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Programma "Visiting Professors"

Barycentre, Minimal Volume and Rigidity

Referenze e background della conferenza
"Dal baricentro euclideo ai teoremi di rigidità"
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1 Minimal (and maximal) volume :

1.1 Definitions :

Let M be a compact connected manifold of dimension $n \geq 2$; its *minimal volume* (denoted by $\mathrm{MinVol}(M)$) has been defined by M. Gromov ([Gr 1]) as being the infimum of the volumes of all Riemannian metrics g on M, whose sectional curvature K_g satisfies $-1 \leq K_g \leq 1$.
Similarly, when the manifold admits some metric with strictly negative sectional curvature, one may define the *maximal volume* of M as the supremum of $\mathrm{Vol}(g)$, for all metrics g whose sectional curvature satisfies $K_g \leq -1$.

1.2 Computation of the minimal and maximal volumes in dimension 2 :

In dimension $n = 2$, one can compute the exact value of the minimal (resp. the maximal) volume of any manifold and, moreover, characterize all the metrics for which this minimum (resp. this maximum) is attained :
In fact, the Gauss-Bonnet formula gives

$$\int_M K_g dv_g = 2\pi\, \chi(M) \quad ,$$

where $\chi(M)$ is the Euler characteristic of the surface M. This immediately implies that

$$\mathrm{MinVol}(M) = 2\pi|\chi(M)| \quad .$$

Similarly, assuming that $\chi(M) < 0$ and $K_g \leq -1$, we get :

$$\mathrm{MaxVol}(M) = 2\pi|\chi(M)|.$$

It is then obvious that the minimal and the maximal volume are both attained for (and only for) metrics with constant sectional curvature -1.

1.3 Is there an analogue in higher dimensions ?

When the dimension n is even, the analogue of the Gauss-Bonnet formula is the Allendœrfer-Chern-Weil one which writes :

$$\chi(M) = \int_M P(R_g) dv_g,$$

where P is a universal polynomial of degree $\frac{n}{2}$ on the exterior algebra, called the *Pfaffian polynomial*, where R_g is the curvature tensor (viewed as a matrix with coefficients in $\wedge^2(T^*M)$) and where $\chi(M)$ is the Euler characteristic of the manifold M , i. e.

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim \left(H^k(M, \mathbb{R}) \right) \quad .$$

An immediate consequence is that, for any metric g on M,

$$|\chi(M)| \leq C_n \, |K_g|_{L^\infty}^{n/2} \, \mathrm{Vol}(M, g),$$

which implies that $\mathrm{MinVol}(M) \geq C_n^{-1}.|\chi(M)|$, where C_n is a universal constant (other lower bounds for the minimal volume may be obtained by using other characteristic classes than the Euler class).

1.4 Objections to these estimates :

Though the above inequality provides a lower bound for the minimal volume in terms of a homology-invariant, this result does not fit our purpose for two reasons, already underlined by M. Gromov :

(1) Except in dimension 2, it is not sharp, so it cannot help in computing the exact value of the minimal volume. Moreover, it gives no information about the fact that this minimal volume may be attained for some metric or not.

(2) The Euler characteristic is a rough invariant, which often vanishes, in particular it vanishes for every odd-dimensional manifold. In these cases, the above inequality is trivial.

2 M. Gromov's approach :

2.1 A bound from below for the minimal volume :

A first main result is that the above inequality remains valid if one replaces the Euler characteristic by another homology-invariant : the simplicial volume (denoted by Simpl Vol). This result writes :

Theorem 2.1 (M. Gromov, [Gr 1]) *For any manifold M, one has :*
$\mathrm{MinVol}(M) \geq C_n$ Simpl $\mathrm{Vol}(M)$*, where* C_n *is a universal constant (only depending on the dimension).*

As illustrated in section 1, this inequality may be seen as the most natural generalization of the Gauss-Bonnet inequality that one may expect (if one wants it to be valid in any dimension).
Notice that, in any dimension, there exist manifolds with non trivial simplicial volume (for instance, as we shall see later, the simplicial volume of any negatively curved compact manifold is non trivial).

2.2 The Simplicial Volume :

Let us recall the definition of the simplicial volume: Every singular chain c with real coefficients is a linear combination of simplices σ_i (i. e. $c = \sum_{i\in I} \lambda_i.\sigma_i$, where $\forall i \ \ \lambda_i \in \mathbb{R}$). The L^1 norm of c is defined by $\|c\|_1 = \sum_{i\in I} |\lambda_i|$; the associated semi-norm on $H_k(M,R)$ is defined, for any $\gamma \in H_k(M,R)$, by

$$\|\gamma\| = \inf\{\|c\|_1 \ : \ c \text{ closed chain such that } [c] = \gamma\} \quad .$$

Definition 2.2 *The simplicial volume of M is defined as $\|[M]\|$, where $[M]$ is the fundamental n-class of M .*

