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# Explicit Formulas for the Geodesics of Homogeneous SO(2)-Isotropic Three-Dimensional Manifolds

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## 1. INTRODUCTION

The aim of this paper is to calculate the geodesics for the simply-connected homogeneous SO(2)-isotropic three-dimensional manifolds. By the Lie group classification due to Milnor [5] and by a result of Kowalski [4], it follows that these spaces have isometry groups of dimension 4 or 6 (cf. Bianchi [1] for the classification of homogeneous three-dimensional manifolds in terms of their isometry group). We will use an expression, depending on 2-parameters $\ell$ and $m$, due to Cartan ([2] and cf. [6, p. 354]), which describes all the homogeneous three-dimensional manifolds with isometry groups of dimension 4 or 6 except those with negative curvature:

$$ds^2 = \frac{dx^2+dy^2}{[1+m(x^2+y^2)]^2} + \left[ dz + \frac{\ell}{2} \frac{y\,dx - x\,dy}{1+m(x^2+y^2)} \right]^2. \tag{1}$$

More precisely, for $\ell = 0$ we have the symmetric spaces: $S^2(4m) \times \mathbb{R}$ if $m > 0$, $H^2(4m) \times \mathbb{R}$ if $m < 0$, and $\mathbb{R}^3$ if $m = 0$. Here $S^2(k)$ and $H^2(k)$ are the two sphere and hyperbolic plane equipped with a metric of constant curvature $k$.

For $\ell \neq 0$ one obtains the metric of symmetric spaces (cf. [6, p. 357]), and precisely one has the unitary quaternions $SU(2)$ if $m > 0$ (and in



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particular $S^3(m)$ if $m=\ell^2/4$), $\widetilde{SL}(2,\mathbb{R})$ (the universal covering space of $SL(2,\mathbb{R})$) if $m<0$, and the Heisenberg group $H_3$ if $m=0$.

The metrics obtained by opposite value of the parameter $\ell$ are equivalent by a reversing orientation isometry, for example the symmetry with respect to the $xy$-plane.

The metrics with negative curvature are not included in the family (1). For the hyperbolic space $H^3(K)$ with negative constant curvature $K$ we use the metric

$$ds^2 = e^{2kz}(dx^2+dy^2)+dz^2, \tag{2}$$

where $k>0$ and $K=-k^2$.

The paper is organized as follows.

In Section 2 we find the geodesics equations for the metrics above by integrating their Euler–Lagrange equation. In Section 3 we write the geodesics equations in a more compact form and we describe their Euclidean shape. Finally, in Section 3 we calculate the volume density function for the family of metrics (1) and (2).

The results obtained in Sections 2 and 3 are summarized at the end of the paper in Appendixes I–V.

## 2. EQUATIONS OF THE GEODESICS

From now on the letters $a, b, c, a', b'$ e $c'$ will be integration constants. The Lagrangian associated to the metrics (2) and (1) are respectively

$$L=\tfrac{1}{2}\left[e^{2kz}(\dot{x}^2+\dot{y}^2)+\dot{z}^2\right] \tag{3}$$

and

$$L=\frac{1}{2}\left\{\frac{\dot{x}^2+\dot{y}^2}{[1+m(x^2+y^2)]^2}+\left[\dot{z}+\frac{\ell}{2}\frac{x\dot{y}-y\dot{x}}{1+m(x^2+y^2)}\right]^2\right\}, \tag{4}$$

and the corresponding Euler–Lagrange equations are

$$\begin{gathered} e^{2kz}\dot{x}=a \\ e^{2kz}\dot{y}=b \\ \ddot{z}-k\,e^{2kz}(\dot{x}^2+\dot{y}^2)=0 \end{gathered} \tag{5}$$

and

$$
\begin{aligned}
&\ddot{x} - \frac{2mx\dot{x}^2 - 2mx\dot{y}^2 + 4my\dot{x}\dot{y}}{1 + m(x^2 + y^2)} + \ell c\dot{y} = 0\\
&\ddot{y} + \frac{2my\dot{x}^2 - 2my\dot{y}^2 - 4mx\dot{x}\dot{y}}{1 + m(x^2 + y^2)} - \ell c\dot{x} = 0\\
&\dot{z} + \frac{\ell}{2}\frac{y\dot{x} - x\dot{y}}{1 + m(x^2 + y^2)} = c.
\end{aligned}
\tag{6}
$$

We will integrate systems (5) and (6) with the initial conditions

$$
\begin{aligned}
&x(0) = 0, \qquad y(0) = 0, \qquad z(0) = 0,\\
&\dot{x}(0) = u, \qquad \dot{y}(0) = v, \qquad \dot{z}(0) = w.
\end{aligned}
\tag{7}
$$

For system (6) we distinguish two cases: if $m = 0$, i.e., in the case of the Heisenberg group $H_3$, system (6) becomes

$$
\begin{aligned}
&\ddot{x} + \ell c\dot{y} = 0\\
&\ddot{y} - \ell c\dot{x} = 0\\
&\dot{z} + \frac{\ell}{2}(y\dot{x} - x\dot{y}) = c.
\end{aligned}
\tag{8}
$$

Let $D = [1 + m(x^2 + y^2)]^{-2}$. If $m \neq 0$, adding the first equation of system (6) multiplied by $\dot{x}D$ one obtains

$$
\frac{d}{dt}\left\{\frac{\dot{x}^2 + \dot{y}^2}{[1 + m(x^2 + y^2)]^2}\right\} = 0.
$$

Adding the first equation of (6) multiplied by $-yD$ to the second equation multiplied by $xD$ one gets

$$
\frac{d}{dt}\left\{\frac{x\dot{y} - y\dot{x}}{[1 + m(x^2 + y^2)]^2} + \frac{\ell c}{2m}\frac{1}{1 + m(x^2 + y^2)}\right\} = 0.
$$

Thus a first integration reduces system (6) to

$$
\begin{aligned}
&\frac{\dot{x}^2 + \dot{y}^2}{[1 + m(x^2 + y^2)]^2} = a\\
&\frac{x\dot{y} - y\dot{x}}{[1 + m(x^2 + y^2)]^2} + \frac{\ell c}{2m}\frac{1}{1 + m(x^2 + y^2)} = b\\
&\dot{z} + \frac{\ell}{2}\frac{y\dot{x} - x\dot{y}}{1 + m(x^2 + y^2)} = c.
\end{aligned}
\tag{9}
$$

If one imposes conditions (7) in (9) one gets the following values for the integration constants: $a = u^2 + v^2$, $b = \ell c/2m$, and $c = w$.

