curves and the distance between two points $x, y \in X$. Let $c : I \to X$ be a smooth curve on X, where I is an open interval of $\mathbb{R}$. We define the length of c from $c(a)$ to $c(b)$, $[a, b] \subset I$, by $\int_a^b (g_{c(t)}(\frac{dc}{dt}, \frac{dc}{dt}))^{\frac{1}{2}} dt$, where $\frac{dc}{dt}$ is the derivative of c with respect to t. We can define a distance $d(x, y)$ between $x, y \in X$ as the infimum of the lengths of all the piecewise smooth curves γ joining x and y. More precisely, one can show that: $d(x, y) = d(y, x)$, $d(x, y) \leq d(x, z) + d(z, y)$ and $d(x, y) \geq 0$ with equality iff $x = y$. Once the distance is defined, the space X becomes a *metric space* which we denote by (X, d).

A smooth curve $\gamma : I \to X$ on a Riemannian manifold (X, g) is a *geodesic* if the covariant derivative of $\gamma'(t)$ (which is a vector field along γ) is zero, i.e. $\frac{D}{dt}\gamma'(t) = 0$.

The geometric interpretation of a geodesic is the following. If we think of $\gamma'(t)$ as the velocity vector of $\gamma(t)$, then $\gamma(t)$ is a geodesic if the component of the acceleration $\gamma^{''}(t)$ on the tangent space $T_{\gamma(t)}X$ of X is zero for all t.

If one writes the equation of a geodesic in local coordinates one obtains a system of ordinary differential equations. Therefore a geodesic is uniquely determined by its initial conditions, namely its starting point and its tangent vector at this point as expressed by the following proposition (see [11] Proposition 2.5 p. 64).

Proposition 2.4 *Given $x \in X$ and $v \in T_xX$, there exists a unique geodesic γ such that $x = \gamma(t_0)$ and $v = \gamma^{'}(t_0)$, for some $t_0 \in I$.*

Every point of a Riemannian manifold (X, g) has an open neighborhood U, called *geodesic convex neighborhood* such that for any pair of points x and y on U there exists a unique minimal geodesic joining them and lying entirely on U (a geodesic joining x and y is said to be minimal if its length equals $d(x, y)$). The previous property is indeed crucial for our main result (see Theorem 3.1).

We conclude this section with the following additional results.

Theorem 2.5 (Tubular Neighborhood Theorem) *Let X be a submanifold of $\mathbb{R}^n$ and let $N(X, \mathbb{R}^n)$ be its normal bundle. There exists a diffeomorphism from an open neighborhood U of X in $\mathbb{R}^n$ onto an open neighborhood of X in $N(X, \mathbb{R}^n)$, which maps each $x \in X$ to the zero vector at x.*

See [13, p. 76] for a proof.

We now introduce a metric considered in the mathematical literature and often referred to as the *Sasakian metric* (see [9] and references therein). This metric will be used in our main result (Theorem 3.1) to make each connected component of the set of critical equilibria *totally geodesic* (see Theorem 2.6). We recall that a smooth submanifold X of a Riemannian manifold (Y, h) is said to be totally geodesic if every geodesic of X is a geodesic of (Y, h). Notice, e.g., that any geodesic of a Riemannian manifold (Y, h) is a one-dimensional totally geodesic submanifold of Y and that the totally geodesic submanifolds of $\mathbb{R}^n$ (with the Euclidean metric) are all the linear affine subspaces of $\mathbb{R}^n$.
Roughly speaking, in order to have a geometric intuition of the use of the Sasakian metric in our setting, we can think that it plays the role of making the set of critical equilibria as a union of (hyper)-planes with respect to the standard Euclidean metric. This implies that a geodesic of the equilibrium manifold, that we can imagine as a line, connecting two regular equilibria cannot belong to these planes (since they are totally geodesic) but must intersect them transversally (see Corollary 2.7).

