Non-free isometric immersions of Riemannian manifolds [1]

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Abstract

Let (V, g) be a Riemannian manifold and let $\mathcal{D}$ be the isometric immersion operator which, to a map $f : (V, g) \to \mathbb{R}^q$, associates the induced metric $\mathcal{D}(f) = g = f^*(\langle \cdot, \cdot \rangle)$ on V, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^q$. By Nash–Gromov implicit function theorem $\mathcal{D}$ is *infinitesimally invertible* over the space of *free* maps. In this paper we study *non-free* isometric immersions $\mathbb{R}^2 \to \mathbb{R}^4$. We show that the operator $\mathcal{D} : C^\infty(\mathbb{R}^2, \mathbb{R}^4) \to \{\mathcal{G}\}$ (where $\{\mathcal{G}\}$ denotes the space of C^∞-smooth quadratic forms on $\mathbb{R}^2$) is infinitesimally invertible over a non-empty open subset of $\mathcal{A} \subset C^\infty(\mathbb{R}^2, \mathbb{R}^4)$ and therefore $\mathcal{D} : \mathcal{A} \to \{\mathcal{G}\}$ is an open map in the respective fine topologies.
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1 Introduction

Let $\{\mathcal{G}\}$ be the space of C^∞-smooth quadratic forms g on a smooth manifold V (that are C^∞-sections of the symmetric square $\mathrm{Sym}^2(T^*V)$ of the cotangent bundle of V) and let $\{\mathcal{F}\}$ be the space of C^∞-maps $f : V \to \mathbb{R}^q$ [2]. Denote by $\mathcal{D} : \{\mathcal{F}\} \to \{\mathcal{G}\}$ the (first order non-linear differential) operator which, to a map f, associates the *induced* (pulled-back)

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[2]Throughout this paper, we assume that all the function spaces involved are endowed with the Whitney topology.

quadratic form g on V. That is, $\mathcal{D}$ is defined by:

$$\mathcal{D} : \{\mathcal{F}\} \to \{\mathcal{G}\} : f \mapsto \mathcal{D}(f) = f^*(\langle \cdot, \cdot \rangle), \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^q$.

The study of the operator $\mathcal{D}$ has at its origin the *isometric immersion problem* whose solution has been given by J. Nash who showed ([11]) that every compact Riemannian manifold V can be isometrically embedded into an arbitrary small domain in the Euclidean space $\mathbb{R}^q$ for $q = q(n) = \frac{3}{2}n(n+1) + 12n$, $n = \dim V$. (The dimension requirement has been significally improved over the years particularly through the work of Gromov who showed the same result for $q(n) = n^2 + 5n + 3$).

The proof of Nash's theorem is based on a powerful existence theorem that he proved for non-linear systems of partial differential equations. One of the main steps in Nash's demonstration amounts to inverting algebraically the linearized operator $\mathcal{D}$. (For the definition of the linearization of the operator $\mathcal{D}$ see the next section). This infinite dimensional implicit function theorem is stated here in a form due to Gromov (compare [7], p. 116) which is specific for partial differential operators. Gromov's results are true for those $\mathcal{D}$ which are *infinitesimally invertible* . According to the general theory, a (non linear) differential operator $\mathcal{D}$ is called infinitesimally invertible on some space of functions $\{\mathcal{F}\}$ where it applies if its linearization say $L = L_f$ is invertible at every $f \in \{\mathcal{F}\}$ by a differential operator $M = M_f$ where invertible means $L_f \circ M_f = Id$ and where the coefficients of M_f are smooth functions which depend on the derivatives of f of some order (cfr. [7] for the formal definition).

Theorem(Nash–Gromov IFT) *Let $\mathcal{D} : \{\mathcal{F}\} \to \{\mathcal{G}\}$ be the operator relating to maps $f : V \to \mathbb{R}^q$ the induced forms $\mathcal{D}(f) = g$. Then, over the space of free maps $V \to \mathbb{R}^q$, this operator $\mathcal{D}$ is infinitesimally invertible.*

Recall that, a map $f : V \to \mathbb{R}^q$ is *free* if all its first and second derivatives are linearly independent. (This notion was introduced by E. Cartan and M. Janet [9] and obviously extends to general Riemannian manifolds). Observe that, by its very definition, a free map is authomatically an immersion. Also notice that the freedom condition for a map $f : V \to \mathbb{R}^q$ automatically limits the dimension q of the ambient space to $q \geq n + \frac{n(n+1)}{2}$.

