Holomorphic Maps of Hartogs Domains into Complex Space Forms

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Abstract

Let H_F be a Hartogs domain with strictly pseudoconvex boundary endowed with its natural Kähler metric g_F (see Sect. 2).

Following Calabi [1] we give necessary and sufficient conditions for (H_F, g_F) to admit a holomorphic and isometric map into a finite or infinite dimensional complex space form. Moreover we prove that, if g_F is Einstein, then (H_F, g_F) is biholomorphically isometric to the unit ball endowed with the hyperbolic metric.

Keywords: Kähler metrics ; Kähler-Einstein metrics, diastasis.
Subj.Class: 53C55, 53C25.

1 Introduction and Preliminaries

The study of holomorphic and isometric immersions of a Kähler manifold (M, g) into a finite or infinite dimensional complex space form started with Calabi [1] to whom we refer for details and further results (see also [2], [4], [5], [7], [8]). There are three types of complex space forms, depending on the sign of (the constant) holomorphic sectional curvature:

(i) the complex Euclidean space $\mathbb{C}^N, N \leq \infty$ with the canonical metric denoted by G_{can} of zero holomorphic sectional curvature;

(ii) the complex projective space $\mathbb{C}P^N_b$ ($b > 0$ and $N \leq \infty$) with the Fubini–Study metric denoted by $G_{FS}(b)$ of positive holomorphic sectional curvature $4b$;

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(iii) the complex hyperbolic space $\mathbb{C}P^N_b$ ($b<0$ and $N \leq \infty$), namely the domain $B \subset \mathbb{C}^N$ given by

$$B = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^N | \sum_{j=1}^N |z_j|^2 < -\frac{1}{b}\}.$$

endowed with the hyperbolic metric denoted by $G_{hyp}(b)$ of negative holomorphic sectional curvature $4b$.

The first important result due to Calabi [1] is the following:

Theorem 1.1 *If a Kähler manifold (M, g) admits a holomorphic and isometric immersion into a complex space form then g is real analytic.*

If a Kähler metric g on M is real analytic, then in a neighborhood of every point $p \in M$, one can introduce a very special Kähler potential D_p for the metric g, which Calabi christened *diastasis*. Recall that a Kähler potential is an analytic function Φ defined in a neighborhood of a point p such that $\omega = \frac{i}{2}\bar{\partial}\partial\Phi$, where ω is the Kähler form associated to g. In a complex coordinate system (z) around p one has:

$$g_{\alpha\bar{\beta}} = 2g(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}) = \frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta}.$$

A Kähler potential is not unique: it is defined up to the sum with the real part of a holomorphic function. By duplicating the variables z and $\bar{z}$ a potential Φ can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighborhood U of the diagonal containing $(p, \bar{p}) \in N \times \bar{N}$ (here $\bar{N}$ denotes the manifold conjugated of N). The *diastasis function* is the Kähler potential D_p around p defined by

$$D_p(q) = \tilde{\Phi}(q, \bar{q}) + \tilde{\Phi}(p, \bar{p}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p}).$$

Since D_p is real analytic one can consider its power series developments:

$$D_p(z, \bar{z}) = \sum_{j,k \geq 0} a_{jk} z^{m_j} \bar{z}^{m_k}, \tag{1}$$

Here we are using the following convention: we arrange every n-tuple of non-negative integers as the sequence $m_j = (m_{1,j}, m_{2,j}, \ldots, m_{n,j})_{j=0,1,\ldots}$ such that $m_0 = (0, \ldots, 0)$, $|m_j| \leq |m_{j+1}|$, with $|m_j| = \sum_{\alpha=1}^n m_{\alpha,j}$ and $z^{m_j} = \prod_{\alpha=1}^n (z_\alpha)^{m_{\alpha,j}}$.

