

Canonical metrics on Hartogs domains ¹

Andrea Loi and Fabio Zuddas

Abstract

An n -dimensional Hartogs domain D_F can be equipped with a natural Kähler metric g_F . This paper contains two results. In the first one we prove that if g_F is an extremal Kähler metric then (D_F, g_F) is holomorphically isometric to an open subset of the n -dimensional complex hyperbolic space. In the second one we prove the same assertion under the assumption that there exists a real holomorphic vector field X on D_F such that (g_F, X) is a Kähler–Ricci soliton.

Keywords: Kähler metrics; Hartogs domain; extremal metrics; Kähler–Ricci solitons; generalized scalar curvatures; canonical metrics.

Subj. Class: 53C55, 32Q15, 32T15.

1 Introduction and statements of the main results

The study of the existence and uniqueness of a preferred Kähler metric on a given complex manifold M is a very interesting and important area of research, both from the mathematical and from the physical point of view. Many definitions of canonical metrics (Einstein, constant scalar curvature, extremal, Kähler–Ricci solitons and so on) have been given both in the compact and in the noncompact case (see e.g. [2], [15] and [24]). In the noncompact case many important questions are still open. For example Yau raised the question on the classification of Bergman Einstein metrics on strongly pseudoconvex domains and S.–Y. Cheng conjectured that if the Bergman metric on a strongly pseudoconvex domain is Einstein, then the domain is biholomorphic to the ball (see [13]).

In this paper we are interested in extremal Kähler metrics and Kähler–Ricci solitons on a particular class of complex domains, the so called *Hartogs domains* (see the next section for their definition and main properties).

Our main results are the following theorems.

Theorem 1.1 *Let (D_F, g_F) be an n -dimensional Hartogs domain. Assume that g_F is an extremal Kähler metric. Then (D_F, g_F) is holomorphically isometric to an open subset of the n -dimensional complex hyperbolic space.*

¹The authors were supported by the M.I.U.R. Project “Geometric Properties of Real and Complex Manifolds”.

Theorem 1.2 *Let (D_F, g_F) be an n -dimensional Hartogs domain and let X be a real holomorphic vector field on D_F such that (g_F, X) is a Kähler–Ricci soliton. Then g_F is Kähler–Einstein. Consequently, (D_F, g_F) is holomorphically isometric to an open subset of the n -dimensional complex hyperbolic space.*

Notice that (compare with Cheng’s conjecture above) the assumptions on the metric g_F in Theorem 1.1 and Theorem 1.2 are weaker than Einstein’s condition. To this regard it is worth pointing out that when g_F equals the Bergman metric on D_F , then (D_F, g_F) is holomorphically isometric to an open subset of the complex hyperbolic space (see Theorem 1.3 in [10] for a proof).

The paper is organized as follows. In the next section, after recalling the definition of Hartogs domains, we analyze their pseudoconvexity, and we prove a lemma regarding their generalized scalar curvatures. Sections 3 and 4 are dedicated to the proofs of Theorem 1.1 and Theorem 1.2 respectively.

2 Hartogs domains

Let $x_0 \in \mathbb{R}^+ \cup \{+\infty\}$ and let $F : [0, x_0) \rightarrow (0, +\infty)$ be a decreasing continuous function, smooth on $(0, x_0)$. The *Hartogs domain* $D_F \subset \mathbb{C}^n$ associated to the function F is defined by

$$D_F = \{(z_0, z_1, \dots, z_{n-1}) \in \mathbb{C}^n \mid |z_0|^2 < x_0, |z_1|^2 + \dots + |z_{n-1}|^2 < F(|z_0|^2)\}.$$

We shall assume that the natural $(1, 1)$ -form on D_F given by

$$\omega_F = \frac{i}{2} \partial \bar{\partial} \log \frac{1}{F(|z_0|^2) - |z_1|^2 - \dots - |z_{n-1}|^2} \quad (1)$$

is a Kähler form on D_F (see Proposition 2.1 below for some conditions on D_F equivalent to this assumption). The Kähler metric g_F associated to the Kähler form ω_F is the metric we will be dealing with in the present paper. Observe that for $F(x) = 1 - x$, $0 \leq x < 1$, D_F equals the n -dimensional complex hyperbolic space $\mathbb{C}H^n$ and g_F is the hyperbolic metric, i.e. $g_F = g_{hyp}$. In the 2-dimensional case this metric has been considered in [11] and [21] in the framework of quantization of Kähler manifolds. In [20], the first author studied the Kähler immersions of (D_F, g_F) into finite or infinite dimensional complex space forms, [9] is concerned with the existence of global symplectic

coordinates on (D_F, ω_F) and [10] deals with the Riemannian geometry of (D_F, g_F) (in particular in this paper one can find necessary and sufficient conditions in terms of F for the completeness of the metric g_F).

Proposition 2.1 *Let D_F be a Hartogs domain in \mathbb{C}^n . Then the following conditions are equivalent:*

- (i) *the $(1, 1)$ -form ω_F given by (1) is a Kähler form;*
- (ii) *the function $-\frac{x F'(x)}{F(x)}$ is strictly increasing, namely $-(\frac{x F'(x)}{F(x)})' > 0$ for every $x \in [0, x_0]$;*
- (iii) *the boundary of D_F is strongly pseudoconvex at all $z = (z_0, z_1, \dots, z_{n-1})$ with $|z_0|^2 < x_0$.*

Proof: (i) \Leftrightarrow (ii) Set

$$A = F(|z_0|^2) - |z_1|^2 - \dots - |z_{n-1}|^2. \quad (2)$$

Then ω_F is a Kähler form if and only if the real-valued function $\Phi = -\log A$ is strictly plurisubharmonic, i.e. the matrix $g_{\alpha\bar{\beta}} = (\frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta})$, $\alpha, \beta = 0, \dots, n-1$ is positive definite, where

$$\omega_F = \frac{i}{2} \sum_{\alpha, \beta=0}^{n-1} g_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta. \quad (3)$$

