Commun. Math. Phys. 289, 825–840 (2009)
Digital Object Identifier (DOI) 10.1007/s00220-009-0808-2

Communications in
Mathematical Physics

TYZ Expansion for the Kepler Manifold

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Received: 8 November 2007 / Accepted: 7 February 2009
Published online: 29 April 2009 –

Abstract: The main goal of the paper is to address the issue of the existence of Kempf's distortion function and the Tian-Yau-Zelditch (TYZ) asymptotic expansion for the Kepler manifold - an important example of non-compact manifold. Motivated by the recent results for compact manifolds we construct Kempf's distortion function and derive a precise TYZ asymptotic expansion for the Kepler manifold. We get an exact formula: finite asymptotic expansion of $n-1$ terms and exponentially small error terms uniformly with respect to the discrete quantization parameter $m \to \infty$ ($\hbar = m^{-1} \to 0$ standing for Planck's constant and $|x| \to \infty$, $x \in \mathbb{C}^n$). Moreover, the coefficients are calculated explicitly and they turned out to be homogeneous functions with respect to the polar radius in the Kepler manifold. We show that our estimates are sharp by analyzing the nonharmonic behaviour of T_m for $m \to +\infty$. The arguments of the proofs combine geometrical methods, quantization tools and functional analytic techniques for investigating asymptotic expansions in the framework of analytic-Gevrey spaces.

1. Introduction and Statements of the Main Results

Let g be a Kähler metric on an n–dimensional complex manifold M. Assume that g is polarized with respect to a holomorphic line bundle L over M, i.e. $c_1(L) = [\omega]$, where ω is the Kähler form associated to g and $c_1(L)$ denotes the first Chern class of L. Let $m \geq 1$ be a non-negative integer and let h_m be a Hermitian metric on $L^m = L^{\otimes m}$ such that its Ricci curvature $\mathrm{Ric}(h_m) = m\omega$. Here $\mathrm{Ric}(h_m)$ is the two–form on M whose local expression is given by

$$\mathrm{Ric}(h_m) = -\frac{i}{2}\partial\bar{\partial} \log h_m(\sigma(x), \sigma(x)), \tag{1.1}$$

The first author was supported in part by the project PRIN (Cofin) n. 2006019457 with M.I.U.R., Italy. The second author was supported in part by the M.I.U.R. Project "Geometric Properties of Real and Complex Manifolds".

for a trivializing holomorphic section $\sigma : U \to L^m \backslash \{0\}$. In the quantum mechanics terminology L^m is called the *quantum line bundle*, the pair (L^m, h_m) is called a *geometric quantization* of the Kähler manifold $(M, m\omega)$ and $\hbar = m^{-1}$ plays the role of Planck's constant (see e.g. [2]). Consider the separable complex Hilbert space $\mathcal{H}_m$ consisting of global holomorphic sections s of L^m such that

$$\langle s, s\rangle_m = \int_M h_m(s(x), s(x))\frac{\omega^n}{n!} < \infty.$$

Let $x \in M$ and let $q \in L^m \backslash \{0\}$ be a fixed point of the fiber over x. If one evaluates $s \in \mathcal{H}_m$ at x, one gets a multiple $\delta_q(s)$ of q, i.e. $s(x) = \delta_q(s)q$. The map $\delta_q : \mathcal{H}_m \to \mathbb{C}$ is a continuous linear functional [9]. Hence from Riesz's theorem, there exists a unique $e_q^m \in \mathcal{H}$ such that $\delta_q(s) = \langle s, e_q^m\rangle_m$, $\forall s \in \mathcal{H}_m$, i.e.

$$s(x) = \langle s, e_q^m\rangle_m q. \tag{1.2}$$

It follows that $e_{cq}^m = \bar{c}^{-1} e_q^m$, $\forall c \in \mathbb{C}^*$. The holomorphic section $e_q^m \in \mathcal{H}_m$ is called the *coherent state* relative to the point q. Thus, one can define a smooth function on M,

$$T_m(x) = h_m(q, q)\|e_q^m\|^2, \quad \|e_q^m\|^2 = \langle e_q^m, e_q^m\rangle, \tag{1.3}$$

where $q \in L^m \backslash \{0\}$ is any point on the fiber of x. If s_j, $j = 0, \dots d_m$, ($d_m + 1 = \dim \mathcal{H}_m \leq \infty$) form an orthonormal basis for $(\mathcal{H}_m, \langle\cdot,\cdot\rangle_m)$ then one can easily verify that

$$T_m(x) = \sum_{j=0}^{d_m} h_m(s_j(x), s_j(x)). \tag{1.4}$$

Notice that when M is compact $\mathcal{H}_m = H^0(L^m)$, where $H^0(L^m)$ denotes the space of global holomorphic sections of L^m. Hence in this case $d_m < \infty$ and (1.4) is a finite sum.

The function T_m has appeared in the literature under different names. The earliest one was probably the η-function of J. Rawnsley [42] (later renamed to ϵ function in [9]), defined for arbitrary Kähler manifolds, followed by the *distortion function* of Kempf [27] and Ji [26], for the special case of Abelian varieties and of Zhang [48] for complex projective varieties. The metrics for which T_m is constant were called *critical* in [48] and *balanced* in [17] (see also [3,32 and 33]). If T_m are constants for all sufficiently large m then the geometric quantization (L^m, h_m) associated to the Kähler manifold (M, g) is called regular. Regular quantization plays a prominent role in the theory of quantization by deformation of Kähler manifolds developed in [9].

