Engliš expansion for Hartogs domains

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Abstract

A 2-dimensional strongly pseudoconvex Hartogs domain D_F can be equipped with a natural Kähler metric g_F. In this paper we prove that if the second term of Engliš's expansion of Rawnsley's epsilon function is constant then (D_F, g_F) is holomorphically isometric to an open subset of the 2-dimensional complex hyperbolic space.

Keywords: Kähler metrics; Hartogs domain; Tian-Yau-Zelditch expansion; Kempf's distortion function.

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1 Introduction and statements of the main results

Let Ω be a strongly pseudoconvex bounded domain in $\mathbb{C}^n$ with real analytic boundary endowed with a real-analytic Kähler metric g admitting a globally defined Kähler potential Φ. This means that $\omega = \frac{i}{2}\partial\bar{\partial}\Phi$, where ω is the Kähler form associated to g (then Φ is a strictly plurisubharmonic function on Ω). For every real number $\alpha > 0$ consider the weighted Bergman space $\mathcal{H}_\alpha$ of all holomorphic functions on Ω square integrable with respect to the measure $e^{-\alpha\Phi}\frac{\omega^n}{n!}$, i.e. f belongs to $\mathcal{H}_\alpha$ iff $\int_\Omega e^{-\alpha\Phi}|f|^2\frac{\omega^n}{n!} < \infty$. Let $K_\alpha(x, y)$ be the reproducing kernel of the Hilbert space $\mathcal{H}_\alpha$, i.e. $K_\alpha(x, y) = \sum_j f_j^\alpha(x)f_j^\alpha(y)$, where f_j^α is an orthonormal basis for $\mathcal{H}_\alpha$. In [9] it is proven that the function

$$\epsilon_\alpha(x) = e^{-\alpha\Phi(x)}K_\alpha(x, x) \tag{1}$$

admits the following asymptotic expansion with respect to α

$$\epsilon_\alpha(x) \sim \sum_{j=0}^{\infty} a_j(x)\alpha^{n-j} \tag{2}$$

where a_j, $j = 0, 1, \ldots$, are smooth coefficients with $a_0(x) = 1$. The function ϵ_α is called Rawnsley's epsilon function (see [3] and [16]). In [10] Engliš computes the coefficients a_j, $j \leq 3$, namely (we do not write the expression of a_3 since it is quite complicated and it will not be used in this paper)

$$\begin{cases} a_0 = 1 \\ a_1 = \frac{1}{2}\rho \\ a_2 = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|\operatorname{Ric}|^2 + 3\rho^2) \end{cases} \tag{3}$$

where (see also [12] and [15]) ρ, R, Ric denote respectively the scalar curvature, the curvature tensor and the Ricci tensor of (Ω, g).

Notice that the terms a_1 was already computed by Berezin in his seminal paper [2] on quantization by deformation. Actually the previous asymptotic expansion is strongly related to Berezin transform and to asymptotic expansion of Laplace integrals (see [9] and [12]).

Expansion (2) is the counterpart of the celebrated Tian-Yau-Zelditch expansion of Kempf's distortion function $T_m(x) \sim \sum_{j=0}^{\infty} b_j(x)m^{n-j}$ for polarized compact Kähler manifolds M, (where m is a non-negative integer) (see Zelditch [18] and [1]), where b_j, $j = 0, 1, \ldots$, are smooth coefficients with $b_0(x) = 1$. Lu [14], by means of Tian's peak section method, proved that each of the coefficients $b_j(x)$ is a polynomial of the curvature and its covariant derivatives at x of the metric g. Such a polynomial can be found by finitely many steps of algebraic operations. Furthermore $b_1(x)$, $b_2(x)$ have the same values of those given by (3), namely $b_1(x) = \frac{1}{2}\rho$ and $b_2(x) = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|\operatorname{Ric}|^2 + 3\rho^2)$ (see also [11] and [12] for the computations of the coefficients b_j's through Calabi's diastasis function). Due to the work of Donaldson ([6], [7]) in the compact case (resp. to the theory of quantization in the noncompact case) it is natural to study metrics with the coefficients b_k's of TYZ expansion (resp. a_k's of Engliš expansion) being prescribed.

In the compact case Lu and Tian [15] proved the following important result.