A straightforward computation gives

$$\frac{\partial^2 \Phi}{\partial z_0 \partial \bar{z}_0} = \frac{F'^2(|z_0|^2)|z_0|^2 - (F''(|z_0|^2)|z_0|^2 + F'(|z_0|^2))A}{A^2},$$

$$\frac{\partial^2 \Phi}{\partial z_0 \partial \bar{z}_\beta} = -\frac{F'(|z_0|^2)\bar{z}_0 z_\beta}{A^2}, \quad \beta = 1, \dots, n-1$$

and

$$\frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta} = \frac{\delta_{\alpha\beta} A + \bar{z}_\alpha z_\beta}{A^2}, \quad \alpha, \beta = 1, \dots, n-1.$$

Then, by setting

$$C = F'^2(|z_0|^2)|z_0|^2 - (F''(|z_0|^2)|z_0|^2 + F'(|z_0|^2))A, \quad (4)$$

one sees that the matrix $h = (g_{\alpha\bar{\beta}}) = (\frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta})_{\alpha, \beta=0, \dots, n-1}$ is given by:

$$h = \frac{1}{A^2} \begin{pmatrix} C & -F' \bar{z}_0 z_1 & \dots & -F' \bar{z}_0 z_\alpha & \dots & -F' \bar{z}_0 z_{n-1} \\ -F' z_0 \bar{z}_1 & A + |z_1|^2 & \dots & \bar{z}_1 z_\alpha & \dots & \bar{z}_1 z_{n-1} \\ \vdots & \vdots & & \vdots & & \vdots \\ -F' z_0 \bar{z}_\alpha & z_1 \bar{z}_\alpha & \dots & A + |z_\alpha|^2 & \dots & \bar{z}_\alpha z_{n-1} \\ \vdots & \vdots & & \vdots & & \vdots \\ -F' z_0 \bar{z}_{n-1} & z_1 \bar{z}_{n-1} & \dots & z_\alpha \bar{z}_{n-1} & \dots & A + |z_{n-1}|^2 \end{pmatrix}. \quad (5)$$

First notice that the $(n-1) \times (n-1)$ matrix obtained by deleting the first row and the first column of h is positive definite. Indeed it is not hard to see that, for all $1 \leq \alpha \leq n-1$,

$$\begin{aligned} \det \begin{pmatrix} A + |z_\alpha|^2 & \bar{z}_\alpha z_{\alpha+1} & \dots & \bar{z}_\alpha z_{n-1} \\ \vdots & \vdots & & \vdots \\ \bar{z}_{n-1} z_\alpha & \bar{z}_{n-1} z_{\alpha+1} & \dots & A + |z_{n-1}|^2 \end{pmatrix} = \\ = A^{n-\alpha} + A^{n-\alpha-1}(|z_\alpha|^2 + \dots + |z_{n-1}|^2) > 0. \end{aligned} \quad (6)$$

On the other hand, by the Laplace expansion along the first row, we get

$$\begin{aligned} \det(h) &= \frac{C}{A^{2n}} [A^{n-1} + A^{n-2}(|z_1|^2 + \dots + |z_{n-1}|^2)] + \\ &+ \frac{F' \bar{z}_0 z_1}{A^{2n}} \det \begin{pmatrix} -F' z_0 \bar{z}_1 & z_2 \bar{z}_1 & \dots & z_{n-1} \bar{z}_1 \\ -F' z_0 \bar{z}_2 & A + |z_2|^2 & \dots & z_{n-1} \bar{z}_2 \\ \vdots & \vdots & & \vdots \\ -F' z_0 \bar{z}_{n-1} & z_2 \bar{z}_{n-1} & \dots & A + |z_{n-1}|^2 \end{pmatrix} + \dots + \\ &+ (-1)^n \frac{F' \bar{z}_0 z_{n-1}}{A^{2n}} \det \begin{pmatrix} -F' z_0 \bar{z}_1 & A + |z_1|^2 & \dots & z_{n-2} \bar{z}_1 \\ -F' z_0 \bar{z}_2 & z_1 \bar{z}_2 & \dots & z_{n-2} \bar{z}_2 \\ \vdots & \vdots & & \vdots \\ -F' z_0 \bar{z}_{n-1} & z_1 \bar{z}_{n-1} & \dots & z_{n-2} \bar{z}_{n-1} \end{pmatrix} = \\ &= \frac{C}{A^{2n}} [A^{n-1} + A^{n-2}(|z_1|^2 + \dots + |z_{n-1}|^2)] + \end{aligned}$$

$$\begin{aligned}
& + \frac{F'^2 |z_0|^2 |z_1|^2}{A^{2n}} \det \begin{pmatrix} -1 & z_2 & \dots & z_{n-1} \\ -\bar{z}_2 & A + |z_2|^2 & \dots & z_{n-1} \bar{z}_2 \\ \vdots & \vdots & & \vdots \\ -\bar{z}_{n-1} & z_2 \bar{z}_{n-1} & \dots & A + |z_{n-1}|^2 \end{pmatrix} + \dots + \\
& + (-1)^n \frac{F'^2 |z_0|^2 |z_{n-1}|^2}{A^{2n}} \det \begin{pmatrix} -\bar{z}_1 & A + |z_1|^2 & \dots & z_{n-2} \bar{z}_1 \\ -\bar{z}_2 & z_1 \bar{z}_2 & \dots & z_{n-2} \bar{z}_2 \\ \vdots & \vdots & & \vdots \\ -1 & z_1 & \dots & z_{n-2} \end{pmatrix} = \\
& \frac{1}{A^{n+2}} [CA + (C - F'^2 |z_0|^2) (|z_1|^2 + \dots + |z_{n-1}|^2)].
\end{aligned}$$

By substituting (2) and (4) into this last equality one gets

$$\det(h) = -\frac{F^2}{A^{n+1}} \left(\frac{x F'}{F} \right)' \Big|_{x=|z_0|^2}. \quad (7)$$

Hence, by (6) and (7), the matrix $(\frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta})$ is positive definite if and only if $\left(\frac{x F'}{F} \right)' < 0$.

