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Kähler immersions into complex space forms

Relatore:
Prof. **Andrea Loi**

Candidato:
Marco Usula

Correlatore:
Prof. **Roberto Frigerio**

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Introduction

In Riemannian geometry, the famous Nash embedding theorem states that every Riemannian manifold admits an isometric embedding into some Euclidean space of sufficiently large dimension, endowed with its standard Riemannian metric. The problem is more complicate when the manifolds are Kähler. A Kähler manifold is a smooth manifold endowed with three mutually compatible structures: an almost complex integrable structure, a symplectic form, and a Riemannian metric. We cannot expect to find a direct analougous of Nash's statement in the Kähler case: in fact, for example, a compact Kähler manifold cannot admit a holomorphic and isometric (from now on Kähler) immersion into any complex Euclidean space, since a holomorphic map defined on a compact complex manifold with complex values must be constant, because of the maximum principle. It is more interesting to investigate on the Kähler immergibility problem when the ambient space is allowed to be a simply connected complex space form. A complex space form is a complete Kähler manifold whose holomorphic sectional curvature is constant: as in the Riemannian case, there is an uniformization theorem, which states that a simply connected complex space form is determined up to holomorphic isometries by its holomorphic sectional curvature. Since a rescalation of the metric of a complex space form provides another complex space form having curvature of the same sign, we have essentially three models of simply connected complex space forms: the complex Euclidean space with its standard Kähler metric (parabolic case, zero curvature), the complex hyperbolic space with its Bergmann metric (hyperbolic case, negative curvature), and the complex projective space with its Fubini-Study metric (elliptic case, positive curvature).

In his doctoral thesis, Eugenio Calabi considered the Kähler immergibility problem into simply connected complex space forms of finite or infinite dimension. He first observed that a Kähler immersed submanifold of a simply connected complex space form must be analytic, i.e. admits an analytic

Kähler potential in a neighborhood of every point. Then, he discovered that every analytic Kähler manifold admits a very special Kähler potential in a neighborhood of every point p_0, called the *diastatic potential* centered at p_0; the diastatic potential centered at p_0 is unique in a neighborhood of p_0, so it is defined a germ of real analytic function called the *germ of the diastasis at* p_0. The fundamental property of the diastasis is that it is preserved by Kähler immersions. Calabi used the diastasis to give a simple criterion of Kähler immergibility of a Kähler manifold into the simply connected complex space form of curvature $b \in \mathbb{R}$ and dimension $N \leq \infty$. Coherently with the fact that, in the "analytic world", local behaviour determines global behaviour, the criterion relies on the properties of the matrix of coefficients of the Taylor expansion of the germ of the diastasis at a point, with respect to some holomorphic chart centered at that point, and needs to be verified at only one point of the manifold. In the first part of the thesis, we present Calabi's diastasis and Calabi's criterion.

Although Calabi's criterion is theoretically impeccable, often it is difficult to compute diastatic potentials, so the criterion is of difficult applicability. However, it is interesting to study particular cases in which this criterion is applicable. In the second part of the thesis, following Calabi's work, we provide an example when the criterion is easily applicable: we provide a complete answer to the question of which simply connected complex space forms can be Kähler immersed into a given simply connected complex space form (of finite or infinite dimension). On the other hand, to study Kähler immersions into a complex projective space endowed with its Fubini-Study form, it is very useful the theory of *geometric quantizations of Kähler manifolds*. A *geometric quantization* of a Kähler manifold is a Hermitian holomorphic line bundle whose Ricci curvature coincides with the Kähler form. Not every Kähler manifold admits a geometric quantization: it turns out that this is equivalent to the integrality of the Kähler form. Geometric quantization is useful for our problem, since the Fubini-Study Kähler form of a complex projective space is integral, and hence a necessary condition on a Kähler manifold to be Kähler immersed into a complex projective space is to have integral Kähler form. In the final part of the thesis, following Loi and Mossa's work, we use geometric quantizations to prove that if the Kähler form of a simply connected and homogeneous Kähler manifold is integral, then, after a suitable rescalation of the metric, we can find a Kähler immersion of that Kähler manifold into a complex projective space of sufficiently large (possibly infinite) dimension, endowed with its Fubini-Study Kähler metric.

Chapter 1

Complex and Kähler geometry

1.1 Complex and almost complex manifolds

Definition 1. Let X be a topological space. A *n-dimensional complex atlas* on X is a family $\{(\varphi_\alpha, U_\alpha)\}$, where:

1. $\{U_\alpha\}$ is an open cover of X;
2. $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ are homeomorphisms onto open subsets of $\mathbb{C}^n$;
3. for every α, β the map $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is holomorphic.

A *n-dimensional complex manifold* is a Hausdorff topological space X with a countable basis, along with a maximal n-dimensional complex atlas on it: this maximal atlas is called the *complex structure* of X. An element of the complex structure of X is called an *holomorphic chart*.

Remark 2. From now on, unless otherwise stated, every manifold is assumed to be connected.

As in the smooth case, a complex atlas on a topological space X extends uniquely to a complex structure on X.

Definition 3. A map $f : X \to Y$ between complex manifolds is called *holomorphic* if, for every $p \in X$, there is a holomorphic chart (φ, U) of X in a neighborhood of p and a holomorphic chart (V, ψ) of Y in a neighborhood of $f(p)$ such that $f(U) \subseteq V$ and $\psi \circ f \circ \varphi^{-1}$ is holomorphic. A *biholomorphism* is a holomorphic map with an holomorphic inverse.

Remark 4. Every complex manifold has a natural smooth structure. More precisely, let X be a n-dimensional complex manifold. Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ through the homeomorphism

$$\begin{aligned} \Psi : \mathbb{C}^n &\to \mathbb{R}^{2n} \\ \left(x^1 + iy^1, ..., x^n + iy^n\right) &\mapsto \left(x^1, y^1, ..., x^n, y^n\right) : \end{aligned}$$

if $\{(\varphi_\alpha, U_\alpha)\}$ is the complex structure of X, then the family $\{(\Psi \circ \varphi_\alpha, U_\alpha)\}$ is clearly a smooth atlas on X, since every holomorphic map is in particular smooth. Now, let $\left(z^1, ..., z^n\right)$ and $\left(\tilde{z}^1, ..., \tilde{z}^n\right)$ be two holomorphic charts of X, and let $\left(x^1, y^1, ..., x^n, y^n\right)$ and $\left(\tilde{x}^1, \tilde{y}^1, ..., \tilde{x}^n, \tilde{y}^n\right)$ be the induced smooth charts. Then the fact that the change of coordinates between $\left(z^1, ..., z^n\right)$ and $\left(\tilde{z}^1, ..., \tilde{z}^n\right)$ is holomorphic is equivalent to the following equations, obtained by Cauchy-Riemann equations:

$$\begin{aligned} \frac{\partial x^k}{\partial \tilde{x}^j} &= \frac{\partial y^k}{\partial \tilde{y}^j} \\ \frac{\partial x^k}{\partial \tilde{y}^j} &= -\frac{\partial y^k}{\partial \tilde{x}^j}. \end{aligned}$$

The following three examples will be of constant use in this thesis:

Example 5. The *n-dimensional complex space* is the n-dimensional complex manifold $\mathbb{C}^n$.

Example 6. The *n-dimensional complex projective space* is the quotient space

$$\mathbb{CP}^n = \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\mathbb{C}^*},$$

i.e. the set of the complex lines of $\mathbb{C}^{n+1}$, with the quotient topology. If $\left(Z^0, ..., Z^n\right) \in \mathbb{C}^{n+1} \setminus \{0\}$, we denote as usual its equivalence class by $\left[Z^0 : \cdots : Z^n\right]$. For $j = 0, ..., n$, define

$$U_j = \left\{[Z] \in \mathbb{CP}^n : Z^j \neq 0\right\}$$

and

$$\begin{aligned} \varphi_j : U_j &\to \mathbb{C}^n \\ [Z] &\mapsto \left(\frac{Z^0}{Z^j}, ..., \frac{Z^{j-1}}{Z^j}, \frac{Z^{j+1}}{Z^j}, ..., \frac{Z^n}{Z^j}\right) : \end{aligned}$$

the family $\{(U_j, \varphi_j)\}$ is a n-dimensional complex atlas on $\mathbb{CP}^n$, that makes it a n-dimensional complex manifold. The complex manifold $\mathbb{CP}^n$ is compact and simply connected. The complex atlas $\{(U_j, \varphi_j)\}$ is called the *canonical complex atlas* of $\mathbb{CP}^n$. We use the notation $0 = [1 : 0 : \cdots : 0]$.

Example 7. The *n-dimensional complex hyperbolic space* has two biholomorphic models. The *ball model* is the open complex submanifold of $\mathbb{C}^n$

$$\mathbb{CB}^n = \{z \in \mathbb{C}^n : |z| < 1\}.$$

The *projective model* is defined as an open complex submanifold of $\mathbb{CP}^n$ in the following way. Define on $\mathbb{C}^{n+1}$ a nondegenerate Hermitian product

$$\begin{aligned} (,) : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} &\to \mathbb{C} \\ \left(\left(Z^0, ..., Z^n\right), \left(W^0, ..., W^n\right)\right) &\mapsto -Z^0\overline{W^0} + \sum_{j=1}^{n} Z^j\overline{W^j} : \end{aligned}$$

this Hermitian product has signature $(n, 1)$. Observe that if $\lambda \in \mathbb{C}^*$, then $(\lambda Z, \lambda Z) = |\lambda|^2 (Z, Z)$ for every $Z \in \mathbb{C}^{n+1}$, so the open subset of $\mathbb{CP}^n$

$$\mathbb{CH}^n = \{[Z] \in \mathbb{CP}^n : (Z, Z) < 0\}$$

is well defined. The complex manifolds $\mathbb{CB}^n$ and $\mathbb{CH}^n$ are biholomorphic: in fact, observe that $\mathbb{CH}^n$ is contained in the open set U_0 of the canonical complex atlas of $\mathbb{CP}^n$, and $\varphi_0 : U_0 \to \mathbb{C}^n$ restricts to a biholomorphism $\mathbb{CH}^n \to \mathbb{CB}^n$.

Example 8. Let S be a subset of $\mathbb{C}^n$ (resp. $\mathbb{CP}^n, \mathbb{CH}^n$). The *subspace generated by S* is:

1. ($\mathbb{C}^n$) the affine subspace of $\mathbb{C}^n$ generated by S;

2. ($\mathbb{CP}^n$) the image of the set span $\left(Z \in \mathbb{C}^{n+1} : [Z] \in S\right) \setminus \{0\} \subseteq \mathbb{C}^{n+1} \setminus \{0\}$ via the canonical submersion $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$;

3. ($\mathbb{CH}^n$) the intersection of $\mathbb{CH}^n$ and the subspace of $\mathbb{CP}^n$ generated by S.

It is easy to verify that every subspace of $\mathbb{C}^n$ (resp. $\mathbb{CP}^n, \mathbb{CH}^n$) is a closed complex submanifold biholomorphic to $\mathbb{C}^{n'}$ (resp. $\mathbb{CP}^{n'}, \mathbb{CH}^{n'}$) for some $n' \leq n$.

It turns out that the most important properties of a complex manifold X are encoded into an automorphism of the tangent bundle of X, called the *almost complex structure* of X. This is defined in the following way. Fix $p \in X$, and let $\left(z^1, ..., z^n\right)$ be a holomorphic chart defined in a neighborhood of p. Let $\left(x^1, y^1, ..., x^n, y^n\right)$ be its induced smooth chart. Define a $\mathbb{R}$-linear map $J_p : T_pX \to T_pX$

$$\begin{aligned} J_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) &= \left.\frac{\partial}{\partial y^j}\right|_p \\ J_p\left(\left.\frac{\partial}{\partial y^j}\right|_p\right) &= -\left.\frac{\partial}{\partial x^j}\right|_p . \end{aligned}$$

In other words, the matrix of J_p with respect to the basis

$$\left(\left.\frac{\partial}{\partial x^n}\right|_p, \left.\frac{\partial}{\partial y^n}\right|_p, ..., \left.\frac{\partial}{\partial x^n}\right|_p, \left.\frac{\partial}{\partial y^n}\right|_p\right)$$

of T_pX is the $2n \times 2n$ block matrix having 2×2 blocks $\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$ along the diagonal and other entries equal to 0. This definition does not depend on the holomorphic chart chosen, because of remark 4. It then follows that J is a well defined automorphism of TX. Moreover, $J^2 = -\operatorname{id}$ by definition.

Definition 9. Let X be a smooth manifold. An *almost complex structure* on X is a bundle morphism $J : TX \to TX$ such that $J^2 = -\operatorname{id}$. An *almost complex manifold* is a smooth manifold along with an almost complex structure on it.

Not every smooth manifold admits an almost complex structure. For example, the following lemma provides an easy necessary condition:

Lemma 10. *An almost complex manifold must have even dimension.*

Proof. Let V be a finite dimensional real vector space. Suppose that there is an automorphism $J : V \to V$ such that $J^2 = -\operatorname{id}$. Then J has only complex eigenvalues $\pm i$, and its characteristic polynomial must be real, so it must be of the form $(\lambda - i)^n (\lambda + i)^n$ for some $n \in \mathbb{N}$, proving that the dimension of V must be $2n$. Now, if (X, J) is a k-dimensional almost complex manifold, then for every $p \in X$ $J_p : T_pX \to T_pX$ is an automorphism such that $J_p^2 = -\operatorname{id}$: it follows that T_pX has even dimension, so X has even dimension. $\square$

Fix a $2n$-dimensional almost complex manifold (X, J).

Definition 11. Define $T^{\mathbb{C}}X := TX \otimes_{\mathbb{R}} \mathbb{C}$ the *complexified tangent bundle* of X, and $\Lambda^k_{\mathbb{C}}X := \Lambda^k X \otimes_{\mathbb{R}} \mathbb{C}$ the *complexified bundle of k-forms*. Denote by $\mathcal{T}^{\mathbb{C}}(X)$ (resp. $\Omega^k_{\mathbb{C}}(X)$) the $C^\infty(X, \mathbb{C})$-module of the smooth sections of $T^{\mathbb{C}}X$ (resp. $\Lambda^k_{\mathbb{C}}X$).

We extend all the usual operators (Lie brackets, exterior product, exterior derivative) to the relevant complexification by $\mathbb{C}$-linearity. By definition, an element of $T^{\mathbb{C}}_pX$ is a formal sum $v + iw$, where $v, w \in T_pX$. Analougously, an element of $\Lambda^k_{\mathbb{C}}X_p$ can be seen as a $\mathbb{R}$-multilinear, alternant map

$$\omega : T_pX \times \cdots \times T_pX \to \mathbb{C}.$$

We often extend k-forms by $\mathbb{C}$-multilinearity to maps $T^{\mathbb{C}}_pX \times \cdots \times T^{\mathbb{C}}_pX \to \mathbb{C}$.

The definition of the complex bundles $T^{\mathbb{C}}X$ and $\Lambda^k_{\mathbb{C}}X$ for an almost complex manifold (X, J) is justified by the fact that J induces on $T^{\mathbb{C}}X$ and $\Lambda^k_{\mathbb{C}}X$ decompositions into direct sums of complex subbundles, that reflect the properties of (X, J).

Lemma 12. *Let (X, J) be an almost complex manifold. Denote by $J^{\mathbb{C}} : T^{\mathbb{C}}X \to T^{\mathbb{C}}X$ the complexified almost complex structure.*

1. *$J^{\mathbb{C}}$ has eigenvalues $\pm i$. If $T^{1,0}X$ (resp. $T^{0,1}X$) is the i (resp. $-i$) eigenbundle of $J^{\mathbb{C}}$, then $T^{1,0}X$ and $T^{0,1}X$ are complex rank n subbundles of $T^{\mathbb{C}}X$, and*
$$T^{1,0}X \oplus T^{0,1}X = TX^{\mathbb{C}}.$$

2. *Let $p \in X$. Then*
$$\begin{aligned} T^{1,0}_pX &= \{v - iJ_pv : v \in T_pX\} \\ T^{0,1}_pX &= \{v + iJ_pv : v \in T_pX\}. \end{aligned}$$

Proof.

1. Since $\left(J^{\mathbb{C}}\right)^2 = -\mathrm{id}_{TX^{\mathbb{C}}}$, $J^{\mathbb{C}}$ must have eigenvalues $\pm i$. Moreover, since J is an automorphism of TX, the characteristic polynomial of $J^{\mathbb{C}}$ (that coincides with the one of J) must be real in every point of X, so it must be the polynomial with constant coefficients $(\lambda - i)^n (\lambda + i)^n \in \mathbb{C}[\lambda]$. For every point $p \in X$ the space $T^{1,0}_pX$ (resp. $T^{0,1}_pX$) has complex

dimension n, since the fact that $\left(J^{\mathbb{C}}\right)^2 = -\operatorname{id}_{TX^{\mathbb{C}}}$ forces $J_p^{\mathbb{C}}$ to be diagonalizable for every $p \in X$. It then follows that $T^{1,0}X$ and $T^{0,1}X$ are rank n complex subbundles of $T^{\mathbb{C}}X$, since they are the kernel of the constant rank bundle morphisms $J^{\mathbb{C}} - i\operatorname{id}_{T^{\mathbb{C}}X} : T^{\mathbb{C}}X \to T^{\mathbb{C}}X$ and $J^{\mathbb{C}} + i\operatorname{id}_{T^{\mathbb{C}}X} : T^{\mathbb{C}}X \to T^{\mathbb{C}}X$ respectively. Since $T^{1,0}X$ and $T^{0,1}X$ are rank n eigenbundles relative to different eigenvalues, it then follows that it must be $T^{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$.

2. For every $v \in T_pX$,

$$\begin{aligned} J_p^{\mathbb{C}}\left(v \mp iJ_pv\right) &= J_pv \mp iJ_p^2v \\ &= J_pv \pm iv \\ &= \pm i\left(v \mp iJ_pv\right) \end{aligned}$$

so we have the inclusions $\supseteq$. On the other hand, if $v + iw \in T_p^{1,0}X$ (resp. $v + iw \in T_p^{0,1}X$), then $J_pv + iJ_pw = \pm i\left(v + iw\right)$, so it must be $w = \mp J_pv$, proving the inclusion $\subseteq$.

□

We can now prove another necessary condition on a smooth manifold to admit an almost complex structure.

Corollary 13. *An almost complex structure (X, J) is canonically orientable.*

Proof. Choose a local frame $(V_1, ..., V_n)$ for $T^{1,0}X$ on a open subset $U \subseteq X$. Then $\left(\overline{V_1}, ..., \overline{V_n}\right)$ is a local frame for $T^{0,1}X$, because of the above lemma. The locally defined real tangent fields $(v_1, w_1, ..., v_n, w_n) \in \mathcal{T}(U)$ defined by

$$\begin{aligned} v_j &= V_j + \overline{V_j} \\ w_j &= i\left(V_j - \overline{V_j}\right) \end{aligned}$$

form clearly a local frame for TX. Let $\left(v^1, w^1, ..., v^n, w^n\right)$ be the local coframe for Λ^1X. We define on the domain of that frame the orientation induced by the differential form $v^1 \wedge w^1 \wedge \cdots \wedge v^n \wedge w^n$. This orientation does not depend on the local frame of $T^{1,0}X$ chosen. In fact, if $\left(\tilde{V}_1, ..., \tilde{V}_n\right)$ is another local frame for $T^{1,0}X$, then

$$\tilde{V}_j = a_j^i V_i$$

for some $a^i_j : U \to \mathbb{C}$ such that, for every $p \in U$, $\left(a^i_j(p)\right) \in \mathrm{GL}(n, \mathbb{C})$. Now, if $(\tilde{v}_1, \tilde{w}_1, ..., \tilde{v}_n, \tilde{w}_n)$ is the local frame for TX defined by

$$\begin{aligned}\tilde{v}_j &= \tilde{V}_j + \overline{\tilde{V}_j}\\ \tilde{w}_j &= i\left(\tilde{V}_j - \overline{\tilde{V}_j}\right),\end{aligned}$$

then we obtain

$$\begin{aligned}\tilde{v}_j &= a^k_j V_k + \overline{a^k_j}\overline{V_k}\\ &= a^k_j \frac{1}{2}(v_k - i w_k) + \overline{a^k_j}\frac{1}{2}(v_k + i w_k)\\ &= \frac{1}{2}\left(a^k_j + \overline{a^k_j}\right) v_k + \frac{i}{2}\left(\overline{a^k_j} - a^k_j\right) w_k\\ &= \Re\left(a^k_j\right) v_k + \Im\left(a^k_j\right) w_k,\end{aligned}$$

and analougously

$$\tilde{w}_j = -\Im\left(a^k_j\right) v_k + \Re\left(a^k_j\right) w_k.$$

The result then follows from the fact that, if $A + iB \in \mathrm{GL}(n, \mathbb{C})$ with A and B real matrix, then the matrix

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

must have strictly positive determinant: in fact, we have

$$\begin{pmatrix} I & \\ -iI & I \end{pmatrix}\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\begin{pmatrix} I & \\ iI & I \end{pmatrix} = \begin{pmatrix} A + iB & B \\ 0 & A + iB \end{pmatrix},$$

and hence

$$\det\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \det(A + iB)^2.$$

□

Lemma 14. *Let (X, J) be an almost complex manifold of dimension $2n$.*

1. *The complexified cotangent bundle $\Lambda^1_{\mathbb{C}}X$ admits a direct sum decomposition in two rank n complex subbundles $\Lambda^{1,0}X, \Lambda^{0,1}X$ defined fibrewise as*

$$\begin{aligned}\Lambda^{1,0}X_p &= \left\{\omega \in \Lambda^1_{\mathbb{C}}X_p : \omega(Z) = 0, \forall Z \in T^{0,1}_p X\right\}\\ \Lambda^{0,1}X_p &= \left\{\omega \in \Lambda^1_{\mathbb{C}}X_p : \omega(Z) = 0, \forall Z \in T^{1,0}_p X\right\}.\end{aligned}$$

2. *The complexified k-form bundle $\Lambda^k_{\mathbb{C}} X$ admits a direct sum decomposition*
$$\Lambda^k_{\mathbb{C}} X = \bigoplus_{u+v=k} \Lambda^{p,q} X,$$
where $\Lambda^{p,q} X$ is the complex subbundle of $\Lambda^{p+q}_{\mathbb{C}} X$ such that, for every $x \in X$, $\Lambda^{p,q} X_x$ is the linear span of elements of the form $\omega^1 \wedge \cdots \wedge \omega^p \wedge \eta^1 \wedge \cdots \wedge \eta^q$, where $\omega^j \in \Lambda^{1,0} X_x$ and $\eta^j \in \Lambda^{0,1} X_x$.

3. *A complex $p+q$-form $\omega \in \Lambda^{u+v}_{\mathbb{C}} X_x$ is an element of $\Lambda^{p,q} X_x$ if and only if $\omega(v_1, ..., v_{p+q}) = 0$ whenever at least $p+1$ of the complex tangent vectors $v_1, ..., v_{p+q}$ are elements of $T^{1,0}_x X$, or at least $q+1$ of the complex tangent vectors $v_1, ..., v_{p+q}$ are elements of $T^{0,1}_x X$.*

Proof.

1. Let $v_1, ..., v_n$ be a basis of $T^{1,0}_p X$. Then $\overline{v_1}, ..., \overline{v_n}$ is a basis of $T^{0,1}_p X$, and $v_1, ..., v_n, \overline{v_1}, ..., \overline{v_n}$ is a basis of $T^{\mathbb{C}}_p X$. Let $\left(v^1, ..., v^n, \overline{v^1}, ..., \overline{v^n}\right)$ be the dual basis of $\Lambda^1_{\mathbb{C}} X_p$: the notation is clearly consistent. We want to prove that $\Lambda^{1,0} X_p$ (resp. $\Lambda^{0,1} X_p$) is spanned by $v^1, ..., v^n$ (resp. $\overline{v^1}, ..., \overline{v^n}$). We prove only the first case, since the second is analougous. Clearly $v^j \in \Lambda^{1,0} X_p$: viceversa, if $\omega \in \Lambda^{1,0} X_p$ is the linear combination $\lambda_j v^j + \mu_j \overline{v^j}$ (we are using the Einstein summation convention), we have
$$\omega(\overline{v_k}) = \mu_k = 0$$
for every k, so ω is a linear combination of $v^1, ..., v^n$. It follows that $\Lambda^{1,0} X$ and $\Lambda^{0,1} X$ are rank n complex vector bundles, since if $V_1, ..., V_n$ is a local frame for $T^{1,0} X$, then $V^1, ..., V^n$ is a local frame for $\Lambda^{1,0} X$ and $\overline{V}^1, ..., \overline{V}^n$ is a local frame for $\Lambda^{0,1} X$. Moreover, the two subbundles $\Lambda^{1,0} X$ and $\Lambda^{0,1} X$ are clearly in direct sum, and by dimensional reasons we have also $\Lambda^1_{\mathbb{C}} X = \Lambda^{1,0} X \oplus \Lambda^{0,1} X$.

