

ON HOMOTHETIC BALANCED METRICS

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ABSTRACT. In this paper we study the set of balanced metrics (in Donaldson's terminology [14]) on a compact complex manifold M which are homothetic to a given balanced one. This question is related to various properties of the Tian-Yau-Zelditch approximation theorem for Kähler metrics. We prove that this set is finite when M admits a non-positive Kähler-Einstein metric, in the case of non-homogenous toric Kähler-Einstein manifolds of dimension ≤ 4 and in the case of the constant scalar curvature metrics found in [3] and [4].

1. INTRODUCTION

A fundamental result of Tian ([38]) states that any Kähler metric on a compact manifold is the limit of projectively induced metrics. Moreover a quantitative and refined version of this result, due to Lu ([29]) and Zelditch ([41]), gives an asymptotic expansion (since then called Tian-Yau-Zelditch expansion) for a suitably chosen sequence of projective metrics. The nature of the coefficients of this expansion is a challenging and intriguing question, in some sense resembling, in a complex form, better known similar problems in Riemannian geometry such as that of isospectral manifolds. This circle of questions turns out to be relevant also in problems coming from the theory of geometric quantization and in the existence problem of constant scalar curvature Kähler metrics ([14]) via the notion of balanced metrics introduced by Donaldson.

Of course it is of particular interest to characterize those Kähler manifolds whose coefficients of the associated TYZ expansion are constants. We will observe in Section 2 that this property is implied by having infinitely many proportional, here called homothetic, balanced metrics (a property already studied in the context of geometric quantization as recalled below) and it is related to another natural question about the characterization of the projectively induced metrics.

These properties are those central in this paper.

To enter more in detail, fix a positive line bundle L over a compact complex manifold M and denote by $\mathcal{B}(L)$ the set of balanced metrics on M

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which are polarized either with respect to L or some of its tensor powers, namely, $g_B \in \mathcal{B}(L)$ iff g_B is balanced and there exists a non-negative integer m_0 such that ω_B , the Kähler form associated to g_B , belongs to $c_1(L^{m_0})$. For a fixed $g_B \in \mathcal{B}(L)$ consider the set of all balanced metrics homothetic to g_B , namely the set $\mathcal{B}_{g_B} = \{mg_B \text{ is balanced} \mid m \in \mathbb{N}^+\}$.

Obviously, $\mathcal{B}_{g_B} \subset \mathcal{B}(L)$ for each $g_B \in \mathcal{B}(L)$. Notice that two balanced metrics in $\mathcal{B}(L)$ are isometric if and only if their associated Kähler forms are cohomologous (see [2] or Theorem 2.2 below) and hence the cardinality of \mathcal{B}_{g_B} is a cohomological invariant. For this reason we consider the quotient, denoted by $\mathcal{B}_c(L)$, of $\mathcal{B}(L)$ by the equivalence relation which identifies two balanced metrics if they belong to the same cohomology class. Observe that if the polarized manifold (M, L) is asymptotically Chow polystable then by a fundamental result of S. Zhang [42] (see the next section) the cardinality of $\mathcal{B}_c(L)$ is infinite. Moreover, by a result in [13] there exist examples where (M, L^m) is not Chow polystable (even of constant scalar curvature) for m large enough and hence, in this case, the cardinality of $\mathcal{B}_c(L)$, and hence that of \mathcal{B}_{g_B} , is finite (possibly zero). On the other hand, it is not hard to verify that any homogeneous integral Kähler metric g on a simply-connected homogeneous compact complex manifold is such that mg is balanced for all sufficiently large non-negative integers and hence the set \mathcal{B}_{g_B} , $g = g_B$, (and a fortiori $\mathcal{B}_c(L)$) has infinite cardinality (see [2]). More generally, given a Kähler metric g_B polarized with respect to L one tries to understand when mg_B is balanced for all non-negative integers (and so our set \mathcal{B}_{g_B} has infinite cardinality in this case). If this happens the corresponding geometric quantization is called regular. Regular quantizations play a fundamental role in the theory of Berezin quantization by deformation developed by M. Cahen, S. Gutt and J. Rawnsley in [7], [8], [9], [10]. A complete classification of regular quantizations is still missing (we refer to [1], [2] and [26] for more details). We believe that the Kähler manifolds which admits a regular quantization or, more generally, those for which \mathcal{B}_{g_B} consist of infinite elements, are in some sense special. More precisely, we believe the validity of the following

Conjecture: Let (M, L) be a polarized manifold. If there exists a balanced metric $g_B \in \mathcal{B}(L)$ such that $\#\mathcal{B}_{g_B} = \infty$ then (M, g_B) is a homogeneous Kähler manifold.

Our main results are the following three results (see Sections 3, 4 and 5 below for details).

The first observation, since Kähler–Einstein metrics with non-positive scalar curvature are never projectively induced, is then the following:

Proposition 1.1. *Let L be a polarization of a compact Kähler–Einstein manifold (M, g) with non-positive scalar curvature. Then $\mathcal{B}_c(L)$ consists of infinitely many balanced metrics such that for each $g_B \in \mathcal{B}(L)$, the set \mathcal{B}_{g_B} is finite.*

On the other hand projectively induced Kähler–Einstein manifolds (M, g) with positive scalar curvature do exist, and it has been repeatedly claimed that only homogeneous manifolds have this property. Unfortunately all these proofs contain fatal errors. The proof of Theorem 1.2 is based on the fact that a Kähler–Einstein metric on toric manifolds of dimension ≤ 4 are not projectively induced. This result, interesting on its own sake, gives the first examples of non projectively induced Kähler–Einstein metrics on compact complex manifolds M , with $c_1(M) > 0$ and large group of isometries.

Theorem 1.2. *Let g be a Kähler–Einstein metric on a toric manifold M of dimension ≤ 4 and let $L = K^*$ be the anticanonical bundle over M . Then $\mathcal{B}_c(L)$ consists of infinitely many balanced metrics. Moreover, there exists $g_B \in \mathcal{B}(L)$ such that \mathcal{B}_{g_B} is infinite if and only if M is either a projective space or a product of projective spaces.*

Passing from Kähler–Einstein to constant scalar curvature (cscK) metrics we prove that in what is at present the greatest source of examples, namely the blow up gluing procedure developed in [3] and [4], the second coefficient of the TYZ asymptotic expansion is never constant, thanks to some special properties of the LeBrun–Simanca model for the gluing procedure. This implies the following:

Theorem 1.3. *Let g be a cscK metric on a compact complex manifold and let g_ε , $\varepsilon > 0$, be a family of cscK metrics constructed as in [3] and [4] on the blow-up $\tilde{M} = Bl_{p_1, \dots, p_k} M$ of M at the points p_1, \dots, p_k of M . Let ε be a sufficiently small rational number, say $\varepsilon = \frac{p}{q}$, and let $L_\varepsilon \rightarrow \tilde{M}$ be a polarization for the Kähler class of the metric qg_ε . Then, for each $g_B \in \mathcal{B}(L_\varepsilon)$, the set \mathcal{B}_{g_B} is finite.*

The authors believe that the computation of the cardinality of \mathcal{B}_{g_B} or the proof of the existence of an upper bound of this cardinality when g_B is varying in $\mathcal{B}(L)$ is a very hard and intriguing problem which could shed some light to the understanding of balanced metrics and of the stability of the polarized manifold (M, L) .

The paper is organized as follows. In the next section we describe the link between balanced and projectively induced Kähler metrics, we recall the TYZ expansion and we use it to prove two lemmata needed in the proof of the main results. Sections 3, 4 and 5 are dedicated to the proofs of Proposition 1.1, Theorem 1.2 and Theorem 1.3 respectively.

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2. BALANCED AND PROJECTIVELY INDUCED METRICS

Let g be a Kähler metric on a compact complex manifold M . In the quantum mechanics terminology (M, g) is said to be *quantizable* if the Kähler

form ω associated to g is integral, i.e. there exists a holomorphic line bundle L over M , called the *quantum line bundle*, whose first Chern class equals the second De-Rham cohomology class of ω , i.e. $c_1(L) = [\omega]_{dR}$. By Kodaira's theory this is equivalent to say that M is a projective algebraic manifold and L is a positive (or ample) line bundle over M . In algebraic-geometric terms L is said to be a *polarization* of M , g a *polarized metric* and the pair (M, L) a *polarized manifold*. Fix a polarization L over M . Then there exists an hermitian metric h on L , defined up to the multiplication with a positive constant, such that its Ricci curvature $\text{Ric}(h) = \omega$ ¹. The pair (L, h) is called a *geometric quantization* of the Kähler manifold (M, ω) . Let s_0, \dots, s_N be an orthonormal basis of $H^0(L)$ (the space of global holomorphic sections of L) with respect to the scalar product

$$\langle s, t \rangle = \int_M h(s(x), t(x)) \frac{\omega^n(x)}{n!}, \quad s, t \in H^0(L).$$

Consider the non-negative smooth function T_g on M given by:

$$T_g(x) = \sum_{j=0}^N h(s_j(x), s_j(x)). \quad (1)$$

As suggested by the notation this function depends only on the Kähler form metric g and not on the orthonormal basis chosen.