We now state, in form of lemma without proof, the basic properties of free maps. (For more details and further references see [7], [8] and [11]).

Lemma 1.1 *Let V be an n-dimensional Riemannian manifold. Denote by*

Free$(V, \mathbb{R}^q)$ *the space of smooth maps* $f : V \to \mathbb{R}^q$ *which satisfy the freedom condition. Then the following F1-F2 hold true.*

F1. Free$(V, \mathbb{R}^q)$ *is a non-empty open subset of* $\{\mathcal{F}\}$*, for* $q \geq n + \frac{n(n+1)}{2}$*;*

F2. Free$(V, \mathbb{R}^q)$ *is dense in* $\{\mathcal{F}\}$*, for* $q \geq 2n + \frac{n(n+1)}{2}$*;*

Remark 1.2 As a combination of Nash–Gromov IFT with property F1 and F2 (see 2.3.1 in [7]) one gets respectively:

F3. the operator $\mathcal{D}$ is infinitesimally invertible over a non-empty open subset in $\{\mathcal{F}\}$ for $q \geq n + \frac{n(n+1)}{2}$;

F4. the operator $\mathcal{D}$ is infinitesimally invertible over a dense subset in $\{\mathcal{F}\}$ for $q \geq 2n + \frac{n(n+1)}{2}$.

Our main purpose in this note is to study the following (still unsolved) problem posed by Gromov in [7] (see $E^{'}$ at p. 162):

Problem: *Show that the isometric immersion operator* $\mathcal{D}$ *is infinitesimally invertible over a dense (or at least non-empty) open subset of maps* $f : V \to \mathbb{R}^q$ *for all* $q > \frac{n(n+1)}{2}$*.*

First note that, for $q > \frac{n(n+1)}{2}$, the system (3) is *underdetermined.* Also observe that, when the integer q is not too far from the critical value $\frac{n(n+1)}{2} + n$, one should expect to have more chance to solve this problem. Immediately after addressing the question, Gromov himself ([7], p. 163), indicates an approach to the problem for the case $q \geq \frac{n(n+1)}{2} + n - \sqrt{\frac{n}{2}}$ based on the application of his "generic theorem" for underdetermined differential operators. The following remark, literally quoted from [12], essentially tells what Gromov' s generic theorem means (for the exact statement, see (2) at the bottom of p. 156 in [7]).

Remark 1.3 (cfr. [12], p. 16) *The existence of an algebraic right inverse for a differential operator is not unusual. In fact, it is generic for underdetermined operators, i.e., with more unknown functions than equations.*

Having this in mind , we have examined the seemingly easiest possible case $n = 2$ and $q = 4$ for non-free immersions $f : V^n \to \mathbb{R}^q$ and, in what follows, we have explicitely solved part of the problem posed before. After explicitely deriving the linearization of the metric inducing operator $\mathcal{D}$, we have established sufficient conditions for its invertibility (Theorem 2.1) thus showing (Theorem 2.3) that *the operator $\mathcal{D}$ is infinitesimally invertible over a non-empty open subset* $\mathcal{A} \subset C^{\infty}(\mathbb{R}^2, \mathbb{R}^4)$. (In this paper $\mathcal{A}$ denote the set of say, "nonfree-regular" maps, later called admissible). It follows that if g is a Riemannian metric induced by a map $f : \mathbb{R}^2 \to \mathbb{R}^4$, $f \in \mathcal{A}$, then in the space $\{\mathcal{G}\}$ (endowed with the Whitney topology) there exists a neighbourhood $\mathcal{U}$ of this g consisting of induced metrics. (Equivalently, every $\tilde{g}$ in this neighbourhood $\mathcal{U}$ is of the form $\tilde{f}^*(\langle\cdot,\cdot\rangle)$ for some smooth map $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^4$).

The major difficulty appears if one wants to fully verify genericity by showing that the subset in question is also *dense*. It happens that the commonly used method of proof, based on an application of Thom's transversality theorem, requires to go to higher order derivatives and this makes significantly more difficult the computation of the dimension of the space of "singular" maps. (On this regard, see the discussion at the end of the paper, where we denote this space by Σ).