System (9) reads in cylindrical coordinates $(\rho, \theta, z)$ as

$$\begin{aligned} \frac{\dot\rho^2 + \rho^2\dot\theta^2}{(1+m\rho^2)^2} &= u^2 + v^2 \\ \frac{\rho^2\dot\theta}{(1+m\rho^2)^2} + \frac{\ell w}{2m}\frac{1}{1+m\rho^2} &= \frac{\ell w}{2m} \\ \dot z - \frac{\ell}{2}\frac{\rho^2\dot\theta}{1+m\rho^2} &= w. \end{aligned} \tag{10}$$

Solving the second equation with respect to $\dot\theta$ and eliminating $\dot\theta$ in the first and third equation one obtains

$$\begin{aligned} \frac{\dot\rho^2}{(1+m\rho^2)^2} + \frac{\ell^2 w^2}{4}\rho^2 &= u^2 + v^2 \\ \dot\theta &= \frac{\ell w}{2}(1+m\rho^2) \\ \dot z - \frac{\ell^2 w}{4}\rho^2 &= w. \end{aligned} \tag{11}$$

We will suppose $w \neq 0$ (if $w = 0$, by (11), one gets that $\theta$ is constant and $z = 0$, and so the geodesic is a line lying in the $xy$-plane).

## 2.1. *Geodesics of the Symmetric Spaces*

In the family of metrics (1) those with constant curvature are: the flat metric $ds^2 = dx^2 + dy^2 + dz^2$ and the metric with positive curvature $m$

$$ds^2 = \frac{dx^2 + dy^2}{[1 + m(x^2+y^2)]^2} + \left[dz + \sqrt{m}\,\frac{y\,dx - x\,dy}{1 + m(x^2+y^2)}\right]^2 \tag{12}$$

whose geodesics can be obtained by setting $4m = \ell^2$ in the equation of the geodesics of $SU(2)$ which will be calculated in the next section (cfr. Appendix II).

To obtain the geodesics of the metrics (2) with constant negative curvature we will integrate the system (5). By initial conditions (7) one gets $a = u$ and $b = v$ and so, by substituting the first two equations in the last equation of (5) one gets

$$\begin{aligned} e^{2kz}\dot x &= u \\ e^{2kz}\dot y &= v \\ \ddot z - ke^{-2kz}(u^2+v^2) &= 0. \end{aligned} \tag{13}$$

Let us integrate the third equation of the system (13). Without loss of generality we can suppose that $\dot{z}$ is not zero, otherwise $u = v = w = 0$ and the geodesic reduces to a point. By multiplying the third equation of (13) by $2\dot{z}$ we obtain

$$\frac{d}{dt}\dot{z}^2 + (u^2 + v^2)\frac{d}{dt}e^{-2kz} = 0$$

and integrating

$$\dot{z}^2 + (u^2 + v^2)\, e^{-2kz} = c \tag{14}$$

where, by (7), the integration constant $c$ has the value $u^2 + v^2 + w^2$ which will be denoted by $r^2$.

Let us suppose $w \geqslant 0$ (if one consider the geodesic through the origin with opposite speed parametrized by $-t$). Equation (14) is equivalent to

$$\frac{1}{kr}\frac{de^{kz}}{\sqrt{e^{2kz} - \dfrac{u^2 + v^2}{r^2}}} = dt$$

whose integration gives

$$\frac{1}{kr}\log\left(e^{kz} + \sqrt{e^{2kz} - \frac{u^2 + v^2}{r^2}}\right) = t + c'. \tag{15}$$

Conditions (7) imply that

$$c' = \frac{1}{kr}\log\left(1 + \frac{w}{r}\right)$$

which, substituted in (15), gives

$$z = \frac{1}{k}\log\frac{(r + w)\, e^{2krt} + r - w}{2r\, e^{krt}}, \tag{16}$$

where $r = \sqrt{u^2 + v^2 + w^2}$.

Let us integrate now the other equations of system (13). By substituting (16) in the first equation of (13) one gets

$$\dot{x} = u\left[\frac{2re^{krt}}{(r + w)\, e^{2krt} + r - w}\right]^2. \tag{17}$$

The latter is equivalent to

$$dx = \frac{2ru}{k(r+w)^2} \frac{de^{2krt}}{\left(e^{2krt} + \dfrac{r-w}{r+w}\right)^2}$$

whose integration gives

$$x = -\frac{2ru}{k(r+w)[(r+w)\,e^{2krt} + r - w]} + a'.$$

The initial conditions (7) imply

$$a' = \frac{u}{k(r+w)}$$

and so

$$x = \frac{u}{k} \frac{e^{2krt} - 1}{(r+w)\,e^{2krt} + r - w}. \tag{18}$$

With a similar procedure by the second equation of (13) one obtains

$$y = \frac{v}{k} \frac{e^{2krt} - 1}{(r+w)\,e^{2krt} + r - w}. \tag{19}$$

Equations (18), (19), and (16) are the equations of the geodesic for $H^3$. Let us observe that these equations remain invariant by the change $(u, v, w, t) \to (-u, -v, -w, -t)$. Thus they are also the equations of the geodesics for $w < 0$.

