# Continuous policies deformations without catastrophes *

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**Abstract**: Let $E(r)$ be the equilibrium manifold associated to a pure exchange smooth economy with fixed total resources $r$ and let $E_c(r) \subset E(r)$ be the set a critical equilibria. In the case of two agents and two commodities we show that, given any two continuous policies connecting two regular equilibria without crossing critical equilibria, it is still possible to continuously deform one policy into the other one without encountering catastrophes. This is equivalent to prove the simply-connectedness of each component of $E(r) \setminus E_c(r)$.

**Keywords:** Equilibrium manifold, regular economies, critical equilibria, catastrophes, transversality.

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# 1 Introduction

Let $S$ be the set of normalized prices under the standard numeraire convention and let $\Omega$ be the space of endowments of a pure exchange economy. Balasko (1988) has shown that the *equilibrium manifold*, i.e., the set of prices and endowments such that aggregate excess demand is zero, denoted by $E = \{(p, \omega) \in S \times \Omega | \sum_i f_i(p, p \cdot \omega_i) = \sum_i \omega_i\}$, is globally diffeomorphic to an Euclidean space. This fact has topological implications with evident economic meaning such as arc-connectedness and simply connectedness of the set $E$. Arc-connectedness means that, given any two equilibria $x$ and $y$ in $E$, it is always possible to connect them with a continuous path. Suppose that there exist more than one path connecting $x$ and $y$. Simply connectedness means that it is possible to continuously deform one path into the other one. A very natural economic interpretation suggested by Balasko is to consider the path as the mathematical formalization of an economic policy (mean) and the endpoints as the goal to be achieved. Simply connectedness formalizes "the possibility of having a consensus solution from a policy of small steps, where there is identity of the goals but divergence of the means" (Balasko, 1988, p. 71).

We recall that a regular equilibrium is a regular point of the *natural projection*, i.e., the mapping $\pi : E \subset S \times \Omega \to \Omega$. A critical point is called a critical equilibrium. The image, via $\pi$, of a regular equilibrium (critical equilibrium) is called regular economy (singular economy). Since singularities of the mapping $\pi$ play a crucial role in determining discontinuities of prices (catastrophes), it is desirable to choose policies which do not intersect critical equilibria. Consider the following situation. Let $x$ and $y$ be two regular equilibria which can be connected by two paths, say $\alpha$ and $\beta$, which do not encounter any critical equilibrium. This means that $x$ and $y$ belong to the same connected component of $E$. Is it still possible to continuously deform $\alpha$ into $\beta$ without crossing singularities? This is equivalent to show the simply connectedness of the connected component. But a deeper understanding of the topology of the set of regular equilibria relies on a crucial question: how is the set of critical equilibria immersed in the equilibrium manifold? In fact, in order to understand the topology of the set of regular equilibria, one needs to investigate the topology of the set of critical equilibria and its immersion in the equilibrium manifold. Unfortunately, this is a very difficult question. In several remarkable contributions, Balasko has deeply studied the singularities of $\pi$ (see Balasko (1988, 1992)). He has shown that that the set of critical equilibria $E_c$ and its image, $\pi(E_c) = \Sigma$, is a closed measure zero subset of the set of regular equilibria and regular economies, respectively. Moreover (see Balasko (1979, 1988)), in a fixed total resources setting ($\sum_i \omega_i = r$) as in the present paper, the set of critical equilibria $E_c(r)$ enjoys the very nice property of being a disjoint union of closed smooth submanifolds of the equilibrium manifold $E(r)$.

In this paper we address this issue considering the case of two consumers and two goods when total resources are fixed. In our main result, Theorem 2.2, we show that each connected component of the set of regular equilibria is simply connected being globally diffeomorphic to $\mathbb{R}^2$. In our proof we use the fact that each stratum of the set of critical equilibria is simply connected, being globally diffeomorphic to $\mathbb{R}$. We hope that our proof can bring new insight for a generalization to an arbitrary number of consumers and goods (see Remark 2.3 below).

# 2 Main result

We consider a smooth pure exchange economy with two agents and two commodities and fixed total resources. All the information we need about the equilibrium manifold $E(r)$ and the set of critical equilibria $E_c(r)$ is summarized in the following lemma. We refer the reader to Balasko (1988) for a complete study of the equilibrium manifold approach.

