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Symplectic coordinates on Kähler manifolds: PART II

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Aim: *In this second part of the talk we show how to construct global symplectic coordinates on HSSNT and on a dense subset of HSSCT. Moreover, we use these coordinates to compute the Gromov width of HSSCT.*

DEFINITION OF HSSNT

An HSSNT (M, ω) is a Kähler manifold, which is holomorphically isometric to a bounded symmetric domain $(M, 0)$ of $M \subset \mathbb{C}^n$ centered at the origin $0 \in \mathbb{C}^n$ equipped with a multiple of the Bergman metric ω_B such that for all $p \in M$ the geodesic symmetry:

$$s_p : \mathrm{exp}_p(v) \mapsto \mathrm{exp}_p(-v), \forall v \in T_p M$$

is a globally defined holomorphic isometry of M.

An HSSNT is a homogenous Kähler manifold (converse not true Pyateskii–Shapiro).

There is a complete classification of irreducible HSSNT, with four classical series, studied by Cartan, and two exceptional cases.

A BASIC EXAMPLE: THE FIRST CARTAN DOMAIN (1)

Let

$$D_I[n] = \{Z \in M_n(\mathbb{C}) \mid I_n - ZZ^* > 0\}$$

be the first Cartan domain equipped with the hyperbolic form

$$\omega_{hyp} = -\frac{i}{2}\partial\bar{\partial} \log \det(I_n - ZZ^*)$$

The compact dual of $D_I[n]$ is $\mathrm{Grass}_n(\mathbb{C}^{2n})$ endowed with the Fubini-Study form

$$\omega_{FS} = P^* \omega_{FS}$$

obtained as follows:

$$D_I[n] \subset M_n(\mathbb{C}) = \mathbb{C}^{n^2} \subset \mathrm{Grass}_n(\mathbb{C}^{2n}) \stackrel{P=Plucker}{\hookrightarrow} \mathbb{C}P^N,$$

$N = \binom{2n}{n} - 1.$

A BASIC EXAMPLE: THE FIRST CARTAN DOMAIN (2)

Theorem: The map

$$\Psi : D_I[n] \to M_n(\mathbb{C}) = \mathbb{C}^{n^2}$$

defined by

$$\Psi(Z) = (I_n - ZZ^*)^{-\frac{1}{2}} Z$$

is a diffeomorphism. Its inverse is given by

$$\Psi^{-1} : \mathbb{C}^{n^2} \to D_I[n],\ X \mapsto (I_n + XX^*)^{-\frac{1}{2}} X.$$

Moreover, Ψ is a *symplectic duality* namely,

$$\boxed{\Psi^*\omega_0 = \omega_{hyp}} \qquad \boxed{\Psi^*\omega_{FS} = \omega_0}$$

where

$$\boxed{\omega_0 = \frac{i}{2}\partial\bar{\partial}\, \mathrm{tr}(ZZ^*)}$$

and

$$\boxed{\omega_{FS} = \frac{i}{2}\partial\bar{\partial} \log\det(I_n + ZZ^*) \text{ on } \mathbb{C}^{n^2} \subset \mathrm{Grass}_n(\mathbb{C}^{2n})}$$

JORDAN TRIPLE SYSTEMS

A **Hermitian Jordan triple system** is a pair $(\mathcal{M}, \{,\ ,\})$, where $\mathcal{M}$ is a complex vector space and $\{,\ ,\}$ is a $\mathbb{R}$-trilinear map

$$\{,\ ,\} : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}, (u, v, w) \mapsto \{u, v, w\}$$

$\mathbb{C}$-bilinear and simmetric in u and w and $\mathbb{C}$-antilinear in v and satisfying the **Jordan identity**:

$$\{x, y, \{u, v, w\}\} - \{u, v, \{x, y, w\}\} =$$

$$= \{\{x, y, u\}, v, w\} - \{u, \{v, x, y\}, w\}.$$

Let $u, v \in \mathcal{M}$, and let $D(u, v) : \mathcal{M} \to \mathcal{M}$ be the operator on $\mathcal{M}$ defined by

$$\boxed{\text{D(u, v)(w)=\{u, v, w\}}}$$

A HJTS is called **positive** and we write **HPJTS** if

$$(u, v) \mapsto \operatorname{tr} D(u, v)$$

is positive definite.

