Riemannian metric and measure of economies with an arbitrary large number of equilibria

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August 2, 2010

Abstract: Balasko [1] has showed that, in a compact subset of the space of economies, not too many economies can have too many equilibria. In this paper we extend Balasko's result to the whole set of economies. We show that there exists a Riemannian metric g on the equlibrium manifold E such that the measure (associated to g) of economies with a large number of equilibria approaches zero as this number tends to infinity.

Keywords: Equilibrium manifold, regular economies, economies with a large number of equilibria.

JEL Classification: D50, D51.

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1 Introduction

The choice of a Riemannian metric on the equilibrium manifold E with "economic meaning" is a delicate and difficult issue (see the Introduction in [5]). Many variables are involved in such a choice and a Riemannian metric itself does depend on the specific problem dealt with. In order to construct a Riemannian metric in the general Arrow-Debreu model, one should enrich this model by adding more "structure" (this occurs in several areas of economic analysis, where the Arrow-Debreu model represents a benchmark to start from for further investigation). Therefore, even if it is natural to start analyzing the metric issue in the arrow-debreu framework, one cannot hope of exhausting it: this issue is so general to leave any answer necessarily ambiguous.

A more fruitful approach is that of investigating what economic properties depend on the choice of a Riemannian metric. Let us start noticing that those economic properties which are topological (e.g. simple connectedness of E, uniqueness and odd number of regular equilibria, finite covering properties, $\ldots$) do not depend on the Riemannian metric chosen. Therefore it would be desirable to consider those metric which are "natural" from the geometric point of view of the natural projection $\pi : E \to \Omega$, so that the differential geometric objects associated to these metrics (such as area, volumes and geodesics) could shed some light on the economic features involved.

This investigation leads us to re-consider economic results in the literature depending on a Riemannian metric and to explore how they can be improved by using such dependence. To the authors' best knowledge, Balasko [1] is the first to consider a Riemannian metric on the equilibrium manifold in order to show that, in a compact subset of the space of economies, not too many economies can have too many equilibria (see also Theorem 3.2 below). In this paper we extend Balasko's result to the whole set of economies. Through the *Alexandroff one-point compactification* [1], we construct the metric $\hat{g}$ on $\Omega = \mathbb{R}^{l(m-1)}$ and on $E \cong \mathbb{R}^{l(m-1)}$ obtained by the restriction of the spherical metric of $S^{l(m-1)} = \mathbb{R}^{l(m-1)} \cup \{\infty\}$ to $\mathbb{R}^{l(m-1)}$. Observe that this metric is strongly related to the geometry of the natural projection $\pi : E \to \Omega$. In fact, since $\pi : E \to \Omega$ is proper, it extends to a continuous map $\hat{\pi} : S^{l(m-1)} \to S^{l(m-1)}$ from the one point compactification of $E \cong \mathbb{R}^{l(m-1)}$ and $\Omega = \mathbb{R}^{l(m-1)}$, respectively. As far as asymptotic properties are concerned, the metric $\hat{g}$ enables us to extend Balasko's Theorem 3.2 (see Section

[1]Roughly speaking, the process of embedding a non compact topological space into compact ones is used in order to enjoy some nice properties compact spaces have. The Alexandroff one-point compactification is obtained by adding one point at infinity, denoted by ∞. In our setting this point represents either an economic equilibrium or an ecomomy . This construction is not new in the theory of regular economies: e.g., [2, Ch. 6] has developed envelope theory in a projective setting by adding economies at infinity.

3) to the whole set of economies (Theorem 3.3)

This paper is organized as follows. Section 2 recalls the economic setting and in Section 3 we prove Theorem 3.3.

2 Economic setting

We consider a pure exchange economy with l goods and m consumers. Let $S = \{p = (p_1, \dots p_l) \mid p_j > 0, j = 1, \dots l, p_l = 1\}$ be the set of normalized prices. Denote by $\Omega = (\mathbb{R}^l)^m$ the space of endowments $\omega = (\omega_1, \dots, \omega_m)$, $\omega_i \in \mathbb{R}^l$. We assume that the standard assumptions of smooth consumer's theory are satisfied (see [2] Chapter 2). The problem of maximizing the smooth utility function $u_i : \mathbb{R}^l \to \mathbb{R}$ subject to the budget constraint $p \cdot \omega_i = w_i$ gives the unique solution $f_i(p, w_i)$, i.e. consumer's i demand. Let E be the closed set consisting of pairs $(p, \omega) \in S \times \Omega$ satisfying the following equations:

