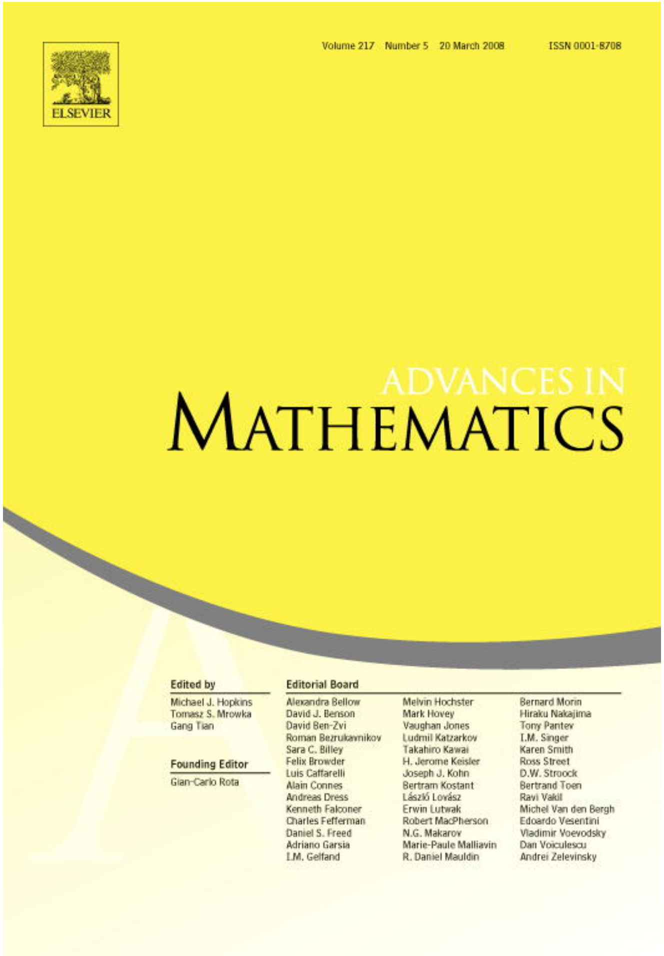
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Symplectic duality of symmetric spaces ☆

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Abstract

Let $M \subset \mathbb{C}^n$ be a complex n-dimensional Hermitian symmetric space endowed with the hyperbolic form ω_{hyp}. Denote by (M^*, ω_{FS}) the compact dual of (M, ω_{hyp}), where ω_{FS} is the Fubini–Study form on M^*. Our first result is Theorem 1.1 where, with the aid of the theory of Jordan triple systems, we construct an explicit *symplectic duality*, namely a diffeomorphism $\Psi_M : M \to \mathbb{R}^{2n} = \mathbb{C}^n \subset M^*$ satisfying $\Psi_M^* \omega_0 = \omega_{hyp}$ and $\Psi_M^* \omega_{FS} = \omega_0$ for the pull-back of Ψ_M, where ω_0 is the restriction to M of the flat Kähler form of the Hermitian positive Jordan triple system associated to M. Amongst other properties of the map Ψ_M, we also show that it takes (complete) complex and totally geodesic submanifolds of M through the origin to complex linear subspaces of $\mathbb{C}^n$. As a byproduct of the proof of Theorem 1.1 we get an interesting characterization (Theorem 5.3) of the Bergman form of a Hermitian symmetric space in terms of its restriction to classical complex and totally geodesic submanifolds passing through the origin.

MSC: 53D05; 58F06

Keywords: Kähler metrics; Bounded domains; Symplectic coordinates; Darboux theorem; Jordan triple systems; Bergman operator

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1. Introduction

In this paper we deal with the symplectic geometry of Hermitian symmetric spaces of noncompact type and their compact dual. We are going to regard such spaces as bounded symmetric domains $(M,0) \subset \mathcal{M}$ centered at the origin of their associated Hermitian positive Jordan triple system $\mathcal{M}$. Furthermore M will be equipped with the hyperbolic form ω_{hyp}. Let (M^*, ω_{FS}) be the compact dual of (M, ω_{hyp}) (see next section). We denote with the same symbol the Kähler form ω_{FS} on $\mathcal{M}$ obtained by the restriction of ω_{FS} via the Borel embedding $\mathcal{M} \subset M^*$. Finally, we denote by *HSSNT* the space of all Hermitian symmetric spaces of noncompact type $(M,0)$ and by $\mathcal{P}$ the set of all diffeomorphisms $\psi : M \to \mathcal{M}$, $M \in HSSNT$, such that $\psi(0) = 0$.

Our main result is the following theorem which establishes a bridge among the symplectic geometry of HSSNT, their duals and the theory of Jordan triple systems.

Theorem 1.1. *Let M be an HSSNT and $B(z,w)$ its associated Bergman operator (see formula* (6) *below). Then the map*

$$\Psi_M : M \to \mathcal{M}, \quad z \mapsto B(z,z)^{-\frac{1}{4}} z, \tag{1}$$

has the following properties:

(D) *Ψ_M is a (real analytic)* diffeomorphism *and its inverse Ψ_M^{-1} is given by:*

$$\Psi_M^{-1} : \mathcal{M} \to M, \quad z \mapsto B(z,-z)^{-\frac{1}{4}} z;$$

(H) *The map $\Psi : HSSNT \to \mathcal{P}$ which takes an $M \in HSSNT$ to the diffeomorphism Ψ_M is* hereditary *in the following sense: for any $(T,0) \overset{i}{\hookrightarrow} (M,0)$ complex and totally geodesic embedded submanifold $(T,0)$ through the origin 0, i.e. $i(0)=0$ one has:*

$$\Psi_{M_{|T}} = \Psi_T.$$

Moreover

$$\Psi_M(T) = \mathcal{T} \subset \mathcal{M}, \tag{2}$$

where $\mathcal{T}$ is the Hermitian positive Jordan triple system associated to T;

(I) *Ψ_M is a (nonlinear)* intertwining *map with respect to the action of the isotropy group $K \subset \mathrm{Iso}(M)$ at the origin, where $\mathrm{Iso}(M)$ is the group of isometries of M, i.e. for every $\tau \in K$*

$$\Psi_M \circ \tau = \tau \circ \Psi_M;$$

(S) *Ψ_M is a* symplectic duality*, i.e. the following holds*

$$\Psi_M^* \omega_0 = \omega_{hyp}, \tag{3}$$

$$\Psi_M^* \omega_{FS} = \omega_0, \tag{4}$$

where ω_0 is (the restriction to M) of the flat Kähler form on $\mathcal{M}$ (see formula (14) *below).*