The function T_g has appeared in the literature under different names. The earliest one was probably the η -function of J. Rawnsley [35] (later renamed to θ function in [7]), defined for arbitrary (not necessarily compact) Kähler manifolds, followed by the *distortion function* of G. R. Kempf [21] and S. Ji [19], for the special case of Abelian varieties and of S. Zhang [42] for complex projective varieties. The metrics for which T_g is constant were called *critical* in [42] and *balanced* in [14].

The two fundamental results about existence and uniqueness of balanced metrics are summarized in the following two theorems.

Theorem 2.1. (*S. Donaldson [14]*) *Let (L, h) be a geometric quantization of a compact Kähler manifold (M, ω) such that the polarized metric g whose associated Kähler form ω has constant scalar curvature. Assume that $\frac{\text{Aut}(M, L)}{\mathbb{C}^*}$ ² is discrete. Then, for all sufficiently large integers m , there exists a unique balanced metric \tilde{g}_m on M , with polarization $L^m = L^{\otimes m}$, such that $\frac{\tilde{g}_m}{m}$ C^∞ -converges to g . Moreover if \tilde{g}_m is a sequence of balanced metrics on M with $\tilde{\omega}_m \in c_1(L^m)$ such that $\frac{\tilde{g}_m}{m}$ C^∞ -converges to a metric g then g has constant scalar curvature.*

¹ $\text{Ric}(h)$ is the two-form on M whose local expression is given by $\text{Ric}(h) = -\frac{i}{2} \partial \bar{\partial} \log h(\sigma(x), \sigma(x))$, for a trivializing holomorphic section $\sigma : U \rightarrow L \setminus \{0\}$.

² $\frac{\text{Aut}(M, L)}{\mathbb{C}^*}$ denotes the group biholomorphisms of M which lift to holomorphic bundles maps $L \rightarrow L$ modulo the trivial automorphism group \mathbb{C}^* .

Besides the uniqueness part which is recalled below, Mabuchi ([31]) extended the above theorem to polarized manifolds with nontrivial automorphisms under certain conditions.

From the GIT (geometric invariant theory) point of view given a polarized manifold (M, L) , with L very ample, there exists a balanced metric g whose associated Kähler form is in the class of $c_1(L)$ if and only if (M, L) is Chow polystable (see [42] for a proof). Since the Chow polystability is equivalent to the Chow stability when $\frac{\text{Aut}(M, L)}{\mathbb{C}^*}$ is discrete, Theorem 2.1 can be equivalently stated by saying that given a polarized manifold (M, L) such that $\frac{\text{Aut}(M, L)}{\mathbb{C}^*}$ is discrete and M admits a constant scalar curvature metric in the class $c_1(L)$ then (M, L) is asymptotically Chow stable (i.e. Chow stable for all m sufficiently large). Notice that the assumption on the automorphism group in Theorem 2.1 cannot be dropped entirely. Indeed, from the point of view of the existence of balanced metrics the recent results of [34] and of A. Della Vedova and the third author [13] show that there exist a large class of polarized manifolds (M, L) such that M admits a constant scalar curvature metric in the class $c_1(L)$ and such that (M, L^m) is not polystable, for all m sufficiently large. Regarding the uniqueness of balanced metrics the first and the second author [2] have shown the following:

Theorem 2.2. *Let g and \tilde{g} be two balanced metrics whose associated Kähler forms are cohomologous. Then g and \tilde{g} are isometric, i.e. there exists $F \in \text{Aut}(M)$ such that $F^*\tilde{g} = g$.*

For a polarization L over (M, g) and every non-negative integer $m \geq 1$ let us consider the Kempf distortion function associated to mg , i.e.

$$T_{mg}(x) = \sum_{j=0}^{d_m} h_m(s_j(x), s_j(x)), \quad (2)$$

where h_m is an hermitian metric on L^m such that $\text{Ric}(h_m) = m\omega$ and s_0, \dots, s_{d_m} , $d_m + 1 = \dim H^0(L^m)$, is an orthonormal basis of $H^0(L^m)$ with respect to the L^2 -scalar product

$$\langle s, t \rangle_m = \int_M h_m(s(x), t(x)) \frac{\omega^n(x)}{n!}, \quad s, t \in H^0(L^m).$$

(In the quantum geometric context m^{-1} plays the role of Planck's constant, see e.g. [1]).

One can give a quantum-geometric interpretation of T_{mg} as follows. Take m sufficiently large such that for each point $x \in M$ there exists $s \in H^0(L^m)$ non-vanishing at x (such an m exists by standard algebraic geometry methods and corresponds to the free-based point condition in Kodaira's theory, see e.g. [22]). Consider the so called *coherent states map*, namely the holomorphic map of M into the complex projective space $\mathbb{C}P^{d_m}$ given by:

$$\varphi_m : M \rightarrow \mathbb{C}P^{d_m}, \quad x \mapsto [s_0(x) : \dots : s_{d_m}(x)]. \quad (3)$$

One can prove (see, e.g. [2]) that

$$\varphi_m^* \omega_{FS} = m\omega + \frac{i}{2} \partial \bar{\partial} \log T_{mg}, \quad (4)$$

where ω_{FS} is the Fubini–Study form on $\mathbb{C}P^{d_m}$, namely the Kähler form which in homogeneous coordinates $[Z_0, \dots, Z_{d_m}]$ reads as $\omega_{FS} = \frac{i}{2} \partial \bar{\partial} \log \sum_{j=0}^{d_m} |Z_j|^2$. Since the equation $\partial \bar{\partial} f = 0$ implies that f is constant, it follows by (4) that mg is balanced if and only if it is projectively induced via the coherent states map. Recall that a polarized Kähler metric g on a complex manifold M with polarization L is *projectively induced* if there exists a basis t_0, \dots, t_N of $H^0(L)$ such that the holomorphic map $\psi : M \rightarrow \mathbb{C}P^N, x \mapsto [t_0(x) : \dots : t_N(x)]$ induced by this basis, satisfies $\psi^*(g_{FS}) = g$ (the author is referred to the seminal paper of E. Calabi [11] for more details on the subject). Notice that there is a large class of projectively induced Kähler metrics which are not balanced. Indeed by Theorem 2.2 the set of balanced metrics on a fixed cohomology class is either empty or in bijection with the automorphism group $\text{Aut}(M)$ of the manifold, while by Calabi’s rigidity theorem (see [11]) the set of projectively induced metrics in the same class are in bijection with $\text{PGL}(N+1)/U(N+1) \times \text{Aut}(M)$ ($N = \dim H^0(L) - 1$).³

Not all Kähler metrics are balanced or projectively induced. Nevertheless, G. Tian [38] and W. D. Ruan [36] solved a conjecture posed by Yau by showing that $\frac{\varphi_m^*(g_{FS})}{m} C^\infty$ -converges to g . In other words, any polarized metric on a compact complex manifold is the C^∞ -limit of (normalized) projectively induced Kähler metrics. S. Zelditch [41] generalized the Tian–Ruan theorem by proving a complete asymptotic expansion in the C^∞ category, namely

$$T_{mg}(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j}, \quad (5)$$

where $a_j(x)$, $j = 0, 1, \dots$, are smooth coefficients with $a_0(x) = 1$. More precisely, for any nonnegative integers r, k the following estimates hold:

$$\|T_{mg}(x) - \sum_{j=0}^k a_j(x) m^{n-j}\|_{C^r} \leq C_{k,r} m^{n-k-1}, \quad (6)$$

where $C_{k,r}$ is a constant depending on k, r and on the Kähler form ω and $\|\cdot\|_{C^r}$ denotes the C^r norm in local coordinates. Later on, Z. Lu [29] (see also [28]), by means of Tian’s peak section method, proved that each of the coefficients $a_j(x)$ in (5) is a polynomial of the curvature and its covariant derivatives at x of the metric g which can be found by finitely many algebraic

³To have an explicit example take the metric $g = \psi^* g_{FS}$ on $\mathbb{C}P^1$, where $\psi : \mathbb{C}P^1 \rightarrow \mathbb{C}P^2$ is the Veronese embedding, i.e. $\psi([z_0, z_1]) = [z_0^2, z_0 z_1, z_1^2]$. Then g is a projectively induced Kähler metric on $\mathbb{C}P^1$ polarized with respect to $O(2)$ and g is not balanced.

operations. Furthermore, he explicitly computes a_j for $j \leq 3$, i.e. (we omit the expression of a_3 since we do not need it in this paper)

$$\begin{cases} a_1(x) = \frac{1}{2}\rho \\ a_2(x) = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|\text{Ric}|^2 + 3\rho^2) \end{cases} \quad (7)$$

where ρ , R , Ric denote respectively the scalar curvature, the curvature tensor and the Ricci tensor of (M, g) . The reader is also referred to [24] and [25] for a recursive formula of the a_j 's and an alternative computations of a_j for $j \leq 3$ using Calabi's diastasis function (see also the recent papers [39] and [40] for a graph-theoretic interpretation of this recursive formula). The expansion (5) is called the *TYZ (Tian–Yau–Zelditch) expansion*. Together with Donaldson's moment maps techniques, it is a key ingredient in the proof of Theorems 2.1 and 2.2 in the Introduction.