Theorem 1.1 *Let ω be a Kähler form cohomologous to the Fubini–Study metric ω_{FS} of the one-dimensional complex projective space $\mathbb{C}P^1$ and let $T_m(x) \sim \sum_{j=0}^{\infty} b_j(x)m^{n-j}$ be the Tian–Yau–Zelditch asymptotic expansion of Kempf's distortion function $T_m(x)$ relative to the metric g associated to ω. Assume that the coefficients $a_k = 0$ for $k \geq 2$. Then, there exists an automorphism $\psi : \mathbb{C}P^1 \to \mathbb{C}P^1$ such that $\psi^*(\omega) = \omega_{FS}$.*

In this paper we address the problem of generalizing the previous result to the noncompact case. Our main result is the following theorem valid for a particular class of bounded domains, the so called Hartogs domains (see next section for their definition), where we characterized the 2-dimensional complex hyperbolic space in terms of the coefficient a_2 (notice that we do not impose any conditions on the terms a_k with $k \geq 3$).

Theorem 1.2 *Let D_F be a 2-dimensional strongly pseudoconvex Hartogs domain in $\mathbb{C}^2$ with real analytic boundary and let g_F be its canonical Kähler metric (see (4) below). Assume that g_F is real analytic (equivalently F is real analytic). If the second term a_2 of Engliš expansion is constant then D_F is holomorphically isometric to an open subset of the complex hyperbolic space $\mathbb{C}H^2$.*

2 Proof of the main results

Given $x_0 \in \mathbb{R}^+ \cup \{+\infty\}$ and a decreasing smooth function $F : [0, x_0) \to (0, +\infty)$ satisfying $-(\frac{xF'}{F})' > 0$, the *Hartogs domain* $D_F \subset \mathbb{C}^2$ associated to the function F is the open subset of $\mathbb{C}^2$ defined by

$$D_F = \{(z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 < x_0,\ |z_1|^2 < F(|z_0|^2)\}.$$

Since D_F is strongly pseudoconvex, the natural $(1,1)$-form on D_F given by

$$\omega_F = \frac{i}{2}\partial\bar{\partial}\log\frac{1}{F(|z_0|^2) - |z_1|^2} \tag{4}$$

is a Kähler form on D_F (see [8]). The Kähler metric g_F associated to the Kähler form ω_F has been considered in [8] and [13] in the framework of quantization of Kähler manifolds, and is the metric we will be dealing with in this paper. (see also [4] and [5]). By setting

$$A = A(z_0, z_1) = F(|z_0|^2) - |z_1|^2 \tag{5}$$

and

$$C = C(z_0, z_1) = |z_0|^2 F'^2(|z_0|^2) - \left(|z_0|^2 F''(|z_0|^2) + F'(|z_0|^2)\right) A, \tag{6}$$

the matrix $g = (g_{\alpha\bar{\beta}})_{\alpha,\beta=0,1}$ of the Kähler metric g_F associated to the Kähler form (4) is given by:

$$g = \begin{pmatrix} g_{0\bar{0}} & g_{0\bar{1}} \\ g_{1\bar{0}} & g_{1\bar{1}} \end{pmatrix} = \frac{1}{A^2}\begin{pmatrix} C & -F'\bar{z}_0 z_1 \\ -F' z_0 \bar{z}_1 & F \end{pmatrix}. \tag{7}$$

Then

$$\det(g) = \frac{|z_0|^2 F'^2 - F(F' + |z_0|^2 F'')}{A^3} \tag{8}$$

and its inverse is given by:

$$g^{-1} = \begin{pmatrix} g^{0\bar{0}} & g^{0\bar{1}} \\ g^{1\bar{0}} & g^{1\bar{1}} \end{pmatrix} = \frac{A}{|z_0|^2 F'^2 - F(F' + |z_0|^2 F'')} \begin{pmatrix} F & F' \bar{z}_0 z_1 \\ F' z_0 \bar{z}_1 & C \end{pmatrix}. \tag{9}$$

Remark 2.1 Notice that the Hartogs domain (D_F, g_F) associated to the function $F(x) = 1 - x$, $x \in [0, 1)$, is the 2-dimensional complex hyperbolic space $\mathbb{C}H^2$, namely the unit ball in $\mathbb{C}^2$ endowed with the hyperbolic metric $g_F = g_{hyp}$.

In the sequel, given a rotation invariant function f (or a tensor) on D_F, namely a function depending only on $|z_0|^2$ and $|z_1|^2$, we will denote by $f(r)$ (resp. $f(s)$) its value at $z_1 = 0$ and $r = |z_0|^2$ (resp. $z_0 = 0$, $s = |z_1|^2$). In the following lemma we compute the functions $a_2(r) = a_2(z_0, 0), r = |z_0|^2$ and $a_2(s) = a_2(0, z_1), s = |z_1|^2$ for a Hartogs domain (D_F, g_F) where a_2 is the second term of Engliš expansion given by (3). This will be enough to prove our Theorem 1.2.