Finally, let us consider the symmetric space $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$ with the metrics of the family (1) where $\ell = 0$, $m > 0$ and $\ell = 0$, $m < 0$, respectively.

For $\ell = 0$, system (11) becomes

$$\begin{aligned} \frac{\dot\rho^2}{(1+m\rho^2)^2} &= u^2 + v^2 \\ \dot\theta &= 0 \\ \dot z &= w. \end{aligned} \tag{20}$$

The integration is immediate and one gets

$$
\begin{aligned}
x &= \frac{u}{\sqrt{m(u^2+v^2)}}\,\mathrm{tg}(\sqrt{m(u^2+v^2)}\;t)\\
y &= \frac{v}{\sqrt{m(u^2+v^2)}}\,\mathrm{tg}(\sqrt{m(u^2+v^2)}\;t)\\
z &= wt.
\end{aligned} \tag{21}
$$

If $w=0$, as we already observed, the geodesics are a line of the $xy$-plane. Equations (21) do not have meaning if $u$ and $v$ both vanish. But this case is easily handled. In fact if $u=v=0$, by (20), one gets $\dot{\rho}=0$ and so $\rho=0$, $\theta=0$, and $z=wt$, i.e., the geodesics reduce to the $z$-axis.

Let observe that $m(u^2+v^2)$ is positive for $m>0$, i.e. in the case of $S^2\times\mathbb{R}$, and negative for $m<0$, i.e., in the case of $H^2\times\mathbb{R}$. Then, since $\mathrm{tg}\; i\alpha = i\;\mathrm{tgh}\;\alpha$ the geodesics equations of $H^2\times\mathbb{R}$ can be written as

$$
\begin{aligned}
x &= \frac{u}{\sqrt{-m(u^2+v^2)}}\,\mathrm{tgh}(\sqrt{-m(u^2+v^2)}\;t)\\
y &= \frac{v}{\sqrt{-m(u^2+v^2)}}\,\mathrm{tgh}(\sqrt{-m(u^2+v^2)}\;t)\\
z &= wt.
\end{aligned} \tag{22}
$$

Later we will see that Eqs. (21) and (22) can be obtained as particular cases of the equation of the geodesics of $SU(2)$ and $\widetilde{SL}(2,\mathbb{R})$ respectively with $4m(u^2+v^2)+\ell^2w^2<0$ (cf. Appendixes II and III).

Let us finally observe that in Eqs. (22) the parameter $t$ can be any real number while in Eqs. (21)

$$
-\frac{\pi}{2\sqrt{m(u^2+v^2)}} < t < \frac{\pi}{2\sqrt{m(u^2)}}.
$$

## 2.2. *Geodesics of $SU(2)$ and of $\widetilde{SL}(2,\mathbb{R})$*

Let us continue the integration of system (11) with the condition $\ell\neq 0$. The first equation is equivalent to

$$
\frac{d\rho}{(1+m\rho^2)\sqrt{u^2+v^2-\dfrac{\ell^2w^2}{4}\,\rho^2}} = \pm dt. \tag{23}
$$

Let us suppose that $C=4m(u^2+v^2)+\ell^2w^2\neq 0$. By integrating (23) one gets

$$\frac{2}{\sqrt{C}}\operatorname{arctg}\frac{\rho\sqrt{C}}{\sqrt{4(u^2+v^2)-\ell^2w^2\rho}}=\pm(t+a'). \tag{24}$$

*Remark* 2.1. In the integration of (23) it has been necessary to impose that $u$ and $v$ do not both vanish. If $u=v=0$ one easily obtains $x=0$, $y=0$, and $z=wt$, but the same result holds also if we put $u=0$ and $v=0$ in (29), (30), and (31).

By imposing the initial conditions (7) in (24) one gets $a'=0$ and so

$$\rho^2=\frac{4(u^2+v^2)\operatorname{tg}^2\left(\frac{\sqrt{C}}{2}t\right)}{C+\ell^2w^2\operatorname{tg}^2\left(\frac{\sqrt{C}}{2}t\right)}. \tag{25}$$

Then by (25) one obtains

$$\rho=\pm\frac{2\sqrt{u^2+v^2}\operatorname{tg}\left(\frac{\sqrt{C}}{2}t\right)}{\sqrt{C+\ell^2w^2\operatorname{tg}^2\left(\frac{\sqrt{C}}{2}t\right)}}, \tag{26}$$

where one takes $+$ for positive $t$ and $-$ for negative $t$.

By substituting (25) in the second equation of system (11) one has

$$\dot{\theta}=\frac{\ell w}{2}\left[1+\frac{4m(u^2+v^2)\operatorname{tg}^2\left(\frac{\sqrt{C}}{2}t\right)}{C+\ell^2w^2\operatorname{tg}^2\left(\frac{\sqrt{C}}{2}t\right)}\right];$$

thus

$$d\theta=\frac{\ell wC}{2}\frac{\left[1+\operatorname{tg}^2\left(\frac{\sqrt{C}}{2}t\right)\right]dt}{C+\ell^2w^2\operatorname{tg}^2\left(\frac{\sqrt{C}}{2}t\right)}$$

and integrating one gets

$$\theta = \operatorname{arctg} \frac{\ell w \operatorname{tg}\left(\frac{\sqrt{C}}{2} t\right)}{\sqrt{C}} + b'. \tag{27}$$

