The function epsilon for complex Tori and Riemann surfaces

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Abstract

In the framework of the quantization of Kähler manifolds carried out in [3], [4], [5] and [6], one can define a smooth function, called the function *epsilon*, which is the central object of the theory. The first explicit calculation of this function can be found in [10].

In this paper we calculate the function *epsilon* in the case of the complex tori and the Riemann surfaces.

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1 Introduction

A quantization of a Kähler manifold (M, ω) is a pair (L, h), where L is a holomorphic line bundle over M and h is a hermitian structure on L such that $\mathrm{curv}(L, h) = -2\pi i\omega$. The curvature $\mathrm{curv}(L, h)$ is calculated with respect to the *Chern connection*, i.e. the unique connection compatible with both the holomorphic and the hermitian structure. Not all manifolds admit such a pair. In terms of cohomology classes, a Kähler manifold admits a quantization if and only if the form ω is integral [7], i.e. its cohomology class $[\omega]_{dR}$ in the de Rham group, is in the image of the natural map $H^2(M, \mathbb{Z}) \hookrightarrow H^2(M, \mathbb{C})$. In particular, when M is compact, the integrality

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of ω implies, by a well-known theorem of Kodaira, that M is a projective algebraic manifold.

In the framework of the quantization of a Kähler manifold (M, ω) one can define a smooth function $\epsilon_{(L,h)}$ on M, depending on the pair (L, h), which is the central object of the theory and which is one of the main ingredients needed to apply a procedure called *quantization by deformation* introduced by Berezin in his foundational paper [1]. The work of Berezin was later developed and generalized in a series of papers [3], [4], [5] and [6] which are the starting point of the present article.

In this paper, we give an explicit calculation of the function *epsilon* in terms of theta functions for the 1-dimensional complex torus (see section 3). We also calculate the function *epsilon* for a Riemann surface of genus $g > 1$ endowed with the hyperbolic metric (see section 4).

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2 Preliminaries

Let (L, h) be a quantization of a Kähler manifold (M, ω). Consider the separable complex Hilbert space $\mathcal{H}_h$ consisting of global holomorphic sections s of L, which are bounded with respect to

$$\langle s, s\rangle_h = \|s\|_h^2 = \int_M h(s(x), s(x)) \frac{\omega^n(x)}{n!}$$

(see [3]). Let $x \in M$ and $q \in L^0$ a point of the fibre over x. If one evaluates $s \in \mathcal{H}_h$ at x, one gets a multiple $\delta_q(s)$ of q, i.e. $s(x) = \delta_q(s)q$. The map $\delta_q : \mathcal{H}_h \to \mathbb{C}$ is a continuous linear functional [3] hence by Riesz's theorem, there exists a unique $e_q \in \mathcal{H}_h$ such that $\delta_q(s) = \langle s, e_q\rangle_h$, i.e.

$$s(x) = \langle s, e_q\rangle_h q. \tag{1}$$

It follows, by (1), that

$$e_{cq} = \overline{c}^{-1} e_q, \ \forall c \in \mathbb{C}^*.$$

Definition 2.1 *The holomorphic section e_q is called the coherent states relative to the point q.*

Then, one can define a real valued function on M by the formula

$$\epsilon_{(L,h)}(x) := h(q,q)\|e_q\|_h^2, \tag{2}$$

where $q \in L^0$ is any point on the fibre of x. Let $(s_0, \ldots, s_N)$ $(N \leq \infty)$ be a unitary basis for $(\mathcal{H}_h, \langle \cdot\,, \cdot \rangle_h)$. Take $\lambda_j \in \mathbb{C}$ such that $s_j(x) = \lambda_j q, j = 0, \ldots, N$. Then

$$s(x) = \sum_{j=0}^{N} \langle s, s_j \rangle_h s_j(x) = \sum_{j=0}^{N} \langle s, s_j \rangle_h \lambda_j q = \langle s, \sum_{j=0}^{N} \bar{\lambda}_j s_j \rangle_h q.$$

By (1) it follows that

$$e_q = \sum_{j=0}^{N} \bar{\lambda}_j s_j, \tag{3}$$

and

$$\epsilon_{(L,h)}(x) = h(q,q)\|e_q\|_h^2 = \sum_{j=0}^{N} h(s_j(x), s_j(x)). \tag{4}$$

One can calculate the function $\epsilon_{(L^k,h^k)}$ for every natural number k. Namely, one considers the Kähler form $k\omega$ on M and (L^k, h^k) the quantum line bundle for $(M, k\omega)$, where L^k is the k-tensor power of L and $h^k := h \otimes \ldots \otimes h$, k-times.