2. The proof is essentially a bundle-theoretic translation of this elementary multilinear algebra fact: if V, W are finite dimensional complex vector spaces, then $\Lambda^k (V \oplus W)$ is canonically isomorphic to the direct sum $\bigoplus_{p+q=k} \Lambda^p V \otimes \Lambda^q W$ via the map
$$\Phi : \bigoplus_{p+q=k} \Lambda^p V \otimes \Lambda^q W \to \Lambda^k (V \oplus W)$$

that extendes by linearity the currespondence

$$\omega \otimes \eta \mapsto \omega \wedge \eta$$

for $\omega \in \Lambda^p V$ and $\eta \in \Lambda^q W$.

3. Let $v_1, ..., v_n$ be a basis of $T_x^{1,0}X$. Then it suffices to observe that the basis vectors $v^{i_1} \wedge \cdots \wedge v^{i_p} \wedge \overline{v^{j_1}} \wedge \cdots \wedge \overline{v^{j_q}}$ of $\Lambda^{p,q}X_x$ satisfies the desired property.

□

Observe that, if X is a complex manifold, then every holomorphic chart $\left(z^1, ..., z^n\right)$ induces canonical local frames for $T^{1,0}X$, $T^{0,1}X$ and $\Lambda^{p,q}X$: in fact, if $\left(x^1, y^1, ..., x^n, y^n\right)$ is the smooth chart induced by $\left(z^1, ..., z^n\right)$, and J is the almost complex structure of X, then by definition of J we have

$$J\frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j},$$

so the complex vector fields

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right)$$

provides a local frame for $T^{1,0}X$, and the complex vector fields

$$\frac{\partial}{\partial \overline{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right)$$

provides a local frame for $T^{0,1}X$. Now, define

$$\begin{aligned} dz^j &= dx^j + idy^j \\ d\overline{z}^j &= dx^j - idy^j : \end{aligned}$$

a direct calculation shows that

$$\begin{aligned} dz^j\left(\frac{\partial}{\partial z^k}\right) &= d\overline{z}^j\left(\frac{\partial}{\partial \overline{z}^k}\right) = \delta^j_k \\ dz^j\left(\frac{\partial}{\partial \overline{z}^k}\right) &= d\overline{z}^j\left(\frac{\partial}{\partial z^k}\right) = 0. \end{aligned}$$

It then follows that $dz^1, ..., dz^n$ (resp. $d\bar{z}^1, ..., d\bar{z}^n$) is a local frame for $\Lambda^{1,0}X$ (resp. $\Lambda^{0,1}X$), and the complex forms

$$dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$$

for $1 \leq i_1 < \cdots < i_p \leq n$, $1 \leq j_1 < \cdots < j_q \leq n$ provides a local frame for $\Lambda^{p,q}X$.

Since every n-dimensional complex manifold is a $2n$-dimensional almost complex manifold with its canonical almost complex structure, a natural question arises: is the converse true? The answer is in general no.

Definition 15. We say that an almost complex structure J on a smooth manifold X is *integrable* if the distribution $T^{1,0}X$ is integrable, i.e. if V, W are two smooth sections of $T^{1,0}X$, then $[V, W]$ is again a smooth section of $T^{1,0}X$.

The following deep theorem shows that an almost complex manifold (X, J) is complex if and only if J satisfies the above integrability condition:

Theorem 16 (Newlander-Nirenberg Theorem)**.** *Let (X, J) be a $2n$ dimensional almost complex manifold. Then J arises from a complex structure if and only if J is integrable.*

Proof. We prove only the implication $\Rightarrow$. For the implication $\Leftarrow$, see e.g. [12].
Let $\left(z^1, ..., z^n; U\right)$ be a holomorphic chart of (X, J): then

$$\frac{\partial}{\partial z^1}, ..., \frac{\partial}{\partial z^n}$$

is a local frame for $T^{1,0}X$. Then, if $f^j, g^j : U \to \mathbb{C}$ are smooth functions for $j = 1, ..., n$, a calculation shows that

$$\left[f^j \frac{\partial}{\partial z^j}, g^k \frac{\partial}{\partial z^k}\right] = \left(f^j \frac{\partial g^k}{\partial z^j} - g^j \frac{\partial f^k}{\partial z^j}\right) \frac{\partial}{\partial z^k}$$

because $\left[\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\right] \equiv 0$ for every j, k. By generality of the holomorphic chart, we obtain the thesis. $\square$

Let (X, J) be a $2n$ dimensional almost complex manifold. Lemma 14 implies that the $C^\infty(X, \mathbb{C})$-module $\Omega^k_{\mathbb{C}}(X)$ decomposes into the direct sum

$$\Omega^k_{\mathbb{C}}(X) = \bigoplus_{p+q=k} \Omega^{p,q}(X).$$

Denote by

$$\pi^{p,q} : \Omega^{p+q}_{\mathbb{C}} (X) \to \Omega^{p,q} (X)$$

the canonical projection. Now, extend by $\mathbb{C}$-linearity the exterior derivative $d : \Omega^k (X) \to \Omega^{k+1} (X)$ to a $\mathbb{C}$-linear map $\Omega^k_{\mathbb{C}} (X) \to \Omega^{k+1}_{\mathbb{C}} (X)$, and define

$$\partial : \Omega^{p,q} (X) \quad \to \quad \Omega^{p+1,q} (X)$$

and

$$\bar{\partial} : \Omega^{p,q} (X) \to \Omega^{p,q+1} (X)$$

as

$$\begin{aligned} \partial &= \pi^{p+1,q} \circ d \\ \bar{\partial} &= \pi^{p,q+1} \circ d. \end{aligned}$$

From the properties of the exterior derivative, it can be verified easily that ∂ and $\bar{\partial}$ are differential operators, i.e. they are $\mathbb{C}$-linear and satisfy

$$\begin{aligned} \partial (\omega \wedge \eta) &= (\partial\omega) \wedge \eta + (-1)^{\deg \omega} \omega \wedge (\partial\eta) \\ \bar{\partial} (\omega \wedge \eta) &= \left(\bar{\partial}\omega\right) \wedge \eta + (-1)^{\deg \omega} \omega \wedge \left(\bar{\partial}\eta\right) \end{aligned}$$

for every $\omega \in \Omega^{p,q} (X) , \eta \in \Omega^{r,s} (X)$.

The ∂ and $\bar{\partial}$ operators provides another characterization of the integrability of an almost complex structure:

Proposition 17. *Let (X, J) be an almost complex manifold. Then the following are equivalent:*

1. *J is integrable;*
2. *$d \left(\Omega^{1,0} (X)\right) \subseteq \Omega^{2,0} (X) \oplus \Omega^{1,1} (X)$;*
3. *$d = \partial + \bar{\partial}$.*

If one of the above properties is satisfied, the ∂ and $\bar{\partial}$ operators satisfy

$$\begin{aligned} \partial^2 &\equiv 0 \\ \bar{\partial}^2 &\equiv 0 \\ \partial\bar{\partial} + \bar{\partial}\partial &\equiv 0. \end{aligned}$$

Proof.

1. $(1 \Rightarrow 2)$ If J is integrable, then by the Newlander-Nirenberg theorem (X, J) is a complex manifold. Then, since d is a local operator, it suffices to prove that if $\left(z^1, ..., z^n; U\right)$ is a holomorphic chart and $f : U \to \mathbb{C}$ is smooth, then $d\left(f dz^j\right) \in \Omega^{2,0}(X) \oplus \Omega^{1,1}(X)$ for every j: an easy calculation shows that

$$d\left(f dz^j\right) = \frac{\partial f}{\partial z^k} dz^k \wedge dz^j + \frac{\partial f}{\partial \overline{z}^k} d\overline{z}^k \wedge dz^j \in \Omega^{2,0}(X) \oplus \Omega^{1,1}(X).$$

2. $(2 \Rightarrow 3)$ We must prove that

$$d\left(\Omega^{p,q}(X)\right) \subseteq \Omega^{p+1,q}(X) \oplus \Omega^{p,q+1}(X):$$

this follows by property 2 and the fact that

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{\deg \omega} \omega \wedge (d\eta).$$

3. $(3 \Rightarrow 1)$ Let $V, W \in \mathcal{T}^{1,0}(X)$. In order to prove that $[V, W] \in \mathcal{T}^{1,0}(X)$, by lemma 14 it suffices to prove that $\omega([V, W]) \equiv 0$ for every $\omega \in \Omega^{0,1}(X)$. By the properties of the exterior derivative, we have

$$d\omega(V, W) = V(\omega(W)) - W(\omega(V)) - \omega([V, W]):$$

moreover, since $\omega \in \Omega^{0,1}(X)$, we have $\omega(W) = \omega(V) \equiv 0$, so

$$d\omega(V, W) = -\omega([V, W]).$$

By property 3, we have $d = \partial + \bar{\partial}$: but $\partial\omega(V, W) \equiv 0$ and $\bar{\partial}\omega(V, W) \equiv 0$ by lemma 14, so we have $\omega([V, W]) \equiv 0$, and by generality of ω we obtain the thesis.

If one of the equivalent properties $1, 2, 3$ is satisfied, then we obtain

$$d^2 = \partial^2 + \bar{\partial}^2 + \left(\partial\bar{\partial} + \bar{\partial}\partial\right) = 0.$$

Since, for every $\omega \in \Omega^{p,q}(X)$, we have $\partial^2\omega \in \Omega^{p+2,q}(X)$, $\bar{\partial}^2\omega \in \Omega^{p,q+2}(X)$ and $\left(\partial\bar{\partial} + \bar{\partial}\partial\right)\omega \in \Omega^{p+1,q+1}(X)$, for degree reasons we obtain $\partial^2 = 0$, $\bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$.

$\square$

Fix a n-dimensional complex manifold X. The following lemma is a complex version of the Poincaré lemma: a proof can be found for example in [11], page 27.

Lemma 18 (Dolbeault lemma)**.** *A $\bar{\partial}$-closed $(0,1)$-form is locally $\bar{\partial}$-exact.*

From the above lemma, we obtain the following proposition, that will be fundamental in our study of Kähler manifolds:

Proposition 19 (the $i\partial\bar{\partial}$-lemma)**.** *Let $\omega \in \Omega^{1,1}(X) \cap \Omega^2(X)$ be a real 2-form of type $(1,1)$. Then ω is closed if and only if, for every $x \in X$, there exists a smooth function $u : U \to \mathbb{R}$ defined in a neighborhood of x such that*

$$\omega = i\partial\bar{\partial}u.$$

Proof. The direction $\Leftarrow$ is obvious, since $d = \partial + \overline{\partial}$ and $\partial^2, \bar{\partial}^2 = 0, \bar{\partial}\partial = -\partial\bar{\partial}$. Let's prove the $\Rightarrow$ direction. Since $d\omega = 0$, by the Poincarè lemma, for every $x \in X$ there exists a locally defined real 1-form τ in a neighborhood U of x such that

$$d\tau = \omega.$$

Let $\tau = \tau^{1,0} + \tau^{0,1}$ be the decomposition of τ into its $(1,0)$-component and its $(0,1)$-component. Since τ is real, we have $\overline{\tau} = \tau$, so

$$\overline{\tau^{0,1}} = \tau^{1,0}.$$

Now, we have

$$\begin{aligned} \omega &= d\tau \\ &= \left(\partial + \overline{\partial}\right)\tau \\ &= \bar{\partial}\tau^{0,1} + \left(\partial\tau^{0,1} + \bar{\partial}\tau^{1,0}\right) + \partial\tau^{1,0} \end{aligned}$$

so since ω is a $(1,1)$-form we obtain $\bar{\partial}\tau^{0,1} = 0, \partial\tau^{1,0} = 0$ and

$$\omega = \partial\tau^{0,1} + \bar{\partial}\tau^{1,0}.$$

By the Dolbeault lemma, by shrinking U we can suppose that there exists a $f : U \to \mathbb{C}$ smooth such that $\tau^{0,1} = \bar{\partial}f$. It then follows that

$$\begin{aligned} \tau^{1,0} &= \overline{\tau^{0,1}} \\ &= \overline{\bar{\partial}f} \\ &= \partial\overline{f}. \end{aligned}$$

We then obtain

$$\begin{aligned}
\omega &= \partial\tau^{0,1} + \bar{\partial}\tau^{1,0} \\
&= \partial\bar{\partial}f + \bar{\partial}\partial\overline{f} \\
&= i\partial\bar{\partial}\left(2\Im\left(f\right)\right)
\end{aligned}$$

so the real smooth function $u : U \to \mathbb{R}$ defined as $u = 2\Im\left(f\right)$ satisfies the desired property. □

1.2 Holomorphic and Hermitian vector bundles

Definition 20. Let X be a complex manifold. A *rank r holomorphic vector bundle* on X is a rank r complex vector bundle $\pi : E \to X$ along with a structure of complex manifold on E such that, for every $p \in X$, there exists an open set $U \ni p$ and a *holomorphic* trivialization for E.

Observe that, by definition, the projection of a holomorphic vector bundle is holomorphic.

Remark 21. An alternative way to define holomorphic vector bundles is the following. Let $\pi : E \to X$ be a rank n complex vector bundle. A *holomorphic trivializing atlas* for E is a trivializing atlas $\{\chi_\alpha, U_\alpha\}$ such that, for every α, β, the transition functions $\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}\left(r, \mathbb{C}\right)$ that satisfies

$$\chi_\alpha \circ \chi_\beta^{-1}\left(p, v\right) = \left(p, \Phi_{\alpha\beta}\left(p\right)v\right)$$

for every $p \in U_\alpha \cap U_\beta$ and $v \in \mathbb{C}^r$, are holomorphic. Clearly, if $p : E \to X$ is a holomorphic vector bundle, then the family of the holomorphic trivializations for E is the unique maximal holomorphic trivializing atlas for E. On the other hand, if we fix a holomorphic trivializing atlas $\{\chi_\alpha, U_\alpha\}$ for E, then this atlas induces on E a unique structure of complex manifold that makes E a holomorphic vector bundle and the maps χ_α holomorphic.

Example 22. On a complex manifold X, the complex bundles $T^{1,0}X$ and $\Lambda^{p,0}X$ are holomorphic: to prove this, it suffices to compute the transition maps for the trivializing atlas induced by the complex atlas of X.

Example 23. Let (X, J) be a n-dimensional complex manifold. Then the structure of complex vector space on T_pX induced by J_p for every $p \in X$ induces on TX a structure of rank n complex vector bundle. TX admits a canonical structure of holomorphic vector bundle. More precisely, one can verify that the map

$$\begin{array}{rcl} F : T^{1,0}X & \to & TX \\ \left.\dfrac{\partial}{\partial z^{\alpha}}\right|_p & \mapsto & \left.\dfrac{\partial}{\partial x^{\alpha}}\right|_p \end{array}$$

is a well defined isomorphism of complex vector bundles, and so the structure of holomorphic vector bundle on $T^{1,0}X$ induces a structure of holomorphic vector bundle on TX such that F is an isomorphism of holomorphic vector bundles.

Definition 24. Let $\pi : E \to X$ be a holomorphic vector bundle. The *bundle of E-valued k-forms* (resp. *(p,q)-forms*) is the complex vector bundle

$$\begin{array}{rcl} \Lambda^k_E X & := & \Lambda^k_{\mathbb{C}} X \otimes_{\mathbb{C}} E \\ \Lambda^{p,q}_E X & := & \Lambda^{p,q} X \otimes_{\mathbb{C}} E. \end{array}$$

We denote by $\Omega^k_E(X)$ (resp. $\Omega^{p,q}_E(X)$) the module of globally defined smooth sections of $\Lambda^k_E X$ (resp. $\Lambda^{p,q}_E X$).

The $C^\infty(X,\mathbb{C})$-module $\Omega^k_E(X)$ is generated by sections of the form $\omega \otimes \sigma$, where $\omega \in \Omega^k_{\mathbb{C}}(X)$ and $\sigma \in \Gamma_E(X)$. The same holds for $\Omega^{p,q}_E(X)$.

Lemma 25. *Let $\pi : E \to X$ be a holomorphic vector bundle. Then there is a unique map*

$$\bar{\partial}^E : \Omega^0_E(X) \to \Omega^{0,1}_E(X)$$

that satisfies the following properties:

1. *it is $\mathbb{C}$-linear;*
2. *$\bar{\partial}^E(f \otimes \sigma) = df \otimes \sigma + f \otimes \bar{\partial}^E \sigma$ for every $f \in \Omega^0_{\mathbb{C}}(X) = C^\infty(X,\mathbb{C})$ and $\sigma \in \Gamma_E(X)$;*
3. *$\bar{\partial}^E \sigma = 0$ for every holomorphic section of E.*

Proof. If $\bar{\partial}^E$ exists then it is a local operator, so it suffices to define it locally. Let U be an open subset of X such that there is a holomorphic frame $\sigma_1, ..., \sigma_r$ for E on U. Then, on U, the unique possible definition is the following:

$$\bar{\partial}^E\left(f^j \otimes \sigma_j\right) = df^j \otimes \sigma_j.$$

$\square$

As usual, we extend $\bar{\partial}^E$ to a unique differential operator

$$\bar{\partial}^E : \Omega_E^{p,q}(X) \to \Omega_E^{p,q+1}(X)$$

that is $\mathbb{C}$-linear and satisfy

$$\bar{\partial}^E (\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^{p+q} \omega \otimes \bar{\partial}^E \sigma$$

for every $\omega \in \Omega^{p,q}(X)$ and $\sigma \in \Gamma_E(X)$. Then, by definition, we have

$$\bar{\partial}^E \circ \bar{\partial}^E = 0.$$

Definition 26. The differential operator $\bar{\partial}^E$ is called the *holomorphic structure* on E.

Definition 27. Let X be a complex manifold, and let $\pi : E \to X$ be a rank r complex vector bundle. A *Hermitian structure* on E is a bundle map $H : E \otimes E \to X \times \mathbb{C}$ such that, for every $p \in X$, the map $H_p : E_p \times E_p \to \mathbb{C}$ is a positive definite Hermitian product on E_p.

Remark 28. Every complex vector bundle $\pi : E \to X$ admits a Hermitian structure. In fact, it suffices to take a trivializing atlas $\{\chi_\alpha, U_\alpha\}$ for E and use a partition of unity subordinate to the open cover $\{U_\alpha\}$ to patch together the Hermitian structure defined on $\pi^{-1}(U_\alpha)$ as the pullback of the standard Hermitian structure on the trivial complex vector bundle $U_\alpha \times \mathbb{C}^r$.

Let $\pi : E \to X$ be a rank r complex vector bundle on a complex manifold X. Recall that a connection on E is a map

$$\nabla : \Omega_E^0(X) \to \Omega_E^1(X)$$

that is $\mathbb{C}$-linear and satisfies

$$\nabla (f \otimes \sigma) = df \otimes \sigma + f \otimes \nabla\sigma$$

for every $f \in \Omega_{\mathbb{C}}^0(X) = C^\infty(X, \mathbb{C})$ and $\sigma \in \Gamma_E(X)$. ∇ can be seen also as a map

$$\mathcal{T}(X) \times \Gamma_E(X) \to \Gamma_E(X)$$

that, to the pair $(V, \sigma) \in \mathcal{T}(X) \times \Gamma_E(X)$, associates the section $\nabla_V \sigma := \nabla\sigma(V)$.

As usual, we extend ∇ to a map $\nabla : \Omega_E^k(X) \to \Omega_E^{k+1}(X)$ that is $\mathbb{C}$-linear and satisfies

$$\nabla (\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^{\deg \omega} \omega \otimes \nabla\sigma$$

for every $\omega \in \Omega^k_{\mathbb{C}}(X)$ and $\sigma \in \Gamma_E(X)$. It can be proved that the map $\nabla^2 : \Omega^0_E(X) \to \Omega^2_E(X)$ is tensorial: this is called the *curvature tensor* of ∇.

Let $(\sigma_1, ..., \sigma_r)$ be a local frame for E in a open set U. Then there are 1-forms $\omega^i_j \in \Omega^1_{\mathbb{C}}(X)$ such that

$$\nabla \sigma_j = \omega^i_j \otimes \sigma_i$$

for every j: those forms are called *connection forms* of ∇ with respect to the local frame chosen. There are also 2-forms $\Omega^i_j \in \Omega^2_{\mathbb{C}}(X)$ such that

$$\nabla^2 \sigma_j = \Omega^i_j \otimes \sigma_i$$

for every j: those forms are called *curvature forms* of ∇ with respect to the local frame chosen. The *Cartan equation* relates the curvature forms with the connection forms:

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j.$$

Definition 29. Let $\pi : E \to X$ be a rank r complex vector bundle on a complex manifold X, and let ∇ be a connection on E.

1. Let H be a Hermitian structure on E. We say that ∇ is *compatible* with H if, for every $\sigma, \tau \in \Gamma_E(X)$, we have

   $$d(H(\sigma, \tau)) = H(\nabla \sigma, \tau) + H(\sigma, \nabla \tau),$$

   where $H(\nabla \sigma, \tau)$ is the E-valued 1-form $\mathcal{T}(X) \to \Gamma_E(X)$ defined by $V \mapsto H(\nabla_V \sigma, \tau)$.

2. Suppose that $\pi : E \to X$ is a holomorphic vector bundle. We say that ∇ is *compatible with the holomorphic structure of* E if

   $$\nabla^{0,1} = \bar{\partial}^E,$$

   where $\nabla^{0,1} : \Omega^0_E(X) \to \Omega^{0,1}_E(X)$ is the $(0,1)$-component of ∇.

Theorem 30. *Let $\pi : E \to X$ be a holomorphic Hermitian vector bundle on a complex manifold X. Then there is a unique connection on E that is both compatible with the Hermitian and the complex structures of E. This connection is called the* Chern connection *of E and is denoted by ∇^{CH}.*

Proof. Let H be the Hermitian structure on E.

1. (uniqueness) Let ∇ be a connection on E compatible with $\bar{\partial}^E$ and H. Let $(s_1, ..., s_k)$ be a local holomorphic frame for E, and let ω_j^i be the connection 1-forms with respect to that frame. Since $\nabla^{0,1} = \bar{\partial}^E$, then $\nabla s^j = \nabla^{1,0} s^j$, because s^j is a holomorphic section and then $\bar{\partial}^E s^j = 0$: this implies that all the ω_j^i are $(1,0)$-forms. Now, compatibility with H implies the following equation:

$$dH_{ij} = \omega_i^u H_{uj} + H_{iu} \overline{\omega_j^u}$$

   where $H_{ij} = H(s_i, s_j)$. From this equation, we obtain

$$\partial H_{ij} = \omega_i^u H_{uj},$$

   and hence

$$\omega_i^u = (\partial H_{ij}) H^{ju} :$$

   this proves that the ω_j^i are characterized by H, and then so is ∇.

2. (existence) By uniqueness, it suffices to define ∇ locally. Let $\{s_1, ..., s_k; U\}$ be a local holomorphic frame for E. Then it suffices to define

$$\omega_i^u = (\partial H_{ij}) H^{ju}$$

   and

$$\nabla : \Omega_E^0 (U) \to \Omega_E^1 (U)$$

   by

$$\nabla \left(f^j \otimes s_j \right) = df^j \otimes s_j + \omega_j^u \otimes s_u.$$

   It is easy to prove that ∇ is a connection on U compatible with both H and $\bar{\partial}^E$.

□

The most important holomorphic Hermitian bundles used in this thesis are those of rank 1, i.e. the *holomorphic Hermitian line bundles.*

Let $\pi : L \to X$ be a holomorphic line bundle on a complex manifold X, and let H be an Hermitian structure on L. Then every local nonvanishing holomorphic section $s : U \to L$ is a local holomorphic frame on L. Let ω, Ω be the connection and curvature forms relative to s, i.e. the ones that satisfies

$$\nabla s = \omega \otimes s$$

and

$$\nabla^2 s = \Omega \otimes s.$$

By the Cartan equations, we have

$$\begin{aligned}\Omega &= d\omega + \omega \wedge \omega \\ &= d\omega.\end{aligned}$$

Moreover, the form Ω does not depend on the choice of the section s: in fact, if $\tilde{s} : U \to L$ is another nonvanishing section, then $\tilde{s} = fs$ for some nonzero smooth map $f : U \to \mathbb{C}$, and $\nabla \tilde{s} = \nabla fs = (df + f\omega) \otimes s = \left(\frac{df}{f} + \omega\right) \otimes \tilde{s}$, so

$$\tilde{\omega} = \frac{df}{f} + \omega$$

and since $d\,(df/f) = 0$, we obtain $d\tilde{\omega} = d\omega$. It follows that Ω can be extended to a globally defined complex 2-form, called the *curvature form* of (L, H).

Lemma 31. *Let (L, H) be a Hermitian holomorphic line bundle on X. Then the curvature form Ω of (L, H) is a closed, purely imaginary $(1,1)$-form. If $s : U \to L$ is a local holomorphic nonvanishing section of L, then the local expression of Ω on U is*

$$\Omega = -\partial\bar{\partial} \log (H (s, s)).$$

Moreover, the cohomology class $\left[\frac{i}{2\pi}\Omega\right] \in H^2_{dR} (X)$ does not depend on H.

