Math. Ann. (2011) 350:145–154
DOI 10.1007/s00208-010-0554-y

Mathematische Annalen

Kähler–Einstein submanifolds of the infinite dimensional projective space

Andrea Loi · Michela Zedda

Received: 15 September 2008 / Revised: 22 June 2010 / Published online: 5 August 2010

Abstract This paper consists of two main results. In the first one we describe all Kähler immersions of a bounded symmetric domain into the infinite dimensional complex projective space in terms of the Wallach set of the domain. In the second one we exhibit an example of complete and non-homogeneous Kähler–Einstein metric with negative scalar curvature which admits a Kähler immersion into the infinite dimensional complex projective space.

Mathematics Subject Classification (2000) 53C55 · 58C25

1 Introduction and statement of the main results

This paper deals with holomorphic and isometric (from now on Kähler) immersions of complete noncompact Kähler–Einstein manifolds into $\mathbb{C}\mathrm{P}^\infty$, the infinite dimensional complex projective space equipped with the Fubini–Study metric g_{FS}. Throughout this paper if a Kähler manifold (M, g) admits a Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$ then we will say either that (M, g) is a *Kähler submanifold* of $\mathbb{C}\mathrm{P}^\infty$ or that g is *projectively induced*. The only known examples of projectively induced Kähler–Einstein metrics are the flat metric on the complex Euclidean space $\mathbb{C}^n$ (see [4]) and the Bergman metric on a bounded homogeneous domain (see [8]). Hence, it is natural to ask if there exists a complete nonhomogeneous Kähler–Einstein submanifold of $\mathbb{C}\mathrm{P}^\infty$. The following

A. Loi (✉) · M. Zedda
Dipartimento di Matematica e Informatica,
Università di Cagliari, Cagliari, Italy
e-mail: loi@unica.it

M. Zedda
e-mail: michela.zedda@gmail.com

theorem, which is the first result of this paper, gives a positive answer to this question (see Sect. 4 for the definition of Cartan–Hartogs domain).

Theorem 1 *There exists a continuous family of homothetic, complete, nonhomogeneous and projectively induced Kähler–Einstein metrics on each Cartan–Hartogs domain based on an irreducible bounded symmetric domain of rank $r \neq 1$.*

Our result should be compared with the compact case. First, it is an open problem to classify the compact Kähler–Einstein manifolds which admit a Kähler immersion into a finite dimensional complex projective space. Actually, the only known examples of such manifolds are homogeneous and it is conjecturally true these are the only ones (see e.g. [3,5,10,11]). Moreover, a family as in the previous theorem cannot exist in the compact case. Indeed if (M, g) admits a Kähler immersion into the finite dimensional complex projective space then, by simple topological reasons, (M, cg) also does iff c is a positive integer.

The proof of Theorem 1 is based on recent results (see [13,14]) about Einstein metrics on Cartan–Hartogs domains and on the following theorem which is the second result of this paper (see next section or [2] for the definition of the Wallach set $W(\Omega)$ of the domain Ω).

Theorem 2 *Let Ω be an irreducible bounded symmetric domain endowed with its Bergman metric g_B. Then (Ω, cg_B) admits a equivariant Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$ if and only if $c\gamma$ belongs to $W(\Omega)\setminus\{0\}$, where γ denotes the genus of Ω.*

The paper is organized as follows. In the next section we recall basic results on Calabi's diastasis function and Calabi's criterion for Kähler immersions into $\mathbb{C}\mathrm{P}^\infty$. In Sect. 3, after describing Calabi's diastasis function for the Bergman metric of a bounded symmetric domain, we prove Theorem 2. The last section is dedicated to the proof of Theorem 1.

2 The diastasis function and Calabi's criterion

In his seminal paper Calabi [4] (to whom we refer for details and further results) gives necessary and sufficient conditions for a n-dimensional Kähler manifold (M, g) to admit a Kähler immersion into a complex space form. The key tool is the introduction of a very particular Kähler potential $\mathrm{D}_p^M(z)$, that Calabi called *diastasis*. Recall that a Kähler potential is a smooth function Φ defined in a neighbourhood of a point p such that $\omega = \frac{i}{2}\bar{\partial}\partial\Phi$, where ω is the Kähler form associated to g. In a complex coordinate system (z) around p one has

$$\mathrm{g}_{\alpha\bar{\beta}} = 2\mathrm{g}\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right) = \frac{\partial^2\Phi}{\partial z_\alpha \partial \bar{z}_\beta}.$$