Observe that Dusa McDuff [17] (see also [5]) proved a global version of Darboux theorem for n-dimensional complete and simply-connected Kähler manifolds with nonpositive sectional curvature. In fact she shows that, for all $p \in M$ there exists a diffeomorphism $\psi_p : M \to \mathbb{R}^{2n} = \mathbb{C}^n$ satisfying $\psi_p(p) = 0$ and $\psi_p^*(\omega_0) = \omega$, where $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$ is the standard symplectic form on $\mathbb{R}^{2n}$. The interest for these kind of questions comes, for example, after Gromov's discovery [9] of the existence of exotic symplectic structures on $\mathbb{R}^{2n}$. Eleonora Ciriza [3] (see also [2,4]) proves that the image $\psi_p(T)$ of any (complete) complex and totally geodesic submanifold T of M passing through p is a complex linear subspace of $\mathbb{C}^n$. It is very important pointing out that our theorem is not just an explicit realization (interesting on its own sake) of McDuff's general theorem for Hermitian symmetric spaces of noncompact type. Indeed, from the point of view of inducing geometric structures, as in Gromov's programme [9], the importance of property (S) relies on the existence of a smooth map (i.e. Ψ_M) which is a simultaneous symplectomorphism with respect to different symplectic structures, namely ω_{hyp} and ω_0 on M and ω_0 and ω_{FS} on $\mathcal{M}$. (We refer the reader to [6] and [7] and the references therein for the case of induction of different pairs like symplectic forms and Riemannian metrics or connections and Riemannian metrics.)

Observe also that property (H) implies the above mentioned property observed by Ciriza for the McDuff map, namely the image via the map Ψ_M of a complex and totally geodesic submanifold $T \subset M$ (through the origin) is sent to a complex linear subspace of $\mathcal{M}$.

The map $\Psi_M : M \to \mathcal{M}$ above was defined, independently from the authors, by Guy Roos in [19] (see Definition VII.4.1 at p. 533). There (see Theorem VII.4.3) he proved the analogues of (S) for volumes, namely $\Psi_M^*(\omega_0^n) = \omega_{hyp}^n$ and $\Psi_M^*(\omega_{FS}^n) = \omega_0^n$ (n is the complex dimension of M) which is, of course a corollary of (S).

The case where M is the first Cartan domain $D_I[n]$ (namely the dual of $\mathrm{Grass}_n(\mathbb{C}^{2n})$) the map Ψ_M was already considered by John Rawnsley in an unpublished work of 1989, where he proved property (3) for this case (see Section 3 below for details). Actually the proof of (S) for classical HSSNT (i.e. those Hermitian spaces which do not contain exceptional factors in theirs de Rham decomposition) follows from the property (S) for $D_I[n]$ (see 4.1 below). Regarding the proof of (S) in the general case we present here two proofs. The first one, presented in Section 5, is actually a "partial proof" since it is obtained by assuming that one already knows that the symplectic forms $\Psi_M^*\omega_0$ and $(\Psi_M^{-1})^*(\omega_0)$ are of type $(1,1)$. The second (and complete) proof, more algebraic in nature, is due to Guy Roos. His proof is, as it often occurs, more or less an adaptation of the proof for matrices into the language of Jordan triples and their operators.

There are two reasons of having included our (partial) proof in this paper. First, it is of geometric nature and second because, the techniques employed, heuristically, have suggested us how to attack and prove Theorem 5.3 below which gives an interesting (and to the authors' knowledge unknown) characterization of the Bergman form in terms of its restriction to classical complex and totally geodesic submanifolds through the origin.

Finally, we remark that in a joint paper [8] with Guy Roos the authors study the unicity of the duality map $\Psi_M : M \to \mathcal{M}$.

The paper is organized as follows. In the next section we collect some basic material about Hermitian positive Jordan triple systems, Hermitian symmetric spaces and their dual. Section 3 is dedicated to the proof of (D) and (S) of Theorem 1.1 for the first Cartan domain. The results of these sections are used in Section 4 to prove (H) and (I) of Theorem 1.1 in the general case and Theorem 1.1 in the classical case. In Section 5, after recalling some basic facts on Jordan algebras we prove (D) and (S) of Theorem 1.1 by reduction to the classical case (property (S) is proved only in the hypothesis mentioned above). Moreover, at the end of this section, we state Theorem 5.3 whose proof can be easily obtained by the same method used in the proof of (S). Finally Section 6 contains Roos's proofs of (D) and (S) of our Theorem 1.1. The paper ends with Appendix A containing two technical results on Hermitian positive Jordan triple systems.

2. Jordan triple systems, Hermitian spaces of noncompact type and their compact dual

2.1. Jordan triple systems and Hermitian spaces

We briefly recall some standard material about Hermitian symmetric spaces of noncompact type and Hermitian positive Jordan triple systems. We refer to [19] for details, notations and further results.

A Hermitian Jordan triple system is a pair $(\mathcal{M}, \{,\ ,\})$, where $\mathcal{M}$ is a complex vector space and $\{,\ ,\}$ is an $\mathbb{R}$-trilinear map

$$\{,\ ,\}: \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}, (u, v, w) \mapsto \{u, v, w\}$$

which is $\mathbb{C}$-bilinear and symmetric in u and w, $\mathbb{C}$-antilinear in v and such that the following *Jordan identity* holds:

$$\big\{x, y, \{u, v, w\}\big\} - \big\{u, v, \{x, y, w\}\big\} = \big\{\{x, y, u\}, v, w\big\} - \big\{u, \{v, x, y\}, w\big\}. \tag{5}$$

For $u, v \in \mathcal{M}$, denote by $D(u, v)$ the operator of $\mathcal{M}$ defined by

$$D(u, v)(w) = \{u, v, w\}.$$

A Hermitian Jordan triple system is called *positive* if the Hermitian form $(u, v) \mapsto \operatorname{tr} D(u, v)$ is positive definite. In the sequel we will write *HPJTS* to denote a Hermitian positive Jordan triple system. We also denote (with a slight abuse of notation) by *HPJTS* the set of all Hermitian positive Jordan triple systems on a fixed complex vector space $\mathcal{M}$. An *HPJTS* is always *semi-simple*, that is a direct sum of finite numbers of simple subsystems with component-wise triple product. An *HPJTS* is called *simple* if it is not the product of two nontrivial Hermitian positive Jordan triple subsystems. The *quadratic representation* $Q: \mathcal{M} \to End_{\mathbb{R}}(\mathcal{M})$ is defined by

$$2Q(u)(v) = \{u, v, u\}, \quad u, v \in \mathcal{M}.$$

The *Bergman operator* is

$$B(u, v) = \mathrm{id} - D(u, v) + Q(u)Q(v), \tag{6}$$

where $\mathrm{id}: \mathcal{M} \to \mathcal{M}$ denotes the identity map of $\mathcal{M}$.