Before proving equivalence (ii) \Leftrightarrow (iii) we briefly recall some facts on complex domains (see e.g. [16]). Let $\Omega \subseteq \mathbb{C}^n$ be any complex domain of \mathbb{C}^n with smooth boundary $\partial\Omega$, and let $z \in \partial\Omega$. Assume that, for some neighbourhood U of z in \mathbb{C}^n , there exists a smooth function $\rho : U \rightarrow \mathbb{R}$ (called *defining function for Ω at z*) satisfying the following: $\rho < 0$ on $U \cap \Omega$, $\rho > 0$ on $U \setminus \bar{\Omega}$ and $\rho = 0$ on $U \cap \partial\Omega$; $\text{grad } \rho \neq 0$ on $U \cap \partial\Omega$. In this case $\partial\Omega$ is said to be *strongly pseudoconvex at z* if the *Levi form*

$$L(\rho, z)(X) = \sum_{\alpha, \beta=0}^{n-1} \frac{\partial^2 \rho}{\partial z_\alpha \partial \bar{z}_\beta}(z) X_\alpha \bar{X}_\beta$$

is positive definite on

$$S_\rho = \{(X_0, \dots, X_{n-1}) \in \mathbb{C}^n \mid \sum_{\alpha=0}^{n-1} \frac{\partial \rho}{\partial z_\alpha}(z) X_\alpha = 0\}$$

(it is easily seen that this definition does not depend on the particular defining function ρ).

(ii) \Leftrightarrow (iii) Let now $\Omega = D_F$ and let us fix $z = (z_0, z_1, \dots, z_{n-1}) \in \partial D_F$ with $|z_0|^2 < x_0$. Then, $|z_1|^2 + \dots + |z_{n-1}|^2 = F(|z_0|^2)$. In this case

$$\rho(z_0, z_1, \dots, z_{n-1}) = |z_1|^2 + \dots + |z_{n-1}|^2 - F(|z_0|^2)$$

is a defining function for D_F at z , the Levi form for D_F reads as

$$L(\rho, z)(X) = |X_1|^2 + \dots + |X_{n-1}|^2 - (F' + F''|z_0|^2)|X_0|^2 \quad (8)$$

and

$$S_\rho = \{(X_0, X_1, \dots, X_{n-1}) \in \mathbb{C}^n \mid -F'\bar{z}_0X_0 + \bar{z}_1X_1 + \dots + \bar{z}_{n-1}X_{n-1} = 0\}. \quad (9)$$

We distinguish two cases: $z_0 = 0$ and $z_0 \neq 0$. At $z_0 = 0$ the Levi form reads as

$$L(\rho, z)(X) = |X_1|^2 + \dots + |X_{n-1}|^2 - F'(0)|X_0|^2$$

which is strictly positive for *any* non-zero vector $(X_0, X_1, \dots, X_{n-1})$ (not necessarily in S_ρ) because F is assumed to be decreasing.

If $z_0 \neq 0$ by (9) we obtain $X_0 = \frac{\bar{z}_1X_1 + \dots + \bar{z}_{n-1}X_{n-1}}{F'\bar{z}_0}$ which, substituted in (8), gives:

$$L(X, z) = |X_1|^2 + \dots + |X_{n-1}|^2 - \frac{F' + F''|z_0|^2}{F'^2|z_0|^2} |\bar{z}_1X_1 + \dots + \bar{z}_{n-1}X_{n-1}|^2. \quad (10)$$

We can assume that $\bar{z}_1X_1 + \dots + \bar{z}_{n-1}X_{n-1} \neq 0$ (which by (9) is equivalent to $X_0 \neq 0$) for otherwise $L(X, z)$ is clearly strictly positive for any non-zero vector $X \in S_\rho$. Therefore we are reduced to show that:

$(xF'/F)' < 0$ for $x \in (0, x_0)$ if and only if $L(X, z)$ is strictly positive for every $(X_1, \dots, X_{n-1}) \neq (0, \dots, 0)$ and every $(z_0, z_1, \dots, z_{n-1}) \in \partial D_F$, $0 < |z_0|^2 < x_0$.

If $(xF'/F)' < 0$ then $(F' + xF'')F < xF'^2$ and, since $F(|z_0|^2) = |z_1|^2 + \dots + |z_{n-1}|^2$, we get:

$$\begin{aligned} L(X, z) &> |X_1|^2 + \dots + |X_{n-1}|^2 - \frac{1}{F(|z_0|^2)} |\bar{z}_1X_1 + \dots + \bar{z}_{n-1}X_{n-1}|^2 = \\ &= \frac{(|X_1|^2 + \dots + |X_{n-1}|^2)(|z_1|^2 + \dots + |z_{n-1}|^2) - |\bar{z}_1X_1 + \dots + \bar{z}_{n-1}X_{n-1}|^2}{|z_1|^2 + \dots + |z_{n-1}|^2} \end{aligned}$$

and the conclusion follows by the Cauchy-Schwarz inequality.

Conversely, assume that $L(X, z)$ is strictly positive for every $(X_1, \dots, X_{n-1}) \neq (0, \dots, 0)$ and each $z = (z_0, z_1, \dots, z_{n-1})$ such that $F(|z_0|^2) = |z_1|^2 + \dots + |z_{n-1}|^2$. By inserting $(X_1, \dots, X_{n-1}) = (z_1, \dots, z_{n-1})$ in (10) we get

$$L(z, z) = F(|z_0|^2) \left(1 - \frac{F' + F''|z_0|^2}{F'^2|z_0|^2} F(|z_0|^2) \right) > 0$$

which implies $(xF'/F)' < 0$. \square

Remark 2.2 Notice that the previous proposition is a generalization of Proposition 3.6 in [11] proved there for the 2-dimensional case.

