Quantization of bounded domains

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Abstract

We consider the quantization of a complex manifold endowed with the Bergman form following the ideas contained in [2], [3], [4] and [5]. In particular we give a geometric interpretation for the quantization to be regular in terms of the Hilbert space of square integrable holomorphic n-forms on M and the Hilbert space of holomorphic n-forms on M bounded with respect to the Liouville element.

Keywords: Kähler metrics ; Quantization.
Subj.Class: 53C55, 58F06.

1 Preliminaries

A geometric quantization of a Kähler manifold (M, ω) is a pair (L, h), where L is a holomorphic line bundle over M and h is a hermitian structure on L such that $\mathrm{curv}(L, h) = -2\pi i\omega$. The curvature $\mathrm{curv}(L, h)$ is calculated with respect to the *Chern connection*. If $\sigma : U \to L^+$ is a trivialising *holomorphic* section (where L^+ denotes the complement of the zero section in L) then the following formula holds

$$\mathrm{curv}(L, h) = -\partial\bar{\partial} \log h(\sigma(x), \sigma(x)). \tag{1}$$

Let $(s_0, \ldots, s_N)$ $(N \leq \infty)$ be a unitary basis of the separable complex Hilbert space $\mathcal{H}_h$ of all global holomorphic sections s of L, which are bounded with respect to the Liouville element defined by

$$\langle s, s\rangle_h = \|s\|_h^2 = \int_M h(s(x), s(x))\frac{\omega^n(x)}{n!}$$

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(see [2]). Define the smooth function on M by

$$\epsilon_{(L,h)}(x) = \sum_{j=0}^{N} h(s_j(x), s_j(x)). \tag{2}$$

Assuming that for all $x \in M$ there exists $s \in \mathcal{H}_h$ such that $s(x)$ is different from zero one can define the map

$$\phi_\sigma : U \to \mathbb{C}^{N+1} \setminus \{0\} : x \mapsto \left(\frac{s_0(x)}{\sigma(x)}, \dots, \frac{s_N(x)}{\sigma(x)}\right), \tag{3}$$

where $\sigma : U \to L^+$ is a trivialising holomorphic section on an open set $U \subset M$. It follows easily that (3) is the local expression of a globally defined map $\phi_{(L,h)} : M \to \mathbb{P}^N(\mathbb{C})$. This is called the *coherent states map.*

Theorem 1.1 ([2]) *The coherent states map is full, i.e. $\phi(M)$ is not contained in $\mathbb{P}^r(\mathbb{C})$ with $r < N$. Furthermore*

$$\phi^*_{(L,h)}\Omega^N_{FS} = \omega + \frac{i}{2\pi}\partial\bar{\partial}\log\epsilon_{(L,h)}. \tag{4}$$

Corollary 1.2 *Let (L,h) be a quantization of a Kähler manifold (M,ω). If $\epsilon_{(L,h)}$ is a positive constant, then ω is projectively induced.*

Recall that a Kähler form ω on a complex manifold M is *projectively induced* if there exists $N \leq \infty$ and a holomorphic map

$$\phi : M \to \mathbb{P}^N(\mathbb{C})$$

such that

$$\phi^*\Omega^N_{FS} = \omega. \tag{5}$$

In the proof of Theorem 2.1 we need

Theorem 1.3 (cfr. [6]) *Let $N, N' \leq \infty$. Let M be a complex manifold, $\phi : M \to \mathbb{P}^N(\mathbb{C})$ and $\psi : M \to \mathbb{P}^{N'}(\mathbb{C})$ full holomorphic maps such that $\phi^*\Omega^N_{FS} = \psi^*\Omega^{N'}_{FS}$ is a Kähler form on M. Then $N = N'$ and there exists a unitary transformation U of $\mathbb{P}^N(\mathbb{C})$ such that $\phi = U \circ \psi$.*

2 The results

Let M be a n-dimensional complex manifold and K its canonical bundle. If α is a holomorphic section of K, i.e. a holomorphic n-form on M then in a complex coordinate system U, endowed with local coordinates $(z_1, \dots, z_n)$, there exists a holomorphic function f_α such that

$$\alpha(z) = f_\alpha(z) dz_1 \wedge \ldots \wedge dz_n, \ \forall z \in U. \tag{6}$$

Let $(\alpha_0, \dots, \alpha_{N'})$ $(N' \leq \infty)$ be an unitary basis for the separable complex Hilbert space $(\mathcal{F}, (\cdot, \cdot))$ of all holomorphic n-forms α bounded with respect to $\|\alpha\|^2 = (\alpha, \alpha) := \frac{i^n}{2^n} \int_M \alpha \wedge \bar{\alpha}$. Let K^* be the smooth function on U given by

$$K^*(z, \bar{z}) = \sum_{j=0}^{N'} f_{\alpha_j}(z) \bar{f}_{\alpha_j}(z). \tag{7}$$

The expression $\partial\bar{\partial} \log K^*$ does not depend on the coordinates. Hence $\omega_B = \frac{i}{2\pi} \partial\bar{\partial} \log K^*$ is a globally defined 2-form on M. In the sequel we will suppose ω_B is a Kähler form . For example this happens for the bounded domains in $\mathbb{C}^N$ (see [7] for details).

