Università degli Studi di Cagliari
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Margulis Lemmas without curvature

Referenze e background della conferenza
"Un Lemma di Margulis senza ipotesi sulla curvatura"
del Visiting Professor Sylvestre Gallot

1 The classical Margulis Lemma :

Theorem 1.1 *Let* (M^n, g) *be a Riemannian manifold whose sectional curvature* σ *satisfies* $-1 \leq \sigma < 0$ *, and let* $\text{inj}(x)$ *be its injectivity radius at the point* x *, then:*

(i) (Margulis, see [6]) $\text{Sup}_{x \in M} (\text{inj}(x)) \geq \varepsilon_0$ *, where* $\varepsilon_0 = 4^{-(n+3)}$.

(ii) (Gromov, see [7]) *if moreover the diameter is bounded from above by a constant* D *, then*

$$\text{Inf}_{x \in M} (\text{inj}(x)) \geq \pi \, \frac{\text{Vol}\,\mathbb{B}^n}{\text{Vol}\,\mathbb{S}^n} \, \frac{\varepsilon_0^n}{(\sinh D)^{n-1}} \quad ,$$

Let $l_g(x)$ be the minimal length of the loops with base-point x which are not homotopic to zero then, under the assumptions of the Theorem1.1, one has $\text{inj}(x) = \frac{1}{2} \, l_g(x)$.

If one splits the manifold into 2 subsets (cf. [5]): the set $\{x \in M^n \ : \ l_g(x) < \frac{\varepsilon_0}{2}\}$, denoted the *thin part*, and its complement, denoted the *thick part* of the manifold M^n, one obtains:

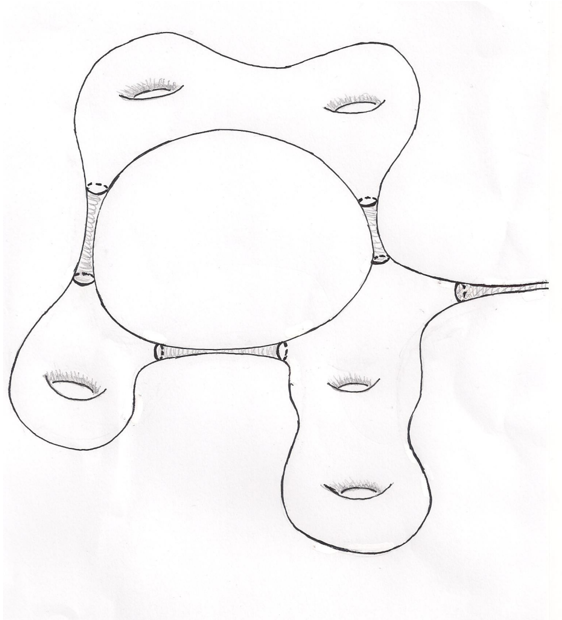


Figure 1: Thick part and thin part (streaked)

Corollary 1.2 *Under the assumptions of the above theorem 1.1,*

- *the thick part is not empty (by Theorem 1.1),*
- *every connected component of the thin part has the homotopy of a "cylinder"* $\mathbb{R}^n/\mathbb{Z}$ *,*
- *The smaller is* $l_g(x)$ *, the longer is this cylinder (more precisely the connected component which contains* x *contains a tubular neighbourhood of a small periodic geodesic with great radius).*

2 Why it is not totally satisfactory :

Any hyperbolic manifold (M^n, g_0) (Figure 2) satisfies the assumptions of the classical Margulis lemma (i. e. its sectional curvature σ satisfies $-1 \leq \sigma < 0$)

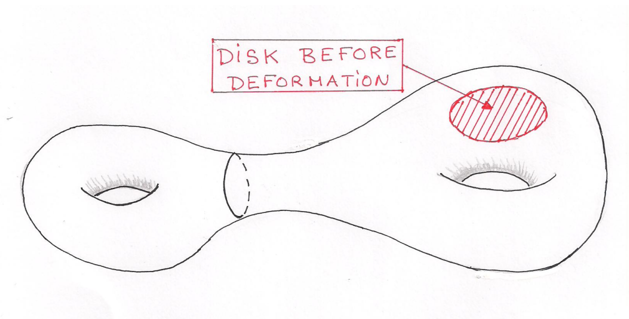


Figure 2: The hyperbolic manifold (M^n, g_0)