The *quadratic representation*

$$Q : \mathcal{M} \to End(\mathcal{M})$$

is defined by

$$2Q(u)(v) = \{u, v, u\}, \;\; u, v \in \mathcal{M}.$$

The **Bergman operator**

$$B(u, v) : \mathcal{M} \to \mathcal{M}$$

is given by the equation

$$\boxed{B(u, v) = Id_{\mathcal{M}} - D(u, v) + Q(u)Q(v)}$$

HPJTS $\longrightarrow$ HSSNT

$(\mathcal{M}, \{,\ ,\}) \longrightarrow (M, 0) = \{u \in \mathcal{M} \mid B(u,u) >> 0\}_0$, where "$>>$" means positive definite w.r.t. $(u,v) \mapsto \operatorname{tr} D(u,v)$.

The **Bergman form** ω_{Berg} of M is defined as:

$$\omega_{Berg} = -\frac{i}{2}\partial\bar{\partial} \log \det B.$$

We also define (in the irreducible case)

$$\boxed{\omega_{hyp} = -\tfrac{i}{2g}\partial\bar{\partial} \log \det B(z,z)}$$

HSSNT $\longrightarrow$ HPJTS

$(M, 0) \longrightarrow (\mathcal{M} = T_0M, \{,\ ,\})$, where

$$\boxed{\{u, v, w\} = -\frac{1}{2}\left(R_0(u, v)w + JR_0(u, Jv)w\right)}$$

(W. Bertram, Lectures Notes in Math. N. 1754)

THE FIRST CARTAN DOMAIN AS HPJTS

Let $\mathcal{M} = M_n(\mathbb{C})$ with the triple product

$$\{u, v, w\} = uv^*w + wv^*u, \ \ u, v, w \in M_n(\mathbb{C})$$

$$\operatorname{tr} D(u, u) = \operatorname{tr}(uu^*)$$

$$B(u, v)(w) = (I_n - uv^*)w(I_n - v^*u)$$

The HSSNT $(M, 0)$ associated to $(M_n(\mathbb{C}), \{,,\})$ is the first Cartan domain

$$D_I[n] = \{Z \in M_n(\mathbb{C}) \mid I_n - ZZ^* > 0\}$$

$$\omega_{hyp} = \frac{\omega_{Berg}}{2n} = -\frac{i}{2}\partial\bar{\partial} \log \det(I_n - ZZ^*)$$

COMPACTIFICATIONS OF HPJTS

Let (M, ω_{hyp}) be an HSSNT and let (M^*, ω_{FS}) be its compact dual equipped with the Fubini–Study form ω_{FS}.

More precisely, one has the following inclusions:

$$(M, 0) \overset{Harish-Chandra}{\subset} \mathcal{M} = T_0M \cong \mathbb{C}^n \overset{Borel}{\subset} M^* \overset{BW}{\hookrightarrow} \mathbb{C}P^N$$

and we set

$$\omega_{FS} = \mathrm{BW}^* \omega_{FS}.$$

Remark: The local expression of ω_{FS} restricted to $\mathcal{M}$ is given (in the irreducible case) by

$$\boxed{\omega_{FS} = \frac{i}{2g} \partial\bar{\partial} \log \det B(z, -z)}$$

$$\boxed{\omega_{hyp} = -\frac{i}{2g} \partial\bar{\partial} \log \det B(z, z)}$$

Theorem (A. J. Di Scala – L. 2008): Let (M, ω_{hyp}) be an HSSNT and (M^*, ω_{FS}) its compact dual. Then the map

$$\boxed{\Psi_M : M \to \mathcal{M} \cong \mathbb{C}^n \subset M^*,\ \ z \mapsto B(z,z)^{-\frac{1}{4}} z}$$

satisfies the following properties:

(D) Ψ_M is a **diffeomorphism** and its inverse is given by

$$\boxed{\Psi_M^{-1} : \mathcal{M} \cong \mathbb{C}^n \subset M^* \to M,\ \ z \mapsto B(z,-z)^{-\frac{1}{4}} z}$$

(S) Ψ_M is a **simplectic duality**, i.e.:

$$\boxed{\Psi_M^* \omega_0 = \omega_{hyp}} \qquad \boxed{\Psi_M^* \omega_{FS} = \omega_0}$$

where $\omega_0 = \frac{i}{2}\partial\bar{\partial}\,\mathrm{tr}\, D(u,u)$ is the flat Kähler form on $\mathcal{M}$.