An element $c \in \mathcal{M}$ is called *tripotent* if $\{c, c, c\} = 2c$. Two tripotents c_1 and c_2 are called *orthogonal* if $D(c_1, c_2) = 0$. A nonzero tripotent c is called *primitive* if it is not the sum of nonzero orthogonal tripotents. Due to the positivity of the Jordan triple system $\mathcal{M}$, each element $x \in \mathcal{M}$ has a unique *spectral decomposition*

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_p c_p, \tag{7}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$ and $(c_1, \ldots, c_p)$ is a system of mutually orthogonal tripotents. Moreover, each $x \in \mathcal{M}$ may also be written as

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_r c_r, \tag{8}$$

with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 0$ and $(c_1, \ldots, c_r)$ is a *frame* (that is a maximal system of mutually orthogonal tripotents). The decomposition (23) is also called *spectral decomposition*; it is unique only for elements x of maximal rank r, which form a Zariski dense open subset of $\mathcal{M}$.

There exist polynomials $m_1, \ldots, m_r$ on $\mathcal{M} \times \overline{\mathcal{M}}$, homogeneous of respective bidegrees $(1,1), \ldots, (r,r)$, such that for $x \in \mathcal{M}$, the polynomial

$$m(T, x, y) = T^r - m_1(x, y)T^{r-1} + \cdots + (-1)^r m_r(x, y)$$

satisfies

$$m(T, x, x) = \prod_{i=1}^{r} \left(T - \lambda_i^2\right),$$

where x is the spectral decomposition of $x = \sum \lambda_j c_j$.

The inhomogeneous polynomial

$$N(x, y) = m(1, x, y)$$

is called the *generic norm*.

Denote by $\mathcal{N}$ and $\mathcal{N}_*$ the associated functions

$$\mathcal{N}(x) = N(x, x) = 1 - m_1(x, x) + \cdots + (-1)^k m_k(x, x) + \cdots + (-1)^r m_r(x, x), \tag{9}$$

$$\mathcal{N}_*(x) = N(x, -x) = 1 + m_1(x, x) + \cdots + m_k(x, x) + \cdots + m_r(x, x). \tag{10}$$

2.2. *HSSNT associated to HPJTS*

M. Koecher [12,13] discovered that to every HPJTS $(\mathcal{M}, \{\ ,\ ,\ \})$ one can associate a Hermitian symmetric space of noncompact type, i.e. a bounded symmetric domain $(M, 0)$ centered at the origin $0 \in \mathcal{M}$. The domain $(M, 0)$ is defined as the connected component containing the origin of the set of all $u \in \mathcal{M}$ such that $B(u, u)$ is positive definite with respect to the Hermitian form $(u, v) \mapsto \operatorname{tr} D(u, v)$. The Bergman form ω_{Berg} on M is given by

$$\omega_{Berg} = -\frac{i}{2\pi} \partial\bar{\partial} \log \det B.$$

The *hyperbolic metric* ω_{hyp} (which appears in the statement of our Theorem 1.1) is given by

$$\omega_{hyp} = -\frac{i}{2\pi} \partial\bar{\partial} \log \mathcal{N}, \tag{11}$$

where $\mathcal{N}(z)$ is given by (9).

Remark 2.1. If M is irreducible, or equivalently $\mathcal{M}$ is simple, then $\det B = \mathcal{N}^g$, where g is the genus of M, and hence, in this case, $\omega_{hyp} = \frac{\omega_{Berg}}{g}$. Observe also that in the rank one case, that is when M is the complex Hermitian ball, the form ω_{hyp} is the standard hyperbolic form (cf. formula (17) in the next section).

The HPJTS $(\mathcal{M}, \{ , , \})$ can be recovered by its associated Hermitian symmetric space of noncompact type $(M, 0)$ by defining $\mathcal{M} = T_0 M$ (the tangent space to the origin of M) and

$$\{u, v, w\} = -\frac{1}{2}\big(R_0(u, v)w + J_0 R_0(u, J_0 v)w\big), \tag{12}$$

where R_0 (resp. J_0) is the curvature tensor of the Bergman metric (resp. the complex structure) of M evaluated at the origin. The reader is referred to Proposition III.2.7 in [1] for the proof of (12) and for some deep and interesting implications of it. For more information on the correspondence between *HPJTS* and *HSSNT* we refer also to p. 85 in Satake's book [21].

2.3. Totally geodesic submanifolds of HSSNT

In the proof of our theorems we need the following result.

Proposition 2.2. *Let $(M, 0)$ be an HSSNT with origin $0 \in M$ and let $\mathcal{M}$ be its associated HPJTS. Then there exists a one-to-one correspondence between (complete) complex totally geodesic submanifolds through the origin and sub-HPJTS of $\mathcal{M}$. This correspondence sends $(T, 0) \subset (M, 0)$ to $\mathcal{T} \subset \mathcal{M}$, where $\mathcal{T}$ denotes the HPJTS associated to T.*

Proof. From the theory of Hermitian symmetric spaces it is well known that there exists a one-to-one correspondence between complex totally geodesic submanifolds through the origin and complex Lie triple systems (see [11, Theorem 4.3, p. 237]). From formula (12) and part (d) of Theorem 2.10 in [15] this correspondence gives rise to a one-to-one correspondence between complex Lie triple systems and sub-*HPJTS* of $\mathcal{M}$. $\square$

2.4. The compact dual of an HSSNT

Let M^* be the compact dual of an HSSNT M. Denote by $\mathrm{BW}: M^* \to \mathbb{C}P^N$ the Borel–Weil (holomorphic) embedding. It is well known (see e.g. [22]) that the pull-back $\mathrm{BW}^*(\omega_{FS})$ of the Fubini–Study form ω_{FS} of $\mathbb{C}P^N$ is a homogeneous Kähler–Einstein form on M^* (ω_{FS} is the Kähler form which, in the homogeneous coordinates $[z_0, \dots, z_N]$ on $\mathbb{C}P^N$, is given by $\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log(|z_0|^2 + \cdots + |z_N|^2)$). In our Theorem 1.1 and in the sequel, we denote (with a slight abuse of notation and terminology) by ω_{FS} the form $\mathrm{BW}^*\omega_{FS}$ and call it the *Fubini–Study form* on M^*. In order to write its local expression, let $p \in M^*$ and assume, without loss of generality, that $\mathrm{BW}(p) = [1, 0, \dots, 0] \in \mathbb{C}P^N$. Let $H_p \subset \mathbb{C}P^N$ be the hyperplane at infinity corresponding to the point $\mathrm{BW}(p)$ and set $Y_p = \mathrm{BW}^{-1}(H_p)$. One can prove (see [23]) that $M^* \setminus Y_p$ is biholomorphic to $\mathcal{M} = T_0 M$ (the *HPJTS* associated to M). Moreover, under the previous biholomorphism, p can be made to correspond to the origin $0 \in M$. Hence we have the following inclusions $M \subset \mathcal{M} \subset M^*$ and one can prove that the restriction to $\mathcal{M}$ of the Kähler form ω_{FS} reads as:

$$\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log\det\mathcal{N}_*, \tag{13}$$

where $\mathcal{N}_*(z)$ is given by (10) (see also [14] for the relations between the two Kähler forms ω_{hyp} and ω_{FS}).