The next section is devoted to the proof of our results (Theorem 2.1 and Theorem 2.3). The idea to adapt our method of proof to the context of Lie equations is extracted from Gromov ([7], p.152).

2 Inversion of $\mathcal{D}$ on the set of non-free maps

We begin by describing the linearization of the isometric immersion operator (1). In local coordinates $x_1, \ldots, x_n$ around a point $v \in V$, the operator $\mathcal{D}$ is expressed by

$$\mathcal{D}(f) = \{\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \rangle\}, i, j = 1, \ldots, n. \tag{2}$$

We call *linearization* of the operator $\mathcal{D}$ at the map $f : V \to \mathbb{R}^q$ the linear operator

$$L(f) : \Gamma(f^*(T\mathbb{R}^q)) \to \{\mathcal{G}\},$$

assigning to each vector field y on $\mathbb{R}^q$ along $f(V)$ the quadratic form g on V. Denote by $\Gamma(f^*(T\mathbb{R}^q))$ the space of smooth sections of the pull-back bundle $f^*(T\mathbb{R}^q) \to V$. This space $\Gamma(f^*(T\mathbb{R}^q))$ can be naturally identified with the space of smooth maps $Y : V \to \mathbb{R}^q$. Now, take a smooth 1-parametric family of maps $f_t : V \to \mathbb{R}^q$, $t \in [0, 1]$ such that $f_0 = f$ and $\frac{df_t}{dt}_{|t=0} = Y$ for a given $Y : V \to \mathbb{R}^q$. Then we have (compare [7] p. 116)

$$L(f)(Y) \stackrel{def}{=} \frac{d}{dt}\mathcal{D}(f_t)_{|t=0}$$

given by:

$$Y \mapsto < \frac{\partial f}{\partial x_i}, \frac{\partial Y}{\partial x_j} > + < \frac{\partial f}{\partial x_j}, \frac{\partial Y}{\partial x_i} > .$$

The study of the linearization of $\mathcal{D}$ then reduces to the study of the system of partial differential equations

$$< \frac{\partial f}{\partial x_i}, \frac{\partial Y}{\partial x_j} > + < \frac{\partial f}{\partial x_j}, \frac{\partial Y}{\partial x_i} >= g_{ij}, i, j = 1, \ldots n, \tag{3}$$

where g_{ij} are the components of a given quadratic form g on V and $Y : V \to \mathbb{R}^q$ are the unknown functions.

Now, we apply a modified version of the original Nash's trick (compare [7], p. 162). After introducing new unknown functions $h_i : V \to \mathbb{R}^q, i = 1, \ldots n$, we add to (3) the n equations

$$< \frac{\partial f}{\partial x_i}, Y >= h_i. \tag{4}$$

By differentiating this system we obtain

$$< \frac{\partial^2 f}{\partial x_i \partial x_j}, Y > + < \frac{\partial f}{\partial x_i}, \frac{\partial Y}{\partial x_j} >= \frac{\partial h_i}{\partial x_j}. \tag{5}$$

Next, alternate i and j and conclude that, under condition (4), system (3) is equivalent to:

$$< \frac{\partial^2 f}{\partial x_i \partial x_j}, Y >= \frac{1}{2}(\frac{\partial h_i}{\partial x_j} + \frac{\partial h_j}{\partial x_i} - g_{ij}). \tag{6}$$

Clearly, it only remains to find conditions on the h_i such that the $\frac{n(n+1)}{2} + n$ linear algebraic equations (4) and (6) are solvable in Y.