Lemma 2.2 *Let (D_F, g_F) be a Hartogs domain. Set*

$$B(r) = \frac{F^2}{rF'^2 - F(F' + rF'')} \tag{10}$$

$$L(r) = \frac{d}{dr}\left[r \frac{d}{dr} \log(rF'^2 - F(F' + rF''))\right]. \tag{11}$$

Then

$$|R|^2(r) = 8 + (BL + 2)^2. \tag{12}$$

$$|Ric|^2(r) = 18 + BL(BL + 6) \tag{13}$$

$$\rho(r) = -6 - BL \tag{14}$$

$$\Delta\rho(r) = B\left(L - r(BL)'' - (BL)'\right) \tag{15}$$

Moreover

$$|R|^2(s) = 12 - 8\frac{F''(0)}{F'(0)^2}(F(0) - s) + 4\frac{F''(0)^2}{F'(0)^4}(F(0) - s)^2 \tag{16}$$

$$|Ric|^2(s) = 18 - 6\frac{L(0)}{F'(0)}(F(0)-s) + \frac{L(0)^2}{F'(0)^2}(F(0)-s)^2 \tag{17}$$

$$\rho(s) = -6 + \frac{(F(0)-s)L(0)}{F'(0)} \tag{18}$$

$$\begin{aligned} \Delta\rho(s) &= \left(\frac{2F(0)F''(0)L(0) - 2F'(0)^2L(0) - F(0)F'(0)L'(0)}{F(0)F'(0)^3}\right)(F(0)-s)^2 \\ &- \frac{L(0)}{F'(0)}(F(0)-s) \end{aligned} \tag{19}$$

Consequently

$$a_2(r) = -\frac{1}{3}Br(BL)^{''} - \frac{1}{3}B(BL)^{'} + BL + 2 \tag{20}$$

$$a_2(s) = c_2(F(0)-s)^2 - \frac{5L(0)F^{'}(0) + 2F^{''}(0)}{6F^{'}(0)^2}(F(0)-s) + 2, \tag{21}$$

where c_2 is a real number whose expression is not important for our purpouses.

Proof: Recall that the curvature tensor and the Ricci tensor for a given Kähler metric $g = (g_{i\bar{j}})$ on an n-dimensional complex manifold are given respectively by:

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{l}}}{\partial z_k \partial \bar{z}_j} + \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{p}}}{\partial z_k} \frac{\partial g_{q\bar{l}}}{\partial \bar{z}_j}. \tag{22}$$

$$Ric_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j}(\log\det(g)) \tag{23}$$

and, in accordance with the general definition of norm of a complex tensor (see e.g. [19], p. 127), we have

$$|R|^2 = \sum_{i,j,k,l,p,q,r,s=1}^{n} \overline{g^{i\bar{p}}} g^{j\bar{q}} \overline{g^{k\bar{r}}} g^{l\bar{s}} R_{i\bar{j}k\bar{l}} \overline{R_{p\bar{q}r\bar{s}}} \tag{24}$$

and

$$|Ric|^2 = \sum_{i,j,k,l=1}^{n} \overline{g^{i\bar{k}}} g^{j\bar{l}} Ric_{i\bar{j}} \overline{Ric_{k\bar{l}}}. \tag{25}$$

Moreover the scalar curvature and its Laplacian are given by:

$$\rho = \sum_{i,j=1}^{n} g^{i\bar{j}} Ric_{i\bar{j}} \tag{26}$$

$$\Delta\rho = \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}. \tag{27}$$

In our case it follows from (7) and (9) that

$$g_{0\bar{0}}(r) = B^{-1},\ g_{0\bar{1}}(r) = g_{1\bar{0}}(r) = 0,\ g_{1\bar{1}}(r) = F^{-1}, \tag{28}$$

$$g^{0\bar{0}}(r) = B,\ g^{0\bar{1}}(r) = g^{1\bar{0}}(r) = 0,\ g^{1\bar{1}}(r) = F. \tag{29}$$

