A note on the structural stability of the equilibrium manifold *

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Abstract: In a smooth pure exchange economy with fixed total resources we investigate whether the smooth selection property holds when endowments are redistributed across consumers through a continuous (non local) redistribution policy. We show that if the policy is regular then there exists a unique continuous path of equilibrium prices which support it.

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1 Introduction

In a smooth pure exchange economy with fixed total resources suppose that endowments are redistributed across agents according to a redistribution policy. Define such a policy by a continuous map $\gamma : [0,1] \to \Omega(r)$, $\gamma(0) = \omega_0$, $\gamma(1) = \omega_1$, where $\Omega(r)$ denotes the set of endowments with fixed total resources $r \in \mathbb{R}^l$ and x (y) denotes the initial (final) allocation, respectively. If we write, using standard vector notation to denote the aggregate excess demand function, the equilibrium condition as $z(p(t), \gamma(t)) = 0$, $t \in [0,1]$, very natural questions are whether $p(t)$ is locally unique and it is changing continuously while the parameter $\gamma(t) \in \Omega(r)$ is varying. As highlighted by Kehoe [6, p. 315], an affirmative answer to these two questions is crucial for comparative statics analysis and planning theory (the advantage of using a continuous policy would be invalidated by discontinuities of prices). In particular, the use of the second welfare theorem becomes problematic with multiplicity and instability of prices (see Garratt and Goenka [5] and references therein).

In order to provide an answer to these questions, the traditional approach uses the implicit function theorem (IFT). Since IFT holds locally, this approach cannot deal with arbitrary variations of regular endowments. Moreover, IFT depends on a regularity condition, the non-singularity of the Jacobian matrix of the aggregrate excess demand function: it cannot be applied if the endowments changed are singular economies. Summarizing: Multiplicity of prices and catastrophes may originate indeterminacy and ambiguity. In Kehoe's words [6, p. 333]: "it does not seem that any sort of comparative statics methodology is applicable in such circumstances".

A geometric understanding of the relationship between endowments and price variations can be provided by the equilibrium manifold approach [2]. If we denote by S the set of normalized prices, the *equilibrium manifold* $E(r) \subset S \times \Omega(r)$ is the set of prices and endowments such that aggregate excess demand is zero. Let us denote by $\pi : E(r) \to \Omega(r)$ the *natural projection*, i.e. the restriction to the equilibrium manifold $E(r)$ of the projection $(p, \omega) \mapsto \omega$ from $S \times \Omega(r)$ into $\Omega(r)$ (see [2] and Section 3). The structure of the equilibrium manifold and the properness property of the natural projection (see Section 3) entail the existence of *smooth selections* of equilibrium prices: i.e., an infinitesimal change of regular endowments implies an infinitesimal change of the corresponding equilibrium price vectors. It is worth noting that this smooth selection property (SSP) is not a theory of equilibrium selection (for the existence of a continuous random selection see [7, p. 348] and [8]).

When endowments are redistributed according to $\gamma(t)$, SSP holds when $\gamma(t)$ is a *local* and *regular* policy. By local we mean that $\gamma(t)$ belongs to a sufficiently small neighborhood of the initial economy $x = \gamma(0)$ and by regular we mean that

$\gamma(t)$ does not cross *singular economies* (see [2] or Section 3). In fact if a continuous local policy $\gamma(t)$ should cross the set of singular economies, it could give rise to *catastrophes* (see [1, 4]), thus determining discontinuities of prices.

In this paper we are interested in whether non local redistributions of endowments across consumers can be supported by continuous price changes. In our main result, Theorem 4.1, we give a solution to this problem when regular endowments are redistributed through a (not necessarily local) policy. This question is deeply related (see Remark 3.2 and Remark 4.2) to the issue, raised and tackled by Balasko, of finding the largest domain of definition of the smooth selection mappings ([2, p. 94 and p. 190]). Our result is achieved by using the *arc lifting property* (ALP), a property known in the mathematics of covering spaces (see Proposition 2.2). ALP formalizes the reason why the connected component is of interest because it extends to the connected component containing the initial endowment what only holds locally by SSP, overcoming the local condition imposed by IFT and providing theoretical foundations to the assumption that when regular endowments are changed smoothly, the corresponding equilibrium price vectors will also change smoothly. Observe that ALP does not hold when the policy is not regular, i.e., when the redistribution concerns singular economies. The existence of a lift on $E(r)$ of a continuous policy becomes an issue in this case.