Let us now deform the metric g_0 in a new metric g_ε as follows: excise small balls $B_0 = B(x_0, \varepsilon)$ and B_1 of radius ε from M^n and from the sphere respectively, glue the two ends of a thin cylinder $[0,1] \times \mathbb{S}^{n-1}(\varepsilon)$ to the boundaries of B_0 and B_1 and smooth the new metric near the gluings (see Figure 3) in such a way that the new metric g_ε and the initial metric g_0 coincide on $M \setminus B(x_0, 2\,\varepsilon)$. The new metric does not satisfy the assumptions of the classical Margulis lemma : in fact its sectional curvature goes to $-\infty$ at the two ends of the thin cylinder when the radius ε of this cylinder goes to zero.

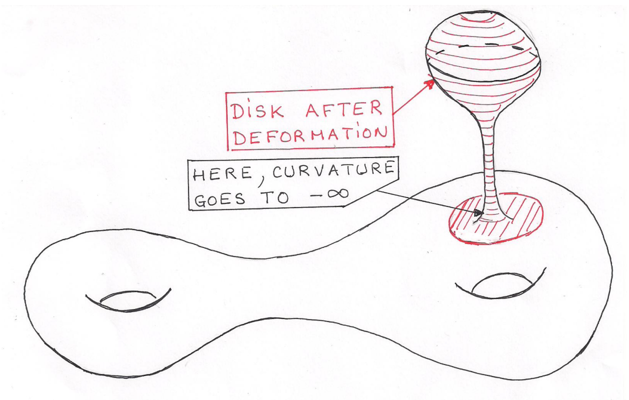


Figure 3: The same manifold, with a modified metric g_ε

However, as any loop of the deformed set (in streaked on the Figure 3) is homotopic to a point, the new metric g_ε still satisfies

$$\sup_x \left(l_{g_\varepsilon}(x)\right) \geq \varepsilon_0 \quad .$$

We should want a new version of the Margulis Lemma such that this inequality could be deduced from it (in other words, we should want the above deformed metric g_ε to satisfy the assumptions of this new lemma). Thus, we have to avoid curvature assumptions.

3 Question :

3.1 A naive formulation of the question :

Question 3.1 *If a manifold (X, g_0) satisfies the assumptions of the classical Margulis Lemma (Theorem 1.1), does any Riemannian manifold (M, g) whose fundamental group $\pi_1(M)$ is isomorphic to $\pi_1(X)$ also satisfy*

(i) $\mathrm{Sup}_{x\in M}\ l_g(x) \geq \varepsilon_0$ *?*

(ii) $\mathrm{Inf}_{x\in M}\ l_g(x) \geq \varepsilon_0'$ *, if moreover* $\mathrm{diameter}(M, g) \leq D$ *?*

where ε_0 and ε_0' are universal strictly positive constants (depending on what bounds on the geometry of (M, g) ? on the geometry of (X, g_0) ?).

In other words, *are Margulis properties (i) and (ii) only depending on algebraic properties of the fundamental group of M and not depending on the metric g on M ?*

Notice that the metric g on M needs to be rescaled, however we should have

$$\mathrm{Sup}_{x\in M}\ l_{\varepsilon^2 . g}(x) = \varepsilon\ \mathrm{Sup}_{x\in M}\ l_g(x) \to 0 \quad \text{when} \quad \varepsilon \to 0_+ \ .$$

We have seen that the rescaling $-1 \leq \sigma_g < 0$ which is assumed in the classical Margulis Lemma (Theorem 1.1) is a too strong assumption (because too much sensitive to local changes of the metric g, see section 2).

The idea is to rescale by the *Volume Entropy* instead of the sectional curvature...

