Kähler Maps of Hermitian Symmetric Spaces into Complex Space Forms

Antonio J. Di Scala
Dipartimento di Matematica – Politecnico di Torino – Italy
e-mail address: antonio.discala@polito.it

and

Andrea Loi
Dipartimento di Matematica e Informatica – Università di Cagliari – Italy
e-mail address: loi@unica.it

Abstract

In this paper we give a complete description of the Kähler immersions of Hermitian symmetric spaces into finite or infinite dimensional complex space forms.
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1 Introduction and preliminaries

In his seminal paper Calabi [2] gave a complete answer to the problem of the existence and uniqueness of holomorphic and isometric immersions (from now on Kähler immersions) of a Kähler manifold (M, g) into a finite or infinite dimensional complex space form.

There are three types, up to homotheties, of complex space forms according to the sign of their constant holomorphic sectional curvature:

(i) the complex Euclidean space $\mathbb{C}^N, N \leq \infty$ with the flat metric denoted by G_0. Here $\mathbb{C}^\infty$ is the complex Hilbert space $l^2(\mathbb{C})$ consisting of sequences $z_j, j = 1 \ldots, z_j \in \mathbb{C}$ such that $\sum_{j=1}^{+\infty} |z_j|^2 < +\infty$.

(ii) the complex projective space $\mathbb{C}P^N, N \leq \infty$, with the Fubini–Study metric G_{FS} of holomorphic sectional curvature 4.

(iii) the complex hyperbolic space $\mathbb{C}H^N$ $N \leq \infty$, namely the unit ball in $\mathbb{C}^N$ endowed with the hyperbolic metric G_{hyp} of constant holomorphic sectional curvature -4.

Calabi's first observation was that if a Kähler immersion of (M, g) into a complex space form exists then the metric g is forced to be real analytic being the pull-back via a holomorphic map of the real analytic metric of a complex space form (see Theorem 1.5 below).

Then in a neighborhood of every point $p \in M$, one can introduce a very special Kähler potential D_p^g for the metric g, which Calabi [2] christened *diastasis*. Recall that a Kähler potential is a function Φ defined in a neighborhood of a point p such that $\omega = \frac{i}{2}\bar{\partial}\partial\Phi$, where ω is the Kähler form associated to g. A Kähler potential is not unique: it is defined up to an addition with the real part of a holomorphic function. By duplicating the variables z and $\bar{z}$ an analytic potential Φ can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighborhood U of the diagonal containing $(p, \bar{p}) \in M \times \bar{M}$ (here $\bar{M}$ denotes the manifold conjugated to M). The *diastasis function* is the Kähler potential D_p^g around p defined by

$$D_p^g(q) = \tilde{\Phi}(q, \bar{q}) + \tilde{\Phi}(p, \bar{p}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p}).$$

Example 1.1 The diastasis function $D_p^{G_0} : \mathbb{C}^N \to \mathbb{R}$ of the Euclidean metric G_0 around $p \in \mathbb{C}^N$ is given by the square of the distance between p and q, i.e. $D_p^{G_0}(q) = \sum_{j=1}^N |p_j - q_j|^2$.

Example 1.2 Let $(Z_0, Z_1, \dots, Z_N)$ be the homogeneous coordinates in $\mathbb{C}P^N$ and let $p = [1, 0, \dots, 0]$. In the affine chart $U_0 = \{Z_0 \neq 0\}$ endowed with coordinates $(z_1, \dots, z_n)$, $z_j = \frac{Z_j}{Z_0}$ the diastasis around p reads as:

$$D_p^{G_{FS}}(z) = \log(1 + \sum_{j=1}^n |z_j|^2). \tag{1}$$

Example 1.3 The globally defined diastasis $D_p^{G_{hyp}} : \mathbb{C}H^N \to \mathbb{R}$ of the complex hyperbolic space $\mathbb{C}H^N$ around $p = (0, \dots 0)$ is given by

$$D_p^{G_{hyp}}(z) = \log(1 - \sum_{j=1}^n |z_j|^2). \tag{2}$$

A very useful characterization of the diastasis can be obtained as follows. Let (z) be a system of complex coordinates in a neighbourhood of p where D_p^g is defined and consider its power series development:

$$D_p^g(z) = \sum_{j,k \geq 0} a_{jk}(g) z^{m_j} \bar{z}^{m_k}. \tag{3}$$

Remark 1.4 Here we are using the following convention: we arrange every n-tuple of nonnegative integers as the sequence $m_j = (m_{1,j}, m_{2,j}, \dots, m_{n,j})_{j=0,1,\dots}$ such that $m_0 = (0, \dots, 0)$, $|m_j| \leq |m_{j+1}|$, with $|m_j| = \sum_{\alpha=1}^{n} m_{\alpha,j}$ and $z^{m_j} = \prod_{\alpha=1}^{n} (z_\alpha)^{m_{\alpha,j}}$. Further, we order all the m_j's with the same $|m_j|$ using the lexicographic order in the variables $(z_1, \dots z_n)$. For example, for $n = 2$ we have, $m_0 = (0, 0) < m_1 = (1, 0) < m_2 = (0, 1) < m_3 = (2, 0) < m_4 = (1, 1) < m_5 = (0, 2)$ and so on.