Recall (see e.g. [18]) that the Ricci curvature and the scalar curvature of a Kähler metric g on an n -dimensional complex manifold (M, g) are given respectively by

$$\text{Ric}_{\alpha\bar{\beta}} = -\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} (\log \det(h)), \quad \alpha, \beta = 0, \dots, n-1 \quad (11)$$

and

$$\text{scal}_g = \sum_{\alpha, \beta=0}^{n-1} g^{\bar{\beta}\alpha} \text{Ric}_{\alpha\bar{\beta}}, \quad (12)$$

where $g^{\bar{\beta}\alpha}$ are the entries of the inverse of $(g_{\alpha\bar{\beta}})$, namely $\sum_{\alpha=0}^{n-1} g^{\bar{\beta}\alpha} g_{\alpha\bar{\gamma}} = \delta_{\beta\gamma}$.

When $(M, g) = (D_F, g_F)$, using (5) it is not hard to check the validity of the following equalities.

$$g^{\bar{0}0} = \frac{A}{B} F, \quad (13)$$

$$g^{\bar{\beta}0} = \frac{A}{B} F' z_0 \bar{z}_\beta, \quad \beta = 1, \dots, n-1, \quad (14)$$

$$g^{\bar{\beta}\alpha} = \frac{A}{B} (F' + F''|z_0|^2) z_\alpha \bar{z}_\beta, \quad \alpha \neq \beta, \quad \alpha, \beta = 1, \dots, n-1, \quad (15)$$

$$g^{\bar{\beta}\beta} = \frac{A}{B} [B + (F' + F''|z_0|^2)|z_\beta|^2], \quad \beta = 1, \dots, n-1, \quad (16)$$

where

$$B = B(|z_0|^2) = F'^2|z_0|^2 - F(F' + F''|z_0|^2).$$

Now, set

$$L(x) = \frac{d}{dx} \left[x \frac{d}{dx} \log(xF'^2 - F(F' + F''x)) \right].$$

A straightforward computation using (7) and (11) gives:

$$\text{Ric}_{0\bar{0}} = -L(|z_0|^2) - (n+1)g_{0\bar{0}}, \quad (17)$$

$$\text{Ric}_{\alpha\bar{\beta}} = -(n+1)g_{\alpha\bar{\beta}}, \quad \alpha > 0. \quad (18)$$

Then, by (12), the scalar curvature of the metric g_F equals

$$\text{scal}_{g_F} = -L(|z_0|^2)g^{\bar{0}0} - (n+1) \sum_{\alpha, \beta=0}^{n-1} g^{\bar{\beta}\alpha} g_{\alpha\bar{\beta}} = -L(|z_0|^2)g^{\bar{0}0} - n(n+1),$$

which by (13) reads as

$$\text{scal}_{g_F} = -\frac{A}{B}FL - n(n+1). \quad (19)$$

We conclude this section with Lemma 2.3 below which will be used in the proof of our results. This lemma is a generalization of a result proved by the first author for 2-dimensional Hartogs domains (see Theorem 4.8 in [21]). We first recall the definition of generalized scalar curvatures. Given a Kähler metric g on an n -dimensional complex manifold M , its *generalized scalar curvatures* are the n smooth functions $\rho_0, \dots, \rho_{n-1}$ on M satisfying the following equation:

$$\frac{\det(g_{\alpha\bar{\beta}} + t\text{Ric}_{\alpha\bar{\beta}})}{\det(g_{\alpha\bar{\beta}})} = 1 + \sum_{k=0}^{n-1} \rho_k t^{k+1}, \quad (20)$$

where $g_{\alpha\bar{\beta}}$ are the entries of the metric in local coordinates. Observe that for $k=0$ we recover the value of the scalar curvature, namely

$$\rho_0 = \text{scal}_g. \quad (21)$$

The introduction and the study of these curvatures (in the compact case) are due to K. Ogiue [23] to whom the reader is referred for further results. In particular, in a joint paper with B.Y. Chen [1], he studies the constancy of one of the generalized scalar curvatures. Their main result is that, under suitable cohomological conditions, the constancy of one of the ρ_k 's, $k=0, \dots, n-1$, implies that the metric g is Einstein.

Lemma 2.3 *Let (D_F, g_F) be an n -dimensional Hartogs domain. Assume that one of its generalized scalar curvatures is constant. Then (D_F, g_F) is holomorphically isometric to an open subset of the n -dimensional hyperbolic space.*

Proof: By (17), (18) we get

$$\frac{\det(g_{\alpha\bar{\beta}} + t\text{Ric}_{\alpha\bar{\beta}})}{\det(g_{\alpha\bar{\beta}})} = (1 - (n+1)t)^n - tL(1 - (n+1)t)^{n-1} \frac{AF}{B}.$$

So the generalized curvatures of (D_F, g_F) are given by

$$\rho_k = (n+1)^k (-1)^{k+1} \binom{n-1}{k} \left[\frac{n(n+1)}{k+1} + \frac{AFL}{B} \right], \quad k = 0, \dots, n-1. \quad (22)$$

Notice that, for $k = 0$, we get $\rho_0 = -\frac{AFL}{B} - n(n+1) = \text{scal}_{g_F}$, (compare with (19)) in accordance with (21).