Fix $m \geq 1$. Under the hypothesis that for each point $x \in M$ there exists $s \in \mathcal{H}_m$ non-vanishing at x, one can give a geometric interpretation of T_m as follows. Consider the holomorphic map of M into the complex projective space $\mathbb{C}P^{d_m}$:

$$\varphi_m : M \to \mathbb{C}P^{d_m} : x \mapsto [s_0(x) : \cdots : s_{d_m}(x)]. \tag{1.5}$$

One can prove that

$$\varphi_m^*(\omega_{FS}) = m\omega + \frac{i}{2}\partial\bar{\partial}\log T_m, \tag{1.6}$$

where ω_{FS} is the Fubini–Study form on $\mathbb{C}P^{d_m}$, namely the form which in homogeneous coordinates $[Z_0, \ldots, Z_{d_m}]$ reads as $\omega_{FS} = \frac{i}{2}\partial\bar{\partial}\log\sum_{j=0}^{d_m}|Z_j|^2$.

Clearly (1.6) leads to

$$\frac{\varphi_m^*(\omega_{FS})}{m} - \omega = \frac{i}{2m}\partial\bar{\partial}\log T_m, \tag{1.7}$$

therefore the term

$$\mathcal{E}_m(x) := \frac{i}{2m}\partial\bar{\partial}\log T_m, \tag{1.8}$$

turns out to play a role of the "error" of the approximation of ω (resp. g) by $\varphi_m^*(\omega_{FS})/m$ (resp. $\varphi_m^*(g_{FS})/m$).

Observe that by (1.6), if there exists m such that mg is a balanced metric, or more generally if T_m is harmonic, then $\mathcal{E}_m(x)$ is identically zero and hence mg is projectively induced via the coherent states map φ_m (see [2,17 and 18] for more details on the link between balanced metrics and quantization of Kähler manifolds). Recall that a Kähler metric g on a complex manifold M is *projectively induced* if there exists a Kähler (i.e. a holomorphic and isometric) immersion $\psi : M \to \mathbb{C}P^N$, $N \leq \infty$, such that $\psi^*(g_{FS}) = g$. Projectively induced Kähler metrics enjoy important geometrical properties and were extensively studied in [8] (see also the beginning of Sect. 4 below). Not all Kähler metrics are balanced or projectively induced. Nevertheless, when M is compact, G. Tian [46] and W. Ruan [43] solved a conjecture posed by Yau by proving that the sequence of metrics $\frac{\varphi_m^*(\omega_{FS})}{m}$ C^∞-converges to ω. In other words, any polarized metric on compact complex manifold is the C^∞-limit of (normalized) projectively induced Kähler metrics. Zelditch [47] generalized the Tian–Ruan theorem by proving a complete asymptotic expansion in the C^∞ category, namely

$$T_m(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j}, \tag{1.9}$$

where a_j, $j = 0, 1, \ldots$, are smooth coefficients with $a_0(x) = 1$, and for any nonnegative integers r, k the following estimates hold:

$$||T_m(x) - \sum_{j=0}^{k} a_j(x) m^{n-j}||_{C^r} \leq C_{k,r} m^{n-k-1}, \tag{1.10}$$

where $C_{k,r}$ are constant depending on k, r and on the Kähler form ω and $||\cdot||_{C^r}$ denotes the C^r norm in local coordinates. (Notice that similar asymptotic expansion were used in [9–12,35 and 36]) to construct the star product on Kähler manifolds.)

Later on, Lu [34], by means of Tian's peak section method, proved that each of the coefficients $a_j(x)$ in (1.9) is a polynomial of the curvature and its covariant derivatives at x of the metric g. Such polynomials can be found by finitely many algebraic operations. Furthermore $a_1(x) = \frac{1}{2}\rho$, where ρ is the scalar curvature of the polarized metric g (see also [30 and 31] for the computations of the coefficients a_j's through Calabi's diastasis function). The expansion (1.9) is called the *TYZ (Tian–Yau–Zelditch) expansion* and is a key ingredient in the investigations of balanced metrics [17].

The aim of the present paper is to address the problem of TYZ expansions for noncompact manifolds (see also the recent paper [22]). Our motivation is two-folded.

First, we are inspired by the investigations of geometric quantization problems, in particular, by the recent works of M. Engliš [20,21,23] (see also the fundamental paper of L. Boutet de Monvel and J. Sjöstrand [7]), where analytical tools have been applied in order to extend Berezin's quantization method (cf. [4,5]) to non-homogeneous complex domains on $\mathbb{C}^n$ (see also [37–39] and the references therein for the study of coherent states and relations to geometric quantization). Secondly, it is a purely geometrical question in the framework of the quantization theory of its own interest.

We choose as a noncompact manifold the Kepler manifold (X, ω), namely the cotangent bundle of the n-dimensional sphere minus its zero section endowed with the standard symplectic form ω (see [45 and 41]).

We summarize the main novelties of our work. First, we compute explicitly Kempf's distortion function $T_m(x)$ for the Kepler manifold (X, ω). Secondly, based on this computation we find an analogue of the results of S. Zelditch and Z. Lu for (X, ω). More precisely, building upon the explicit representation of T_m as an action of "singular derivatives" and using precise estimates of nonlinear compositions in functional spaces, we show that the TYZ expansion for the Kepler manifold has two remarkable features:

- the TYZ expansion is *finite*. More precisely, it consists of $n-1$ terms,

$$T_m(x) = m^n + \frac{(n-2)(n-1)}{2|x|} m^{n-1} + \sum_{k=2}^{n-2} \frac{2a_k}{|x|^k} m^{n-k} + R_m(|x|),$$

  where $a_k,\ k \geq 2$ can be computed explicitly by recursive formulas,
- the *remainder term has an exponentially small decay* $O(e^{-c|x|m})$ as $m \to \infty$ uniformly with respect to $|x| \geq \delta > 0$.

We stress that these constructions on non compact manifolds might be of some interest from the numerical point of view. Our approach should be compared with the recent quantization numerical results of S. Donaldson [19], obtained by projectively induced metrics on compact manifolds.