(H) the map

$$\Psi : HSSNT \to \mathrm{Diff}_0(M, \mathcal{M}),\ M \mapsto \Psi_M$$

is **hereditary**, i.e.: for all $(T, 0) \overset{i}{\hookrightarrow} (M, 0)$ complete, complex and totally geodesic submanifold one has

$$\Psi_M\,|_T = \Psi_T,\ \Psi_M(T) = \mathcal{T} \subset \mathcal{M},$$

where $\mathcal{T}$ is the HPJTS associated to T.

Idea of the proof: show the result for classical Cartan's domains and then use Jordan algebras to extend the result to the exceptional domains.

Remark: One can give an alternative proof of the theorem (A. J. Di Scala, G. Roos, L., *The bisymplectomorphism group of a bounded symmetric domain*, 2008) using spectral decomposition tools of HPJTS.

THE GROMOV WIDTH

The Gromov width (1985) of a $2n$-dimensional symplectic manifold (M, ω) is defined as

$$c_G(M, \omega) = \text{sup}\{\pi r^2 \mid B^{2n}(r) \text{ symplectically embeds into } (M, \omega)\},$$

where

$$B^{2n}(r) = \{(x, y) \in \mathbb{R}^{2n} \mid \sum_{j=1}^{n} x_j^2 + y_j^2 < r^2\}$$

is the open ball of radius r in $(\mathbb{R}^{2n}, \omega_0)$.

Remarks

1. $c_G > 0$ by Darboux theorem.

2. M compact $\Rightarrow c_G(M, \omega) < \infty$.

SYMPLECTIC CAPACITIES

A map c from the class of all symplectic manifolds of dimension $2n$ to $[0, +\infty]$ is called a *symplectic capacity* (H. Hofer, E. Zehnder, 1990) if it satisfies the following conditions:

- (**monotonicity**) if there exists a symplectic embedding $(M_1, \omega_1) \to (M_2, \omega_2)$ then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$;
- (**conformality**) $c(M, \lambda\omega) = |\lambda| c(M, \omega)$, for every $\lambda \in \mathbb{R} \setminus \{0\}$;
- (**nontriviality**) $c(B^{2n}(r), \omega_0) = \pi r^2 = c(Z^{2n}(r), \omega_0)$,

where

$$Z^{2n}(r) = B^2(r) \times \mathbb{R}^{2n-2} = \{(x, y) \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 < r^2\}.$$

REMARKS ON SYMPLECTIC CAPACITIES

- When $n = 1$ (2-dimensional symplectic manifolds)
$$c(M,\omega) := |\int_M \omega|$$
defines a symplecitc capacity which agrees with the Lebesgue measure in $(\mathbb{R}^2, \omega_0)$.

- In contrast, when $n > 1$, $c(M,\omega) := \left(\int_M \frac{\omega^n}{n!}\right)^{\frac{1}{n}}$ does not define a symplectic capacity since $Z^{2n}(r)$ has infinite volume.

- (monotonicity) $\Rightarrow$ symplectic capacities are symplectic invariants.

- for every open set $U \subset \mathbb{R}^n$ such that $B^{2n}(r) \subset U \subset Z^{2n}(r) \Rightarrow c(U) = \pi r^2$.

- It is hard to prove the existence of a symplectic capacity.

THE GROMOV WIDTH AS A SYMPLECTIC CAPACITY

Theorem The Gromov width c_G is a symplectic capacity. Moreover

$$c_G(M, \omega) \leq c(M, \omega)$$

for every capacity c.

The (nontriviality) for c_G, i.e. $c_G(B^{2n}(r), \omega_0) = \pi r^2 = c_G(Z^{2n}(r), \omega_0)$, follows by the celebrated:

Theorem (Gromov nonsqueezing theorem) There exists a symplectic embedding $B^{2n}(r) \hookrightarrow Z^{2n}(R)$ iff $r \leq R$ (Gromov, 1985).

Remark Assuming the existence of any symplectic capacity c one easily deduces Gromov's nonsqueezing theorem. Indeed, let $\varphi : B^{2n}(r) \to Z^{2n}(R)$ be a symplectic embedding. Then (monotonicity)+(nontriviality) $\Rightarrow$

$$\pi r^2 = c(B^{2n}(r), \omega_0) \leq c(Z^{2n}(R), \omega_0) = \pi R^2.$$

SOME KNOWN RESULTS ON THE GROMOV WIDTH

- Computation of the Gromov width of the complex Grassmannian (Y. Karshon, S. Tolman (2005) and G. Lu, (2006))

- Computation of the Gromov width of the 4-dimensional torus (J. Latschev, D. McDuff and F. Schlenk, (2013)).