We say that a quantization (L,h) of a Kähler manifold (M,ω) is *regular* if, for any natural number k, $\epsilon_{(L^k,h^k)}$ is constant. If a manifold (M,ω) admits a regular quantization then on can define a $*$-product on $C^\infty(M)$ the algebra of smooth functions on the manifold M (see [3], [4], [5] and [6]). One of the main tool in constructing this $*$-product is the following Rawnsley's result [10], saying that, if the above regulariy condition is satisifed, then the Kähler forms $k\omega$ are projectively induced i.e. for every natural number k there exists a natural number $N(k)$ and a holomorphic map into the complex $N(k)$-dimensional projective space

$$\phi_k : M \to \mathbb{P}^{N(k)}(\mathbb{C})$$

such that $\phi_k^*\Omega_k = k\omega$, for Ω_k the Fubini-Study form on $\mathbb{P}^{N(k)}(\mathbb{C})$.

3 Quantization of complex tori

Let $M = V/\Lambda$ be an n-dimensional complex torus, where V is an n-dimensional complex vector space and Λ is a $2n$-lattice on V. Let H be a hermitian form on V and

$$\omega := \frac{i}{2}\partial\bar{\partial}H.$$

Since ω is invariant by translations it descends to a globally defined Kähler form ω on M which makes (M, ω) into a homogeneous Kähler manifold. It is well-known [9] that ω is integral iff the imaginary part of H takes integral values on Λ, i.e. $\Im H(\Lambda, \Lambda) \subset \mathbb{Z}$. Under this hypothesis it follows by [7] that (M, ω) admits a quantization (L, h). On the other hand a theorem in [11] asserts that ω can not projectively induced and so by the discussion at the end of the previous section the quantization (L, h) can not be regular.

An explicit description of the line bundle L and of the hermitian structure h can be found in [9, Chapter 1] to whom we refer for the proof of the following assertions. First of all the global holomorphic sections of L can be seen as holomorphic functions θ on V satisfying

$$\theta(v + \lambda) = A(\lambda, v)\theta(v), \tag{5}$$

where

$$A(\lambda, v) = \chi(\lambda)e^{\pi H(v,\lambda)+\frac{\pi}{2}H(\lambda,\lambda)}$$

and $\chi : \Lambda \to S^1$ belongs to the group of semicharacter of H, i.e.

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)e^{\pi i \Im H(\lambda,\mu)},\ \forall \lambda, \mu \in \Lambda. \tag{6}$$

Given θ a holomorphic section of L define

$$h(\theta(v), \theta(v)) = e^{-\pi H(v,v)}|\theta(v)|^2.$$

It follows easily by (5) that the function h is invariant under the action of the lattice, i.e.

$$h(\theta(v + \lambda), \theta(v + \lambda)) = h(\theta(v), \theta(v))\ \forall \lambda \in \Lambda,$$

and so it defines a hermitian structure on L. Furthermore,

$$\mathrm{curv}(L, h) = -\partial\bar{\partial}\log h = \pi\partial\bar{\partial}H = -2\pi i\omega,$$

which shows that (L, h) is a quantization for $(V/\Lambda, \omega)$.

3.1 The function *epsilon* for the 1-dimensional complex torus

Let

$$\Lambda = \{p + iq \mid \ p, q \in \mathbb{Z}\}$$

be the lattice in $\mathbb{C}$ generated by $(1,0)$ and $(0,1)$ and $\mathbb{C}/\Lambda$ be the 1-dimensional complex torus. Let $H(z, w) = z\bar{w}$ be the standard hermitian form on $\mathbb{C}$ and

$$\omega = \frac{i}{2}\partial\bar{\partial}|z|^2 = \frac{i}{2}dz \wedge d\bar{z}$$

the flat Kähler form on $\mathbb{C}/\Lambda$. A simple calculation shows that

$$\Im H(\lambda, \mu) = mq - pn, \ \forall \lambda = p + iq, \mu = m + in,$$

i.e. H is integral on the lattice. By the previous section there exists a holomorphic line bundle L whose global holomorphic sections can be identified with the holomorphic functions θ on $\mathbb{C}$ such that

$$\theta(z + \lambda) = A(\lambda, z)\theta(z) = e^{i\pi pq} e^{\pi z\bar{\lambda} + \frac{\pi}{2}|\lambda|^2}\theta(z), \ \forall \lambda = p + iq \in \Lambda,$$

where we choose

$$\chi(\lambda) = e^{i\pi pq}, \ \forall \lambda = p + iq \in \Lambda$$

as a semicharacter of H.