Thus, ρ_k is constant for some (equivalently, for any) $k = 0, \dots, n-1$ if and only if $\frac{AFL}{B}$ is constant. Since $A = F(|z_0|^2) - |z_1|^2 - \dots - |z_{n-1}|^2$ depends on z_1, \dots, z_{n-1} while $\frac{LF}{B}$ depends only on z_0 , this implies that $L = 0$, i.e.

$$\frac{d}{dx} \left[x \frac{d}{dx} \log(xF'^2 - F(F' + F''x)) \right]_{x=|z_0|^2} \equiv 0.$$

Now, we continue as in the proof of Theorem 4.8 in [21] and conclude that $F(x) = c_1 - c_2x$, $x = |z_0|^2$, with $c_1, c_2 > 0$, which implies that D_F is holomorphically isometric to an open subset of the complex hyperbolic space $\mathbb{C}H^n$ via the map

$$\phi : D_F \rightarrow \mathbb{C}H^n, \quad (z_0, z_1, \dots, z_{n-1}) \mapsto \left(\frac{z_0}{\sqrt{c_1/c_2}}, \frac{z_1}{\sqrt{c_1}}, \dots, \frac{z_{n-1}}{\sqrt{c_1}} \right).$$

□

3 Proof of Theorem 1.1

Extremal metrics were introduced and christened by Calabi [4] in the compact case as the solution for the variational problem in a Kähler class defined

by the square integral of the scalar curvature. Therefore they are a generalization of constant scalar curvature metrics. Calabi himself constructs nontrivial (namely with nonconstant scalar curvature) metrics on some compact manifolds. Only recently extremal Kähler metrics were rediscovered by several mathematicians due to their link with the stability of complex vector bundles (see e.g. [3], [8], [14], [19] and [22]). Obviously extremal metrics cannot be defined in the noncompact case as the solutions of a variational problem involving some integral on the manifold. Nevertheless they can be alternatively defined (also in the noncompact case) as those metrics such that the (1,0)-part of the Hamiltonian vector field associated to the scalar curvature is holomorphic. Therefore, in local coordinates an extremal metric must satisfy the following system of PDE's (see [4]):

$$\frac{\partial}{\partial \bar{z}_\gamma} \left(\sum_{\beta=0}^{n-1} g^{\bar{\beta}\alpha} \frac{\partial \text{scal}_g}{\partial \bar{z}_\beta} \right) = 0, \quad (23)$$

for every $\alpha, \gamma = 0, \dots, n-1$. Notice that in the noncompact case, the existence and uniqueness of such metrics are far from being understood. For example, only recently in [6] (see also [7]), there has been shown the existence of a nontrivial extremal and complete Kähler metric in a complex one-dimensional manifold.

Proof of Theorem 1.1 In order to use equations (23) for (D_F, g_F) we write the entries $g^{\bar{\beta}\alpha}$ by separating the terms depending only on z_0 from the other terms. More precisely, (13), (14), (15) and (16) can be written as follows.

$$g^{\bar{0}0} = P_{00} + Q_{00}(|z_1|^2 + \dots + |z_{n-1}|^2),$$

$$g^{\bar{0}\alpha} = \bar{z}_0 z_\alpha [P_{0\alpha} + Q_{0\alpha}(|z_1|^2 + \dots + |z_{n-1}|^2)], \quad \alpha = 1, \dots, n-1,$$

$$g^{\bar{\alpha}\alpha} = F + P_{aa}|z_\alpha|^2 - (1 + Q_{aa}|z_\alpha|^2) \sum_{k \neq \alpha} |z_k|^2 - R_{aa}|z_\alpha|^4, \quad \alpha = 1, \dots, n-1,$$

$$g^{\bar{\beta}\alpha} = \bar{z}_\beta z_\alpha [P_{ab} + Q_{ab}(|z_1|^2 + \dots + |z_{n-1}|^2)], \quad \alpha \neq \beta, \quad \alpha, \beta = 1, \dots, n-1,$$

where

$$\begin{aligned}
P_{00} &= \frac{F^2}{B}, & Q_{00} &= -\frac{F}{B}, \\
P_{0a} &= \frac{F'F}{B}, & Q_{0a} &= -\frac{F'}{B}, \\
P_{aa} &= \frac{F(F' + F''|z_0|^2)}{B} - 1, & Q_{aa} = R_{aa} &= \frac{F' + F''|z_0|^2}{B}, \\
P_{ab} &= \frac{F(F' + F''|z_0|^2)}{B}, & Q_{ab} &= -\frac{F' + F''|z_0|^2}{B}
\end{aligned}$$

are all functions depending only on $|z_0|^2$.

We also have (cfr. (19))

$$\text{scal}_{g_F} = -n(n+1) + G(F - |z_1|^2 - \cdots - |z_{n-1}|^2) \quad (24)$$

where

$$G = G(|z_0|^2) = -\frac{L(|z_0|^2)F(|z_0|^2)}{B(|z_0|^2)}.$$

Assume that g_F is an extremal metric, namely equation (23) is satisfied. We are going to show that scal_{g_F} is constant and hence by Lemma 2.3 (D_F, g_F) is holomorphically isometric to an open subset of $(\mathbb{C}H^n, g_{hyp})$. In order to do that, fix $i \geq 1$ and let us consider equation (23) when $g = g_F$ for $\alpha = 0$, $\gamma = i$.

We have

$$\begin{aligned}
\frac{\partial \text{scal}_{g_F}}{\partial \bar{z}_0} &= G' z_0 (F - |z_1|^2 - \cdots - |z_{n-1}|^2) + z_0 G F' \\
\frac{\partial \text{scal}_{g_F}}{\partial \bar{z}_i} &= -G z_i.
\end{aligned}$$

So, equation (23) gives

$$\begin{aligned}
\frac{\partial}{\partial \bar{z}_i} \left\{ \left[P_{00} + Q_{00} \sum_{k=1}^{n-1} |z_k|^2 \right] \left[G' z_0 (F - \sum_{k=1}^{n-1} |z_k|^2) + z_0 G F' \right] - \right. \\
\left. - z_0 G \left[P_{0a} + Q_{0a} \sum_{k=1}^{n-1} |z_k|^2 \right] \sum_{k=1}^{n-1} |z_k|^2 \right\} = 0,
\end{aligned}$$

namely

$$Q_{00}z_i \left[G'z_0(F - \sum_{k=1}^{n-1} |z_k|^2) + z_0GF' \right] - G'z_0z_i \left[P_{00} + Q_{00} \sum_{k=1}^{n-1} |z_k|^2 \right] - \\ - z_0GQ_{0a}z_i \sum_{k=1}^{n-1} |z_k|^2 - z_0z_iG \left[P_{0a} + Q_{0a} \sum_{k=1}^{n-1} |z_k|^2 \right] = 0.$$