Example 1.2 Let p be the origin in $\mathbb{C}^N$. Then the diastasis at p is given by:

$$D_p(q) = |p - q|^2, \;\; \forall q \in \mathbb{C}^N$$

Example 1.3 Let $(Z_0, Z_1, \dots, Z_N)$ be the homogeneous coordinates in $\mathbb{C}P^N_b, b > 0$ and let $p = [1, 0, \dots, 0]$. In the affine chart $U_0 = \{Z_0 \neq 0\}$ endowed with coordinates $(z_1, \dots, z_n), z_j = \frac{Z_j}{Z_0}$ the diastasis at p reads as:

$$D_p(z_j, \bar{z}_j) = \frac{1}{b} \log(1 + b \sum_{j=1}^{n} |z_j|^2). \tag{2}$$

If one takes $b < 0$, formula (2) define the diastasis at p of the complex hyperbolic space $\mathbb{C}P^N_b, b < 0$.

We are now ready to state the general criterium due to Calabi [1] for a Kähler manifold to admit a holomorphic and isometric immersion into a complex space form. This is expressed by Theorem 1.5 and Theorem 1.6 below. First we need the following:

Definition 1.4 *Let (M, g) be a real analytic Kähler manifold an let p be a point in M. We say that the Kähler metric g is resolvable of rank N at p if the $\infty \times \infty$ matrix a_{jk} given by formula (1) is positive semidefinite and of rank N. If $N = \infty$ we say that the Kähler metric g is resolvable of infinite rank.*

Theorem 1.5 (see Calabi [1]) *Let (M, g) be a real analytic Kähler manifold.*

(i) if g is resolvable of rank N at $p \in M$ then it is resolvable of rank N at every point in M;

(ii) suppose that M is simply-connected. Then (M, g) admits a holomorphic and isometric immersion into $\mathbb{C}^N$ if and only if g is resolvable of rank at most N;

(iii) let $\varphi : M \to \mathbb{C}^N$ be a holomorphic and isometric immersion which is full (i.e. the image $\varphi(M)$ is not contained in any hyperplane of $\mathbb{C}^N$), then N is determined by the metric g and two such immersions are congruent under the unitary group $U(N)$.

Now, we consider the case of holomorphic immersions into $\mathbb{C}_b^N$. Let D_p be the diastasis relative to a point $p \in M$. Consider the "modified diastasis" $\frac{1}{b}(e^{bD_p} - 1)$ and its power series development:

$$\frac{1}{b}(e^{bD_p} - 1) = \sum_{j,k \geq 0} b_{jk} z^{m_j} \bar{z}^{m_k}. \tag{3}$$

We say that the metric g is *b-resolvable* of rank N at p, if the $\infty \times \infty$ matrix b_{jk} given by formula (3) is positive semidefinite and of rank N.

Theorem 1.6 (see Calabi [1]) *Let (M, g) be a real analytic Kähler manifold and let b a real number different from 0.*

(i) if g is b-resolvable of rank N at $p \in M$ then it is resolvable of rank N at every point in M;

(ii) suppose that M is simply-connected. Then (M, g) admits a holomorphic and isometric immersion into $\mathbb{C}P_b^N$ if and only if g is b-resolvable of rank at most N;

(iii) let $\varphi : M \to \mathbb{C}P_b^N$ be a holomorphic and isometric immersion which is full (i.e. the image $\varphi(M)$ is not contained in any hyperplane of $\mathbb{C}P_b^N$). Then N is determined by the metric g and the constant b and two such immersions are congruent under the isometry group of $\mathbb{C}P_b^N$.

In this paper we study the holomorphic and isometric immersions of a Hartogs domain (H_F, g_F) (see Sect. 2) into a complex space form. The main results of this paper are contained in Sect. 2 and 3. In Section 2 we give a necessary and sufficient condition for (H_F, g_F) to admit a holomorphic and isometric immersion into a complex space form (see Theorem 2.1.1 and Theorem 2.2.1). Moreover, we prove that (H_F, g_F) cannot be isometrically immersed either into $\mathbb{C}^N$ or $\mathbb{C}P_b^N$ for $b > 0$ and N finite (see Corollaries 2.1.2 and 2.2.2). The previous result can be considered as an extension of a result of Calabi [1] (see Remark 2.2.5). In Section 3 we prove that if g_F satisfies the Einstein condition then (H_F, g_F) is biholomorphically isometric to $\mathbb{C}P_{-1}^2$.