Theorem 2.6 *Let X be a k-dimensional submanifold of $\mathbb{R}^n$ and denote by $E = N(X, \mathbb{R}^n)$ its normal bundle. Then there exists a metric g_E on E such that $X \subset (E, g_E)$ is totally geodesic.*

Proof: Let $\xi = (x, n) \in E$ ($x \in X$ and $n \in T_x^{\perp}X$) and let Ξ and $\tilde{\Xi}$ be two vectors in $T_\xi E$. Take two smooth curves $\xi : [0,1] \to E, \xi(t) = (x(t), n(t))$ and $\tilde{\xi} : [0,1] \to E, \tilde{\xi}(t) = (\tilde{x}(t), \tilde{n}(t))$ such that $\xi(0) = \tilde{\xi}(0) = \xi$ and $\xi'(0) = \Xi$ and $\tilde{\xi}'(0) = \tilde{\Xi}$. The desired metric g_E on E is defined as follows:

$$(g_E)_\xi(\Xi, \tilde{\Xi}) = h_x(d\pi_\xi(\Xi), d\pi_\xi(\tilde{\Xi})) + k_x(n'(0), \tilde{n}'(0)), \tag{1}$$

where $d\pi_\xi : T_\xi E \to T_x X$ is the differential of the projection $\pi : E \to X, (x, n) \mapsto x$, and h_x (resp. k_x) denotes the restriction of the Euclidean metric g_{can} (see Example 2.1 and 2.2) of $\mathbb{R}^n$ on $T_x X$ (resp. $T_x^{\perp}X$). Finally, let $f : E \to E$ be the smooth map defined by $f(\xi) = f((x, n)) = (x, -n)$. It is easily seen that f is an isometry of (E, g_E) and X is the fixed points set for f. By a standard argument in Riemannian geometry X is then totally geodesic in (E, g_E). $\square$

Theorem 2.6 combined with Lemma 2.8 in [15] yields the following

Corollary 2.7 *Let (E, g_E) be the normal bundle of a k-dimensional submanifold X of $\mathbb{R}^n$ equipped with the Sasakian metric. Then any geodesic of E joining two points x and y in the complement of X in E intersects X in a finite number of points.*

Figure 1 gives a geometrical perspective for $X = S^1$, where $N(S^1, \mathbb{R}^2) \approxeq \mathbb{R}^2$.

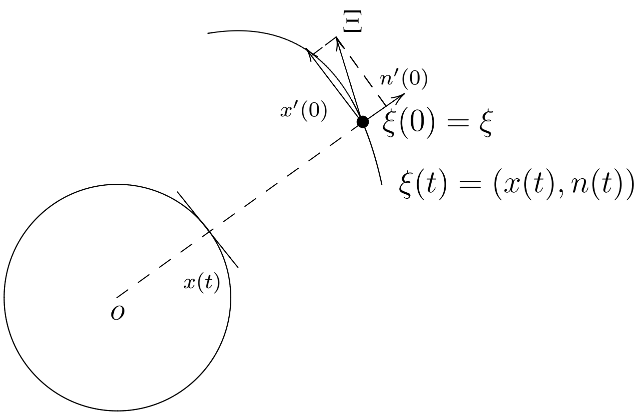

Figure 1: Sasakian metric

In the example above the Sasakian metric is $(g_E)_\xi(\Xi, \Xi) = \|x'(0)\|^2 + \|n'(0)\|^2$, where $\|\cdot\|$ denotes the standard norm on $\mathbb{R}^2$. In this case X is totally geodesic for g_E and the Sasakian metric equals the standard Euclidean metric on $\mathbb{R}^2$, i.e. $g_E = g_{can}$.

3 Main result

We consider a pure exchange economy with l goods, m consumers and fixed total resources. We denote with $S = \{p = (p_1, \dots p_l) | \, p_j > 0, j = 1, \dots l, p_l = 1\}$ the set of normalized prices, with $r \in \mathbb{R}^l$ the vector representing the total resources of the economy and with $\Omega(r) = \{\omega \in \mathbb{R}^{lm} | \sum_{i=1}^{m} \omega_i = r\}$ the space of endowments. We assume that the standard assumptions of smooth consumer's theory are satisfied (see [4, Chapter 2]). The equilibrium manifold in the fixed total resources setting is defined as (see [4])