We stress that the approach for other noncompact manifolds in [7,20,21 and 23]) provides good approximations of the Bergman kernel via Fourier integral operators. For the Kepler manifold we rely on an explicit computation.

We also demonstrate uniform analytic–Gevrey regularity estimates for T_m keeping the exponential decay for $m \to \infty$, $|x| \to \infty$, which resemble the simultaneous analytic–Gevrey estimates and exponential decay for solitary waves via the use of global analytic–Gevrey pseudodifferential operators in $\mathbb{R}^n$ (cf. [13,14] and the references therein). We mention also the recent works [15,16], where exponentially small error terms in the framework of Gevrey spaces appear in the study of oscillatory integrals with non–Morse functions and divergent normal forms.

Observe that as for the compact case our expansion shows that g (the metric g associated to the Kepler manifold (M, ω)) is the C^∞-limit of (suitable normalized) projectively induced Kähler metrics, namely $\lim_{m\to\infty} \frac{1}{m}\varphi_m^*(g_{FS}) = g$, where $\varphi_m : X \to \mathbb{C}P^\infty$ is the coherent states map. A geometric construction is proposed showing that our estimates are sharp. Indeed, we show that g is not projectively induced, i.e. there cannot exist *any* Kähler immersion of (X, ω) into a finite or infinite dimensional complex projective space. The arguments use Calabi's tools which provide necessary and sufficient conditions for a Kähler metric to be projectively induced.

The paper is organized as follows. We propose an explicit construction of Kempf's distortion function T_m for the Kepler manifold (X, ω) in Sect. 2. In Sect. 3 we derive an exact TYZ asymptotic expansion when $m \to \infty$. In Sect. 4 we prove (see Theorem 4.4) that our estimate is sharp.

2. Kempf's Distortion Function for the Kepler Manifold

The (regularized) Kepler manifold [45] is (may be identified with) the $2n$-dimensional symplectic manifold (X, ω), where $X = T^*S^n \backslash 0$ is the cotangent bundle of the n-dimensional sphere minus its zero section endowed with the standard symplectic form ω. This may further be identified with

$$X = \{(e, x) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid e \cdot e = 1, x \cdot e = 0, x \neq 0\}.$$

Here $\cdot$ denotes the standard scalar product on $\mathbb{R}^{n+1}$. J. Souriau [45] showed that the Kepler manifold admits a natural complex structure. He proved, by introducing $z = |x|e + ix = |x|(e + is) \in \mathbb{C}^{n+1}$, $s = \frac{x}{|x|} \in S^n$, that X is diffeomorphic to the isotropic cone

$$C = \{z \in \mathbb{C}^{n+1} \mid z \cdot z = z_1^2 + \cdots + z_{n+1}^2 = 0, z \neq 0\} \subset \mathbb{C}^{n+1},$$

and hence X inherits the complex structure of C via this diffeomorphism. Later on J. Rawnsley [42] observed that the symplectic form ω is indeed a Kähler form with respect to this complex structure and can be written (up to a factor) as

$$\omega = \frac{i}{2}\partial\bar{\partial}|x|. \tag{2.11}$$

We denote by g the Kähler metric metric induced by ω.

Remark 2.1. Although the Kepler manifold is not complete (cf. Remark 4.1) and hence not homogeneous, its isometry group is very large, being equal to $S^1 \times O(n+1)$ (see the appendix in [41]). The radial symmetries of the metric has the great advantage that its diastasis function and Kempf's distortion function can be computed explicitly as function of the polar radius in $\mathbb{C}^n$.

Another interesting Kähler metric g_G on T^*S^n using a Grauert tube function ρ has been studied by Guillemin and Stenzel [24,25], Lempert and Szöke [28,29] and Patrizio and Wong [40]). The metric g_G is uniquely determined by a Kähler potential ρ such that $\sqrt{\rho}$ satisfies the Monge–Ampère equation and the metric induced by g_G on S^n (viewed as the zero section of T^*S^n) equals the round metric on S^n. We can show, taking advantage of the conic structure of the Kepler manifold, the singularity at the zero section and the radial symmetry of g, that there is no smooth map $f : N_G \backslash S^n \to N_K \backslash S^n$ such that $f^*(g_G) = g$, where N_K (resp. N_G) is an arbitrary open neighborhood of $S^n \subset T^*S^n$.

Moreover, since ω is exact, it is trivially integral and hence there exists a holomorphic line bundle L over X such that $c_1(L) = [\omega]$.

For $n \geq 3$, X is simply-connected, so L^m is holomorphically trivial ($L^m = X \times \mathbb{C}$) and we can identify $H^0(L^m)$ with the set of holomorphic functions on X. Furthermore, we can define a Hermitian metric h_m on $L^m = X \times \mathbb{C}$ by

$$h_m(\sigma(z), \sigma(z)) = e^{-m|x|}, \tag{2.12}$$

where $\sigma : X \to X \times \mathbb{C}$, is the global holomorphic section such that $\sigma(z) = (z, 1)$. It follows by (1.1) above that the pair (L^m, h_m) is a geometric quantization of the Kepler manifold (X, ω). Then the Hilbert space $\mathcal{H}_m$ consists of the set of holomorphic functions f of X such that

$$\|f\|_m^2 := \int_X |f(z)|^2 e^{-m|x|} d\mu(z) < \infty, \qquad d\mu(z) = \frac{\omega^n(z)}{n!} = (\frac{i}{2}\partial\bar{\partial}|x|)^n.$$