By substituting (25) in the third equation of system (11) one has

$$\dot{z} - \frac{\ell^2 (u^2 + v^2)\, w \operatorname{tg}^2\left(\frac{\sqrt{C}}{2} t\right)}{C + \ell^2 w^2 \operatorname{tg}^2\left(\frac{\sqrt{C}}{2} t\right)} = w. \tag{28}$$

Equation (28) is equivalent to the following

$$dz = w\, dt + \frac{\ell^2 (u^2 + v^2)\, w \operatorname{tg}^2\left(\frac{\sqrt{C}}{2} t\right) dt}{C + \ell^2 w^2 \operatorname{tg}^2\left(\frac{\sqrt{C}}{2} t\right)}$$

whose integration gives

$$z = wt + \frac{\ell^2 w}{2m\sqrt{C}} \left[ -\frac{\sqrt{C}}{2} t + \frac{\sqrt{C}}{\ell w} \operatorname{arctg} \frac{\ell w \operatorname{tg}\left(\frac{\sqrt{C}}{2} t\right)}{\sqrt{C}} \right] + c'.$$

Hence by the initial conditions (7), $c' = 0$ and then

$$z = w\left(1 - \frac{\ell^2}{4m}\right) t + \frac{\ell}{2m} \operatorname{arctg}\left[\frac{\ell w}{\sqrt{C}} \operatorname{tg}\left(\frac{\sqrt{C}}{2} t\right)\right], \tag{29}$$

where we recall that $C = 4m(u^2 + v^2) + \ell^2 w^2$.

Let us obtain the value of the constant of integration $b'$. Setting

$$T = \operatorname{arctg}\left[\frac{\ell w}{\sqrt{C}} \operatorname{tg}\left(\frac{\sqrt{C}}{2} t\right)\right]$$

one has $\theta = T + b'$. Then

$$\begin{aligned}
x &= \rho \cos(T + b'), \\
y &= \rho \sin(T + b'), \\
\dot{x} &= \dot{\rho} \cos(T + b') - \rho \sin(T + b')\, \dot{T}, \\
\dot{y} &= \dot{\rho} \sin(T + b') + \rho \cos(T + b')\, \dot{T}.
\end{aligned}$$

By imposing the initial conditions (7) one gets

$$\cos b' = \pm \frac{u}{\sqrt{u^2+v^2}} \qquad \text{and} \qquad \sin b' = \pm \frac{v}{\sqrt{u^2+v^2}},$$

with the same convention as before. One obtains

$$x = \frac{2\,\mathrm{tg}\left(\frac{\sqrt{C}}{2}t\right)}{\sqrt{C+\ell^2 w^2\,\mathrm{tg}^2\left(\frac{\sqrt{C}}{2}t\right)}}(u\cos T - v\sin T), \tag{30}$$

$$y = \frac{2\,\mathrm{tg}\left(\frac{\sqrt{C}}{2}t\right)}{\sqrt{C+\ell^2 w^2\,\mathrm{tg}^2\left(\frac{\sqrt{C}}{2}t\right)}}(v\cos T + u\sin T). \tag{31}$$

Equations (30), (31), and (29) are the geodesics equations relative to the metrics (1) with $m \neq 0$ and $C \neq 0$ (cf. (21) and Appendix II).

In the case $C < 0$, it is convenient to write (30), (31), and (29) in the form

$$\begin{aligned} x &= \frac{2\,\mathrm{tgh}\left(\frac{\sqrt{C'}}{2}t\right)}{\sqrt{C'+\ell^2 w^2\,\mathrm{tgh}^2\left(\frac{\sqrt{C'}}{2}t\right)}}(u\cos T' - v\sin T') \\ y &= \frac{2\,\mathrm{tgh}\left(\frac{\sqrt{C'}}{2}t\right)}{\sqrt{C'+\ell^2 w^2\,\mathrm{tgh}^2\left(\frac{\sqrt{C'}}{2}t\right)}}(v\cos T' + u\sin T') \\ z &= w\left(1-\frac{\ell^2}{4m}\right)t + \frac{\ell}{2m}T', \end{aligned} \tag{32}$$

where

$$C' = -C = -[4m(u^2+v^2)+\ell^2 w^2]$$

$$T' = \mathrm{arctg}\left[\frac{\ell w}{\sqrt{C'}}\,\mathrm{tgh}\left(\frac{\sqrt{C'}}{2}t\right)\right].$$

*Remark* 2. Equations (30), (31), and (29) with $C>0$ are defined for

$$-\frac{\pi}{\sqrt{4m(u^2+v^2)+\ell^2w^2}}<t<\frac{\pi}{\sqrt{4m(u^2+v^2)+\ell^2w^2}}.$$

Let us set $L=\pi/\sqrt{4m(u^2+v^2)+\ell^2w^2}$. If $\ell\neq 0$ the following limits are finite and

$$\begin{aligned} \lim_{t\to -L} x &= \lim_{t\to L} x, & \lim_{t\to -L} \dot{x} &= \lim_{t\to L} \dot{x},\\ \lim_{t\to -L} y &= \lim_{t\to L} y, & \lim_{t\to -L} \dot{y} &= \lim_{t\to L} \dot{y},\\ -\lim_{t\to -L} z &= \lim_{t\to L} z, & \lim_{t\to -L} \dot{z} &= \lim_{t\to L} \dot{z}. \end{aligned}$$

Hence the entire geodesics, corresponding to all values of the parameter $t$, can be obtained by translating the geodesic given by (30), (31), and (29) in the direction of the $z$-axis.

Finally, we have to integrate (11) in the case where $\ell\neq 0$ and $4m(u^2+v^2)+\ell^2w^2=0$. Since $m<0$ we will find the geodesics equations of the group $\widetilde{SL}(2,\mathbb{R})$.