More generally, given any natural number k let L^k be the k-th tensor power of L.

The global holomorphic sections of L^k, can be seen as the holomorphic functions θ on $\mathbb{C}$ satisfying

$$\theta(z + \lambda) = e^{ki\pi pq} e^{k\pi z\bar{\lambda} + \frac{k\pi}{2}|\lambda|^2}\theta(z), \ \forall \lambda = p + iq \in \Lambda, \tag{7}$$

and the hermitian structure h^k such that $\mathrm{curv}(L^k, h^k) = -2\pi k i\omega$ is given by

$$h^k(\theta(z), \theta(z)) = e^{-k\pi|z|^2}|\theta(z)|^2, \forall \theta \in H^0(L^k).$$

By the Riemann-Roch theorem $\mathcal{H}_{h^k}$ is k-dimensional. Given $j = 0, \ldots, k-1$ define

$$\theta_j(z) = e^{k\frac{\pi}{2}z^2} \sum_{m\in\mathbb{Z}} e^{\frac{-\pi}{k}(km+j)^2 + 2\pi i(km+j)z}$$

It is not hard to see that the functions θ_j's satisfy the functional equation (7). Furthermore

Proposition 3.1 $\left\{(\frac{2}{k})^{\frac{1}{4}}\theta_0, \ldots, (\frac{2}{k})^{\frac{1}{4}}\theta_{k-1}\right\}$ *form a unitary basis for* $(\mathcal{H}_{h^k}, \langle \cdot , \cdot \rangle_{h^k})$.

Proof: For $a, b = 0, 1, \ldots, k-1$

$$\langle \theta_a, \theta_b \rangle_{h^k} = \sum_{m,p \in \mathbb{Z}} e^{\frac{-\pi}{k}((km+a)^2+(kp+b)^2)} \int_{\mathbb{C}/\Lambda} e^{-k\pi|z|^2} e^{\frac{k\pi}{2}(z^2+\bar{z}^2)} e^{2\pi i(km+a)z} e^{-2\pi i(km+a)\bar{z}} k\omega.$$

If $z = x + iy$, the previous integral can be written as

$$\sum_{m,p \in \mathbb{Z}} e^{\frac{-\pi}{k}((km+a)^2+(kp+b)^2)} \int_0^1 \int_0^1 e^{-2k\pi y^2} e^{2\pi i(k(m-p)+(a-b))x} e^{-2\pi(k(m+p)+(a+b))y} k dx \wedge dy.$$

Integrating with respect to x we obtain

$$\int_0^1 e^{2\pi i(k(m-p)+(a-b))x} dx = \delta_{0k(m-p)+b-a} = \delta_{mp}\delta_{ab},$$

where the last equality follows from the fact that $b-a$ is divisible by k if and only if $b = a$. Thus,

$$\langle \theta_a, \theta_b \rangle_{h^k} = k\delta_{ab} \sum_{m \in \mathbb{Z}} e^{\frac{-\pi}{k}((km+a)^2+(km+b)^2)} \int_0^1 e^{-2k\pi y^2} e^{-4\pi(km+\frac{a+b}{2})y} dy.$$

Therefore the θ_j's form an orthogonal basis for $(\mathcal{H}_{h^k}, \langle \cdot , \cdot \rangle_{h^k})$. For $a = b = j$ one gets:

$$\begin{aligned} \|\theta_j\|^2_{h^k} &= k \int_0^1 e^{-2k\pi y^2} \sum_{m \in \mathbb{Z}} e^{\frac{-2\pi}{k}(km+j)^2} e^{-4\pi(km+j)y} dy \\ &= k \sum_{m \in \mathbb{Z}} \int_0^1 e^{-2k\pi(y+m+\frac{j}{k})^2} dy. \end{aligned}$$

By the change of variable $t = y + m + \frac{j}{k}$ one obtains:

$$\|\theta_j\|^2_{h^k} = k \int_{-\infty}^{+\infty} e^{-2k\pi t^2} dt = \sqrt{\frac{k}{2}}.$$