Notice that in this case

$$T_m(z) = e^{-m|x|} K^{(m)}(z, z),$$

where $K^{(m)}(z, z)$ is the reproducing Kernel for the Hilbert space $\mathcal{H}_m$. At p. 412 in [42] J. Rawnsley explicitly computed $K(z, z) = K^{(1)}(z, z)$ (the reproducing kernel for $\mathcal{H} = \mathcal{H}_1$) and hence the corresponding Kempf's distortion function, which in our notations is read:

$$T_1(z) = e^{-|x|}\mathrm{K}(z, z) = 2^{n-1} e^{-|x|} \sum_{j=0}^{\infty} \frac{(j + n - 2)!}{(2j + n - 2)!} \frac{|x|^{2j}}{j!}, \qquad 2|x|^2 = z \cdot \bar{z}. \tag{2.13}$$

Next, we compute Kempf's distortion functions $T_m(z)$, $m \in \mathbb{Z}_+$. The change of variable $mz = w$ yields

$$\|f\|_m^2 = \int_X |f(w/m)|^2 e^{-|\operatorname{Im} w|} m^{-n} d\mu(w).$$

Consequently, the operator

$$T : \ Tf(w) := m^{-n/2} f(w/m) \tag{2.14}$$

is a unitary isomorphism from $\mathcal{H}_m$ onto $\mathcal{H}$. Denote by $\mathrm{K}^{(m)}(w, z) \equiv \mathrm{K}_z^m(w)$ the reproducing kernel of $\mathcal{H}_m$, $m \geq 1$ (and write $\mathrm{K}(w, z) \equiv K_z(w)$ if $m = 1$). We have, on the one hand,

$$f(z) = \langle f, \mathrm{K}_z^{(m)} \rangle_m = \langle Tf, T\mathrm{K}_z^{(m)} \rangle$$

for any $f \in \mathcal{H}_m$, while, on the other hand,

$$f(z) = m^{\frac{n}{2}} Tf(mz) = \langle Tf, m^{\frac{n}{2}} \mathrm{K}_{mz} \rangle.$$

Thus $T\,\mathrm{K}_z^{(m)} = m^{\frac{n}{2}} \mathrm{K}_{mz}$, and $\mathrm{K}_z^{(m)}(w) = m^{n/2} T^{-1} \mathrm{K}_{mz}(w) = m^n \mathrm{K}_{mz}(mw)$, i.e.,

$$\mathrm{K}^{(m)}(w, z) = m^n \mathrm{K}(mw, mz).$$

Substituting this into Rawnsley's formula (2.13), we thus get

$$T_m(z) = e^{-m|x|}\mathrm{K}^{(m)}(z, z) = 2^{n-1} m^n e^{-m|x|} \sum_{j=0}^{\infty} \frac{(j + n - 2)!}{(2j + n - 2)!} \frac{(m|x|)^{2j}}{j!}. \tag{2.15}$$

Remark 2.1. From (2.15) one sees that $T_m(x) = m^n T_1(mx)$. In the compact case the relationship between T_m and T_1 is in general unknown. This is also true in the Bargman Fock case. In fact in [44] one can find a general result which explains this kind of relation in the case of Bergman metrics.

Notice that the growth of $T_m(z)$ as $m \to \infty$ is not clear from representation (2.15). The following proposition gives us important analytic information about T_m as $m \to \infty$.

Proposition 2.2. *Kempf's distortion function for the Kepler manifold can be written in the following two forms:*

$$T_m(z) = 2m^n e^{-m|x|} \sum_{j=0}^{\infty} (1+\tau_j) \frac{(m|x|)^{2j}}{(2j)!}, \tag{2.16}$$

with

$$\tau_j = 1 - \frac{(j+1)\cdots(j+n-2)}{(j+1/2)\cdots(j+(n-2)/2)} \longrightarrow 0 \quad \text{for } j \to \infty,$$

and

$$T_m(z) = 2m^n e^{-\xi_m} \left(\frac{1}{\xi_m}\frac{\partial}{\partial \xi_m}\right)^{n-2} \left[\xi_m^{n-2}\left(\frac{e^{\xi_m} + (-1)^{n-2}e^{-\xi_m}}{2} + Q(\xi_m)\right)\right], \tag{2.17}$$

where $\xi_m = m|x|$, $Q(\xi_m)$ is a polynomial of degree $\leq n-4$ in the variable ξ_m.

Proof. From (2.15) one gets

$$\begin{aligned}
T_m(z) = e^{-m|x|}\mathrm{K}^{(m)}(z,z) &= 2^{n-1}m^n e^{-m|x|} \sum_{j=0}^{\infty} \frac{(j+n-2)!}{(2j+n-2)!}\frac{(m|x|)^{2j}}{j!} \\
&= 2^{n-1}m^n e^{-m|x|} \sum_{j=0}^{\infty} \frac{(j+n-2)!(2j)!}{j!(2j+n-2)!}\frac{(m|x|)^{2j}}{(2j)!} \\
&= 2m^n e^{-m|x|} \sum_{j=0}^{\infty} \frac{(j+1)\cdots(j+n-2)}{(j+1/2)\cdots(j+(n-2)/2)}\frac{(m|x|)^{2j}}{(2j)!} \\
&= 2m^n e^{-m|x|} \sum_{j=0}^{\infty} (1+\tau_j)\frac{(m|x|)^{2j}}{(2j)!}.
\end{aligned} \tag{2.18}$$

In order to prove (2.17) set

$$y_m^2 = m|x| = \xi_m, \;\; y_m, \xi_m \in \mathbb{R}\backslash\{0\}.$$

Then, since $\frac{\partial}{\partial y_m} = \frac{1}{2\xi_m}\frac{\partial}{\partial \xi_m}$ one gets

$$\begin{aligned}
T_m(z) &= 2^{n-1}m^n e^{-\xi_m} \sum_{j=0}^{\infty} \frac{(j+n-2)!}{(2j+n-2)!}\frac{y_m^j}{j!} \\
&= 2^{n-1}m^n e^{-\xi_m} \left(\frac{\partial}{\partial y_m}\right)^{n-2} \sum_{j=0}^{\infty} \frac{y_m^{j+n-2}}{(2j+n-2)!} \\
&= 2m^n e^{-\xi_m} \left(\frac{1}{\xi_m}\frac{\partial}{\partial \xi_m}\right)^{n-2} \left[\xi_m^{n-2}\sum_{j=0}^{\infty} \frac{\xi_m^{2j+n-2}}{(2j+n-2)!}\right].
\end{aligned}$$

