A NOTE ON THE L^2-NORM OF THE SECOND FUNDAMENTAL FORM OF ALGEBRAIC MANIFOLDS

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Abstract. Let $M \stackrel{f}{\hookrightarrow} \mathbb{C}\mathrm{P}^n$ be an algebraic manifold of complex dimension d and let σ_f be its second fundamental form. In this paper we address the following conjecture: *if* $||\sigma_f||^2_{\mathrm{L}^2} < 2\,d\,\mathrm{vol}(\mathbb{C}\mathrm{P}^d)$ *then* M *is totally geodesic and equality holds iff* f *is congruent to the standard embedding of the complex quadric* Q_d *into* $\mathbb{C}\mathrm{P}^n$. We prove the conjecture in the following three cases: (i) $d=1$; (ii) M is a complete intersection; (iii) the scalar curvature of M is constant.

1. Introduction and statement of main result

In [5] M. Gromov conjectures that every *smooth* immersion $f : M \to \mathbb{C}\mathrm{H}^n/G$ of a compact manifold M of dimension d into a compact quotient of the complex hyperbolic space $\mathbb{C}\mathrm{H}^n/G$, whose second fundamental form σ_f is "small", is homotopic to a totally geodesic submanifold.

In [2] G. Besson, G. Courtois and S. Gallot give an answer to this problem in terms of the L^2 and L^{2d} norms of the second fundamental form σ_f, when the immersion is a *holomorphic* map:

Theorem 1. *Let $f : M \to \mathbb{C}\mathrm{H}^n/G$ be a holomorphic immersion of a compact Kähler manifold M of complex dimension d. If $||\sigma_f||^2_{\mathrm{L}^2}$ and $||\sigma_f||^2_{\mathrm{L}^{2d}}$ are smaller than a constant depending only on n, then M is totally geodesic.*

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It is natural to ask what happens if the ambient space is replaced by its compact dual, namely the complex projective space $\mathbb{C}\mathrm{P}^n$ endowed with the Fubini–Study metric g_{FS} of holomorphic sectional curvature 1. So, let $M \stackrel{f}{\hookrightarrow} \mathbb{C}\mathrm{P}^n$ be a complex d-dimensional algebraic manifold (f is a holomorphic injective immersion) and denote by σ_f the second fundamental form of f, by $\|\sigma_f\|^2$ its length and by

$$\|\sigma_f\|^2_{\mathrm{L}^2} = \int_M \|\sigma_f\|^2 \frac{\omega^d}{d!}$$

its L^2-norm, where ω is the Kähler form associated to the induced metric $g = f^* g_{FS}$. Observe that

$$||\sigma_f||^2 = \sum_{j,k=1}^{2d} g_{FS}\left(\sigma_f(e_j, e_k), \sigma_f(e_j, e_k)\right),$$

where $\{e_1, \dots, e_d, Je_1, \dots, Je_d\}$ is an orthonormal basis for T_xM (here J denotes the complex structure on M). If $\{e_1, \dots, e_d, Je_1, \dots, Je_d\}$ is a basis which diagonalizes the quadratic form

$$\tilde{\sigma}_f(X, Y) = \sum_{j=1}^{2d} g_{FS}\left(\sigma_f(e_j, X), \sigma_f(e_j, Y)\right), \quad X, Y \in T_xM,$$

and $\eta_1^2, \dots, \eta_{2d}^2$ are its eigenvalues, then we can write

$$||\sigma_f||^2 = \sum_{j=1}^{2d} \eta_j^2.$$

Observe that by $\sigma_f(X, JY) = \sigma_f(JX, Y) = J\sigma_f(X, Y)$ for all $X, Y \in T_xM$ it follows that $\eta_j^2 = \eta_{j+d}^2$ for $j = 1, \dots, d$.

In this paper we address the problem of finding the optimal constant $c(d)$ (depending only on d) such that if $||\sigma_f||^2_{\mathrm{L}^2} < c(d)$ then M is totally geodesic. Similar questions for $||\sigma_f||^2$ have been addressed and studied by several mathematicians (cfr. [3], [4], [7], [8], [9]). In particular, in the next section we recall the result by J. Cheng [3] which proves a long standing conjecture posed by K. Ogiue [7].