2 The Main Results

Let $F : [0, x_0) \to (0, +\infty]$ be a non increasing C^2 function from the interval $[0, x_0) \subset \mathbb{R}$ to the extended positive reals $(0, +\infty]$ (the case $x_0 = +\infty$ is not excluded). The Hartogs domain corresponding to the function F is the 2-complex dimensional manifold $H_F \subset \mathbb{C}^2$ defined as:

$$H_F = \{(z_1, z_2) \in \mathbb{C}^2 | \ |z_1|^2 < x_0, |z_2|^2 < F(|z_1|^2)\} \tag{4}$$

In the hypothesis that $F(0) < \infty$, one can define a real 2-form on H_F by

$$\omega_F = \frac{i}{2}\partial\bar{\partial}\log\frac{1}{F(|z_1|^2) - |z_2|^2}. \tag{5}$$

Theorem 2.1 *(cf. [3]) The following conditions are equivalent:*

(i) ω_F *is a Kähler form*

(ii) $\left(\frac{xF'}{F}\right)' < 0,\ \forall x \in [0, x_0)$, *(where F' denotes the first derivative of F).*

(iii) ∂H_F, *the boundary of H_F, is strictly pseudoconvex.*

Proof: Let $\omega_F = \frac{i}{2}\sum_{j,k=1}^{2} g_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$ be the expression of the Kähler form ω_F in the (global) coordinates (z_1, z_2). A simple calculation shows that

$$\begin{aligned} g_{1\bar{1}} &= \tfrac{-HF' - H|z_1|^2F'' + |z_1|^2F'^2}{H^2} \ |_{x=|z_1|^2}, \\ \bar{g}_{1\bar{2}} &= g_{2\bar{1}} = \tfrac{-F'}{H^2} z_1\bar{z}_2 \ |_{x=|z_1|^2}, \\ g_{2\bar{2}} &= \tfrac{F}{H^2} \ |_{x=|z_1|^2}, \end{aligned}$$

where H is the real valued function on H_F defined by $H(z_1, z_2) = F(|z_1|^2) - |z_2|^2$. It follows that:

$$\det g_{\alpha\bar{\beta}} = g_{1\bar{1}}g_{2\bar{2}} - |g_{1\bar{2}}|^2 = -\frac{F^2}{H^3}\left(\frac{xF'}{F}\right)' |_{x=|z_1|^2}. \tag{6}$$

The form ω_F is Kähler if and only if the matrix $g_{\alpha\bar{\beta}}$ is positive definite and, since $g_{2\bar{2}} > 0$, this is the case if and only if $\det g_{\alpha\bar{\beta}} > 0$. By (6) this condition turns out to be equivalent to condition (ii) in Proposition 2.1 . This shows the equivalence between (i) and (ii). The equivalence between (ii) and (iii) can be found in [3]. □

In the sequel we will suppose ω_F is a Kähler form and will denote by g_F the corresponding Kähler metric on H_F. Furthermore, we will suppose that g_F is real analytic. Let $p = (0, 0)$ be the origin in $\mathbb{C}^2$. Then the diastasis at p, globally defined in $H_F \times \overline{H_F}$, is given by:

$$D_p(z, \bar{z}) = \log\frac{1}{F(|z_1|^2) - |z_2|^2}. \tag{7}$$

2.1 Holomorphic immersions into $\mathbb{C}^N$

Define

$$C(\rho_1, \rho_2) = \log \frac{1}{F(\rho_1) - \rho_2}. \tag{8}$$

Since by hypothesis F is real analytic function it follows that the function C is real analytic in the open set

$$\{(\rho_1, \rho_2) \in \mathbb{R}^2 \mid \ \rho_1 < \sqrt{x_0}, \rho_2 < \sqrt{F(\rho_1)}\}.$$

Hence (8) can be expanded in power series

$$C(\rho_1, \rho_2) = \sum_{j,k=0}^{+\infty} c_{jk} \rho_1^j \rho_2^k = \sum_{j,k=0}^{+\infty} \frac{\partial^{j+k} C}{\partial \rho_1^j \rho_2^k}(p) \rho_1^j \rho_2^k. \tag{9}$$