If n is even then

$$T_m(z) = 2m^n e^{-\xi_m}\left(\frac{1}{\xi_m}\frac{\partial}{\partial \xi_m}\right)^{n-2}[\xi_m^{n-2}(\cosh\xi_m - P(\xi_m))],$$

with

$$P(\xi_m) = \sum_{j=0}^{\frac{n-4}{2}} \frac{\xi_m^{2j}}{(2j)!},$$

while n odd leads to

$$T_m(z) = 2m^n e^{-\xi_m}\left(\frac{1}{\xi_m}\frac{\partial}{\partial \xi_m}\right)^{n-2}[\xi_m^{n-2}(\sinh\xi_m - R(\xi_m))],$$

where

$$R(\xi_m) = \sum_{j=0}^{\frac{n-5}{2}} \frac{\xi_m^{2j+1}}{(2j+1)!}.$$

Hence we get (2.17). $\square$

3. TYZ Expansion for the Kepler Manifold

The key ingredient to find the TYZ expansion of T_m is (2.17). Clearly we have

$$T_m(z) = 2m^n F(m|x|), \tag{3.19}$$

where

$$F(y) = e^{-y}\left(\frac{1}{y}\frac{d}{dy}\right)^{n-2}\left(y^{n-2}\left(\frac{e^y + (-1)^{n-2}e^{-y}}{2} + Q(y)\right)\right), \quad y \in \mathbb{R}. \tag{3.20}$$

The explicit representation (3.19)–(3.20) of $T_m(z)$ for the Kepler manifold has a remarkable feature, namely, it is defined by a generating function $F(y)$ depending on one variable. Note that in fact $T_m(z)$ is independent of the base variables $e \in S^n$.

We show the first main result for the TYZ expansion for the Kepler manifold.

Theorem 3.1. *Let F satisfy (3.20). Then the following representation holds:*

$$F(y) = \sum_{j=0}^{n-2} \frac{b_j}{y^j} + \Phi(y) + \Psi(y), \tag{3.21}$$

where

$$\Phi(y) = e^{-2y}\sum_{j=0}^{n-2} \frac{p_j}{y^j}, \tag{3.22}$$

$$\Psi(y) = e^{-y}\sum_{j=0}^{n-2} \frac{r_j}{y^j}, \tag{3.23}$$

and the constants a_j, p_j, r_j are written explicitly. The functions $\Phi(y)$, $\Psi(y)$ and therefore, $F(y)$ as well, can be extended to meromorphic functions in $\mathbb{C}$. In particular, by (3.19) *and* (3.21) *we get*

$$T_m(z) = \sum_{j=0}^{n-2} a_j(x) m^{n-j} + 2m^n \Phi(m|x|) + 2m^n \Psi(m|x|), \, m \in \mathbb{N}, \tag{3.24}$$

where

$$a_j(x) = \frac{2b_j}{|x|^j}, \qquad j = 0, 1, \ldots, n-2, \tag{3.25}$$

and

$$a_0(x) = 1, \tag{3.26}$$

$$a_1(x) = \frac{(n-2)(n-1)}{2|x|}. \tag{3.27}$$

Moreover, there exists an absolute constant $C_0 > 0$ such that for every $\delta \in]0, 1]$,

$$\sup_{|x| \geq \delta} |D_x^\alpha \Theta_m(x)|, \leq C_0^{\alpha+1} \frac{\alpha!}{\delta^\alpha} e^{-m\delta/2} \tag{3.28}$$

for all $m \in \mathbb{N}$, where $\Theta = \Phi, \Psi$. Therefore, we have the following estimates

$$|D_x^\alpha \left(T_m - \sum_{j=0}^{n-2} a_j(x) m^{n-j} \right)| \leq C_0^{\alpha+1} \frac{\alpha!}{\delta^\alpha} e^{-m\delta/2} \tag{3.29}$$

for all $|x| \geq \delta$, $\alpha \in \mathbb{Z}_+^n$.

Proof. We recall the well known Faà di Bruno type formula for the derivative of $g \circ \varphi$, namely, for a given $\alpha \in \mathbb{N}$ we have

$$D_t^\alpha(g(\varphi(t))) = D_y^\alpha(g(\varphi(y)))|_{y=t}$$

$$= \sum_{j=1}^{\alpha} \frac{g^{(j)}(\varphi(t))}{j!} D_x^\alpha \left((\varphi(x) - \varphi(t))^j \right) |_{x=t} \tag{3.30}$$

$$= \sum_{j=1}^{\alpha} \frac{g^{(j)}(\varphi(t))}{j!} \sum_{\substack{\alpha_1 + \cdots + \alpha_j = \alpha \\ \alpha_1 \geq 1, \cdots, \alpha_j \geq 1}} \frac{\alpha!}{\alpha_1! \ldots \alpha_j!} \varphi^{(\alpha_1)}(t) \ldots \varphi^{(\alpha_j)}(t), \tag{3.31}$$

where $\varphi^{(k)}(t)$ stands for $D_t^k \varphi(t)$.