We end this section with two lemmata needed in the proofs of our theorems.

Lemma 2.3. *Let g be any (not necessarily balanced) polarized metric on a compact complex manifold M . Assume that the set*

$$\mathcal{B}_g = \{mg \text{ is balanced} \mid m \in \mathbb{N}\}$$

consists of infinite elements. Then the coefficients $a_j(x)$ of the TYZ expansion of the Kempf distortion function T_{mg} are constants for all $j = 0, 1, \dots$.

Proof. Assume that there exists an increasing sequence $\{m_s\}_{s=1,2,\dots}$ of non-negative integers such that $m_s g$ is a balanced metric, i.e. $T_{m_s g}(x) = T_{m_s}$ for some positive constants T_{m_s} . We argue by induction on j . We already know that $a_0 = 1$ is a constant, so assume that the $a_j(x)$'s are constants, say a_j , for $j = 0, \dots, k-1$. By (6) we have

$$|T_{s,k,n} - a_k(x)m_s^{n-k}| \leq C_k m_s^{n-k-1}$$

for some constant C_k , where $T_{s,k,n}$ is the constant (depending on s, k and n) equal to $T_{m_s} - \sum_{j=0}^{k-1} a_j m_s^{n-j}$. Hence $|m_s^{k-n} T_{s,k,n} - a_k(x)| \leq C_k m_s^{-1}$ and letting $s \rightarrow \infty$ we get that $m_s^{k-n} T_{s,k,n}$ tends to $a_k(x)$ which is then forced to be a constant. \square

As a simple consequence of the previous lemma we get the following result which shows the validity of the Conjecture in the Introduction in the one-dimensional case.

Corollary 2.4. *Let (M, L) be a polarized manifold and M have complex dimension 1. Assume that there exists $g_B \in \mathcal{B}(L)$ such that $\sharp \mathcal{B}_{g_B} = \infty$. Then M is biholomorphic to the Riemann sphere \mathbb{CP}^1 .*

Proof. Assume $\sharp \mathcal{B}_{g_B} = \infty$ for some $g_B \in \mathcal{B}(L)$. Then, by Lemma 2.3, the coefficients a_j^B of the TYZ expansion of T_{mg_B} are constants. In particular $a_1^B = \rho_B/2$ is constant, where ρ_B is the scalar curvature of g_B (cfr. (7)). On the other hand the flat metric on an elliptic curve and the hyperbolic metric on a Riemann surface of genus ≥ 2 cannot be projectively induced

(see [27] for a proof) and hence M is forced to be biholomorphic to $\mathbb{C}P^1$ (and g_B isometric to an integer multiple of the Fubini–Study metric). \square

Lemma 2.5. *Let g be a cscK metric on a compact complex manifold M polarized with respect to a holomorphic line bundle L . Assume g satisfies one of the following conditions:*

- (1) *mg is not projectively induced for all m ;*
- (2) *there is at least a non-constant coefficient a_{j_0} , with $j_0 \geq 2$, of the TYZ expansion (5) of the Kempf distortion function $T_{mg}(x)$.*

Then, for any $g_B \in \mathcal{B}(L)$, the set \mathcal{B}_{g_B} consists of finitely many elements.

Proof. Let $g_B \in \mathcal{B}(L)$, this means that g_B is a balanced metric on M such that its associated Kähler form ω_B belongs to $c_1(L^{m_0})$ for some m_0 . Assume by a contradiction that \mathcal{B}_{g_B} has infinite elements. Then, by Lemma 2.3, the coefficients of the TYZ expansion of g_B , denoted by a_j^B , are constants for all $j = 0, 1, \dots$. In particular, by the first of (7), g_B is cscK. Since ω_B is cohomologous to $m_0\omega$ and by assumption g has constant scalar curvature it follows by a theorem of X. X. Chen and G. Tian [12] that there exists an automorphism F of M such that $F^*g_B = m_0g$. Since g_B is projectively induced and the a_j^B 's are constants for all $j = 0, 1, \dots$ we get that: (a) m_0g is projectively induced; (b) the coefficients a_j 's of the TYZ expansion of T_{mg} are constants for all $j = 0, 1, \dots$. Since (a) and (b) are in contrast with (1) and (2) respectively this yields the desired contradiction and concludes the proof of the lemma. \square

It is interesting pointing out that there are examples, even in complex dimension 1, of cscK metrics such that all the coefficients of the associated TYZ expansion are constant and mg is not projectively induced for all non negative integer m . Take for example a compact Riemann surface Σ with the hyperbolic metric g_{hyp} which is polarized with respect to the anticanonical bundle. Then, being (Σ, g_{hyp}) locally homogeneous all the coefficients a_j of TYZ expansion are constant, more precisely a_1 is half of the constant scalar curvature and one can show that $a_k = 0$, for $k \geq 2$. On the other hand, as we have already pointed out, mg_{hyp} is not projectively induced (see [27] for a proof).

Finally, notice that prescribing the values of the coefficients of the TYZ expansion gives rise to interesting elliptic PDE as shown by Z. Lu and G. Tian [30]. The main result obtained there is that if the log term of the Szegő kernel of the unit disk bundle over M vanishes then $a_k = 0$, for $k > n$. Hence, in the light of the previous lemma and considerations, we believe that the link between the Szegő kernel and our results deserves further study.

3. THE PROOF OF PROPOSITION 1.1

In this section we assume that L is a polarization of a compact complex manifold M which admits a non-positively curved Kähler–Einstein metric g

in $c_1(L)$. Obviously if $c_1(M) < 0$ we can take $L = K$, where K is the canonical bundle over M , while when $c_1(M) = 0$ the polarization could not exist, take for example a complex torus which is not an abelian variety. Moreover, in both cases the existence of a Kähler–Einstein metric with negative or zero scalar curvature is guaranteed by Yau’s solution of Calabi’s conjecture. Notice also that in both cases the manifold (M, L) is asymptotically Chow polystable. Indeed, when $c_1(M) < 0$, $\text{Aut}(M)$ is finite and hence the assertion follows by Donaldson’s Theorem 2.1 above. On the other hand, if $c_1(M) = 0$, it is well-known that the set $h_0(M)$ of holomorphic fields on M with zeros is trivial. Since the Lie algebra of the identity component of $\text{Aut}(M, L)$ is exactly $h_0(M)$ (see, for example, [18], Prop. 7.1.2), we conclude by applying Theorem 2.1 again.

Therefore the set $\mathcal{B}_c(L)$ defined in the previous section is infinite. On the other hand the metric mg is not projectively induced for any m as it follows by a Theorem of D. Hulin [20] which asserts that the scalar curvature of a projectively induced Kähler–Einstein metric is strictly positive. Combining this fact with Lemma 2.5 the proof of Proposition 1.1 is immediate.