3.2 The (Volume) Entropy :

Definition 3.2 (see for example [8], [9], [1] and [10]) *Let (M, g) be a compact Riemannian manifold and let $\pi : \left(\widetilde{M}, \tilde{g}\right) \to (M, g)$ be its Riemannian universal covering (i. e. $\tilde{g} = \pi^* g$); if $\widetilde{B}(\tilde{x}, R)$ denotes the geodesic ball in $\left(\widetilde{M}, \tilde{g}\right)$, one defines the Volume Entropy of (M, g) (that we shall call "Entropy" for the sake of simplicity and denote by* $\mathrm{Ent}(M, g)$ *) as*

$$\mathrm{Ent}(M, g) := \lim_{R\to+\infty} \left(\frac{1}{R}\ Log \left[Vol\, \widetilde{B}(\tilde{x}, R) \right] \right)$$

This limit exists and does not depend on the choice of the center $\tilde{x}$ of the balls.

Examples : If Λ and Γ are uniform lattices in the Euclidean space $\mathbb{R}^n$ and in the Hyperbolic space $\mathbb{H}^n$ respectively, then

- $\mathrm{Ent}(\mathbb{S}^n, g) = 0$ for every metric g (obvious),
- $\mathrm{Ent}(\mathbb{R}^n/\Lambda, g) = 0$ for every metric g (because of the polynomial growth of the volume of balls, see [8]),
- $\mathrm{Ent}(\mathbb{H}^n/\Gamma, can) = n - 1$ and $\mathrm{Ent}(\mathbb{H}^n/\Gamma, g) > n - 1$ for any other metric g with the same volume as the canonical hyperbolic metric *can* (cf. [1]).

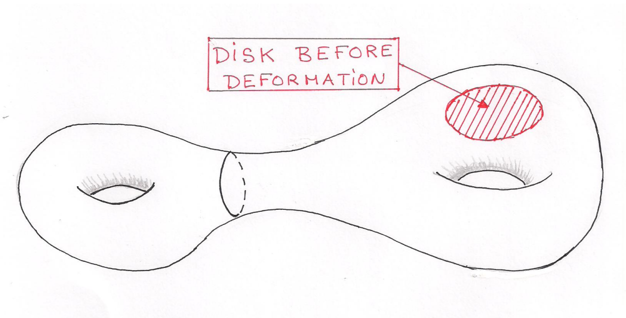
3.3 Why do we choose to rescale by the Entropy ?

a) **The assumption "Bounded Entropy" is much weaker than the assumption "curvature bounded from below":** In fact, for every Riemannian manifold (M^n, g),

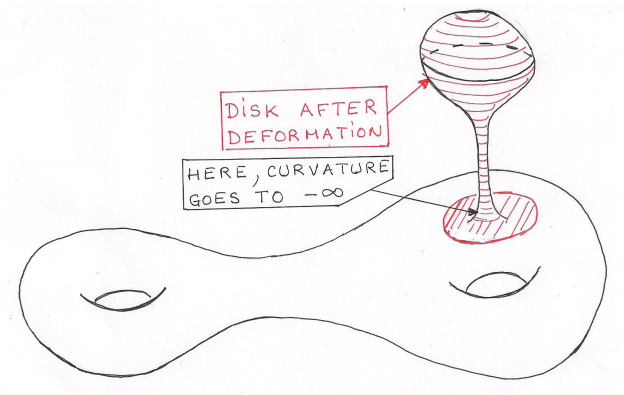
$$\sigma \geq -K^2 \implies \text{Ricci} \geq -(n-1)K^2 \implies \text{Ent}(M,g) \leq (n-1)K$$

where "Ricci" denotes the Ricci curvature[1].

b) **The entropy is not sensitive to local changes of the metric** (even when these changes are drastic) :
Let us for example consider any hyperbolic manifold (M^n, g_0)



and let us deform the metric g_0 in a new metric g_ε as in the section 2.