A Kähler potential is not unique: it is defined up to an addition with the real part of a holomorphic function. If (M, g) admits a Kähler immersion into a complex space form then g is real analytic (see Theorem 4 below). In this case by duplicating the variables z and $\bar{z}$ a potential Φ can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighbourhood U of the diagonal containing $(p, \bar{p}) \in M \times \bar{M}$ (here $\bar{M}$ denotes the manifold conjugated to M). The *diastasis function* is the Kähler potential D_p^M around p defined by

$$\mathrm{D}_p^M(q) = \tilde{\Phi}(q, \bar{q}) + \tilde{\Phi}(p, \bar{p}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p}).$$

Observe that $\mathrm{D}_p^M(q)$ is symmetric in p and q and $\mathrm{D}_p^M(p) = 0$. The following theorem provides us with a very useful characterization of the diastasis function.

Theorem 3 (characterization of the diastasis) *Among all the Kähler potentials the diastasis is characterized by the fact that in every coordinate system (z) centered in p, the $\infty \times \infty$ matrix of coefficients (a_{jk}) in its power expansion around the origin*

$$\mathrm{D}_p^M(z, \bar{z}) = \sum_{j,k=0}^{\infty} a_{jk}(z)^{m_j}(\bar{z})^{m_k}, \tag{1}$$

satisfies $a_{j0} = a_{0j} = 0$ for every nonnegative integer j.

Here we are using the following convention: we arrange every n-tuple of nonnegative integers as the sequence $m_j = (m_{j,1}, \ldots, m_{j,n})$ with nondecreasing order, that is $m_0 = (0, \ldots, 0)$, $|m_j| \leq |m_{j+1}|$, with $|m_j| = \sum_{\alpha=1}^{n} m_{j,\alpha}$, and $(z)^{m_j}$ denotes the monomial in n variables $\prod_{\alpha=1}^{n} z_\alpha^{m_{j,\alpha}}$. Further, we order all the m_j's with the same $|m_j|$ using the lexicographic order in the variables $(z_1, \ldots z_n)$. For example, for $n = 2$ we have, $m_0 = (0, 0) < m_1 = (1, 0) < m_2 = (0, 1) < m_3 = (2, 0) < m_4 = (1, 1) < m_5 = (0, 2)$ and so on.

The importance of the diastasis function for our purposes is expressed by the following three theorems due to Calabi. Recall that the Fubini–Study metric g_{FS} on the infinite dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$ is the metric whose Kähler form ω_{FS} in homogeneous coordinates $Z_0, \ldots Z_j, \ldots$ is given by $\omega_{FS} = \frac{i}{2}\partial\bar{\partial}\log(\sum_{j=0}^{\infty}|Z_j|^2)$. If $p_0 = [1, 0, 0, \ldots] \in U_0 = \{Z_0 \neq 0\}$, then in the affine coordinates $z_j = \frac{Z_j}{Z_0}$ the diastasis $\mathrm{D}_{p_0}^\infty : U_0 \to \mathbb{R}$ around p_0 is given by

$$\mathrm{D}_{p_0}^\infty(z) = \log\left(1 + \sum_{j=1}^{\infty}|z_j|^2\right).$$

Theorem 4 (hereditary property) *Let $f : (M, \mathrm{g}) \to \mathbb{C}\mathrm{P}^\infty$ be a Kähler immersion such that $f(p) = p_0$. Then the metric* g *is real analytic and $\mathrm{D}_p^M = \mathrm{D}_{p_0}^\infty \circ f : M \setminus f^{-1}(H_0) \to \mathbb{R}$, where $H_0 = \mathbb{C}\mathrm{P}^\infty \setminus U_0$.*

Theorem 5 (Calabi's rigidity) *Let $f_1, f_2 : (M, g) \to \mathbb{C}\mathrm{P}^\infty$ be two full Kähler immersions. Then there exists a unitary transformation U of $\mathbb{C}\mathrm{P}^\infty$ such that $f_2 = U \circ f_1$.*

Recall that a holomorphic map $f : (M, \mathrm{g}) \to \mathbb{C}\mathrm{P}^\infty$ is said to be *full* if $f(M)$ is not contained in any complex totally geodesic submanifold of $\mathbb{C}\mathrm{P}^\infty$.