Equation (23) becomes

$$\frac{d\rho}{\sqrt{u^2+v^2}\,\sqrt{(1+m\rho^2)^3}}=dt$$

which, when integrated, gives

$$\frac{1}{\sqrt{u^2+v^2}}\frac{\rho}{\sqrt{1+m\rho^2}}=(t+a'). \tag{33}$$

As in the previous case we can suppose that $u$ and $v$ do not both vanish (cf. Remark 4.1). By the initial conditions (7) $a'=0$ and by (32), since $4m(u^2+v^2)=-\ell^2w^2$, one gets

$$\rho=\pm\frac{2\sqrt{u^2+v^2}\,t}{\sqrt{4+\ell^2w^2t^2}}. \tag{34}$$

By substituting (34) in the second equation of system (11) one has

$$\dot{\theta}=\frac{\ell w}{2}\left[1+\frac{4m(u^2+v^2)\,t^2}{4+\ell^2w^2t^2}\right];$$

thus

$$d\theta=\frac{2\ell w\,dt}{4+\ell^2w^2t^2}$$

which, when integrated, gives

$$\theta = \operatorname{arctg}\left(\frac{\ell w}{2}\, t\right) + b'. \tag{35}$$

By substituting (34) in the third equation of system (11):

$$\dot{z} - \frac{\ell^2(u^2+v^2)\, wt^2}{4+\ell^2 w^2 t^2} = w,$$

so

$$dz = w\, dt + \frac{\ell^2(u^2+v^2)^2\, wt^2\, dt}{4+\ell^2 w^2 t^2}$$

and integrating one gets

$$z = wt + \frac{u^2+v^2}{w}\left[t - \frac{2}{\ell w}\operatorname{arctg}\left(\frac{\ell w}{2}\, t\right)\right] + c'.$$

By the initial conditions (7) $c'=0$ and since we imposed the condition $4m(u^2+v^2) = -\ell^2 w^2$, one has

$$z = w\left(1 - \frac{\ell^2}{4m}\right) t + \frac{\ell}{2m}\operatorname{arctg}\left(\frac{\ell w}{2}\, t\right). \tag{36}$$

As in the previous case one gets

$$\cos b' = \pm\frac{u}{\sqrt{u^2+v^2}} \qquad \text{and} \qquad \sin b' = \pm\frac{v}{\sqrt{u^2+v^2}}.$$

Setting $T = \operatorname{arctg}(\ell w\, t/2)$ one obtains

$$x = \frac{2t}{\sqrt{4+\ell^2 w^2 t^2}}\,(u\cos T - v\sin T), \tag{37}$$

$$y = \frac{2t}{\sqrt{4+\ell^2 w^2 t^2}}\,(v\cos T + u\sin T). \tag{38}$$

Equations (37), (38), and (36) are the geodesics equations of $\widetilde{SL}(2,\mathbb{R})$ relative to the family of metrics (1) where $\ell \neq 0$, $m<0$ and the condition $4m(u^2+v^2)+\ell^2 w^2 = 0$ (cf. Appendix IV).

Let us observe that these equations can be obtained from Eqs. (30), (31), and (29) for $C \to 0$.

### 2.3. *Geodesics of $H_3$*

Let us integrate system (8) by imposing $\ell \neq 0$. By (7) one gets $c = w$.

If $w = 0$ by (8) one gets $x = ut$, $y = vt$, and $z = 0$ and hence the geodesics are a line in the $xy$-plane. Then let us suppose that $w \neq 0$. The solutions of the first two equations of (8) are of the form

$$\begin{aligned} x &= a \sin(\ell wt) + b \cos(\ell wt) + c, \\ y &= a' \sin(\ell wt) + b' \cos(\ell wt) + c'. \end{aligned} \tag{39}$$

Since (39) satisfy (8) one has $a' = b$ and $b' = -a$ and by imposing (7) one gets $a = c' = u/\ell w$ and $b = -c = v/\ell w$. Then

$$x = \frac{u}{\ell w} \sin(\ell wt) + \frac{v}{\ell w} \cos(\ell wt) - \frac{v}{\ell w}, \tag{40}$$

and

$$y = \frac{v}{\ell w} \sin(\ell wt) - \frac{u}{\ell w} \cos(\ell wt) + \frac{u}{\ell w}. \tag{41}$$

By integrating the last equation of system (8) and by substituting the solution of (40) and (41) one finds

$$\dot{z} = w + \frac{\ell}{2}\left[\frac{u^2 + v^2}{\ell w} - \frac{u^2 + v^2}{\ell w} \cos(\ell wt)\right] \tag{42}$$

which, when integrated, gives

$$z = w\,t + \frac{u^2 + v^2}{2w}\,t - \frac{u^2 + v^2}{2\ell w} \sin(\ell wt) + c', \tag{43}$$

where $c' = 0$.

Equations (40), (41), and (43) are the geodesics equations of the Heisenberg group $H_3$ with the family of metrics (1) where $\ell \neq 0$ and $m = 0$ (cf. Appendix V). These equations can be obtained as $m$ goes to zero from (30), (31), and (29), respectively.

## 3. SIMPLIFIED EQUATIONS OF THE GEODESICS AND THEIR EUCLIDEAN SHAPE

We want to show that all the geodesics we found lie either in the cylinder of equation

$$x^2 + y^2 + \frac{2v}{\ell w}\,x - \frac{2u}{\ell w}\,y = 0 \tag{44}$$

or in the plane

$$vx - uy = 0. \tag{45}$$

The geodesics of $H^3$ have equations

$$\begin{aligned} x &= \frac{u}{k} \frac{e^{2krt} - 1}{(r+w)\, e^{2krt} + r - w} \\ y &= \frac{v}{k} \frac{e^{2krt} - 1}{(r+w)\, e^{2krt} + r - w} \\ z &= \frac{1}{k} \log \frac{(r+w)\, e^{2krt} + r - w}{2r\, e^{krt}}, \end{aligned} \tag{46}$$

where $r = \sqrt{u^2 + v^2 + w^2}$; then they lie in plane (45).