The initial manifold (M^n, g_0) has curvature and entropy bounded. We have seen in section 2 that the modified manifolds (M^n, g_ε) do not satisfy the assumption "curvature bounded" of the classical Margulis Lemma. On the contrary, the modified manifolds (M^n, g_ε) satisfy the assumption "entropy bounded", because of the following:

Fact : *The entropy of (M^n, g_ε) is almost equal to the entropy of (M^n, g_0).*

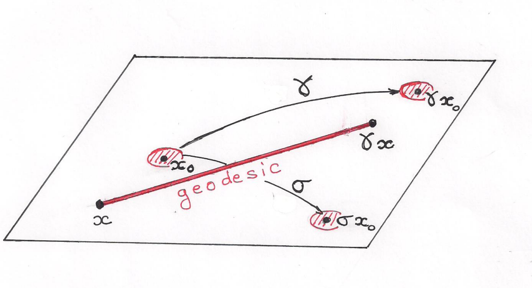


Figure 4: The universal covering of (M^n, g_0)

Idea of the proof of this fact: If the (Riemannian) universal covering of the initial manifold (M, g_0) looks like this:

Then the (Riemannian) universal covering of the (locally) modified manifold (M, g_ε) looks like this :

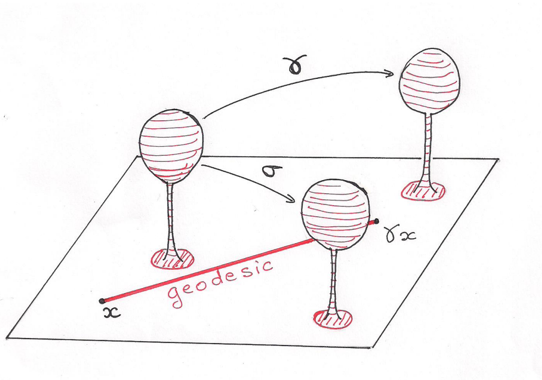


Figure 5: The universal covering of (M^n, g_ε)

More precisely, let $\pi : \left(\widetilde{M}, \tilde{g}_0\right) \to (M, g_0)$ be the Riemannian universal covering of (M^n, g_0) ; let $B = B(x_0, 2\varepsilon)$ be the geodesic ball of (M^n, g_0) which is the support of the deformation (i. e. such that the two metrics g_0 and g_ε coincide on $M^n \setminus B(x_0, 2\varepsilon)$). We fix a point in $\pi^{-1}(x_0)$, that we shall still denote by x_0 for the sake of simplicity. When ε is sufficiently small, $\pi^{-1}(B)$ is equal to the disjointed union $\cup_{x_i \in \pi^{-1}(x_0)} \widetilde{B}(x_i, 2\,\varepsilon)$, where the $\widetilde{B}(x_i, 2\,\varepsilon)$'s are geodesic balls of $\left(\widetilde{M}, \tilde{g}_0\right)$. As the fundamental group Γ of M^n acts on $\left(\widetilde{M}, \tilde{g}_0\right)$ by isometries, one may rewrite (cf. Figure 4):

$$\pi^{-1}(B) = \cup_{\gamma \in \Gamma} \widetilde{B}(\gamma\, x_0, 2\,\varepsilon) = \cup_{\gamma \in \Gamma} \gamma \left(\widetilde{B}(x_0, 2\,\varepsilon)\right) .$$

The Riemannian universal covering $\left(\widetilde{M}, \tilde{g}_\varepsilon\right)$ of the modified manifold (M, g_ε) may

[1]The first implication comes from the fact that the Ricci curvature is a trace of the curvature tensor, the second implication from the fact that the Ricci curvature controls the volumes of balls by the celebrated theorem of R. L. Bishop.

be constructed as follows : we first modify the metric $\tilde{g}_0$ in the neighbourhood $\widetilde{B}(x_0, 2\,\varepsilon)$ of $x_0 \in \widetilde{M}$ by excising ε-balls $\widetilde{B}_0 = \widetilde{B}(x_0, \varepsilon)$ and B_1 from $(\widetilde{M}, \tilde{g}_0)$ and from the sphere respectively, by gluing the two ends of a thin cylinder $[0,1] \times \mathbb{S}^{n-1}(\varepsilon)$ to the boundaries of $\widetilde{B}_0$ and B_1 and smoothing the new metric near the gluings : we thus construct a "mushroom" in the neighbourhood of x_0. We then repeat this process, constructing a similar "mushroom" in the neighbourhood of each point x_i of $\pi^{-1}(x_0)$: we thus obtain a new metric $\tilde{g}_\varepsilon$ such that the action of each element γ of the fundamental group Γ on $(\widetilde{M}, \tilde{g}_\varepsilon)$ is still isometric (see Figure 5).
Let $\tilde{d}_0$ and $\tilde{d}_\varepsilon$ be the distances associated to the Riemannian metrics $\tilde{g}_0$ and $\tilde{g}_\varepsilon$ on $\widetilde{M}$, there exists a point $x \in \widetilde{M} \setminus \pi^{-1}(B)$ such that [2]:

$$\tilde{d}_0(x, \gamma\, x) \leq \tilde{d}_\varepsilon(x, \gamma\, x) \leq (1 + C\,\varepsilon)\, \tilde{d}_0(x, \gamma\, x) \ ,$$

and thus we get

$$\mathrm{Entropy}(M, d_0) \geq \mathrm{Entropy}(M, d_\varepsilon) \geq \frac{1}{1 + C\,\varepsilon}\ \mathrm{Entropy}(M, d_0) \ .$$

c) **The entropy may be defined in the general setting of metric measured spaces :** This is quite evident : in the definition 3.2, just replace the Riemannian distance by the current distance and the Riemannian measure by the current measure (see [10] for more details). We shall see an example of such a generalization of the Entropy when we shall define the Algebraic Entropy of a discrete group.
This extension of the notion of Entropy fits very well with the convergence theory, because a Cauchy sequence of Riemannian manifolds converges to a metric measured space (with respect to the Gromov-Hausdorff distance between metric spaces) and because the entropy passes to the limit (in fact, it is a uniformly continuous function of the metric with respect to the Gromov-Hausdorff distance between metric spaces, see [10]).

3.4 Thus the "naive" question becomes :

Question 3.3 *Let (X, g_0) be a manifold which satisfies the assumptions of the Margulis Lemma, does any Riemannian manifold (M, g), with bounded entropy (for the sake of simplicity, rescaling eventually g, let us suppose that $\mathrm{Ent}(M, g) \leq 1$) and whose fundamental group $\pi_1(M)$ is isomorphic to $\pi_1(X)$, also satisfy*

(i) $\mathrm{Sup}_{x \in M}\ l_g(x) \geq \varepsilon_0(X)$.

(ii) $\mathrm{Inf}_{x \in M}\ l_g(x) \geq \varepsilon_0'(X, D)$ *, if moreover* $\mathrm{diameter}(M, g) \leq D$.

The answer is YES (see later).
Notice that **no curvature assumption** is made on (M, g). The problem is to get $\varepsilon_0(X)$ as independant on X as possible; ideally it should only depend on algebraic properties of $\pi_1(X)$.

[2]The first inequality comes from the fact that $\tilde{g}_0 \leq \tilde{g}_\varepsilon$, the second inequality comes from the fact that, if $\tilde{c}$ is a $\tilde{g}_0$-minimizing-geodesic from x to γx, there exists a path from x to γx in $\widetilde{M} \setminus \pi^{-1}(B)$ whose lengths with respect to the two metrics $\tilde{g}_\varepsilon$ and $\tilde{g}_0$ are the same and are smaller than $(1 + C\,\varepsilon)\,\mathrm{length}(\tilde{c})$, (see Figure 5).

4 A first result :

Theorem 4.1 *(Besson, Courtois, G., see [2]) Let* (X^p, g_0) *be a compact Riemannian manifold with strictly negative curvature (* $\sigma \leq -a^2 < 0$ *). Let* $\delta = a.\mathrm{inj}(g_0)$ *.*
Let M^n *be a compact manifold such that there exists an injective morphism :* $\pi_1(M) \to \pi_1(X)$ *with non abelian image.*
Then, for any metric g *on* M^n *such that* $\mathrm{Ent}(M^n, g) \leq 1$ *, one has :*