Let us now concentrate on a specific example and consider a map $f : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$f(x, y) = (\alpha(x, y), \beta(x, y), \gamma(x, y), \delta(x, y)), \tag{7}$$

where $\alpha, \beta, \gamma, \delta$ are smooth real valued functions on $\mathbb{R}^2$. The equations (4) and (6) thus become the following five linear equations in the four unknown functions $y_i : \mathbb{R}^2 \to \mathbb{R}^4, i = 1, 2, 3, 4$

$$\begin{cases} \alpha_x y_1 + \beta_x y_2 + \gamma_x y_3 + \delta_x y_4 = h_1 \\ \alpha_y y_1 + \beta_y y_2 + \gamma_y y_3 + \delta_y y_4 = h_2 \\ \alpha_{xy} y_1 + \beta_{xy} y_2 + \gamma_{xy} y_3 + \delta_{xy} y_4 = \frac{1}{2}(\frac{\partial h_2}{\partial x} + \frac{\partial h_1}{\partial y} - g_{12}) \\ \alpha_{xx} y_1 + \beta_{xx} y_2 + \gamma_{xx} y_3 + \delta_{xx} y_4 = \frac{\partial h_1}{\partial x} - \frac{1}{2} g_{11} \\ \alpha_{yy} y_1 + \beta_{yy} y_2 + \gamma_{yy} y_3 + \delta_{yy} y_4 = \frac{\partial h_2}{\partial y} - \frac{1}{2} g_{22} \end{cases} \tag{8}$$

In order to find the right non-degeneracy conditions, we shall ask our map $f : \mathbb{R}^2 \to \mathbb{R}^4$ to satisfy three assumptions.

Assumption 1: *The derivatives f_x, f_y, f_{xx}, f_{yy} are linearly independent on $\mathbb{R}^2$.*

The above implies that

$$D = \begin{vmatrix} \alpha_x & \alpha_y & \alpha_{xx} & \alpha_{yy} \\ \beta_x & \beta_y & \beta_{xx} & \beta_{yy} \\ \gamma_x & \gamma_y & \gamma_{xx} & \gamma_{yy} \\ \delta_x & \delta_y & \delta_{xx} & \delta_{yy} \end{vmatrix} \neq 0. \tag{9}$$

Therefore, there exist smooth real valued functions c_1, c_2, d_1, d_2 on $\mathbb{R}^2$ such that

$$f_{xy} = c_1 f_x + c_2 f_y + d_1 f_{xx} + d_2 f_{yy}.$$

Equivalently,

$$\begin{cases} c_1 \alpha_x + c_2 \alpha_y + d_1 \alpha_{xx} + d_2 \alpha_{yy} = \alpha_{xy} \\ c_1 \beta_x + c_2 \beta_y + d_1 \beta_{xx} + d_2 \beta_{yy} = \beta_{xy} \\ c_1 \gamma_x + c_2 \gamma_y + d_1 \gamma_{xx} + d_2 \gamma_{yy} = \gamma_{xy} \\ c_1 \delta_x + c_2 \delta_y + d_1 \delta_{xx} + d_2 \delta_{yy} = \delta_{xy} \end{cases} \tag{10}$$

and, by Cramer's rule

$$c_1 = \frac{\Delta_1}{D} \quad c_2 = \frac{\Delta_2}{D} \quad d_1 = \frac{\Delta_1^{'}}{D} \quad d_2 = \frac{\Delta_2^{'}}{D} \tag{11}$$

where

$$\Delta_1 = \begin{vmatrix} \alpha_{xy} & \alpha_y & \alpha_{xx} & \alpha_{yy} \\ \beta_{xy} & \beta_y & \beta_{xx} & \beta_{yy} \\ \gamma_{xy} & \gamma_y & \gamma_{xx} & \gamma_{yy} \\ \delta_{xy} & \delta_y & \delta_{xx} & \delta_{yy} \end{vmatrix} \quad \Delta_2 = \begin{vmatrix} \alpha_x & \alpha_{xy} & \alpha_{xx} & \alpha_{yy} \\ \beta_x & \beta_{xy} & \beta_{xx} & \beta_{yy} \\ \gamma_x & \gamma_{xy} & \gamma_{xx} & \gamma_{yy} \\ \delta_x & \delta_{xy} & \delta_{xx} & \delta_{yy} \end{vmatrix} \tag{12}$$

$$\Delta_1^{'} = \begin{vmatrix} \alpha_x & \alpha_y & \alpha_{xy} & \alpha_{yy} \\ \beta_x & \beta_y & \beta_{xy} & \beta_{yy} \\ \gamma_x & \gamma_y & \gamma_{xy} & \gamma_{yy} \\ \delta_x & \delta_y & \delta_{xy} & \delta_{yy} \end{vmatrix} \quad \Delta_2^{'} = \begin{vmatrix} \alpha_x & \alpha_y & \alpha_{xx} & \alpha_{xy} \\ \beta_x & \beta_y & \beta_{xx} & \beta_{xy} \\ \gamma_x & \gamma_y & \gamma_{xx} & \gamma_{xy} \\ \delta_x & \delta_y & \delta_{xx} & \delta_{xy} \end{vmatrix}. \tag{13}$$