Therefore,

$$D_p(z, \bar{z}) = C(|z_1|^2, |z_2|^2) = \sum_{j,k=0}^{+\infty} c_{jk} |z_1|^{2j} |z_2|^{2k}.$$

Consequently, the $\infty \times \infty$ matrix a_{jk} given by formula (1) is diagonal, more precisely $a_{jk} = \delta_{jk} c_{m_j}$, where $m_j = (m_{1,j}, m_{2,j})$ (with the notation at page 2). Since H_F is simply-connected (even contractible) by Theorem 1.5 one easily gets:

Theorem 2.1.1 *The Hartogs domain H_F endowed with the Kähler metric g_F admits a holomorphic and isometric full immersion into $\mathbb{C}^N, N \leq \infty$ iff N among the c_{jk}'s, given by (9), are positive and all other are zero.*

Corollary 2.1.2 *The Hartogs domain (H_F, g_F) cannot admit a holomorphic and isometric map into $\mathbb{C}^N$ for N finite.*

Proof: Suppose that there exists a holomorphic and isometric immersion of (H_F, g_F) into $\mathbb{C}^N$ with N finite. Then, by Theorem 2.1.1 only finitely many c_{jk}'s would be strictly greater than zero. On the other hand,

$$c_{0k} = \frac{\partial^k C}{\partial \rho_2^k}(p) = (F(0))^{-k} > 0 \ \forall k,$$

which gives the desired contradiction. $\square$

Remark 2.1.3 Theorem 2.1.1 gives an infinite number of conditions which involve the derivatives of all orders of the function F at $x = 0$. For example $c_{10} \geq 0$ is equivalent to $\frac{\partial C}{\partial \rho_1}(p) = -\frac{F'(0)}{F(0)} \geq 0$, which is automatically satisfied being $F(0) > 0$ and being F a non increasing function. The first non trivial condition comes from $c_{20} \geq 0$. In fact

$$c_{20} = \frac{\partial^2 C}{\partial \rho_1^2}(p) = \frac{(F'(0))^2 - F''(0)F(0)}{F(0)^2} \geq 0,$$

i.e.

$$F''(0) \leq \frac{(F'(0))^2}{F(0)}. \tag{10}$$

Example 2.1.4 Let $F(x) = e^{-x}, x \in [0, +\infty)$. It is immediate to verify that condition (ii) in Proposition 2.1 is satisfied and hence ω_F is a Kähler form on H_F. This domain is considered also in [3, p. 451] and it is called the *Spring domain*.

The function C given by (8) reads, in this case, as:

$$C(\rho_1, \rho_2) = -\log(e^{-\rho_1} - \rho_2) = \rho_1 + \sum_{j=0}^{+\infty} \sum_{k=1}^{+\infty} \frac{k^{j-1}}{j!} \rho_1^j \rho_2^k.$$

Then $c_{00} = c_{j0} = 0, \forall j > 2$, $c_{10} = 1$, and

$$c_{jk} > 0, \forall j \geq 0, \forall k > 1.$$

Therefore, by Theorem 2.1.1, the Spring domain admits a holomorphic and isometric immersion into $\mathbb{C}^\infty$.

Example 2.1.5 Consider the function $F(x) = e^{-x} + 2$, $x \in [0, 1)$. Since

$$\left(\frac{xF'}{F}\right)' = -\frac{1 + 2e^x(1 - x)}{(1 + 2e^x)^2} < 0, \ \forall x \in [0, 1),$$

it follows that the condition (ii) in Proposition 2.1 is satisfied and g_F is a Kähler metric on H_F. On the other hand,

$$F''(0) = 1 > \frac{1}{3} = \frac{(F'(0))^2}{F(0)}.$$

Therefore condition (10) is not satisfied, and so (H_F, g_F) cannot be holomorphically and isometrically immersed into $\mathbb{C}^N$ for any $N \leq \infty$.