Theorem 6 (Calabi's criterion) *A Kähler manifold* (M, g) *admits a local full Kähler immersion into* $\mathbb{C}\mathrm{P}^\infty$ *if and only if the* $\infty \times \infty$ *matrix of coefficients* (b_{jk}) *in the power expansion*

$$e^{\mathrm{D}_p^M(z,\bar{z})} - 1 = \sum_{j,k=0}^{\infty} b_{jk}(z)^{m_j}(\bar{z})^{m_k}, \tag{2}$$

is positive semidefinite of infinite rank. Moreover, if the manifold M *is assumed to be simply connected a local Kähler immersion can be extended to a global one.*

3 The diastasis function of bounded symmetric domains and the proof of Theorem 2

In the following proposition we describe the diastasis of a bounded symmetric domain and one of its important features which will be a key ingredient for the proof of our results (see also [9] for the global aspects of the diastasis function for these domains). Recall that the Bergman metric g_B is a Kähler metric on Ω whose associated Kähler form ω_B is given by $\omega_B = \frac{i}{2}\partial\bar{\partial}\log \mathrm{K}(z, z)$, where K is the reproducing kernel for the Hilbert space of holomorphic L^2-functions on Ω, namely those $f \in Hol(\Omega)$ such that $\int_\Omega |f|^2 d\mu(z) < \infty$, where $d\mu(z)$ is the standard Lebesgue measure on $\mathbb{C}^n$.

Proposition 7 *Let* Ω *be a bounded symmetric domain. Then the diastasis for its Bergman metric* g_B *around the origin is*

$$\mathrm{D}_0^\Omega(z) = \log(V(\Omega)\mathrm{K}(z, z)), \tag{3}$$

where $V(\Omega)$ *denotes the total volume of* Ω *with respect to the Euclidean measure of the ambient complex Euclidean space. Moreover the matrix* (b_{jk}) *given by* (2) *for* D_0^Ω *satisfies* $b_{jk} = 0$ *whenever* $|m_j| \neq |m_k|$.

Proof $\mathrm{D}_0^\Omega(z)$ is centered at the origin, in fact by the reproducing property of the kernel we have

$$\frac{1}{\mathrm{K}(0, 0)} = \int_\Omega \frac{1}{\mathrm{K}(\zeta, 0)}\mathrm{K}(\zeta, 0)d\mu,$$

hence $\mathrm{K}(0, 0) = 1/V(\Omega)$, and substituting in (3) we obtain $\mathrm{D}_0^\Omega(0) = 0$. By the circularity of Ω, that is $z \in \Omega$, $\theta \in \mathbb{R}$ imply $e^{i\theta}z \in \Omega$, rotations around the origin are automorphisms and hence isometries, that leave D_0^Ω invariant. Thus we have $\mathrm{D}_0^\Omega(z) = \mathrm{D}_0^\Omega(e^{i\theta}z)$ for any $0 \leq \theta \leq 2\pi$, that is, each time we have a monomial $(z)^{m_j}(\bar{z})^{m_k}$ in $\mathrm{D}_0^\Omega(z)$, we must have

$$(z)^{m_j}(\bar{z})^{m_k} = e^{i|m_j|\theta}(z)^{m_j}e^{-i|m_k|\theta}(\bar{z})^{m_k} = (z)^{m_j}(\bar{z})^{m_k}e^{(|m_j|-|m_k|)i\theta},$$

implying $|m_j| = |m_k|$. This means that every monomial in the expansion of $\mathrm{D}_0^\Omega(z)$ has holomorphic and antiholomorphic part with the same degree. Hence, by Theorem 3, $\mathrm{D}_0^\Omega(z)$ is the diastasis for g_B. By the chain rule the same property holds true for $e^{\mathrm{D}_0^\Omega(z)} - 1$ and the second part of the proposition follows immediately. □