Our result should be compared with the work of [5] which deal with the connected component containing the efficient allocations. In their work they want to decentralize with certainty any efficient allocation: their redistribution policies are assumed to belong to an open, connected set of regular economies with a unique equilibrium. This strong assumption is needed to overcome some problems related to multiplicity and instability of prices while using the second welfare theorem (for example, counter-intuitive utility changes may arise when a tax policy encounters singularities).

The structure of this paper is the following. Sections 2 and 3 recall some mathematical results and the economic model. In Section 4 we prove our main result.

2 Mathematical preliminaries

We start this section by introducing some mathematical results on covering spaces and lifting properties. Unless otherwise specified all the topological spaces involved are subsets of Euclidean spaces (and therefore Hausdorff and locally connected) and all maps between them are assumed to be smooth. We refer the reader to [9] for the standard material on coverings. Let $\tilde{X}$ and X be two (not necessarily connected) topological spaces. A map $p : \tilde{X} \to X$ is called a *covering map* if it satisfies the following conditions:

(a) p is surjective;

(b) each $x \in X$ has an open neighbourhood U such that $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by p diffeomorphically onto U.

The neighbourhood U is said to be *well-covered* for p and the set $p^{-1}(x)$ is called the *fiber* of x. If U is connected the cardinality of $p^{-1}(x)$ does not depend on the point $x \in U$. Observe that this number could be infinite as for the covering map $\exp : \mathbb{R} \to S^1$ from the real numbers to the unit circle $S^1 \subset \mathbb{R}^2$ defined by $\exp(t) = (\cos t, \sin t)$.

Notice that property (b) in the definition above immediately implies that p is a local diffeomorhism and hence an open map. Thus an injective covering map is a diffeomorphism. Observe also that there exist surjective local diffeomorphisms which are not a covering map. For example, if one restricted the map $\exp : \mathbb{R} \to S^1$ above to the interval $(0, 4\pi)$, one would obtain a surjective local diffeomorphism but the point $(1, 0)$ does not admit a well-covered neighbourhood. This example also shows that the restriction of a covering $p : \tilde{X} \to X$ to a subset of its domain $\tilde{X}$ is not in general a covering map. Nevertheless one has the following proposition.

Proposition 2.1 *Let $p : \tilde{X} \to X$ be a covering map. Then for any subset $Y \subset X$ the restriction $p_{|p^{-1}(Y)} : p^{-1}(Y) \to Y$ is still a covering map, where Y and $p^{-1}(Y)$ are equipped with induced topology of X and $\tilde{X}$ respectively.*

Proof: The map $p_{|p^{-1}(Y)}$ is surjective and smooth since p is surjective and smooth. Let $x \in Y$ and let U be a well-covered open neighbourhood of x for p. Then it is straightforward to verify that $Y \cap U$ is well-covered open neighbourhood of x for $p_{|p^{-1}(Y)}$. □

Let now $p : \tilde{X} \to X$ be a smooth map (not necessarily a covering) and let Y be a topological space. A *lift* of a map $f : Y \to X$ is a map $\tilde{f} : Y \to \tilde{X}$ such that $p\tilde{f} = f$. We recall that an arc on X is a map $\alpha : I \to X$, where $I = [0, 1]$. The points $\alpha(0)$ and $\alpha(1)$ are called the starting and final points of α. The following proposition is the key ingredient for the proof of Theorem 4.1 (the reader is referred to [9, Lemma 3.1 on p. 123] for a proof).

Proposition 2.2 (ALP: arc lifting property) *Given a covering space $p : \tilde{X} \to X$, let $\alpha : I \to X$ be an arc with starting point x_0 and let $\tilde{x}_0$ be any point in the fiber of x_0. There exists a unique lift $\tilde{\alpha} : I \to \tilde{X}$ of α with starting point $\tilde{x}_0$.*

A natural question is to understand when a surjective local diffeomorphism is a covering map. It is interesting to notice that a local diffeomorphism $p : \tilde{X} \to X$ such that every arc on X admits a lift in $\tilde{X}$ is in fact a covering map. Since this property is generally impossible to handle, one can impose some sufficient topological conditions in order to have a covering map as in the following proposition.