2.2 Holomorphic immersions into $\mathbb{C}P_b^N$

Define the function

$$C(\rho_1, \rho_2) = \frac{1}{b}(F(\rho_1) - \rho_2)^{-b} - 1, \tag{11}$$

which is real analytic on the open set

$$\{(\rho_1, \rho_2) \in \mathbb{R}^2 \mid \ \rho_1 < \sqrt{x_0}, \rho_2 < \sqrt{F(\rho_1)}\}.$$

It follows that

$$D_p(z, \bar{z}) = \sum_{j,k=1}^{+\infty} c_{jk} |z_1|^{2j} |z_1|^{2k}$$

where $c_{jk} = \frac{\partial C^{j+k}}{\partial \rho_1^j \rho_2^k}(p)$. Consequently the $\infty \times \infty$ matrix b_{jk} given by formula (3) is diagonal, more precisely $b_{jk} = \delta_{jk} c_{m_j}$ where $m_j = (m_{1j}, m_{2j})$. Since H_F is simply-connected by Theorem 1.6 one gets:

Theorem 2.2.1 *The Hartogs domain H_F endowed with the Kähler metric g_F admits a holomorphic and isometric full immersion into $\mathbb{C}P_b^N, N \leq \infty$ iff N among the c_{jk}'s, given by formula (3), are positive and all other are zero.*

Corollary 2.2.2 *The Hartogs domain (H_F, g_F) cannot admit a holomorphic and isometric immersion into the finite dimensional complex projective space, $\mathbb{C}P_b^N$ ($b > 0$ and N finite).*

Proof: Suppose that there exists a holomorphic and isometric immersion of (H_F, g_F) into the complex projective space $\mathbb{C}P_{b>0}^N$ with N finite. Then, by Theorem 2.2.1 only finitely many c_{jk}'s would be strictly greater than zero. On the other hand, it is immediate to verify that $c_{0k} = \frac{\partial^k C}{\partial \rho_2^k}(p) > 0, \ \forall k$, the desired contradiction. $\square$

Example 2.2.3 Let $b = 1$ and $F(x) = e^{-x}, x \in [0, +\infty)$. The function C given by (11) reads as:

$$C(\rho_1, \rho_2) = \frac{1}{e^{-\rho_1} - \rho_2} - 1 = \sum_{j,k=0}^{+\infty} \frac{(k+1)^j}{j!} \rho_1^j \rho_2^k.$$

Thus $c_{jk} > 0, \forall j, k$ and, by Theorem 2.2.1, the Spring domain admits a holomorphic and isometric map in $\mathbb{C}P_1^\infty$.

Remark 2.2.4 Let $b = -1$. The function C given by (11) reads as:

$$C(\rho_1, \rho_2) = 1 + \rho_2 - F(\rho_1) = 1 + \rho_2 - \sum_{j=0}^{+\infty} F_j \rho_1^j,$$

where

$$F_j = \frac{\partial^j F}{\partial x^j}(0).$$

Then the matrix b_{jk} given by formula (3) is positive semidefinite iff $F_j \leq 0$. So, for example the Spring domain cannot admit a holomorphic and isometric immersion into the hyperbolic space $\mathbb{C}P^N_{b<0}$ for any $N \leq \infty$, since the second derivative of e^{-x} at 0 is negative.

Remark 2.2.5 Observe that if $F(x) = 1 - x$ then (H_F, g_F) is equal to the 2-dimensional hyperbolic space $\mathbb{C}P^2_{-1}$. Thus, Corollaries 2.1.2 and 2.2.2 can be considered as a generalization of a result due to Calabi [1, Theorem 13] which asserts that $\mathbb{C}P^2_{-1}$ cannot admit a holomorphic and isometric immersion into $\mathbb{C}^N$ and $\mathbb{C}^N_b$ for $b > 0$ and N finite.