Deriving again with respect to \bar{z}_i , we get

$$-2Q_{00}G'z_0z_i^2 - 2GQ_{0a}z_0z_i^2 = 0.$$

Let us assume $z_0z_i \neq 0$. This implies $Q_{00}G' + GQ_{0a} = 0$, i.e. $GF' + FG' = 0$ or, equivalently, $G = \frac{c}{F}$ for some constant $c \in \mathbb{R}$. The proof of Theorem 1.1 will be completed by showing that $c = 0$. In fact, in this case $G = 0$ on the open and dense subset of D_F consisting of those points such that $z_0z_i \neq 0$ and therefore, by (24), scal_{g_F} is constant on D_F . In order to prove that $c = 0$, let us now consider equation (23) for $\alpha = i$, $\gamma = i$.

$$\frac{\partial}{\partial \bar{z}_i} \left\{ \bar{z}_0z_i \left[G'z_0(F - \sum_{k=1}^{n-1} |z_k|^2) + GF'z_0 \right] \left[P_{0a} + Q_{0a} \sum_{k=1}^{n-1} |z_k|^2 \right] - \right. \\ \left. - Gz_i \left[F + P_{aa}|z_i|^2 - (1 + Q_{aa}|z_i|^2) \sum_{k \neq 0,i} |z_k|^2 - R_{aa}|z_i|^4 \right] - \right. \\ \left. - Gz_i \sum_{k \neq 0,i} |z_k|^2 \left[P_{ab} + Q_{ab} \sum_{k=1}^{n-1} |z_k|^2 \right] \right\} = 0.$$

This implies

$$-G'|z_0|^2z_i^2 \left[P_{0a} + Q_{0a} \sum_{k=1}^{n-1} |z_k|^2 \right] + \bar{z}_0z_i^2Q_{0a} \left[G'z_0(F - \sum_{k=1}^{n-1} |z_k|^2) + GF'z_0 \right] - \\ - P_{aa}Gz_i^2 + Gz_i^2Q_{aa} \sum_{k \neq 0,i} |z_k|^2 + 2Gz_i^3\bar{z}_iR_{aa} - Gz_i^2Q_{ab} \sum_{k \neq 0,i} |z_k|^2.$$

If we divide by z_i^2 (we are assuming $z_i \neq 0$) and derive again the above expression with respect to \bar{z}_i we get

$$-G'|z_0|^2 Q_{0a} + GR_{aa} = 0.$$

By the definitions made at page 10 this is equivalent to

$$\frac{G'F'|z_0|^2 + G(F' + F''|z_0|^2)}{B} = 0,$$

i.e. $(GF'x)' = 0, x = |z_0|^2$. Substituting $G = \frac{c}{F}$ in this equality we get $c(\frac{F'x}{F})' = 0$. Since $(\frac{F'x}{F})' < 0$ (by (ii) in Proposition 2.1) c is forced to be zero, and this concludes the proof. \square

4 Proof of Theorem 1.2

A *Kähler–Ricci soliton* on a complex manifold M is a pair (g, X) consisting of a Kähler metric g and a real holomorphic vector field X on M such that

$$\text{Ric}_g = \lambda g + L_X g, \quad (25)$$

for some $\lambda \in \mathbb{R}$, where $L_X g$ is the Lie derivative of g along X , i.e.

$$(L_X g)(Y, Z) = X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]), \quad (26)$$

for Y, Z vector fields on M . A *real holomorphic vector field* X is the real part of a holomorphic vector field, namely, in local complex coordinates (z_0, \dots, z_{n-1}) on an open subset $U \subset M$,

$$X = \sum_{k=0}^{n-1} \left(f_k \frac{\partial}{\partial z_k} + \bar{f}_k \frac{\partial}{\partial \bar{z}_k} \right), \quad (27)$$

for some holomorphic functions $f_k, k = 0, \dots, n-1$ on U .

We refer the reader to [5], [25], [26] for the existence and uniqueness of Kähler–Ricci solitons on compact manifolds and to [12] for the noncompact case. Kähler–Ricci solitons generalize Kähler–Einstein metrics. Indeed any Kähler–Einstein metric g on a complex manifold M gives rise to a trivial Kähler–Ricci soliton by choosing $X = 0$ or X Killing with respect to g . Obviously if the automorphism group of M is discrete then a Kähler–Ricci soliton (g, X) is nothing but a Kähler–Einstein metric g .

Our Theorem 1.2 asserts that a Kähler–Ricci soliton (g_F, X) on a Hartogs domain D_F is necessarily trivial. Notice that the automorphism group of D_F is not discrete (see also [17]).

Proof of Theorem 1.2 Let (g_F, X) be a Kähler–Ricci soliton. By applying both sides of (25) to the pair $(\frac{\partial}{\partial z_0}, \frac{\partial}{\partial \bar{z}_0})$ and taking into account (17) one gets:

$$-L(|z_0|^2) = \gamma g_{0\bar{0}} + \sum_{k=0}^{n-1} \left(f_k \frac{\partial g_{0\bar{0}}}{\partial z_k} + \bar{f}_k \frac{\partial g_{0\bar{0}}}{\partial \bar{z}_k} \right) + \sum_{k=0}^{n-1} \left(\frac{\partial f_k}{\partial z_0} g_{k\bar{0}} + \frac{\partial \bar{f}_k}{\partial \bar{z}_0} g_{0\bar{k}} \right) \quad (28)$$

where

$$\gamma = \lambda + (n+1). \quad (29)$$

By (5), we have

$$\tilde{C} = \sum_{k=0}^{n-1} C_k (f_k \bar{z}_k + \bar{f}_k z_k) + C(\phi_0 + \bar{\phi}_0) - F' \sum_{k=1}^{n-1} \left(z_0 \bar{z}_k \frac{\partial f_k}{\partial z_0} + \bar{z}_0 z_k \frac{\partial \bar{f}_k}{\partial \bar{z}_0} \right) \quad (30)$$

where we have set $\tilde{C} = -A^2 L - \gamma C$, $C_k = A^2 \frac{\partial g_{0\bar{0}}}{\partial x_k}$ ($x_k = |z_k|^2$) and $\phi_0 = \frac{\partial f_0}{\partial z_0}$ (A and C are given by (2) and (4) respectively).