Next, we straighten $y^{-1} D_y$ into D_t via the singular change of the variable $y = y(t) = \sqrt{2t}$, $t = t(y) = y^2/2$. Therefore, setting

$$G(t) = F(\sqrt{2t}), \; t > 0, \qquad F(y) = G\left(\frac{y^2}{2}\right), \; y > 0, \tag{3.32}$$

we get by (3.20),

$$F(y) = G(t) = e^{-\sqrt{2t}} \left(\frac{d}{dt}\right)^{n-2} (2t)^{(n-2)/2} \left(\frac{e^{\sqrt{2t}} + (-1)^{n-2} e^{-\sqrt{2t}}}{2} + Q(\sqrt{2t}) \right). \tag{3.33}$$

The next assertion is instrumental in the proof.

Lemma 3.2. *Let $N \in \mathbb{N}$, $c \in \mathbb{R}$, and $r > 0$. Then*

$$\begin{aligned} \psi_N^{c,r}(y) &:= e^{-y} \left(\frac{1}{y}\frac{d}{dy}\right)^N (y^r e^{cy}) \\ &= \psi_N^{c,r}(\sqrt{2t}) =: \varphi_N^{c,r}(t) = e^{-\sqrt{2t}} \left(\frac{d}{dt}\right)^N ((2t)^{r/2} e^{c\sqrt{2t}}) \end{aligned} \tag{3.34}$$

has the following representation:

$$\varphi_N^{c,r}(t) = e^{-(1-c)\sqrt{2t}} (2t)^{(r-N)/2} \sum_{s=0}^{N} \frac{\varkappa_s}{(2t)^{s/2}}, \tag{3.35}$$

i.e.

$$\psi_N^{c,r}(y) = e^{-(1-c)y} y^{r-N} \sum_{s=0}^{N} \frac{\varkappa_s}{y^s}, \tag{3.36}$$

where

$$\begin{aligned} \varkappa_s = {} & \frac{c^{N-s}}{(N-s)!} \sum_{\ell=N-s}^{N} \binom{N}{\ell} \left(\prod_{q=0}^{N-\ell-1} \left(\frac{r}{2} - q\right) \right) 2^{N-r/2} (-1)^{\ell+s-N} \\ & \times \sum_{\substack{\ell_1+\cdots+\ell_{N-s}=\ell \\ \ell_1 \ge 1, \cdots, \ell_{N-s} \ge 1}} \frac{\ell!}{\ell_1! \ldots \ell_{N-s}!} \prod_{q_1}^{\ell_1-1} \left(\frac{1}{2} - q_1\right) \ldots \prod_{q_{N-s}}^{\ell_{N-s}-1} \left(\frac{1}{2} - q_{N-s}\right) \end{aligned} \tag{3.37}$$

for $s = 0, \ldots, N-1$ and

$$\varkappa_N = 2^{N-r/2} \prod_{q=0}^{N-1} (\frac{r}{2} - q). \tag{3.38}$$

Proof. By Faà di Bruno type formula (3.31) we derive

$$\begin{aligned} \Theta_N^{r,c}(t) = \left(\frac{d}{dt}\right)^N (t^{r/2} e^{c\sqrt{2t}}) &= \sum_{\ell=0}^{N} \binom{N}{\ell} D_t^{N-\ell}(t^{r/2}) D_t^{\ell}(e^{c\sqrt{2t}}) \\ &= D_t^N(t^{r/2}) e^{c\sqrt{2t}} + \sum_{\ell=1}^{N} \binom{N}{\ell} \left(\prod_{q=0}^{N-\ell-1} (\frac{r}{2} - q) \right) t^{r/2-N+\ell} e^{c\sqrt{2t}} \sum_{j=1}^{\ell} \frac{c^j 2^{j/2}}{j!} \\ & \quad \times \sum_{\substack{\ell_1+\cdots+\ell_j=\ell \\ \ell_1 \ge 1, \ldots, \ell_j \ge 1}} \frac{\ell!}{\ell_1! \cdots \ell_j!} D_t^{\ell_1}(t^{1/2}) \cdots D_t^{\ell_j}(t^{1/2}) \end{aligned} \tag{3.39}$$

with the convention $\prod_{q=0}^{-1} \ldots = 1$. Since

$$D_t^{\mu}(t^{1/2}) = \frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-\mu+1\right)t^{1/2-\mu} = (-1)^{\mu-1}\frac{(2\mu-3)!!}{2^{\mu}}t^{1/2-\mu} \tag{3.40}$$

for all positive integers μ, with $(-1)!! := 1$, $(2\mu-3)!! := 1\cdots(2\mu-3)$ if $\mu \geq 2$, combining (3.39) and (3.40), we obtain

$$\sum_{\substack{\ell_1+\cdots+\ell_j=\ell \\ \ell_1\geq 1,\ldots,\ell_j\geq 1}} \frac{\ell!}{\ell_1!\cdots\ell_j!} D_t^{\ell_1}(t^{1/2})\cdots D_t^{\ell_j}(t^{1/2}) = (-1)^{\ell-j}\Gamma^{\ell,j}\frac{t^{j/2-\ell}}{2^{\ell}} \tag{3.41}$$

with

$$\Gamma^{\ell,j} := \sum_{\substack{\ell_1+\ldots+\ell_j=\ell \\ \ell_1\geq 1,\ldots,\ell_j\geq 1}} \frac{\ell!}{\ell_1!\cdots\ell_j!}(2\ell_1-3)!!\cdots(2\ell_j-3)!!. \tag{3.42}$$