3 The Einstein condition

Theorem 3.1 *Let H_F be a Hartogs domain with strictly pseudoconvex boundary endowed with its Kähler metric g_F given by Theorem 2.1. Suppose that g_F is Einstein. Then (H_F, g_F) is biholomorphically isometric to the 2-complex hyperbolic space $\mathbb{C}P^2_{-1}$.*

We first prove an elementary lemma

Lemma 3.2 *Let ϕ be a holomorphic function on an open set $U \subset \mathbb{C}$ containing the origin. Suppose that there exists a real analytic function $f : (-x_0, x_0) \to \mathbb{R}$ such that $|\phi(z)|^2 = f(|z|^2)$ and $f(0) \neq 0$. Then $\phi(z)$ reduces to the constant $\phi(0)$.*

Proof: Let $\phi(z) = \sum_{j=0}^{+\infty} a_j z^j$ be the power series expansion of ϕ at the origin, and $f(x) = \sum_{l=0}^{+\infty} b_l x^l$ be the Taylor expansion of f at the origin. By hypothesis,

$$\sum_{j,k=0}^{+\infty} a_j \bar{a}_k z^j \bar{z}^k = \sum_{l=0}^{+\infty} b_l |z|^{2l},$$

which implies that all the terms of the form $a_0\bar{a}_k\bar{z}^k$ with $k \neq 0$, are zero. It follows that $a_k = 0$ for $k > 0$, and so the result. $\square$

Proof of Theorem 3.1: If g_F is Kähler-Einstein, then

$$\rho_{\omega_F} = -i\partial\bar{\partial}\log\det g_{\alpha\bar{\beta}} = \lambda\omega_F = \lambda\frac{i}{2}\partial\bar{\partial}\log\frac{1}{H} = -\frac{i}{2}\partial\bar{\partial}\log H^{\lambda}, \qquad (12)$$

where λ is the scalar curvature and ρ_{ω_F} is the Ricci form (see [6]). Thus

$$\partial\bar{\partial}\log(H^{-\frac{\lambda}{2}}\det g_{j\bar{k}}) = 0.$$

Since the domain H_F is simply connected there exists a holomorphic function ϕ on H_F such that

$$H^{-\frac{\lambda}{2}}\det g_{j\bar{k}} = |\phi|^2.$$

Therefore, by formula (6) above, one gets:

$$|\phi|^2 = -\frac{F^2}{H^{\frac{\lambda}{2}+3}}\left(\frac{xF'}{F}\right)'|_{x=|z_1|^2} = -\frac{(F' + |z_1|^2F'')F - |z_1|^2F'^2}{H^{\frac{\lambda}{2}+3}}|_{x=|z_1|^2}.$$

Since the Kähler metric g_F is Einstein it is also real analytic and hence the function F is real analytic in $(-x_0, x_0)$. By Lemma 3.2, being ϕ holomorphic, one can deduce that the function ϕ equals a constant, say C. Therefore

$$\frac{(F' + |z_1|^2F'')F - |z_1|^2F'^2}{H^{\frac{\lambda}{2}+3}} = -C^2. \qquad (13)$$

Observe that the numerator of (13) depends only on $|z_1|^2$, while the denominator depends also on $|z_2|^2$. Then formula (13) makes sense if and only if $\lambda = -6$ and

$$(F' + xF'')F - xF'^2 = -C^2, \ \forall x \in (-x_0, x_0). \qquad (14)$$

Taking the first derivative of (14) at zero one gets $2F(0)F''(0) = 0$. Since $F(0) \neq 0$, it follows that $F''(0) = 0$. Taking the higher order derivatives of (14) at zero one obtains

$$0 = \frac{\partial^k((F' + xF'')F - xF'^2)}{\partial x^k}(0) = (k+1)F(0)\frac{\partial^k F}{\partial x^k}(0), \ k \geq 1,$$

and so $\frac{\partial^k F}{\partial x^k}(0) = 0$. Using again the analyticity of F one immediately obtains that $F(x) = \alpha - \beta x$, where α and β are positive constants. Then the map

$$\varphi : H_F \to \mathbb{C}P^2_{-1} : (z_1, z_2) \mapsto (\sqrt{\frac{\beta}{\alpha}}z_1, \sqrt{\frac{1}{\alpha}}z_2)$$

is the desired biholomorphism satisfying

$$\varphi^*(G_{hyp}(-1)) = g_F.$$

$\square$

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