$\square$

By (4) and 3.1, the function epsilon can be calculated as

$$\epsilon_{(L^k,h^k)}(z) = e^{-k\pi|z|^2}\sqrt{\frac{2}{k}\sum_{j=0}^{k-1}|\theta_j(z)|^2}.$$

Remark 3.2 *The previous calculation can be generalized to the case where Λ is a general lattice in $\mathbb{C}$. Similar calculations can be found in [2].*

4 Quantization of Riemann surfaces

Let Σ_g be a compact Riemann surface of genus $g \geq 2$. One can realize Σ_g as the quotient $\mathbb{D}/\Gamma$ of the unit disk $\mathbb{D} \subset \mathbb{C}$ under the fractional linear transformations of a Fuchsian subgroup Γ of

$$\mathrm{SU}(1,1) = \{\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mid \ |a|^2 - |b|^2 = 1\}.$$

Here the action of $\gamma = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right) \in \Gamma$ is given by $z \mapsto \gamma(z) = \frac{az+b}{\bar{b}z+\bar{a}}$. It is immediate to check that the Kähler form

$$\omega_{hyp} = \frac{i}{\pi}\frac{dz \wedge d\bar{z}}{(1 - z\bar{z})^2}$$

is invariant under fractional linear transformations, so it defines a Kähler form on Σ_g, denoted by the same symbol ω_{hyp}. Let L be the canonical bundle over Σ_g, i.e. the holomorphic line bundle whose global holomorphic sections are the holomorphic forms of type $(1,0)$ on Σ_g. Let $p : \mathbb{D} \to \mathbb{D}/\Gamma$ be the natural projection map. The line bundle $p^*(L)$ is holomorphically trivial and its global holomorphic sections are the form of type $(1,0)$ on $\mathbb{D}$, i.e. $f(z)dz$ where $f(z)$ is a holomorphic function on $\mathbb{D}$. Hence, the global holomorphic sections of L can be seen as the forms $s = fdz$ invariant by the action of Γ, i.e.

$$f(\gamma(z))d(\gamma(z)) = f(\gamma(z))\gamma'(z)dz = f(z)dz, \forall\gamma \in \Gamma, \tag{8}$$

where $\gamma'(z)$ denotes the derivative of $\gamma(z)$ with respect to z (if $\gamma(z) = \frac{az+b}{\bar{b}z+\bar{a}}$ then $\gamma'(z) = (\bar{b}z + \bar{a})^{-2}$). In other words if

$$\sigma : \mathbb{D} \to \mathbb{D} \times \mathbb{C} : z \to (z, 1)$$

is the section of the trivial bundle over $\mathbb{D}$, then the space of holomorphic sections of L can be identify with the space of all $s = f\sigma$, where f is a holomorphic function on $\mathbb{D}$ such that

$$f(\gamma(z)) = (\gamma'(z))^{-1} f(z).$$

More generally, given k a natural number, one can show that the global holomorphic sections of L^k can be seen as $s = f\sigma$, where f is holomorphic function on $\mathbb{D}$, such that

$$f(\gamma(z)) = (\gamma'(z))^{-k} f(z). \tag{9}$$

Given such a section $s = f\sigma$ define

$$h^k(s(z), s(z)) = (1 - |z|^2)^{2k} |f(z)|^2.$$

One can easily check that

$$(1 - |\gamma(z)|^2)^{2k} = |\gamma'(z)|^{2k}(1 - |z|^2)^{2k}, \tag{10}$$

so

$$h^k(s(\gamma(z)), s(\gamma(z))) = h^k(s(z), s(z)), \forall \gamma \in \Gamma.$$

Therefore h^k defines a hermitian structure on L^k. Moreover

$$\operatorname{curv}(L, h) = -2\partial\bar{\partial} \log(1 - |z|^2) = \frac{2dz \wedge d\bar{z}}{(1 - |z|^2)^2} = -2\pi i \omega_{hyp}, \tag{11}$$

which shows that the pair (L^k, h^k) is a quantization for $(\Sigma_g, k\omega_{hyp})$.