- Computation of the Gromov width of the first Cartan domain and upper and lower bounds for the classical ones (G. Lu, H. Ding, Q. Zhang, Int. Math. Forum 2, (2007)).

MAIN RESULTS ON THE GROMOV WIDTH OF HSSNT and HHSCT

Theorem 1 (R. Mossa - L - F. Zuddas, 2015): Let (M^*, ω_{FS}) be an irreducible HSSCT endowed with the canonical symplectic (Kähler) form ω_{FS}. Then

$$c_G(M^*, \omega_{FS}) = \pi.$$

Remark Theorem 1 extends (to the case of HSSCT) the result of Y. Karshon, S. Tolman (2005), when M^* is the complex Grassmannian.

Idea of the proof of Theorem 1

The upper bound $c_G(M^*, \omega_{FS}) \leq \pi$ is obtained by the computations of some genus-zero three-points Gromov-Witten invariants for irreducible HSSCT and through nonsqueezing theorem techinques using and extending the ideas in Y. Karshon, and S. Tolman for the complex Grassmannian.

The lower bound $c_G(M, \omega_{FS}) \geq \pi$ is obtained as follows.

Let (M, ω_0), $M \subset \mathbb{C}^n$, be the HSSNT dual of (M^*, ω_{FS}) equipped with the canonical symplectic form ω_0 of $\mathbb{R}^{2n}$. Then

$$(B^{2n}(1), \omega_0) \hookrightarrow (M, \omega_0) \stackrel{\Phi_M}{\to} (\mathbb{C}^n, \omega_{FS}) \stackrel{Borel}{\subset} (M^*, \omega_{FS})$$

$\Rightarrow c_G(M, \omega_{FS}) \geq \pi$

Theorem 2 (R. Mossa - L - F. Zuddas, 2015): Let $M \subset \mathbb{C}^n$ be a HSSNT. Then

$$c_G(M, \omega_0) = \pi.$$

Remark Theorem 2 extends the results of G. Lu, H. Ding, Q. Zhang (2007), valid for classical Cartan domains.

Proof:

$$(B^{2n}(1), \omega_0) \hookrightarrow (M, \omega_0) \overset{\Phi_M}{\to} (\mathbb{C}^n, \omega_{FS}) \overset{Borel}{\subset} (M^*, \omega_{FS})$$

$$c_G(B^{2n}(1), \omega_0) = c_G(M^*, \omega_{FS}) \overset{Th1}{=} \pi \Rightarrow c_G(M, \omega_0) = \pi$$

Theorem 3 (R. Mossa - L - F. Zuddas, 2015): Let $M \subset \mathbb{C}^n$ be a bounded symmetric domain. Then

$$c(M, \omega_0) = \pi$$

for any symplectic capacity c.

Proof One can prove that

$$B^{2n}(1) \subset M \subset Z^{2n}(1).$$

Hence the conclusion follows by (monotonicity)+(nontriviality) of c.

BIRAN'S CONJECTURE

Conjecture: Let (M, ω) be a compact symplectic manifold with $[\omega] \in H^2(M, \mathbb{Z})$. Then $c_G(M, \omega) \geq 1$

Theorem (Kaveh, 2016) Let X be a smooth complex projective variety embedded in a complex projective space $\mathbb{C}P^N$. Then

$$c_G(X, \frac{\omega_{FS}}{\pi}) \geq 1$$

where ω_{FS} denotes the restriction to X of the Fubini–Study Kähler form of $\mathbb{C}P^N$.

Theorem (- L, F. Zuddas, 2016): Let (M, ω) be a compact homogeneous simply-connected Kähler manifold with $[\omega] \in H^2(M, \mathbb{Z})$. Then $c_G(M, \omega) \geq 1$.

Corollary Let (M, ω) be a compact homogeneous simply connected Kähler manifold with $[\omega] \in H^2(M, \mathbb{Z})$ and $b_2(M) = 1$. Then $c_G(M, \omega) = 1$.

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Thank you for your attention!