Before proving Theorem 2 we recall the definition of the *Wallach set* of an irreducible bounded symmetric domain Ω of genus γ, referring the reader to [2,7] and [12] for more details and results. This set, denoted by $W(\Omega)$, consists of all $\lambda \in \mathbb{C}$ such that there exists a Hilbert space $\mathcal{H}_\lambda$ whose reproducing kernel is $\mathrm{K}^{\frac{\lambda}{\gamma}}$. This is equivalent to the requirement that $\mathrm{K}^{\frac{\lambda}{\gamma}}$ is positive definite, i.e. for all n-tuples of points $x_1, \dots, x_n$ belonging to Ω the $n \times n$ matrix $(\mathrm{K}(x_\alpha, x_\beta)^{\frac{\lambda}{\gamma}})$, is positive *semidefinite*. It turns out (see Corollary 4.4 p. 27 in [2] and references therein) that $W(\Omega)$ consists only of real numbers and depends on two of the domain's invariants, denoted by a (strictly positive real number) and r (the rank of Ω). More precisely we have

$$W(\Omega) = \left\{0, \frac{a}{2}, 2\frac{a}{2}, \dots, (r-1)\frac{a}{2}\right\} \cup \left((r-1)\frac{a}{2}, \infty\right). \tag{4}$$

The set $W_d = \left\{0, \frac{a}{2}, 2\frac{a}{2}, \dots, (r-1)\frac{a}{2}\right\}$ and the interval $W_c = \left((r-1)\frac{a}{2}, \infty\right)$ are called respectively the *discrete* and *continuous* part of the Wallach set of the domain Ω.

Remark 8 When Ω has rank $r = 1$, namely Ω is the complex hyperbolic space $\mathbb{C}\mathrm{H}^d$, then $g_B = (d+1)g_{hyp}$, where g_{hyp} is the hyperbolic metric on $\mathbb{C}\mathrm{H}^d$. In this case (and only in this case) $W_d = \{0\}$ and $W_c = (0, \infty)$. Therefore our Theorem 2 is asserting that, for all positive constants c, $(\mathbb{C}\mathrm{H}^d, cg_{hyp})$ admits a full Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$. This is well-known and can also be proved by using Calabi's criterion (see Theorem 13 in [4]).

Proof of Theorem 2 Let $f : (\Omega, cg_B) \to \mathbb{C}\mathrm{P}^\infty$ be a Kähler immersion, we want to show that $c\gamma$ belongs to $W(\Omega)$, i.e. K^c is positive definite. Since Ω is contractible it is not hard to see that there exists a sequence f_j, $j = 0, 1\dots$ of holomorphic functions defined on Ω, not vanishing simultaneously, such that the immersion f is given by $f(z) = [\dots, f_j(z), \dots]$, $j = 0, 1\dots$, where $[\dots, f_j(z), \dots]$ denotes the equivalence class in $\ell^2(\mathbb{C})$ (two sequences are equivalent iff they differ by the multiplication by a nonzero complex number). Let $x_1, \dots, x_n \in \Omega$. Without loss of generality (up to unitary transformation of $\mathbb{C}P^\infty$) we can assume that $f(0) = e_1$, where e_1 is the first vector of the canonical basis of $\ell^2(\mathbb{C})$, and $f(x_j) \notin H_0$, $\forall\, j = 1, \dots, n$. Therefore, by Theorem 4 and Proposition 7, we have

$$c\mathrm{D}_0^\Omega(z) = \log[V(\Omega)^c\, \mathrm{K}^c(z, z)] = \log\left(1 + \sum_{j=1}^\infty \frac{|f_j(z)|^2}{|f_0(z)|^2}\right), \quad z \in \Omega \backslash f^{-1}(H_0).$$

$$V(\Omega)^c\, \mathrm{K}^c(x_\alpha, x_\beta) = 1 + \sum_{j=1}^\infty g_j(x_\alpha)\overline{g_j(x_\beta)}, \quad g_k = \frac{f_k}{f_0}.$$

Thus for every $(v_1, \dots, v_n) \in \mathbb{C}^n$ one has

$$\sum_{\alpha,\beta=1}^{n} v_\alpha \mathrm{K}^c(x_\alpha, x_\beta)\bar{v}_\beta = \frac{1}{V(\Omega)^c}\sum_{k=0}^{\infty}|v_1 g_k(x_1) + \cdots + v_n g_k(x_n)|^2 \geq 0, \quad g_0 = 1,$$