Next, consider the following p.d.e. in the unknown functions h_1 and h_2

$$c_1h_1 + c_2h_2 + d_1(\frac{\partial h_1}{\partial x} - \frac{1}{2}g_{11}) + d_2(\frac{\partial h_2}{\partial y} - \frac{1}{2}g_{22}) = \frac{1}{2}(\frac{\partial h_2}{\partial x} + \frac{\partial h_1}{\partial y} - g_{12}). \tag{14}$$

Thus, (9) together with (10) tell us that the existence of (four smooth functions) y_1, y_2, y_3, y_4 satisfying (8) is equivalent to the existence of a solution of (14).

Let us now write (14) in the form

$$\tilde{L}_1(h_1) + \tilde{L}_2(h_2) = \tilde{g}, \tag{15}$$

where $\tilde{L}_1, \tilde{L}_2 : C^\infty(\mathbb{R}^2, \mathbb{R}) \to C^\infty(\mathbb{R}^2, \mathbb{R})$ denote the first order (smooth differential) operators given by:

$$\tilde{L}_1 = d_1\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial}{\partial y} + c_1,\ \tilde{L}_2 = -\frac{1}{2}\frac{\partial}{\partial x} + d_2\frac{\partial}{\partial y} + c_2.$$

Here the functions $c_i, i = 1, 2$ are thought of as the multiplication operator $f \mapsto c_i f$, and $\tilde{g} = \frac{1}{2}(d_1g_{11} + d_2g_{22} - g_{12})$. By setting:

$$a = 2\Delta_1^{'},\ \ b = -D,\ \ d = 2\Delta_2^{'},\ \ A_1 = 2\Delta_1,\ \ A_2 = 2\Delta_2,\ \ \hat{g} = 2D\tilde{g} \tag{16}$$

and using (11), equation (15) becomes equivalent to the following p.d.e. equation:

$$\hat{L}_1(h_1) + \hat{L}_2(h_2) = \hat{g}, \tag{17}$$

(in the same unknown functions h_1 and h_2), where $\hat{L}_1$ and $\hat{L}_2$ are the first order differential operators on $C^\infty(\mathbb{R}^2, \mathbb{R})$ defined by

$$\hat{L}_1 = L_1 + A_1,\ \hat{L}_2 = L_2 + A_2 \tag{18}$$

with

$$L_1 = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y},\quad L_2 = b\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}. \tag{19}$$

Assumption 2: *The vector fields L_1 and L_2 are linearly independent on $\mathbb{R}^2$, i.e.*

$$ad - b^2 \neq 0 \tag{20}$$

Clearly, this implies that the bracket $[L_1, L_2]$ can be written as

$$[L_1, L_2] = L_1L_2 - L_2L_1 = r_1L_1 + r_2L_2, \tag{21}$$

with r_1 and r_2 smooth real valued functions on U. Hence, by (19),

$$\begin{cases} ar_1 + br_2 = ab_x + bb_y - a_xb - a_yd \\ br_1 + dr_2 = ad_x + bd_y - bb_x - b_yd \end{cases} \tag{22}$$

and by Cramer's rule one gets

$$r_1 = \frac{ab_xd - a_xbd + 2bb_yd - a_yd^2 - abd_x + b^2b_x - b^2d_y}{ad - b^2} \tag{23}$$

$$r_2 = \frac{a^2d_x - 2abb_x + abd_y - ab_yd + a_xb^2 - b^2b_y + a_ybd}{ad - b^2} \tag{24}$$

Assumption 3: By setting

$$\hat{\Delta} = \hat{L}_1(r_1) + \hat{L}_2(r_2) - L_1(A_2) + L_2(A_1), \tag{25}$$

we assume $\hat{\Delta} \neq 0$ for all points of $\mathbb{R}^2$.