These and (22) give

$$R_{0\bar{0}0\bar{0}}(r) = B^{'}B^{-2} - rB^{-3}(B^{'})^2 + rB^{''}B^{-2} = -B^{-2}(BL+2), \tag{30}$$

$$R_{0\bar{0}1\bar{1}}(r) = R_{0\bar{1}1\bar{0}}(r) = -\frac{rF'^2 - F(F' + rF'')}{F^3} = -B^{-1}F^{-1}, \tag{31}$$

$$R_{1\bar{1}1\bar{1}}(r) = -2F(r)^{-2}, \tag{32}$$

where the second equality of (30) follows by

$$BL = -2 - B' - rB^{''} + rB^{-1}(B^{'})^2,$$

which is a straight consequence of the definitions of B and L (cf. (10) and (11)). By the symmetries of the curvature tensor, we have $R_{1\bar{1}0\bar{0}}(r) = R_{0\bar{0}1\bar{1}}(r)$ and $R_{1\bar{0}0\bar{1}}(r) = R_{0\bar{1}1\bar{0}}(r)$, while the remaining components of the curvature tensor are easily seen to vanish. Thus, by (24) and (29) one gets

$$\begin{aligned} |R|^2(r) &= (g^{0\bar{0}}(r))^4 |R_{0\bar{0}0\bar{0}}(r)|^2 + 2(g^{0\bar{0}}(r))^2 (g^{1\bar{1}}(r))^2 |R_{0\bar{0}1\bar{1}}(r)|^2 \\ &+ 2(g^{0\bar{0}}(r))^2 (g^{1\bar{1}}(r))^2 |R_{0\bar{1}1\bar{0}}(r)|^2 + (g^{1\bar{1}}(r))^4 |R_{1\bar{1}1\bar{1}}(r)|^2 \\ &= 8 + (BL+2)^2, \end{aligned}$$

namely (12). Let us consider now the component of the Ricci tensor. A simple computation using (8) and (23) gives:

$$Ric_{0\bar{0}} = -L(|z_0|^2) - 3g_{0\bar{0}} \tag{33}$$

$$Ric_{0\bar{1}} = -3g_{0\bar{1}},\ Ric_{1\bar{0}} = -3g_{1\bar{0}},\ Ric_{1\bar{1}} = -3g_{1\bar{1}}. \tag{34}$$

By (28) one has:

$$Ric_{0\bar{0}}(r) = -L(r) - 3B^{-1},\ Ric_{0\bar{1}}(r) = Ric_{1\bar{0}}(r) = 0,\ Ric_{1\bar{1}}(r) = -3F^{-1}. \tag{35}$$

Thus by (25) and (29) one gets

$$\begin{aligned} |Ric|^2(r) &= (g^{0\bar{0}}(r))^2|Ric_{0\bar{0}}(r)|^2 + 2g^{0\bar{0}}(r)g^{1\bar{1}}(r)|Ric_{0\bar{1}}(r)|^2 \\ &+ (g^{1\bar{1}}(r))^2|Ric_{1\bar{1}}(r)|^2 = 18 + BL(BL+6), \end{aligned}$$

namely (13). Now we calculate $\rho(r)$ and $\Delta\rho(r)$. By (9), (26), (33) and (34) one obtains:

$$\rho(z_0, z_1) = -6 - \frac{A(z_0, z_1)B(|z_0|^2)L(|z_0|^2)}{F(|z_0|^2)}, \tag{36}$$

where A and B are given respectively by (5) and (10). Hence, (14) is immediate and

$$\Delta\rho(r) = g^{0\bar{0}}(r)\frac{\partial^2\rho}{\partial z_0\partial\bar{z}_0}(r) + g^{1\bar{1}}(r)\frac{\partial^2\rho}{\partial z_1\partial\bar{z}_1}(r) = B(L - r(BL)'' - (BL)'),$$

namely (15)), follows by a straightforward computation.

Formulae (16)–(19) are obtained in a similar manner by long but straightforward computations (which can also be obtained with the help of Mathematica) after noticing that:

$$g_{0\bar{0}}(s) = -\frac{F'(0)}{F(0)-s},\ g_{0\bar{1}}(s) = g_{1\bar{0}}(s) = 0,\ g_{1\bar{1}}(s) = \frac{F(0)}{(F(0)-s)^2}, \tag{37}$$

$$g^{0\bar{0}}(s) = -\frac{F(0)-s}{F'(0)},\ g^{0\bar{1}}(s) = g_{1\bar{0}}(s) = 0,\ g^{1\bar{1}}(s) = \frac{(F(0)-s)^2}{F(0)} \tag{38}$$