Among all the potentials the diastasis is characterized by the fact that in every coordinates system (z) centered in p the coefficents $a_{jk}(g)$ of the expansion (3) satisfy $a_{j0}(g) = a_{0j}(g) = 0$ for every nonnegative integer j.

The diastasis function is the key tool for studying Kähler immersions between Kähler manifolds due to his hereditary property:

Theorem 1.5 *(Calabi [2]) Let (M, g) be a Kähler manifold which admits a Kähler immersion $f : (M, g) \to (S, G)$ into a real analytic Kähler manifold (S, G). Then the metric g is real analytic. Let $D^g_p : U \to \mathbb{R}$ and $D^G_{f(p)} : V \to \mathbb{R}$ be the diastasis functions of (M, g) and (S, G) around p and $f(p)$ respectively. Then $f^{-1}(D^G_{f(p)}) = D^g_p$ on $f^{-1}(V) \cap U$.*

In the case the ambient space (S, G) is a complex space form it follows by previous theorem and the previous examples that the diastasis function of a Kähler manifold (M, g) which admits a Kähler immersion into (S, G) has a very special form. This is expressed in Theorem 1.8 which goes deeply to the heart of the problem we are dealing with.

In order to state it we need some definitions due to Calabi.

Definition 1.6 *A Kähler immersion f of (M, g) into $\mathbb{C}^N$ (resp. $\mathbb{C}P^N$ or $\mathbb{C}H^N$) is said to be full if $f(M)$ is not contained in any complex totally geodesic hypersurface of $\mathbb{C}^N$ (resp. $\mathbb{C}P^N$ or $\mathbb{C}H^N$).*

Definition 1.7 *We say that the Kähler metric g on a complex manifold M is* resolvable *of rank N at p if the $\infty \times \infty$ matrix $a_{jk}(g)$ given by formula (3) is positive semidefinite and of rank N. Consider the function $e^{D^g_p} - 1$ (resp. $1 - e^{-D^g_p}$) and its power series development: $e^{D^g_p} - 1 = \sum_{j,k \geq 0} b_{jk}(g) z^{m_j} \bar{z}^{m_k}$ (resp. $1 - e^{-D^g_p} = \sum_{j,k \geq 0} c_{jk}(g) z^{m_j} \bar{z}^{m_k}$). The metric g is said to be* 1-resolvable *(resp.* -1-resolvable*) of rank N at p, if the $\infty \times \infty$ matrix $b_{jk}(g)$ (resp. $c_{jk}(g)$) is positive semidefinite and of rank N. If in the previous defintions $N = \infty$ we say that the Kähler metric g is resolvable (resp. ± 1-resolvable) of infinite rank.*

Theorem 1.8 (Calabi) *Let M be a complex manifold endowed with a real analytic Kähler metric g.*

(i) If g is resolvable (resp. 1-resolvable or -1-resolvable) of rank N at $p \in M$ then it is resolvable (resp. 1-resolvable or -1-resolvable) of rank N at every point in M.

(ii) A neighbourhood of a point p admits a (full) Kähler immersion into $\mathbb{C}^N$ (resp. $\mathbb{C}P^N$ or $\mathbb{C}H^N$) if and only if g is resolvable (resp. 1-resolvable or -1-resolvable) of rank at most (exactly) N at p.

(iii) Two full Kähler immersions into $\mathbb{C}^N$ (resp. $\mathbb{C}P^N$ or $\mathbb{C}H^N$) are congruent under the isometry group of $\mathbb{C}^N$ (resp. $\mathbb{C}P^N$ or $\mathbb{C}H^N$).

It follows by the previous theorem that if there exists a neighborhood of a point $p \in M$ which admits a Kähler immersion into a given complex space form (S, G) then any other point admits such a neighborhood. Moreover if the manifold M is assumed to be simply-connected these local immersions glue together as expressed by the following theorem (see [2] for a proof).