Proposition 2.3 *Let $\tilde{X}$ and X be two topological spaces. Let $p : \tilde{X} \to X$ be a proper surjective local diffeomorphism. Then p is a finite covering map.*

Proof: Recall that a continuous map $p : \tilde{X} \to X$ is proper if the preimage of any compact set of X is a compact set of $\tilde{X}$. Let $x \in X$. The set $p^{-1}(x)$ is compact and discrete (since p is a local diffeomorphism) and hence finite. Let $p^{-1}(x) = \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_k\}$ and consider disjoint open neighborhoods $W_1, \dots, W_k$, $x_j \in W_j = 1, \dots, k$ such that

$$p_{|W_j} : W_j \to U_j = p(W_j),\ j = 1, \dots, k$$

is a diffeomorphism. The set $U = U_1 \cap U_2 \cap \dots \cap U_k \setminus p(\tilde{X} \setminus (W_1 \cup \dots \cup W_k))$ is a well-covered open neighborhood of x. Indeed, since a proper map is closed, $p(\tilde{X} \setminus (W_1 \cup \dots \cup W_k))$ is closed and hence U is open. Moreover, $p^{-1}(U) = \cup_j p^{-1}(U) \cap W_j$ and

$$p_{|p^{-1}(U)\cap W_j} : p^{-1}(U) \cap W_j \to U,\ j = 1, \dots, k$$

is a diffeomorphism. In fact, the map $p_{|p^{-1}(U)\cap W_j}$ is injective, since $p_{|W_j}$ is, and surjective, since

$$p(p^{-1}(U) \cap W_j) = p(p^{-1}(U)) \cap p(W_j) = U \cap p(W_j) = U \cap U_j = U.$$

$\square$

3 The model

We refer the reader to [2] for the economic set-up briefly described in this section. We consider a pure exchange economy with fixed total resources. Let m and l be, respectively, the (finite) number of agents and commodities. Let $S = \{p \in \mathbb{R}^l \mid p_i \geq 0, i = 1, 2, \dots, l-1, p_l = 1\}$ be the set of prices normalized by the numeraire convention. Let $r \in \mathbb{R}^l$ be the vector of fixed total resources and denote by $\Omega(r)$ the set of endowments with fixed total resources, i.e., $\Omega(r) = \{\omega \in \mathbb{R}^{lm} | \sum_{i=1}^m \omega_i = r\}$. Under the standard smooth assumptions on preferences (see [2, chap. 2]), the problem of maximizing the smooth utility function $u_i : \mathbb{R}^l \to \mathbb{R}$ subject to the budget constraint $p \cdot \omega_i = w_i$ gives the unique solution $f_i(p, w_i)$, i.e. consumer i's demand. Define the equilibrium manifold, denoted by $E(r)$, the set of pairs of prices and endowments such that aggregate net demand is zero, i.e.,

$$E(r) = \{(p, \omega) \in S \times \Omega(r) \mid \sum_{i=1}^m f_i(p, p \cdot \omega_i) = r\}.$$

The set $E(r)$ is globally diffeomorphic to $\mathbb{R}^{l(m-1)}$ (see [2, Ch. 5]). Let $\pi : E(r) \to \Omega(r)$ be the *natural projection*, i.e. the restriction to $E(r)$ of the projection $S \times \Omega(r) \to \Omega(r)$, such that $(p, \omega) \mapsto \omega$. The map π is smooth, proper and surjective. One can define the set of *critical equilibria*, denoted by $E_c(r)$, as the pairs $(p, \omega) \in E(r)$ such that the derivative of π is not onto [1]. The set $E_c(r)$ is a closed subset of measure zero of the equilibrium manifold $E(r)$ [3]. The set of *singular economies*, denoted by Σ, is the image via π of the set $E_c(r)$. The set Σ is a closed (by properness of π) and a measure zero set in $\Omega(r)$ (by Sard's theorem). Let us define the regular economies $\mathcal{R} = \Omega(r) \setminus \Sigma$ as the regular values of the map π.
We state as a theorem the following important result due to Balasko.