We note that

$$\Gamma^{\ell,\ell} = \ell!, \tag{3.43}$$

$$\Gamma^{\ell,\ell-1} = \frac{\ell-1}{2}\ell!. \tag{3.44}$$

Therefore, by (3.39)–(3.41),

$$\begin{aligned}
\Theta_N^{r,c}(t) &= \left(\prod_{q=0}^{N-1}\left(\tfrac{r}{2}-q\right)\right)t^{r/2-N}e^{c\sqrt{2t}} \\
&\quad + \sum_{\ell=1}^{N}\binom{N}{\ell}\left(\prod_{q=0}^{N-\ell-1}\left(\frac{r}{2}-q\right)\right)2^{N-\ell-r/2}(2t)^{r/2-N+\ell} \\
&\quad \times \sum_{j=1}^{\ell}\frac{c^j}{j!}(-1)^{\ell-j}\Gamma^{\ell,j}(2t)^{j/2-\ell} \\
&= (2t)^{r/2-N/2}e^{c\sqrt{2t}}\frac{2^{N-r/2}\prod_{q=0}^{N-1}\left(\tfrac{r}{2}-q\right)}{(2t)^{N/2}} \\
&\quad + (2t)^{r/2-N/2}e^{c\sqrt{2t}}\sum_{j=1}^{N}\frac{c^j}{j!(2t)^{(N-j)/2}}\sum_{\ell=j}^{N}\binom{N}{\ell} \\
&\quad \times \left(\prod_{q=0}^{N-\ell-1}\left(\frac{r}{2}-q\right)\right)2^{N-\ell-r/2}(-1)^{\ell-j}\Gamma^{\ell,j}
\end{aligned}$$

$$
\begin{aligned}
&= (2t)^{r/2-N/2} e^{c\sqrt{2t}} \frac{2^{N-r/2} \prod_{q=0}^{N-1} \left(\frac{r}{2}-q\right)}{(2t)^{N/2}} \\
&\quad + (2t)^{r/2-N/2} e^{c\sqrt{2t}} \sum_{s=0}^{N-1} \frac{c^{N-s}}{(N-s)!(2t)^{s/2}} \sum_{\ell=N-s}^{N} \binom{N}{\ell} \\
&\quad \times \left(\prod_{q=0}^{N-\ell-1} \left(\frac{r}{2}-q\right) \right) 2^{N-\ell-r/2} (-1)^{\ell+s-N} \Gamma^{\ell,N-s} \\
&= (2t)^{r/2-N/2} e^{c\sqrt{2t}} \sum_{s=0}^{N} \frac{\varkappa_s}{(2t)^{s/2}},
\end{aligned} \tag{3.45}
$$

where $\varkappa_s$ is defined by

$$
\varkappa_s := \frac{c^{N-s}}{(N-s)!} \sum_{\ell=N-s}^{N} \binom{N}{\ell} \left(\prod_{q=0}^{N-\ell-1} \left(\frac{r}{2}-q\right) \right) 2^{N-\ell-r/2} (-1)^{\ell+s-N} \Gamma^{\ell,N-s}.
$$

In view of the definition of $\Gamma^{\ell,j}$ with the convention $\Gamma^{\ell,0} = 1$, it is equivalent to (3.37), (3.38). This ends the proof of the lemma. □

We conclude the proof of the theorem by applying the previous lemma for $z = m|x|$ and obtain the value of $a_s = \varkappa_s/2$ by setting $c = 1$, $r = N = (n-2)$; $p_s = (-1)^{n-2}\varkappa_s/2$ by setting $c = -1$, $r = N = n-2$, and

$$
r_s = q_{n-2-s} \prod_{\ell=1}^{n-2} (2n-2-s-2\ell),
$$

provided $Q(z) = \sum_{j=0}^{n-3} q_j z^j$. □

Remark 3.2. In view of (3.24), we have

$$
T_m(z) = m^n + \frac{(n-2)(n-1)}{2|x|} m^{n-1} + \sum_{k=2}^{n-2} \frac{2b_k}{|x|^k} m^{n-k} + R_m(|x|), \tag{3.46}
$$

with $R_m(x)$ being exponentially small $e^{-c|x|m}$ away from the origin $x = 0$.

Remark 3.3. The novelty of the theorem above is twofold. First, our TYZ type expansion is finite, i.e., $a_j = 0$ for $j \geq n-1$ (compare (1.10)). Secondly, the remainder is exponentially small. Moreover, the coefficients a_j can be computed explicitly.

We also mention that our approach allows us to investigate the asymptotic behaviour of the obstruction term

$$
\mathcal{E}_m(z) = \sum_{j,\ell=1}^{n+1} \mathcal{E}_m^{j,\ell}(z) dz_j \wedge d\bar{z}_\ell
$$

in (1.8) and to prove that the coefficients decay polynomially of the type m^{-2}. More precisely, for some $C > 0$, they behave like

$$\frac{C}{m^2|z|^3}(1 + o(1)) \qquad m \to \infty, \tag{3.47}$$

uniformly for $|z|$ away from the origin in $\mathbb{C}^n$. In fact, we are able to show an abstract theorem for the asymptotic behaviour of obstruction terms similar to (1.8) on conic manifolds of Kepler type. The proof is based on a suitable choice of global singular coordinates parametrizing the Kepler manifold and the use of implicit function theorem arguments. Consequently, by (1.6), the metric g associated to ω can be approximated by suitable normalized projectively induced Kähler metrics with an error of the type m^{-2}, $m \to \infty$. The details are to be found in another work.

4. Proof that our Estimate is Sharp

As a consequence of Theorem 3.1 the Kähler form g on the Kepler manifold X is the C^∞-limit of suitable normalized projectively induced Kähler metrics, namely

$$\lim_{m\to\infty} \frac{1}{m}\varphi_m^*(g_{FS}) = g,$$

where $\varphi_m : X \to \mathbb{C}P^\infty$ is the coherent states map. In this section we show that g is not projectively induced (via any map) and then that our estimate in Theorem 3.1 is sharp.

We need to recall briefly some results about Calabi's diastasis function referring the reader to [8] for details and further results.