Theorem 1.9 *(***Calabi***) Let (M, g) be a simply-connected Kähler manifold. If a neighbourhood of a point $p \in M$ can be Kähler immersed into a complex space form (S, G) then the whole (M, g) admits a Kähler immersion into (S, G).*

Remark 1.10 Theorem 1.8 can be applied when one has the explicit expression of the metric and hence of its diastasis function. In some very special cases like for the complex space forms [2] or for Hartogs domains [9] the matrices $a_{jk}(g)$, $b_{jk}(g)$ or $c_{jk}(g)$ in Definition 1.7 are diagonal and hence Calabi's criterion is easy to apply. Even for Einstein or homogeneous metrics this criterion is very hard to be handled (we refer the reader to [6], [7] and [17] for the Einstein case and to [10] and the reference therein for the homogeneous case). The authors believe that the resolvability and ± 1-resolvability of a Kähler metric deserve further investigation.

In this paper we study the Kähler immersions of Hermitian symmetric spaces of noncompact type endowed with their Bergman metrics into the infinite dimensional hyperbolic space $\mathbb{C}H^\infty$ or the infinite dimensional Euclidean space $l^2(\mathbb{C})$. Our main result is Theorem 3.3 where we show that *among all the Hermitian symmetric spaces of noncompact type the hyperbolic space (or the product of hyperbolic spaces) is the only one which admits* *a Kähler immersion into* $\mathbb{C}H^\infty$ *or* $l^2(\mathbb{C})$. The proof of Theorem 3.3 relies on Theorem 1.8 and on the explicit knowledge of the diastasis function of the Hermitian symmetric spaces of noncompact type. Our result together with some known results, summarized in Theorems 1–4 below, give us a complete description of those Hermitian symmetric spaces (not necessarily of noncompact type) which admit a Kähler immersion into a given complex space form (see also the table at the end of the paper). Observe, finally, that every Hermitian symmetric space of noncompact type (D, g) admits isometric and equivariant immersions into $l^2(\mathbb{C})$ (cfr. e.g. [18]). From this point of view, our result shows that one of these immersions can be holomorphic only when (D, g) is the hyperbolic space (or product of hyperbolic spaces).

The paper is organized as follows. In the next section we recall the basic material on the Bergman kernel of an Hermitian symmetric space of noncompact type and we describe their diastasis. Section 3 is dedicated to the proof of our Theorem 3.3.

2 The diastasis of the Bergman metrics

Every Hermitian symmetric space is a product of irreducible ones. Irreducible ones are of two types: compact and noncompact.

In this section we are interested only to the Hermitian symmetric spaces of noncompact type. We refer the reader to [10] for details.

An irreducible Hermitian symmetric space of *noncompact type* is, up to multiply the metric for a positive constant, biholomorphically isometric to (D, g), where D is a bounded symmetric domain in $\mathbb{C}^n$ which can be chosen circular (i.e. $z \in D, \theta \in \mathbb{R}$ imply $e^{i\theta} z \in D$) and g is the Bergman metric and D. Therefore a (globally defined) potential for the metric g is given by $\Phi(z) = \log K_D(z, z)$, where $K_D(z, z)$ is the Bergman kernel function on D. The Kähler form associated to g is $\omega = \frac{i}{2}\partial\bar{\partial}\log K_D$. Observe that the circularity of D implies that $K_D(z, 0) = K_D(0, z) = \frac{1}{\mathrm{vol}(D)}$, where $\mathrm{vol}(D)$ is the Euclidean volume. Therefore the diastasis function $D_0^g : D \to \mathbb{R}$ around 0 is given by:

$$D_0^g(z) = \log[\mathrm{vol}(D) K_D(z, z)]. \qquad (4)$$

The Hermitian symmetric space of noncompact type can be divided into two categories: the classical and the exceptional one. We start with the classical domains (cfr. pp. 33-36 in [12]). There are four kind of classical domains which can be described in terms of complex matrices as follows (m an n are nonnegative integers):

- $D_1[m \cdot n] = \{Z \in M_{m,n}(\mathbb{C})| \ I_m - ZZ^* > 0\}$
- $D_2[\frac{n(n+1)}{2}] = \{Z \in M_n(\mathbb{C})| \ Z = Z^T, \ \ I_n - ZZ^* > 0\}$
- $D_3[\frac{n(n-1)}{2}] = \{Z \in M_n(\mathbb{C})| \ Z = -Z^T, \ \ I_n - ZZ^* > 0\}$
- $D_4[n] = \{Z = (z_1, \dots, z_n) \in \mathbb{C}^n)| \ \sum_{j=1}^n |z_j|^2 < 1, \ \ 1 + |\sum_{j=1}^n z_j^2|^2 - 2\sum_{j=1}^n |z_j|^2 > 0\}$,

where I_m (resp. I_n) denotes the $m \times m$ (resp. $n \times n$) identity matrix, and $A > 0$ means that A is positive definite. We simply write $D_\alpha, \ \alpha = 1, 2, 3, 4$ instead of $D_1[m \cdot n]$, $D_2[\frac{n(n+1)}{2}]$, $D_3[\frac{n(n-1)}{2}]$, $D_4[n]$ when the dimension of the domain (i.e. the number between the square brackets) we are working with is clear from the context.