We believe in the validity of the following:

Conjecture. *Let $M \overset{f}{\hookrightarrow} \mathbb{C}\mathrm{P}^n$ be as above. If $||\sigma_f||^2_{\mathrm{L}^2} < 2\,d\,\mathrm{vol}(\mathbb{C}\mathrm{P}^d)$ then M is totally geodesic and equality holds iff f is congruent to the standard embedding of the complex quadric*

$$Q_d = \{[Z_0, \dots, Z_{d+1}], Z_0^2 + \cdots + Z_{d+1}^2 = 0\} \subset \mathbb{C}\mathrm{P}^{d+1} \overset{i}{\hookrightarrow} \mathbb{C}\mathrm{P}^n,$$

where i is the natural inclusion.

Remark 2. Recall that $M \overset{f}{\hookrightarrow} \mathbb{C}\mathrm{P}^n$ is totally geodesic, i.e. $\sigma_f \equiv 0$, if and only if M is biholomorphic to $\mathbb{C}\mathrm{P}^d$ and $f = A \circ i$, where $A \in \mathrm{Aut}(\mathbb{C}\mathrm{P}^n)$ and $i\colon \mathbb{C}\mathrm{P}^d \hookrightarrow \mathbb{C}\mathrm{P}^n$ is the natural inclusion, i.e. $i([Z_0, \dots, Z_d]) = [Z_0, \dots, Z_d, 0, \dots, 0]$. Furthermore, observe that for $d = 1$, $Q_1 = (\mathbb{C}\mathrm{P}^1, 2g_{FS})$ and f is (congruent to) the Veronese embedding

$$[Z_0, Z_1] \mapsto [Z_0^2, Z_0Z_1, Z_1^2, 0, \dots, 0].$$

Here is the main result of the present paper:

Theorem 3. *Let $M \overset{f}{\hookrightarrow} \mathbb{C}\mathrm{P}^n$ be an algebraic manifold of complex dimension d which satisfies one of the following conditions:*

(i) *$d = 1$;*

(ii) *M is a complete intersection;*

(iii) *the scalar curvature ρ of M is constant.*

If

$$||\sigma_f||^2_{\mathrm{L}^2} < 2\,d\,\mathrm{vol}(\mathbb{C}\mathrm{P}^d)$$

then M is totally geodesic and, if equality holds, i.e. $||\sigma_f||^2_{\mathrm{L}^2} = 2\,d\,\mathrm{vol}(\mathbb{C}\mathrm{P}^d)$, then f is congruent to the standard embedding of the complex quadric Q_d.

The paper contains two other sections. In the next one we summarize the background material, while the last one is entirely dedicated to the proof of Theorem 3.

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2. Preliminaries

Let $\{e_1, \dots, e_d, Je_1, \dots, Je_d\}$ be an ortonormal basis of T_xM as in the previous section and let us denote $Je_j = e_{d+j}$, $j = 1, \dots, d$. From the Gauss–Codazzi formula (see e.g. [6, Prop. 9.5, Ch. IX])

$$\mathrm{Ric}_g(X, X) = \frac{1}{2}(d+1)g(X, X) - \sum_{j=1}^{2d} g_{FS}\left(\sigma_f(e_j, X), \sigma_f(e_j, X)\right), \qquad (1)$$

we obtain (cfr. [2])

$$\mathrm{Ric}_g = \frac{1}{2}\sum_{j=1}^{d}\left(d+1-2\eta_j^2\right)\left(e_j^* \otimes e_j^* + (Je_j)^* \otimes (Je_j)^*\right). \qquad (2)$$

If ρ is the scalar curvature for M, namely the smooth function on M defined by

$$\rho = \sum_{j=1}^{2d} \mathrm{Ric}_g(e_j, e_j),$$

then by (2) we get

$$\rho = d(d+1) - ||\sigma_f||^2. \qquad (3)$$

This formula together with the inequality

$$\int_M \left(\rho - d^2\right)\left(\rho - d(d+1)\right)\frac{\omega^d}{d!} \geq 0,$$

which is obtained by using algebro-geometric machinery, are the key ingredients for the proof of the following result needed in the proof of Theorem 3:

Lemma 4 (J. Cheng [3])**.** *Let $M \stackrel{f}{\hookrightarrow} \mathbb{C}\mathrm{P}^n$ be as above. If $\|\sigma_f\|^2 < d$ then M is totally geodesic and equality holds iff f is congruent to the standard embedding of the complex quadric Q_d.*

The proof of Theorem 3 relies on the concept of degree $\deg(f)$ of $M \stackrel{f}{\hookrightarrow} \mathbb{C}\mathrm{P}^n$. Given a holomorphic immersion $f\colon M \to \mathbb{C}\mathrm{P}^n$, if $\dim(M) = d < n$ by Sard's Theorem there exists a point $q \notin f(M)$. Up to unitary transformation of $\mathbb{C}\mathrm{P}^n$ we can suppose q to be the point of coordinates $[1, 0, \dots, 0]$. Consider the projection $p_n\colon \mathbb{C}\mathrm{P}^n \setminus \{q\} \to \mathbb{C}\mathrm{P}^{n-1}$, $p_n([Z_0, \dots, Z_n]) = [Z_1, \dots, Z_n]$ and define the map $F\colon M \to \mathbb{C}\mathrm{P}^d$ by $F = \tilde{p} \circ f$, where $\tilde{p} = p_{d+1} \circ \cdots \circ p_n$. The degree $\deg(f)$ of f is by definition the degree $\deg(F)$ of the map F, which is the integer number such that

$$< F^*\alpha, [M] >= \deg F < \alpha, [\mathbb{C}\mathrm{P}^d] >, \tag{4}$$

where $[\alpha] \in H^{2d}(\mathbb{C}\mathrm{P}^d, \mathbb{R})$ and

$$< \alpha, [\mathbb{C}\mathrm{P}^d] >= \int_{\mathbb{C}\mathrm{P}^d} \alpha, \qquad < F^*\alpha, [M] >= \int_M F^*\alpha.$$

What we need about $\deg(f)$ is summarized in the following:

Lemma 5 (W. Wirtinger [10], M. Barros, A. Ros, [1])**.** *The degree* $\deg(f)$ *is a positive integer such that*

$$\mathrm{vol}(M) = \deg(f)\mathrm{vol}(\mathbb{C}\mathrm{P}^d), \tag{5}$$

where $\mathrm{vol}(M) = \int_M \frac{\omega^d}{d!}$ *and* $\mathrm{vol}(\mathbb{C}\mathrm{P}^d) = (4\pi)^d/d!$. *Moreover,* $\deg(f) = 1$ *iff M is totally geodesic and* $\deg(f) = 2$ *iff f is congruent to the standard embedding of Q_d.*

Observe that (5) follows easily by the definition of $\deg(f)$ above. In fact, if we denote by $\omega_{FS}(n)$ (resp. $\omega_{FS}(d)$) the Fubini–Study metric on $\mathbb{C}\mathrm{P}^n$ (resp. $\mathbb{C}\mathrm{P}^d$), we have

$$< f^*\omega_{FS}^d(n), [M] >= \int_M \omega^d = d!\,\mathrm{vol}(M).$$

Since the map $\Psi\colon \mathbb{C}\mathrm{P}^n \times [0,1] \to \mathbb{C}\mathrm{P}^n$,

$$\Psi([Z_0, \ldots, Z_n], t) = [tZ_0, \ldots, tZ_{n-d-1}, Z_{n-d}, \ldots, Z_n]$$

is a homotopy between the identity map of $\mathbb{C}\mathrm{P}^d$, and $i \circ \tilde{p}$, where $i\colon \mathbb{C}\mathrm{P}^d \to \mathbb{C}\mathrm{P}^n$ is the canonical inclusion (cfr. Remark 2), we get

$$\begin{aligned} d!\,\mathrm{vol}(M) &= < f^*\omega_{FS}^d(n), [M] >=< (i \circ F)^*\omega_{FS}^d(n), [M] >=< F^*(i^*\omega_{FS}^d(n)), [M] > \\ &= < F^*(\omega_{FS}^d(d)), [M] >= \deg(F) < \omega_{FS}^d(d), [\mathbb{C}\mathrm{P}^d] > \\ &= \deg(f)\, d!\,\mathrm{vol}(\mathbb{C}\mathrm{P}^d). \end{aligned}$$