**Lemma 2.1** *In a smooth pure exchange economy with two agents and two goods:*

*(i)* $E(r)$ *is a smooth manifold globally diffeomorphic to* $\mathbb{R}^2$*;*

*(ii)* $E_c(r)$ *is a disjoint and countable union of smooth submanifolds* $\mathcal{S}_1^j$ *each of which is globally diffeomorphic to* $\mathbb{R}$*.*

**Proof:** The proof of (i) is given in Balasko (1988) while that of (ii) easily follows by the results contained in Balasko (1979) applied to the case when the number $l$ of commodities equals two and the number $m$ of agents equals two. Since we do not need the technical details concerning duality theory in the sequel of the paper, for reader's convenience we give a concise proof of (ii) taking notations and definitions for granted. We refer the reader to Balasko (1979). Balasko shows that the set $\mathcal{S}_i$ of critical equlibria of type $i$ is an open subset of a manifold $P\mathcal{S}_i$ which is a fiber bundle on $G_i \cong \mathbb{R}^{m-1} \times \mathrm{Gr}_{m-1,i}$, with typical fibers $\mathrm{Gr}_{l+m-2-i,l-1-i}$, where $\mathrm{Gr}_{h,k}$ is the Grassmannian of $k$-planes in $\mathbb{R}^h$. For $l = m = 2$, we then have $G_1 = P\mathcal{S}_1 = \mathbb{R}$ and $P\mathcal{S}_i = \mathcal{S}_i = \emptyset$ for $i \geq 2$. If one writes $\mathcal{S}_1 = \cup_j S_1^j$ as a countable union of its connected components then each $\mathcal{S}_1^j$ is an open and connected subset of $\mathbb{R}$ and hence diffeomorphic to $\mathbb{R}$. $\square$

We will refer to the decomposition $E_c(r) = \mathcal{S}_1 = \cup_j \mathcal{S}_1^j$ as *Balasko's stratification* of the set of critical equilibria $E_c(r)$ and, for fixed $j$, $\mathcal{S}_1^j$ will be called the $j$-th *stratum.*

We can now state and prove the main result of this paper.

**Theorem 2.2** *Each connected component of* $E(r) \setminus E_c(r)$ *is simply connected and hence globally diffeomorphic to* $\mathbb{R}^2$*.*

**Proof:** Let $U \subset E(r) \setminus E_c(r)$ be a connected component of $E(r) \setminus E_c(r)$. In order to prove that $U$ is simply connected we have to show that any continuous map $\gamma : S^1 \to U$ from the unit circle $S^1 \subset \mathbb{R}^2$ into $U$ can be extended to a continuous map $\Gamma : D^2 \to U$, where $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ denotes the closed disk in $\mathbb{R}^2$ whose boundary $\partial D^2 = S^1$. Since any continuous map between open subsets of Euclidean spaces is homotopic to a smooth map (see e.g. Madsen and Tornehave (1997, p. 40)) we can assume, up to an homotopy $F : S^1 \times [0, 1] \to U$, $\gamma$ is smooth. Let $\tilde{\Gamma} : D^2 \to E(r)$ be a continuous extension of $\gamma$ whose existence is guaranteed by the simply-connectedness of $E(r)$ (part (i) of Lemma 2.1). Without loss of generality (see Boothby (1986, p. 197)) we can assume that $\tilde{\Gamma}$ is smooth and (since $D^2$ is compact and the strata $\mathcal{S}_1^j$ are disjoints manifolds) that there exist a finite number, say $\mathcal{S}_1^1, \cdots, \mathcal{S}_1^k$, of strata of $E_c(r) = \mathcal{S}_1$ such that $\tilde{\Gamma}$ is transversal to each $\mathcal{S}_1^h$, $h = 1, \ldots, k$ (see Guillemin and Pollack (1974, pp. 72-73) for this standard transversality theorem). Thus, for each $h$, $C_h = \tilde{\Gamma}^{-1}(\mathcal{S}_1^h)$ is a smooth and closed 1-dimensional submanifold of $D^2$ without boundary (lying in the interior of $D^2$). From the classification of one-dimensional manifolds (see e.g. Milnor (1997)), each $C_h$ is diffeomorphic to the unit circle $S^1$. Denote by $\tilde{\gamma}_h : C_h \to \mathcal{S}_1^h$, $h = 1, \ldots, k$ the restriction of $\tilde{\Gamma}$ to $C_h$ and by $D_h$ the closed disk in $D^2$ whose boundary is $C_h$. Then one can find $\alpha$ integers $\{s_1, \ldots, s_\alpha\} \subset \{1, \ldots, k\}$ such that each $D_{s_a}$, $a = 1, \ldots, \alpha$, is an outtermost disk, namely the inclusion $D_{s_a} \subset D_h$ can occur for some index $h = 1, \ldots, k$ only if $D_{s_a} = D_h$. Since each $\mathcal{S}_1^{s_a}$ is simply–connected (by (ii) of Lemma 2.1) the map $\tilde{\gamma}_{s_a} : C_{s_a} \to \mathcal{S}_1^{s_a}$ admits a continuous extension $\tilde{\Gamma}_{s_a} : D_{s_a} \to \mathcal{S}_1^{s_a}$. Consider the continuous map $\hat{\Gamma} : D^2 \to E(r)$ which coincides with $\tilde{\Gamma}$ on $D^2 \setminus \cup_{a=1}^{\alpha}(D_{s_a} \setminus C_{s_a})$ and with $\tilde{\Gamma}_{s_a}$ on $D_{s_a}$. This map is a continuous extension of $\gamma$ and satisfies $\hat{\Gamma}(D_{s_a}) \subset \mathcal{S}_{s_a}$, for each $a = 1, \ldots, \alpha$, and $\hat{\Gamma}(D^2 \setminus \cup_{a=1}^{\alpha} D_{s_a}) \subset U$ (in particular $\hat{\Gamma}(D^2 \setminus \cup_{a=1}^{\alpha} D_{s_a}) \cap E_c(r) = \emptyset$. In order to find a continuous extension $\Gamma : D^2 \to U$ of $\gamma$ one needs to perturbe $\hat{\Gamma}$ inside each disk $D_{s_a}$ in such a way that $\Gamma(D_{s_a}) \subset U$ . This can be achieved as follows. By the Jordan–Brouwer separation theorem (see Loi and Matta (2009, Theorem 3.2)), $E(r) \setminus \mathcal{S}_1^{s_\alpha}$, $a = 1, \ldots, \alpha$ consists of two connected components. We denote the components containing $\gamma(S^1)$ by $E_\gamma^\alpha$. Notice that $U \subset E_\gamma^\alpha$. Since the $\mathcal{S}_1^{s_a}$'s are disjoints we can find tubular neighbourhoods $T_a$ of $\mathcal{S}_1^{s_a}$ in $E(r)$ such that $\cap_{a=1}^{\alpha} T_a = \emptyset$, $T_a \cap \gamma(S^1) = \emptyset$ and $T_a \cap E_c(r) = \mathcal{S}_1^{s_a}$. Let $t_a$ be the radius of $T_a$. For each $a$, let $n_a : \mathcal{S}_1^{s_a} \to E(r)$ be a normal and unitary vector field along $\mathcal{S}_1^{s_a}$ pointing towards the component $E_\gamma^a$. Then $\epsilon_a n_a : \mathcal{S}_1^{s_a} \to E(r)$, with $0 < \epsilon_a < t_a$, is such that $x + \epsilon_a n_a(x)$ (where "+" denotes the sum in $E(r) \cong \mathbb{R}^2$) belongs to $T_a \cap E_\gamma^a$, for all $x \in \mathcal{S}_1^{s_a}$. Denote by $\tilde{n}_a : T_a' \to E(r)$ an extension of $n_a$ to an open tubular neighbourhood of $T_a'$ of $\mathcal{S}_1^{s_a}$ of radius $t_a' < t_a$ such that $x + \tilde{n}_a(x)$ belongs to $T_a' \cap E_\gamma^a$ for all $x \in T_a'$. For each $a$ consider disjoint closed disks $D_a'' \subset D_a' \subset D^2$ such that $D_{s_a} \subset D_a'' \subset D_a' \subset \hat{\Gamma}^{-1}(T_a)' \subset D^2$ and a smooth map $\rho_\alpha : D^2 \to \mathbb{R}$

