$\boxed{\text{Jussieu, 23 February 2018}}$

Symplectic coordinates on Kähler manifolds: PART I

Andrea Loi, University of Cagliari

joint works with:

A. J. Di Scala, R. Mossa, M. Zedda, F. Zuddas

SYMPLECTIC COORDINATES

Let (M^{2n}, ω) be a symplectic manifold and let $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$ be the standard symplectic form on $\mathbb{R}^{2n}$.

Theorem (Darboux): Given $p \in M$ there exist an open set $U_p \subset M$ and a diffeomorphism

$$\psi : U_p \to \psi_p(U_p) \subset \mathbb{R}^{2n}$$

such that

$$\psi^* \omega_0 = \omega_{|U_p}$$

Questions: How large U_p can be taken? When $U_p = M$ or U_p is dense in M?

Theorem (Gromov, Inv.Math. 1985): There exists a symplectic form ω on $\mathbb{R}^{2n}, n \geq 2$, such that $(\mathbb{R}^{2n}, \omega)$ cannot be symplectically embedded into $(\mathbb{R}^{2n}, \omega_0)$.

SYMPLECTIC COORDINATES: THE KÄHLER CASE

Theorem (D. McDuff, J. Diff. Geom. 1988): Let (M, ω) be a Kähler manifold. Assume that $\pi_1(M) = \{1\}$, M is complete and $K \leq 0$. Given $p \in M$ there exists a diffeomorphism

$$\psi_p : M \to \mathbb{R}^{2n}, \ \psi_p(p) = 0$$

satisfying $\psi_p^* \omega_0 = \omega$.

Theorem (E. Ciriza, Diff. Geom. Appl. 1993): Let $T \subset M$ be a complex and totally geodesic submanifold of M passing through p. Then, $\psi_p(T) = \mathbb{C}^k \subset \mathbb{C}^n \cong \mathbb{R}^{2n}, \dim_{\mathbb{C}} T = k$.

PROBLEMS

1. Find explicit global symplectic coordinates on Kähler manifolds satisfying the assumptions of McDuff's theorem.

We will analyze the case of **Hermitian symmetric spaces of noncompact type** (**HSSNT) and their compact duals.**

2. Find an example of complete Kähler manifold (M, ω) admitting global coordinates not satisfying McDuff's assumptions.

DEFINITION OF HSSNT

An HSSNT (M, ω) is a Kähler manifold, which is holomorphically isometric to a bounded symmetric domain $(M, 0)$ of $M \subset \mathbb{C}^n$ centered at the origin $0 \in \mathbb{C}^n$ equipped with a multiple of the Bergman metric ω_B such that for all $p \in M$ the geodesic symmetry:

$$s_p : \mathrm{exp}_p(v) \mapsto \mathrm{exp}_p(-v), \forall v \in T_pM$$

is a globally defined holomorphic isometry of M.

An HSSNT is a homogenous Kähler manifold (converse not true Pyateskii–Shapiro).

There is a complete classification of irreducible HSSNT, with four classical series, studied by Cartan, and two exceptional cases.

THE CASE OF THE UNIT DISK (1)

$\mathbb{C}H^1 = \{z \in \mathbb{C} \mid |z|^2 < 1\},\ \omega = \omega_{hyp} = \frac{i}{2}\frac{dz\wedge d\bar{z}}{(1-|z|^2)^2} = -\frac{i}{2}\partial\bar{\partial}\log(1-|z|^2) = \frac{i}{2}\frac{dz\wedge d\bar{z}}{(1-|z|^2)^2}$

We look for a map

$$\psi : \mathbb{C}H^1 \to \mathbb{R}^2, \psi(0) = 0$$

such that

$$\psi^*\omega_0 = \omega_{hyp},\ \omega_0 = dx \wedge dy = \frac{i}{2}dz \wedge d\bar{z}$$

Assume $\psi(z) = f(r)z,\ r = |z|^2$.