3. Proof of Theorem 3

Assume (i) holds. Then $\rho = 2K$, where K is the Gaussian curvature of M. Hence Gauss–Bonnet theorem yields

$$\int_M \rho \frac{\omega^d}{d!} = 4\pi\, \chi(M),$$

where $\chi(M) = 2 - 2\gamma$ denotes the Euler characteristic of M.

By (3) we have

$$\int_M \rho \frac{\omega^d}{d!} = \int_M (2 - ||\sigma_f||^2) \frac{\omega^d}{d!} = 2\mathrm{vol}(M) - ||\sigma_f||^2_{L^2},$$

thus

$$||\sigma_f||^2_{L^2} = 2\mathrm{vol}(M) - 4\pi\, \chi(M).$$

If $||\sigma_f||^2_{\mathrm{L}^2} < 8\pi$, then $2\mathrm{vol}(M) - 4\pi\, \chi(M) < 8\pi$. By (5) one gets

$$\deg(f) < 1 + \frac{\chi(M)}{2} = 2 - \gamma.$$

It follows by Lemma 5 that $\deg(f) = 1$ and so $\gamma = 0$ and M is totally geodesic.

If $||\sigma_f||^2_{\mathrm{L}^2} = 8\pi$ then $\deg(f) = 2$, $\gamma = 0$ and again by Lemma 5 f is congruent to the Veronese embedding (cfr. Remark 2).

Assume (ii) holds. Let $a_1, \dots, a_p$, $p = n - d$, be the degrees of the hypersurfaces defining M. Then, by [7, Th. 7.1], we have

$$\int_M \rho \frac{\omega^d}{d!} = d \left(d + p + 1 - \sum_{j=1}^{p} a_j \right) \left(\prod_{j=1}^{p} a_j \right) \mathrm{vol}(\mathbb{C}\mathrm{P}^d),$$

and, since $\deg(f) = \prod_{j=1}^{p} a_j$, by (3) and (5) we gets

$$||\sigma_f||^2_{\mathrm{L}^2} = d \left(\sum_{j=1}^{p} a_j - p \right) \left(\prod_{j=1}^{p} a_j \right) \mathrm{vol}(\mathbb{C}\mathrm{P}^d).$$

If $||\sigma_f||^2_{\mathrm{L}^2} < 2\, d\, \mathrm{vol}(\mathbb{C}\mathrm{P}^d)$, we have

$$\left(\sum_{j=1}^{p} a_j - p \right) \left(\prod_{j=1}^{p} a_j \right) < 2,$$

and since each a_j's is an integer greater than or equals to 1, we get $a_j = 1$ for all $j = 1, \ldots, p$. So $\deg(f) = 1$ and by Lemma 5 M is totally geodesic.

If $||\sigma_f||^2_{\mathrm{L}^2} = 2\, d \operatorname{vol}(\mathbb{C}\mathrm{P}^d)$ we get

$$\left(\sum_{j=1}^{p} a_j - p\right)\left(\prod_{j=1}^{p} a_j\right) = 2.$$

Thus $\deg(f) = \prod_{j=1}^{p} a_j = 2$ and the conclusion follows once again by the last part of Lemma 5.

Finally, assume (iii) holds which, by (3), implies $\|\sigma_f\|^2$ is constant. If $||\sigma_f||^2 < d$ (resp. $||\sigma_f||^2 = d$) then f is totally geodesic (resp. congruent to the quadric) by Lemma 4. If $||\sigma_f||^2 > d$ then

$$d \operatorname{vol}(M) < ||\sigma_f||^2_{\mathrm{L}^2} < 2\, d \operatorname{vol}(\mathbb{C}\mathrm{P}^d)$$

which, by (5), implies $\deg(f) = 1$, i.e. M is totally geodesic. $\square$

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