The Bergman Kernels of these domains, are given by:

$$K_1(Z) = \frac{1}{V(D_1)}[\det(I_m - ZZ^*)]^{-(m+n)} \tag{5}$$

$$K_2(Z) = \frac{1}{V(D_2)}[\det(I_n - ZZ^*)]^{-(n+1)} \tag{6}$$

$$K_3(Z) = \frac{1}{V(D_3)}[\det(I_n - ZZ^*)]^{-(n-1)} \tag{7}$$

$$K_4(z) = \frac{1}{V(D_4)}(1 + |\sum_{j=1}^n z_j^2|^2 - 2\sum_{j=1}^n |z_j|^2), \tag{8}$$

where $V(D_\alpha), \alpha = 1, \dots, 4$ is the total volume of D_α with respect to the Euclidean measure of the ambient complex Euclidean space.

We denote by $g_\alpha, \alpha = 1 \dots 4$ the Bergman metric of these domains, namely the metric whose associated Kähler form is given by: $\omega_\alpha = \frac{i}{2}\partial\bar{\partial}\log K_\alpha$, $\alpha = 1 \dots 4$.

Notice that for some values of nonnegative integers m and n the domains above are complex hyperbolic spaces or product of hyperbolic spaces. More precisely, up to biholomorphic isometries, we have:

$$(D_1[m \cdot 1], g_1) = (\mathbb{C}H^m, (m+1)G_{hyp}) \tag{9}$$

$$(D_1[1 \cdot n], g_1) = (\mathbb{C}H^n, (n+1)G_{hyp}) \tag{10}$$

$$(D_2[1], g_2) = (D_3[1], g_3) = (D_4[1], g_4) = (\mathbb{C}H^1, 2G_{hyp}) \tag{11}$$

$$(D_3[3], g_3) = (\mathbb{C}H^3, 4G_{hyp}) \tag{12}$$

$$(D_4[2], g_4) = (\mathbb{C}H^1 \times \mathbb{C}H^1, 2(G_{hyp} \oplus G_{hyp})). \tag{13}$$

Equalities (9), (10), (11), (12) follow easily from the definition of (D_α, g_α) for $\alpha = 1, 2, 3$, while equality (13) follows from the fact that the biholomorphism

$$\varphi : D_4[2] \to \mathbb{C}H^1 \times \mathbb{C}H^1, (z_1, z_2) \mapsto (z_1 + iz_2, z_1 - iz_2)$$

satisfies

$$\varphi^*(2(G_{hyp} \oplus G_{hyp})) = g_4.$$

We now describe what we need about exceptional domains. Recall that from E. Cartan's classification there are just two exceptional hermitian symmetric spaces of noncompact type. Namely, $EIII^d$ and $EVII^d$. The notation is justified by the fact that these spaces are the dual of $EIII = E_6/SO(2).Spin(10)$ and $EVII = E_7/SO(2).E_6$ the two Hermitian symmetric spaces of compact type. From tables VII and VIII and Theorem 7.2 in [3] we get the following inclusions of totally geodesic and complex submanifolds:

$$DIII(5) \subset EIII \subset EVI,$$

where $DIII(5) = SO(10)/U(5)$. (In the terminology of [3] each totally geodesic submanifold in the above sequence is *a polar* M_+ of the respective ambient manifold). Since complex and totally geodesic submanifolds of symmetric spaces are characterized by Lie triples the above inclusions are also valid for the non-compact duals. Namely,

$$DIII(5)^d = SO^*(10)/U(5) \subset EIII^d \subset EVII^d. \tag{14}$$

Observe finally that $DIII(5)^d = D_3[15] = \{Z \in M_6(\mathbb{C}) |\ Z = -Z^T,\ I_6 - ZZ^* > 0\}$ (see e.g. [13, pp. 74]).