$\Rightarrow\ \psi^*\omega_0 = (2rf\frac{\partial f}{\partial r} + f^2)dx \wedge dy = \frac{\partial}{\partial r}(rf^2)dx \wedge dy = \omega_{hyp} = \frac{1}{(1-r)^2}dx \wedge dy$

$\Rightarrow\ \frac{\partial}{\partial r}(rf^2) = \frac{1}{(1-r)^2} \Rightarrow rf^2 = (1-r)^{-1} + C \Rightarrow C = -1,\ rf^2 = r(1-r)^{-1} \Rightarrow f(r) = (1-r)^{-\frac{1}{2}}$

Hence

$$\boxed{\psi(z) = \frac{z}{\sqrt{1-|z|^2}}}$$

THE CASE OF THE UNIT DISK (2)

Let $\mathbb{C}P^1$ be the one-dimensional complex projective space, (namely the compact dual of $\mathbb{C}H^1$) endowed with the Fubini–Study form ω_{FS}. Then

$$\mathbb{R}^2 \cong \mathbb{C} \cong U_0 = \{z_0 \neq 0\} \subset \mathbb{C}P^1$$

and

$$\omega_{FS}|_{U_0} = \frac{i}{2}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$$

and it is easily seen that $\psi : \mathbb{C}H^1 \subset \mathbb{C} \subset \mathbb{C}P^1 \to \mathbb{C}, z \mapsto \frac{z}{\sqrt{1-|z|^2}}$ satisfies:

$$\psi^*\omega_{FS} = \omega_0,$$

where ω_0 is the restriction of ω_0 to $\mathbb{C}H^1 \subset \mathbb{C}$.

Summarizing we have proved a "symplectic duality" between $(\mathbb{C}H^1, \omega_{hyp})$ and $(\mathbb{C}P^1, \omega_{FS})$, namely there exists a diffeomorphism

$$\psi : \mathbb{C}H^1 \to \mathbb{R}^2 \cong \mathbb{C} \cong U_0 \subset \mathbb{C}P^1$$

satisfying:

$$\boxed{\psi^*\omega_0 = \omega_{hyp}} \qquad \boxed{\psi^*\omega_{FS} = \omega_0}$$

In the second part of this talk we show that the previous example extends to all HSSNT and we also compute teh Gromov width of HSSCT

THE CASE OF ROTATION INVARIANT KÄHLER METRICS

Lemma Let M be a complex domain in $\mathbb{C}^n$, such that $0 \in M$ and let $\omega = \frac{i}{2}\partial\bar{\partial}\Phi$ be a Kähler form on M rotation invariant, i.e. $\Phi : M \to \mathbb{R}$ only depends on $x_1 = |z_1|^2, \ldots, x_n = |z_n|^2$. Let $x = (x_1, \ldots, x_n)$ and assume that

$$\boxed{\frac{\partial\Phi}{\partial x_j} > 0 \quad \forall j = 1, \ldots, n} \qquad \boxed{\lim_{\|x\| \to \partial M} \sum_{j=1}^{n} \frac{\partial\Phi}{\partial x_j} x_j = +\infty}$$

Then the map $\Psi : M \to \mathbb{C}^n = \mathbb{R}^{2n}$ given by:

$$\boxed{\Psi(z) = \left(\sqrt{\frac{\partial\Phi}{\partial x_1}}\; z_1, \cdots, \sqrt{\frac{\partial\Phi}{\partial x_n}}\; z_n\right)}$$

is a global symplectomorphism, i.e. Ψ is a diffeomorphism satisfying $\Psi^*\omega_0 = \omega$.

Proof: simple computation and the condition of properness.