2.5. *The flat Kähler form on $\mathcal{M}$*

The *flat Kähler form* on $\mathcal{M}$ is defined by

$$\omega_0 = \frac{i}{2\pi}\partial\bar{\partial} m_1(x, x), \tag{14}$$

where $m_1(x, x)$ is the polynomial appearing in (9).

Remark 2.3. Observe that if $\mathcal{M}$ is simple then $\operatorname{tr} D(x, y) = g m_1(x, y)$ and hence $\omega_0 = \frac{i}{2g\pi}\partial\bar{\partial} D(x, x)$. Notice also that in the rank-one case ω_0 is the standard Euclidean form on $\mathcal{M} = \mathbb{C}^n$ (cf. formula (20) below).

3. The proof of (D) and (S) of Theorem 1.1 for the first Cartan's domain

Let $M = D_1[n]$ be the complex noncompact dual of $M^* = G_n(\mathbb{C}^{2n})$, where $G_n(\mathbb{C}^{2n})$ is the complex Grassmannian of complex n subspaces of $\mathbb{C}^{2n}$. In its realization as a bounded domain, $D_1[n]$ is given by

$$D_1[n] = \{ Z \in M_n(\mathbb{C}) \mid I_n - ZZ^* \gg 0 \}. \tag{15}$$

The triple product on $\mathbb{C}^{n^2}$ making it an HPJTS is

$$\{U, V, W\} = UV^*W + WV^*U, \quad U, V, W \in M_n(\mathbb{C}). \tag{16}$$

Hence the Bergman operator is given by

$$B(U, V)W = (I_n - UV^*)W(I_n - V^*U).$$

A simple computation shows that the hyperbolic form and the map $\Psi_M : D_I[n] \to M_n(\mathbb{C}) = \mathbb{C}^{n^2}$ of Theorem 1.1 are:

$$\omega_{hyp} = -\frac{i}{2\pi}\partial\bar{\partial}\log\det(I_n - ZZ^*) \tag{17}$$

and

$$\Psi_M(Z) = (I_n - ZZ^*)^{-\frac{1}{2}} Z, \tag{18}$$

respectively. In this case the Borel–Weil embedding is the Plücker embedding $G_n(\mathbb{C}^{2n}) \hookrightarrow \mathbb{C}P^N$, $N = \binom{2n}{n} - 1$ and the local expression (13) of ω_{FS} on $\mathcal{M} = \mathbb{C}^{2n}$ reads as

$$\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log\det(I_n + XX^*), \tag{19}$$

with $X \in M_n(\mathbb{C})$. Moreover the flat Kähler form (14) is given by

$$\omega_0 = \frac{i}{2\pi} \partial\bar{\partial}\,\mathrm{tr}(XX^*). \tag{20}$$

By using the equality

$$XX^*(I_n + XX^*)^{\frac{1}{2}} = (I_n + XX^*)^{\frac{1}{2}} XX^*$$

it is easy to verify that the map

$$\Phi_M : \mathbb{C}^{n^2} \to D_1[n], \quad X \mapsto (I_n + XX^*)^{-\frac{1}{2}} X \tag{21}$$

is the inverse of Ψ_M.

We are now ready to prove (S), namely the equalities

$$\Psi_M^* \omega_0 = \omega_{hyp}, \tag{22}$$

$$\Psi_M^* \omega_{FS} = \omega_0. \tag{23}$$

As we already pointed out, the proof of Eq. (22), is due to J. Rawnsley (unpublished). Here we present his proof. First of all observe that we can write

$$\begin{aligned}
\omega_{hyp} &= -\frac{i}{2\pi} \partial\bar{\partial} \log\det(I_n - ZZ^*) = \frac{i}{2\pi}\, \mathrm{d}\,\partial \log\det(I_n - ZZ^*) \\
&= \frac{i}{2\pi}\, \mathrm{d}\,\partial\, \mathrm{tr}\log(I_n - ZZ^*) = \frac{i}{2\pi}\, \mathrm{d}\,\mathrm{tr}\,\partial \log(I_n - ZZ^*) \\
&= -\frac{i}{2\pi}\, \mathrm{d}\,\mathrm{tr}\big[Z^*(I_n - ZZ^*)^{-1} dZ\big],
\end{aligned}$$

where we use the decomposition $\mathrm{d} = \partial + \bar{\partial}$ and the identity $\log\det A = \mathrm{tr}\log A$. By substituting $X = (I_n - ZZ^*)^{-\frac{1}{2}} Z$ in the last expression one gets:

$$-\frac{i}{2\pi}\, \mathrm{d}\,\mathrm{tr}\big[Z^*(I_n - ZZ^*)^{-1} dZ\big] = -\frac{i}{2\pi}\, \mathrm{d}\,\mathrm{tr}(X^* dX) + \frac{i}{2\pi}\, \mathrm{d}\,\mathrm{tr}\big\{X^* d\big[(I_n - ZZ^*)^{-\frac{1}{2}}\big] Z\big\}.$$

Observe now that $-\frac{i}{2\pi}\, \mathrm{d}\,\mathrm{tr}(X^* dX) = \omega_0$ and the 1-form $\mathrm{tr}[X^*\, \mathrm{d}(I_n - ZZ^*)^{-\frac{1}{2}} Z]$ on $\mathbb{C}^{n^2}$ is exact being equal to $\mathrm{d}\,\mathrm{tr}(\frac{C^2}{2} - \log C)$, where $C = (I_n - ZZ^*)^{-\frac{1}{2}}$. Therefore ω_{hyp} in the X-coordinates equals ω_0 and this concludes the proof of equality (22).

The proof of (23) follows the same line. Indeed, by (19) we get

$$\begin{aligned}
\omega_{FS} &= \frac{i}{2\pi} \partial\bar{\partial} \log\det(I_n + XX^*) = -\frac{i}{2\pi}\, \mathrm{d}\,\mathrm{tr}\,\partial \log(I_n + XX^*) \\
&= -\frac{i}{2\pi}\, \mathrm{d}\,\mathrm{tr}\big[X^*(I_n + XX^*)^{-1} dX\big].
\end{aligned}$$

By substituting $Z = (I_n + XX^*)^{-\frac{1}{2}} X$ in the last expression one gets:

$$-\frac{i}{2\pi}\,\mathrm{d}\,\mathrm{tr}\big[X^*(I_n+XX^*)^{-1}dX\big]=-\frac{i}{2\pi}\,\mathrm{d}\,\mathrm{tr}(Z^*dZ)+\frac{i}{2\pi}\,\mathrm{d}\,\mathrm{tr}\big\{Z^*d\big[(I_n+XX^*)^{-\frac{1}{2}}\big]X\big\}$$
$$=\omega_0+\frac{i}{2\pi}\,\mathrm{d}^2\,\mathrm{tr}\bigg(\log D-\mathrm{tr}\,\frac{D^2}{2}\bigg)=\omega_0,$$

where $D=(I_n+XX^*)^{-\frac{1}{2}}$ and this concludes the proof of (S) for $D_I[n]$.