4. KÄHLER-EINSTEIN METRICS ON LOW-DIMENSIONAL TORIC MANIFOLDS

Let us briefly recall that a compact, complex manifold M of complex dimension n is said to be *toric* if it contains a complex torus $(\mathbb{C}^*)^n$ as a dense open subset, together with a holomorphic action $(\mathbb{C}^*)^n \times M \rightarrow M$ that extends the natural action of $(\mathbb{C}^*)^n$ on itself. A basic fact in the theory of toric manifolds is that any such M is determined by the combinatorial data encoded in a *fan of cones* in \mathbb{R}^n , that is a set of convex linear cones satisfying some properties which the interested reader can find, for example, in [15]. Moreover, any ample linear bundle L on M corresponds to a polytope $\Delta_L = \{x \in \mathbb{R}^n \mid \langle x, u_i \rangle \leq \lambda_i, \ i = 1, \dots, d\}$, where $u_i, \ i = 1, \dots, d$, are integral vectors which generate the edges of the cones in the fan and $\lambda_i \in \mathbb{Z}$. In particular, when M is Fano the anticanonical bundle K^* corresponds to the choice $\lambda_i = 1, \ i = 1, \dots, d$. This correspondence is such that in the coordinates $z = (z_1, \dots, z_n) \in (\mathbb{C}^*)^n$ defined on the open dense subset of M diffeomorphic to the complex torus, a basis of the space $H^0(L)$ of global sections of L is given by $S = \{z^{J_0}, \dots, z^{J_N}\}$, where $\{J_0, \dots, J_N\} = \Delta_L \cap \mathbb{Z}^n$ and, for any $J = (j_1, \dots, j_n) \in \mathbb{Z}^n$, we set $z^J = z_1^{j_1} \dots z_n^{j_n}$.

Since the fan associated to a toric manifold is determined up to the action of $SL(n, \mathbb{Z})$, we can always assume that it contains the n -dimensional cone generated by $-e_1, \dots, -e_n$, where $\{e_1, \dots, e_n\}$ is the canonical basis of \mathbb{R}^n . Moreover, it is known ([5]) that polytopes which differ by a translation via a vector $v \in \mathbb{Z}^n$ represent isomorphic line bundles on the same toric manifold. Then, it follows that the anticanonical bundle can be represented by a polytope Δ which satisfies the following

Assumption 1. Δ contains the origin $(0, \dots, 0)$ as vertex and the edge at this vertex is generated by $+e_1, \dots, +e_n$.

Lemma 4.1. *Under Assumption 1, a projectively induced toric metric $\omega \in c_1(K^*)$ writes in the coordinates z_1, \dots, z_n on the open dense subset diffeomorphic to $(\mathbb{C}^*)^n$ as $\omega = \frac{i}{2} \partial \bar{\partial} \log F$ where $F = \sum_I a_I x^I$ is a polynomial in $x = (x_1 = |z_1|^2, \dots, x_n = |z_n|^2)$ such that $a_I \geq 0$ and $a_I > 0$ if and only if $I \in \Delta \cap \mathbb{Z}^n$.*

Moreover, ω is Kähler-Einstein if and only if F satisfies the following polynomial equality

$$\det(A) = cF^{2n-1}, \quad A_{ij} = (FF_{ij} - F_i F_j) \bar{z}_i z_j + FF_j \delta_{ij}, \quad c \in \mathbb{R}^+ \quad (8)$$

where $F_i = \frac{\partial F}{\partial x_i}$, $F_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}$, $i, j = 1, \dots, n$.

Proof. In the coordinates z_1, \dots, z_n , a generic element v_i of a basis of $H^0(K^*)$ writes $\sum_k f_{ik} z^{J_k}$, where $\{J_0, \dots, J_N\} = \Delta \cap \mathbb{Z}^n$ and $(f_{jk}) \in GL(N+1, \mathbb{C})$, so a projectively induced toric metric (i.e. invariant by the action of the real torus $\mathbb{T}^n = \{(e^{i\theta_1}, \dots, e^{i\theta_n})\}$ on M given on $(\mathbb{C}^*)^n$ by

$$(z_1, \dots, z_n) \mapsto (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$$

can be written as

$$\omega = \frac{i}{2} \partial \bar{\partial} \log F(x_1, \dots, x_n),$$

where $x_i = |z_i|^2$ and F is a linear combination with real coefficients of the functions $x^I = x_1^{i_1} \cdots x_n^{i_n}$, where $I = (i_1, \dots, i_n) \in \Delta \cap \mathbb{Z}^n$. Notice that, since $(f_{jk}) \in GL(N+1, \mathbb{C})$, we have that the coefficient of x^{J_l} in F is given by $\sum_i |f_{il}|^2 > 0$. By Assumption 1 and the convexity of Δ it follows that $\Delta \subseteq \bigcap \{x_i \geq 0\}$, so F is in fact a polynomial. Notice that $\{e_1, \dots, e_n\} \subseteq \Delta \cap \mathbb{Z}^n$, so that $F = 1 + \alpha_1 x_1 + \cdots + \alpha_n x_n +$ (terms of higher order), with $\alpha_i > 0$ for each $i = 1, \dots, n$.

Now, observe that the matrix g_ω of the metric associated to ω has entries $g_{i\bar{j}} = \frac{(FF_{ij} - F_i F_j) \bar{z}_i z_j + FF_j \delta_{ij}}{F^2}$, so that $\det(g_\omega) = \frac{\det(A)}{F^{2n}}$, where A_{ij} is given by (8). By the well-known formula for the Ricci form $\rho_\omega = -i \partial \bar{\partial} \log \det(g_\omega)$ we then see that the Einstein condition is equivalent to the equation $\partial \bar{\partial} \Phi = 0$, where $\Phi = \log \left(\frac{\det(A)}{F^{2n-1}} \right)$. Now, since $\Phi = \Phi(x_1, \dots, x_n)$, this is equivalent to the equations

$$\frac{\partial \Phi}{\partial x_i} + \frac{\partial^2 \Phi}{\partial x_i^2} x_i = 0, \quad i = 1, \dots, n, \quad \frac{\partial^2 \Phi}{\partial x_i \partial x_j} = 0, \quad i \neq j.$$

The last set of equations implies that $\frac{\partial \Phi}{\partial x_i}$ depends only on x_i , for every $i = 1, \dots, n$, so that the first n equations become ordinary differential equations whose general solution is $\frac{\partial \Phi}{\partial x_i} = c_i/x_i$, for $c_i \in \mathbb{R}$, and $\Phi = c_i \log x_i + f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ for some real function f . Since this holds true for every $i = 1, \dots, n$, one concludes that

$$\Phi = c_1 \log x_1 + \cdots + c_n \log x_n + c_{n+1} = \log(x_1^{c_1} \cdots x_n^{c_n}) + c_{n+1}, \quad (9)$$

for $c_i \in \mathbb{R}$. We claim that $c_i = 0$ for $i \leq n$. Indeed, since $F = 1 + \alpha_1 x_1 + \cdots + \alpha_n x_n +$ (terms of higher order), $\alpha_i > 0$, by $\Phi = \log \left(\frac{\det(A)}{F^{2n-1}} \right)$ and by

noticing that at $z_1 = \dots = z_n = 0$ we have $A = \text{diag}(FF_1, \dots, FF_n)$, we see that when $x_1, \dots, x_n \rightarrow 0$ then $\Phi \rightarrow \log(\alpha_1 \dots \alpha_n) \in \mathbb{R}$. On the other hand, by (9) one easily checks that $\Phi \rightarrow \pm\infty$ for $x_1, \dots, x_n \rightarrow 0$ along suitable curves in $(\mathbb{R}^+)^n$ if at least one among the c_i 's, $i = 1, \dots, n$, does not vanish (for example, if $c_i \neq 0$ take $x_i = t^k, x_j = t$ for $j \neq i$, for k large enough). This proves the claim and then (8), for $c = e^{c_{n+1}}$. \square

We are now ready to prove the following proposition interesting on its own sake.

Proposition 4.2. *If M is a smooth, compact toric n -dimensional manifold, $n \leq 4$, then M does not admit any projectively induced Kähler-Einstein metric unless it is a projective space or the product of projective spaces.*

Proof. Suppose, contrary to the claim, that there exists a projectively induced Kähler-Einstein metric ω on M . Let us first assume that $\omega \in c_1(K^*)$.