Example 1

As a simple application of the Lemma we obtain the complex hyperbolic space $(\mathbb{C}H^n, \omega_{hyp})$, namely the unit ball $B^{2n}(1) = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n | \ \sum_{j=1}^n |z_j|^2 < 1\}$ in $\mathbb{C}^n$ endowed with the hyperbolic form $\omega_{hyp} = -\frac{i}{2}\partial\bar{\partial}\log(1 - \sum_{j=1}^n |z_j|^2)$ is globally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. An explicit global symplectomorphism $\Psi : B^{2n}(1) \to \mathbb{R}^{2n}$ is given by:

$$(z_1, \dots, z_n) \mapsto \left(\frac{z_1}{\sqrt{1 - \sum_{i=1}^n |z_i|^2}}, \dots, \frac{z_n}{\sqrt{1 - \sum_{i=1}^n |z_i|^2}} \right). \tag{1}$$

Example 2: Complete Reinhardt domains

Let $x_0 \in \mathbb{R}^+ \cup \{+\infty\}$ and let $F : [0, x_0) \to (0, +\infty)$ be a non-increasing smooth function. Consider the domain

$$D_F = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 < x_0,\ |z_2|^2 < F(|z_1|^2)\}$$

endowed with the 2-form

$$\omega_F = \frac{i}{2}\partial\bar{\partial}\Phi,\ \Phi = \log \frac{1}{F(|z_1|^2) - |z_2|^2}.$$

If the function $A(x) = -\frac{xF'(x)}{F(x)}$ satisfies $A'(x) > 0$ for every $x \in [0, x_0)$, then ω_F is a Kähler form on D_F and (D_F, ω_F) is called the *complete Reinhardt domain* associated with F.

Example Let F be the real-valued, stricly decreasing smooth function on $[0,1)$ defined by:

$$F : [0,1) \to \mathbb{R} : x \mapsto (1-x)^p, \ p > 0.$$

Its associated complete Reinhardt domain is given by:

$$D_F = \{z \in \mathbb{C}^2 | \ |z_1|^2 + |z_2|^{\frac{2}{p}} < 1\}.$$

The map

$\Psi : D_F \to \mathbb{R}^4$ given by:

$$(z_1, z_2) \mapsto \left(\left(\frac{p(1-|z_1|^2)^{p-1}}{(1-|z_1|^2)^p - |z_2|^2} \right)^{\frac{1}{2}} z_1, \left(\frac{1}{(1-|z_1|^2)^p - |z_2|^2} \right)^{\frac{1}{2}} z_2 \right)$$

is an explicit global symplectomorphism.

Example Let $F(x) = e^{-x}$ in the interval $[0, +\infty)$. The map

$\Psi : D_F \to \mathbb{R}^4$ given by:

$$(z_1, z_2) \mapsto \left(\left(\frac{e^{-|z_1|^2}}{e^{-|z_1|^2} - |z_2|^2} \right)^{\frac{1}{2}} z_1, \left(\frac{1}{e^{-|z_1|^2} - |z_2|^2} \right)^{\frac{1}{2}} z_2 \right)$$

defines global symplectic coordinates on the Spring domain.

The Taub-NUT metric

C. LeBrun constructes the following family of Kähler forms on $\mathbb{C}^2$ defined by $\omega_m = \frac{i}{2}\partial\bar{\partial}\Phi_m$, where

$$\Phi_m(u,v) = u^2 + v^2 + m(u^4 + v^4),\ \ m \geq 0$$

and u and v are implicitly defined by

$$|z_1| = e^{m(u^2-v^2)}u,\ \ |z_2| = e^{m(v^2-u^2)}v.$$

For $m = 0$ one gets the flat metric, while for $m > 0$ each of the metrics of this family represents the first example of complete Ricci flat (non-flat) metric on $\mathbb{C}^2$ having the same volume form of the flat metric ω_0, namely $\omega_m \wedge \omega_m = \omega_0 \wedge \omega_0$. Moreover, for $m > 0$, these metrics are isometric (up to dilation and rescaling) to the Taub-NUT metric.