(i) $\mathrm{Sup}_{x\in M}\ l_g(x) \geq \dfrac{\delta\, Log\, 2}{4+\delta}$,

(ii) let $\varepsilon_0 = \dfrac{\delta\, Log\, 2}{4+\delta}$ *. If one defines the "thin part" of* (M^n, g) *as the set* $\{x \ :\ l_g(x) < \varepsilon_0/2\ \}$ *, then every connected component* M_{ε_0} *of this thin part "has the homotopy of a cylinder", i. e. for* $m \in M_{\varepsilon_0}$ *, the canonical injection* $i\ :\ M_{\varepsilon_0} \to M$ *induces* $i_*\ :\ \pi_1(M_{\varepsilon_0}, m) \to \pi_1(M, m)$ *whose image is isomorphic to* $\mathbb{Z}$ *,*

(iii) every connected component M_{ε_0} *of the "thin part" contains (at least) one periodic geodesic (denoted* c *), whose length is the minimum on* M_{ε_0} *of the fonction* $x \mapsto l_g(x)$ *.*
Let us denote by ε *the length of* c *and define*

$$R_\varepsilon = \frac{\delta}{2(4+\delta)}\ Log\left(\frac{1}{\varepsilon}\right)\ -\ Log\left(\frac{4+\delta}{\delta}\right)\ -\ \frac{\varepsilon}{2}\ ,$$

then the R_ε*-tubular neighbourhood of* c *is isometric to the the* R_ε*-tubular neighbourhood of a periodic geodesic in a quotient of the universal covering* $(\widetilde{M}, \tilde{g})$ *by a subgroup isomorphic to* $\mathbb{Z}$ *in* $\pi_1(M)$ *,*

(iv) if moreover $\mathrm{diameter}(M, g) \leq D$ *, then*

$$\mathrm{Inf}_{x\in M}\, [\, l_g(x)\,] \geq \frac{\delta}{4+\delta}\ e^{-\dfrac{2(4+\delta)}{\delta}D}\ .$$

We shall see that this theorem deduces from the properties of the fundamental group of M (denoted by Γ) acting on the Riemannian universal covering $(\widetilde{M}, \tilde{g})$ by isometries.

Applications of this theorem are compactness and precompactness results on manifolds and finiteness results on Einstein manifolds ([2]). Proving a version of this theorem generalized to other groups Γ (in particular to Gromov hyperbolic groups, limit groups and amalgamated products) and to the case where M is the Cayley graph of a discrete group, F. Zuddas obtained finiteness results for groups ([11]).

4.1 Before doing the proof :

The main idea : The fundamental group Γ of M acts by isometries on the universal covering $(\widetilde{M}, \tilde{g})$ of (M^n, g), but also on the universal covering $(\widetilde{X}, \widetilde{g}_0)$ of (X, g_0), because any element γ of Γ may be considered both as an element of the fundamental group of M and as an element of the fundamental group of X (via the morphism $\pi_1(M) \to \pi_1(X)$). As the curvature of (X, g_0) satisfies $\sigma \leq -a^2$, the idea is that Γ inherits from its action on $(\widetilde{X}, \widetilde{g}_0)$ some algebraic property which is sufficient to prove that the other action of Γ on $(\widetilde{M}, \tilde{g})$ also enjoys the conclusions of the Margulis Lemma.

This algebraic property is a **quantified Tits alternative** :

Let Σ be a finite system of generators of Γ : for any $\gamma \in \Gamma$ and any $\gamma' \in \Gamma$, one can write $\gamma^{-1} \cdot \gamma'$ as a word $\sigma_{i_1} \cdot \sigma_{i_2} \cdot ... \cdot \sigma_{i_p}$ whose letters σ_{i_1}, σ_{i_2}, ..., σ_{i_p} belong to $\Sigma \cup \Sigma^{-1}$: by definition the algebraic distance $d_\Sigma(\gamma, \gamma')$ is the minimal length of all the words $\sigma_{i_1} \cdot \sigma_{i_2} \cdot ... \cdot \sigma_{i_p}$ equal to $\gamma^{-1} \cdot \gamma'$.