$$E(r) = \{(p, \omega) \in S \times \Omega(r) | \sum_{i=1}^{m} f_i(p, p \cdot \omega_i) = r\},$$

where $f_i(p, p \cdot \omega_i)$ represents consumer's i demand function. Let $\pi : E(r) \to \Omega(r)$ be the *natural projection*, i.e. the smooth map defined by the restriction to $E(r)$ of $(p, \omega) \mapsto \omega$. Let $E_c(r)$ be the set of critical equilibria, namely the pairs $(p, \omega) \in E(r)$ such that the derivative of π at (p, ω) is not onto. We recall that Balasko [3, 4] has shown that the set $E_c(r)$ is a disjoint union of closed smooth connected submanifolds S_j (observe that in [15] the authors were only dealing with those S_j of codimension one).
The following theorem represents our main result.

Theorem 3.1 *Let g_{eco} be a given metric on E(r) with economic meaning and let $E_c(r) = \cup_j S_j$ be the set of critical equilibria. Then there exists a Riemannian metric g on $E(r)$ which coincides with g_{eco} outside an arbitrary small neighborhood of $E_c(r)$ in $E(r)$ and satisfies the following properties:*

1. *a geodesic joining any two regular equilibria intersects $E_c(r)$ in a finite number of points;*
2. *every regular equilibrium admits an open neighborhood U, disjoint from $E_c(r)$, such that every pair of regular equilibria in U can be joined by a unique and minimal geodesic which lies entirely on U;*
3. *every critical equilibrium admits an open neighborhood V such that every pair of regular equilibria in V can be joined by a unique and minimal geodesic lying on V and intersecting $E_c(r)$ in at most one point.*

Proof: Since ($E(r)$ is (globally) diffeomorphic to $\mathbb{R}^{l(m-1)}$ (see [4]), we can assume, without loss of generality, that $E(r) = \mathbb{R}^{l(m-1)}$.

Let U_j be an open neighborhood of S_j in $E(r)$ diffeomorphic to an open neighborhood N_j of S_j in its normal bundle $E_j = N(S_j, \mathbb{R}^{l(m-1)})$ whose existence is guaranteed by Theorem 2.5. Denote this diffeomorphism by $T_j : U_j \to N_j$. We can assume, by shrinking the U_j's if necessary, that $U_j \cap U_k = \emptyset$ for all $j \neq k$.

Let $V_j \subset U_j$ be a closed neighborhood of S_j and let $M_j \subset N_j \subset E_j$ be the image of V_j under T_j, i.e. $M_j = T_j(V_j)$. Let g_{E_j} be the Sasakian metric on E_j and denote with the same symbol, g_{E_j}, its restriction to M_j. Furthermore, let $\tilde{g}_j = T_j^*(g_{E_j})$ be the pull-back metric (see Example 2.3 above) on V_j induced by the map $T_j : V_j \to M_j$. Now we glue together the metrics $\tilde{g}_j$ and g_{eco}. Therefore, fix j and consider a *partition of unity* $\lambda^j_\alpha : E(r) \to [0,1], \alpha = 1, 2$, subordinate to the open cover of $E(r)$ given by two open sets $U^j_1 = E(r) \setminus V_j$ and $U^j_2 = U_j$. This means to take two smooth real valued functions $\lambda^j_\alpha : E(r) \to [0,1], \alpha = 1, 2$, such that:

- $\lambda^j_1(x) + \lambda^j_2(x) = 1, \ \forall x \in E(r)$;
- $\operatorname{supp} \lambda^j_\alpha \subset U^j_\alpha, \ \alpha = 1, 2$, namely each λ^j_α vanishes outside a closed subset contained in U^j_α for all $\alpha = 1, 2$.

Consider now the Riemannian metric g on $E(r)$ given by

$$g_j = \lambda^j_1 g_{eco} + \lambda^j_2 \tilde{g}_j$$

on U_j and equal to g_{eco} on $E(r) \setminus \cup_j \operatorname{supp}(\lambda^j_2)$. Since we can choose the U_j arbitrary small this proves the first assertion of the theorem.