Now, by applying the operator $\frac{\partial^4}{\partial z_i^2 \partial \bar{z}_i^2}$ ($i = 1, \dots, n-1$) to both sides of this equation we get

$$\begin{aligned} -4L &= \sum_{k=0}^{n-1} \frac{\partial^4 C_k}{\partial z_i^2 \partial \bar{z}_i^2} (f_k \bar{z}_k + \bar{f}_k z_k) + 2 \sum_{k=0}^{n-1} \frac{\partial^3 C_k}{\partial z_i^2 \partial \bar{z}_i} \left(f_k \delta_{ik} + \frac{\partial \bar{f}_k}{\partial \bar{z}_i} z_k \right) + \\ &+ 2 \sum_{k=0}^{n-1} \frac{\partial^3 C_k}{\partial z_i \partial \bar{z}_i^2} \left(\bar{f}_k \delta_{ik} + \frac{\partial f_k}{\partial z_i} \bar{z}_k \right) + 4 \sum_{k=0}^{n-1} \frac{\partial^2 C_k}{\partial z_i \partial \bar{z}_i} \left(\frac{\partial f_k}{\partial z_i} \delta_{ik} + \frac{\partial \bar{f}_k}{\partial \bar{z}_i} \delta_{ik} \right) + \\ &+ \sum_{k=0}^{n-1} \frac{\partial^2 C_k}{\partial z_i^2} \frac{\partial^2 \bar{f}_k}{\partial \bar{z}_i^2} z_k + \sum_{k=0}^{n-1} \frac{\partial^2 C_k}{\partial \bar{z}_i^2} \frac{\partial^2 f_k}{\partial z_i^2} \bar{z}_k + 2 \sum_{k=0}^{n-1} \frac{\partial C_k}{\partial z_i} \frac{\partial^2 \bar{f}_k}{\partial \bar{z}_i^2} \delta_{ik} + \\ &+ 2 \sum_{k=0}^{n-1} \frac{\partial C_k}{\partial \bar{z}_i} \frac{\partial^2 f_k}{\partial z_i^2} \delta_{ik} + \frac{\partial^4 C}{\partial z_i^2 \partial \bar{z}_i^2} (\phi_0 + \bar{\phi}_0) + 2 \frac{\partial^3 C}{\partial z_i^2 \partial \bar{z}_i} \frac{\partial \bar{\phi}_0}{\partial \bar{z}_i} + \\ &+ 2 \frac{\partial^3 C}{\partial z_i \partial \bar{z}_i^2} \frac{\partial \phi_0}{\partial z_i} + \frac{\partial^2 C}{\partial \bar{z}_i^2} \frac{\partial^2 \phi_0}{\partial z_i^2} + \frac{\partial^2 C}{\partial z_i^2} \frac{\partial^2 \bar{\phi}_0}{\partial \bar{z}_i^2}. \end{aligned}$$

Since C and C_k are rotation invariant, by evaluating the previous expression at $z_1 = \dots = z_{n-1} = 0$ and taking into account that

$$\frac{\partial^4 C_0}{\partial z_i^2 \partial \bar{z}_i^2} \big|_{\{z_1=\dots=z_{n-1}=0\}} = -8x \frac{F'^3}{F^3},$$

$$\frac{\partial^2 C_i}{\partial z_i \partial \bar{z}_i} \big|_{\{z_1=\dots=z_{n-1}=0\}} = 2x \frac{F'^2}{F^2},$$

$$\frac{\partial^4 C}{\partial z_i^2 \partial \bar{z}_i^2} \big|_{\{z_1=\dots=z_{n-1}=0\}} = 0,$$

we have

$$L = 2x \frac{F'^3}{F^3} (f_0 \bar{z}_0 + \bar{f}_0 z_0) - 2x \frac{F'^2}{F^2} (\phi_i + \bar{\phi}_i), \quad (31)$$

where $\phi_i = \frac{\partial f_i}{\partial z_i}$.

Now, let $i = 1, \dots, n-1$. By applying both sides of (25) to the pair $(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i})$ one gets

$$-\gamma g_{i\bar{i}} = \sum_{k=0}^{n-1} \left(f_k \frac{\partial g_{i\bar{i}}}{\partial z_k} + \bar{f}_k \frac{\partial g_{i\bar{i}}}{\partial \bar{z}_k} \right) + \sum_{k=0}^{n-1} \left(\frac{\partial f_k}{\partial z_i} g_{k\bar{i}} + \frac{\partial \bar{f}_k}{\partial \bar{z}_i} g_{i\bar{k}} \right) \quad (32)$$

where γ is given by (29). By (5) and (18) this means

$$\begin{aligned} -\gamma \frac{A + |z_i|^2}{A^2} &= \sum_{k=1}^{n-1} A^{-3} [2|z_i|^2 + A(1 + \delta_{ik})] (f_k \bar{z}_k + \bar{f}_k z_k) - \\ &- A^{-3} F' (2|z_i|^2 + A) (f_0 \bar{z}_0 + \bar{f}_0 z_0) - \frac{F'}{A^2} \left(\frac{\partial f_0}{\partial z_i} \bar{z}_0 z_i + \frac{\partial \bar{f}_0}{\partial \bar{z}_i} z_0 \bar{z}_i \right) + \\ &+ \frac{1}{A^2} \sum_{k=1}^{n-1} \left[\frac{\partial f_k}{\partial z_i} (\bar{z}_k z_i + \delta_{ki} A) + \frac{\partial \bar{f}_k}{\partial \bar{z}_i} (z_k \bar{z}_i + \delta_{ki} A) \right]. \end{aligned} \quad (33)$$