which equals 1 on $D_{s_a}$ and vanish outside $D^{''}_a$. The desired continuous extension $\Gamma : D^2 \to U$ of $\gamma$ is defined by $\Gamma(\xi) = \hat{\Gamma}(\xi) + \sum_{a=1}^{\alpha} \rho_a(\xi) \times \tilde{n}_a(\hat{\Gamma}(\xi))$ for $\xi \in D^{'}_a$ and by $\hat{\Gamma}$ on $D^2 \setminus \cup_{a=1}^{\alpha} D^{''}_a$. The last part of the theorem, namely $U \cong \mathbb{R}^2$, follows by the fact that any noncompact simply-connected surface is diffeomorphic to $\mathbb{R}^2$ (see Chapter I in Ahlfors and Sario (1960)).

□

**Remark 2.3** The understanding of the topology of $E(r) \setminus E_c(r)$ in higher dimensions (namely for $l + m \geq 5$) is a difficult task. The main issue is represented by the fact that one does not know the topology of $\mathcal{S}_1$. If one knew that each connected component $\mathcal{S}_1^j$ of $\mathcal{S}_1$ is simply connected, then one could show that each connected component of $E(r) \setminus E_c(r)$ is simply connected. Indeed let $U$ be such a component. Then there exists a connected component $V$ of $E(r) \setminus \mathcal{S}_1$ such that $U = V \setminus \cup_{i\geq 2} \mathcal{S}_i$. Hence (using the simply connectedness of $\mathcal{S}_1^j$) it is not hard to see that the proof of Theorem 2.2 extends to higher dimension yelding the simply connectedness of $V$. Finally, since the codimension of $\mathcal{S}_i$ in $E(r)$ is greater than 3 (it is equals to $i^2 \geq 4$, see Balasko (1979)), a transversality argument (Guillemin and Pollack , 1974, pp. 72-73) similar to that used in the proof of Theorem 2.2 also yields that $U = V \setminus \cup_{i\geq 2} \mathcal{S}_i$ is simply connected.

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