Let us recall ([6]) that, for $n \leq 4$, the toric manifolds M which admit a Kähler-Einstein metric are completely classified. More precisely, if $n = 2$, M is either \mathbb{CP}^2 , $\mathbb{CP}^1 \times \mathbb{CP}^1$ or the blow-up of \mathbb{CP}^2 at three points. This last case is associated to the fan in \mathbb{R}^2 whose set of edges is generated by

$$\pm e_1, \pm e_2, \pm(e_1 - e_2). \quad (10)$$

If $n = 3$, either M is \mathbb{CP}^3 , or can be decomposed into a product of lower dimensional manifolds or it is the manifold associated to the fan in \mathbb{R}^3 whose cones have the following generators:

$$e_1, e_2, \pm e_3, -(e_1 + e_3), -(e_2 - e_3). \quad (11)$$

Finally, if $n = 4$, either M is \mathbb{CP}^4 , or can be decomposed into a product of lower dimensional manifolds or it is the manifold associated to one of the following fans in \mathbb{R}^4 (given by the generators of their cones):

$$\pm e_1, \dots, \pm e_4, \pm(e_1 + \dots + e_4); \quad (12)$$

$$e_1, e_2, \pm e_3, \pm e_4, \pm(e_3 + e_4), -(e_1 - e_3), -(e_2 + e_3); \quad (13)$$

$$e_1, \dots, e_4, -(e_1 + \dots + e_4), -(e_1 + e_2), -(e_3 + e_4), e_1 + e_3, e_2 + e_4. \quad (14)$$

In order to prove the Proposition, we first apply to each of the above fans a suitable transformation $A \in SL(n, \mathbb{Z})$ so that the anticanonical bundle of the corresponding manifold can be represented by a polytope Δ satisfying the properties given in Assumption 1 and Lemma 4.1 can be applied. The case (10) already meets the required condition, while one easily verifies that the following matrices

$$-I_3, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

have this property, respectively for (11), (12), (13), (14). Then the anti-canonical bundles are represented respectively by the polytopes $\Delta =$

$$\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 2, -1 \leq x - y \leq 1\}; \quad (15)$$

$$\{(x, y, z) \in \mathbb{R}^3 \mid x, y \geq 0, 0 \leq z \leq 2, x + z \leq 3, y - z \leq 1\}; \quad (16)$$

$$\{(x, y, z, w) \in \mathbb{R}^4 \mid 0 \leq x, y, z, w \leq 2, -1 \leq x + y - z - w \leq 1\}; \quad (17)$$

$$\{(x, y, z, w) \in \mathbb{R}^4 \mid x, y \geq 0, 0 \leq z, w \leq 2, -1 \leq z - w \leq 1, x - z \leq 1, y + z \leq 3\}; \quad (18)$$

$$\{(x, y, z, w) \in \mathbb{R}^4 \mid x, y, z, w \geq 0, x - z \leq 1, y - w \leq 1, z + w \leq 3, x + y \leq 1, z + w - x - y \leq 1\}. \quad (19)$$

We are now going to treat each of the cases (15)-(19) above separately. Let F be the polynomial given in Lemma 4.1. We shall use the following notation:

$$F_{x_1^{i_1} \dots x_n^{i_n}} = \frac{\partial^{i_1 + \dots + i_n} F}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} \Big|_{x_1 = \dots = x_n = 0}, \quad \tilde{F}_{x_1^{i_1} \dots x_n^{i_n}} = \frac{F_{x_1^{i_1} \dots x_n^{i_n}}}{(F_{x_1})^{i_1} \dots (F_{x_n})^{i_n}}.$$

In order to prove the Proposition, we check that equation (8) implies in each of the cases (15)-(19) that $\tilde{F}_{x_1^{i_1} \dots x_n^{i_n}} = 0$ for some $(i_1, \dots, i_n) \in \Delta \cap \mathbb{Z}^n$, against the first part of the statement of Lemma 4.1.

Case (15): the left-hand side $\det(A)$ in equation (8) can be written more explicitly as

$$\begin{aligned} & [(F_{11}F - F_1^2)(F_{22}F - F_2^2) - (F_{12}F - F_1F_2)^2]x_1x_2 + F_2F(F_{11}F - F_1^2)x_1 + \\ & + F_1F(F_{22}F - F_2^2)x_2 + F_1F_2F^2. \end{aligned} \quad (20)$$

By a straight calculation one gets the following equalities at $x_1 = x_2 = 0$:

$$\det(A) = F_{x_1}F_{x_2}, \quad \frac{\partial \det(A)}{\partial x_1} = F_{x_1}^2F_{x_2} + F_{x_1}F_{x_1x_2}, \quad \frac{\partial \det(A)}{\partial x_2} = F_{x_2}^2F_{x_1} + F_{x_2}F_{x_1x_2},$$

$$\frac{\partial^2 \det(A)}{\partial x_1^2} = 2F_{x_1}^2F_{x_1x_2} + F_{x_1^2x_2}F_{x_1}, \quad \frac{\partial^2 \det(A)}{\partial x_2^2} = 2F_{x_2}^2F_{x_1x_2} + F_{x_1x_2^2}F_{x_2}.$$

By comparing with the corresponding derivatives of the right-hand side of equation (8), one gets $\tilde{F}_{x_1x_2} = \tilde{F}_{x_1^2x_2} = \tilde{F}_{x_1x_2^2} = 2$. One then gets a contradiction by substituting these values into

$$\frac{\partial^2 \det(A)}{\partial x_1 \partial x_2} \Big|_{x_1=x_2=0} = 4F_{x_1}F_{x_2}F_{x_1x_2} + 2F_{x_2}F_{x_1^2x_2} + 2F_{x_1}F_{x_1x_2^2}$$

and comparing with $\frac{\partial^2 (cF^3)}{\partial x_1 \partial x_2} \Big|_{x_1=x_2=0}$.

Case (16) : by calculating the derivatives of both sides of equation (8) with respect to x_2 one gets the following system

$$\begin{cases} \tilde{F}_{x_1x_2} + \tilde{F}_{x_2x_3} = 3 \\ 2\tilde{F}_{x_1x_2}\tilde{F}_{x_2x_3} + \tilde{F}_{x_2^2x_3} = 6 \\ \tilde{F}_{x_1x_2}\tilde{F}_{x_2^2x_3} = 2 \end{cases}$$

which has as unique solution $\tilde{F}_{x_1x_2} = 1, \tilde{F}_{x_2x_3} = 2, \tilde{F}_{x_2^2x_3} = 2$.

By calculating the derivatives of both sides of equation (8) with respect to x_3 one gets

$$\begin{cases} \tilde{F}_{x_1x_3} + \tilde{F}_{x_2x_3} + 2\tilde{F}_{x_3^2} = 3 \\ \tilde{F}_{x_1x_3^2} + 2\tilde{F}_{x_1x_3}\tilde{F}_{x_2x_3} + \tilde{F}_{x_2^2x_3} + 4\tilde{F}_{x_3^2}(\tilde{F}_{x_1x_3} + \tilde{F}_{x_2x_3}) + \tilde{F}_{x_3^3} = 6 + 3\tilde{F}_{x_3^2} \\ \tilde{F}_{x_1x_3^2}\tilde{F}_{x_2x_3} + \tilde{F}_{x_2x_3^2}\tilde{F}_{x_1x_3} + 2\tilde{F}_{x_3^2}(\tilde{F}_{x_1x_3^2} + 2\tilde{F}_{x_1x_3}\tilde{F}_{x_2x_3} + \tilde{F}_{x_2^2x_3}) + \tilde{F}_{x_3^3}(\tilde{F}_{x_1x_3} + \tilde{F}_{x_2x_3}) = 2 + 6\tilde{F}_{x_3^2} \\ \tilde{F}_{x_1x_3^2}\tilde{F}_{x_2x_3^2} + 4\tilde{F}_{x_3^2}(\tilde{F}_{x_1x_3^2}\tilde{F}_{x_2x_3} + \tilde{F}_{x_2x_3^2}\tilde{F}_{x_1x_3}) + \tilde{F}_{x_3^3}(\tilde{F}_{x_1x_3^2} + 2\tilde{F}_{x_1x_3}\tilde{F}_{x_2x_3} + \tilde{F}_{x_2^2x_3}) = 6\tilde{F}_{x_3^2} + 3\tilde{F}_{x_3^3} \\ 2\tilde{F}_{x_1x_3^2}\tilde{F}_{x_2x_3^2} + \tilde{F}_{x_1x_3^2}\tilde{F}_{x_2x_3} + \tilde{F}_{x_2^2x_3}\tilde{F}_{x_1x_3} = 3\tilde{F}_{x_3^2} \\ \tilde{F}_{x_1x_3^2}\tilde{F}_{x_2x_3^2} = \tilde{F}_{x_3^3} \end{cases}$$

which, solved by substitution from the last equation, implies $\tilde{F}_{x_3^2} = \frac{1}{2}$ and then, from the first equation and by the previous system, $\tilde{F}_{x_1x_3} = 0$, which contradicts $(1, 0, 1) \in \Delta \cap \mathbb{Z}^n$.

Case (17) : by calculating the derivatives of both sides of equation (8) with respect to x_4 one gets the following system

$$\begin{cases} \tilde{F}_{x_1x_4^2}\tilde{F}_{x_2x_4^2} = 4 \\ \tilde{F}_{x_2x_4^2}\tilde{F}_{x_1x_4^2} + \tilde{F}_{x_1x_4}\tilde{F}_{x_2x_4^2} = 8 \\ \tilde{F}_{x_1x_4^2} + 2\tilde{F}_{x_1x_4}\tilde{F}_{x_2x_4} + \tilde{F}_{x_2x_4^2} = 12 \\ \tilde{F}_{x_1x_4} + \tilde{F}_{x_2x_4} = 4 \end{cases}$$

which is seen to have as unique solution $\tilde{F}_{x_1x_4} = \tilde{F}_{x_2x_4} = \tilde{F}_{x_1x_4^2} = \tilde{F}_{x_2x_4^2} = 2$.