It follows by Theorem 2.6 that (S_j, h_j) is totally geodesic in $(E(r), g)$. Hence, by Corollary 2.7, we deduce that every geodesic joining two regular equilibria intersects each S_j transversally, and hence in a finite number of points. This proves Property 1. The proof of Property 2 and 3 follows the same reasoning of the proof in [15]. Property 2 follows by taking a geodesic convex neighborhood U (see Section 2) around the regular equilibrium x such that U does not intersect $E_c(r)$. Finally, let V be a geodesic convex neighborhood around a critical equilibrium $y \in S_j$ such that $V \cap S_k = \emptyset$, $\forall k \neq j$. Let γ be the (unique) minimal geodesic joining two regular equilibria $\gamma(t_1) = x_1 \in V$, $\gamma(t_2) = x_2 \in V$, $t_1, t_2 \in [0, 1]$. Suppose, by contradiction, that γ intersects $E_c(r)$ in more than one point. By property 1, γ must intersect S_j in a finite number of points, say $\{y_1, \dots y_k\}$, $k \geq 2$. Let $\gamma(t) = x$ be a point in $\gamma([0, 1])$, $t_1 < t < t_2$, such that $\gamma([t_1, t]) \cap S_j = \{y_1, y_2\}$. Let $\sigma : [0, 1] \to S_j$ be the minimal geodesic of S_j joining $y_1 = \gamma(s_1)$ and $y_2 = \gamma(s_2)$, $t_1 < s_1 < s_2 < t_2$. Since S_j is totally geodesic in $E(r)$, σ is also a geodesic of $E(r)$. But then we have $\sigma([0, 1]) = \gamma([s_1, s_2])$ because V is a geodesic convex neighborhood. This gives the desired contradiction since S_j is totally geodesic in $E(r)$ (again by Corollary 2.7). $\square$

Remark 3.2 The fact that each $\mathcal{S}_j$ is a smooth submanifold of $E(r)$ is deeply related to the assumptions made on the utility function. This is not in general true for the critical set, say S, of a smooth map from $\mathbb{R}^n$ to itself even if S is of measure zero in $\mathbb{R}^n$. Consider, for example, the smooth map $\pi : \mathbb{R}^n \to \mathbb{R}^n, n \geq 2$ defined by

$$\pi(x_1, x_2, \ldots, x_n) = (e^{-\frac{1}{x_1^2}} \sin \frac{1}{x_1}, x_2, \ldots, x_n).$$

One can show that the set S of critical points of π is given by a countable union of hyperplanes parallel to the hyperplane $x_1 = 0$ and which accumulate to it. Therefore S is a measure zero set in $\mathbb{R}^n$ but it is not a smooth submanifold of $\mathbb{R}^n$.

Remark 3.3 The proof of the previous theorem is based on the manifold structure of the set $E_c(r)$. Without this property one does not have any chance to prove similar results, even if the set of points not to be intersected (i.e., the set of critical equilibria) is of measure zero (cf. [5] and Remark 2.2 in [15]). Consider, for example, the map $\pi : \mathbb{R}^n \to \mathbb{R}^n$ of Remark 3.2. Given any two points $x = (x_1, \ldots x_n), y = (y_1, \ldots y_n) \in \mathbb{R}^n \setminus S$ with $x_1 > 0$ and $y_1 < 0$ any (continuous) curve joining them intersects S in an infinite number of points. In particular there does not exist a Riemannian metric on $\mathbb{R}^n$ whose geodesic intersects S in a finite number of points.

Remark 3.4 One can observe that $\omega(t) = \pi(\gamma(t))$, i.e. the projection of a geodesic onto the Euclidean space of endowments $\Omega(r)$, will not generically be a straight line. Nevertheless it minimizes length and catastrophes as seen from the perspective of the equilibrium manifold. As such, it conveys relevant information not available when dealing with $\Omega(r)$. Moreover (see Remark 3.3), the set of singular economies may behave very badly compared with the set of critical equilibria (unfortunately the usual two-dimensional representations does not allow us to get a complete understanding of the phenomena connected with singularities).

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