Now, with the aid of Lemma, we prove that for every m the Kähler manifold $(\mathbb{C}^2, \omega_m)$ admits global symplectic coordinates. Set $u^2 = U$, $v^2 = V$. Then

$$\frac{\partial \Phi_m}{\partial x_1} = \frac{\partial \Phi_m}{\partial U}\frac{\partial U}{\partial x_1} + \frac{\partial \Phi_m}{\partial V}\frac{\partial V}{\partial x_1},$$

$$\frac{\partial \Phi_m}{\partial x_2} = \frac{\partial \Phi_m}{\partial U}\frac{\partial U}{\partial x_2} + \frac{\partial \Phi_m}{\partial V}\frac{\partial V}{\partial x_2},$$

where $x_j = |z_j|^2, j = 1, 2$. In order to calculate $\frac{\partial U}{\partial x_j}$ and $\frac{\partial V}{\partial x_j}, j = 1, 2$, let us consider the map

$$G : \mathbb{R}^2 \to \mathbb{R}^2,\ (U, V) \mapsto (x_1 = e^{2m(U-V)}U,\ x_2 = e^{2m(V-U)}V)$$

and its Jacobian matrix

$$J_G = \begin{pmatrix} (1 + 2mU)\, e^{2m(U-V)} & -2mU\, e^{2m(U-V)} \\ -2mV\, e^{2m(V-U)} & (1 + 2mV)\, e^{2m(V-U)} \end{pmatrix}.$$

We have $det J_G = 1 + 2m(U + V) \neq 0$, so

$$J_G^{-1} = J_{G^{-1}} = \frac{1}{1 + 2m(U + V)} \begin{pmatrix} (1 + 2mV)e^{2m(V-U)} & 2mUe^{2m(U-V)} \\ 2mVe^{2m(V-U)} & (1 + 2mU)e^{2m(U-V)} \end{pmatrix}.$$

Since $J_{G^{-1}} = \begin{pmatrix} \frac{\partial U}{\partial x_1} & \frac{\partial U}{\partial x_2} \\ \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{pmatrix}$, by a straightforward calculation we get

$$\frac{\partial \Phi_m}{\partial x_1} = (1 + 2mV)e^{2m(V-U)} > 0, \ \frac{\partial \Phi_m}{\partial x_2} = (1 + 2mU)e^{2m(U-V)} > 0,$$

and

$$\lim_{\|x\|\to+\infty} \left(\frac{\partial \Phi_m}{\partial x_1} x_1 + \frac{\partial \Phi_m}{\partial x_2} x_2\right) = \lim_{\|x\|\to+\infty} (U + V + 4mUV) = +\infty, .$$

Hence, by the Lemma the map

$$\Psi : \mathbb{C}^2 \to \mathbb{C}^2, (z_1, z_2) \mapsto \left((1 + 2mV)^{\frac{1}{2}} e^{m(V-U)} z_1, (1 + 2mU)^{\frac{1}{2}} e^{m(U-V)} z_2\right)$$

is a global symplectomorphism from $(\mathbb{C}^2, \omega_m)$ into $(\mathbb{R}^4, \omega_0)$.

Kähler-Ricci solitons

The Cigar metric on $\mathbb{C}$ whose associated Kähler form reads:

$$\omega_C = \frac{i}{2}\frac{dz \wedge d\bar{z}}{1+|z|^2},$$

which was introduced by Hamilton as the first example of Kähler–Ricci soliton on non-compact manifolds. Observe that a Kähler potential for ω_C is given by

$$\Phi_C(z) = \int_0^{|z|} \frac{\log(1+s^2)}{s} ds.$$

Furthermore, in this case the Riemannian curvature reads:

$$R = \frac{1}{(1+|z|^2)^3}.$$

It is interesting to observe that the Kähler metric $\omega_{C,n} = \frac{i}{2}\partial\bar{\partial}\Phi_{C,n}$ on $\mathbb{C}^n$ defined as product of n copies of Cigar metric ω_C, satisfies $\Phi_{C,n} = \Phi_C \oplus \ldots \oplus \Phi_C$ and it is still a complete and positively curved (i.e. with non-negative sectional curvature) gradient Kähler–Ricci soliton