Remark: It has to be noticed that (as formulas (23), (24), (9), (12),(13) and (16) tell us) the (smooth) functions a, b, c, d, A_1, A_2, as well as r_1, r_2 and $\hat{\Delta}$, all depend on the map f given by (7).

We are now ready to state

Theorem 2.1 *Let $\mathcal{D} = \mathcal{D}(f)$ be the isometric immersion operator associated to a map $f : \mathbb{R}^2 \to \mathbb{R}^4$ satisfying Assumptions 1, 2 and 3. Then $\mathcal{D}$ is infinitesimally invertible.*

Proof: The above discussion shows that since the map f satisfies (9) our system (8) admits a solution iff the p.d.e. equation (17) is solvable in h_1 and h_2. We introduce a new unknown function h_3 and look for a solution of the following p.d.e.

$$\hat{L}_1(h_1) + \hat{L}_2(h_2) + \hat{L}_3(h_3) = \hat{g}, \tag{26}$$

where

$$\hat{L}_3 = [\hat{L}_1, \hat{L}_2] = \hat{L}_1\hat{L}_2 - \hat{L}_2\hat{L}_1 = [L_1, L_2] + L_1(A_2) - L_2(A_1).$$

Take $x_1 = \frac{r_1\hat{g}}{\hat{\Delta}}$, $x_2 = \frac{r_2\hat{g}}{\hat{\Delta}}$ and $x_3 = -\frac{\hat{g}}{\hat{\Delta}}$. By Assumption 3 these are well-defined. We claim that they are solution of (26). Indeed,

$$\begin{aligned}
\hat{L}_1(x_1) + \hat{L}_2(x_2) + \hat{L}_3(x_3) &= L_1(\tfrac{r_1\hat{g}}{\hat{\Delta}}) + A_1 r_1 \tfrac{\hat{g}}{\hat{\Delta}} + L_2(\tfrac{r_2\hat{g}}{\hat{\Delta}}) + A_2 r_2 \tfrac{\hat{g}}{\hat{\Delta}} + \\
&- [L_1, L_2](\tfrac{\hat{g}}{\hat{\Delta}}) - L_1(A_2)\tfrac{\hat{g}}{\hat{\Delta}} + L_2(A_1)\tfrac{\hat{g}}{\hat{\Delta}} = \\
&= \tfrac{\hat{g}}{\hat{\Delta}}[L_1(r_1) + A_1 r_1 + L_2(r_2) + A_2 r_2 + \\
&- L_1(A_2) + L_2(A_1)] + (r_1 L_1 + r_2 L_2 + \\
&- [L_1, L_2])(\tfrac{\hat{g}}{\hat{\Delta}}) = \tfrac{\hat{g}}{\hat{\Delta}}\hat{\Delta} = \hat{g},
\end{aligned}$$

where the last equality follows by (25). From this it is easily checked that $\tilde{x}_1 = x_1 + \hat{L}_2(x_3)$ and $\tilde{x}_2 = x_2 - \hat{L}_1(x_3)$ satisfy equation (17). □

The above suggests the following:

Definition 2.2 *Call a smooth map $f = (\alpha, \beta, \gamma, \delta) : \mathbb{R}^2 \to \mathbb{R}^4$* admissible *if it satisfies Assumption 1, 2 and 3.*

Admissible maps do exist. Indeed, the set $\mathcal{A} \subset C^\infty(\mathbb{R}^2, \mathbb{R}^4)$ of admissible maps is a non-empty subset of $C^\infty(\mathbb{R}^2, \mathbb{R}^4)$. Obviously $\mathcal{A}$ is open. Moreover, we claim that the map $f = (x, e^y, x^2, \frac{e^x cosh(e^y)}{2})$ belongs to $\mathcal{A}$. Indeed, notice first that by substituting

$$\alpha = x, \ \beta = e^y, \ \gamma = x^2, \ \delta = \frac{e^x \cosh(e^y)}{2} \tag{27}$$

in (9) one easily gets

$$D = e^{x+3y}\cosh(e^y) \neq 0,$$

and hence Assumption 1 is satisfied. Secondly, by inserting the values (27) into (12) and (13), one gets:

$$\Delta_1 = 0, \quad \Delta_2 = -e^{x+2y}\sinh(e^y), \quad \Delta_1^{'} = 0, \quad \Delta_2^{'} = e^{x+2y}\sinh(e^y). \tag{28}$$