3 The main results

This section is dedicated to the description of all the Hermitian symmetric spaces equipped with their Bergman metric g, which admit a Kähler immersion into a given finite or infinite dimensional complex space form (S, G). The following Theorems $1 - 4$ allow us to restrict ourself to the case of Kähler immersions of n-dimensional symmetric spaces (D, g) into complex space forms (S, G) of the following kind:

1. (D, g) is of noncompact type and there is not any $\lambda \in \mathbb{R}^+$ such that (D, g) is biholomorphically isometric to $(\mathbb{C}H^n, \lambda G_{hyp})$ or to the product of such factors, namely $(\mathbb{C}H^{n_1} \times \cdots \times \mathbb{C}H^{n_l}, \lambda_1 G_{hyp} \oplus \cdots \oplus \lambda_l G_{hyp})$, $n_1 + \ldots n_l = n$ and $\lambda_j \in \mathbb{R}^+$.

2. (S, G) is either the infinite dimensional hyperbolic space $\mathbb{C}H^\infty$ or the infinite dimensional Euclidean space $l^2(\mathbb{C})$.

Theorem 1 *(cfr. [2]) The n-dimensional Euclidean space $\mathbb{C}^n$ admits a full Kähler embedding into $\mathbb{C}P^\infty$. On the other hand an open subset of $\mathbb{C}^n$ cannot be Kähler immersed into $\mathbb{C}P^N$ for N finite and into $\mathbb{C}H^N$ for $N \leq \infty$.*

Proof: It is not hard to see that the map $f : \mathbb{C}^n \to \mathbb{C}P^\infty : z = (z_1, \ldots z_n) \mapsto (\ldots, \sqrt{\frac{1}{m_j!}} z^{m_j}, \ldots)$, where $m_j = m_{1,j}! \cdots m_{n,j}!$ (cfr. Remark 1.4) is a full Kähler immersion $\mathbb{C}^n$ into $\mathbb{C}P^\infty$, namely $f^*(G_{FS}) = G_0$. By (iii) in Theorem 1.8 this implies that $\mathbb{C}^n$ cannot be Kähler immersed into $\mathbb{C}P^N$ for N finite. Finally, it is immediate to see that G_0 is not -1-resolvable of any rank and the conclusion follows by Theorem 1.8. $\square$

Theorem 2 *An Hermitian symmetric space of compact type (C, g) admits a full Kähler embedding into $\mathbb{C}P^N$. On the other hand, an open subset of (C, g) cannot be Kähler immersed into $\mathbb{C}^N$ or $\mathbb{C}H^N$ for any $N \leq \infty$.*

Proof: The first part is well-known since the work of Borel and Weil (see [10] or [16] for a proof). To prove the second part assume that there exists an open subset of C as in the statement. Since C is simply-connected it follows by Theorem 1.9 that the whole (C, g) would admit a Kähler immersion into $\mathbb{C}^N$ or $\mathbb{C}H^N$ which is impossible due to the compactness of C. $\square$

Theorem 3 *A n-dimensional Hermitian symmetric space of noncompact type (D, g) admits a full Kähler embedding into $\mathbb{C}P^\infty$. Assume that an open subset of (D, g) admits a Kähler immersion into a finite dimensional complex space form (S, G). Then $(S, G) = \mathbb{C}H^N$ and $(D, g) = \mathbb{C}H^n$.*

Proof: The first part is an immediate consequence of the definition of the Bergman metric (see [10] and [11] for details). By (iii) in Theorem 1.8 this implies that it cannot exist a Kähler immersion of (D, g) into $\mathbb{C}P^N$ with N finite. We refer the reader to [5], [15] or [17] for the proof that (D, g) cannot be Kähler immersed into $\mathbb{C}^N$ with N finite. $\square$

Theorem 4 *(cfr. [2])* $(\mathbb{C}H^n, dG_{hyp})$ *and* $(\mathbb{C}H^{n_1} \times \cdots \times \mathbb{C}H^{n_l}, d_1 G_{hyp} \oplus \cdots \oplus d_l G_{hyp})$ *admit a full Kähler embedding into* $l^2(\mathbb{C})$*.*

Proof: First, observe that if two Kähler manifolds (M_1, g_1) and (M_2, g_2) admit Kähler immersions, say f_1 and f_2 into $l^2(\mathbb{C})$ then the Kähler manifold $(M_1 \times M_2, g_1 \oplus g_2)$ admits a Kähler immersion into $l^2(\mathbb{C})$ obtained by mapping $(z_1, z_2) \in M_1 \times M_2$ to $(\lambda_1 f_1(z_1), \lambda_2 f_2(z_2)) \in l^2(\mathbb{C})$. On the other hand the map

$$f : \mathbb{C}H^n \to l^2(\mathbb{C}) : z = (z_1, \dots z_n) \mapsto (\dots, \sqrt{\frac{(|m_j| - 1)!}{m_j!}} z^{m_j}, \dots) \qquad (15)$$

is a Kähler embedding of $\mathbb{C}H^n$ into $l^2(\mathbb{C})$ (see [2]). $\square$

Notice that the observation made at the begining of the proof of the previous theorem can be reversed as expressed by the following lemma which will be used in the proof of Theorem 3.3.