Proof. Choose a nonvanishing local section $s : U \to L$. Let $\omega \in \Omega^{1,0}(U)$ be the connection form with respect to s. Define $h : U \to \mathbb{C}$ by $h = H (s, s)$: since H is a Hermitian structure, h is a strictly positive real smooth function. From the proof of the above theorem, we have

$$\begin{aligned}\omega &= \frac{\partial h}{h} \\ &= \partial \log h.\end{aligned}$$

Moreover, if Ω is the curvature form, since $\Omega = d\omega$ and $d\omega = d\partial \log h = \bar{\partial}\partial \log h$, by $\partial\bar{\partial} + \bar{\partial}\partial = 0$ we obtain

$$\Omega = -\partial\bar{\partial} \log h.$$

This proves that Ω is a closed and purely imaginary $(1,1)$-form, so $\frac{i}{2\pi}\Omega$ is a closed, real $(1,1)$-form. Now, let H' be another Hermitian structure on L, and let $h' = H'(s,s)$: since h and h' are both smooth and strictly positive real valued functions, we can write $h' = e^f h$ for some $f : U \to \mathbb{R}$ smooth. Then, if ω' and Ω' are respectively the connection and curvature forms relative to this new Hermitian structure, we obtain

$$\begin{aligned} \Omega' &= -\partial\bar{\partial}\log h' \\ &= -\partial\bar{\partial} f - \partial\bar{\partial}\log h \\ &= -\partial\bar{\partial} f + \Omega. \end{aligned}$$

Since $-\partial\bar{\partial} f = d\partial f$, it follows that $\left[\frac{i}{2\pi}\Omega'\right] = \left[\frac{i}{2\pi}\Omega\right]$ in $H^2_{dR}(X)$. □

1.3 Hermitian manifolds

Definition 32. A *Hermitian manifold* is a complex manifold (X, J) together with a Riemannian metric g such that

$$g(JV, JW) = g(V, W)$$

for every $V, W \in \mathcal{T}(X)$. If g satisfies the above property, we say that g is *compatible* with J.

Remark 33. Every complex manifold (X, J) admits a structure of Hermitian manifold. In fact, let $\tilde{g}$ be a Riemannian metric on X, and define $g : \mathcal{T}(X) \times \mathcal{T}(X) \to C^\infty(X)$ by

$$g(V, W) = \tilde{g}(V, W) + \tilde{g}(JV, JW) :$$

then g is a Riemannian metric on X, and it is compatible with J.

Definition 34. Let (X, J, g) be a Hermitian manifold. The tensor

$$\begin{aligned} \omega : \mathcal{T}(X) \times \mathcal{T}(X) &\to C^\infty(X) \\ (V, W) &\mapsto g(JV, W) \end{aligned}$$

is called the *fundamental form* of the Hermitian manifold.

Lemma 35. *The fundamental form ω of a Hermitian manifold (X, J, g) is a real $(1,1)$-form.*

Proof. First of all, ω is a differential form, since

$$\begin{aligned}\omega(W,V) &= g(JW,V) \\ &= g\left(J^2W,JV\right) \\ &= -g(JV,W) \\ &= -\omega(V,W)\end{aligned}$$

for every $V,W \in \mathcal{T}(X)$. ω is real by definition. Now, let $\left(z^1,...,z^n\right)$ be a holomorphic chart for X: observe that the complexified metric satisfies

$$\begin{aligned}g\left(\frac{\partial}{\partial z^j},\frac{\partial}{\partial z^k}\right) &= g\left(J\frac{\partial}{\partial z^j},J\frac{\partial}{\partial z^k}\right) \\ &= -g\left(\frac{\partial}{\partial z^j},\frac{\partial}{\partial z^k}\right) \\ &= 0\end{aligned}$$

for every j,k, and analougously

$$g\left(\frac{\partial}{\partial \overline{z}^j},\frac{\partial}{\partial \overline{z}^k}\right) = 0$$

for every j,k. By definition of ω, the same holds for ω, so by lemma 14 ω is a $(1,1)$-form. □

Remark 36. The fundamental form ω of a Hermitian manifold (X,g) characterizes the metric, because if $V,W \in \mathcal{T}(X)$ then

$$g(V,W) = \omega(V,JW).$$

A real $(1,1)$-form ω such that the symmetric tensor g defined by the above equation is a Riemannian metric is called a *positive* form: in this case, (X,g) is a Hermitian manifold having fundamental form ω. For this reason, we often define Hermitian manifolds as complex manifolds along with a real, positive $(1,1)$-form ω.

Remark 37. If (X,g,ω) is a Hermitian manifold, then the tensor $h = g - i\omega$ is an Hermitian structure on the tangent bundle (TX,J) with the structure of complex bundle induced by J. This explains the choice of the word "Hermitian".

Let (X,ω,g) be a n-dimensional Hermitian manifold, and let $\left(z^1,...,z^n\right)$ be a holomorphic chart for X. Then the components of the complexified metric tensor and the fundamental form with respect to the local frames induced by $\left(z^1,...,z^n\right)$ have a special behaviour.

Notation. We use capital latin letters $A, B, \ldots$ to denote elements of the set $\left\{1, \ldots, n, \overline{1}, \ldots, \overline{n}\right\}$, and greek letters $\alpha, \beta, \ldots$ to denote elements of the set $\{1, \ldots, n\}$. We use the convention $\overline{\overline{A}} = A$, and we define

$$\begin{aligned} \frac{\partial}{\partial z^{\overline{\alpha}}} &= \frac{\partial}{\partial \overline{z}^{\alpha}} \\ dz^{\overline{\alpha}} &= d\overline{z}^{\alpha}. \end{aligned}$$

Now, denote

$$\begin{aligned} g_{AB} &= g\left(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}\right) \\ \omega_{AB} &= \omega\left(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}\right). \end{aligned}$$

Obviously, $g_{AB} = g_{BA}$ and $\omega_{AB} = -\omega_{BA}$; moreover, we have already seen that

$$\begin{aligned} g_{\alpha\beta} &= g_{\overline{\alpha}\overline{\beta}} = 0 \\ \omega_{\alpha\beta} &= \omega_{\overline{\alpha}\overline{\beta}} = 0. \end{aligned}$$

By definition, we have

$$\omega_{\alpha\overline{\beta}} = i g_{\alpha\overline{\beta}}.$$

Finally, it is easy to verify that

$$\begin{aligned} \overline{g_{\alpha\overline{\beta}}} &= g_{\overline{\alpha}\beta} \\ \overline{\omega_{\alpha\overline{\beta}}} &= \omega_{\overline{\alpha}\beta}. \end{aligned}$$

1.4 Kähler manifolds

Definition 38. A *Kähler manifold* is a Hermitian manifold (X, ω) whose fundamental form ω is closed. In this case, the fundamental form (resp. the Riemannian metric) is called a *Kähler form* (resp. *Kähler metric*).

Kähler manifolds have a very rich structure: Kähler geometry is, in a certain sense, the intersection point of Riemannian geometry, symplectic geometry and complex geometry. In fact, a Kähler manifold can be seen as a smooth manifold X along with three mutually compatible structures: an integrable almost complex structure J, a symplectic form ω, and a Riemannian metric g, which satisfies:

1. $g(JV, JW) = g(V, W)$ for all $V, W \in \mathcal{T}(X)$;
2. $\omega(JV, JW) = \omega(V, W)$ for all $V, W \in \mathcal{T}(X)$;
3. $\omega(V, W) = g(JV, W)$.

One of the most important features of Kähler manifolds is that its Kähler form (and hence also its Kähler metric) arises locally from a real valued function, as explained in the following way:

Definition 39. Let (X, ω) be a Kähler manifold. A *Kähler potential* for X is a smooth function $u : U \to \mathbb{R}$ defined in an open subset of X such that

$$\frac{i}{2}\partial\bar{\partial}u = \omega.$$

Remark 40. If $u : U \to \mathbb{R}$ is a Kähler potential for (X, ω), then u gives the local coordinates of the complexified metric tensor $g_{\alpha\overline{\beta}}$:

$$g_{\alpha\overline{\beta}} = \frac{1}{2}\frac{\partial^2 u}{\partial z^\alpha \partial \overline{z}^\beta}.$$

Kähler potentials do exists in a neighborhood of every point. In fact, if (X, ω) is a Kähler manifold and $x \in X$, then since ω is a closed real $(1,1)$-form, by the $i\partial\bar{\partial}$-lemma there is a smooth function $u : U \to \mathbb{R}$ defined in a neighborhood of x such that $\frac{i}{2}\partial\bar{\partial}u = \omega$. On the other hand, if (X, ω) is a Hermitian manifold, and for every $x \in X$ there is a smooth function $u : U \to \mathbb{R}$ in a neighborhood of x such that $\frac{i}{2}\partial\bar{\partial}u = \omega$, then ω is obviously closed since $d = \partial + \bar{\partial}$, so (X, ω) is Kähler.

Kähler potentials are not unique:

Lemma 41. *Let (X, ω) be a Kähler manifold, and let $u : U \to \mathbb{R}$ be a Kähler potential. Then, for every holomorphic function $f : U \to \mathbb{C}$, the function $u + \Re(f)$ is a Kähler potential for (X, ω). Moreover, every Kähler potential defined on U is locally of this form.*

Proof. Since f is holomorphic, we have $\partial\bar{\partial}f = 0$. This implies that $\partial\bar{\partial}\Re(f) = -i\partial\bar{\partial}\Im(f)$: but the right hand term is a real form, and the left hand term is a purely imaginary form: this implies that $\partial\bar{\partial}\Re(f) = 0$, and hence

$$\frac{i}{2}\partial\bar{\partial}(u + \Re(f)) = \frac{i}{2}\partial\bar{\partial}u,$$

proving that $u + \Re(f)$ is a Kähler potential. Now, let $u, u' : U \to \mathbb{R}$ be two Kähler potentials for (X, ω). Define $f_1 = u - u'$. Then $i\left(\bar{\partial} f_1 - \partial f_1\right)$ is a closed real 1-form, since $d = \partial + \overline{\partial}$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$. It follows that, given a $x \in U$, there exists a neighborhood $V \subseteq U$ of x and a real valued smooth function f_2 such that $df_2 = i\left(\bar{\partial} f_1 - \partial f_1\right)$. This implies that $\bar{\partial} f_2 = i\bar{\partial} f_1$, and hence the function $f_1 + if_2$ defined on V is holomorphic. It follows then that $u - u'$ is the real part of a holomorphic function on V, and by generality of x we obtain the thesis. □

Definition 42. Let (X, ω) and (Y, Ω) be two Kähler manifolds. A *Kähler immersion* is a holomorphic map $f : X \to Y$ such that $f^*\Omega = \omega$.

By definition of fundamental form, a holomorphic map $f : (X, \omega) \to (Y, \Omega)$ is a Kähler immersion if and only if it is an isometric immersion. Moreover, if $f : X \to Y$ is a Kähler immersion, then for every Kähler potential $v : V \to \mathbb{R}$ of Y, the composition $v \circ f : f^{-1}(V) \to \mathbb{R}$ is a Kähler potential for X, since

$$\begin{aligned}
\frac{i}{2}\partial\bar{\partial}(v \circ f) &= \frac{i}{2}\partial\bar{\partial}(f^*(v)) \\
&= f^*\left(\frac{i}{2}\partial\bar{\partial} v\right) \\
&= f^*\Omega \\
&= \omega.
\end{aligned}$$

Conversely, if $f : X \to Y$ is holomorphic and for every $x \in X$ there is a Kähler potential $v : V \to \mathbb{R}$ of Y in a neighborhood of $f(x)$ such that $v \circ f$ is again a Kähler potential for X in a neighborhood of x, then we have $f^*\Omega = \omega$ in a neighborhood of x, so f is a Kähler immersion by generality of x.

Example 43. Consider the n-dimensional complex space $\mathbb{C}^n$. The differential form

$$\omega_0 = \frac{i}{2}\sum_{j=1}^{n} dz^j \wedge d\bar{z}^j$$

is a real, positive and closed $(1,1)$-form, so it defines a Kähler metric on $\mathbb{C}^n$ denoted by g_0. The Kähler manifold $(\mathbb{C}^n, \omega_0)$ admits a global Kähler potential

$$\begin{aligned}
u : \mathbb{C}^n &\to \mathbb{R} \\
z &\mapsto |z|^2 .
\end{aligned}$$

Observe that, by identifying $\mathbb{C}^n$ with its underlining smooth manifold $\mathbb{R}^{2n}$, the Kähler metric g_0 coincides with the usual flat metric on $\mathbb{R}^{2n}$.

Example 44. Let Λ be a lattice of $\mathbb{C}^n$, i.e. a discrete subgroup of $\mathbb{C}^n$ of rank $2n$. The group Λ can be seen as a 0-dimensional complex Lie group acting properly and freely on $\mathbb{C}^n$ by holomorphic isometries, so the quotient space $\mathbb{C}^n/\Lambda$ has a natural structure of Kähler manifold induced by the proper and free action of Λ on $\mathbb{C}^n$. The projection $\pi : \mathbb{C}^n \to \mathbb{C}^n/\Lambda$ is a holomorphic covering map that preserves the Kähler metric. We call $\mathbb{C}^n/\Lambda$ a *complex torus* of dimension n, and we call the metric induced by the standard metric of $\mathbb{C}^n$ the *flat metric* on the complex torus. Observe that a complex torus is compact, since it is the image of a compact set via π.

Example 45. Consider the ball model of the n-dimensional complex hyperbolic space $\mathbb{CB}^n$. Then the global form

$$\omega_{hyp} = -\frac{i}{2\pi}\partial\bar{\partial}\log\left(1-|z|^2\right)$$

is a real, positive and closed $(1,1)$-form that defines a Kähler metric on $\mathbb{CB}^n$ denoted by g_{hyp}. Also in this case, we have a globally defined Kähler potential

$$\begin{array}{rcl} u : \mathbb{CB}^n & \to & \mathbb{R} \\ z & \mapsto & -\dfrac{1}{\pi}\log\left(1-|z|^2\right). \end{array}$$

In the projective model $\mathbb{CH}^n$, the same Kähler potential reads as

$$\begin{array}{rcl} u\circ\varphi_0 : \mathbb{CH}^n & \to & \mathbb{R} \\ [Z] & \mapsto & -\dfrac{1}{\pi}\log\left(-\dfrac{(Z,Z)}{Z^0\overline{Z^0}}\right), \end{array}$$

where $(,)$ is the Hermitian product on $\mathbb{C}^{n+1}$ defined in example 7.

Example 46. Consider the n-dimensional complex projective space $\mathbb{CP}^n$. Consider the function

$$\begin{array}{rcl} u : \mathbb{C}^n & \to & \mathbb{R} \\ z & \mapsto & \dfrac{1}{\pi}\log\left(1+|z|^2\right), \end{array}$$

and define $u_j = u\circ\varphi_j$ for $j=0,...,n$. Then the real $(1,1)$-form

$$\omega_j = \frac{i}{2}\partial\bar{\partial}u_j$$

is positive and closed. The locally defined 2-forms ω_j are actually the restriction of a globally defined 2-form: in fact, if we define

$$\begin{array}{rcl} v : \mathbb{C}^{n+1} \backslash \{0\} & \to & \mathbb{R} \\ Z & \mapsto & \frac{1}{\pi} \log \left(|Z|^2 \right), \end{array}$$

then, if we denote by $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{CP}^n$ the canonical projection, it is easy to prove that

$$\pi^* \omega_j = \frac{i}{2} \partial \bar{\partial} v,$$

and since π is a holomorphic submersion, the locally defined 2-forms ω_j must be pairwise compatible. Patching together the 2-forms ω_j, we obtain a Kähler form ω_{FS}, called the *Fubini Study form*, having Kähler potentials u_j. Observe that the explicit expression of u_j in homogeneous coordinates is

$$u_j([Z]) = \frac{1}{\pi} \log \left(\frac{\langle Z, Z \rangle}{Z^j \overline{Z^j}} \right).$$

Remark 47. In literature, the Kähler forms ω_{hyp} and ω_{FS} often differs from our definition up to a multiplication by some positive constant. With our choice, the Kähler forms ω_{hyp} and ω_{FS} are *integral* (see definition **??**).

Definition 48. A Kähler manifold X is called (complex) *homogeneous* if there is a group acting transitively on X by holomorphic isometries.

Lemma 49. *The Kähler manifolds* $(\mathbb{C}^n, \omega_0)$*,* $(\mathbb{CP}^n, \omega_{FS})$ *and* $(\mathbb{CH}^n, \omega_{hyp})$ *are homogeneous.*

Proof.

1. Denote by $\mathrm{E}(n)$ the group generated by the group of translations of $\mathbb{C}^n$ and the unitary group of the canonical positive definite Hermitian product $\langle , \rangle$ on $\mathbb{C}^n$, called $\mathrm{U}(n)$. It is immediate to verify that this group acts by holomorphic isometries on $\mathbb{C}^n$, and this action is transitive and faithful.

2. The group $\mathrm{U}(n+1)$ acts transitively by isometries on $\mathbb{CP}^n$. In fact, if $A \in \mathrm{U}(n+1)$, by $\mathbb{C}$-linearity and by the fact that A preserves the $\langle , \rangle$-norms, A induces a biholomorphism $\tilde{A} : \mathbb{CP}^n \to \mathbb{CP}^n$, which is an

isometry: in fact, since

$$\begin{aligned}\pi^*\left(\tilde{A}^*\omega_{FS}\right) &= (\pi\circ A)^*(\omega_{FS})\\ &= A^*(\pi^*\omega_{FS})\\ &= A^*\left(\frac{i}{2}\partial\bar{\partial}v\right)\\ &= \frac{i}{2}\partial\bar{\partial}(v\circ A)\\ &= \frac{i}{2}\partial\bar{\partial}v\\ &= \pi^*\omega_{FS},\end{aligned}$$

and since π is a surjective submersion, it follows that

$$\tilde{A}^*\omega_{FS}=\omega_{FS}.$$

The action is transitive since the action of $\mathrm{U}(n+1)$ on the set of unitary vectors of $\mathbb{C}^n$ is transitive. Observe that, for $A\in\mathrm{U}(n+1)$, we have $\tilde{A}=\mathrm{id}$ if and only if $A\in\mathrm{U}(1)\subseteq\mathrm{U}(n+1)$, so the action of $\mathrm{U}(n+1)$ induces an action of $\mathrm{PU}(n+1)=\mathrm{U}(n+1)/\mathrm{U}(1)$ by holomorphic isometries, which is transitive and faithful.

3. Let $\mathrm{U}(n,1)$ be the unitary group of the Hermitian product $(,)$ on $\mathbb{C}^{n+1}$. Then, with nearly the same proof of point 2, it can be proved that $\mathrm{U}(n,1)$ acts transitively by holomorphic isometries on $\mathbb{CH}^n$, and that this action induces a transitive and faithful action of the group $\mathrm{PU}(n,1)=\mathrm{U}(n,1)/\mathrm{U}(1)$ by holomorphic isometries.

$\square$

Remark 50. It can be proved that the groups $\mathrm{E}(n)$, $\mathrm{PU}(n+1)$ and $\mathrm{PU}(n,1)$ are actually the holomorphic isometry groups of $(\mathbb{C}^n,\omega_0)$, $(\mathbb{CP}^n,\omega_{FS})$, and $(\mathbb{CH}^n,\omega_{hyp})$ respectively.

Definition 51. A Kähler manifold X is called (complex) *isotropic* at a point $p\in X$ if, for every $v,w\in T_pX$ having the same norm, there exists a holomorphic isometry $f:X\to X$ fixing p such that $df_p(v)=w$.

It is clear that an homogeneous Kähler manifold is isotropic at a point if and only if it is isotropic at every point.

Lemma 52. *The Kähler manifolds* $(\mathbb{C}^n, \omega_0)$*,* $(\mathbb{CP}^n, \omega_{FS})$ *and* $(\mathbb{CH}^n, \omega_{hyp})$ *are isotropic.*

Proof. By homogeneity, it suffices to verify isotropy at one point.

1. The stabilizer of $0 \in \mathbb{C}^n$ in $\mathrm{E}(n)$ is precisely the unitary group $\mathrm{U}(n)$, and this group clearly acts transitively on the set of unit vectors of $T_0\mathbb{C}^n \equiv \mathbb{C}^n$.

2. The stabilizer of $0 \in \mathbb{CB}^n$ in $\mathrm{PU}(1,n)$ is again the unitary group $\mathrm{U}(n)$, embedded in $\mathrm{PU}(1,n)$ as the set of the equivalence classes of unitary transformations $A : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ that fix $(1, 0, ..., 0)$. Since the metric g_{hyp} coincides with g_0 in the point 0, as in the previous point the group $\mathrm{U}(n)$ acts transitively on the set of unit vectors of $T_0\mathbb{CB}^n$.

3. The stabilizer of $0 \in \mathbb{CP}^n$ in $\mathrm{PU}(n+1)$ is again the unitary group $\mathrm{U}(n)$ embedded in $\mathrm{PU}(n+1)$ as above. In local coordinates given by the chart of the canonical atlas (φ_0, U_0) centered at 0, the metric g_{FS} coincides with g_0 in 0, so as above $\mathrm{U}(n)$ acts transitively on the set of unit vectors of $T_0\mathbb{CP}^n$.

□

1.5 Curvature tensors in Kähler manifolds

In this section, we recall some of the most important simmetries of the curvature tensors in a Kähler manifold. We make use of the Einstein convention.

Let (X, g) be a n-dimensional Kähler manifold. We extend the curvature tensor $R \in \mathcal{T}_3^1(X)$, the curvature endomorphism $Rm \in \mathcal{T}_4(X)$, the Ricci tensor $Ric \in \mathcal{T}_2(X)$ and the Levi Civita connection $\nabla : \mathcal{T}(X) \times \mathcal{T}(X) \to \mathcal{T}(X)$ by $\mathbb{C}$-linearity. We now compute the components of R, Rm, Ric and the Christoffel symbols of ∇ in terms of a generic local holomorphic chart $\left(z^1, ..., z^n\right)$. The first observation to make is that

$$\begin{aligned} \overline{R_{ABC}^{D}} &= R_{\overline{ABC}}^{\overline{D}} \\ \overline{Rm_{ABCD}} &= Rm_{\overline{ABCD}} \\ \overline{Ric_{AB}} &= Ric_{\overline{AB}} \\ \overline{\Gamma_{AB}^{C}} &= \Gamma_{\overline{AB}}^{\overline{C}} \end{aligned}$$

and the above coefficients satisfy the usual symmetries of the Riemannian case. Other simmetries that reflects the Kähler structure can be obtained by the following lemma: a proof can be found e.g. in [20], theorem 11.5.

Lemma 53. *The tensor J is ∇-parallel, i.e. $\nabla_V JW = J\nabla_V W$ for every $V, W \in \mathcal{T}(X)$.*

From the above lemma, it follows immediately that, for every complex vector field $V \in \mathcal{T}^{\mathbb{C}}(X)$, the map $\nabla_V : \mathcal{T}^{\mathbb{C}}(X) \to \mathcal{T}^{\mathbb{C}}(X)$ preserves the decomposition $\mathcal{T}^{\mathbb{C}}(X) = \mathcal{T}^{1,0}(X) \oplus \mathcal{T}^{0,1}(X)$. In local coordinates, this property is reflected by the equation

$$\Gamma^{\gamma}_{A\overline{\beta}} = \Gamma^{\overline{\gamma}}_{A\beta} = 0.$$

Since ∇ is symmetric, it follows that the only nonvanishing Christoffel symbols of ∇ are

$$\Gamma^{\gamma}_{\alpha\beta} \text{ and } \Gamma^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}}.$$

Now, by compatibility of ∇ with g, we obtain

$$\begin{aligned} \frac{\partial g_{\beta\overline{\delta}}}{\partial z^{\alpha}} &= \frac{\partial}{\partial z^{\alpha}} g\left(\frac{\partial}{\partial z^{\beta}}, \frac{\partial}{\partial \overline{z}^{\delta}}\right) \\ &= g\left(\nabla_{\frac{\partial}{\partial z^{\alpha}}} \frac{\partial}{\partial z^{\beta}}, \frac{\partial}{\partial \overline{z}^{\delta}}\right) + g\left(\frac{\partial}{\partial z^{\beta}}, \nabla_{\frac{\partial}{\partial z^{\alpha}}} \frac{\partial}{\partial \overline{z}^{\delta}}\right) \\ &= g\left(\nabla_{\frac{\partial}{\partial z^{\alpha}}} \frac{\partial}{\partial z^{\beta}}, \frac{\partial}{\partial \overline{z}^{\delta}}\right) \\ &= \Gamma^{\gamma}_{\alpha\beta} g_{\gamma\overline{\delta}}. \end{aligned}$$

If we define coefficients g^{AB} such that

$$\begin{aligned} \overline{g^{AB}} &= g^{\overline{AB}} \\ g^{AB} &= g^{BA} \\ g^{\alpha\overline{\beta}} g_{\overline{\beta}\gamma} &= \delta^{\alpha}_{\gamma}, \end{aligned}$$

then we obtain

$$\Gamma^{\gamma}_{\alpha\beta} = g^{\gamma\overline{\delta}} \frac{\partial g_{\beta\overline{\delta}}}{\partial z^{\alpha}}.$$

Again from the above lemma, we obtain

$$R^{\delta}_{AB\overline{\gamma}} = R^{\overline{\delta}}_{AB\gamma} = 0,$$

and hence, using the definition of Rm and the simmetries of R and Rm, the only nonvanishing components of R and Rm are

$$R^{\delta}_{\alpha\overline{\beta}\gamma}, R^{\overline{\delta}}_{\alpha\overline{\beta}\overline{\gamma}}, R^{\delta}_{\overline{\alpha}\beta\gamma}, R^{\overline{\delta}}_{\overline{\alpha}\beta\overline{\gamma}}$$
$$Rm_{\alpha\overline{\beta}\gamma\overline{\delta}}, Rm_{\alpha\overline{\beta}\overline{\gamma}\delta}, Rm_{\overline{\alpha}\beta\overline{\gamma}\delta}, Rm_{\overline{\alpha}\beta\gamma\overline{\delta}} \quad .$$

Using the definition of Rm and the above identities, after some easy computations we obtain

$$Rm_{\alpha\overline{\beta}\gamma\overline{\delta}} \quad = \quad g^{w\overline{v}}\frac{\partial g_{\gamma\overline{v}}}{\partial z^{\alpha}}\frac{\partial g_{w\overline{\delta}}}{\partial\overline{z}^{\beta}} - \frac{\partial^2 g_{\gamma\overline{\delta}}}{\partial z^{\alpha}\partial\overline{z}^{\beta}}. \tag{1.5.1}$$

1.6 Complex space forms

Recall that, if (M, g) is a Riemannian manifold, the *sectional curvature of M in a tangent 2-plane $\sigma_p \subseteq T_pM$*, denoted by $K(\sigma_p)$, is the Gaussian curvature at p of the embedded surface defined locally as the image of a sufficiently small neighborhood of O_p in σ_p via the restriction to σ_p of the exponential map $\exp_p$. Explicitly, if u, v is a basis of σ_p, then

$$K(\sigma_p) = \frac{Rm_p(u, v, v, u)}{g_p(u, u)\, g_p(v, v) - g_p(u, v)^2}.$$

Definition 54. Let (X, J, g) be a Kähler manifold, and let σ_p be a complex line of (T_pX, J_p). The *holomorphic sectional curvature of X at σ_p* is the sectional curvature $K(\sigma_p)$.