Let M be a complex manifold endowed with a real analytic Kähler metric g. Then, in a neighborhood of every point $p \in M$, one can introduce a special Kähler potential D_p^g (*diastasis*, cf. [8]) for the Kähler form ω associated to g. Recall that a Kähler potential is an analytic function Φ defined in a neighborhood of a point p such that $\omega = \frac{i}{2}\bar{\partial}\partial\Phi$. A Kähler potential is not unique: it is defined up to addition to the real part of a holomorphic function. By duplicating the variables z and $\bar{z}$, a potential Φ can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighborhood U of the diagonal containing $(p, \bar{p}) \in M \times \bar{M}$ (here $\bar{M}$ denotes the manifold conjugated to M). The *diastasis function* is the Kähler potential D_p^g around p defined by

$$D_p^g(q) = \tilde{\Phi}(q, \bar{q}) + \tilde{\Phi}(p, \bar{p}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p}).$$

Observe that the diastasis does not depend on the potential chosen, $D_p^g(q)$ is symmetric in p and q and $D_p^g(p) = 0$.

The diastasis function is the key tool for studying the Kähler immersions of a Kähler manifold into another Kähler manifold as expressed by the following lemma.

Lemma 4.1. (Calabi [8]) *Let (M, g) be a Kähler manifold which admits a Kähler immersion $\varphi : (M, g) \to (S, G)$ into a real analytic Kähler manifold (S, G). Then g is real analytic. Let $D_p^g : U \to \mathbb{R}$ and $D_{\varphi(p)}^G : V \to \mathbb{R}$ be the diastasis functions of (M, g) and (S, G) around p and $\varphi(p)$, respectively. Then $\varphi^{-1}(D_{\varphi(p)}^G) = D_p^g$ on $\varphi^{-1}(V) \cap U$.*

When (S, G) is the N-dimensional complex projective space $S = \mathbb{C}P^N$ equipped with the Fubini–Study metric $G = g_{FS}$, one can show that for all $p \in \mathbb{C}P^N$ the diastasis function $D_p^{g_{FS}}$ around p is globally defined except in the cut locus H_p of p where it blows up. Moreover, $e^{-D_p^{g_{FS}}}$ is globally defined (and smooth) on $\mathbb{C}P^N$ (see [8] for details).

Then, by Lemma 4.1 one immediately gets the following:

Lemma 4.2. *Let g be a projectively induced Kähler metric on a complex manifold M. Then, $e^{-D_p^g}$ is globally defined on all M.*

Corollary 4.3. *Let g_* be the Kähler metric on $\mathbb{C}^*$ whose associated Kähler form is given by $\omega_* = \frac{i}{2}\partial\bar{\partial}|\eta|$, $\eta = x + iy$. Then g_* is not projectively induced.*

Proof. Fix any point $\alpha \in \mathbb{C}^*$. A globally defined Kähler potential Φ for the Kähler metric g_* around α is given by $\Phi(\eta) = |\eta|$ and Calabi's diastasis function around α reads:

$$D_\alpha^{g_*} : U \to \mathbb{R},\ \eta \mapsto |\eta| + |\alpha| - \sqrt{\eta\bar{\alpha}} - \sqrt{\bar{\eta}\alpha},$$

where $U \subset \mathbb{C}^*$ is a suitable simply-connected open subset of $\mathbb{C}^*$ around α (as a maximal domain of definition of $D_\alpha^{g_*}$ one can take $U = \mathbb{C}^* \backslash L$, where L is any half-line starting from the origin of $\mathbb{C} = \mathbb{R}^2$ such that $\alpha \notin L$). Neither the function $D_\alpha^{g_*}$ nor the function $e^{-D_\alpha^{g_*}}$ can be extended to all $\mathbb{C}^*$. Hence we are done by Lemma 4.2. □

We are now in the position to prove that our estimate is sharp.

Theorem 4.4. *Let g be the Kähler metric on the Kepler manifold X whose associated Kähler form is given by (2.11). Then g is not projectively induced.*

Proof. First observe that the map

$$j : (\mathbb{C}^*, g_*) \to (X, g) \tag{4.48}$$

defined by $j(\eta) = (\eta, i\eta, 0, \ldots, 0)$ is a Kähler immersion satisfying $j^*(g) = g_*$, with g_* as in Corollary 4.3. Assume by contradiction that g is projectively induced, namely there exists $N \leq \infty$ and a Kähler immersion $\varphi : (X, g) \to (\mathbb{C}P^N, g_{FS})$. Then the map $\varphi \circ j : (\mathbb{C}^*, g_*) \to (\mathbb{C}P^N, g_{FS})$ would be a Kähler immersion contradicting Corollary 4.3. □

Remark 4.1. From the proof of the previous theorem one can easily see that the metric g of the Kepler manifold X is not complete. Indeed, if it were complete the same would be true for the metric $g_* = \frac{1}{4}\frac{dx^2+dy^2}{\sqrt{x^2+y^2}}$ on $\mathbb{C}^* = \mathbb{R}^2\backslash\{0\}$ since the map (4.48) is totally geodesic. On the other hand the length of the geodesic segment $\{(t, 0)\ |0 < t \leq 1\}$ from the origin to $(1, 0)$ is finite since it is given by: $\frac{1}{4}\int_0^1 \frac{1}{\sqrt{x}}dx = \frac{1}{8} < \infty$ and thus g_* is not complete.

Acknowledgements. The authors are indebted to the anonymous referees for their critical remarks and suggestions which led to improvements of the revised version of the paper. The authors thank Simon Donaldson for the encouragement to study TYZ type asymptotic expansions for noncompact manifolds and Miroslav Engliš for useful discussions on reproducing kernels of complex domains and for pointing out the idea of using an isomorphism (2.14) for computing T_m from T_1. Thanks are also due to Antonio J. Di Scala, Ivailo Mladenov, Vassil Tsanov and Cornelis Van Der Mee for useful discussions and for providing references related to the subject of the paper.

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Communicated by S. Zelditch