and hence the matrix $(\mathrm{K}^c(x_\alpha, x_\beta))$ is positive semidefinite. Conversely, assume that $c\gamma \in W(\Omega)$. Then, by the very definition of Wallach set, there exists a Hilbert space $\mathcal{H}_{c\gamma}$ whose reproducing kernel is $\mathrm{K}^c = \sum_{j=0}^{\infty}|f_j|^2$, where f_j is an orthonormal basis of $\mathcal{H}_{c\gamma}$. Then the holomorphic map $f : \Omega \to \ell^2(\mathbb{C}) \subset \mathbb{C}\mathrm{P}^\infty$ constructed by using this orthonormal basis satisfies $f^*(g_{FS}) = cg_B$. In order to prove that this map is equivariant write $\Omega = G/K$ where G is the simple Lie group acting holomorphically and isometrically on Ω and K is its isotropy group. For each $h \in G$ the map $f \circ h : (\Omega, cg_B) \to \mathbb{C}\mathrm{P}^\infty$ is a full Kähler immersion and therefore by Calabi's rigidity (Theorem 5) there exists a unitary transformation U_h of $\mathbb{C}\mathrm{P}^\infty$ such that $f \circ h = U_h \circ f$ and we are done. □

Remark 9 In [2] it is proven that if λ belongs to $W(\Omega)\backslash\{0\}$ then G admits a representation in the Hilbert space $\mathcal{H}_\lambda$. This is in accordance with our result. Indeed if $c\gamma$ belongs to $W(\Omega)\backslash\{0\}$ then the correspondence $h \mapsto U_h, h \in G$ defined in the last part of the proof of Theorem 2 is a representation of G.

We conclude this section with some remarks regarding Kähler immersions of bounded symmetric domains into other complex space forms different from $\mathbb{C}\mathrm{P}^\infty$.

Remark 10 The multiplication of the Bergman metric by c in Theorem 2 is harmless when one studies the Kähler immersions of a Kähler manifold (M, g) into the infinite dimensional complex Euclidean space $\ell^2(\mathbb{C})$ equipped with the flat metric g_0. Indeed if $f : M \to \ell^2(\mathbb{C})$ satisfies $f^*(g_0) = g$ then $(\sqrt{c}f)^*(g_0) = cg$. It is worth pointing out that the only bounded symmetric domain which admits a Kähler immersion into $\ell^2(\mathbb{C})$ has rank one, i.e. it is the product of complex hyperbolic spaces (see [6] for a proof). Actually the authors believe the validity of the following conjecture: *A complete Kähler manifold with negative scalar curvature which admits a Kähler immersion into $\ell^2(\mathbb{C})$ is a bounded symmetric domain of rank one.*

Remark 11 For the case of Kähler immersions of bounded symmetric domains of noncompact type into the finite (resp. infinite) dimensional complex hyperbolic space $\mathbb{C}\mathrm{H}^N$ (resp. $\mathbb{C}\mathrm{H}^\infty$), namely the unit ball in $\mathbb{C}^n$ (resp. $\ell^2(\mathbb{C})$) equipped with the hyperbolic metric g_{hyp}, we can prove the following theorem: *if a n-dimensional bounded symmetric domain (Ω, cg_B) admits a Kähler immersion into $\mathbb{C}\mathrm{H}^N$ (resp. $\mathbb{C}\mathrm{H}^\infty$) then $\Omega = \mathbb{C}\mathrm{H}^n$, $g_B = g_{hyp}$ and $c = 1$ (resp. $c \leq 1$).* The proof follows easily by Theorem 17 in [1] where it is shown that a Kähler manifold which admits a Kähler immersion into a complex hyperbolic space is locally irreducible (for the values of c see Theorem 13 in [4]).

4 Cartan–Hartogs domains and the proof of Theorem 1

In order to prove Theorem 1 we briefly recall some recent results about Einstein metrics on Cartan–Hartogs domains.

Let Ω be an irreducible bounded symmetric domain of complex dimension d and genus γ. For all positive real numbers μ consider the family of Cartan–Hartogs domains

$$\mathrm{M}_\Omega(\mu) = \left\{(z, w) \in \Omega \times \mathbb{C},\ |w|^2 < \mathrm{N}_\Omega^\mu(z, z)\right\}, \tag{5}$$

where $\mathrm{N}_\Omega(z, z)$ is the *generic norm* of Ω namely,

$$N_\Omega(z, z) = (V(\Omega)K(z, z))^{-\frac{1}{\gamma}}.$$

The domain Ω is called the *base* of the Cartan–Hartogs domain $\mathrm{M}_\Omega(\mu)$ (one also says that $\mathrm{M}_\Omega(\mu)$ is based on Ω). Consider on $\mathrm{M}_\Omega(\mu)$ the metric $\mathrm{g}(\mu)$ whose globally defined Kähler potential around the origin is given by

$$\mathrm{D}_0(z, w) = -\log(\mathrm{N}_\Omega^\mu(z, z) - |w|^2). \tag{6}$$

The following theorem summarizes what we need about these domains (see [13] and [14] for a proof.)