4.1 The function epsilon for the Riemann surfaces

Given a natural number k define a function on $\mathbb{D} \times \mathbb{D}$ by the formula

$$e^k(z, w) = \frac{2k - 1}{2k} \sum_{\gamma \in \Gamma} (1 - \gamma(z)\bar{w})^{-2k} (\gamma^{'}(z))^k. \tag{12}$$

Classical theorems going back to Poincare (see [8, pp. 101-104]) assert that the series (12) converges almost uniformly for all $z \in \mathbb{D}$. It is easily seen that for every $w \in \mathbb{D}$

$$e^k(\gamma(z), w) = (\gamma^{'}(z))^{-k} e^k(z, w), \ \forall \gamma \in \Gamma. \tag{13}$$

Hence $e^k_{\sigma(w)}(z) := e^k(z,w)\sigma(z)$ is a holomorphic section of L^k. Let U be a fundamental domain in $\mathbb{D}$ for the action of Γ. Given any $s = f\sigma$ a holomorphic section for L^k it follows by (9) and (13) that

$$\begin{aligned}
\langle s, e^k_{\sigma(w)}\rangle_{h^k} &= \int_{\Sigma_g} f(z)\overline{e^k(z,w)}(1-|z|^2)^{2k} k\omega_{hyp}(z) \\
&= \frac{2k-1}{2k}\sum_{\gamma\in\Gamma}\int_U f(z)(1-\overline{\gamma(z)}w)^{-2k}(\overline{\gamma'(z))^k}(1-|z|^2)^{2k}k\omega_{hyp}(z) \\
&= \frac{2k-1}{2k}\sum_{\gamma\in\Gamma}\int_U f(\gamma(z))(1-\overline{\gamma(z)}w)^{-2k}(1-|\gamma(z)|^2)^{2k}k\omega_{hyp}(z) \\
&= \int_{\mathbb{D}} f(z)(1-\bar{z}w)^{-2k}(1-|z|^2)^{2k}k\omega_{hyp}(z) = f(w),
\end{aligned}$$

where the last equality follows by a direct calculation (cfr. [5, p10]). Hence

$$\langle s, e^k_{\sigma(w)}\rangle_{h^k}\sigma(w) = f(w)\sigma(w),$$

i.e. $e^k_{\sigma(w)}$ is the coherent state relative to $\sigma(w)$. By the very definition of coherent states one has $\|e^k_{\sigma(z)}\|^2_{h^k}\sigma(z) = e^k(z,z)\sigma(z)$ and by (2)

$$\epsilon_{(L^k,h^k)}(z) = \|e^k_{\sigma(z)}\|^2_{h^k} h^k(\sigma(z),\sigma(z)) = \frac{2k-1}{2k}(1-|z|^2)^{2k}\sum_{\gamma\in\Gamma}(1-\gamma(z)\bar{z})^{-2k}(\gamma'(z))^k.$$

References

[1] F.A. Berezin, *Quantization*, Math. USSR Izvestija 8 (1974), pp. 1109-1165.

[2] M. Bordemann, J. Hoppe, P. Schaller and M. Schlichenmaier, *$gl(\infty)$ and Geometric Quantization*, Comm. Math. Phys. 138 (1991), pp. 209-244.

[3] M. Cahen, S. Gutt, J. H. Rawnsley, *Quantization of Kähler manifolds I: Geometric interpretation of Berezin's quantization*, JGP 7 (1990), pp. 45-62.

[4] M. Cahen, S. Gutt, J. H. Rawnsley, *Quantization of Kähler manifolds II*, Trans. Amer. Math. Soc. 337 (1993), pp. 73-98.

[5] M. Cahen, S. Gutt, J. H. Rawnsley, *Quantization of Kähler manifolds III*, Lett. Math. Phys. 30 (1994), pp. 291-305.

[6] M. Cahen, S. Gutt, J. H. Rawnsley, *Quantization of Kähler manifolds IV*, Lett. Math. Phys. 34 (1995), pp. 159-168.

[7] B. Kostant, *Quantization and Unitary Representation*, Lecture notes in Mathematics 170.

[8] I. Kra, *Automorphic Functions and Klenian Groups*, Benjamin, Reading, Mass., (1972).

[9] H. Lange and C. Birkenhake, *Complex Abelian Varieties*, Springer-Verlag (1992).

[10] J. H. Rawnsley, *Coherent states and Kähler manifolds*, The Quarterly Journal of Mathematics (1977), pp. 403-415.

[11] M. Takeuchi, *Homogeneous Kähler Manifolds in Complex Projective Space*, Japan J. Math. vol. 4 (1978), pp. 171-219.