Let σ_p be a complex line of (T_pX, J_p). Let $v \in \sigma_p$ be a nonzero vector. Let $v = v^{1,0} + v^{0,1}$ be its unique decomposition into a sum of a vector of type $(1, 0)$ and a vector of type $(0, 1)$. Since v is real, by conjugation we obtain $v^{0,1} = \overline{v^{1,0}}$. We then have

$$\begin{aligned} g_p(v, J_pv) &= g_p\left(v^{1,0} + \overline{v^{1,0}}, iv^{1,0} - i\overline{v^{1,0}}\right) \\ &= -ig_p\left(v^{1,0}, \overline{v^{1,0}}\right) + ig_p\left(\overline{v^{1,0}}, v^{1,0}\right) \\ &= 0. \end{aligned}$$

Since v is nonzero, this implies that v and J_pv are linearly independent as real vectors of T_pX. It then follows that the holomorphic sectional curvature is given explicitly by

$$K(\sigma_p) = \frac{Rm_p(v, J_pv, J_pv, v)}{g_p(v, v)^2}. \tag{1.6.1}$$

Definition 55. A complete Kähler manifold with constant holomorphic sectional curvature is called a *complex space form*.

As in the Riemannian case, there are three possible simply connected complex space forms up to holomorphic conformal equivalences (a conformal equivalence is a biholomorphism $f : (X, \omega) \to (Y, \Omega)$ such that $f^*\Omega = \lambda\omega$ for some real valued positive smooth map λ).

Proposition 56. *The Kähler manifolds* $(\mathbb{C}^n, \omega_0)$, $(\mathbb{CP}^n, \omega_{FS})$ *and* $(\mathbb{CB}^n, \omega_{hyp})$ *are simply connected complex space forms of holomorphic sectional curvature respectively* 0, 4π, -4π.

Proof. All the three above Kähler manifolds are simply connected, homogeneous and isotropic. They are then complete, since they are in particular homogeneous Riemannian manifolds, and a homogeneous Riemannian manifold is always complete. Now we prove that every homogeneous and isotropic Kähler manifold (X, J, g) has constant holomorphic sectional curvature. Let $\sigma_p \subseteq T_pX$ and $\sigma_q \subseteq T_qX$ be two complex lines: let $v \in \sigma_p$ and $w \in \sigma_q$ be two unit vectors. Then, since X is homogeneous and isotropic, there is an isometry $f : X \to X$ such that

$$\begin{aligned} f(p) &= q \\ df_p(v) &= w. \end{aligned}$$

We then have

$$\begin{aligned} K(\sigma_p) &= Rm_p(v, J_pv, J_pv, v) \\ &= Rm_q(df_p(v), (df_p \circ J_p)(v), (df_p \circ J_p)(v), df_p(v)) \\ &= Rm_q(w, J_qw, J_qw, w) \\ &= K(\sigma_q) \end{aligned}$$

where we have used that $df_p \circ J_p = J_q \circ df_p$ by holomorphy of f.
It then follows that $(\mathbb{C}^n, \omega_0)$, $(\mathbb{CP}^n, \omega_{FS})$ and $(\mathbb{CB}^n, \omega_{hyp})$ are all simply connected complex space forms. The curvature can then be easily computed by applying formulas 1.5.1 and 1.6.1. $\square$

Analougously to the Riemannian case, the above simply connected complex space forms are essentially the unique possibilities:

Theorem 57 (uniformization theorem)**.** *Let* X, Y *be two simply connected complex space forms. If they have the same holomorphic sectional curvature, then they are holomorphically isometric.*

Proof. See [13], theorem 7.9. □

The above theorem, together with lemma 56, implies that the only n-dimensional flat (resp. elliptic, hyperbolic) simply connected complex space form is $(\mathbb{C}^n, \omega_0)$ (resp. a rescalation of $(\mathbb{CP}^n, \omega_{FS})$, a rescalation of $(\mathbb{CB}^n, \omega_{hyp})$).

1.7 Infinite dimensional Kähler manifolds

Smooth (resp. complex) manifolds are locally modeled on open sets of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$), with smooth (resp. holomorphic) changes of local coordinates. There are no difficults in generalizing this definition to manifolds locally modeled on the separable Hilbert space $\ell^2(\mathbb{R})$ (resp. $\ell^2(\mathbb{C})$):

Definition 58. Let X be a topological space. A *infinite dimensional smooth* (resp. *complex*) *atlas* on X is a family $\{(\varphi_\alpha, U_\alpha)\}$, where:

1. $\{U_\alpha\}$ is an open cover of X;
2. φ_α are homeomorphisms of U_α onto open subsets of $\ell^2(\mathbb{R})$ (resp. $\ell^2(\mathbb{C})$);
3. for every α, β the map $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is smooth (resp. holomorphic).

A *infinite dimensional smooth* (resp. *complex*) *manifold* is a Hausdorff topological space X with a countable basis, along with a maximal infinite dimensional smooth (resp. complex) atlas on it.

The definition of smooth (resp. holomorphic) map between smooth (resp. complex) manifolds of finite or infinite dimension is the usual one. We can generalize all the basic concepts of differential topology to the infinite dimensional case: tensor bundles are defined almost exactly as in the finite dimensional case, except for the fact that in the infinite dimensional case they are not finite rank vector bundles anymore, but vector bundles with fibres having a structure of separable Hilbert space. All the concepts developed until now in this thesis can be extended to the infinite dimensional case with no difficulties. Unfortunately, to the author's knowledge, there is not a systematic treatise about infinite dimensional complex and Kähler geometry. Good references about infinite dimensional Riemannian manifolds are e.g. [17, 27, 16].

Chapter 2

Kähler immersions into complex space forms

The famous Nash embedding Theorem states that every Riemannian manifold can be isometrically immersed in a sufficiently large Euclidean space. Of course, there is not an immediate Kähler analougous of this theorem: in fact, for example, every compact Kähler manifold cannot even be *holomorphically* immersed in some $\mathbb{C}^n$, since a holomorphic map from a compact complex manifold to $\mathbb{C}$ must be constant because of the maximum principle. The isometric immergibility problem can be generalized by allowing the ambient manifold to be a finite or infinite dimensional simply connected complex space form. From now on, we denote by $F(b, N)$ ($b \in \mathbb{R}$, $N \leq \infty$) the N-dimensional simply connected complex space form of curvature $4b$, i.e.

$$F(b, N) \equiv \begin{cases} \left(\mathbb{C}^N, g_0\right) & b = 0 \\ \left(\mathbb{CB}^N, \frac{\pi}{-b} g_{hyp}\right) \simeq \left(\mathbb{CH}^n, \frac{\pi}{-b} g_{hyp}\right) & b < 0 \\ \left(\mathbb{CP}^N, \frac{\pi}{b} g_{FS}\right) & b > 0 \end{cases} .$$

Problem. Given a finite dimensional Kähler manifold (X, ω) and a simply connected complex space form $F(b, N)$, is there a Kähler immersion $(X, \omega) \to F(b, N)$?

A necessary and sufficient condition on (X, ω) to admit a Kähler immersion into some $F(b, N)$ was found by Eugenio Calabi in his doctoral thesis [8]. In this chapter, we present this work.

2.1 Calabi's diastasis

Since we are interested in studying Kähler immersions, we are interested in properties of Kähler manifolds that are preserved by Kähler immersions. The first one is the *analyticity* of a Kähler manifold:

Definition 59. A Kähler manifold (X, ω) is called *analytic* if, for every $p \in X$, there exists a Kähler potential $u : U \to \mathbb{R}$ which is a real analytic function.

Remark 60. Notice that, if (X, ω) is an analytic Kähler manifold, then *every* Kähler potential of (X, ω) is a real analytic function, since two Kähler potentials differs locally by the real part of a holomorphic function, which is real analytic. It follows that a Kähler potential $u : U \to \mathbb{R}$ of (X, ω) is determined in a neighborhood of a point $p \in U$ by its germ.

Analyticity of a Kähler manifold passes to its immersed Kähler submanifolds:

Lemma 61. *Let $f : (X, \omega) \to (Y, \Omega)$ be a Kähler immersion between Kähler manifolds. Then, if (Y, Ω) is analytic, also (X, ω) is analytic.*

Proof. For a $x \in X$, consider an analytic Kähler potential $v : V \to \mathbb{R}$ for (Y, Ω) defined on a neighborhood of $f(x)$. Since f is a Kähler immersion, the map $v \circ f : f^{-1}(V) \to \mathbb{R}$ is an analytic Kähler potential for (X, ω) in a neighborhood of x. By generality of x we obtain the thesis. □

From the above lemma and the fact that every simply connected complex space form is clearly analytic, we obtain the following

Corollary 62. *If (X, ω) can be Kähler immersed in some simply connected complex space form, then (X, ω) is analytic.*

Because of this corollary, we concentrate from now on to analytic Kähler manifolds. Although in a generic Kähler manifold a Kähler potential is defined locally up to the sum with the real part of a holomorphic function, if (X, ω) is an analytic Kähler manifold, then for every $p_0 \in X$ there is a very special Kähler potential in a neighborhood of p_0.

Notation. If $\mathbf{m} = (m^1, ..., m^n)$ is a n-multiindex (an element of $(\mathbb{N} \cup \{0\})^n$), then we use symbols

$$\mathbf{z}^{\mathbf{m}} \equiv (z^1)^{m^1} \cdots (z^n)^{m_n^n}$$

and

$$\partial \mathbf{z}^{\mathbf{m}} \equiv \partial \left(z^1\right)^{m^1} \cdots \partial \left(z^n\right)^{m^n}.$$

We define $|\mathbf{m}| = m^1 + \cdots + m^n$. We define a total order on the set of n-multiindexes: we say that $\mathbf{m} < \mathbf{h}$ if and only if $|\mathbf{m}| < |\mathbf{h}|$ or $|\mathbf{m}| = |\mathbf{h}|$ and if j is the first index such that $m^j \neq h^j$, then $m^j > h^j$. For example, we have

$$(1,0,0) < (0,1,0) < (2,0,0) < (1,1,0) < (4,0,0) < (3,1,0) < (3,0,1).$$

We denote by $\{\mathbf{m}_j\}_{j\in\mathbb{N}}$ the increasing sequence of n-multiindexes with respect to this order. Now, if $f : Y \to \mathbb{C}$ is a smooth map, and $\left(z^1, ..., z^n\right)$ are holomorphic coordinates in a neighborhood of p_0, we express the Taylor series of f at p_0 in local coordinates in the following form:

$$\sum_{k,h=0}^{\infty} a_{kh} \mathbf{z}^{\mathbf{m}_k} \overline{\mathbf{z}}^{\mathbf{m}_h}.$$

If we define $\mathbf{m}_j! = m_j^1! \cdots m_j^n!$, then

$$a_{kh} = \frac{1}{\mathbf{m}_k!} \frac{1}{\mathbf{m}_h!} \frac{\partial^{|\mathbf{m}_k| + |\mathbf{m}_h|} f}{\partial \mathbf{z}^{\mathbf{m}_k} \partial \overline{\mathbf{z}}^{\mathbf{m}_h}} (p).$$

We call the $\infty \times \infty$ Hermitian matrix $(a_{kh})_{k,h=0}^{\infty}$ the *Taylor matrix* of f at p_0 with respect to $\left(z^1, ..., z^n\right)$.

Let $u : U \to \mathbb{R}$ be a Kähler potential defined in a connected neighborhood of p_0. By possibly shrinking U, we can suppose that there exists an *off-diagonal* holomorphic extension of u, i.e. a holomorphic map $\tilde{u} : U \times \overline{U} \to \mathbb{C}$ (where $\overline{U}$ denotes the conjugate manifold of U) such that

$$\tilde{u}(p, \overline{p}) = u(p)$$

for every $p \in U$. In fact, if we choose local holomorphic coordinates $\left(z^1, ..., z^n\right)$ centered at p_0, then the Taylor expansion of u at p_0 is

$$u(p) = \sum_{k,h=0}^{\infty} b_{kh} \mathbf{z}^{\mathbf{m}_k}(p) \overline{\mathbf{z}^{\mathbf{m}_h}(p)},$$

and is convergent in a neighborhood of p_0 (that we can suppose to be U) because u is analytic. By possibly shrinking again U, the map

$$\tilde{u}(p, \overline{q}) = \sum_{k,h=0}^{\infty} b_{kh} \mathbf{z}^{\mathbf{m}_k}(p) \overline{\mathbf{z}^{\mathbf{m}_h}(q)}$$

is holomorphic on $U\times\overline{U}$ and satisfies $\tilde{u}(p,\overline{p})=u(p)$ for every $p\in U$. Observe that $\tilde{u}$ is uniquely determined by u: in fact, if $\tilde{u}$ is holomorphic and $\tilde{u}(p,\overline{p})=u(p)$ in a neighborhood of p_0, then if we choose a holomorphic chart $\left(z^1,...,z^n\right)$ centered at p_0, and we denote by $\left(z^1,...,z^n,\overline{z}^1,...,\overline{z}^n\right)$ the induced chart of $X\times\overline{X}$, the Taylor matrix of $\tilde{u}$ at $(p_0,\overline{p}_0)$ depends only on u, and since U is connected, the Taylor matrix of $\tilde{u}$ at $(p_0,\overline{p_0})$ determines $\tilde{u}$.

It then follows that the map

$$\begin{aligned} D^X_{p_0,U}:U &\to \mathbb{R} \\ q &\mapsto \tilde{u}(p_0,\overline{p_0})+\tilde{u}(q,\overline{q})-\tilde{u}(p_0,\overline{q})-\tilde{u}(q,\overline{p_0}) \end{aligned} \tag{2.1.1}$$

is a well defined Kähler potential for (X,ω), because the map $\tilde{u}(p_0,\cdot)$ is antiholomorphic and hence $D^X_{p_0,U}$ is the sum of the Kähler potential u and of the real part of a holomorphic function $u(p_0)+\tilde{u}(p_0,\cdot)+\overline{\tilde{u}(p_0,\cdot)}$. The map $D^X_{p_0,U}$ does not depend on u in a neighborhood of p_0: in fact, if $v:U\to\mathbb{R}$ is another Kähler potential, by possibly shrinking U we can suppose that $v=u+\phi+\overline{\phi}$ for some holomorphic map $\phi:U\to\mathbb{C}$; this implies that v admits an unique off-diagonal holomorphic extension $\tilde{v}(p,\overline{q})=\tilde{u}(p,\overline{q})+\phi(p)+\overline{\phi(q)}$ on U, and hence it is an easy calculation to show that

$$\tilde{u}(p_0,\overline{p_0})+\tilde{u}(q,\overline{q})-\tilde{u}(p_0,\overline{q})-\tilde{u}(q,\overline{p_0})=\tilde{v}(p_0,\overline{p_0})+\tilde{v}(q,\overline{q})-\tilde{v}(p_0,\overline{q})-\tilde{v}(q,\overline{p_0}).$$

The above argument shows that the following definition is well posed:

Definition 63. The *germ of the diastasis of X at p_0* is the germ at p_0 of the function $D^X_{p_0,U}$ defined above, and it is denoted by $\mathbf{D}^X_{p_0}$.

Observe that, by definition, $\mathbf{D}^X_{p_0}(p_0)=0$.

Definition 64. Let U be a domain of X containing p_0. If there exists a representative of the germ $\mathbf{D}^X_{p_0}$ defined on U, then we denote it by $D^X_{p_0,U}$ and we call it the *diastatic potential*[1] *of X centered at p_0 defined on U*.

The above definition is clearly coherent with the one used in equation 2.1.1. A diastatic potential $D^X_{p_0,U}$ is actually a Kähler potential for (X,ω): in fact, by equation 2.1.1, $\frac{i}{2}\partial\bar{\partial}D^X_{p_0,U}=\omega$ in a neighborhood of p_0, and since the Kähler manifold is analytic, by connectedness of U we obtain $\frac{i}{2}\partial\bar{\partial}D^X_{p_0,U}=\omega$ on U.

[1] In his article, Calabi called it the *diastasis centered at p_0*. In this thesis, instead, we use the term "diastasis" to denote the function D^X defined on a domain of $X\times X$ such that for fixed $p\in X$ the map $D^X(p,\cdot)$ is the maximally defined diastatic potential centered at p. This function does not exists always: see 65 for a counterexample.

Example 65. Consider the 1-dimensional Kähler manifold $X = \mathbb{C}^*$ with the global Kähler potential

$$\begin{array}{rcl} u : \mathbb{C}^* & \to & \mathbb{R} \\ z & \mapsto & |z| \,. \end{array}$$

The Kähler potential u does not admit any off-diagonal holomorphic extension defined in $\mathbb{C}^*$. However, define $U = \mathbb{C}^* \setminus (0, +\infty)$ and $V = \mathbb{C}^* \setminus (-\infty, 0)$. Observe that the intersection $U \cap V$ has two connected components $A = \{z \in \mathbb{C} : \Im(z) > 0\}$ and $B = \{z \in \mathbb{C} : \Im(z) < 0\}$. The locally defined potentials $u_{|U}$ and $u_{|V}$ do admits off-diagonal holomorphic extensions: if we choose two branches of the square root $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$, such that

$$f(p) = \begin{cases} g(p) & p \in A \\ -g(p) & p \in B \end{cases},$$

then the map

$$\begin{array}{rcl} U \times \overline{U} & \to & \mathbb{C} \\ (p, \overline{q}) & \mapsto & f(p)\,\overline{f(q)} \end{array}$$

is the holomorphic extension of $u_{|U}$, and the map

$$\begin{array}{rcl} V \times \overline{V} & \to & \mathbb{C} \\ (p, \overline{q}) & \mapsto & g(p)\,\overline{g(q)} \end{array}$$

is the holomorphic extension of $u_{|V}$. Then, for every $p_0 \in U$, the map

$$\begin{array}{rcl} D^X_{p_0,U} : U & \to & \mathbb{R} \\ q & \mapsto & f(p_0)\,\overline{f(p_0)} + f(q)\,\overline{f(q)} - f(p_0)\,\overline{f(q)} - f(q)\,\overline{f(p_0)} \end{array}$$

is the diastatic potential of X centered at p_0 defined on U, and for every $p_0 \in V$ the map

$$\begin{array}{rcl} D^X_{p_0,V} : V & \to & \mathbb{R} \\ q & \mapsto & g(p_0)\,\overline{g(p_0)} + g(q)\,\overline{g(q)} - g(p_0)\,\overline{g(q)} - g(q)\,\overline{g(p_0)} \end{array}$$

is the diastatic potential of X centered at p_0 defined on V. If $p_0 \in A$, then $D^X_{p_0,U}(q) \neq D^X_{p_0,V}(q)$ for every $q \in B$, so the two local diastatic potentials $D^X_{p_0,U}$ and $D^X_{p_0,V}$ cannot be patched together to a global one. Observe that $D^X_{p_0,U}$ and $D^X_{p_0,V}$ do not admit any real analytic extension to a larger domain.

Definition 66. For every $p_0 \in X$, we define the *maximal domain of analyticity* of $\mathbf{D}^X_{p_0}$ as the domain of points $q \in X$ such that the germ $\mathbf{D}^X_{p_0}$ admits an analytic continuation along a curve joining p_0 to q. We denote it by X_{p_0}. When the diastatic potential centered at p_0 is defined on X_{p_0}, we denote it simply by $D^X_{p_0}$. If it does happen for every p_0, we define the *diastasis* of X as the function defined in the connected neighborhood of Δ_X

$$\mathcal{U} = \{(p,q) \in X \times X : q \in X_p\}$$

by

$$D^X(p,q) = D^X_p(q).$$

The last example shows that there are analytic Kähler manifolds (X, ω) such that the diastatic potential of X centered at p_0 is not defined on the whole X_{p_0} for some $p_0 \in X$. However, in simply connected complex space forms (and, as we will see, in every Kähler immersed submanifold of them), the above issues does not occur:

Example 67. For every $b \in \mathbb{R}$ and $N \leq \infty$, and for every point $p \in F(b,N)$, the diastatic potential of $F(b,N)$ centered at p is defined in its maximal domain of analyticity, so we can define the diastasis function.

1. $(b = 0)$ In this case $F(b,N) = \left(\mathbb{C}^N, g_0\right)$, and the diastasis of $F(b,N)$ is defined as

$$\begin{array}{rcl} D^{F(0,N)} : \mathbb{C}^N \times \mathbb{C}^N & \to & \mathbb{R} \\ (p,q) & \mapsto & |p-q|^2 : \end{array}$$

   Observe that $D^{F(0,N)}(p,q)$ coincides with the squared Riemannian distance between p and q.

2. $(b < 0)$ In this case $F(b,N) = \left(\mathbb{CB}^N, \frac{\pi}{-b} g_{hyp}\right) \simeq \left(\mathbb{CH}^N, \frac{\pi}{-b} g_{hyp}\right)$. In order to describe its diastasis, it is easier to use the projective model. The diastasis is then globally defined as

$$\begin{array}{rcl} D^{F(b,N)} : \mathbb{CH}^N \times \mathbb{CH}^N & \to & \mathbb{R} \\ ([Z],[W]) & \mapsto & \frac{1}{b} \log\left(\frac{(Z,Z)(W,W)}{(Z,W)(W,Z)}\right). \end{array}$$

   It can be proved that the Riemannian distance of $F(b,N)$ satisfies the formula

$$d([Z],[W]) = \frac{1}{\sqrt{-b}} \cosh^{-1}\left(\sqrt{\frac{(Z,Z)(W,W)}{(Z,W)(W,Z)}}\right).$$

It follows that $D^{F(b,N)}([Z],[W])$ depends only on the Riemannian distance between $[Z]$ and $[W]$.