4. Proof of (H) and (I) and the proof of Theorem 1.1 for classical domains

Let $(M,0)$ be any HSSNT. Since the map Ψ_M depends only on the triple product $\{,\ ,\}$ properties, (H) is a straightforward consequence of Proposition 2.2 above. Let $\mathcal{M}$ be the HPJTS associated to M. As usual, let us write $M=G/K$, where G is the isometry group of M and $K\subset G$ is the (compact) isotropy subgroup of the origin $0\in M$. Due to a theorem of E. Cartan (see [18, p. 63]) the group K consists entirely of linear transformations, i.e. $K\subset \mathrm{GL}(\mathcal{M})$. In order to prove (I) of Theorem 1.1, observe that the Bergman operator associated to $\mathcal{M}$ is invariant by the group of isometry of M, namely for every isometry $\tau\in K$

$$B\big(\tau(u),\tau(v)\big)\big(\tau(w)\big)=\tau\big(B(u,v)(w)\big),\quad \forall u,v,w\in\mathcal{M},$$

which implies that

$$B\big(\tau(z)\,,\,\tau(z)\big)^{-1/4}(\cdot)=\tau\big(B(z,z)^{-1/4}\big(\tau^{-1}(\cdot)\big)\big),\quad \forall z\in M.$$

Hence

$$\Psi_M\circ\tau=\tau\circ\Psi_M$$

for all $\tau\in K$ and we are done. □

4.1. *Proof of Theorem 1.1 for classical HSSNT*

Observe that since now we have proved properties (H), (I) for any HSSNT and properties (D) and (S) for $D_I[n]$. Let $(M,0)$ be a classical HSSNT and let $\mathcal{M}$ be its associated *HPJTS*. It is known that $(M,0)$ can be complex and totally geodesic embedded into $D_I[n]$, for n sufficiently large. (Indeed, this is obviously true for the domains D_I, D_{II} and D_{III}, while for the domain D_{IV} (associated to the so-called Spin-factor) the explicit embedding can be found at the bottom of p. 42 in [16]. We can assume that this embedding takes the origin $0\in M$ to the origin $0\in D_I[n]$. Therefore, by Proposition 2.2, the HPJTS $\mathcal{M}$ is a sub-HPJTS of $(\mathbb{C}^{n^2},\{,\ ,\})$. Hence properties (D), (S) for M are consequences of property (H) and the fact (proved in Section 3) that these properties hold true for $D_I[n]$. □

5. Proof of Theorem 1.1 by reduction to the classical case

Let $(M,0)$ be an exceptional *HSSNT*. Properties (H) and (I) were proved in the previous section. In this section we prove (D) and (S) in the general case, namely when M is not necessarily of classical type. In order to prove them, we pause to state Lemmata 5.1, 5.2 below which will be the bridges between the exceptional case and the classical one. Before doing this, let us briefly recall the concept of *Jordan algebras* (see e.g. [19] for details). A Jordan algebra (over $\mathbb{R}$ or $\mathbb{C}$) is a (real or complex) vector space $\mathcal{A}$ endowed with a commutative bilinear product

$$\circ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (a, b) \mapsto a \circ b$$

satisfying the following identity:

$$a \circ \left(a^2 \circ b\right) = a^2 \circ (a \circ b), \quad \forall a, b \in \mathcal{A},$$

where $a^2 = a \circ a$. Given a Jordan algebra $\mathcal{A}$ over $\mathbb{C}$ the triple product given by

$$\{x, y, z\} = 2\big((x \circ \bar{y}) \circ z + (z \circ \bar{y}) \circ x - (x \circ z) \circ \bar{y}\big)$$

defines a structure of Jordan triple system on $\mathcal{A}$ (cf. Proposition II.3.1 at p. 459 and formula (6.18) at p. 514 in [19]). Not all HPJTS $\mathcal{M}$ arise from a Jordan algebra. If this happens the HSSNT associated to $\mathcal{M}$ is called of *tube type*. Nevertheless we have the following result.

Lemma 5.1. *Let $(M, 0)$ be an HSSNT and let $\mathcal{M}$ be its associated HPJTS. Then, there exists an HSSNT $(\widetilde{M}, 0)$ such that:*

(i) $(M, 0) \hookrightarrow (\widetilde{M}, 0)$ *complex and totally geodesically embedded.*
(ii) *The HPJTS $\widetilde{\mathcal{M}}$ associated to $(\widetilde{M}, 0)$ arises from a Jordan algebra.*

Proof. Assume first that M is irreducible. Then (i) and (ii) follow easily from the classification of HSSNT given in 4.12 of [15] and Proposition 2.2 above. More precisely, if M is of classical type take a suitable n and a complex and totally geodesic embedding $(M, 0) \hookrightarrow D_I[n]$. The HPJTS associated to $D_I[n]$ comes from a Jordan algebra and so, by Proposition 2.2, the lemma is proved for classical HSSNT. If M is of exceptional type, it follows by [15] that the HPJTS associated to E_6 (the exceptional HSSNT of dimension 16) is a sub-HPJTS of the HPJTS $(H_3(\mathbb{O}_{\mathbb{C}}), \{,,\})$ of dimension 27 associated to the exceptional HSSNT E_7. Since $(H_3(\mathbb{O}_{\mathbb{C}}), \{,,\})$ arises from a Jordan algebra (i.e. E_7 is of tube type), the proof of the lemma follows again by Proposition 2.2.

For a reducible HSSNT one simply takes the product of the Jordan algebras associated to each factor. □

Lemma 5.2. *Let M be an HSSNT. Let p be a point of M, $a, b \in T_pM = \mathcal{M}$ be two nonzero vectors and $\pi \subset T_pM$ be the complex subspace generated by these vectors. Then there exists a classical HSSNT $C \hookrightarrow M$ complex and totally geodesically embedded in M passing through p such that $\pi \subset T_pC$.*

Proof. Without loss of generality we can assume that p is equal the origin 0 of M. Consider the Jordan subalgebra $\mathcal{C}_{ab} \subset \widetilde{\mathcal{M}}$ generated by a and b, where $\widetilde{\mathcal{M}}$ is the Jordan algebra given by the previous lemma. A deep result due to Jacobson–Shirsov [10] asserts that this Jordan algebra is special, namely its associated HSSNT is of classical type. Therefore, by (i) of the previous lemma, the HSSNT $C \hookrightarrow M \hookrightarrow \widetilde{M}$ associated to the HPJTS $\mathcal{C}_{ab} \cap \mathcal{M} \subset \mathcal{M}$ satisfies the desired properties. □