Furthermore it is immediate to verify that the geodesics equations of the Heisenberg group

$$\begin{aligned} x &= \frac{u}{\ell w} \sin(\ell w t) + \frac{v}{\ell w} \cos(\ell w t) - \frac{v}{\ell w} \\ y &= \frac{v}{\ell w} \sin(\ell w t) - \frac{u}{\ell w} \cos(\ell w t) + \frac{u}{\ell w} \\ z &= w\, t + \frac{u^2 + v^2}{2w}\, t - \frac{u^2 + v^2}{2\ell w} \sin(\ell w t). \end{aligned} \tag{47}$$

satisfy the equation of cylinder (44).

All the geodesics of the metrics (1) except for the case $m = 0$ and the case $\ell \neq 0$, $m < 0$, and $C = 4m(u^2 + v^2) + \ell^2 w^2 = 0$ can be written in the form

$$\begin{aligned} x &= \frac{2 \operatorname{tg}\left(\frac{\sqrt{C}}{2} t\right)}{\sqrt{C + \ell^2 w^2 \operatorname{tg}^2\left(\frac{\sqrt{C}}{2} t\right)}} (u \cos T - v \sin T) \\ y &= \frac{2 \operatorname{tg}\left(\frac{\sqrt{C}}{2} t\right)}{\sqrt{C + \ell^2 w^2 \operatorname{tg}^2\left(\frac{\sqrt{C}}{2} t\right)}} (v \cos T + u \sin T) \\ z &= w \left(1 - \frac{\ell^2}{4m}\right) t + \frac{\ell}{2m} T, \end{aligned} \tag{48}$$

where

$$C = 4m(u^2 + v^2) + \ell^2 w^2,$$
$$T = \operatorname{arctg}\left[\frac{\ell w}{\sqrt{C}} \operatorname{tg}\left(\frac{\sqrt{C}}{2} t\right)\right].$$

By squaring and adding the first two equations one gets

$$x^2 + y^2 = \frac{4(u^2 + v^2) \operatorname{tg}^2\left(\frac{\sqrt{C}}{2} t\right)}{C + \ell^2 w^2 \operatorname{tg}^2\left(\frac{\sqrt{C}}{2} t\right)}. \tag{49}$$

We can simplify Eqs. (48) as follows. Since $T = \operatorname{arctg}[\ell w \operatorname{tg}(\sqrt{C}\, t/2)/\sqrt{C}]$ is an angle between $-\pi/2$ and $\pi/2$ one has

$$\cos T = \frac{1}{\sqrt{1 + \operatorname{tg}^2 T}} \qquad \text{and} \qquad \sin T = \frac{\operatorname{tg} T}{\sqrt{1 + \operatorname{tg}^2 T}}. \tag{50}$$

By substituting (50) in the first two equations of system (48) one obtains

$$\begin{aligned} x &= \frac{2\sqrt{C} \operatorname{tg}\left(\frac{\sqrt{C}}{2} t\right)}{C + \ell^2 w^2 \operatorname{tg}^2\left(\frac{\sqrt{C}}{2} t\right)} \left[u - \frac{\ell v w}{\sqrt{C}} \operatorname{tg}\left(\frac{\sqrt{C}}{2} t\right)\right] \\ y &= \frac{2\sqrt{C} \operatorname{tg}\left(\frac{\sqrt{C}}{2} t\right)}{C + \ell^2 w^2 \operatorname{tg}^2\left(\frac{\sqrt{C}}{2} t\right)} \left[v - \frac{\ell u w}{\sqrt{C}} \operatorname{tg}\left(\frac{\sqrt{C}}{2} t\right)\right]. \end{aligned} \tag{51}$$

By (51) one gets

$$\frac{2v}{\ell w} x - \frac{2u}{\ell w} y = -\frac{4(u^2 + v^2) \operatorname{tg}^2\left(\frac{\sqrt{C}}{2} t\right)}{C + \ell^2 w^2 \operatorname{tg}^2\left(\frac{\sqrt{C}}{2} t\right)}. \tag{52}$$

By comparing (49) and (52) one finds that the geodesics of $SU(2)$, $S^3$ and $\widetilde{SL}(2, \mathbb{R})$ (except for the case $4m(u^2+v^2)+\ell^2 w^2=0$) are contained in cylinder (44).

By observing Eqs. (21) of the geodesics of $S^2 \times \mathbb{R}$ and Eqs. (22) of the geodesics of $H^2 \times \mathbb{R}$ it is immediate to verify that they lie in the plane (45). In a similar way we can write geodesics equations (37), (38), and (36) for $\widetilde{SL}(2, \mathbb{R})$ with $4m(u^2+v^2)+\ell^2 w^2=0$ as

$$
\begin{aligned}
x &= \frac{4t}{4+\ell^2 w^2 t^2}\left(u - \frac{\ell v w}{2}\, t\right) \\
y &= \frac{4t}{4+\ell^2 w^2 t^2}\left(v + \frac{\ell u w}{2}\, t\right) \\
z &= w\left(1 - \frac{\ell^2 w}{4m}\right) t + \frac{\ell}{2m} \operatorname{arctg}\left(\frac{\ell w}{2}\, t\right),
\end{aligned}
\tag{53}
$$

which are easily seen to be contained in cylinder (44).