3. $(b>0)$ In this case $F(b,N)=\left(\mathbb{CP}^N,\frac{\pi}{b}g_{FS}\right)$. Contrarily to the two previous cases, the diastasis function is not globally defined. Define

$$\mathcal{U}=\left\{([Z],[W])\in\mathbb{CP}^N\times\mathbb{CP}^N:\langle Z,W\rangle\neq 0\right\},$$

and define

$$\begin{aligned} D^{F(b,N)}:\mathcal{U} &\rightarrow \mathbb{R} \\ ([Z],[W]) &\mapsto \frac{1}{b}\log\left(\frac{\langle Z,Z\rangle\langle W,W\rangle}{\langle Z,W\rangle\langle W,Z\rangle}\right). \end{aligned}$$

Then, for every $[Z]\in\mathbb{CP}^N$ the diastatic potential centered at $[Z]$ of $F(b,N)$ is defined on $\mathcal{U}_{[Z]}=\left\{[W]\in\mathbb{CP}^N:([Z],[W])\in\mathcal{U}\right\}$ by

$$D^{F(b,N)}_{[Z]}([W])=D^{F(b,N)}([Z],[W]).$$

Observe that, for every $[Z]\in\mathbb{CP}^N$, the set $P_{[Z]}=\mathcal{U}\backslash\mathcal{U}_{[Z]}$ is a codimension 1 subspace of $\mathbb{CP}^n$, called the *dual hyperplane* of $[Z]$. It can be proved that $P_{[Z]}$ is the cut locus of $[Z]$ in $F(b,N)$. The domain $\mathcal{U}_{[Z]}$ is the maximal domain of analyticity of $\mathbf{D}^{F(b,N)}_{[Z]}$: in fact,

(a) $D^{F(b,N)}_{[Z]}$ does not extend properly to any analytic function, because when $[W]$ approaches $P_{[Z]}$, $D^{F(b,N)}_{[Z]}([W])$ approaches $+\infty$;

(b) if the diastatic potential centered at $[Z]$ is defined on a domain U, then $U\cap\mathcal{U}_{[Z]}=U\backslash P_{[Z]}$ is connected since $P_{[Z]}$ cannot disconnect U because it has real codimension 2: it follows that the two definitions of diastatic potentials centered at $[Z]$ on U and on $\mathcal{U}_{[Z]}$ coincides in $U\cap\mathcal{U}_{[Z]}$, so by point (a) it must be $U\subseteq\mathcal{U}_{[Z]}$.

It follows that $D^{F(b,N)}$ is actually the diastasis function of $F(b,N)$. It can be proved that the Riemannian distance of $F(b,N)$ satisfies the formula

$$d([Z],[W])=\frac{1}{\sqrt{b}}\cos^{-1}\left(\sqrt{\frac{\langle Z,W\rangle(W,Z)}{\langle Z,Z\rangle\langle W,W\rangle}}\right),$$

so again $D^{F(b,N)}([Z],[W])$ depends only on the Riemannian distance between $[Z]$ and $[W]$.

It seems to be interesting (and difficult) to investigate on the global behaviour of diastatic potentials and diastasis functions. In special cases there are partial results: for example, it has been proved in [24] that if X is a Hermitian symmetric space of compact type, then for every $p \in X$ the diastatic potential centered at p_0 is defined on X_p, and the set $X \backslash X_p$ coincides with the cut locus of p. In general, one cannot expect too much: quoting Marcel Berger from his overview article on Calabi's work in honour of Calabi's seventieth birthday ([2], pp. 21-25),

> Calabi wrote to us he can prove that, generically, the locus where the diastasis blows up can be as close as one wishes to any set given a priori in the manifold.

Diastatic potentials are the key tool for studying Kähler immersions:

Notation. Denote by $\mathcal{G}^X$ the set of germs of real valued analytic functions on X, and for every $p \in X$ denote by $\mathcal{G}_p^X$ the set of germs at p. For every holomorphic map $f : X \to Y$ between complex manifolds, and for every $x \in X$, denote by

$$f_x^* : \mathcal{G}_{f(x)}^Y \to \mathcal{G}_x^X$$

the map such that, if $\mathbf{v}$ is the germ at $f(x)$ of an analytic function $v : V \to \mathbb{R}$ with $f(x) \in V \subseteq Y$ open, then $f_x^*(\mathbf{v})$ is the germ of $v \circ f : f^{-1}(V) \to \mathbb{R}$ at x.

Theorem 68 (Fundamental property of the diastasis)**.** *Let X, Y be analytic Kähler manifolds. Let $f : X \to Y$ be a holomorphic map. Then the following are equivalent:*

1. *f is a Kähler immersion;*
2. *for every $x \in X$, $f_x^*\left(\mathbf{D}_{f(x)}^Y\right) = \mathbf{D}_x^X$;*
3. *there is a $x \in X$ such that $f_x^*\left(\mathbf{D}_{f(x)}^Y\right) = \mathbf{D}_x^X$.*

Moreover, if the diastatic potential $D_{f(x),V}^Y$ is defined on a domain $V \ni f(x)$, then the diastatic potential $D_{x,U}^X$ is defined for every domain $U \ni x$ such that $f(U) \subseteq V$, and

$$D_{x,U}^X(p) = D_{f(x),V}^Y(f(p))$$

for every $p \in U$.

Proof. $(1 \Rightarrow 2)$ Choose $x \in X$, and choose a Kähler potential $v : V \to \mathbb{R}$ in a domain $V \ni f(x)$ such that there exists an off-diagonal holomorphic extension $\tilde{v} : V \times \overline{V} \to \mathbb{C}$. Then the diastatic potential $D^Y_{f(x),V}$ is defined as in equation 2.1.1. Choose a domain $U \subseteq f^{-1}(V)$ containing x. Then the map $u : U \to \mathbb{R}$ defined by $u = v \circ f$ admits the off-diagonal holomorphic extension $\tilde{u} = \tilde{v} \circ \left(f \times \overline{f}\right)$: the diastatic potential $D^X_{x,U}$ is then again defined as in equation 2.1.1, and by definition satisfies

$$D^X_{x,U}(p) = D^Y_{f(x),V}(f(p))$$

for every $p \in U$. It follows that $f^*_x\left(\mathbf{D}^Y_{f(x)}\right) = \mathbf{D}^X_x$.
$(2 \Rightarrow 3)$ Obvious.
$(3 \Rightarrow 1)$ By hypotesis, there are domains $U \ni x$ and $V \ni f(x)$ such that the diastatic potentials $D^X_{x,U}$ and $D^Y_{f(x),V}$ are defined, and $D^X_{x,U}(p) = D^Y_{f(x),V}(f(p))$ for every $p \in U$. Let ω and Ω be the Kähler forms of X and Y respectively: then

$$\frac{i}{2}\partial\bar{\partial}D^Y_{f(x),V} \equiv \Omega$$

and

$$\frac{i}{2}\partial\bar{\partial}D^X_{x,U} \equiv \omega,$$

so $f^*(\Omega) \equiv \omega$ on U. But ω is analytic, so by connectedness of X we have $f^*(\Omega) \equiv \omega$ on X, and hence f is a Kähler immersion.
Suppose now that $f : X \to Y$ is a Kähler immersion, and that the diastatic potential of Y centered at $f(x)$ is defined on a domain V. Let $U \ni x$ be a domain such that $f(U) \subseteq V$. Define

$$\begin{aligned} \mathcal{D} : U &\to \mathbb{R} \\ p &\mapsto D^Y_{f(x),V}(f(p)) : \end{aligned}$$

this is analytic since $D^Y_{f(x),V}$ is analytic; moreover, since $f^*_x\left(\mathbf{D}^Y_{f(x)}\right) = \mathbf{D}^X_x$, it follows that the germ of $\mathcal{D}$ in x is $\mathbf{D}^X_x$. Since U is connected, it follows that $\mathcal{D} = D^X_{x,U}$. □

Diastatic potentials in a neighborhood of p_0 can be seen as "normalizations" of Kähler potentials in a neighborhood of p_0:

Lemma 69. *Fix $p_0 \in X$. If u is a Kähler potential in a neighborhood of p_0, $\left(z^1, ..., z^n\right)$ are holomorphic coordinates centered at p_0, and the Taylor series* *of u at p_0 with respect to $\left(z^1, ..., z^n\right)$ is*

$$\sum_{k,h=0}^{\infty} b_{kh} \mathbf{z}^{\mathbf{m}_k} \overline{\mathbf{z}}^{\mathbf{m}_h},$$

then the Taylor series of $\mathbf{D}_{p_0}^X$ at p_0 with respect to $\left(z^1, ..., z^n\right)$ is the series

$$\sum_{k,h=1}^{\infty} b_{kh} \mathbf{z}^{\mathbf{m}_k} \overline{\mathbf{z}}^{\mathbf{m}_h}$$

obtained by canceling every pure monomial in $\mathbf{z}$ or in $\overline{\mathbf{z}}$ from the above series.

Proof. We can suppose that the coordinates $\left(z^1, ..., z^n\right)$ are defined in a domain $U \ni p_0$ such that the potential $u : U \to \mathbb{R}$ admits an off-diagonal extension $\tilde{u} : U \times \overline{U} \to \mathbb{C}$. Then, by formula 2.1.1 and by the definition of $\tilde{u}$, we have

$$\begin{aligned} D_{p_0,U}^X(q) &= \sum_{k,h=0}^{\infty} b_{kh} [\mathbf{z}^{\mathbf{m}_k}(p_0) \overline{\mathbf{z}^{\mathbf{m}_h}(p_0)} + \mathbf{z}^{\mathbf{m}_k}(q) \overline{\mathbf{z}^{\mathbf{m}_h}(q)} \\ &\quad - \mathbf{z}^{\mathbf{m}_k}(p_0) \overline{\mathbf{z}^{\mathbf{m}_h}(q)} - \mathbf{z}^{\mathbf{m}_k}(q) \overline{\mathbf{z}^{\mathbf{m}_h}(p_0)}] \end{aligned}$$

in a neighborhood of p_0, that we can suppose to be U by shrinking it. It can be verified by hand that every term with $k = 0$ or $h = 0$ vanishes; moreover, since $\left(z^1, ..., z^n\right)$ is centered at p_0, every monomial that contains $\mathbf{z}^{\mathbf{m}}(p_0)$ or $\overline{\mathbf{z}^{\mathbf{m}}(p_0)}$ with $\mathbf{m} \neq 0$ vanishes: it follows that every term with $k \neq 0$ and $h \neq 0$ equals $b_{kh} \mathbf{z}^{\mathbf{m}_k}(q) \overline{\mathbf{z}^{\mathbf{m}_h}(q)}$, so the thesis follows. □

In example 67, the diastatic potential $D_{p_0}^X$ depends only on the Riemannian distance from p_0. Although this is not always the case, there is actually a relation between diastatic potentials and Riemannian distance. In order to explain this relation, we need some special holomorphic coordinates, namely *Bochner coordinates* (appeared for the first time in Bochner's paper [3]): those are local holomorphic coordinates that have some properties in common with normal coordinates in Riemannian geometry.

Proposition 70. *Let (X, J, g) be a n-dimensional Kähler manifold. Then, for every $p_0 \in X$, there exists a holomorphic chart $\left(z^1, ..., z^n; U\right)$ centered at p_0 such that, if $g_{\alpha\overline{\beta}}$ $(1 \leq \alpha, \beta \leq n)$ are the coefficients of the complexified metric tensor, then the following holds:*

1. $g_{\alpha\overline{\beta}}(p_0) = \frac{1}{2}\delta_{\alpha\beta}$;
2. $\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^{\gamma}}(p_0) = \frac{\partial g_{\alpha\overline{\beta}}}{\partial \overline{z}^{\gamma}}(p_0) = 0$.

Such coordinates are called Bochner coordinates.

Proof. Choose local holomorphic coordinates $(u, U) \equiv (u^1, ..., u^n; U)$ centered at p_0. Let $V_1, ..., V_n \in TX_{p_0}$ be n tangent vectors in p_0 such that $V_1, ..., V_n, JV_1, ..., JV_n$ is a orthonormal basis of TX_{p_0}, and let $b_j = du_{p_0}(V_j) \in \mathbb{C}^n$. Now, let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation such that $A^{-1}(e_j) = b_j$, where $e_1, ..., e_n$ is the canonical basis of $\mathbb{C}^n$. Then, if we define $w = A \circ u$, we obtain holomorphic coordinates $(w^1, ..., w^n; V)$ such that, if $G_{\alpha\overline{\beta}}$ are the coefficients of the complexified metric tensor in those coordinates, then

$$G_{\alpha\overline{\beta}}(p_0) = \frac{1}{2}\delta_{\alpha\beta}.$$

Now, consider the holomorphic map $T : \mathbb{C}^n \to \mathbb{C}^n$ that associates to the vector $(w^1, ..., w^n)$ the vector $(z^1, ..., z^n)$ defined by

$$z^{\alpha} = w^{\alpha} + \frac{1}{2}b_{uv}{}^{\alpha} w^u w^v$$

for some complex coefficients $b_{uv}{}^{\alpha}$ symmetric in u, v to be fixed later. This holomorphic map sends 0 in 0, and for the Inverse Function Theorem it is a biholomorphism in a neighborhood of 0: by shrinking U, we can suppose that T is a biholomorphism in $w(U)$. Now, fix the holomorphic coordinates $(z^1, ..., z^n; U)$ obtained by defining $z = T \circ w$, and denote by $g_{\alpha\overline{\beta}}$ the coefficients of the complexified metric tensor in those coordinates. Then we have

$$\frac{\partial^2 w^u}{\partial z^{\alpha} \partial z^{\gamma}} = b_{\alpha\gamma}{}^{u}$$

and

$$\begin{aligned} \frac{\partial w^{\alpha}}{\partial z^{\beta}}(p_0) = \frac{\partial \overline{w}^{\alpha}}{\partial \overline{z}^{\beta}}(p_0) &= \delta^{\alpha}_{\beta} \\ \frac{\partial G_{u\overline{v}}}{\partial z^{\gamma}}(p_0) &= \frac{\partial G_{u\overline{v}}}{\partial w^{\gamma}}(p_0); \end{aligned}$$

moreover, since $G_{u\overline{v}}(p_0) = \frac{1}{2}\delta_{uv}$, we have

$$\begin{aligned} \frac{\partial g_{\alpha\overline{\beta}}}{\partial w^{\gamma}} &= b_{\alpha\gamma}{}^{u}\frac{1}{2}\delta_{uv}\delta^{v}_{\beta} + \delta^{u}_{\alpha}\frac{\partial G_{u\overline{v}}}{\partial w^{\gamma}}(p_0)\,\delta^{v}_{\beta} \\ &= b_{\alpha\gamma}{}^{\beta} + \frac{\partial G_{\alpha\overline{\beta}}}{\partial w^{\gamma}}(p_0). \end{aligned}$$

Since $\frac{\partial G_{\alpha\overline{\beta}}}{\partial w^{\gamma}} = \frac{\partial G_{\gamma\overline{\beta}}}{\partial w^{\alpha}}$, we can choose $b_{\alpha\gamma}{}^{\beta} = -\frac{\partial G_{\alpha\overline{\beta}}}{\partial w^{\gamma}}(p_0)$, and we obtain the thesis. $\square$

We can now explain the relation between diastatic potentials and Riemannian distance.

Theorem 71. *Fix $p_0 \in X$. Then, for $U \ni p_0$ sufficiently small, we have*

$$D^X_{p_0,U}(q) = d^2_{p_0}(q) + O\left(d^4_{p_0}(q)\right),$$

where $d_{p_0} : X \to \mathbb{R}$ is the Riemannian distance from p_0.

Proof. By lemma 69 and by the above proposition, we can choose U such as the domain of definition of Bochner coordinates centered at p_0 in which the Taylor expansion

$$D^X_{p_0,U}(q) = \sum_{k,h=1}^{\infty} b_{kh}\mathbf{z}^{\mathbf{m}_k}(q)\overline{\mathbf{z}^{\mathbf{m}_h}(q)}$$

converges. Since $D^X_{p_0,U}$ is a Kähler potential, for $1 \le k, h \le n$ we have

$$b_{kh} = \frac{\partial^2 D^X_{p_0,U}}{\partial z^k \partial \overline{z}^h}(p_0) = 2g_{k\overline{h}}(p_0) = \delta_{kh};$$

moreover, all the degree 3 monomials in the Taylor series vanishes because their coefficients are multiples of coefficients of the form

$$\frac{\partial^3 D^X_{p_0,U}}{\partial z^{\alpha}\partial\overline{z}^{\beta}\partial z^C}(p_0) = 2\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^C}(p_0) = 0$$

for some $C \in \left\{1, ..., n, \overline{1}, ..., \overline{n}\right\}$. It follows that

$$D^X_{p_0,U}(q) = |\mathbf{z}(q)|^2 + O\left(|\mathbf{z}(q)|^4\right).$$

By expanding the squared Riemannian distance $d^2_{p_0}$ in Taylor series in p_0 with respect to $\left(z^1, ..., z^n\right)$, we obtain again

$$d^2_{p_0}(q) = |\mathbf{z}(q)|^2 + O\left(|\mathbf{z}(q)|^4\right):$$

the thesis follows. $\square$

2.2 Kähler immergibility into $\left(\mathbb{C}^N, \omega_0\right)$

In this section, we approach the Kähler immergibility problem of a finite dimensional Kähler manifold into a flat space $\left(\mathbb{C}^N, \omega_0\right)$ for $N \leq \infty$. Unless stated otherwise, the complex manifold $\mathbb{C}^N$ $(N \leq \infty)$ is always equipped with its Kähler form ω_0.

Let X be a finite dimensional analytic Kähler manifold. As we already said, X is not always Kähler immergible in $\mathbb{C}^N$ for some $N \leq \infty$. A necessary condition arises immediately from the fundamental property of the diastasis:

Lemma 72. *If X is Kähler immergible in $\mathbb{C}^N$ for some $N \leq \infty$, then for every $p_0 \in X$ the diastatic potential centered at p_0 is globally defined. In particular, the diastasis of X is defined on $X \times X$, and if $f : X \to \mathbb{C}^N$ is a Kähler immersion, then*

$$D^X(p,q) = |f(p) - f(q)|^2 .$$

Proof. Let $f : X \to \mathbb{C}^N$ be a Kähler immersion. Then, since $D^{\mathbb{C}^N}_{f(p_0)}$ is globally defined and X is connected, by the fundamental property of the diastasis, the map

$$\begin{aligned} D^X_{p_0}(q) &= D^{\mathbb{C}^N}_{f(p_0)}(f(q)) \\ &= |f(p_0) - f(q)|^2 \end{aligned}$$

is the globally defined diastatic potential of X centered at p_0. It then follows that $D^X(p,q) = D^{\mathbb{C}^N}(f(p), f(q))$ is the diastasis of X. □

Definition 73. Let X be a complex manifold. A holomorphic map $f : X \to \mathbb{C}^N$ is called *full* if the affine subspace of $\mathbb{C}^N$ generated by $f(X)$ coincides with $\mathbb{C}^N$.

Observe that being full is a local property, meaning that if $f : X \to \mathbb{C}^N$ is full and U is a domain of X, then $f_{|U} : U \to \mathbb{C}^N$ is again full: in fact, if $f_{|U}$ were not full, then there would be $v \in \mathbb{C}^N$ and $\lambda \in \mathbb{C}$ nontrivial, such that

$$\langle f(p), v \rangle = \lambda$$

for every $p \in U$; but the map

$$\begin{aligned} X &\to \mathbb{C} \\ p &\mapsto \langle f(p), v \rangle - \lambda \end{aligned}$$

is holomorphic, and hence if it vanishes on a nonempty open subset of X, by connectedness of X it must vanish in X, implying that $f(X)$ would be contained in the affine hyperplane of $\mathbb{C}^N$ of equation $\langle z, v\rangle = \lambda$. This ensures that the following definition is well posed:

Definition 74. We say that a germ of holomorphic map with values in $\mathbb{C}^N$ is *full* if it defines a full map in a neighborhood of its base point.

Moreover, observe that if the Kähler manifold (X, ω) admits a Kähler immersion $f : (X, \omega) \to (\mathbb{C}^N, \omega_0)$, then it admits a *full* Kähler immersion into $\left(\mathbb{C}^{N'}, \omega_0\right)$: in fact, the subspace of $\mathbb{C}^N$ generated by $f(X)$ is a complex submanifold of $\mathbb{C}^N$ biholomorphic to $\mathbb{C}^{N'}$ for some $N' \leq N$, and it is easy to verify that the Kähler metric of $\mathbb{C}^N$ induces on that subspace the flat Kähler metric of $\mathbb{C}^{N'}$, so the restriction of f to that subspace is a full Kähler immersion of (X, ω) into $\mathbb{C}^{N'}$ with the flat metric.

Let $\mathcal{O}(X, \mathbb{C}^N)$ be the sheaf of germs of holomorphic maps from X with values in $\mathbb{C}^N$. The following lemma gives a characterization of the fullness of a germ $\mathbf{f} \in \mathcal{O}(X, \mathbb{C}^N)$ in terms of its Taylor matrix. By the Taylor matrix of a germ of holomorphic function $\mathbf{f} \in \mathcal{O}_{p_0}(X, \mathbb{C}^N)$ at p_0 with respect to a holomorphic chart $(z^1, ..., z^n)$ centered at p_0, we mean the $\infty \times N$ complex matrix such that the i-th column contains the coefficients of the Taylor expansion of the i-th component of $\mathbf{f}$ at p_0 with respect to those local coordinates.

Lemma 75. *Let $\mathbf{f} \in \mathcal{O}_{p_0}(X, \mathbb{C}^N)$ be a germ at p_0 such that $\mathbf{f}(p_0) = 0$. Then the following are equivalent:*

1. *$\mathbf{f}$ is full;*
2. *for every holomorphic chart centered at p_0, if A denotes the Taylor matrix A of $\mathbf{f}$ with respect to that chart, then $Au = 0$ implies $u = 0$ for every vector $u \in \mathbb{C}^N$;*
3. *for some holomorphic chart centered at p_0, if A denotes the Taylor matrix A of $\mathbf{f}$ with respect to that chart, then $Au = 0$ implies $u = 0$ for every vector $u \in \mathbb{C}^N$.*

Proof. Let A be the matrix of $\mathbf{f}$ with respect to some chart centered at p_0. Then $Au = 0$ for some $u \in \mathbb{C}^N$ if and only if, if U is a domain in which the Taylor expansion of $\mathbf{f}$ converges in U, and $f : U \to \mathbb{C}^N$ denotes its sum, then

$$\langle f(p), \overline{u}\rangle = 0$$

for every $p \in U$. The thesis then follows. $\square$

Definition 76. Let B be a $\infty \times \infty$ complex matrix $(b_{kh})_{0\leq k,h<\infty}$. Denote by B_n the $n+1 \times n+1$ principal submatrix of B. Then B is called *positive semidefinite* if every B_n is positive semidefinite, and the rank of B is defined by

$$\mathrm{rk}\,(B) = \sup_n \left(\mathrm{rk}\,(B_n)\right).$$

We can now define the key property used to obtain our immergibility criterion:

Definition 77. We say that X is 0*-resolvable of rank* $N \leq \infty$ in p_0 if the Taylor matrix of $\mathbf{D}^X_{p_0}$ with respect to some holomorphic chart centered at p_0 is positive semidefinite of rank N.

Remark 78. Using holomorphic changes of coordinates, it is easy to prove that the above definition is well posed, i.e. it does not depend on the holomorphic chart.

The following theorem shows why 0-resolvability of rank N is so important when studying Kähler immersions into $\mathbb{C}^N$:

Theorem 79 (local 0-resolvability criterion)**.** *Let p_0 be a point of X. Then X is* 0*-resolvable of rank $N \leq \infty$ in p_0 if and only if there exists a full germ of Kähler immersion into $\mathbb{C}^N$ at the point p_0.*

Before proving the theorem, we need a technical lemma about infinite matrix:

Lemma 80. *Let B be a $\infty \times \infty$ complex matrix. Then B is positive semidefinite of rank $N \leq \infty$ if and only if there is a $\infty \times N$ complex matrix A such that:*

1. *every row $a_{k:}$ of A is an element of $\mathbb{C}^N$ (so the matrix product $AA^\dagger$ is defined);*
2. *$AA^\dagger = B$;*
3. *if $Au = 0$ for some $\mathbb{C}^N$, then $u = 0$.*

Proof.

1. $(\Rightarrow)$ Let $(k_j)_{j=1}^N$ the strictly increasing sequence of nonnegative integers such that $\left|B_{k_j}\right| \neq 0$. Observe that, since B is Hermitian and positive

semidefinite, $\left|B_{k_j}\right| > 0$ for $j = 1, ..., N$. Now, inductively on $j = 1, ..., N$, define

$$a_{k_j j} = \sqrt{\left|B_{k_j}\right|}$$

and

$$a_{k_j+s,j} = \begin{cases} 0 & s < 0 \\ \frac{b_{k_j+s,k_j} - \sum_{t=1}^{j-1} a_{k_j+s,t}\overline{a}_{k_j t}}{a_{k_j j}} & s > 0 \end{cases}.$$

The definition is well posed since every element of A depends only on previously defined elements. Moreover, the $\infty \times N$ matrix $A = (a_{kj})_{0\leq k<\infty, 0\leq j\leq N}$ is in column echelon form, so it satisfies properties 1 and 2. By construction, A verifies $AA^\dagger = B$. Since A is in column echelon form, $Au = 0$ implies $u = 0$.