By the symmetry of the equations defining the polytope, we also get $\tilde{F}_{x_1x_3} = \tilde{F}_{x_1x_3^2} = \tilde{F}_{x_2x_3} = \tilde{F}_{x_2x_3^2} = \tilde{F}_{x_1^2x_3} = \tilde{F}_{x_1^2x_4} = \tilde{F}_{x_2^2x_3} = \tilde{F}_{x_2^2x_4} = 2$.

By considering the $\frac{\partial}{\partial x_1}, \frac{\partial^2}{\partial x_1\partial x_2}, \frac{\partial^3}{\partial x_1^2\partial x_2}, \frac{\partial^4}{\partial x_1^2\partial x_2^2}$ derivatives of both sides of equation (8) and taking into account the above data, we get the equations $\tilde{F}_{x_1x_2x_3} + \tilde{F}_{x_1x_2x_4} = 4, \tilde{F}_{x_1x_2x_3}\tilde{F}_{x_1x_2x_4} = 4$ which immediately yield $\tilde{F}_{x_1x_2x_3} = \tilde{F}_{x_1x_2x_4} = 2$. Again by the symmetry of the equations of the polytope, we have also $\tilde{F}_{x_1x_3x_4} = \tilde{F}_{x_2x_3x_4} = 2$.

Now, notice that $\Delta \cap \{y = z = 0\}$ gives the 2-dimensional polytope of Case (15). It follows that we can use the calculations made in that case in order to get the $\frac{\partial^2}{\partial x_1\partial x_4}$ derivative of both sides of equation (8), evaluated at $x_2 = x_3 = 0$. By replacing in the result obtained the values found above, one easily gets a contradiction.

Case (18) : by calculating the derivatives of both sides of equation (8) with respect to x_3 one gets the following system

$$\begin{cases} \tilde{F}_{x_1x_3}\tilde{F}_{x_2x_3}\tilde{F}_{x_3^2x_4} = 2 \\ 2\tilde{F}_{x_1x_3}\tilde{F}_{x_2x_3}\tilde{F}_{x_3x_4} + (\tilde{F}_{x_1x_3} + \tilde{F}_{x_2x_3})\tilde{F}_{x_3^2x_4} = 8 \\ 2\tilde{F}_{x_1x_3}\tilde{F}_{x_2x_3} + 2\tilde{F}_{x_1x_3}\tilde{F}_{x_3x_4} + 2\tilde{F}_{x_2x_3}\tilde{F}_{x_3x_4} + \tilde{F}_{x_3^2x_4} = 12 \\ \tilde{F}_{x_1x_3} + \tilde{F}_{x_2x_3} + \tilde{F}_{x_3x_4} = 4 \end{cases}$$

which is seen to have as unique solution $\tilde{F}_{x_1x_3} = \tilde{F}_{x_2x_3} = 1, \tilde{F}_{x_3x_4} = \tilde{F}_{x_3^2x_4} = 2$. By exchanging the role of x_3 and x_4 , one finds an analogous system from which one similarly gets $\tilde{F}_{x_1x_4} = \tilde{F}_{x_2x_4} = 1, \tilde{F}_{x_3x_4} = \tilde{F}_{x_3x_4^2} = 2$. By substituting these data into the equations $\tilde{F}_{x_1x_2} + \tilde{F}_{x_1x_3} + \tilde{F}_{x_1x_4} = 4, 2\tilde{F}_{x_2^2} + \tilde{F}_{x_1x_2} + \tilde{F}_{x_2x_3} + \tilde{F}_{x_2x_4} = 4$ (which arise from equation (8) derivated with respect to x_1 and x_2 respectively) one gets $\tilde{F}_{x_2^2} = 0$, which contradicts $(0, 2, 0, 0) \in \Delta \cap \mathbb{Z}^n$.

Case (19) : as in the previous case, by calculating the derivatives of both sides of equation (8) with respect to x_1 one gets the system

$$\begin{cases} \tilde{F}_{x_1x_2}\tilde{F}_{x_1x_4}\tilde{F}_{x_1^2x_3} = 2 \\ 2\tilde{F}_{x_1x_2}\tilde{F}_{x_1x_3}\tilde{F}_{x_1x_4} + (\tilde{F}_{x_1x_2} + \tilde{F}_{x_1x_4})\tilde{F}_{x_1^2x_3} = 8 \\ 2\tilde{F}_{x_1x_2}\tilde{F}_{x_1x_3} + 2\tilde{F}_{x_1x_2}\tilde{F}_{x_1x_4} + 2\tilde{F}_{x_1x_4}\tilde{F}_{x_1x_3} + \tilde{F}_{x_1^2x_3} = 12 \\ \tilde{F}_{x_1x_2} + \tilde{F}_{x_1x_3} + \tilde{F}_{x_1x_4} = 4 \end{cases}$$

which is seen as above to have as unique solution $\tilde{F}_{x_1x_2} = \tilde{F}_{x_1x_4} = 1, \tilde{F}_{x_1x_3} = \tilde{F}_{x_1^2x_3} = 2$. By exchanging the roles of x_3 and x_4 and of x_1 and x_2 one finds an analogous system from which one similarly gets $\tilde{F}_{x_2x_3} = \tilde{F}_{x_1x_2} = 1, \tilde{F}_{x_2x_4} = \tilde{F}_{x_2^2x_4} = 2$. By substituting these data into the equation $\tilde{F}_{x_1x_3} + \tilde{F}_{x_2x_3} = 4$ (which arises from equation (8) derivated with respect to x_3) one gets a contradiction.

In order to conclude the proof of Proposition 4.2, we need to consider the general case when the embedding inducing the Kähler-Einstein metric is given by a basis of the space $H^0((K^*)^\alpha)$ of the global sections of a power $(K^*)^\alpha$ of the anticanonical bundle K^* . From the theory of toric manifolds it is known that a polytope representing $(K^*)^\alpha$ is obtained from the polytope Δ representing K^* by applying the homothety of constant α . Now, in each of the cases (15)-(19) one sees that Δ has e_i as vertex, for some $i = 1, \dots, n$, so $\alpha\Delta$ is an integral polytope if and only if α is a positive integer (this means that K^* is indivisible). As above, we have that any projectively induced Kähler-Einstein metric representing $c_1((K^*)^\alpha)$ writes in the coordinates z_1, \dots, z_n as $\frac{i}{2}\partial\bar{\partial}G$, where $G = G(x_1, \dots, x_n)$ is now a polynomial with positive coefficients in the monomials $x_1^{i_1} \cdots x_n^{i_n}$, $(i_1, \dots, i_n) \in \alpha\Delta \cap \mathbb{Z}^n$. Since the Einstein equation reads now $\rho_\omega = \frac{2}{\alpha}\omega$, arguing as above one gets the following equation (with notations as above)

$$\det(A) = cG^{2n-\frac{1}{\alpha}}, \quad A_{ij} = (GG_{ij} - G_iG_j)\bar{z}_iz_j + GG_j\delta_{ij} \quad (21)$$

which, G being a polynomial, cannot be true unless $G = F^\alpha$ for some polynomial $F = F(x_1, \dots, x_n)$. So $\frac{i}{2}\partial\bar{\partial}G = \alpha\frac{i}{2}\partial\bar{\partial}F$, and $\frac{i}{2}\partial\bar{\partial}F$ is a projectively induced, Kähler-Einstein metric belonging to $c_1(K^*)$, which contradicts what we have obtained in the first part of the proof. \square

We can now prove Theorem 1.2.