Theorem 12 (G. Roos, A. Wang, W. Yin, L. Zhang, W. Zhang) *Let* $\mu_0 = \gamma/(d + 1)$. *Then* $(\mathrm{M}_\Omega(\mu_0), \mathrm{g}(\mu_0))$ *is a complete Kähler–Einstein manifold which is homogeneous if and only if the rank of* Ω *equals* 1*, i.e.* $\Omega = \mathbb{C}\mathrm{H}^d$.

Remark 13 Observe that when $\Omega = \mathbb{C}\mathrm{H}^d$, we have $\mu_0 = 1$, $\mathrm{M}_\Omega(1) = \mathbb{C}\mathrm{H}^{d+1}$ and $\mathrm{g}(1) = g_{hyp}$ (cfr. Remark 8).

In the following proposition, interesting on its own sake, we describe the Kähler immersions of a Cartan–Hartogs domain into $\mathbb{C}\mathrm{P}^\infty$ in terms of its base.

Proposition 14 *The potential* $\mathrm{D}_0(z, w)$ *given by* (6) *is the diastasis around the origin of the metric* $\mathrm{g}(\mu)$*. Moreover,* $c\mathrm{g}(\mu)$ *is projectively induced if and only if* $(c+m)\frac{\mu}{\gamma}\mathrm{g}_B$ *is projectively induced for every integer* $m \geq 0$.

Proof The power expansion around the origin of $\mathrm{D}_0(z, w)$ can be written as

$$\mathrm{D}_0(z, w) = \sum_{j,k=0}^{\infty} A_{jk}(zw)^{m_j}(\bar{z}\bar{w})^{m_k}, \tag{7}$$

where m_j are ordered $(d + 1)$-tuples of integer and

$$(zw)^{m_j} = z_1^{m_{j,1}} \cdots z_d^{m_{j,d}} w^{m_{j,d+1}}.$$

In order to prove that $\mathrm{D}_0(z,w)$ is the diastasis for $\mathrm{g}(\mu)$ we need to verify that $A_{j0} = A_{0j} = 0$ (see Theorem 3). This is straightforward. Indeed if we take derivatives with respect either to z or $\bar{z}$ is the same as deriving the function $-\log(\mathrm{N}^{\mu}_{\Omega}(z,z)) = \frac{\mu}{\gamma}\mathrm{D}^{\Omega}_0(z)$ that is the diastasis of $(\Omega, \frac{\mu}{\gamma}\mathrm{g}_B)$, thus we obtain 0. If we take derivatives with respect either to w or $\bar{w}$ we obtain zero no matter how many times we derive with respect to z or $\bar{z}$, since $\mathrm{D}_0(z,w)$ is radial in w.

In order to prove the second part of the proposition take the function

$$e^{c\mathrm{D}_0(z,w)} - 1 = \frac{1}{(\mathrm{N}^{\mu}_{\Omega}(z,\bar{z}) - |w|^2)^c} - 1, \tag{8}$$

and using the same notations as in (7) write the power expansion around the origin as

$$e^{c\mathrm{D}_0(z,w)} - 1 = \sum_{j,k=0}^{\infty} B_{jk}(zw)^{m_j}(\bar{z}\bar{w})^{m_k}.$$

Since $M_{\Omega}(\mu)$ is simply connected (even contractible), by Calabi's criterion (Theorem (6)), $c\mathrm{g}(\mu)$ is projectively induced if and only if $B = (B_{jk})$ is positive semidefinite of infinite rank. The generic entry of B is given by

$$B_{jk} = \frac{1}{m_j! \cdot m_k!}\frac{\partial^{|m_j|+|m_k|}}{\partial(zw)^{m_j}\partial(\bar{z}\bar{w})^{m_k}}\left(\frac{1}{(\mathrm{N}^{\mu}_{\Omega}(z,\bar{z}) - |w|^2)^c} - 1\right)\Bigg|_0,$$