Observe that circular cylinder (44) has generatrix parallel to the $z$-axis and its center and ray (for a fixed metric) depends on the the initial speed of the geodesics. The geodesics contained in the cylinder have an elicoidal behavior in all cases except $\widetilde{SL}(2, \mathbb{R})$ with the condition $4m(u^2+v^2)+\ell^2 w^2 \leqslant 0$. The geodesics of $SU(2)$, $S^3$, and $\widetilde{SL}(2, \mathbb{R})$ with the condition $4m(u^2+v^2)+\ell^2 w^2>0$ wrap around the cylinder with period

$$
\left(1 - \frac{\ell^2}{4m}\right) \frac{2\pi w}{\sqrt{4m(u^2+v^2)+\ell^2 w^2}} + \frac{\pi \ell}{2m},
$$

while those of $H_3$ have period

$$
\left(w + \frac{u^2+v^2}{2w}\right) \frac{2\pi}{\ell w}.
$$

In the case of $\widetilde{SL}(2, \mathbb{R})$ with the condition $4m(u^2+v^2)+\ell^2 w^2 \leqslant 0$ the geodesics do not wrap around the cylinder but they tend asymptotically to two symmetric generatrices of the cylinder which in the case that $4m(u^2+v^2)+\ell^2 w^2=0$ coincide with the generatrix of the cylinder opposite the $z$-axis.

*Remark* 3.1. Observe that for all the metrics of the family (1) and (2) the rotation around the $z$-axis is an isometry and so the surface obtained by this rotation is made by geodesics. Hence by the previous result every geodesic can be written as an intersection either of cylinder (44) or of plane (45) with a surface of revolution entirely made by geodesics. Furthermore in the case of the family (1) with parameter $\ell$ and $m$, this intersection with the cylinder gives rise to two curves: the geodesic itself and a curve which is a geodesic with respect to the metric with parameters $m$ and $-\ell$.

## 4. NORMAL COORDINATES AND VOLUME DENSITY FUNCTION

If one puts $t=1$ in the geodesics equations one gets the link between the coordinates $(x, y, z)$ and the normal coordinates $(u, v, w)$. One can then calculate the expression of the *volume density function* $\vartheta$. If we indicate by $J$ the Jacobian matrix of the change of coordinates $(u, v, w) \to (x, y, z)$ one gets

$$\vartheta(u, v, w) = \det J\, \vartheta(x, y, z).$$

One easily obtains

$$\vartheta(x, y, z) = [1 + m(x^2 + y^2)]^{-2}.$$

The calculation of the Jacobian is more complicated.

For the hyperbolic space $H^3$ endowed with metric (2) and geodesics (46) one finds

$$\vartheta_{H^3} = \frac{(e^{2kr} - 1)^2}{4k^2 r^2 e^{2kr}} \tag{54}$$

which is equivalent to the expression

$$\vartheta = \left(\frac{\sinh kr}{kr}\right)^2 \tag{55}$$

(cf. [3]).

Take now into consideration Eqs. (48).

Setting

$$R = \frac{2\,\mathrm{tg}\left(\frac{\sqrt{C}}{2}\right)}{\sqrt{C + \ell^2 w^2\, \mathrm{tg}^2\left(\frac{\sqrt{C}}{2}\right)}},$$

$$T = \mathrm{arctg}\left[\frac{\ell w}{\sqrt{C}}\, \mathrm{tg}\left(\frac{\sqrt{C}}{2}\right)\right].$$

After a straightforward (but long) calculation, one gets that the Jacobian is given by

$$\det J = R\left(1 - \frac{\ell^2}{4m} + \frac{\ell}{2m}\frac{\partial T}{\partial w}\right)\left(u\frac{\partial R}{\partial u} + v\frac{\partial R}{\partial v} + R\right) - \frac{\ell}{2m}R\frac{\partial R}{\partial w}\left(u\frac{\partial T}{\partial u} + v\frac{\partial T}{\partial v}\right).$$

By calculating the partial derivatives one obtains

$$\vartheta_{m,\ell} = \frac{4\ell^2 r^2 \sin^2(\sqrt{C}/2)}{C^2} + \frac{(4m - \ell^2)(u^2+v^2)\sin\sqrt{C}}{C\sqrt{C}}. \tag{56}$$

This is the volume density function for the all the metrics (1) except for the case of $\widetilde{SL}(2, \mathbb{R})$ with $C = 0$, and the case of $H_3$ $(m = 0)$. Nevertheless, for these two cases, we know that the geodesics can be obtained by (48) for $C \to 0$ and for $m \to 0$, respectively. This holds also for their volume density functions and one finds

$$\vartheta_{m,\ell} = 1 - \tfrac{1}{12}(4m - \ell^2)(u^2+v^2)$$

and

$$\vartheta_{H_3} = \frac{4r^2 \sin^2(\ell w/2)}{\ell^2 w^4} - \frac{(u^2+v^2)\sin(\ell w)}{\ell w^3},$$

respectively.

## APPENDIX I

$$ds^2 = e^{2kz}(dx^2 + dy^2) + dz^2,$$

$$H^3 \qquad k > 0,$$

$$x = \frac{u}{k}\frac{e^{2krt} - 1}{(r+w)\,e^{2kr\,t} + r - w}$$

$$y = \frac{v}{k}\frac{e^{2krt} - 1}{(r+w)\,e^{2kr\,t} + r - w}$$

$$z = \frac{1}{k}\log\frac{(r+w)\,e^{2krt} + r - w}{2r\,e^{krt}},$$

where

$$r = \sqrt{u^2 + v^2 + w^2};$$

$$\vartheta_{H^3} = \frac{(e^{2kr} - 1)^2}{4k^2 r^2 e^{2kr}}.$$

## APPENDIX II

$$ds^2 = \frac{dx^2 + dy^2}{[1 + m(x^2 + y^2)]^2} + \left[ dz + \frac{\ell}{2} \frac{y\,dx - x\,dy}{1 + m(x^2 + y^2)} \right]^2,$$