Definition 4.2 *(see for example [8] and [11]) The algebraic entropy of* Γ *associated to* Σ *is the entropy of the measured metric space* Γ *, endowed with the distance* d_Σ *and with the counting measure, more precisely :*

$$\mathrm{Ent}_\Sigma(\Gamma) = \lim_{R \to \infty} \frac{1}{R} \, Log\left(\#\{\gamma \ : \ d_\Sigma(e, \gamma) < R\}\right)$$

And the quantified Tits alternative writes

Definition 4.3 *(see [3] and [4]) Let* $\alpha > 0$ *; the group* Γ *has the property* $\mathrm{Tits}(\alpha)$ *iff it is not virtually nilpotent and if, for any finite subset* $\Sigma' \subset \Gamma - \{e\}$ *, the subgroup* $< \Sigma' >$ *generated by* Σ' *is*

- *either virtually nilpotent,*
- *either satisfying* $\mathrm{Ent}_{\Sigma'}(< \Sigma' >) \geq \alpha$ *.*

4.2 A sketch of the proof of the part (i) of Theorem 4.1:

Lemma 4.4 *The fundamental group* Γ *of a compact Riemannian manifold* (X^p, g_0) *whose sectional curvature and injectivity radius satisfy* $\sigma \leq -1$ *and* $\mathrm{inj}(g_0) \geq \delta$ *has the property* $\mathrm{Tits}(\alpha)$ *for* $\alpha = \dfrac{\delta \, Log\, 2}{4 + \delta}$ *.*

For the proof of this Lemma, see [2].

From now on, we suppose that Γ has the property $\mathrm{Tits}(\alpha)$.

Lemma 4.5 *Considering the action of* Γ *on* $(\widetilde{M}, \tilde{g})$ *, let us denote by* $\Gamma_\alpha(\tilde{x})$ *the subgroup of* Γ *generated by those* $\gamma \in \Gamma$ *such that* $d_{\widetilde{M}}(\tilde{x}, \gamma\,\tilde{x}) < \alpha$ *. Then, for every* $\tilde{x} \in \widetilde{M}$ *, the subgroup* $\Gamma_\alpha(\tilde{x})$ *is abelian (thus, here, it is isomorphic to* $\mathbb{Z}$ *or* $\{0\}$ *).*

Proof : Let $\Sigma_\alpha(\tilde{x})$ be the (finite) set whose elements are those $\gamma \in \Gamma - \{e\}$ such that $d_{\widetilde{M}}(\tilde{x}, \gamma\,\tilde{x}) < \alpha$, then $\Gamma_\alpha(\tilde{x})$ is the subgroup generated by $\Sigma_\alpha(\tilde{x})$. Suppose that there exists a pair of elements σ_1 and σ_2 in $\Sigma_\alpha(\tilde{x})$ which do not commute, they thus satisfy $L < \alpha$, where $L = \mathrm{Max}\left[d_{\widetilde{M}}(\tilde{x}, \sigma_1\,\tilde{x}); d_{\widetilde{M}}(\tilde{x}, \sigma_2\,\tilde{x})\right]$.
Let $\Gamma' =< \sigma_1 \, , \sigma_2 >$ endowed with the system of generators $\{\sigma_1 \, , \sigma_2\}$ and with the corresponding algebraic distance d_{alg}. The geometric distance on Γ' beeing defined by $d_{geom}(e, \gamma) = d_{\widetilde{M}}(\tilde{x}, \gamma\,\tilde{x})$, the triangle inequality gives : $d_{geom} \leq L \, . \, d_{alg}$ and thus :

$$1 \geq \mathrm{Ent}(M, g) \geq \mathrm{Ent}(\Gamma'.\tilde{x}, d_{\widetilde{M}}) = \mathrm{Ent}(\Gamma', d_{geom})$$

$$\geq \frac{1}{L}\ \mathrm{Ent}(\Gamma', d_{alg}) = \frac{1}{L}\ \mathrm{Ent}_{\{\sigma_1\,,\sigma_2\}}(\Gamma') \geq \frac{\alpha}{L}\quad ,$$

where the last inequality deduces from the Lemma 4.4, from the definition 4.3 and from the fact that σ_1 and σ_2 do not commute which, in the present case, implies that Γ' is not virtually nilpotent. This last series of inequalities is in contradiction with the supposition $L < \alpha$, and thus all the elements of $\Sigma_\alpha(\tilde{x})$ commute, which implies that $\Gamma_\alpha(\tilde{x})$ is abelian.