where $m_j! = m_{j,1}!\cdots m_{j,d+1}!$ and $\partial(zw)^{m_j} = \partial z_1^{m_{j,1}}\cdots\partial z_d^{m_{j,d}}\partial w^{m_{j,d+1}}$. By Proposition (7) we have

$$m_{j,1} + \cdots + m_{j,d} \neq m_{k,1} + \cdots + m_{k,d} \Longrightarrow B_{jk} = 0, \tag{9}$$

and since (8) is radial in w we also have

$$m_{j,d+1} \neq m_{k,d+1} \Longrightarrow B_{jk} = 0. \tag{10}$$

Thus, B is a $\infty \times \infty$ matrix of the form

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & E_2 & 0 & 0 & \dots \\ 0 & \vdots & 0 & E_3 & 0 & \dots \\ 0 & & \vdots & 0 & \ddots & \end{pmatrix},$$

where the generic block E_i contains derivatives $\partial(zw)^{m_j}\ \partial(\bar{z}\bar{w})^{m_k}$ of order $2i$, $i = 1, 2, \dots$ such that $|m_j| = |m_k| = i$. We can further write

$$E_i = \begin{pmatrix} F_{z(i)}(0) & 0 & 0 \\ 0 & F_{w(i)}(0) & 0 \\ 0 & 0 & F_{(z,w)(i)}(0) \end{pmatrix}, \tag{11}$$

where $F_{z(i)}(0)$ (resp. $F_{w(i)}(0)$, $F_{(z,w)(i)}(0)$) contains derivatives $\partial(zw)^{m_j}\ \partial(\bar{z}\bar{w})^{m_k}$ (of order $2i$ with $|m_j| = |m_k| = i$) such that $m_{j,d+1} = m_{k,d+1} = 0$ (resp. $m_{j,d+1} = m_{k,d+1} = i, m_{j,d+1}, m_{k,d+1} \neq 0, i$). (Notice also that we have 0 in all the other entries because of (9) and (10)). Since the derivatives are evaluated at the origin, deriving (8) with respect to $\partial(zw)^{m_j}\ \partial(\bar{z}\bar{w})^{m_k}$ with $|m_j| = |m_k| = i$ and $m_{j,d+1} = m_{k,d+1} = 0$ is the same as deriving the function

$$\frac{1}{(\mathrm{N}_\Omega^\mu(z,z))^c} - 1 = e^{c\frac{\mu}{\gamma}\mathrm{D}_0^\Omega(z)} - 1. \tag{12}$$

Thus, by Calabi's criterion, all the blocks $F_{z(i)}(0)$ are positive semidefinite if and only if $c\frac{\mu}{\gamma}g_B$ is projectively induced. Observe that the blocks $F_{w(i)}(0)$ are semipositive definite without extras assumptions. Indeed if we consider derivatives $\partial(zw)^{m_j}$ $\partial(\bar{z}\bar{w})^{m_k}$ of (8) with $|m_j| = |m_k| = i$ and $m_{j,d+1} = m_{k,d+1} = i$, since $\mathrm{N}_\Omega^\mu(z,z)$ evaluated in 0 is equal to 1, it is the same as deriving the function $1/(1-|w|^2)^c - 1 = \left(\sum_{j=0}^\infty |w|^{2j}\right)^c - 1$ and the claim follows. Finally, consider the block $F_{(z,w)(i)}(0)$. It can be written as

$$F_{(z,w)(i)}(0) = \begin{pmatrix} H_{z(i-1),w(1)}(0) & 0 & 0 & 0 \\ 0 & H_{z(i-2),w(2)}(0) & 0 & 0 \\ \vdots & & \ddots & \\ 0 & 0 & 0 & H_{z(1),w(i-1)}(0) \end{pmatrix},$$

where the generic block $H_{z(i-m),w(m)}(0)$, $1 \leq m \leq i-1$, contains derivatives $\partial(zw)^{m_j}$ $\partial(\bar{z}\bar{w})^{m_k}$ of order $2i$ such that $|m_j| = |m_k| = i$ and $m_{j,d+1} = m_{k,d+1} = m$ evaluated at zero (as before, by (9) and (10) all entries outside these blocks are 0). Now it is not hard to verify that these blocks can be obtained by taking derivatives $\partial(zw)^{m_j}\ \partial(\bar{z}\bar{w})^{m_k}$ of order $2(i-m)$ such that $|m_j| = |m_k| = 2(i-m)$ and $m_{j,d+1} = m_{k,d+1} = 0$ of the function