Proof of Theorem 1.2. It is known that the polarized manifold (M, L) is asymptotically Chow polystable and hence the set $\mathcal{B}_c(L)$ has infinite elements. For the reader's convenience, let us outline a proof here. By the Futaki's reformulation of the results proved in [31], (M, L) is asymptotically Chow polystable provided the Lie algebra characters $F_i : \text{Lie}(\text{Aut}(M)) \rightarrow \mathbb{C}$, $i = 1, \dots, n$, introduced in [16], vanish (see also [17]). Now, the Kähler-Einstein toric n -dimensional manifolds, $n \leq 4$, are *symmetric*, i.e. the only character on the algebraic torus $G = (\mathbb{C}^*)^n \subseteq \text{Aut}(M)$ which is invariant by the action by conjugation of the normalizer $N(G)$ of G in $\text{Aut}(M)$ is the trivial one (see, for example, [6]). One then sees that the restrictions of the F_i 's to the Lie algebra $\text{Lie}(G)$ of G must vanish. By Matsushima's results on Kähler-Einstein manifolds, $\text{Aut}(M)$ is reductive so that $\text{Lie}(\text{Aut}(M)) = [\text{Lie}(\text{Aut}(M)), \text{Lie}(\text{Aut}(M))] + Z(\text{Lie}(\text{Aut}(M)))$, where $Z(\text{Lie}(\text{Aut}(M)))$ denotes the center of $\text{Lie}(\text{Aut}(M))$. So, by $Z(\text{Lie}(\text{Aut}(M))) \subseteq \text{Lie}(G)$ (G is a maximal torus) and the fact that the F_i 's are Lie algebra characters, we see that the F_i 's vanish on all $\text{Lie}(\text{Aut}(M))$ and this proves the claim by Futaki's above-mentioned result.

Now, let $g_B \in \mathcal{B}(L)$. If M is either a projective space or the product of projective spaces then it is not hard to see that the cardinality of \mathcal{B}_{g_B} is infinite (see Section 4 in [2] for a proof). Otherwise, it follows by Proposition 4.2 and part (1) of Lemma 2.5 that \mathcal{B}_{g_B} consists of a finite numbers of balanced metrics. \square

Remark 4.3. The main difficulty in extending this theorem to arbitrary dimensions is that there is not a classification of toric manifolds for $n \geq 5$.

5. THE CONSTANT SCALAR CURVATURE CASE

Recall the following from [3] and [4]:

Theorem 5.1. *Let (M, g, ω) be a constant scalar curvature Kähler metric on a compact n -dimensional complex manifold. Assume that there is no nonzero holomorphic vector field vanishing somewhere on M . Then, given finitely many points $p_1, \dots, p_k \in M$ and positive numbers $b_1, \dots, b_k > 0$, there exists $\varepsilon_0 > 0$ such that the blow up $\tilde{M} = \text{Bl}_{p_1, \dots, p_k} M$ of M at p_1, \dots, p_k carries constant scalar curvature Kähler forms*

$$\omega_\varepsilon \in \pi^*[\omega] - \varepsilon^2(b_1 PD[E_1] + \dots + b_k PD[E_k]),$$

where the $PD[E_j]$ are the Poincaré duals of the $(2n-2)$ -homology classes of the exceptional divisors of the blow up at p_j and $\varepsilon \in (0, \varepsilon_0)$. Moreover, the sequence of metrics g_ε converges to g in $C^\infty(M \setminus \{p_1, \dots, p_k\})$.

If the scalar curvature of g is not zero then the scalar curvatures of g_ε (the metric associated to ω_ε) and of g have the same signs.

Let us denote by \mathfrak{h} the space of hamiltonian holomorphic vector fields and by

$$\xi_\omega : M \mapsto \mathfrak{h}^*$$

the *moment* map which is defined by requiring that if $\Xi \in \mathfrak{h}$, the function $\zeta_\omega := \langle \xi_\omega, \Xi \rangle$ is a (complex valued) Hamiltonian for the vector field Ξ , namely the unique solution of

$$-\bar{\partial}\zeta_\omega = \frac{1}{2}\omega(\Xi, -),$$

which is normalized by

$$\int_M \zeta_\omega d\text{vol}_g = 0.$$

With these notations, the result obtained in [4] reads :

Theorem 5.2. *Assume that (M, g, ω) is a cscK compact complex manifold and that $p_1, \dots, p_k \in M$ and $b_1, \dots, b_k > 0$ are chosen so that :*

- (i) $\xi_\omega(p_1), \dots, \xi_\omega(p_k)$ span \mathfrak{h}^*
- (ii) $\sum_{j=1}^k b_j^{n-1} \xi_\omega(p_j) = 0 \in \mathfrak{h}^*$.

Then there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, there exists on $\tilde{M} = Bl_{p_1, \dots, p_k} M$ a cscK metric g_ε associated to the Kähler form

$$\omega_\varepsilon \in \pi^*[\omega] - \varepsilon^2 (b_{1,\varepsilon} PD[E_1] + \dots + b_{k,\varepsilon} PD[E_k]),$$

where

$$|b_{j,\varepsilon} - b_j| \leq c \varepsilon^{\frac{2}{2n+1}}. \quad (22)$$

Moreover, the sequence of metrics g_ε converges to g in $C^\infty(M \setminus \{p_1, \dots, p_k\})$.

Therefore, in the presence of nontrivial hamiltonian holomorphic vector fields, the number of points which can be blown up, their position, as well as the possible Kähler classes on the blown up manifold have to satisfy some constraints.

Despite the fact we do not know explicitly the metrics given by the above constructions, we want to show, that, at least for ε sufficiently small, they cannot have the second coefficient a_2 in the TYZ expansion equal to a constant.

Proposition 5.3. *For any family of metrics g_ε constructed either by Theorem 5.1 or Theorem 5.2, there exists $\varepsilon_1 > 0$ such that the coefficient $a_2(g_\varepsilon)$ in TYZ expansion (see (7) above) is not constant for $\varepsilon < \varepsilon_1$.*

Proof. We first argue by contradiction on a_2 : suppose there exists a sequence $\varepsilon_j \rightarrow 0$ s.t. $a_2(g_{\varepsilon_j})$ are indeed constants. Being cscK metrics this is equivalent to say that $|R_{g_{\varepsilon_j}}|^2 - 4|Ric_{g_{\varepsilon_j}}|^2 = C_{\varepsilon_j}$, for some constant C_{ε_j} . Scaling the metrics g_ε around a point on any of the exceptional divisors by a factor $\frac{1}{\varepsilon^2}$, we know that $\frac{1}{\varepsilon^2}g_\varepsilon$ converges smoothly (as $\varepsilon \rightarrow 0$) to the LeBrun-Simanca g_{lbs} metric on $Bl_0\mathbb{C}^n$. In particular we would have $|R_{g_{lbs}}|^2 - 4|Ric_{g_{lbs}}|^2 = \lim_{\varepsilon_j \rightarrow 0} \varepsilon_j^4 C_{\varepsilon_j}$. This would imply that $|R_{g_{lbs}}|^2 - 4|Ric_{g_{lbs}}|^2$ is constant on $Bl_0\mathbb{C}^n$, contradicting the following lemma. \square

Lemma 5.4. *Having called (v_1, \dots, v_n) the standard euclidean coordinates on $Bl_0\mathbb{C}^n \setminus K$, for some compact K containing the exceptional divisor, then*

$$(|R_{g_{lbs}}|^2 - 4|Ric_{g_{lbs}}|^2)(v_1, 0) = -\frac{113}{|v_1|^8} + o(|v_1|^{-8}) \quad (23)$$

for $n = 2$ as v_1 goes to infinity, and

$$(|R_{g_{lbs}}|^2 - 4|Ric_{g_{lbs}}|^2)(v_1, 0, \dots, 0) = -\frac{a}{|v_1|^{4n}} + o(|v_1|^{-4n}) \quad (24)$$

with $a > 0$, for $n \geq 3$ as v_1 goes to infinity.

Proof. Recall the following properties of the g_{lbs} : by construction, the Kähler form ω_{lbs} is invariant under the action of $U(n)$. If $v = (v_1, \dots, v_n)$ are complex coordinates in $\mathbb{C}^n - \{0\}$, the Kähler form ω_{lbs} can be written as

$$\omega_{lbs} := \frac{i}{2} \partial \bar{\partial} \left(\frac{1}{2} |v|^2 + E_n(|v|) \right) \quad (25)$$

More precisely

$$\omega_{lbs} = \frac{i}{2} \partial \bar{\partial} \left(\frac{1}{2} |v|^2 + \log |v| \right) \quad (26)$$

in dimension $n = 2$. In dimension $n \geq 3$, even though there is no explicit formula, we have

$$\omega_{lbs} = \frac{i}{2} \partial \bar{\partial} \left(\frac{1}{2} |v|^2 - |v|^{4-2n} + o(|v|^{2-2n}) \right) \quad (27)$$

Having in complex dimension 2 an explicit Kähler potential, the proof of the theorem reduces to estimating the relevant quantities. For this purpose we get an explicit formula for the matrix G_{lbs} which represents the metric g_{lbs} , namely:

$$G_{lbs} = \frac{1}{|v|^4} \begin{pmatrix} 1 + |v_1|^2 + |v|^4 & \bar{v}_1 v_2 \\ v_1 \bar{v}_2 & 1 + |v_2|^2 + |v|^4 \end{pmatrix}.$$

Observe that $\det(\omega_{lbs}) = 1 + \frac{1}{|v|^2} + \text{higher order terms}$, as $|v|$ goes to infinity. From now on we will write $f \simeq h$ for two functions (or two matrixes) which agree up to higher order terms as $|v|$ goes to infinity.