Lemma 3.1 *A Kähler map* $f : M \times N \to l^2(\mathbb{C})$ *from a product* $M \times N$ *of two Kähler manifolds is a product, i.e.* $f(p, q) = (f_1(p), f_2(q))$ *where* $f_1 : M \to l^2(\mathbb{C})$ *and* $f_2 : M \to l^2(\mathbb{C})$ *are Kähler maps.*

Proof: Let $\alpha(X, Y)$ be the second fundamental form of the Kähler map f. In order to show that f is a product it is enough to show that $\alpha(TM, TN) \equiv 0$, see [14] and [5, Lemma 2.5.]. The Gauss equation implies the following equation for the holomorphic bisectional curvature of $M \times N$, see [8, Prop. 9.2, pp. 176]:

$$-\langle R_{X,JX} JY, Y \rangle = 2\|\alpha(X, Y)\|^2,$$

where R is the curvature tensor of $M \times N$. Thus, if $X \in TM$ and $Y \in TN$ we get that $\alpha(X, Y) = 0$. $\square$

Remark 3.2 It is interesting to remark that there are not Kähler maps from a product $M \times N$ of Kähler manifolds into $\mathbb{C}H^\infty$, see [1, Thm.2.11].

We are now in the position to state and prove our main result.

Theorem 3.3 *If an open subset* U *of a* n*-dimensional Hermitian symmetric space of noncompact type* (D, g) *admits a Kähler immersion into* $\mathbb{C}H^\infty$ *then* $(D, g) = \mathbb{C}H^n$*. If* U *admits a Kähler immersion into* $l^2(\mathbb{C})$ *then* (D, g) *is* *either* $(\mathbb{C}H^n, \lambda G_{hyp})$ *for some* $\lambda \in \mathbb{R}^+$ *or* $(\mathbb{C}H^{n_1} \times \cdots \times \mathbb{C}H^{n_l}, \lambda_1 G_{hyp} \oplus \cdots \oplus \lambda_l G_{hyp})$. *Furthermore, if* $f : \mathbb{C}H^{n_1} \times \cdots \times \mathbb{C}H^{n_l} \to l^2(\mathbb{C})$ *is a Kähler immersion then, up to unitary transformation of* $l^2(\mathbb{C})$, f *is the product of* l *maps i.e.* $f = (f_1, \ldots, f_l)$ *where each* $f_j : \mathbb{C}H^{n_j} \to l^2(\mathbb{C})$ *is given by (15) with* $n = n_j$.

Proof: Let $f : U \to \mathbb{C}H^\infty$ be a Kähler immersion of on open subset $U \subset (D_\alpha, g_\alpha)$, where (D_α, g_α) is a classical bounded domain. Without loss of generality we can assume that the origin belongs to U and that f is an embedding. It follows by formula (4) and Theorem 1.5 that the diastasis around the origin of the restriction of g_α to U (which we still denote by g_α) is given by the restriction to U of the function

$$D_0^{g_\alpha}(Z) = \log[V(D_\alpha)K_\alpha(Z)], \ \alpha = 1 \ldots, 4. \tag{16}$$

In order to study the -1-resolvability of g_α consider the function $1 - e^{-D_0^{g_\alpha}}$ on U and its power expansion

$$1 - e^{-D_0^{g_\alpha}} = 1 - (V(D_\alpha)K_\alpha(Z))^{-1} = \sum_{j,k=0}^{\infty} c_{jk}(g_\alpha) Z^{m_j} \overline{Z}^{m_k}$$

in the coordinates $Z = (Z_{pq})$, where Z_{pq} are the entries of the matrix Z in definition of the domain D_α. By the explicit expression of the Bergman kernels K_α (see (5), (6), (7), (8) above) the matrix $c_{jk}(g_\alpha)$ is such that $c_{jk}(g_\alpha) = 0$ for j, k sufficiently large. In particular this matrix cannot have infinite rank. Thus, by Theorem 1.8, there exists a nonnegative integer N such that $f(U) \subset \mathbb{C}H^N \subset \mathbb{C}H^\infty$ and by Theorem 3 $(D, g) = \mathbb{C}H^n$. This concludes the proof of the first part of the theorem for the case of classical domains. The prove for the exceptional domains follows immediatly from the inclusions (14). Indeed, a Kähler map of an exceptional domains into $\mathbb{C}H^\infty$ would give rise to a Kähler map of the classical domain $D_3[5]$ into $\mathbb{C}H^\infty$ contradicting what we just proved for the classical domains.