Lemma 4.6 *There exists a point $\widetilde{x}_0 \in \widetilde{M}$ such that $\Gamma_\alpha(\widetilde{x}_0)$ reduces to $\{e\}$.*

Proof : Let $\Sigma_\alpha(\tilde{x})$ still be the set whose elements are those $\gamma \in \Gamma$ such that $d_{\widetilde{M}}(\tilde{x}, \gamma\,\tilde{x}) < \alpha$. Suppose that the conclusion of the Lemma 4.6 is false then, for every $\tilde{x} \in \widetilde{M}$, $\Gamma_\alpha(\tilde{x})$ is a *non trivial abelian* subgroup. Let us define the following equivalence relation $\equiv$ on $\Gamma^* = \Gamma - \{e\}$ by:

$\gamma_1 \equiv \gamma_2$ iff γ_1 and γ_2 commute.

and let $C(\tilde{x})$ be the equivalence class which contains $\Gamma_\alpha(\tilde{x})$. When $\tilde{x}$ is fixed and when $\tilde{y}$ lies in a sufficiently small neighbourhood of $\tilde{x}$, then $d_{\widetilde{M}}(\tilde{x}, \gamma\,\tilde{x}) < \alpha$ implies $d_{\widetilde{M}}(\tilde{y}, \gamma\,\tilde{y}) < \alpha$ (here we use the fact that $\Sigma_\alpha(\tilde{x})$ is finite), and thus $\Sigma_\alpha(\tilde{x}) \subset \Sigma_\alpha(\tilde{y})$, and thus $C(\tilde{y}) = C(\tilde{x})$. We have proved that *the map $\tilde{x} \mapsto C(\tilde{x})$ is locally constant, and thus constant.* Let us denote $C_0 = C(\tilde{x})$. Then $\Gamma_0 = C_0 \cup \{e\}$ is an abelian group containing $\Sigma_\alpha(g\,\tilde{y}) = g\,\Sigma_\alpha(\tilde{y})\,g^{-1}$ for every g , thus $\Gamma_0 = g\,\Gamma_0\,g^{-1}$ *is an abelian normal subgroup.* In the present case, this implies that $\Gamma_0 = \Gamma$ or $\{e\}$, in contradiction with the fact that Γ is non abelian and that Γ_0 contains $\Sigma_\alpha(\tilde{y}) \neq \{e\}$.

End of the proof of the part (i) of the Theorem 4.1 :
By the Lemma 4.6, one can find some point $\widetilde{x}_0 \in \widetilde{M}$ such that $\Gamma_\alpha(\widetilde{x}_0) = \{e\}$; this implies that $\Sigma_\alpha(\widetilde{x}_0) = \emptyset$ and thus that, for every $\gamma \in \Gamma^*$, $d_{\widetilde{M}}(\widetilde{x}_0, \gamma\,\widetilde{x}_0) \geq \alpha$.
As $l_g(x_0) = \inf_{\gamma\in\Gamma^*}\ d_{\widetilde{M}}(\widetilde{x}_0, \gamma\,\widetilde{x}_0)$ (for any projection x_0 of $\widetilde{x}_0$), one gets

$$l_g(x_0) \geq \alpha = \frac{\delta\,Log\,2}{4+\delta}\quad .$$

5 Coming back to the assumptions of Theorem 4.1:

- A good point is the fact that ε_0 does not depend on M nor on g.
- A bad point is that it depends on a lower bound of the injectivity radius of (X, g_0) .

We get rid of this last dependance by the

Proposition 5.1 *(see [3] and [4]) Let (X^n, g) be a Cartan-Hadamard manifold whose sectional curvature satisfies $-1 \leq \sigma \leq -a^2$, let Γ be a finitely generated discrete, non virtually nilpotent group of isometries, then Γ satisfies the property* Tits(α) *for $\alpha = \varepsilon_0(n, a)$.*

Work in progress : a generalization of Theorems 5.1 and 4.1 to hyperbolic groups (improving previous results of F. Zuddas on hyperbolic groups, [11], [12]) and, more generally to groups acting on Gromov's δ-hyperbolic spaces.

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