Theorem 3.1 (Balasko [2]) *The map $\pi_{|_{\pi^{-1}(R)}} : \pi^{-1}(R) \to R$ is a finite covering.*

Proof: By the inverse function theorem $\pi_{|_{\pi^{-1}(R)}}$ is a local diffeomorphism. Since $\pi_{|_{\pi^{-1}(V)}} : \pi^{-1}(V) \to V$ is still proper and surjective for any subset $V \subset \Omega(r)$, the result follows by Proposition 2.3. $\square$

The economic meaning of this theorem, known as smooth selection property (SSP) [2, p. 94], is that in a neighborhood of a regular economy, smooth changes of the parameter ω imply smooth changes of the corresponding equilibrium price vectors, namely there exists a supporting equilibrium price vector sufficiently close to the initial one. More precisely, if $\omega \in \Omega(r)$ is a regular economy such that the cardinality of its fiber $\pi^{-1}(\omega)$ is n, then it follows by Theorem 3.1 that there exists a well-covered open and connected neighbourhood U of ω and smooth mappings $s_i : U \to S$, $i = 1, \ldots n$, called the *selections of price equibria*, such that $\cup_i s_i(\bar{\omega})$ is the set of the equilbrium price vectors associated with the allocation $\bar{\omega}$ for all $\bar{\omega} \in U$.

Remark 3.2 By Balasko's Theorem 3.1 and by Proposition 2.1 one can deduce that if U is the connected component of the set of regular economies containing $\omega \in \mathcal{R}$ and W is a connected component of $\pi^{-1}(U)$ then $\pi_{|W} : W \to U$ is a finite covering map. The natural question raised by Balasko in [2] is to understand when U is a well-covered neighbourhood of ω, i.e. when the map $\pi_{|W} : W \to U$ is a global diffeomorphism for *all connected components* W of $\pi^{-1}(U)$. In economic terms, one wants to understand when one can take U as a largest domain of definition of selections of price equilibria supporting ω. Balasko [2, p. 191] shows that if there are two commodities or two consumers then the previous question has a positive answer. In the general case (see Theorem 7.3.10 by Balasko [2, p. 192]) he shows that there exists an open open and dense subset $\mathcal{R}'$ such that one can take as the largest domain of definition of selection of prices the connnected component $U' \subset U$ of $\mathcal{R}'$ containing ω ($U' = U$ for $l = 2$ or $m = 2$).

4 Main result

In the sequel we denote a redistribution policy of endowments as a continuous map $\gamma : [0,1] \to \Omega(r)$, where $\omega_0 = \gamma(0)$ and $\omega_1 = \gamma(1)$. Since we are interested in whether arbitrary redistributions of endowments across consumers can be supported by continuous price changes, we define a redistribution policy $\gamma(t)$:

- *regular* if $\gamma(t) \subset \mathcal{R}$;
- *singular* if $\gamma(t) \in \Sigma$ for some $t \in [0,1]$.

The following theorem extends SSP to the whole connected component containing the endowments redistributed by a regular policy.

Theorem 4.1 *Let $\gamma : I \to \mathcal{R}$ be a regular policy connecting $\omega_0 = \gamma(0)$ and $\omega_1 = \gamma(1)$ and let p_0 be the supporting equilibrium price vector associated with ω_0. Then there exists a unique lift $\tilde{\gamma} : I \to \pi^{-1}(\mathcal{R})$ of γ.*

Proof: It follows by Proposition 2.2 and Theorem 3.1. $\square$

Remark 4.2 The proof of Theorem 4.1 does not need ALP in the case $l = 2$ or $m = 2$. Indeed, by Remark 3.2, it follows that the lift of γ is given by $\pi_{|W}^{-1} \circ \gamma$ where W is the connected component of $\pi^{-1}(U)$ containing p_0 (and $\pi_{|W} : W \to U$ is a diffeomorphism). In the general case (for $l, m \geq 3$) our Theorem 4.1 provides a further insight of the structural stability of the equilibrium manifold. In fact, our result works for any regular policy in U, even if the domain of definition of selections of price equilibria cannot be eventually extended to all U.

Since the existence of a lift relies on properties of covering maps (see Proposition 2.2), Theorem 4.1 cannot be extended to singular policies.

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