We substitute these in (16) and obtain:

$$a = 2\Delta_1^{'} = 0, \ b = -D = -e^{x+3y}\cosh(e^y), \ d = 2\Delta_2^{'} = 2e^{x+2y}\sinh(e^y) \tag{29}$$

and

$$A_1 = 2\Delta_1 = 0, \;\; A_2 = 2\Delta_2 = -2e^{x+2y}\sinh(e^y). \tag{30}$$

Hence

$$ad - b^2 = -b^2 = -D^2 = -e^{2x+6y}\cosh^2(e^y) \neq 0,$$

so that Assumption 2 holds true.

In order to verify that the map $f = (x, e^y, x^2, \frac{e^x cosh(e^y)}{2})$ satisfies Assumption 3, we first observe that, by taking into account (29) and (30), (18), (19) (23) and (24) become respectively:

$$\hat{L}_1 = L_1 = -e^{x+3y}\cosh(e^y)\frac{\partial}{\partial y}$$

$$\hat{L}_2 = -e^{x+3y}\cosh(e^y)\frac{\partial}{\partial x} + 2e^{x+2y}\sinh(e^y)\frac{\partial}{\partial y} - 2e^{x+2y}\sinh(e^y).$$

$$r_1 = e^{x+2y}[3e^y\cosh(e^y) - 4\sinh(e^y)(2 + e^y\tanh(e^y))]$$

$$r_2 = \frac{\partial r_2}{\partial x} = -e^{x+3y}\left(3\cosh(e^y) + e^y\sinh(e^y)\right).$$

After substituting these in (25) a long (but straightforward) computation gives

$$\hat{\Delta} = 4e^{2x+7y}\tanh(e^y) \neq 0,$$

and hence also Assumption 3 is fullfilled.

Now, we come to the following:

Theorem 2.3 *The operator $\mathcal{D} : C^\infty(\mathbb{R}^2, \mathbb{R}^4) \to \{\mathcal{G}\}$ is infinitesimally invertible over the non-empty open subset $\mathcal{A} \subset C^\infty(\mathbb{R}^2, \mathbb{R}^4)$ of admissible maps. Hence,*

$$\mathcal{D} : \mathcal{A} \to \{\mathcal{G}\}$$

is an open map in the respective fine topologies.

Finally, observe that one can express the admissibility of a smooth map $f : \mathbb{R}^2 \to \mathbb{R}^4$ in terms of its jets of order 4. Let $J^4(\mathbb{R}^2, \mathbb{R}^4)$ be the space of 4-jets of our maps $f : \mathbb{R}^2 \to \mathbb{R}^4$. (This is a trivial bundle based on $\mathbb{R}^2 \times \mathbb{R}^4$ with fiber $J^4_{v,w}$, $v \in \mathbb{R}^2$, $w \in \mathbb{R}^4$). Let $\Sigma_{v,w} \subset J^4_{v,w}$ be the set consisting of the 4-jets of non-admissible maps. (The jet J^4_f of a given map $f : \mathbb{R}^2 \to \mathbb{R}^4$ at the point v is given by its derivatives up to order four). It is not difficult to show (see, for e.g., [5]) that $\Sigma_{v,w}$ is a smooth stratified (in the sense of Whitney) space of codimension 1 in $J^4_{v,w}$. Hence, the "singularity" set corresponding to non admissibility is given by

$$\Sigma = \cup_{(v,w)\in V\times W}\Sigma_{v,w} \subset J^4(\mathbb{R}^2, \mathbb{R}^4). \tag{31}$$

Clearly, Σ is a stratified space of codimension 1 in $J^4(\mathbb{R}^2, \mathbb{R}^4)$ which fibers over $\mathbb{R}^2 \times \mathbb{R}^4$. By all this, it is immediately seen that a smooth map $f : \mathbb{R}^2 \to \mathbb{R}^4$ is admissible iff $J^4_f(\mathbb{R}^2) \cap \Sigma = \emptyset$. This explains why one cannot establish that admissible maps are generic by applying standard transversality theorems as this would demand $\operatorname{codim}\Sigma \geq 3$ (see, e.g. Corollary D' p. 33 in [7]).

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