Assume now that there exists a Kähler immersion f from $U \subset (D, g)$ into $l^2(\mathbb{C})$. Observe that the natural inclusions

$$D_2[2] \subset D_2[n], n \geq 3 \qquad D_2[2] \subset D_1[m \cdot n], m, n \geq 2$$

$$D_3[4] \subset D_3[n], n \geq 5 \qquad D_4[3] \subset D_4[n], n \geq 4$$

are indeed Kähler embedding with respect to the Bergman metrics. Therefore by (9), (10), (11) (12), (13) and inclusions (14) in order to prove the second part of our theorem we can restrict ourself to show that $(D_2[2], g_2)$, $(D_3[4], g_3)$, $(D_4[3], g_4)$ cannot be (locally) immersed into $l^2(\mathbb{C})$. By Theorem 1.8 and Theorem 1.5 this is equivalent to prove that diastasis around the origin of these three domains is not resolvable of infinite rank.

We start from $(D_2[2], g_2)$ and $(D_4[3], g_4)$ which are both three-dimensional. Using the expression of (6), (8) and (16) one can write down the expression of the diastasis of the metric g_2 of $D_2[2]$ and g_4 on $D_4[3]$ around the origin, namely:

$$D_0^{g_2}(z) = -3\log(1-|z_1|^2-2|z_2|^2-|z_3|^2+|z_1|^2|z_3|^2+|z_2|^4-z_1z_3\bar{z}_2^2-z_2^2\bar{z}_1\bar{z}_3)$$

and

$$D_0^{g_4}(z) = -\log[1-2(|z_1|^2+|z_2|^2+|z_3|^2)+|z_1^2+z_2^2+z_3^2|^2].$$

By taking the power series expansions $D_0^{g_2}(z) = \sum_{j,k=0}^{\infty} a_{jk}(g_2)z^{m_j}\bar{z}^{m_k}$ and $D_0^{g_4}(z) = \sum_{j,k=0}^{\infty} a_{jk}(g_4)z^{m_j}\bar{z}^{m_k}$ in the coordinates z_1, z_2, z_3 and their complex conjugate $\bar{z}_1, \bar{z}_2, \bar{z}_3$ one gets:

$$D_0^{g_2}(z) = 3(|z_1|^2+2|z_2|^2+|z_3|^2+\frac{|z_1|^4}{2}+|z_1|^2|z_2|^2+z_1z_3\bar{z}_2^2+z_2^2\bar{z}_1\bar{z}_3+2|z_2|^2|z_3|^2+\frac{|z_3|^2}{2}+\cdots),$$

and

$$\begin{aligned} D_0^{g_4}(z) &= 2|z_1|^2+2|z_2|^2+2|z_3|^2+|z_1|^4+|z_2|^4+|z_3|^4+4|z_1|^2|z_2|^2+4|z_1|^2|z_3|^2 \\ &+ 4|z_2|^2|z_3|^2-z_1^2\bar{z}_2^2-z_2^2\bar{z}_1^2-z_1^2\bar{z}_3^2-z_3^2\bar{z}_1^2-z_2^2\bar{z}_3^2-z_3^2\bar{z}_2^2+\cdots, \end{aligned}$$

respectively. From these expansions and by our convention on the order of the m_j (see Remark 1.4 above) the matrices $A_2 = (a_{jk}(g_2))_{j,k=1,\ldots 8}$ and $A_4 = (a_{jk}(g_4))_{j,k=1,\ldots 10}$ are given by:

$$A_2 = 3\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&2&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&\frac{1}{2}&0&0&0\\ 0&0&0&0&0&2&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&1 \end{pmatrix} A_4 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&2&0&0&0&0&0&0&0&0\\ 0&0&2&0&0&0&0&0&0&0\\ 0&0&0&2&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&-1&0&-1\\ 0&0&0&0&0&4&0&0&0&0\\ 0&0&0&0&0&0&4&0&0&0\\ 0&0&0&0&-1&0&0&1&0&-1\\ 0&0&0&0&0&0&0&0&4&0\\ 0&0&0&0&-1&0&0&-1&0&1 \end{pmatrix}$$

These matrices are not semipositive definite. Indeed, A_2 (resp. A_4) has a negative eigenvalue, namely $3(1-\sqrt{5})$ (resp. -1). Hence the whole matrices $a_{jk}(g_2)$, $a_{jk}(g_4)$ $j,k=1\dots\infty$ cannot be semipositive definite and the diastasis of g_2 and g_4 is not resolvable (of any rank).