which together with (22), (33) and (34) provide the following values for the curvature tensor and the Ricci tensor at $z_0 = 0$ and $s = |z_1|^2$:

$$R_{0\bar{0}0\bar{0}}(s) = -\frac{2F'(0)^2}{(F(0)-s)^2} + \frac{2F''(0)}{F(0)-s},$$

$$R_{0\bar{0}1\bar{1}}(s) = R_{0\bar{1}1\bar{0}}(s) = \frac{F(0)F'(0)}{(F(0)-s)^3},\ R_{1\bar{1}1\bar{1}}(s) = -\frac{2F(0)^2}{(F(0)-s)^4},$$

$$Ric_{0\bar{0}}(s) = -L(0) + 3\frac{F'(0)}{F(0)-s},\ Ric_{0\bar{1}}(s) = Ric_{1\bar{0}}(s) = 0,\ Ric_{1\bar{1}}(s) = -3\frac{F(0)}{(F(0)-s)^2}$$

(notice again that $R_{1\bar{1}0\bar{0}}(s) = R_{0\bar{0}1\bar{1}}(s)$, $R_{1\bar{0}0\bar{1}}(s) = R_{0\bar{1}1\bar{0}}(s)$ and that the remaining components $R_{i\bar{j}k\bar{l}}(s)$ of the curvature tensor vanish.)

Proof of Theorem 1.2 Assume that $a_2(z_0, z_1)$ is a constant, say K. Then, in particular, $a_2(r) = K$ and $a_2(s) = K$. Hence formula (21) yields:

$$c_2(F(0)-s)^2 - \frac{5L(0)F^{'}(0) + 2F^{''}(0)}{6F^{'}(0)^2}(F(0)-s) = K - 2$$

which can hold for every s (with $s < F(0)$) if and only if $c_2 = 0$,

$$L(0) = -\frac{2F^{''}(0)}{5F^{'}(0)} \tag{39}$$

and $K = 2$. Therefore, by setting $\phi(r) = BL$, equation (20) gives

$$-rB\phi'' - B\phi' + 3\phi = 0. \tag{40}$$

Now, we prove by induction that, for every integer $n \geq 0$, the derivative $\phi^{(n)}(0)$ vanishes. This will complete the proof of Theorem 1.2 since, being ϕ analytic (since F is analytic by assumption), it implies that $\phi = BL$ identically vanishes. Being $B \neq 0$, it follows that $L = 0$, and then we continue as in the proof of Theorem 4.8 in [13] concluding that $F(x) = \alpha_1 - \alpha_2 r,\ r = |z_0|^2$, with $\alpha_1, \alpha_2 > 0$. Thus D_F is holomorphically isometric to an open subset of the hyperbolic space $\mathbb{C}H^2$ via the map

$$\psi : D_F \to \mathbb{C}H^2,\ (z_0, z_1) \mapsto \left(\frac{z_0}{\sqrt{\alpha_1/\alpha_2}}, \frac{z_1}{\sqrt{\alpha_1}}\right).$$

In order to prove that $\phi(0) = 0$, let us notice that, by the very definition of $L(r)$ one gets $L(0) = 2\frac{F''(0)}{F'(0)}$. This together with (39) implies $L(0) = 0$ and then $\phi(0) = B(0)L(0) = 0$. Now, let us derivate equation (40) $n \geq 0$ times. By the chain rule we get

$$3\phi^{(n)} - \sum_{k=0}^{n}\binom{n}{k}B^{(k)}\phi^{(n-k+1)} - \sum_{k=0}^{n}\binom{n}{k}r^{(k)}(B\phi'')^{(n-k)} = 0.$$

which evaluated at $r=0$ gives

$$\begin{aligned}
3\phi^{(n)}(0) &- \sum_{k=0}^{n}\binom{n}{k}B^{(k)}(0)\phi^{(n-k+1)}(0) - n(B\phi'')^{n-1}(0) \\
&= 3\phi^{(n)}(0) - \sum_{k=0}^{n}\binom{n}{k}B^{(k)}(0)\phi^{(n-k+1)}(0) \\
&- n\sum_{k=0}^{n-1}\binom{n-1}{k}B^{(k)}(0)\phi^{(n-k+1)}(0) = 0.
\end{aligned}$$

Finally, assume that $\phi^{(k)}(0) = 0$ for $k \leq n$. Then by the previous equality we immediately get $-(n+1)B(0)\phi^{(n+1)}(0) = 0$ which implies $\phi^{n+1}(0) = 0$ and this ends the proof of the theorem. □

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