Theorem Let $(\mathbb{C}^n, \omega_{C,n})$ be the product of n copies of the Cigar soliton. Then there exists a simplectomorphism $\Psi_{C,n} : (\mathbb{C}^n, \omega_{C,n}) \to (\mathbb{R}^{2n}, \omega_0)$, with $\Psi_{C,n}(0) = 0$, taking complete complex totally geodesic submanifolds through the origin to complex linear subspaces of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

Proof: The existence of a global symplectic coordinates, namely of a symplectomorphism $\Psi : (\mathbb{C}^n, \omega_{C,n}) \to (\mathbb{R}^{2n}, \omega_0)$ is guaranteed by the Lemma. Indeed the map:

$$\Psi_{C,n} \colon (\mathbb{C}^n, \omega_{C,n}) \to (\mathbb{R}^{2n}, \omega_0), \quad z = (z_1, \dots, z_n) \mapsto (\psi_1(z_1)z_1, \dots, \psi_n(z_n)z_n), \quad (2)$$

with

$$\psi_j = \sqrt{\frac{\log(1 + |z_j|^2)}{|z_j|^2}},$$

is a global symplectomorphism. Moreover, the complex totally geodesic submanifolds T of complex dimension k of $(\mathbb{C}^n, \omega_{C,n})$ are given by $(\mathbb{C}^k, \omega_{C.k})$ and so $\psi(T) = \mathbb{C}^k \subset \mathbb{C}^n$.

Calabi's inhomogeneous Kähler–Einstein metric on tubular domains

The complex tubular domain $M = D_a \oplus i\mathbb{R}^n \subset \mathbb{C}^n$, $n \geq 2$, where $D_a \subset \mathbb{R}^n$ is the open ball of $\mathbb{R}^n$ centered at the origin and of radius a. Let g be the metric on $M \subset \mathbb{C}^n$ whose associated Kähler form is given by:

$$\omega = \frac{i}{2}\partial\bar{\partial} f(z_1 + \bar{z}_1, \ldots, z_n + \bar{z}_n)$$

where $f : D_a \to \mathbb{R}$ is a radial function $f(x_1, \ldots, x_n) = Y(r)$, being $r = (\sum_{j=1}^n x_j^2)^{1/2}$ and $x_j = (z_j + \bar{z}_j)/2$, $y_j = (z_j - \bar{z}_j)/2i$, that satisfies the differential equation:

$$(Y'/r)^{n-1} Y'' = e^Y,$$

with initial conditions:

$$Y'(0) = 0, \; Y''(0) = e^{Y(0)/n}.$$

Moreover, it satisfies:

$$\lim_{r \to a} Y'(r) \to \infty$$

Theorem: For all $n \geq 2$, the Kähler manifold (M, ω) is globally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$ via the map:

$$\Psi\colon M \to \mathbb{R}^n \oplus i\mathbb{R}^n \simeq \mathbb{R}^{2n},\ (x, y) \mapsto (gradf, y)\,,$$

where $f : D_a \to \mathbb{R}, x = (x_1, \dots x_n) \mapsto f(x)$ is a Kähler potential for ω, i.e. $\omega = \frac{i}{2}\partial\bar{\partial}f$, and $gradf = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$.

Proof: The map Ψ satisfies $\Psi^*\omega_0 = \omega$. In order to simplify the notation we write $\partial f/\partial x_j = f_j$ and $\partial^2 f/\partial x_j \partial x_k = f_{jk}$. The pull-back of ω_0 through Ψ reads:

$$\Psi^*\omega_0 = \sum_{j=1}^{n} df_j \wedge dy_j = \sum_{j,k=1}^{n} f_{jk}\, dx_k \wedge dy_j = \frac{i}{2} \sum_{j,k=1}^{n} f_{jk}\, dz_j \wedge d\bar{z}_k,$$

Ψ is a proper map, i.e.

$$\lim_{(x,y)\to\partial M} \Psi(x, y) = \infty$$

or equivalently:

$$\lim_{x\to\partial D_a} ||gradf(x)|| = \infty.$$

This readily follows by $f_j(x) = \frac{x_j}{r} Y'(r)$ and the fact that $Y'(r)$ tends to infinity as $r \to a$.