If we evaluate both sides of this equation at $z_1 = \dots = z_{n-1} = 0$ we get

$$-\gamma F = -F' (f_0 \bar{z}_0 + \bar{f}_0 z_0) + F (\phi_i + \bar{\phi}_i). \quad (34)$$

Moreover, by multiplying equation (33) by A^2 , by applying the operator $\frac{\partial^2}{\partial z_i \partial \bar{z}_i}$ to both sides and evaluating at $z_1 = \dots = z_{n-1} = 0$ one gets

$$0 = -\frac{F'}{F} (f_0 \bar{z}_0 + \bar{f}_0 z_0) + (\phi_i + \bar{\phi}_i). \quad (35)$$

Finally, by comparing (31) with (35), one gets $L = 0$ and hence, by the proof of Lemma 2.3, (D_F, g_F) is holomorphically isometric to an open subset of (CH^n, g_{hyp}) and we are done. (Notice that equations (34) and (35) yield $\gamma = 0$ and by (25) with $g_F = g_{hyp}$ one gets that X is a Killing vector field with respect to g_{hyp} , as expected). \square

References

- [1] B.Y. Chen and K. Ogiue, *Compact Kähler manifolds with constant generalized scalar curvature*, J. Diff. Geometry 11 (1976), 317-319.
- [2] Besse, Arthur L., *Einstein manifolds*, Springer-Verlag, Berlin, 1987.
- [3] D. Burns and P. De Bartolomeis, *Stability of vector bundles and extremal metrics*, Inv. Math. 92 (1988), 403-407.
- [4] E. Calabi, *Extremal Kähler metrics*, Seminar on Differential Geometry, Ann. of Math. Stud., 102, Princeton University Press, 1982, 259-290.
- [5] H.-D. Cao, G. Tian, X. Zhu, *Kähler-Ricci solitons on compact complex manifolds with $C_1(M) > 0$* , Geom. Funct. Anal. 15 (2005), no. 3, 697-719.
- [6] S.-C. Chang, *On the existence of nontrivial extremal metrics on complete non-compact surfaces*, Math. Ann. 324 (2002), 465-490.
- [7] S.-C. Chang and C.-T.-Wu, *On the existence of extremal metrics on complete noncompact 3-manifolds*, Indiana Univ. Math. J. 53 (2004), no. 1, 243-268.
- [8] X. Chen and G. Tian, *Uniqueness of extremal Kähler metrics*, C. R. Math. Acad. Sci. Paris 340 (2005), no. 4, 287-290.
- [9] F. Cuccu and A. Loi, *Global symplectic coordinates on complex domains*, J. Geom. and Phys. 56 (2006), 247-259.
- [10] A. J. Di Scala, A. Loi and F. Zuddas, *Riemannian geometry of Hartogs domains*, to appear in Internat. J. Math.
- [11] M. Engliš, *Berezin quantization and reproducing kernels on complex domains*, Trans. Amer. Math. Soc. 348 (1996), no. 2, 411-479.
- [12] M. Feldman, T. Imanen, D. Knopf, *Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons*, J. Diff. Geom. 65 (2003), no. 2, 169-209.
- [13] S. Fu and B. Wong, *On strictly pseudoconvex domains with Kähler-Einstein Bergman metrics*, Math. Res. Lett. 4, no. 5 (1997), pp. 697-703.
- [14] A. Futaki, *Canonical metrics in Kähler geometry and stability in GIT*, Trans. Amer. Math. Soc. 348 (1996), no. 2, 411-479.

- [15] P. Gauduchon, *Calabi's extremal Kähler metrics*, in preparation.
- [16] R.C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall Series in Modern Analysis, Prentice-Hall Inc. (1965).
- [17] A. V. Isaev and S. G. Krantz, *Domains with non-compact automorphism group: a survey*, Adv. Math. 146 (1999), no. 1, 1–38.
- [18] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry vol. II*, John Wiley and Sons Inc. (1967).
- [19] M. Levine, *A remark on extremal Kähler metrics*, J. Diff. Geom. 21 (1985), 73–77.
- [20] A. Loi, *Holomorphic maps of Hartogs domains into complex space forms*, Riv. Mat. Univ. Parma (7) vol. 1 (2002), 103–113.
- [21] A. Loi, *Regular quantizations of Kähler manifolds and constant scalar curvature metrics*, J. Geom. Phys. 53 (2005), 354–364.
- [22] T. Mabuchi, *Uniqueness of extremal Kähler metrics for an integral Kähler class*, Internat. J. Math. 15 (2004), 531–546.
- [23] K. Ogiue, *Generalized scalar curvatures of cohomological Einstein Kähler manifolds*, J. Diff. Geom. 10 (1975), 201–205.
- [24] G. Tian, *Canonical Metrics in Kähler Geometry*, Lectures in Mathematics, ETH Zürich.
- [25] G. Tian and X. Zhu, *A new holomorphic invariant and uniqueness of Kähler-Ricci solitons*, Comment. Math. Helv. 77 (2002), no. 2, 297–325.
- [26] G. Tian and X. Zhu, *Uniqueness of Kähler-Ricci solitons*, Acta Math. 184 (2000), no. 2, 271–305.

Andrea Loi and Fabio Zuddas

Dipartimento di Matematica e Informatica – Università di Cagliari – Italy

e-mail address: loi@unica.it, fzuddas@unica.it