$$\sum_{i=1}^{m} f_i(p, p \cdot \omega_i) = \sum_{i=1}^{m} \omega_i.$$

The set E is a smooth submanifold of $S \times \Omega$ globally diffeomorphic to $\mathbb{R}^{lm}$ [2, p. 73]. Let $\pi : E \to \Omega$ be the *natural projection*, i.e. the smooth map defined by the restriction to E of $(p, \omega) \mapsto \omega$. Let E_c be the set of critical equilibria, namely the pairs $(p, \omega) \in E$ such that the derivative of π at (p, ω) is not onto. The set of *singular economies*, denoted by Σ, is the image via π of the set E_c. It is a closed and measure zero subset of Ω. The set of *regular economies* $\mathcal{R} = \Omega \setminus \Sigma$ represent the regular values of the map π. The map $\pi_{|_{\pi^{-1}(R)}} : \pi^{-1}(R) \to R$ is a finite covering [2, p. 91].

3 Main result

There is a natural connection between Riemannian metric and measure. Given a Riemannian manifold (M, g), one can associate to the metric g a measure $\mu_g : \mathcal{B} \to [0, +\infty)$ on M, where $\mathcal{B}$ represent the Borel sets of M, $\mu_g(B) = \int_B dv_g$, $B \in \mathcal{B}$, where dv_g is the Riemannian volume element associated to g (see e.g. [4]). For example, if $M = \mathbb{R}^n$ and $g = g_{eucl}$ denotes the Euclidean metric, then $\mu_{g_{eucl}}(A) = \mu_L(A)$, where μ_L denotes the standard Lebesge measure of $\mathbb{R}^n$. For an arbitrary Riemannian metrics g on $\mathbb{R}^n$ one has

$$dv_g = f dx_1, \dots, dx_n, \tag{1}$$

where f is a smooth positive function on $\mathbb{R}^n$ ($f = 1$ when $g = g_{eucl}$). Observe that while $\mu_L(\mathbb{R}^n) = \infty$, the measure $\mu_g(\mathbb{R}^n)$ can have any positive real value depending on the behavior of f. On the contrary, the fact that a set has zero measure does not depend on the metric chosen as shown by the following lemma.

Lemma 3.1 *Let g_1 and g_2 be two Riemannian metrics on $\mathbb{R}^n$ and let A be a measurable subset of $\mathbb{R}^n$. Then $\mu_{g_1}(A) = 0 \iff \mu_{g_2}(A) = 0$.*

Proof: Being $\mathbb{R}^n$ σ-compact, i.e., it is union of a countable union of compacts sets $K_1, \dots, K_n, \dots$, where $K_1 \subset K_2 \subset \dots$, we can write $A = \cup_{j=1}^{+\infty} A_j$, where $A_j = A \cap K_j$ and $A_1 \subset A_2 \dots \subset A_n$. By standard measure theory (see e.g. [6]), $\mu_{g_i}(A) = \lim_{j\to\infty} \mu_{g_i}(A_j)$, $i = 1, 2$. It is then sufficient to show that $\mu_{g_1}(A_j) = 0 \iff \mu_{g_2}(A_j) = 0$ for a fixed j. By (1) we have $dv_{g_1} = f_1 dx_1 \dots dx_n$, $dv_{g_2} = f_2 dx_1 \dots dx_n$ for some smooth and positive functions f_1 and f_2 on $\mathbb{R}^n$. Due to the compactness of K_j there exist positive constants m_1, m_2, M_1 and M_2 such that $m_i \le f_i \le M_i$, $i = 1, 2$. Hence $\frac{m_2}{m_1}\mu_{g_1}(A_j) \le \mu_{g_2}(A_j) \le \frac{M_2}{M_1}\mu_{g_1}(A_j)$ and $\frac{m_1}{m_2}\mu_{g_2}(A_j) \le \mu_{g_1}(A_j) \le \frac{M_1}{M_2}\mu_{g_2}(A_j)$ and the conclusion easily follows. $\square$

Balasko [1] considers on the equilibrium manifold E the metric g_E obtained by the restriction of the Euclidean metric on $S \times \Omega \subset \mathbb{R}^{l-1} \times \mathbb{R}^{lm}$. By using: 1) the inequality

$$\mu_L(\pi(H)) \le \mu_{g_E}(H)$$

valid for any measurable set $H \subset E$, where μ_{g_E} is the measure associated to g_E and $\mu_L = \mu_{g_{eucl}}$ is the Lebesgue measure on Ω and 2) the ramified structure of the natural projection $\pi : E \to \Omega$, Balasko proved the following result, which can be considered "an asympotic version of the fact that the set of economies with an infinite number of equilibria has Lebesgue measure zero in Ω" [3, p. 43].