Finally let us consider the six-dimensional bounded domain $(D_3[4], g_3)$. Using (7) and (16) one gets (after some computations) the following expression of the diastasis of the metric g_3 of $D_3[4]$ around the origin:

$$\begin{array}{rcl} D_0^{g_3}(z) & = & -3[\log(1-\sum_{j=1}^6 |z_j|^2+|z_1|^2|z_6|^2+|z_2|^2|z_5|^2+|z_3|^2|z_4|^2 \\ & - & z_1z_6\overline{z}_2\overline{z}_5 - z_2z_5\overline{z}_1\overline{z}_6 + z_1z_6\overline{z}_3\overline{z}_4 + z_3z_4\overline{z}_1\overline{z}_6 - z_2z_5\overline{z}_3\overline{z}_4 - z_3z_4\overline{z}_2\overline{z}_5))] \end{array}$$

By taking its power expansion $D_0^{g_3}(z)=\sum_{j,k=0}^{\infty} a_{jk}(g_3)z^{m_j}\overline{z}^{m_k}$ in $z_1,\dots,z_6$ and their complex conjugate $\bar{z}_1,\dots,\bar{z}_6$ one gets:

$$\begin{array}{rcl} D_0^{g_3}(z) & = & 3[\sum_{j=1}^6(|z_j|^2+\frac{|z_j|^4}{2})+|z_1|^2\sum_{j=2}^5|z_j|^2+|z_2|^2(|z_3|^2+|z_4|^2+|z_6|^2) \\ & + & (|z_3|^2+|z_4|^2)(|z_5|^2+|z_6|^2)+|z_5|^2|z_6|^2+z_1z_6\overline{z}_2\overline{z}_5+z_2z_5\overline{z}_1\overline{z}_6 \\ & - & z_1z_6\overline{z}_3\overline{z}_4 - z_3z_4\overline{z}_1\overline{z}_6 + z_2z_5\overline{z}_3\overline{z}_4 + z_3z_4\overline{z}_2\overline{z}_5 + \cdots] \end{array}$$

A long but straightforward calculation, using the previous expansion (and again our convention on the order of the multi-indexes m_j in Remark 1.4) shows that the 28×28 matrix $A_3=(a_{jk}(g_3))_{j,k=1,\dots 28}$ has the following form:

$$A_3 = 3\begin{pmatrix} B_1 & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & B_2 \end{pmatrix},$$

where B_1 (resp. B_2) is the 12×12 (resp. 8×8) diagonal matrix whose diagonal elements are $\{0,1,1,1,1,1,1,\frac{1}{2},1,1,1,1\}$ (resp. $\{1,1,1,1,1,\frac{1}{2},1,\frac{1}{2}\}$) and

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Observe now that M has a negative eigenvalue namely -2 and then the matrix $a_{jk}(g_3)$ is not semipositive definite, i.e. the metric g_3 is not resolvable (of any rank). Finally, the proof of the last part of our theorem follows by Lemma 3.1 and (iii) of Theorem 1.8. □

An immediate corollary of our theorem is:

Corollary 3.4 *If an irreducible Hermitian symmetric space of noncompact type admits a Kähler immersion into $l^2(\mathbb{C})$ or $\mathbb{C}H^\infty$ then it has rank one.*

For the reader convenience we conclude this article with the following table where we summarize the complete picture obtained so far (N is a sufficiently large nonnegative integer).

$\exists$ Kähler immersion	$\mathbb{C}^N$	$\mathbb{C}P^N$	$\mathbb{C}H^N$	$l^2(\mathbb{C})$	$\mathbb{C}P^\infty$	$\mathbb{C}H^\infty$
$\mathbb{C}^n$	yes	no	no	yes	yes	no
(C, g) compact type	no	yes	no	no	yes	no
$\mathbb{C}H^n$	no	no	yes	yes	yes	yes
$D_1[m \cdot n]$ $(m, n \geq 2)$	no	no	no	no	yes	no
$D_2[\frac{n(n+1)}{2}]$ $(n \geq 2)$	no	no	no	no	yes	no
$D_3[\frac{n(n-1)}{2}]$ $(n \geq 3)$	no	no	no	no	yes	no
$D_4[n]$ $(n \geq 3)$	no	no	no	no	yes	no
Exceptional domains	no	no	no	no	yes	no

Observe that the values of m, n not included in the table are covered by the equalities (9)-(13) at the end of the previous section. Observe also that our results extend without any substantial change to the case when the ambient space is the complex projective (resp. complex hyperbolic) space equipped with a metric of positive (resp. negative) constant holomorphic sectional curvature, not necessarily of value 4 (resp. -4).

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