$$\frac{(m+c-1)!}{(c-1)!\ m!\ \mathrm{N}_\Omega^{\mu(c+m)}(z,z)} - 1 = e^{(c+m)\frac{\mu}{\gamma}\mathrm{D}_0^\Omega(z)} - 1,$$

and evaluating at $z = \bar{z} = 0$. Thus, again by Calabi's criterion, $F_{(z,w)(i)}(0)$ is positive semidefinite iff $(c+m)\frac{\mu}{\gamma}g_B$, $m \geq 1$ is projectively induced and this ends the proof of the proposition. □

We are now in the position to prove Theorem 1.

Proof of Theorem 1 Take $\mu = \mu_0 = \gamma/(d+1)$ in (5) and $\Omega \neq \mathbb{C}\mathrm{H}^d$. By Theorem 12 $(\mathrm{M}_\Omega(\mu_0), cg(\mu_0))$ is Kähler–Einstein, complete and nonhomogeneous for all positive real number c. By Proposition 14 $cg(\mu_0)$ is projectively induced if and only if $\frac{c+m}{d+1}\mathrm{g}_B$ is projectively induced, for all nonnegative integer m. By Theorem 2 this happens if $\frac{(c+m)}{d+1} \geq \frac{(r-1)a}{2\gamma}$. Hence $cg(\mu_0)$ with $c \geq \frac{(r-1)(d+1)a}{2\gamma}$ is the desired family of projectively induced Kähler–Einstein metrics. □

Notice that by choosing $0 < c < \frac{a(d+1)}{2\gamma}$ (and $r \neq 1$) one also gets the existence of a continuous family of complete, nonhomogeneous and Kähler–Einstein metrics which are not projectively induced.

Acknowledgments The authors would like to thank Antonio J. Di Scala for his very useful comments and remarks.

References

1. Alekseevsky, D., Di Scala, A.J.: The normal holonomy group of Kähler submanifolds. Proc. Lond. Math. Soc. (3) **89**(1), 193–216 (2004)
2. Arazy, J.: A survey of invariant Hilbert spaces of analytic functions on bounded symmetric domains. Contemp. Math. **185**, 7–65 (1995)
3. Arezzo, C., Loi, A.: A note on Kähler–Einstein metrics and Bochner's coordinates. Abh. Math. Sem. Univ. Hamburg **74**, 49–55 (2004)
4. Calabi, E.: Isometric imbedding of complex manifolds. Ann. Math. **58**(1), 1–23 (1953)
5. Chern, S.S.: On Einstein hypersurfaces in a Kähler manifold of constant sectional curvature. J. Differ. Geom. **1**, 21–31 (1967)
6. Di Scala, A.J., Loi, A.: Kähler maps of Hermitian symmetric spaces into complex space forms. Geom. Dedicata **125**, 103–113 (2007)
7. Faraut, J., Koranyi, A.: Function spaces and reproducing kernels on bounded symmetric domains. J. Funct. Anal. **88**, 64–89 (1990)
8. Kobayashi, S.: Geometry of bounded domains. Trans. Am. Math. Soc. **92**, 267–290 (1996)
9. Loi, A.: Calabi's diastasis function for Hermitian symmetric spaces. Differ. Geom. Appl. **24**, 311–319 (2006)
10. Takeuchi, M.: Homogeneous Kähler submanifolds in projective spaces. Jpn. J. Math **4**, 171–219 (1978)
11. Tsukada, K.: Einstein–Kähler Submanifolds with codimension two in a complex space form. Math. Ann. **274**, 503–516 (1986)
12. Upmeier, H.: Index theory for Toeplitz operators on bounded symmetric domains. Reprsentations des groupes et analyse complexe (Luminy, 1986), pp. 89–94. Journes SMF, 24. Univ. Poitiers, Poitiers (1986)
13. Wang, A., Yin, W., Zhang, L., Roos, G.: The Kähler–Einstein metric for some Hartogs domains over bounded symmetric domains. Sci. China **49**, 1175–1210 (2006)
14. Wang, A., Yin, W., Zhang, L., Zhang, W.: The Einstein–Kähler metric with explicit formulas on some non-homogeneous domains. Asian J. Math. **8**(1), 39–49 (2004)