2. ($\Leftarrow$) Since $B = AA^\dagger$, then $B^\dagger = \left(AA^\dagger\right)^\dagger = AA^\dagger = B$, so B is Hermitian. Moreover, if x is a $n+1$ row vector, then

$$\begin{aligned} xB_n x^\dagger &= \sum_{k,h=0}^{n} x_k \left(B_n\right)_{kh} \overline{x}_h \\ &= \sum_{k,h=0}^{n} x_k \left\langle a_{k:}, a_{h:}\right\rangle \overline{x}_h \\ &= \left\langle \sum_{k=0}^{n} x_k a_{k:}, \sum_{k=0}^{n} x_h a_{h:} \right\rangle \\ &\geq 0 \end{aligned}$$

so B is positive semidefinite.
Now, suppose for the sake of contradiction that B has rank $k < N$. Denote by A_n the $n \times N$ matrix of the first $n+1$ rows of A. Then $B_n = A_n A_n^\dagger$, and $B_n x = 0$ if and only if $A_n^\dagger x = 0$: this implies that the rank of B_n equals the dimension of the image of $A_n^\dagger : \mathbb{C}^n \to \mathbb{C}^N$. But $\operatorname{im}\left(A_n^\dagger\right) \subseteq \operatorname{im}\left(A_{n+1}^\dagger\right)$, so if $\operatorname{rk} B = k < N$ then there is a subspace of dimension k of $\mathbb{C}^N$ such that $\operatorname{im}\left(A_n^\dagger\right)$ is contained in that subspace for every n. Since $k < N$, there is a nonvanishing bounded linear map $f : \mathbb{C}^N \to \mathbb{C}$ such that $f \circ A_n^\dagger = 0$ for every n: but f is of the form $\langle \cdot, \overline{u}\rangle$ for some $u \in \mathbb{C}^N$, so $f \circ A_n^\dagger = 0$ for every n means that $Au = 0$; by hypotesis on A, we obtain $u = 0$, and this is in contradiction with the fact that f is nonvanishing.

□

We can now prove theorem 68:

Proof. ($\Leftarrow$) Let $\mathbf{f} \in \mathcal{O}_{p_0}\left(X, \mathbb{C}^N\right)$ be a germ of full Kähler immersion, and let $\left(z^1 ..., z^n\right)$ be a holomorphic chart centered in p_0. By composing with a translation we obtain again a germ of full Kähler immersion, so we can assume that $\mathbf{f}(p_0) = 0$. Denote by A the Taylor matrix of $\mathbf{f}$ with respect to $\left(z^1, ..., z^n\right)$, and choose a domain U in which the Taylor series of $\mathbf{f}$ and $\mathbf{D}^X_{p_0}$ with respect to $\left(z^1, ..., z^n\right)$ converges both. By lemma 72, the diastatic potential of X centered at p_0 is globally defined, and if $f : U \to \mathbb{C}^N$ denotes the sum of the Taylor series of $\mathbf{f}$, since f is a Kähler immersion we have

$$\begin{aligned} D^X_{p_0}(p) &= \left|f(p) - f(p_0)\right|^2 \\ &= \left|f(p)\right|^2 \\ &= \left\langle \sum_{k=0}^{\infty} a_{k:}\mathbf{z}^{\mathbf{m}_k}(p), \sum_{k=0}^{\infty} a_{k:}\mathbf{z}^{\mathbf{m}_k}(p) \right\rangle \\ &= \sum_{k,h=0}^{\infty} \left\langle a_{k:}, a_{h:}\right\rangle \mathbf{z}^{\mathbf{m}_k}(p)\,\overline{\mathbf{z}^{\mathbf{m}_h}(p)} \end{aligned}$$

for every $p \in U$, so

$$b_{kh} = \left\langle a_{k:}, a_{h:}\right\rangle,$$

or in matricial notation

$$AA^\dagger = B.$$

Since A is the Taylor matrix of the full germ $\mathbf{f}$ such that $\mathbf{f}(p_0) = 0$, by lemma 75 we have that $Au = 0$ implies $u = 0$ for every $u \in \mathbb{C}^N$. It then follows from lemma 80 that B is positive semidefinite of rank N.
($\Rightarrow$) Choose a holomorphic chart $\left(z^1, ..., z^n; U\right)$ centered at p_0. By hypotesis, the Taylor matrix $\infty \times \infty$ of $\mathbf{D}^X_{p_0}$, which we call B, is positive semidefinite of rank N. Let A be the $\infty \times N$ matrix defined in the proof of the ($\Rightarrow$) implication of lemma 80, such that $B = AA^\dagger$. The power series

$$f(p) = \sum_{k=0}^{\infty} a_{k:}\mathbf{z}^{\mathbf{m}_k}(p)$$

has positive radius of convergence: in fact,

$$\begin{aligned} \limsup_{n\to\infty} \|a_{n:}\|^{\frac{1}{n}} &= \limsup_{n\to\infty} \left(b_{nn}\right)^{\frac{1}{n}} \\ &< +\infty \end{aligned}$$

because the Taylor matrix of $\mathbf{D}_{p_0}^X$ converges in a neighborhood of p_0. By shrinking U, we can suppose that it converges in U: then, its sum is a holomorphic map $f : U \to \mathbb{C}^N$. Moreover, f is also a Kähler immersion, since

$$\begin{aligned}
D_{p_0,U}^X(p) &= \sum_{k,h=0}^{\infty} b_{kh} \mathbf{z}^{\mathbf{m}_k}(p) \overline{\mathbf{z}^{\mathbf{m}_h}(p)} \\
&= \left\langle \sum_{k=0}^{\infty} a_{k:} \mathbf{z}^{\mathbf{m}_k}(p), \sum_{h=0}^{\infty} a_{h:} \mathbf{z}^{\mathbf{m}_h}(p) \right\rangle \\
&= |f(p)|^2 \\
&= |f(p) - f(p_0)|^2 \\
&= D_{f(p_0)}^{\mathbb{C}^N}(f(p))
\end{aligned}$$

and so

$$f_{p_0}^* \left(\mathbf{D}_{f(p_0)}^{\mathbb{C}^N} \right) = \mathbf{D}_{p_0}^X.$$

The germ $\mathbf{f}$ of f in p_0 is then a germ of Kähler immersion. This is a full germ: this follows from lemma 75 by observing that the matrix A is in column echelon form and so $Au = 0$ implies $u = 0$ for every $\mathbb{C}^N$, and since B has vanishing first row and column, the first row of A is 0, meaning that $\mathbf{f}(p_0) = 0$. □

Although 0-resolvability of rank N in a point is *a priori* a local property, the following theorem shows that 0-resolvability of rank N is actually a *global* property of X:

Theorem 81 (global 0-resolvability criterion)**.** *X is 0-resolvable of rank $N \leq \infty$ in a point if and only if it is 0-resolvable of rank N at every point.*

Proof. The proof is fairly technical, and we refer to [8], theorem 4, for it. □

If we put together the local and global 0-resolvability criterions, we obtain the following

Corollary 82. *If X is 0-resolvable of rank N, then for every $p_0 \in X$ there exists a full germ of Kähler immersion into $\mathbb{C}^N$ at the point p_0.*

Can we patch together those germs into a globally defined full Kähler immersions? The key tool for passing from local to global is the *rigidity theorem*:

Theorem 83 (rigidity theorem)**.** *Let $f, g : X \to \mathbb{C}^N$ be two Kähler immersions. Then there exists a $\Phi \in E(N)$ such that $f = \Phi \circ g$.*

Proof. Up to translations, we can suppose that $f(p_0) = g(p_0) = 0$ for some $p_0 \in X$. It follows that the real (resp. complex) closed affine subspaces generated by $f(X)$ and $g(X)$ are real (resp. complex) linear subspaces of $\mathbb{C}^N$. Now, since f and g are full, the closed complex subspaces of $\mathbb{C}^N$ generated by $f(X)$ and $g(X)$ are both $\mathbb{C}^N$. It is true also that the closed *real* subspaces of $\mathbb{C}^N \equiv \mathbb{R}^{2N}$ generated by $f(X)$ and $g(X)$ are both $\mathbb{C}^N$: in fact, if that were not true for f, there would be a $u \in \mathbb{C}^N$ nonzero, and a $v \in \mathbb{C}^N$, such that

$$\langle f, u \rangle - \left\langle \overline{f}, v \right\rangle = 0.$$

The map $p \mapsto \langle f(p), u \rangle$ is holomorphic, and the map $p \mapsto \left\langle \overline{f(p)}, v \right\rangle$ is antiholomorphic, so by $\langle f, u \rangle = \left\langle \overline{f}, v \right\rangle$ it would follow that $p \mapsto \langle f(p), u \rangle$ would be both holomorphic and antiholomorphic, implying that $\langle f, u \rangle = 0$ because $f(p_0) = 0$. this is a contradiction, since f is full.
Now, consider the map

$$\begin{array}{rcl} \phi : f(U) & \to & g(U) \\ f(p) & \mapsto & g(p) : \end{array}$$

then ϕ preserves the distances, so for a well known property of real Hilbert spaces (formulated and proved e.g. in [26], theorem 11.4) the map ϕ extends to a unique orthogonal transformation (with respect to the canonical positive definite inner product of $\mathbb{R}^{2N}$) Φ from the real closed subspace generated by $f(U)$ to the real closed subspace generated by $g(U)$; from the above considerations, those are both $\mathbb{C}^N$, so $\Phi : \mathbb{C}^N \equiv \mathbb{R}^{2N} \to \mathbb{C}^N \equiv \mathbb{R}^{2N}$ is a orthogonal transformation. Now we prove that Φ is a unitary transformation, i.e. that it is $\mathbb{C}$-linear. It is easy to prove that Φ, being a bounded $\mathbb{R}$-linear map, admits a unique decomposition as $\Phi = \Phi_1 + \Phi_2$, where Φ_1 is a bounded, $\mathbb{C}$-linear map, and Φ_2 is a bounded, $\mathbb{C}$-antilinear map. But this implies that Φ_2 is antiholomorphic; now, it follows by $\Phi \circ f = g$ that

$$\Phi_1 \circ f - g = -\Phi_2 \circ f :$$

but this implies $\Phi_2 \circ f = 0$, since this is both holomorphic and antiholomorphic, and vanishes at p_0. Since f is full, it must be $\Phi_2 \equiv 0$, so $\Phi = \Phi_1$ is $\mathbb{C}$-linear. □

We now apply the 0-resolvability criterions and the rigidity theorem to answer our initial question.

Theorem 84. *Let X be a finite dimensional, simply connected analytic Kähler manifold. Then X is 0-resolvable of rank N if and only if there exists a full Kähler immersion of X into $\mathbb{C}^N$.*

Proof. The implication $\Leftarrow$ is an immediate consequence of the 0-resolvability criterion. Let's prove the $\Rightarrow$ implication.
Fix $p_0 \in X$. Since X is resolvable of rank N, there is a full germ of Kähler immersion $\mathbf{f}_0$ at p_0. We claim that $\mathbf{f}_0$ admits an analytic continuation along every simple curve γ starting from p_0: then, by generality of γ and since X is simply connected, it follows from the monodromy theorem that $\mathbf{f}_0$ is the germ of a globally defined full Kähler immersion of X into $\mathbb{C}^N$.
Choose a simple curve $\gamma : [0,1] \to X$ starting from p_0. Let $\{(f_i, U_i)\}_{i\in I}$ be the family of locally defined full Kähler immersions $f_i : U_i \to \mathbb{C}^N$ defined on a domain: since X is 0-resolvable of rank N, $\{U_i\}_{i\in I}$ covers X. By Lebesgue's number lemma, we can select a sequence $0 = t_0 < \cdots < t_{k+1} = 1$ and a sequence $(f_0, U_0), ..., (f_k, U_k)$ of full Kähler immersions such that:

1. $\gamma([0,1]) \subseteq U_0 \cup \cdots \cup U_k$;
2. U_j contains $\gamma(t_j)$ and $\gamma(t_{j+1})$;
3. f_0 has germ $\mathbf{f}_0$ in $\gamma(t_0) = p_0$;
4. $U_j \cap U_{j+1}$ is connected;
5. $U_j \cap U_{j+h} = \emptyset$ if $h \neq 0, 1$.

Now, by the rigidity theorem, there exists a $\Phi_1 \in \mathrm{E}(N)$ such that $\Phi_1 \circ f_1 \equiv f_0$ in $U_0 \cap U_1$; analougously, there exists a $\Phi_2 \in \mathrm{E}(N)$ such that $\Phi_2 \circ f_2 = \Phi_1 \circ f_1$ in $U_1 \cap U_2$; proceeding in this way, we obtain an analytic continuation of $\mathbf{f}_0$ along γ, as claimed.

$\square$

Theorem 85. *Let X be a finite dimensional Kähler manifold. Then X admits a full Kähler immersion in $\mathbb{C}^N$ if and only if X verifies the following properties:*

1. *X is analytic;*

2. *X is 0-resolvable of rank N;*

3. *for every $p_0 \in X$, the diastatic potential of X centered at p_0 is defined on its maximal domain of analiticity X_{p_0}.*

Moreover, the Kähler immersion is injective if and only if, for every $p, q \in X$, $D^X(p,q) = 0 \iff p = q$.

Proof. ($\Rightarrow$) We already observed that since $\mathbb{C}^N$ is analytic, then X must be analytic. By the 0-resolvability criterions, X is 0-resolvable of rank N. Finally, by lemma 72 property 3 trivially holds.
($\Leftarrow$) Let $\pi : \tilde{X} \to X$ be the universal cover of X. Since π is a local isometry and X is resolvable of rank N, also $\tilde{X}$ is analytic and resolvable of rank N, so by previous theorem there exists a full Kähler immersion $\tilde{f} : \tilde{X} \to \mathbb{C}^N$. Again by lemma 72, the diastatic potential of $\tilde{X}$ centered at any point is globally defined by

$$D_{\tilde{p}_0}^{\tilde{X}}(\tilde{q}) = \left| \tilde{f}(\tilde{p}_0) - \tilde{f}(\tilde{q}) \right|^2.$$

We want to prove that $\tilde{f}$ is the lift of a full Kähler immersion $f : X \to \mathbb{C}^N$. Since π is a surjective local isometry, it is sufficient to prove that $\tilde{f}$ descends to the quotient, i.e. if $\pi(\tilde{p}_0) = \pi(\tilde{p}_1)$ then $\tilde{f}(\tilde{p}_0) = \tilde{f}(\tilde{p}_1)$: by the expression of the diastasis of $\tilde{X}$, it suffices to show that if $\pi(\tilde{p}_0) = \pi(\tilde{p}_1)$ then $D_{\tilde{p}_0}^{\tilde{X}}(\tilde{p}_1) = 0$.
Let $\tilde{p}_0, \tilde{p}_1 \in \tilde{X}$ such that $\pi(\tilde{p}_0) = \pi(\tilde{p}_1) = p$. Consider a curve $\tilde{\gamma} : [0,1] \to \tilde{X}$ joining $\tilde{p}_0$ and $\tilde{p}_1$: the curve $\gamma = \pi \circ \tilde{\gamma}$ is then a loop with base point p, and since π is a local biholomorphism and $\pi_{\tilde{p}_0}^*\left(\mathbf{D}_p^X\right) = \mathbf{D}_{\tilde{p}_0}^{\tilde{X}}$, the diastatic potential $D_{\tilde{p}_0}^{\tilde{X}}$ induces an analytic continuation $\{\mathbf{u}_t\}_{t\in[0,1]}$ of $\mathbf{D}_p^X$ along γ such that, for every $t \in [0,1]$, $\pi_{\tilde{\gamma}(t)}^*(\mathbf{u}_t)$ is the germ of the diastatic potential $D_{\tilde{p}_0}^{\tilde{X}}$ at $\tilde{\gamma}(t)$. But, by hypothesis 3 on X, it must be $\mathbf{u}_1(p) = 0$. This implies that $D_{\tilde{p}_0}^{\tilde{X}}(\tilde{p}_1) = 0$, proving our assertion.
Suppose now that, for every $p, q \in X$, $D^X(p,q) = 0 \Rightarrow p = q$. Suppose that $f(p) = f(q)$ for some $p, q \in X$. Then we have $D^{\mathbb{C}^n}(f(p), f(q)) = 0$, so since f is a Kähler immersion we obtain $D^X(p,q) = 0$, and hence $p = q$ by hypothesis. By generality of $p, q \in X$, we obtain the injectivity of f. $\square$

2.3 Kähler immergibility in $\left(\mathbb{CB}^N, \lambda\omega_{hyp}\right)$ and $\left(\mathbb{CP}^N, \lambda\omega_{FS}\right)$

In the same fashion of the previous section, we now approach the Kähler immergibility problem for a finite dimensional Kähler manifold into a finite or infinite dimensional, elliptic or hyperbolic, simply connected complex space form.

Exactly as in the flat case, diastatic potentials of a Kähler immersed submanifold of an hyperbolic simply connected complex space form are well behaved.

Lemma 86. *If there exists a Kähler immersion of X into $F(b,N)$ for some $b<0$ and $N \leq \infty$, then for every $p_0 \in X$ the diastatic potential of X centered at p_0 is globally defined. Moreover, X admits a globally defined diastasis D^X, and if $f : X \to F(b,N)$ is a Kähler immersion, then*

$$D^X(p,q) = D^{F(b,N)}(f(p), f(q))$$

for every $p, q \in X$.

Proof. The proof of lemma 72 applies also in this case with no changes. $\square$

A little different property holds if the ambient space is an elliptic simply connected complex space form. We first need the concept of full holomorphic map with values in $\mathbb{CP}^N$ or $\mathbb{CH}^N$:

Definition 87. Let X be a complex manifold. A holomorphic map $f : X \to \mathbb{CP}^N$ (resp. $f : X \to \mathbb{CH}^N$) is called *full* if the subspace of $\mathbb{CP}^N$ (resp. $\mathbb{CH}^N$) generated by $f(X)$ coincides with $\mathbb{CP}^N$ (resp. $\mathbb{CH}^N$).

Analougously to the flat case, we can prove that being full is a local property, so we can speak of germs of full holomorphic maps. Moreover, if there exists a Kähler immersion $f : (X,\omega) \to F(b,N)$, then there exists a *full* Kähler immersion of (X,ω) in some $F(b,N')$ with $N' \leq N$, because the subspaces of $F(b,N)$ with the induced metric are again holomorphic isometric copies of $F(b,N')$ for some $N' \leq N$.

Lemma 88. *Suppose that X is fully Kähler immergible in $F(b,N)$ for some $b>0$ and $N \leq \infty$. Then, for every $p \in X$, the diastatic potential of X centered at p is defined in the maximal domain of analiticity X_p of $\mathbf{D}_p^X$. Moreover, the set $X \backslash X_p$ is either empty or a codimension 1 complex analytic subvariety of X, called the* polar variety *of p. More precisely, if $f : X \to F(b,N)$ is a full Kähler immersion, and $\mathcal{U}$ denotes the domain of the diastasis of $F(b,N)$, then the domain of the diastasis of X is $\mathcal{V} = (f \times f)^{-1}(\mathcal{U})$, and*

$$D^X(p,q) = D^{F(b,N)}(f(p), f(q))$$

for every $(p,q) \in \mathcal{V}$.

Proof. Let $\mathcal{U}$ be the domain of the diastasis of $F(b,N)$. For every $p_0 \in X$, the set $\mathcal{V}_{p_0} = f^{-1}\left(\mathcal{U}_{f(p_0)}\right)$ is open and contains p_0: we want to prove that $X_{p_0} = \mathcal{V}_{p_0}$. In order to prove that $\mathcal{V}_{p_0}$ is connected, it suffices to prove that $X \backslash \mathcal{V}_{p_0}$ is either empty or a codimension 1 complex subvariety of X, and then it cannot disconnect X. Since $\mathbb{CP}^N \backslash \mathcal{U}_{f(p_0)} = P_{f(p_0)}$ is a codimension 1 complex submanifold of $\mathbb{CP}^N$, it is locally the zero set of some holomorphic function with complex values, so the same holds for $X \backslash \mathcal{V}_{p_0} = f^{-1}\left(P_{f(p_0)}\right)$. Moreover, if $\varphi : U \to \mathbb{C}$ is such a function, then it cannot be $\varphi \circ f \equiv 0$: in fact, if that were true, then $f(U)$ would be contained in a dual hyperplane $P_{[Z]}$ for some open subset U of X, and this is in contrast with fullness of f since being full is a local property. It then follows that, if $X \backslash \mathcal{V}_{p_0} \neq \emptyset$, then $X \backslash \mathcal{V}_{p_0}$ is a codimension 1 complex analytic subvariety of X. It then follows from the fundamental property of the diastasis that the diastatic potential of X centered at p_0 is defined on $\mathcal{V}_{p_0}$ by

$$D_{p_0}^X(q) = D_{f(p_0)}^{F(b,N)}(f(q)).$$

We can prove that $\mathcal{V}_{p_0}$ coincides with the maximal domain of analyticity of $\mathbf{D}_{p_0}^X$ exactly as in the proof, given in example 67, that the diastatic potential of $F(b,N)$ centered at $[Z]$ is defined in its maximal domain of definition $\mathbb{CP}^N \backslash P_{[Z]}$. It follows then that the diastasis of X is defined on $\mathcal{V} = \{(p,q) \in X \times X : q \in \mathcal{V}_p\}$ by

$$D^X(p,q) = D^{F(b,N)}(f(p), f(q)),$$

and hence, when $q \in \mathcal{V}_{p_0}$ approaches $X \backslash \mathcal{V}_{p_0}$, then $D_{p_0}^X(q)$ approaches $+\infty$. $\square$

Let X be an analytic Kähler manifold, and let $b \in \mathbb{R} \backslash \{0\}$. Then, for every $p_0 \in X$, we can associate the germ of real analytic function

$$\frac{e^{b\mathbf{D}_{p_0}^X} - 1}{b}:$$

we call it the *b-stereographic projection of* $\mathbf{D}_{p_0}^X$. The germ $\frac{e^{b\mathbf{D}_{p_0}^X}-1}{b}$ is the germ of the diastatic potential centered at p_0 of a new Kähler metric defined in a neighborhood of p_0: in fact, if $\left(z^1, ..., z^n\right)$ are holomorphic coordinates centered at p_0, then

$$\begin{aligned}
\frac{\partial^2 \frac{e^{b\mathbf{D}^X_{p_0}}-1}{b}}{\partial z^\alpha \partial \bar{z}^\beta} &= e^{b\mathbf{D}^X_{p_0}} \left(\frac{\partial^2 \mathbf{D}^X_{p_0}}{\partial z^\alpha \partial \bar{z}^\beta} + b \frac{\partial \mathbf{D}^X_{p_0}}{\partial z^\alpha} \frac{\partial \mathbf{D}^X_{p_0}}{\partial \bar{z}^\beta} \right) \\
&= e^{b\mathbf{D}^X_{p_0}} \frac{\partial^2 \mathbf{D}^X_{p_0}}{\partial z^\alpha \partial \bar{z}^\beta}
\end{aligned}$$

where we used the fact that, by lemma 69, $\frac{\partial \mathbf{D}^X_{p_0}}{\partial z^\alpha} = \frac{\partial \mathbf{D}^X_{p_0}}{\partial \bar{z}^\beta} = 0$. Then, since the matrix $\left(\frac{\partial^2 \mathbf{D}^X_{p_0}}{\partial z^\alpha \partial \bar{z}^\beta} \right)$ is positive definite, so is the matrix $\left(\frac{\partial^2 \frac{e^{b\mathbf{D}^X_{p_0}}-1}{b}}{\partial z^\alpha \partial \bar{z}^\beta} \right)$.

The following example explains the choice of the word "stereographic":

Example 89. Fix $b \neq 0$.

1. Suppose $b < 0$. Then $F(b, N) = \left(\mathbb{CB}^N, \frac{\pi}{-b} \right)$ has a canonical Bochner chart centered at 0:

$$\begin{aligned}
\psi_0 : F(b, N) &\to \frac{1}{\sqrt{-b}} \mathbb{CB}^N \subseteq \mathbb{C}^N \\
z &\mapsto \frac{1}{\sqrt{-b}} z.
\end{aligned}$$

   We call it the *stereographic projection* of $F(b, N)$ centered at 0. It is easy to verify that

$$\frac{e^{b\mathbf{D}_0^{F(b,N)}} - 1}{b} = \psi_{p_0}^* \left(\mathbf{D}_0^{\mathbb{C}^N} \right) :$$

   in particular, $\frac{e^{b\mathbf{D}_0^{F(b,N)}}-1}{b}$ defines on the whole complex manifold $\mathbb{CB}^N$ a Kähler potential for a new Kähler metric, isometric to $\frac{1}{\sqrt{-b}} \mathbb{CB}^N$ with the flat metric induced by $\mathbb{C}^N$, through the map ψ_0.

2. Suppose $b > 0$. Then $F(b, N) = \left(\mathbb{CP}^N, \frac{\pi}{b} g_{FS} \right)$ has a canonical Bochner chart centered at 0 and defined in the domain U_0:

$$\begin{aligned}
\psi_0 : U_0 &\to \mathbb{C}^N \\
[Z] &\mapsto \frac{1}{\sqrt{b}} \left(\frac{Z^0}{Z^j}, \frac{Z^1}{Z^j}, ..., \frac{\hat{Z}^j}{Z^j}, ... \right).
\end{aligned}$$

We call it the *stereographic projection* of $F(b,N)$ centered at 0. Again, it is easy to verify that

$$\frac{e^{b\mathbf{D}_0^{F(b,N)}}-1}{b}=\psi_j^*\left(\mathbf{D}_0^{\mathbb{C}^N}\right):$$

in particular, $\frac{e^{b\mathbf{D}_0^{F(b,N)}}-1}{b}$ defines on the open subset U_0 a Kähler potential for a new Kähler metric on U_0, isometric to $\mathbb{C}^N$ with the flat metric, through the map ψ_0.