5.1. Proof of (D)

First we prove that Ψ_M is a local diffeomorphism. Let $p \in M$ and $a \in T_pM$, $a \neq 0$. Assume $p \neq 0$ and let γ be the one-dimensional real subspace of $\mathcal{M}$ spanned by p. It follows by the standard properties of the Bergman metric that $\tilde{\gamma} = \gamma \cap M$ is (as a subset) a geodesic of M. Fix any $q \in \tilde{\gamma}$, $q \neq p$, and let C be a classical (complete) complex and totally geodesic submanifold passing through p and such that a and $\vec{pq}$ belong to T_pC (the existence of C is guaranteed by Lemma 5.2). Since C is totally geodesic it follows that $\tilde{\gamma} \subset C$ and so $0 \in C$. Then, we can apply (H) which, combined with the results of Section 4 above, implies that

$$(d\Psi_M)_p(a) = (d\Psi_C)_p(a) \neq 0.$$

Therefore, by the inverse function theorem, Ψ_M is a local diffeomorphism. If $p = 0$ one can apply the previous argument by taking $b \in T_pM$ instead of $\vec{pq}$.

The injectivity of Ψ_M follows the same line. More precisely, let $p, q \in M$, $p \neq q$. Assume $p \neq 0$ and $q \neq 0$. By Lemma 5.2 there exists a classical (complete) complex totally geodesic submanifold C of M passing through the origin and such that $\vec{0p}$ and $\vec{0q}$ belong to T_0C. Since C is totally geodesic, p and q belong to C. Then, property (H) and the injectivity of Ψ_C immediately imply that Ψ_M is injective (if $p = 0$, then one can apply the previous argument by taking any $b \in T_pM$ in place of $\vec{0p}$).

In order to prove the surjectivity of Ψ_M, let $q \in \mathcal{M}$ be an arbitrary point. We can assume that $q \neq 0$, since $\Psi_M(0) = 0$. We have to show that there exists $p \in M$ such that $\Psi_M(p) = q$. Let γ be the one-dimensional real subspace of $\mathcal{M}$ generated by q and let $\tilde{\gamma} = \gamma \cap M$ be the corresponding geodesic of M. Let C be a classical complex and totally geodesic submanifold of M containing $\widetilde{\gamma}$ given by Lemma 5.2. Notice that by Proposition 2.2 the point q belongs to $\mathcal{C}$ (the HPJTS associated to C). Since C is classical, we know (by Section 4 and property (H)) that there exists $p \in C \subset M$ such that $q = \Psi_C(p) = \Psi_M(p)$. □

5.2. Proof of (S) under the assumptions that $\Psi_M^*\omega_0$ and $(\Psi_M^{-1})^*\omega_0$ are of type $(1,1)$

We only give a proof of (3) since (4) is obtained in a similar manner by applying the following argument to the map Ψ_M^{-1}.

First of all notice that if we set $\omega_{\Psi_M} = \Psi_M^*\omega_0$ equality (3) is equivalent to the validity of the following equations

$$(\omega_{\Psi_M})_p(a, Ja) = (\omega_{hyp})_p(a, Ja), \tag{24}$$

$$(\omega_{\Psi_M})_p(Ja, Jb) = (\omega_{hyp})_p(Ja, Jb), \tag{25}$$

for all $p \in M$, $a, b \in T_pM$, where J denotes the almost complex structure of M evaluated at the point p. Eq. (25) is precisely our assumption that $\Psi_M^*\omega_0$ is of type $(1,1)$. Thus, it remains to prove (24). Fix $p \in M$ and $a \in T_pM$. As in the proof of (D) above (using Lemma 5.2), we can find a classical complex and totally geodesic submanifold $C \subset M$ through the origin containing p and such that $a \in T_pC$. Since C is complex also $J(a)$ belongs to T_pC. Therefore, if we denote by $\omega_{hyp,C}$ and $\omega_{0,\mathcal{C}}$ the hyperbolic form on C and the flat Kähler form on $\mathcal{C}$ (the HPJTS associated to C) respectively, we get:

$$(\omega_{\Psi_M})_p(a, Ja) = \big(\Psi_C^*(\omega_{0,\mathcal{C}})\big)_p(a, Ja) = (\omega_{hyp,C})_p(a, Ja) = (\omega_{hyp})_p(a, Ja),$$

where the first and third equalities follow by the hereditary property (H) and the fact that the embedding $(C,0) \hookrightarrow (M,0)$ is a Kähler embedding while the second equality is true since C is of classical type (and hence $\Psi_C^*(\omega_{0,\mathcal{C}}) = \omega_{hyp,C}$ by Section 4). $\square$

As a byproduct of the previous proof one immediately gets the following theorem.

Theorem 5.3. *Let $(M,0)$ be an HSSNT equipped with its Bergman form $\omega_{Berg,M}$. Let ω be a 2-form of type $(1,1)$ on M. Assume that the restriction of ω to all classical complex and totally geodesic submanifolds $C \subset M$ passing through the origin equals the Bergman form of C. Then $\omega = \omega_{Berg,M}$.*

6. Roos' proof of properties (D) and (S)

This section is dedicated to Roos' proofs of properties (D) and (S) of our Theorem 1.1.

Let $\mathcal{M}$ be an HPJTS of rank r. In this section we denote by $\Psi = \Psi_M : M \to \mathcal{M}$, $z \mapsto B(z,z)^{-1/4} z$ the duality map (1). Let

$$z = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_r c_r$$

be a spectral decomposition of $z \in M$. As (see [19, Proposition V.4.2, (5.8)])

$$B(z,z)c_j = \left(1-\lambda_j^2\right)^2 c_j,$$
$$D(z,z)c_j = 2\lambda_j^2 c_j,$$

we have

$$\Psi(z) = \sum_{j=1}^{r} \frac{\lambda_j}{(1-\lambda_j^2)^{1/2}} c_j \tag{26}$$

and

$$\left(\mathrm{id} - \frac{1}{2} D(z,z)\right) c_j = \left(1-\lambda_j^2\right) c_j. \tag{27}$$

Thus

$$\Psi(z) = \left(\mathrm{id} - \frac{1}{2} D(z,z)\right)^{-1/2} z = (\mathrm{id} - z \,\square\, z)^{-1/2} z, \tag{28}$$

where we use the operator

$$z \,\square\, z = \frac{1}{2} D(z,z). \tag{29}$$

From the previous equation, it is easily seen that Ψ is bijective and that the inverse map $\Psi^{-1} : \mathcal{M} \to M$ is given by

$$\Psi^{-1}(u) = \sum_{j=1}^{r} \frac{\mu_j}{(1+\mu_j^2)^{1/2}} c_j, \tag{30}$$

if $u=\sum_{j=1}^r \mu_j c_j$ is the spectral decomposition of $u\in V$. The relation (30) is equivalent to

$$\Psi^{-1}(u)=B(u,-u)^{-1/4}u \quad (u\in V), \tag{31}$$

so that Ψ is a diffeomorphism. Therefore (D) in Theorem 1.1 is proved.