Clearly $G_{lbs}^{-1} \simeq Id$. A straightforward computation now shows that

$$\frac{\partial g_{i\bar{l}}}{\partial v_k} = \bar{v}_l |v|^{-4} \delta_{ik} - 2|v|^{-6} \bar{v}_k (\delta_{il} + v_i \bar{v}_l)$$

and

$$\begin{aligned} \frac{\partial^2 g_{i\bar{l}}}{\partial v_k \partial \bar{v}_j} &= |v|^{-4} \delta_{lj} \delta_{ik} - 2|v|^{-6} (\delta_{il} \delta_{kj} + \bar{v}_l v_j \delta_{ik} + \bar{v}_k v_i \delta_{lj} + \bar{v}_l v_i \delta_{kj}) + \\ &\quad + 6|v|^{-8} (\bar{v}_k v_j \delta_{il} + \bar{v}_l \bar{v}_k v_i v_j) . \end{aligned}$$

Hence, restricting on the complex line $v_2 = 0$, we get that the only non zero contributions among the second derivatives of the metric are given by

$$\begin{aligned} \frac{\partial^2 g_{1\bar{1}}}{\partial v_1 \partial \bar{v}_1} &= -\frac{5}{|v_1|^4}, \\ \frac{\partial^2 g_{2\bar{1}}}{\partial v_1 \partial \bar{v}_2} &= -\frac{1}{|v_1|^4}, \\ \frac{\partial^2 g_{1\bar{1}}}{\partial v_2 \partial \bar{v}_2} &= -\frac{2}{|v_1|^4}, \\ \frac{\partial^2 g_{2\bar{2}}}{\partial v_2 \partial \bar{v}_2} &= \frac{1}{|v_1|^4}, \end{aligned}$$

Now recall that

$$|R|^2 = \sum_{i,j,k,l,p,q,r,s=1}^n \overline{g^{i\bar{p}} g^{j\bar{q}} g^{k\bar{r}} g^{l\bar{s}}} R_{i\bar{j}k\bar{l}} \overline{R_{p\bar{q}r\bar{s}}} \quad (28)$$

and

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{l}}}{\partial v_k \partial \bar{v}_j} + \sum_{p,q=1}^n g^{p\bar{q}} \frac{\partial g_{i\bar{p}}}{\partial v_k} \frac{\partial g_{q\bar{l}}}{\partial \bar{v}_j}.$$

This readily implies that the lowest order terms of the contributions of the Riemann tensor are

$$\begin{aligned} R_{1\bar{1}1\bar{1}} &\simeq \frac{5}{|v_1|^4}, \\ R_{1\bar{2}2\bar{1}} &\simeq \frac{2}{|v_1|^4}, \\ R_{1\bar{2}1\bar{2}} &\simeq \frac{1}{|v_1|^4}, \\ R_{2\bar{2}2\bar{2}} &\simeq -\frac{1}{|v_1|^4}, \end{aligned}$$

hence $|R|^2 \simeq \frac{31}{|v_1|^8}$ on the line $v_2 = 0$. Recall that

$$\begin{aligned} |Ric|^2 &= \sum_{i,j,k,l=1}^n \overline{g^{i\bar{k}} g^{j\bar{l}}} Ric_{i\bar{j}} \overline{Ric_{k\bar{l}}} \quad \text{and} \\ Ric_{k\bar{p}} &= g^{i\bar{l}} R_{i\bar{l}k\bar{p}}. \end{aligned} \quad (29)$$

Hence, restricting on the complex line $v_2 = 0$, we get

$$|Ric|^2 \simeq Ric_{1\bar{1}} \overline{Ric_{1\bar{1}}} + Ric_{2\bar{2}} \overline{Ric_{2\bar{2}}} \simeq \frac{36}{|v_1|^8}.$$

In summary we have proved that

$$a_2(g_{lbs})|_{v_2=0} = [|R|^2 - 4|Ric|^2]|_{v_2=0} \simeq -\frac{113}{|v_1|^8}$$

which is exactly (23).

Not having an explicit expression for the LeBrun-Simanca metric when $n > 2$ we can give only estimates instead of precise formulae, though the line of the argument will be the same as in complex dimension 2.

This time

$$\omega_{lbs} \simeq (1 + |v|^{2-2n}\delta_{ij} + v_i\bar{v}_j|v|^{-2n})dv_i \wedge d\bar{v}_j,$$

$$\frac{\partial g_{i\bar{p}}}{\partial v_k} \simeq (1-n)\bar{v}_k|v|^{-2n}\delta_{ip} + |v|^{-2n}\bar{v}_p\delta_{ik} - nv_i\bar{v}_p\bar{v}_k|v|^{-2-2n}$$

and

$$\begin{aligned} \frac{\partial^2 g_{i\bar{p}}}{\partial v_k \partial \bar{v}_j} &\simeq |v|^{-2n}[(1-n)\delta_{kj}\delta_{il} + \delta_{jl}\delta_{ik}] + |v|^{-2-2n}[-n(1-n)v_j\bar{v}_k\delta_{il} - \\ &n\bar{v}_l v_j\delta_{ik} - nv_i\bar{v}_k\delta_{jl} - nv_i\bar{v}_l\delta_{kj}] + |v|^{-4-2n}[n(1+n)v_i v_j \bar{v}_l \bar{v}_k]. \end{aligned}$$

Hence, restricting on the complex line $v_2 = \dots = v_n = 0$, we get that the only non zero contributions among the second derivatives of the metric are given by

$$\begin{aligned} \frac{\partial^2 g_{1\bar{1}}}{\partial v_1 \partial \bar{v}_1} &\simeq \frac{2(n-1)^2}{|v_1|^{2n}}, \\ \frac{\partial^2 g_{1\bar{r}}}{\partial v_1 \partial \bar{v}_r} &\simeq \frac{1-n}{|v_1|^{2n}}, \\ \frac{\partial^2 g_{1\bar{1}}}{\partial v_r \partial \bar{v}_r} &\simeq \frac{1-2n}{|v_1|^{2n}}, \\ \frac{\partial^2 g_{r\bar{r}}}{\partial v_1 \partial \bar{v}_1} &\simeq \frac{(1-n)^2}{|v_1|^{2n}}, \\ \frac{\partial^2 g_{r\bar{r}}}{\partial v_r \partial \bar{v}_r} &\simeq \frac{2-n}{|v_1|^{2n}}, \end{aligned}$$

where r is an index ranging from 2 to n . This immediately gives, on the line v_1 ,

$$|R|^2 \simeq |v_1|^{-4n}[4(n-1)^4 + (n-1)(1-2n)^2 + (1-n)^2 + (1-n)^2 + (2-n)^2],$$

and

$$|Ric|^2 \simeq |v_1|^{-4n}[-4(n-1)^4 + (n-1)(2-n)^2]$$

hence

$$\begin{aligned} |R|^2 - 4|Ric|^2 &\simeq \\ &\simeq |v_1|^{-4n}\{(n-1)[-12(n-1)^3 + (1-2n)^2 + 2(1-n) - 4(2-n)^2] + (2-n)^2\} \end{aligned}$$

which is readily seen to be negative for $n \geq 3$. This ends the proof of claim (24) in all dimensions. \square

We are now in the position to prove Theorem 1.3. which in the previous notation reads:

Theorem 5.5. *Let $\varepsilon = \frac{p}{q} < \varepsilon_1$ with ε_1 as in Proposition 5.3 and let $L_\varepsilon \rightarrow M$ be a polarization for the Kähler class of the metric qg_ε . Then, for each $g_B \in \mathcal{B}(L_\varepsilon)$, the set \mathcal{B}_{g_B} is finite.*

Proof. The proof follows by combining Proposition 5.3 with part (2) of Lemma 2.5. \square

Remark 5.6. We do not know if in the previous theorem \mathcal{B}_{g_B} is empty. For example, in [13] it is shown that there exist cscK polarizations L on the blow up M of $\mathbb{C}P^2$ at four points (all but one aligned) constructed from Theorem 5.2 such that (M, L^m) is not asymptotically Chow polystable for m large enough, so that $\mathcal{B}_c(L)$ is finite. One could try to use part (1) instead of part (2) of Lemma 2.5 to prove Theorem 5.5, namely to show that mg_ε is not projectively induced for all m .

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