Theorem 3.2 (Balasko) *Let K be a compact set of Ω and denote by $\Omega_n(K)$ the set of economies $\omega \in K$ having at least n equilibria. Then there exists a constant $c(K)$, depending only on K, such that*

$$\mu_L(\Omega_n(K)) \le \frac{c(K)}{n}$$

for any $n \ge 1$. Therefore the Lebesgue measure of $\Omega_n(K)$ goes to zero as n tends to infinity.

We now extend this result to the whole set of economies Ω (without restricting to compact sets) by changing the measure on Ω. We equip $\hat{\Omega} = \Omega \cup \{\infty\} \simeq S^{lm}$, the Alexandroff one-point compactification of $\Omega = \mathbb{R}^{lm}$, with the standard spherical metric $\hat{g}$ of S^{lm}, namely the restriction to $S^{lm} \subset \mathbb{R}^{lm+1}$ of the Euclidean metric of $\mathbb{R}^{lm+1}$.

Theorem 3.3 *Let $\hat{\mu} = \mu_{\hat{g}}$ be the measure on Ω associated to the metric $\hat{g}$ and denote by $\Omega_n \subset \Omega$ the set of economies having at least n equlibria. Then* $\lim_{n\to\infty} \hat{\mu}(\Omega_n) = 0$.

Proof: If $\hat{\mu}(\Omega_n) = \emptyset$ for n sufficiently large then there is nothing to prove. So assume that for any n_0 there exists $n \geq n_0$ such that $\hat{\mu}(\Omega_n) \neq \emptyset$. Set $\hat{\Sigma} = \Sigma \cup \{\infty\}$, where Σ is the set of singular economies, and let U_j be a countable family of open subsets of $\hat{\Omega} \simeq S^{lm}$ such that:

- $\hat{\Sigma} \subset U_j$ for all j;
- $U_k \subset U_j$ for $k > j$;
- $\cap_{j=1}^{+\infty} U_j = \hat{\Sigma}$.

For fixed j consider the function $f_j : \hat{\Omega} \setminus U_j \to \mathbb{N}$ which associates to $x \in \hat{\Omega} \setminus U_j$ the cardinality of $p^{-1}(x)$. This function is locally constant due to the ramified structure of the projection $\pi : E \to \Omega$. Thus, due to the compactness of $\hat{\Omega} \setminus U_j$, one can define a natural number n_j as the maximum of the function f_j on $\hat{\Omega} \setminus U_j$. It follows that (for fixed j) there exists a natural number n_j such that $\Omega_n \subset U_j$ for all $n > n_j$ and hence $\hat{\mu}(\Omega_n) < \hat{\mu}(U_j)$ for $n > n_j$.

Observe that since $\hat{\mu}(\Omega_n) \neq \emptyset$ one has that $j \to \infty$ implies $n_j \to \infty$. So the desired equality $\lim_{n\to\infty} \hat{\mu}(\Omega_n) = 0$ will be achieved if one proves that $\lim_{j\to\infty} \hat{\mu}(U_j) = 0$. This is obtained as follows:

$$\lim_{j\to\infty} \hat{\mu}(U_j) = \hat{\mu}(\cap_j^{\infty} U_j) = \hat{\mu}(\hat{\Sigma}) = \hat{\mu}(\Sigma) = \mu_L(\Sigma) = 0,$$

where the first equality is a consequence of $\cap_{j=1}^{+\infty} U_j = \hat{\Sigma}$ and standard measure theory while equality $\hat{\mu}(\Sigma) = \mu_L(\Sigma)$ follows by Lemma 3.1. $\square$

The proof of Theorem 3.3 (and that of Balasko's Theorem 3.2), can be extended without any substantial change to other settings where total resources are fixed (or where utiliy functions are price dependent, see [3, Ch. 6]).

Remark 3.4 By a suitable normalization, the metric $\hat{g}$ in Theorem 3.3 can be chosen in such a way that $\hat{\mu}(\Omega) = 1$ (one simply multiplies $\hat{g}$ by $1/\operatorname{vol}(S^{lm})$) and so $\hat{\mu}$ defines a probabilty measure on Ω. Therefore our result can be formulated in a suggestive way by saying that *the probability of economies with a large number of equilibria approaches zero as this number tends to infinity.*

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