The choice of words "stereographic projection" is due to the fact that the stereographic projection of $F(1,1)$ (that is, the Riemann sphere) with respect to 0 coincides with the usual stereographic projection.

As in the flat case, we have a resolvability property giving us informations about the Kähler immergibility of an analytic Kähler manifold into $F(b,N)$ locally:

Definition 90. Let $b\in\mathbb{R}\setminus\{0\}$. X is called *b-resolvable of rank N at p_0* if the locally defined Kähler metric induced by $\frac{e^{b\mathbf{D}_{p_0}^X}-1}{b}$ is 0-resolvable of rank N at p_0.

Lemma 91. *Let p_0 be a point of X. Let $\mathbf{f}\in\mathcal{O}_{p_0}(X,F(b,N))$ be a germ such that $\mathbf{f}(p_0)=0$. Then $\mathbf{f}$ is a full germ if and only if $\psi_0\circ\mathbf{f}\in\mathcal{O}_{p_0}\left(X,\mathbb{C}^N\right)$ is a full germ, where ψ_0 is the stereographic projection of $F(b,N)$ centered at 0 defined in example 89.*

Proof. ($\Rightarrow$) Suppose that $\mathbf{f}$ is full. If $\psi_0\circ\mathbf{f}$ were not full, then there would be a nontrivial $u\in\mathbb{C}^N$ such that, for every $p\in U$,

$$\langle(\psi_0\circ f)(p),u\rangle=0.$$

This means that, if $[Z]\in f(X)$, then

$$\sum_{j=1}^N\frac{Z^j}{Z^0}\overline{u}^j=0,$$

or equivalently

$$\langle Z,(0,u)\rangle=0.$$

This is a contradiction since $\mathbf{f}$ is full. ($\Leftarrow$) The proof is analougous to the one of implication $\Rightarrow$. $\square$

As in the flat case, b-resolvability of rank N at p_0 is equivalent to the existence of a germ in p_0 of full Kähler immersion into $F(b,N)$:

Theorem 92 (local b-resolvability criterion)**.** *Let $p_0 \in X$. Then X is b-resolvable of rank N at p_0 if and only if there exists a full germ of Kähler immersion $\mathbf{f} \in \mathcal{O}_{p_0}(X, F(b,N))$.*

Proof. The following are equivalent:

1. X is b-resolvable of rank N at p_0;
2. the locally defined Kähler metric induced by $\frac{e^{b\mathbf{D}^X_{p_0}}-1}{b}$ in a neighborhood of p_0 is 0-resolvable of rank N at p_0;
3. there exists a domain $U \ni p_0$ in which $\frac{e^{b\mathbf{D}^X_{p_0}}-1}{b}$ defines a new Kähler metric, and there exists a full Kähler immersion (with respect to this new Kähler metric) $f : U \to \mathbb{C}^N$ that sends p_0 in 0;
4. there exists a domain $U \ni p_0$ and a full Kähler immersion $f : U \to F(b,N)$ that sends p_0 to 0.

$1 \iff 2$ is the definition of b-resolvability, and $2 \iff 3$ is the local 0-resolvability criterion. We now prove $3 \iff 4$.

1. ($3 \Rightarrow 4$) If $b < 0$, we can shrink U and suppose that $f(U) \subseteq \frac{1}{\sqrt{-b}}\mathbb{CB}^N$. Then it is defined the map $\psi_0^{-1} \circ f : U \to \mathbb{C}^N$. This is a full map because of lemma 91; moreover, this is a Kähler immersion, since
$$\begin{aligned}\left(\psi_0^{-1} \circ f\right)^*_{p_0}\left(\frac{e^{b\mathbf{D}_0^{F(b,N)}}-1}{b}\right) &= f^*_{p_0}\left(\mathbf{D}_0^{\mathbb{C}^N}\right) \\ &= \frac{e^{b\mathbf{D}^X_{p_0}}-1}{b}\end{aligned}$$
implies
$$\left(\psi_0^{-1} \circ f\right)^*_{p_0}\left(\mathbf{D}_0^{F(b,N)}\right) = \mathbf{D}^X_{p_0}.$$
If $b > 0$ the proof is identical, except that we don't need to shrink U since ψ_0 has image $\mathbb{C}^N$ for $b > 0$.

2. ($4 \Rightarrow 3$) Suppose $b < 0$. We can shrink U and suppose that the germ $\frac{e^{b\mathbf{D}^X_{p_0}}-1}{b}$ defines a Kähler metric on U. Then it is defined the map $\psi_0 \circ f : U \to \mathbb{C}^N$. This is a full map because of lemma 91; moreover, this is a Kähler immersion with respect to the new metric on U, since

$$f^*_{p_0}\left(\mathbf{D}_0^{F(b,N)}\right) = \mathbf{D}^X_{p_0}$$

implies

$$\begin{aligned}(\psi_0 \circ f)^*_{p_0}\left(\mathbf{D}_0^{\mathbb{C}^N}\right) &= f^*_{p_0}\left(\frac{e^{b\mathbf{D}_O^{F(b,N)}}-1}{b}\right) \\ &= \frac{e^{b\mathbf{D}^X_{p_0}}-1}{b}.\end{aligned}$$

If $b > 0$, the proof is identical, except that we must shrink U to have $f(U) \subseteq U_0$ in order to compose f with the stereographic projection $\psi_0 : U_0 \to \mathbb{C}^N$ of $F(b,N)$.

□

As in the flat case, b-resolvability of rank N is actually a global property:

Theorem 93 (global b-resolvability criterion)**.** *X is b-resolvable of rank N in a point if and only if it is b-resolvable of rank N in every point.*

Proof. As in the case $b = 0$, we refer to [8] for a proof. □

The local and global b-resolvability criterions together imply the following

Corollary 94. *X is b-resolvable of rank N if and only if, for every $p_0 \in X$, there is a full germ of Kähler immersion $\mathbf{f} \in \mathcal{O}_{p_0}(X, F(b,N))$.*

As in the flat case, we have a rigidity theorem that allows us to patch together locally defined full Kähler immersions.

Lemma 95. *Let Φ be an element of $PU(N+1)$ (resp. $PU(N,1)$). Then Φ fixes 0 if and only if $\psi_0 \circ \Phi \circ \psi_0^{-1}$ is (the restriction of) an element of $U(N)$.*

Proof. A $N+1 \times N+1$ Hermitian matrix A fixes the vector $(1,0,0,...)$ if and only if has the form

$$A = \begin{pmatrix} 1 & \\ & A' \end{pmatrix},$$

where A' is a Hermitian matrix. It follows that, since $\Phi([Z]) = [A(Z)]$ for some $A \in \mathrm{U}(N+1)$ (resp. $A \in \mathrm{U}(N,1)$), $\Phi(0) = \Phi(0)$ if and only if Φ has a representant of the form

$$A = \begin{pmatrix} 1 & \\ & A' \end{pmatrix},$$

where $A' \in \mathrm{U}(N)$. This representant is unique, since any other representant of Φ differs from A by the multiplication for a constant diagonal unitary matrix. Now, suppose that $\Phi(0) = 0$: then it is easy to verify that

$$\psi_0 \circ \Phi \circ \psi_0^{-1} = A'.$$

On the other hand, if $\psi_0 \circ \Phi \circ \psi_0^{-1} \equiv A' \in \mathrm{U}(N)$ in its domain of definition, then

$$\begin{aligned} \Phi\left(\left[1 : z^1 : z^2 : \cdots\right]\right) &= \psi_0^{-1} \circ A' \circ \psi_0\left(\left[1 : z^1 : z^2 : \cdots\right]\right) \\ &= \psi_0^{-1} \circ \frac{1}{\sqrt{|b|}} A' z \\ &= \psi_0^{-1}\left(\frac{1}{\sqrt{|b|}}\left(w^1, w^2, ...\right)\right) \\ &= \left[1 : w^1 : w^2 : \cdots\right], \end{aligned}$$

where $w = A'z$, so $\Phi(0) = 0$. □

Theorem 96 (rigidity theorem)**.** *Let $f, g : X \to F(b,N)$ be two full Kähler immersions. Then there exists a $\Phi \in PU(N+1)$ (resp. $\Phi \in PU(1,N)$) such that $\Phi \circ f = g$.*

Proof. Suppose without loss of generality that $f(p_0) = g(p_0) = 0$ for some $p_0 \in X$. Consider a domain $U \ni p_0$ such that the germ $\frac{e^{b\mathbf{D}^X_{p_0}} - 1}{b}$ defines a Kähler metric on U, and $f(U), g(U)$ are contained in the domain of the stereographic projection ψ_0: it follows that $\psi_0 \circ f, \psi_0 \circ g : U \to \mathbb{C}^N$ are full Kähler immersions with respect to the new metric on U, and so by the rigidity theorem for $\mathbb{C}^N$ there is a unique $A' \in \mathrm{E}(N)$ such that $A' \circ \psi_0 \circ f = \psi_0 \circ g$.

But $\psi_0^{-1} \circ A' \circ \psi_0$ is then (the resctriction of) an element Φ of $\mathrm{PU}(N+1)$ (resp. $\mathrm{PU}(1,N)$) such that $\Phi \circ f = g$: by the above lemma, we obtain the thesis. $\square$

The b-resolvability criterions and the rigidity theorem can be used to answer our initial question.

Theorem 97. *Let X be simply connected. Then X is b-resolvable of rank N if and only if there exists a full Kähler immersion of X in $F(b,N)$.*

Proof. The proof is identical to the one of theorem 84. $\square$

Theorem 98. *Let X be a Kähler manifold. Then X can be fully Kähler immersed in $F(b,N)$ if and only if:*

1. *X is analytic;*
2. *X is b-resolvable of rank N;*
3. *for every $p_0 \in X$, the diastatic potential of X centered at p_0 is defined on its maximal domain of analiticity X_{p_0}.*

Moreover, the Kähler immersion is injective if and only if, for every (p,q) in the domain of definition of D^X, $D^X(p,q) = 0 \iff p = q$.

Proof. The proof is identical to the one of theorem 85 in the case $b < 0$, but in the case $b > 0$ we need some extra care in the implication $(\Leftarrow)$ because the diastasis of $F(b,N)$ is not globally defined for $b > 0$.
Let $\pi : \tilde{X} \to X$ be the universal cover of X. Since π is a local isometry and X is b-resolvable of rank N, also $\tilde{X}$ is analytic and b-resolvable of rank N, so by previous theorem there exists a full Kähler immersion $\tilde{f} : \tilde{X} \to F(b,N)$. We want to prove that $\tilde{f}$ is the lift of a full Kähler immersion $f : X \to F(b,N)$. Since π is a surjective local isometry, it is sufficient to prove that $\tilde{f}$ descends to the quotient, i.e. if $\pi(\tilde{p}_0) = \pi(\tilde{p}_1)$ then $\tilde{f}(\tilde{p}_0) = \tilde{f}(\tilde{p}_1)$. Let $\tilde{p}_0, \tilde{p}_1 \in \tilde{X}$ such that $\pi(\tilde{p}_0) = \pi(\tilde{p}_1) = p$. We would do the same of the proof of theorem 85, except for the fact that, a priori, if $\pi(\tilde{p}_0) = \pi(\tilde{p}_1)$, it is not automatically true that the diastatic potential of $\tilde{X}$ centered at $\tilde{p}_0$ is defined on $\tilde{p}_1$. This is actually true: in fact, suppose for the sake of contradiction that this is not true. Since the polar variety of $\tilde{p}_0$ is a codimension 1 complex analytic subvariety of $\tilde{X}$ by lemma 88, we can find a curve $\tilde{\gamma} : [0,1] \to \tilde{X}$ joining $\tilde{p}_0$ with $\tilde{p}_1$ such that $D^{\tilde{X}}_{\tilde{p}_0}$ is defined on $\tilde{\gamma}(t)$ for $t \in [0,1)$. Define $\gamma = \pi \circ \tilde{\gamma}$: since

π is a local biholomorphism and $\pi^*_{\tilde{p}_0}\left(\mathbf{D}^X_p\right) = \mathbf{D}^{\tilde{X}}_{\tilde{p}_0}$, the diastatic potential $D^{\tilde{X}}_{\tilde{p}_0}$ induces an analytic continuation $\{\mathbf{u}_t\}_{t\in[0,1)}$ of $\mathbf{D}^X_p$ along γ such that

$$\begin{aligned}\lim_{t\to 1}\mathbf{u}_t\left(\gamma\left(t\right)\right) &= \lim_{t\to 1} D^{\tilde{X}}_{\tilde{p}_0}\left(\tilde{\gamma}\left(t\right)\right)\\ &= +\infty\end{aligned}$$

because $\tilde{\gamma}(t) \to \tilde{p}_1$ and $\tilde{p}_1$ is in the polar variety of $\tilde{p}_0$. But the hypotesis 3 implies that the first limit must be 0, contradiction. It follows that the diastatic potential of $\tilde{X}$ centered at $\tilde{p}_0$ is defined at $\tilde{p}_1$.
Now, since the diastatic potential of $F(b,N)$ centered at $\tilde{p}_0$ vanishes only on $\tilde{p}_0$, we have $\tilde{f}(\tilde{p}_0) = \tilde{f}(\tilde{p}_1)$ if and only if $D^{\tilde{X}}_{\tilde{p}_0}(\tilde{p}_1) = 0$. Consider a curve $\tilde{\gamma} : [0,1] \to \tilde{X}$ joining $\tilde{p}_0$ and $\tilde{p}_1$: the curve $\gamma = \pi\circ\tilde{\gamma}$ is then a loop with base point p, and since π is a local biholomorphism and $\pi^*_{\tilde{p}_0}\left(\mathbf{D}^X_p\right) = \mathbf{D}^{\tilde{X}}_{\tilde{p}_0}$, the diastatic potential $D^{\tilde{X}}_{\tilde{p}_0}$ induces an analytic continuation $\{\mathbf{u}_t\}_{t\in[0,1]}$ of $\mathbf{D}^X_p$ along γ such that, for every $t\in[0,1]$, $\pi^*_{\tilde{\gamma}(t)}(\mathbf{u}_t)$ is the germ of the diastatic potential $D^{\tilde{X}}_{\tilde{p}_0}$ at $\tilde{\gamma}(t)$. But, by hypotesis 3 on X, it must be $\mathbf{u}_1(p) = 0$: this implies that $D^{\tilde{X}}_{\tilde{p}_0}(\tilde{p}_1) = 0$, proving our assertion.
The second part of the theorem is similar to the one of theorem 85. Suppose that, for every (p,q) in the domain of D^X, we have $D^X(p,q) = 0 \Rightarrow p = q$. Suppose now that, for some $p,q\in X$, we have $f(p) = f(q)$: then, since the domain of D^X is the preimage under $f\times f$ of the domain of the diastasis of $F(b,N)$, which contains the diagonal, it follows that q is in the domain of the diastatic potential of X centered at p. Since $f(p) = f(q)$, we have $D^{F(b,N)}(f(p), f(q)) = 0$: but this implies that $D^X(p,q) = 0$ since f is a Kähler immersion, so by hypotesis we obtain $p = q$. By generality of $p,q\in X$, we obtain that f is injective. $\square$

Chapter 3

Applications and further results

Theoretically, Calabi's criterions solves completely the Kähler immergibility problem 2, but concretely it is easy to apply only in very special cases. In this chapter, we use Calabi's criterion to solve completely the following

Problem. For which $b, b' \in \mathbb{R}$ and $n < \infty, N' \leq \infty$ there is a Kähler immersion of $F(b,n)$ into $F(b',N')$?

In addition to Calabi's criterion, in order to study Kähler immersions into $\left(\mathbb{CP}^N, \omega_{FS}\right)$ for some $N \leq \infty$, there is another tool available: at the end of the thesis, using the powerful tool of *quantizations of Kähler manifolds* developed in [4, 5, 6, 7], we will prove that if (X,ω) is a simply connected homogeneous Kähler manifold whose Kähler form ω is integral, then $(X, k\omega)$ can be Kähler immersed into $\left(\mathbb{CP}^N, \omega_{FS}\right)$ for some $N \leq \infty$ and $k \in \mathbb{N}$.

3.1 Simply connected complex space forms into simply connected complex space forms

In this section we provide a complete answer to the following

Problem. For which $b, b' \in \mathbb{R}$ and $n < \infty, N \leq \infty$ there is a Kähler immersion of $F(b,n)$ into $F(b',N)$?

By theorems 84 and 97, we need only to verify b'-resolvability of some rank $\leq N$ of $F(b,n)$. In order to verify b'-resolvability, calculations are easy if we choose as base point the point 0 and as local coordinates the stereographic projection of $F(b,n)$ centered at 0. With those choices, the diastatic potential centered at 0 in local coordinates is

$$D_0^{F(b,n)}(z) = \begin{cases} |z|^2 & b = 0 \\ \frac{1}{b}\log\left(1 + b|z|^2\right) & b \neq 0 \end{cases}$$

and is defined in the domain of the stereographic projection ψ_0. We want to check b'-resolvability at 0, so if $b' \neq 0$ we need to compute the Taylor matrix at 0 of

$$\frac{e^{b' D_0^{F(b,n)}(z)} - 1}{b'} = \begin{cases} \frac{e^{b'|z|^2}-1}{b'} & b = 0 \\ \frac{\left(1+b|z|^2\right)^{b'/b}-1}{b'} & b \neq 0 \end{cases};$$

observe that

$$\lim_{b' \to 0} \frac{e^{b' D_0^{F(b,n)}(z)} - 1}{b'} = D_0^{F(b,n)}(z),$$

so our considerations for $b' \neq 0$ easily extends to the case $b' = 0$ by taking the limit for $b' \to 0$. Computations shows that the Taylor expansion at 0 of the above functions is

$$\sum_{j=1}^{\infty} C_j \mathbf{z}^{\mathbf{m}_j} \overline{\mathbf{z}}^{\mathbf{m}_j},$$

with

$$C_j = \frac{\prod_{k=1}^{|\mathbf{m}_j|-1} \left(b' - kb\right)}{\mathbf{m}_j!}$$

(here a product of an empty set of number is considered equal to 1). It follows that the Taylor matrix is diagonal, and so it is easy to compute its rank and verify its definite positiveness.

1. If $F(b,n)$ is b'-resolvable of some finite rank, then it must be $C_j = 0$ for j sufficiently large: this means that $b' = kb$ for some $k \in \mathbb{N}$, and hence it is easy to verify that $C_j \neq 0$ only for $1 \leq j \leq \binom{n+k}{k} - 1$. We then obtain that $F(b,n)$ is b'-resolvable of rank $\binom{n+k}{k} - 1$ if and only if $b' - hb > 0$ for every $h < k$, i.e. $(k-h)b > 0$ for every $1 \leq h < k$. This is verified in the following two subcases:

   (a) $k = 1$ and $b \in \mathbb{R}$;

   (b) $k > 1$ and $b > 0$.

2. If $F(b,n)$ is b'-resolvable of infinite rank, then it must be $C_j > 0$ for every $j \in \mathbb{N}$. This is verified if and only if $b \leq 0$ and $b' > b$.

The above argument can be summarized in the following

Theorem 99. *The simply connected complex space form $F(b,n)$ $(n < \infty)$ can be Kähler immersed into the simply connected complex space form $F(b',N)$ $(N \leq \infty)$ if and only if it is verified one of the following conditions:*

1. *$N < \infty$ and either*
   (a) *$b = b'$ and $n \leq N$, or*
   (b) *$b > 0$, $b' = kb$ for some $k \in \mathbb{N}$ and $\binom{n+k}{k} - 1 \leq N$;*
2. *$N = \infty$ and either*
   (a) *$b \leq 0$ and $b' \geq b$, or*
   (b) *$b > 0$ and $b' = kb$ for some $k \in \mathbb{N}$.*

Now we turn to a negative result using Calabi's criterions. It is important to point out that, if (X,ω) is *compact*, then it cannot be fully Kähler immersed into a infinite dimensional elliptic simply connected complex space form:

Lemma 100. *Let (X,ω) be a compact Kähler manifold. If (X,ω) can be Kähler immersed into $(\mathbb{CP}^{\infty},\omega_{FS})$, then it can be Kähler immersed also into $(\mathbb{CP}^{n},\omega_{FS})$ for some $n \in \mathbb{N}$.*

Proof. Suppose for the sake of a contradiction that this were not true. Then there would be a full Kähler immersion $f : (X,\omega) \to (\mathbb{CP}^{\infty},\omega_{FS})$. Hence, we could find a basis $\{Z_j\}_{j\geq 0}$ of $\mathbb{C}^{\infty}$ such that $[Z_j] \in S$ for every j. Let $\Phi : \mathbb{C}^{\infty} \to \mathbb{C}^{\infty}$ be the isomorphism that sends the basis $\{Z_j\}_{j\geq 0}$ in the canonical basis of $\mathbb{C}^{\infty}$: the map Φ induces a biholomorphism $[\Phi] : \mathbb{CP}^{\infty} \to \mathbb{CP}^{\infty}$ defined by $[\Phi]([Z]) = [\Phi(Z)]$. Since $[\Phi]$ is continuous, $[\Phi](S)$ is compact, and hence the sequence $[1:0:0:\cdots], [0:1:0:0:\cdots], \ldots$ would have a subsequence that converges to an element of $[\Phi](S)$: in particular, this set would have a limit point, and this is a contradiction, since this set is a discrete subspace of $\mathbb{CP}^{\infty}$. $\square$

This result, together with Calabi's criterions, leads to an example of an analytic Kähler manifold that cannot be Kähler immersed into *any* simply connected complex space form of finite or infinite dimension: this underlines the strongly different behaviour of Kähler manifolds compared with Riemannian manifolds.

Theorem 101. *A flat complex torus cannot be Kähler immersed into any simply connected complex space form of finite or infinite rank.*

Proof. Let T be a flat complex torus. Since T is compact, it cannot be Kähler immersed into $F(b, N)$ for any $b \leq 0$ and $N \leq \infty$. Suppose for the sake of a contradiction that T can be Kähler immersed into $(\mathbb{CP}^N, \lambda\omega_{FS})$ for some $\lambda > 0$ and $N \leq \infty$. The above lemma ensures that we can suppose $N < \infty$. On the other hand, since T is a complex flat torus, the universal covering map $\pi : (\mathbb{C}^n, \omega_0) \to T$ is Kähler, and hence by composing with a Kähler immersion of T into $(\mathbb{CP}^N, \lambda\omega_{FS})$ we obtain a Kähler immersion of $(\mathbb{C}^n, \omega_0)$ into $(\mathbb{CP}^N, \lambda\omega_{FS})$. This is a contradiction by the above classification theorem. □

3.2 Homogeneous Kähler manifolds into complex projective spaces

Definition 102. Let (X, ω) be a Kähler manifold. The Kähler form ω is said to be *integral* if its de Rham class is contained in the image of the canonical morphism $H^2(X, \mathbb{Z}) \to H^2_{dR}(X)$.

Definition 103. Let (X, ω) be a Kähler manifold. A *quantization* of (X, ω) is a holomorphic Hermitian line bundle (L, H) such that, if Ω denotes the curvature form of (L, H), then

$$\omega = \frac{i}{2\pi}\Omega.$$

How are integral forms related to quantizations? The answer is explained in the following theorem:

Theorem 104. *A Kähler manifold (X, ω) admits a quantization if and only if ω is integral.*

Proof. We refer to [15] for a proof of the symplectic version of the theorem, and to [19], theorem 1.4.6, for its Kähler version. □

From the above theorem, we obtain a first immediate necessary condition for a Kähler manifold (X, ω) to admit a Kähler immersion into some $\left(\mathbb{CP}^N, \omega_{FS}\right)$:

Corollary 105. *If (X, ω) can be Kähler immersed into $\left(\mathbb{CP}^N, \omega_{FS}\right)$ for some $N \leq \infty$, then ω is integral.*

Proof. Let $f : (X, \omega) \to \left(\mathbb{CP}^N, \omega_{FS}\right)$ be a Kähler immersion. Since $f^*\omega_{FS} = \omega$, it suffices to show that ω_{FS} is integral, or equivalently that $\left(\mathbb{CP}^N, \omega_{FS}\right)$ admits a quantization. Define on $\mathbb{CP}^N$ a holomorphic line bundle, called the *tautological line bundle* and denoted by $\mathcal{O}(-1)$: the total space of $\mathcal{O}(-1)$ is the subset of $\mathbb{CP}^N \times \mathbb{C}^{N+1}$ of pairs $([Z], Z)$, and the projection $\pi : \mathcal{O}(-1) \to \mathbb{CP}^N$ sends $([Z], Z)$ to $[Z]$. A trivializing atlas for $\mathcal{O}(-1)$ is given by

$$\begin{array}{rcl} \chi_j : \pi^{-1}(U_j) & \to & U_j \times \mathbb{C} \\ ([Z], Z) & \mapsto & \left([Z], Z^j\right). \end{array}$$

The transition maps $\Phi_{ij} : U_i \cap U_j \to \mathbb{C}^*$ are given by $\Phi_{ij}([Z]) = \frac{Z^i}{Z^j}$, so $\mathcal{O}(-1)$ is actually a holomorphic line bundle over $\mathbb{CP}^N$. Now, denote by $\mathcal{O}(1)$ its dual bundle, called the *hyperplane line bundle*. Define on $\mathcal{O}(1)$ an Hermitian metric in the following way: if $f, g \in \mathcal{O}(1)_{[Z]}$, then

$$H(f, g) = \frac{f(Z)\overline{g(Z)}}{\langle Z, Z\rangle}.$$

It is easy to show that H is an Hermitian structure on $\mathcal{O}(1)$. Now, for every $0 \leq j \leq n$, there is a canonical nonzero section $s_j : U_j \to \mathcal{O}(1)$ that associates to $[Z]$ the map $\langle Z\rangle \to \mathbb{C}$ given by $W \mapsto W^j$. If we define $h_j = H(s_j, s_j) : U_j \to \mathbb{R}$, we have

$$h_j([Z]) = \frac{Z^j\overline{Z^j}}{\langle Z, Z\rangle}.$$

It follows that, in the open subset U_j, the curvature form of $(\mathcal{O}(1), H)$ is given by

$$\Omega = -\partial\bar{\partial}\log\left(\frac{Z^j\overline{Z^j}}{\langle Z, Z\rangle}\right),$$

and then, on U_j, we have

$$\begin{array}{rcl} \frac{i}{2\pi}\Omega & = & \frac{i}{2\pi}\partial\bar{\partial}\log\left(\frac{\langle Z, Z\rangle}{Z^j\overline{Z^j}}\right) \\ & = & \omega_{FS}. \end{array}$$

It follows that $(\mathcal{O}(1), H)$ is a quantization for $\left(\mathbb{CP}^N, \omega_{FS}\right)$, so ω_{FS} is integral. $\square$

The converse of the above corollary is not true. To find some sufficient conditions, we need to focus on Kähler manifolds with additional structure: in this section, we study Kähler immersions of *homogeneous* Kähler manifolds into complex projective spaces.