Formula

$$h(\alpha, \alpha) := \frac{|f_\alpha|^2}{K^*}, \ \forall \alpha \in H^0(K). \tag{8}$$

defines a hermitian structure on K and it is immediate to verify that the pair (K, h) is a geometric quantization for (M, ω_B).

Furthermore it follows by (2) that

$$\epsilon_{(K,h)} = \sum_{j=0}^{N} |f_{s_j}|^2 / K^*, \tag{9}$$

where $s_j = f_{s_j} dz_1 \wedge \ldots \wedge dz_n, j = 0, \dots, N$ is a unitary basis for $(\mathcal{H}_h, \langle \cdot, \cdot \rangle_h)$.

Theorem 2.1 *Let M be a complex manifold such that ω_B is a Kähler form. Then $\epsilon_{(K,h)}$ equals a positive constant λ if and only if the following conditions are satisified*

(i) the complex dimension of $\mathcal{F}$ is equal to the complex dimension of $\mathcal{H}_h$;

(ii) $(\cdot, \cdot) = \lambda \langle \cdot, \cdot \rangle_h$.

Proof: Suppose that $\dim \mathcal{H}_h = \dim \mathcal{F} = N+1$ and $\langle \cdot , \cdot \rangle_h = \lambda(\cdot,\cdot)$. Let α_j and s_j, $j = 0,1,\ldots N$ be unitary bases for $(\mathcal{F}, (\cdot , \cdot))$ and $(\mathcal{H}_h, \langle \cdot , \cdot \rangle_h)$ respectively. Then there exist a $(N+1)\times(N+1)$ unitary matrix u_{jk} and a complex number C such that

$$f_{s_j} = C \sum_{k=0}^{N} u_{jk} f_{\alpha_k}, j = 0, \ldots, N, \tag{10}$$

and $|C|^2 = \lambda$. Hence, by (9),

$$\epsilon_{(K,h)} = \sum_{j=0}^{N} |f_{s_j}|^2 / K^* = \sum_{j=0}^{N} |f_{s_j}|^2 / \sum_{j=0}^{N} |f_{\alpha_j}|^2 = \lambda.$$

Conversely, suppose that $\epsilon_{(K,h)} = \lambda$ and let $N+1$ be the dimension of $\mathcal{H}_h$. By Theorem 1.1 the coherent states map

$$\phi_{(L,h)} : M \to \mathbb{P}^N(\mathbb{C}) : x \mapsto [(f_{s_0}(x), \ldots, f_{s_N}(x))],$$

is a full holomorphic map and $\phi^*_{(L,h)}\Omega^N_{FS} = \omega_B$. On the other hand ω_B is projectively induced via the full holomorphic map

$$j : M \to \mathbb{P}^{N'}(\mathbb{C}) : x \mapsto [(f_{\alpha_0}(x), \ldots, f_{\alpha_{N'}}(x))].$$

(see [7] for details). By Theorem 1.3 $N = N'$, i.e. $\dim \mathcal{H}_h = \dim \mathcal{F} = N+1$. Moreover, there exist a $(N+1)\times(N+1)$ unitary matrix u_{jk} and a complex number C such that 10 holds, i.e. $(\cdot,\cdot) = |C|^2 \langle \cdot , \cdot \rangle_h$. and, by the proof of the first part, $|C|^2 = \lambda$. $\square$

Theorem 2.2 *Let M be a simply connected complex manifold such that ω_B is Kähler-Einstein with scalar curvature -1 and $\epsilon_{(K,h)}$ is a positive constant, say λ. Then $K^* = \lambda \det(g_{j\bar{k}})$ (see [7, p. 274] for a discussion of this condition).*

Proof: Let $\omega_B = \frac{i}{2} \sum_{j,\bar{k}=1}^{n} g_{j\bar{k}} dz_j \wedge d\bar{z}_{\bar{k}}$ be the expression of the Bergman form in local coordinates $(z_1, \ldots, z_n)$. If ω_B is Kähler-Einstein then

$$\omega_B = \frac{i}{2\pi}\partial\bar{\partial}\log K^* = \frac{i}{2\pi}\partial\bar{\partial}\log\det(g_{j\bar{k}}) = -\rho_{\omega_B}, \tag{11}$$

where ρ_{ω_B} is the Ricci form on M (see [1])

This is equivalent to $\partial\bar{\partial}\log\frac{K^*}{\det(g_{j\bar{k}})} = 0$. It follows by the simply connectness of M that there exists a holomorphic function k on M such that $\frac{K^*}{\det(g_{j\bar{k}})} = e^{\Re(k)}$.

Since $\epsilon_{(K,h)}$ is constant if follows by (2.1) that $\dim\mathcal{H}_h = \dim\mathcal{F} = N+1$. Then it is not hard to see that $s_j := e^{\frac{k}{2}}\alpha_j$ is a unitary basis for $(\mathcal{H}_h, \langle\cdot,\cdot\rangle_h)$, where $(\alpha_0,\ldots,\alpha_N)$ is a unitary basis for $(\mathcal{F}, (\cdot,\cdot))$. By formula (9) and by hypothesis $\epsilon_{(K,h)} = e^{\Re(k)} = \lambda$. $\square$

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