$$\begin{array}{llll} SU(2) & \ell \neq 0 & m > 0 & 4m \neq \ell^2, \\ S^3 & \ell \neq 0 & m > 0 & 4m = \ell^2, \\ S^2 \times \mathbb{R} & \ell = 0 & m > 0, & \\ \widetilde{SL}(2, \mathbb{R}) & \ell \neq 0 & m < 0 & 4m(u^2 + v^2) + \ell^2 w^2 > 0, \end{array}$$

$$x = \frac{2\,\mathrm{tg}\left(\frac{\sqrt{C}}{2}\,t\right)}{\sqrt{C + \ell^2 w^2\,\mathrm{tg}^2\left(\frac{\sqrt{C}}{2}\,t\right)}} (u \cos T - v \sin T)$$

$$y = \frac{2\,\mathrm{tg}\left(\frac{\sqrt{C}}{2}\,t\right)}{\sqrt{C + \ell^2 w^2\,\mathrm{tg}^2\left(\frac{\sqrt{C}}{2}\,t\right)}} (v \cos T + u \sin T)$$

$$z = w\left(1 - \frac{\ell^2}{4m}\right) t + \frac{\ell}{2m}\, T,$$

where

$$C = 4m(u^2 + v^2) + \ell^2 w^2,$$

$$T = \mathrm{arctg}\left[\frac{\ell w}{\sqrt{C}}\,\mathrm{tg}\left(\frac{\sqrt{C}}{2}\,t\right)\right];$$

$$\vartheta_{m,\ell} = \frac{4\ell^2 r^2 \sin^2(\sqrt{C}/2)}{C^2} + \frac{(4m - \ell^2)(u^2 + v^2) \sin\sqrt{C}}{C\sqrt{C}}.$$

## APPENDIX III

$$ds^2 = \frac{dx^2 + dy^2}{[1 + m(x^2 + y^2)]^2} + \left[ dz + \frac{\ell}{2} \frac{y\,dx - x\,dy}{1 + m(x^2 + y^2)} \right]^2,$$

$$\widetilde{SL}(2, \mathbb{R}) \qquad \ell \neq 0 \qquad m < 0 \qquad 4m(u^2 + v^2) + \ell^2 w^2 < 0,$$

$$H^2 \times \mathbb{R} \qquad \ell = 0 \qquad m < 0,$$

$$x = \frac{2 \operatorname{tgh}\left(\frac{\sqrt{C}}{2} t\right)}{\sqrt{C + \ell^2 w^2 \operatorname{tgh}^2\left(\frac{\sqrt{C}}{2} t\right)}} (u \cos T - v \sin T)$$

$$y = \frac{2 \operatorname{tgh}\left(\frac{\sqrt{C}}{2} t\right)}{\sqrt{C + \ell^2 w^2 \operatorname{tgh}^2\left(\frac{\sqrt{C}}{2} t\right)}} (v \cos T + u \sin T)$$

$$z = w\left(1 - \frac{\ell^2}{4m}\right) t + \frac{\ell}{2m} T,$$

where

$$C = -[4m(u^2 + v^2) + \ell^2 w^2],$$

$$T = \operatorname{arctg}\left[\frac{\ell w}{\sqrt{C}} \operatorname{tgh}\left(\frac{\sqrt{C}}{2} t\right)\right];$$

$$\vartheta_{m,\ell} = -\frac{4\ell^2 r^2 \sinh^2(\sqrt{C}/2)}{C^2} - \frac{(4m - \ell^2)(u^2 + v^2) \sinh\sqrt{C}}{C\sqrt{C}}.$$

## APPENDIX IV

$$ds^2 = \frac{dx^2 + dy^2}{[1 + m(x^2 + y^2)]^2} + \left[ dz + \frac{\ell}{2} \frac{y\,dx - x\,dy}{1 + m(x^2 + y^2)} \right]^2,$$

$$\widetilde{SL}(2, \mathbb{R}) \qquad \ell \neq 0 \qquad m < 0 \qquad 4m(u^2 + v^2) + \ell^2 w^2 = 0,$$

$$x = \frac{2t}{\sqrt{4 + \ell^2 w^2 t^2}} (u \cos T - v \sin T)$$

$$y = \frac{2t}{\sqrt{4 + \ell^2 w^2 t^2}} (v \cos T + u \sin T)$$

$$z = w\left(1 - \frac{\ell^2}{4m}\right) t + \frac{\ell}{2m} T,$$

where

$$T = \operatorname{arctg}\left(\frac{\ell w}{2}\, t\right);$$

$$\vartheta_{m,\ell} = 1 - \frac{1}{12}(4m - \ell^2)(u^2 + v^2).$$

## APPENDIX V

$$ds^2 = dx^2 + dy^2 + \left[dz + \frac{\ell}{2}(y\,dx - x\,dy)\right]^2,$$

$$H_3 \qquad \ell \neq 0 \qquad m = 0$$

If $w \neq 0$

$$x = \frac{u}{\ell w}\sin(\ell w t) + \frac{v}{\ell w}[\cos(\ell w t) - 1]$$

$$y = \frac{v}{\ell w}\sin(\ell w t) - \frac{u}{\ell w}[\cos(\ell w t) - 1]$$

$$z = wt + \frac{u^2 + v^2}{2w}\, t - \frac{u^2 + v^2}{2\ell w}\sin(\ell w t)$$

$$\vartheta_{H_3} = \frac{4r^2 \sin^2(\ell w/2)}{\ell^2 w^4} - \frac{(u^2 + v^2)\sin(\ell w)}{\ell w^3}.$$

If $w = 0$

$$x = ut$$

$$y = vt$$

$$z = 0$$

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