In order to prove (S) of Theorem 1.1, set $p_1(z)=m_1(z,z)$. We then have $\bar\partial p_1=m_1(z,\mathrm{d}z)$ and

$$\begin{aligned}\Psi^*(\bar\partial p_1)&=m_1\big(\Psi(z),\mathrm{d}\Psi(z)\big)\\&=m_1\big(\Psi(z),\big(\mathrm{d}(\mathrm{id}-z\,\Box\,z)^{-1/2}\big)z\big)\\&\quad+m_1\big(\Psi(z),\big((\mathrm{id}-z\,\Box\,z)^{-1/2}\big)\,\mathrm{d}z\big),\end{aligned} \tag{32}$$

where we have used the identity

$$\mathrm{d}\Psi(z)=\big(\mathrm{d}(\mathrm{id}-z\,\Box\,z)^{-1/2}\big)z+(\mathrm{id}-z\,\Box\,z)^{-1/2}\,\mathrm{d}z.$$

As $z\,\Box\,z$ is self-adjoint with respect to the Hermitian metric m_1, we have

$$\begin{aligned}m_1\big(\Psi(z),\big((\mathrm{id}-z\,\Box\,z)^{-1/2}\big)\,\mathrm{d}z\big)&=m_1\big((\mathrm{id}-z\,\Box\,z)^{-1/2}z,\big((\mathrm{id}-z\,\Box\,z)^{-1/2}\big)\,\mathrm{d}z\big)\\&=m_1\big((\mathrm{id}-z\,\Box\,z)^{-1}z,\mathrm{d}z\big).\end{aligned}$$

If $z=\lambda_1c_1+\lambda_2c_2+\cdots+\lambda_rc_r$ is a spectral decomposition of $z\in\mathcal{M}$, we have

$$(\mathrm{id}-z\,\Box\,z)^{-1}z=\sum_{j=1}^r\frac{\lambda_j}{1-\lambda_j^2}c_j=z^z, \tag{33}$$

where z^z denotes the *quasi-inverse* in the Jordan triple system $\mathcal{M}$.

Using (A.1), (33) and Lemma A.1 in Appendix A, we get the last term in (32), namely

$$m_1\big(\Psi(z),\big((\mathrm{id}-z\,\Box\,z)^{-1/2}\big)\,\mathrm{d}z\big)=-\frac{\bar\partial\mathcal{N}}{\mathcal{N}}. \tag{34}$$

Applying Lemma A.2 in Appendix A, we get

$$\begin{aligned}m_1\big(\Psi(z),\big(\mathrm{d}(\mathrm{id}-z\,\Box\,z)^{-1/2}\big)z\big)&=m_1\left(\Psi(z),\frac12(\mathrm{id}-z\,\Box\,z)^{-3/2}\big(\mathrm{d}(z\,\Box\,z)\big)z\right)\\&=\frac12 m_1\big((\mathrm{id}-z\,\Box\,z)^{-2}z,\big(\mathrm{d}(z\,\Box\,z)\big)z\big).\end{aligned}$$

We finally obtain, using this last result and (34):

$$\Psi^*(\bar\partial p_1)=-\frac{\bar\partial\mathcal{N}}{\mathcal{N}}+\frac12 m_1\big((\mathrm{id}-z\,\Box\,z)^{-2}z,\big(\mathrm{d}(z\,\Box\,z)\big)z\big). \tag{35}$$

Along the same lines, one proves

$$\left(\Psi^{-1}\right)^*(\bar\partial p_1) = \frac{\bar\partial \mathcal{N}_*}{\mathcal{N}_*} - \frac{1}{2} m_1\left((\mathrm{id} + z \,\square\, z)^{-2} z, \left(\mathrm{d}(z \,\square\, z)\right) z\right). \tag{36}$$

In view of (11) and (13), in order to prove (S) of Theorem 1.1, one needs to check that the forms

$$\beta(z) = m_1\left((\mathrm{id} - z \,\square\, z)^{-2} z, \left(\mathrm{d}(z \,\square\, z)\right) z\right),$$
$$\beta_*(z) = m_1\left((\mathrm{id} + z \,\square\, z)^{-2} z, \left(\mathrm{d}(z \,\square\, z)\right) z\right)$$

are d-closed (or d-exact, as M and $\mathcal{M}$ are simply connected). We verify it for $\beta(z)$ in the following proposition (the proof for $\beta_*(z)$ is similar).

Proposition 6.1. *Let G be the analytic function defined on $]-1, +1[$ by*

$$G(t) = \frac{1}{t} \int_0^t \frac{u}{(1-u)^2}\, du$$

and $\gamma : M \to \mathbb{R}$ the function defined by

$$\gamma(z) = m_1\left(G(z \,\square\, z) z, z\right).$$

Then $\beta(z) = d\gamma(z)$.

Proof. By using Lemma A.2, in Appendix A one has

$$\mathrm{d}\gamma(z) = m_1\left(G'(z \,\square\, z)\left(\mathrm{d}(z \,\square\, z)\right) z, z\right) + m_1\left(G(z \,\square\, z)\,\mathrm{d}z, z\right) + m_1\left(G(z \,\square\, z) z, \mathrm{d}z\right).$$

Using the identity $G(t) + tG'(t) = \frac{t}{(1-t)^2}$, we get

$$\begin{aligned}
\mathrm{d}\gamma(z) = {} & m_1\left(G'(z \,\square\, z)\left(\mathrm{d}(z \,\square\, z)\right) z, z\right) \\
& - m_1\left(G'(z \,\square\, z)(z \,\square\, z)\,\mathrm{d}z, z\right) - m_1\left(G'(z \,\square\, z)(z \,\square\, z) z, \mathrm{d}z\right) \\
& + m_1\left((\mathrm{id} - z \,\square\, z)^{-2}(z \,\square\, z)\,\mathrm{d}z, z\right) + m_1\left((\mathrm{id} - z \,\square\, z)^{-2}(z \,\square\, z) z, \mathrm{d}z\right).
\end{aligned}$$