A fundamental tool we use to study homogeneous Kähler manifolds is an important theorem known as the *fundamental conjecture for homogeneous Kähler manifolds*: we refer to [10] for a proof of it.

Theorem 106. *A homogeneous Kähler manifold (X, ω) is a* complex *product $\Omega \times \mathcal{F}$, where Ω is a homogeneous bounded domain, and $\mathcal{F}$ is, with the Kähler structure induced by ω, a Kähler product*

$$\mathcal{F} = \mathcal{C} \times \frac{\mathbb{C}^n}{\Gamma},$$

where $\mathcal{C}$ is a simply connected compact homogeneous Kähler manifold, Γ is a discrete subgroup of $\mathbb{C}^n$, and $\mathbb{C}^n/\Gamma$ is endowed with its standard flat Kähler metric induced by the one of $\mathbb{C}^n$.

We can now prove a nontrivial necessary condition for the Kähler immergibility of a homogeneous Kähler manifold into some $\left(\mathbb{CP}^N, \omega_{FS}\right)$:

Theorem 107. *If a homogeneous Kähler manifold (X, ω) can be Kähler immersed into some $\left(\mathbb{CP}^N, \omega_{FS}\right)$, then X is simply connected.*

Proof. Since a homogeneous bounded domain is simply connected, we know from the fundamental conjecture on Kähler manifolds that (X, ω) can be identified as a Kähler product $\mathcal{S} \times \frac{\mathbb{C}^n}{\Gamma}$, where $\mathcal{S}$ is a simply connected Kähler manifold. It follows that if (X, ω) can be Kähler immersed into $\left(\mathbb{CP}^N, \omega_{FS}\right)$, then so does $\mathbb{C}^n/\Gamma$. Let $\pi : \mathbb{C}^n \to \mathbb{C}^n/\Gamma$ be the canonical projection, and let $f : \mathbb{C}^n/\Gamma \to \left(\mathbb{CP}^N, \omega_{FS}\right)$ be a Kähler immersion: then, since π is a Kähler immersion with respect to the flat Kähler metrics, it follows that $f \circ \pi : (\mathbb{C}^n, \omega_0) \to \left(\mathbb{CP}^N, \omega_{FS}\right)$ is a Kähler immersion. Now, we proved using Calabi's criterions that it must be $N = \infty$: by the rigidity theorem 96 and the fact that $(\mathbb{C}^n, \omega_0)$ admits an *injective* Kähler immersion into $(\mathbb{CP}^\infty, \omega_{FS})$ by theorem 98, we obtain that the map $f \circ \pi$ must be injective. This implies that π must be injective, meaning that Γ must be the trivial subgroup $\{0\}$

of $\mathbb{C}^n$. It then follows that X is a product of simply connected spaces, and then it is simply connected too.

It is now natural to ask if the three hypotesis of integrality of the Kähler form, homogeneity and simply connectedness of a Kähler manifold (X, ω) are together sufficient to admit a Kähler immersion into some $\left(\mathbb{CP}^N, \omega_{FS}\right)$: the answer is that, if we admit the possibility of rescaling ω by a sufficiently large positive integer k, we can find a Kähler immersion of $(X, k\omega)$ into some $\left(\mathbb{CP}^N, \omega_{FS}\right)$. To find such a Kähler immersion, we will use the theory of quantizations of Kähler manifolds. □

Fix a n-dimensional Kähler manifold (X, ω) with ω integral. To every quantization (L, H) of (X, ω), we can associate a Hilbert space $\mathcal{H}_{(L,H)}$ defined in the following way. If $\mathrm{Hol}_L(X)$ denotes the complex vector space of global holomorphic sections of L, we define

$$\mathcal{H}_{(L,H)} = \left\{ s \in \mathrm{Hol}_L(X) : \int_X H(s, s) \frac{\omega^n}{n!} < \infty \right\} :$$

the inner product

$$\langle s, t \rangle_H = \int_X H(s, t) \frac{\omega^n}{n!}$$

endows $\mathcal{H}_{(L,H)}$ with the structure of a separable complex Hilbert space (see [4]). Now, for every $q \in L$, we have

$$s(\pi(q)) = \delta_q(s)\, q$$

for some $\delta_q(s) \in \mathbb{C}$: this defines for every $q \in L$ a map

$$\delta_q : \mathcal{H}_{(L,H)} \to \mathbb{C}.$$

The map δ_q is a bounded linear map (see [4]), and hence it is an element of the dual of $\mathcal{H}_{(L,H)}$: thus, by the Riesz theorem, there is a unique $e_q \in \mathcal{H}_{(L,H)}$ such that

$$\delta_q(s) = \langle s, e_q \rangle_H$$

or equivalently

$$(s \circ \pi)(q) = \langle s, e_q \rangle_H\, q.$$

Definition 108. The holomorphic section $e_q \in \mathcal{H}_{(L,H)}$ is called the *coherent state* relative to the point $q \in L$.

Observe that, if $c \in \mathbb{C}$, then $\langle s, e_q \rangle_H q = (s \circ \pi)(q) = (s \circ \pi)(cq) = \langle s, e_{cq} \rangle_H cq = \langle s, \bar{c} e_{cq} \rangle_H q$, so by the generality of s we obtain

$$e_q = \bar{c} e_{cq}$$

for every $c \in \mathbb{C}$. This implies that the following definition is well posed:

Definition 109. The *epsilon function* associated to the Hermitian line bundle (L, H) is the smooth function

$$\epsilon_{(L,H)} : X \quad \to \quad \mathbb{R}$$

that associates to $x \in X$ the real number $H_x(q, q) \langle e_q, e_q \rangle_H$, where q is a nonzero element of L_x.

We can express the epsilon function of a Hermitian line bundle in another useful way:

Lemma 110. *Let $N_{(L,H)} + 1 \leq \infty$ be the dimension of $\mathcal{H}_{(L,H)}$, and let $\{s_j\}_{j=0,\dots,N_{(L,H)}}$ be an orthonormal basis for $\mathcal{H}_{(L,H)}$. Then the epsilon function $\epsilon_{(L,H)}$ satisfies*

$$\epsilon_{(L,H)}(x) = \sum_{j=0}^{N_{(L,H)}} H_x(s_j(x), s_j(x))$$

for every $x \in X$.

Proof. Given a $x \in X$, choose $q \in L_x$ nonzero. Then, for every j, we have

$$s_j(x) = \lambda_j q$$

for some $\lambda_j \in \mathbb{C}$: moreover, if $s \in \mathcal{H}_{(L,H)}$, then

$$\begin{aligned} s(x) &= \sum_{j=0}^{N_{(L,H)}} \langle s, s_j \rangle_H s_j(x) \\ &= \sum_{j=0}^{N_{(L,H)}} \langle s, s_j \rangle_H \lambda_j q \\ &= \left\langle s, \sum_{j=0}^{N_{(L,H)}} \overline{\lambda}_j s_j \right\rangle_H q, \end{aligned}$$

so since $s(x) = \langle s, e_q \rangle_H q$, we obtain

$$e_q = \sum_{j=0}^{N_{(L,H)}} \overline{\lambda}_j s_j.$$

It then follows that

$$\begin{aligned} \epsilon_{(L,H)}(x) &= H_x(q,q) \langle e_q, e_q \rangle_H \\ &= \sum_{j=0}^{N_{(L,H)}} \lambda_j \overline{\lambda}_j H_x(q,q) \\ & \sum_{j=0}^{N_{(L,H)}} H_x(s_j(x), s_j(x)). \end{aligned}$$

□

The epsilon function $\epsilon_{(L,H)}$ induces a holomorphic map of an open subset of X into $\mathbb{CP}^{N_{(L,H)}}$:

Definition 111. Let $Z_{(L,H)} \subseteq X$ be the set of zeros of $\epsilon_{(L,H)}$. The *coherent states map* is the map

$$\begin{aligned} f_{(L,H)} : X \backslash Z_{(L,H)} &\to \mathbb{CP}^{N_{(L,H)}} \\ x &\mapsto [s_0(x) : s_1(x) : \cdots]. \end{aligned}$$

Remark 112. A precisation is needed here: $s_j(x)$ is not a complex number, but an element of L_x, so we should read the above definition in the following way. Fix $x \in X$. If $\chi : \pi^{-1}(U) \to U \times \mathbb{C}$ is a local holomorphic trivialization for L with $x \in U$, then we have $\chi \circ s_j(x) = (x, \xi_j(x))$ for some $\xi_j : U \to \mathbb{C}$ holomorphic. Now, if $\epsilon_{(L,H)}(x) \neq 0$, then some $s_j(x)$ must be $\neq 0$, and so $[\xi_0(x) : \xi_1(x) : \cdots]$ is a well defined element of $\mathbb{CP}^{N_{(L,H)}}$. This definition does not depend on the trivialization chosen: in fact, if (χ', U') is another holomorphic trivialization for L with $x \in U'$, then on $U \cap U'$ we have $\xi'_j = \Phi \xi_j$ for some holomorphic map $\Phi : U \cap U' \to \mathbb{C}^*$, and hence $[\xi_0(x) : \xi_1(x) : \cdots] = [\xi'_0(x) : \xi'_1(x) : \cdots]$. We then can define $[s_0(x) : s_1(x) : \cdots] = [\xi_0(x) : \xi_1(x) : \cdots]$, and by generality of x we obtain that the map $f_{(L,H)}$ is a well defined holomorphic map.

The coherent states map $f_{(L,H)}$ is globally defined if and only if $\epsilon_{(L,H)}$ is strictly positive: in this case, $f_{(L,H)}$ is a holomorphic map of X into $\mathbb{CP}^{N_{(L,H)}}$. It could be not possible to find a quantization of (X,ω) having strictly positive epsilon function. However, if X is compact and ω is rescaled by a sufficiently large positive integer, we can actually find such a quantization: this is the famous Kodaira embedding theorem (for a proof, see [14]):

Theorem 113 (Kodaira embedding theorem)**.** *Let (X,ω) be a compact Kähler manifold with ω integral. Let (L,H) be a quantization of (X,ω). For every $k\in\mathbb{N}$, denote by $\left(L^k,H^k\right)$ the holomorphic Hermitian line bundle $\otimes^k(L,H)$, which is a quantization of $(X,k\omega)$. Then, for $k\in\mathbb{N}$ sufficiently large, the coherent states map $f_{\left(L^k,H^k\right)}$ provides a holomorphic embedding of X into $\mathbb{CP}^{N_{\left(L^k,H^k\right)}}$.*

Let us return to the general not necessarily compact case. Suppose that the epsilon function $\epsilon_{(L,H)}$ is strictly positive, so the coherent states map $f_{(L,H)}$ is globally defined. It is natural to ask what is the relation between the Kähler form ω of X and the pullback $f^*_{(L,H)}\omega_{FS}$ of the Fubini-Study metric. The following theorem proves that $f^*_{(L,H)}$ is actually an "approximation" of ω, and the "error" depends on the epsilon function of (L,H):

Theorem 114. *Suppose that $\epsilon_{(L,H)}$ is everywhere positive. Then $f_{(L,H)}: X\to\mathbb{CP}^{N_{(L,H)}}$ is a holomorphic immersion satisfying*

$$f^*_{(L,H)}\omega_{FS}=\omega+\frac{i}{2\pi}\partial\bar{\partial}\log\epsilon_{(L,H)}.$$

Proof. Fix an orthonormal basis $\{s_j\}$ of $\mathcal{H}_{(L,H)}$. Then the open set $U=f^{-1}_{(L,H)}(U_0)$ coincides with the set of the $x\in X$ such that $s_0(x)\neq 0$. This implies that the section $s_0:U\to L$ is nonvanishing, and hence

$$\varphi_0\circ f_{(L,H)}(x)=\sum_{j=1}^{N_{(L.H)}}\frac{s_j(x)}{s_0(x)}\overline{\frac{s_j(x)}{s_0(x)}}.$$

Now, on U, we have

$$\begin{aligned}
f^*_{(L,H)}\omega_{FS} &= \frac{i}{2\pi}\partial\bar{\partial}\log\left(1+\left|\varphi_0\circ f_{(L,H)}\right|^2\right)\\
&= \frac{i}{2\pi}\partial\bar{\partial}\log\left(1+\sum_{j=1}^{N_{(L.H)}}\frac{s_j(x)}{s_0(x)}\overline{\frac{s_j(x)}{s_0(x)}}\right).
\end{aligned}$$

Observe that, for every j, we have

$$\frac{s_j(x)}{s_0(x)}\overline{\frac{s_j(x)}{s_0(x)}}H_x(s_0(x), s_0(x)) = H_x(s_j(x), s_j(x)),$$

so since for every $x \in U$ we have $s_0(x) \neq 0$, in U we have

$$\begin{aligned} 1 + \sum_{J=1}^{N_{(L,H)}} \frac{s_j(x)}{s_0(x)}\overline{\frac{s_j(x)}{s_0(x)}} &= 1 + \sum_{j=1}^{N_{(L,H)}} \frac{H_x(s_j(x), s_j(x))}{H_x(s_0(x), s_0(x))} \\ &= \frac{\epsilon_{(L,H)}(x)}{H_x(s_0(x), s_0(x))}. \end{aligned}$$

It follows that

$$f^*_{(L,H)}\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\left(\log \epsilon_{(L,H)}\right) - \frac{i}{2\pi}\partial\bar{\partial}\log H(s_0, s_0);$$

since $s_0 : U \to L$ is a nonvanishing section of L, the form $-\partial\bar{\partial}\log H(s_0, s_0)$ coincides on U with the fundamental form Ω of (L, H): since (L, H) is a quantization of (X, ω), we obtain $\frac{i}{2\pi}\partial\bar{\partial}\log H(s_0, s_0) = \omega$, and hence

$$f^*_{(L,H)}\omega_{FS} = \omega + \frac{i}{2\pi}\partial\bar{\partial}\log \epsilon_{(L,H)}$$

on U. The same argument can be applied to every open set U_j of the canonical atlas of $\mathbb{CP}^{N_{(L,H)}}$, so the thesis follows. □

By the Kodaira embedding theorem, if (X, ω) is compact with ω integral, then for every quantization (L, H) of (X, ω) and for every sufficiently large $k \in \mathbb{N}$, the map $f_{(L^k,H^k)}$ is a holomorphic embedding of X into $\mathbb{CP}^{N_{(L^k,H^k)}}$, and hence it provides an approximation of ω

$$\omega_k = \frac{1}{k}f^*_{(L^k,H^k)}\omega_{FS} = \omega + \frac{i}{2k\pi}\partial\bar{\partial}\log \epsilon_{(L^k,H^k)}.$$

The sequence ω_k actually approximates ω, as proved by Tian in [25]:

Theorem 115 (Tian). *Let (X, ω) be a compact Kähler manifold with ω integral. Let (L, H) be a quantization of (X, ω), and let ω_k be defined as above for $k \in \mathbb{N}$ sufficiently large. Then the sequence ω_k C^2-converges to ω.*

In [23], Ruan improved the above theorem showing that the convergence of the sequence ω_k to ω is actually C^∞.

In the general (not necessarily compact) case, theorem 114 implies immediately that a quantization (L,H) of (X,ω) with *constant* positive epsilon function $\epsilon_{(L,H)}$ induces a Kähler immersion $f_{(L,H)} : (X,\omega) \to \left(\mathbb{CP}^{N_{(L,H)}}, \omega_{FS}\right)$. Those quantizations deserve a special name:

Definition 116. Let (X,ω) be a Kähler manifold with ω integral, and let (L,H) be a quantization of (X,ω). We say that the quantization (L,H) is *balanced* if $\epsilon_{(L,H)}$ is a positive constant.

By the above argument, if (X,ω) admits a balanced quantization then it can be Kähler immersed into some $\left(\mathbb{CP}^N, \omega_{FS}\right)$. We are especially interested in finding balanced quantizations of *homogeneous* and *simply connected* Kähler manifolds.

Definition 117. Two quantizations (L,H) and (L',H') of a Kähler manifold (X,ω) are said to be *equivalent* if there is an isomorphism of holomorphic line bundles $\Phi : L \to L'$ such that $\Phi^* H' = H$, i.e. $H_x(v,w) = H'_x(\Phi v, \Phi w)$ for every $x \in X$ and $v, w \in L_x$.

It is easy to verify that the above defined relation is actually an equivalence relation. It is important to observe that the epsilon function and the coherent states map of a quantization depends only on its equivalence relation:

Lemma 118. *If two quantizations (L,H) and (L',H') of a Kähler manifold (X,ω) are equivalent, then $\epsilon_{(L,H)} \equiv \epsilon_{(L',H')}$ and $f_{(L,H)} \equiv f_{(L',H')}$.*

Proof. Let $\Phi : L \to L'$ be as in the definition of equivalence of Hermitian line bundles. First of all, observe that the map

$$\begin{array}{rcl} \mathcal{H}_{(L,H)} & \to & \mathcal{H}_{(L',H')} \\ s & \mapsto & s' := \Phi \circ s \end{array}$$

is an isomorphism of Hilbert spaces. This implies that, if $\{s_j\}_{j=0}^N$ is an orthonormal basis of $\mathcal{H}_{(L,H)}$, then $\left\{s'_j\right\}_{j=0}^N$ is an orthonormal basis of $\mathcal{H}_{(L',H')}$, and hence

$$\begin{aligned}\epsilon_{(L,H)}(x) &= \sum_{j=0}^{N} H_x\left(s_j(x), s_j(x)\right) \\ &= \sum_{j=0}^{N} H'_x\left(s'_j(x), s'_j(x)\right) \\ &= \epsilon_{(L',H')}(x)\end{aligned}$$

for every $x \in X$. Thus, the coherent states maps $f_{(L,H)}$ and $f_{(L',H')}$ have the same domain. Moreover, it is immediate to verify that $f_{(L,H)} \equiv f_{(L',H')}$: for every $x \in X$, if we choose a nonvanishing section $\tau : U \to L$ in a neighborhood U of x, then $\tau' := \Phi \circ \tau$ is a nonvanishing section of L' defined on U, and if we use τ, τ' to define local trivializing coordinates of L and L' then $s_j(x)$ and $s'_j(x)$ have the same local coordinates, so $[s_0(x) : s_1(x) : \cdots]$ and $[s'_0(x) : s'_1(x) : \cdots]$ represents the same point of $\mathbb{CP}^N$, as defined in remark 112. $\square$

When the Kähler manifold is simply connected, as in our special case of interest, the equivalence relation between quantizations becomes trivial (it is a straightforward consequence of the second part of theorem 1.4.6 in [19]):

Theorem 119. *Let (X, ω) be a simply connected Kähler manifold. Then, two quantizations of (X, ω) are equivalent.*

It follows that, if a Kähler manifold (X, ω) is simply connected and ω is integral, then we can refer to *the epsilon function* ϵ_ω and *the coherent states map* f_ω of (X, ω), without referring to a specific quantization. Moreover, we can say that (X, ω) is *balanced* if ϵ_ω is a positive constant.

We can finally prove the main theorem of this section:

Theorem 120. *Let (X, ω) be a homogeneous and simply connected Kähler manifold, with ω integral. Then, for $k \in \mathbb{N}$ sufficiently large, $(X, k\omega)$ is balanced, and hence it is Kähler immergible into some $\left(\mathbb{CP}^N, \omega_{FS}\right)$.*

Proof. First we prove that the epsilon function ϵ_ω is constant. Let (L, H) be a quantization of (X, ω). Let Φ be an holomorphic isometry of (X, ω). We can pullback the holomorphic line bundle $\pi : L \to X$ to a holomorphic line bundle $\pi' : L' \to X$, and by definition there is a holomorphic vector bundle isomorphism $\tilde{\Phi} : L' \to L$ that lifts Φ. Denote by H' the pullback of the

Hermitian structure H on L': since $\Phi^*\omega = \omega$ and (L, H) is a quantization of (X, ω), it is easy to verify that (L', H') is another quantization of (X, ω). Now, it is easy to prove that the map

$$\begin{array}{rcl} \mathcal{H}_{(L,H)} & \to & \mathcal{H}_{(L',H')} \\ s & \mapsto & s' := \tilde{\Phi}^{-1} \circ s \circ \Phi \end{array}$$

is an isomorphism of Hilbert spaces: then, if $\{s_j\}_{j=0}^N$ is an orthonormal basis of $\mathcal{H}_{(L,H)}$, then the family $\left\{s'_j\right\}_{j=0}^N$ is an orthonormal basis of $\mathcal{H}_{(L',H')}$. Moreover, we have

$$\begin{aligned} \epsilon_{(L,H)}\left(\Phi\left(x\right)\right) &= \sum_{j=0}^{N} H_{\Phi(x)}\left(s_j\left(\Phi\left(x\right)\right), s_j\left(\Phi\left(x\right)\right)\right) \\ &= \sum_{j=0}^{N} H'_x\left(\tilde{\Phi}^{-1}\left(s_j\left(\Phi\left(x\right)\right)\right), \tilde{\Phi}^{-1}\left(s_j\left(\Phi\left(x\right)\right)\right)\right) \\ &= \sum_{j=0}^{N} H'_x(s'_j(x), s'_j(x)) \\ &= \epsilon_{(L',H')}\left(x\right) \end{aligned}$$

so $\epsilon_{(L',H')} = \epsilon_{(L,H)} \circ \Phi$. But (L, H) and (L', H') are both quantizations of a simply connected Kähler manifold, so by the above theorem, we have $\epsilon_{(L,H)} \equiv \epsilon_{(L',H')} \equiv \epsilon_\omega$, and hence

$$\epsilon_\omega = \epsilon_\omega \circ \Phi$$

for every holomorphic isometry Φ of (X, ω). Since (X, ω) is homogeneous, this implies that ϵ_ω is constant. Since every rescalation $(X, k\omega)$ is obviously simply connected and homogeneous, we obtain that every $\epsilon_{k\omega}$ is constant for every $k \in \mathbb{N}$. Now, we observe that ϵ_ω is a *positive* constant if and only if $\mathcal{H}_{(L,H)} \neq \{0\}$ for some quantization (L, H) of (X, ω): in fact, this is equivalent to say that $\mathcal{H}_{(L,H)}$ has a nonempty orthonormal basis $\{s_j\}_{j=0}^N$, and then since

$$\int_X H\left(s_j, s_j\right) \frac{\omega^n}{n!} = 1$$

for every j, we must have $H_x\left(s_j\left(x\right), s_j\left(x\right)\right) > 0$ for some $x \in X$, implying $\epsilon_\omega\left(x\right) = \epsilon_{(L,H)}\left(x\right) > 0$. The same obviously hold for $\epsilon_{k\omega}$ and the quantization (L^k, H^k) of $(X, k\omega)$. We can now use the following lemma from [22]:

Lemma. *Let (X, ω) be a homogeneous and simply connected Kähler manifold, with ω integral. If (L, H) is a quantization of (X, ω), then for every $k \in \mathbb{N}$ sufficiently large we have $\mathcal{H}_{\left(L^k, H^k\right)} \neq \{0\}$.*

By the above considerations, this lemma ensures that $(X, k\omega)$ is balanced for every $k \in \mathbb{N}$, so by theorem 114 it follows that $(X, k\omega)$ can be Kähler immersed into some $\left(\mathbb{CP}^N, \omega_{FS}\right)$.

$\square$

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