Now

$$\begin{aligned}
m_1\left(G'(z \,\square\, z)\left(\mathrm{d}(z \,\square\, z)\right) z, z\right) &= m_1\left(G'(z \,\square\, z)(\mathrm{d}z \,\square\, z) z, z\right) + m_1\left(G'(z \,\square\, z)(z \,\square\, \mathrm{d}z) z, z\right) \\
&= m_1\left(G'(z \,\square\, z)(z \,\square\, z)\,\mathrm{d}z, z\right) + m_1\left(G'(z \,\square\, z)(z \,\square\, \mathrm{d}z) z, z\right)
\end{aligned}$$

and using the commutativity between $z \,\square\, z$ and $Q(z)$ and the identity (4.55) in [19, p. 495], one gets

$$\begin{aligned} m_1\big(G'(z\,\square\,z)(z\,\square\,\mathrm{d}z)z, z\big) &= m_1\big(Q(z)\,\mathrm{d}z, G'(z\,\square\,z)z\big) \\ &= \overline{m_1\big(\mathrm{d}z, Q(z)G'(z\,\square\,z)z\big)} \\ &= \overline{m_1\big(\mathrm{d}z, G'(z\,\square\,z)Q(z)z\big)} \\ &= m_1\big(G'(z\,\square\,z)(z\,\square\,z)z, \mathrm{d}z\big). \end{aligned}$$

So we get

$$\mathrm{d}\gamma(z) = m_1\big((\mathrm{id} - z\,\square\,z)^{-2}(z\,\square\,z)\,\mathrm{d}z, z\big) + m_1\big((\mathrm{id} - z\,\square\,z)^{-2}(z\,\square\,z)z, \mathrm{d}z\big).$$

By the same argument as before, with G' replaced by $F'(t) = \frac{1}{(1-t)^2}$, we have

$$\begin{aligned} \beta(z) = m_1\big((\mathrm{id} - z\,\square\,z)^{-2}z, \big(\mathrm{d}(z\,\square\,z)\big)z\big) &= m_1\big((\mathrm{id} - z\,\square\,z)^{-2}(z\,\square\,z)\,\mathrm{d}z, z\big) \\ + m_1\big((\mathrm{id} - z\,\square\,z)^{-2}(z\,\square\,z)z, \mathrm{d}z\big) &= \mathrm{d}\gamma(z). \qquad \square \end{aligned}$$

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Appendix A. Some technical results on HPJTS

The following general result holds in Jordan triple systems (see Lemma 2 in [20]):

Lemma A.1. *Let $\mathcal{M}$ be a Hermitian positive Jordan triple system with generic trace m_1 and generic norm N. Let $\mathcal{N}(z) = N(z,z)$ and $\mathcal{N}_*(z) = N(z,-z)$. Then*

$$\frac{\bar\partial\mathcal{N}}{\mathcal{N}} = -m_1\big(z^z, dz\big), \tag{A.1}$$

$$\frac{\bar\partial\mathcal{N}_*}{\mathcal{N}_*} = m_1\big(z^{-z}, dz\big), \tag{A.2}$$

where z^z denotes the quasi inverse (see definition at p. 467 in [19]) in the Jordan triple system $\mathcal{M}$.

Lemma A.2. *Let $f :]-1,1[\to \mathbb{R}$ and $F :]-1,1[\to \mathbb{R}$ be real-analytic functions. Then*

$$m_1\big(f(z\,\square\,z)z, \big(dF(z\,\square\,z)\big)z\big) = m_1\big(f(z\,\square\,z)z, F'(z\,\square\,z)d(z\,\square\,z)z\big). \tag{A.3}$$

Proof. It suffices to prove (A.3) for $f = t^p$ and $F = t^k$. For $k > 0$, we have

$$\big(\mathrm{d}\big((z\,\square\,z)^k\big)\big)z = \sum_{j=0}^{k-1}(z\,\square\,z)^{k-1-j}\,\mathrm{d}(z\,\square\,z)(z\,\square\,z)^j z.$$

Recall that the *odd powers* $z^{(2j+1)}$ in a Hermitian Jordan triple system are defined recursively by

$$z^{(1)} = z, \qquad z^{(2j+1)} = Q(z)z^{(2j-1)} \tag{A.4}$$

and that they satisfy the identity

$$z^{(2j+1)} = (z \,\Box\, z)^j z. \tag{A.5}$$

Using the commutativity between $z \,\Box\, z$ and $Q(z)$ we then have

$$\begin{aligned}
\mathrm{d}(z \,\Box\, z)Q(z) &= (\mathrm{d}\,z \,\Box\, z + z \,\Box\, \mathrm{d}\,z)Q(z) = Q(z)\,\mathrm{d}(z \,\Box\, z),\\
\mathrm{d}(z \,\Box\, z)(z \,\Box\, z)^j z &= \mathrm{d}(z \,\Box\, z)Q(z)^j z = Q(z)^j\,\mathrm{d}(z \,\Box\, z)z,\\
\big(\mathrm{d}\big((z \,\Box\, z)^k\big)\big)z &= \sum_{j=0}^{k-1}(z \,\Box\, z)^{k-1-j}Q(z)^j\,\mathrm{d}(z \,\Box\, z)z.
\end{aligned}$$

As $z \,\Box\, z$ is self-adjoint with respect to m_1, we have

$$m_1\big(z^{(2p+1)}, \big(\mathrm{d}(z \,\Box\, z)^k\big)z\big) = \sum_{j=0}^{k-1} m_1\big((z \,\Box\, z)^{p+k-1-j}z, Q(z)^j\,\mathrm{d}(z \,\Box\, z)z\big). \tag{A.6}$$

Using the identity (4.55) in [19, p. 495], we obtain (denoting by τ the conjugation of complex numbers)

$$\begin{aligned}
m_1\big(z^{(2p+1)}, \big(\mathrm{d}(z \,\Box\, z)^k\big)z\big) &= \sum_{j=0}^{k-1}\tau^j m_1\big(Q(z)^j(z \,\Box\, z)^{p+k-1-j}z, \mathrm{d}(z \,\Box\, z)z\big)\\
&= \sum_{j=0}^{k-1}\tau^j m_1\big((z \,\Box\, z)^{p+k-1}z, (z \,\Box\, z)\,\mathrm{d}\,z + Q(z)\,\mathrm{d}\,z\big).
\end{aligned}$$

But

$$\begin{aligned}
m_1\big((z \,\Box\, z)^{p+k-1}z, \mathrm{d}(z \,\Box\, z)z\big) &= m_1\big((z \,\Box\, z)^{p+k-1}z, (z \,\Box\, z)\,\mathrm{d}\,z + Q(z)\,\mathrm{d}\,z\big)\\
&= m_1\big(z^{(2p+2k+1)}, \mathrm{d}\,z\big) + \tau m_1\big(z^{(2p+2k+1)}, \mathrm{d}\,z\big)
\end{aligned}$$

is real, so that

$$\begin{aligned}
m_1\big(z^{(2p+1)}, \big(\mathrm{d}(z \,\Box\, z)^k\big)z\big) &= k m_1\big((z \,\Box\, z)^{p+k-1}z, \mathrm{d}(z \,\Box\, z)z\big)\\
&= m_1\big((z \,\Box\, z)^p z, k(z \,\Box\, z)^{k-1}\,\mathrm{d}(z \,\Box\, z)z\big),
\end{aligned}$$

which is precisely (A.3) for $f = t^p$, $F = t^k$. $\Box$

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