Interpretation : If the coefficients where integers, the simplicial volume could be interpreted as the minimal number of n-simplices in a simplicial triangulation of M. As the coefficients are real (or, equivalently, rational), one accepts triangulations covering the manifold p-times and may see ([Gr 1]) the simplicial volume as

$$\inf_{p\in N} \frac{1}{p} \ (\text{Minimal number of simplices in a triangulation of } p.[M]) \quad .$$

2.3 Computation of some simplicial volumes :

2.3.1 Manifolds whose simplicial volume is zero :

Lemma 2.3 *Let $f : Y \to X$ be a continuous map between two compact manifolds of the same dimension, then* $\text{Simpl Vol}(Y) \geq |\deg f|.\text{Simpl Vol}(X)$.

Proof : If the homology class $[Y]$ is represented by the chain $\sum_{i\in I} \lambda_i\sigma_i$, then the homology class $[X] = \dfrac{1}{\deg f} \ f_*([Y])$ is represented by the chain $\sum_{i\in I} \dfrac{\lambda_i}{\deg f}(f\circ\sigma_i)$. This implies that $\|[X]\| \leq \sum_{i\in I} \dfrac{|\lambda_i|}{|\deg f|}$. $\square$

This provides a method for proving that some simplicial volumes are trivial, for example one has the following immediate corollary

Corollary 2.4 *If there exists a continuous map $f : X \to X$ whose degree is different from -1 , 0 or $+1$, then* $\text{Simpl Vol}(X) = 0$.

Proof: As $|\deg f| \geq 2$, the previous property implies that $\text{Simpl Vol}(Y) \geq 2\,\text{Simpl Vol}(X)$, which implies that $\text{Simpl Vol}(X) = 0$. $\square$

For instance, this proves that the *simplicial* volumes of $\mathbb{S}^n$ and $\mathbb{T}^n$ are trivial for any n .

Are the *minimal* volumes of $\mathbb{S}^n$ and $\mathbb{T}^n$ also trivial ?

The minimal volume of $\mathbb{T}^n$ is trivial : one proof consists in writing $\mathbb{T}^n$ as $\mathbb{T} \times \mathbb{T}^{n-1}$ and in multiplying the metric of the first factor by ε^2 , then the volume of $(\varepsilon\, \mathbb{T}) \times \mathbb{T}^{n-1}$ goes to zero, but the curvature of $(\varepsilon\, \mathbb{T}) \times \mathbb{T}^{n-1}$ is equal to zero, and thus bounded.

The minimal volume of $\mathbb{S}^{2p+1}$ is also trivial (just see it as the total space of the Hopf-fibration with fiber $\mathbb{S}^1$ and multiply the metric of the fiber by ε : the volume goes to zero and the curvature remains bounded; these examples are from M. Berger, cf. [C-E] p.70).

Is the minimal volume of $\mathbb{S}^{2p}$ also trivial ?
Notice that, in this case, the theorem 2.1 does not help, because (the simplicial volume of $\mathbb{S}^{2p}$ beeing trivial) it only provides the trivial inequality $\mathrm{MinVol}(S^{2p}) \geq 0$. On the contrary, the Allendœrfer-Chern-Weil formulas (cf section 1.3) show that the minimal volume of $\mathbb{S}^{2p}$ is non zero.

2.3.2 Manifolds whose simplicial volume is non zero :

On the contrary, the simplicial volume of a compact manifold is non zero when this manifold admits a metric with strictly negative curvature (see [Gr 1]). When it admits a hyperbolic metric, one has the following exact computation of the simplicial volume :

Theorem 2.5 (M. Gromov and W. Thurston, see [Gr 1]) .– *Let X be a compact Riemannian manifold whose sectional curvature is constant and equal to -1 , the Simplicial Volume of X is equal to $T_n^{-1}\, Vol(X)$, where T_n is the supremum of the volumes of all geodesic n-simplices*[1] *on the real hyperbolic space-form.*

One interpretation of the theorem 2.5 is that all the (eventually) possible volumes of compact hyperbolic manifolds are n-homology invariants. For example, any 4-dimensional compact hyperbolic manifold satisfies

$$Vol(X) = \frac{\chi(X)}{2} . Vol(S^4) \ ,$$

where $\chi(X)$ is the Euler characteristic of the manifold X .

Proof: As X and S^4 are Einstein manifolds, the Allendœrfer-Chern-Weil formula writes, in this case :

$$8\pi^2\, \chi(X) = \int_X \|R_X\|^2,$$
$$8\pi^2\, \chi(S^4) = \int_{S^4} \|R_{S^4}\|^2,$$

where the curvature tensors R_X and R_{S^4} of the two manifolds X and S^4 satisfy $\|R_X\| = \|R_{S^4}\| =$ Constant , because both manifolds have constant curvature ± 1 . This gives $\dfrac{Vol(X)}{Vol(S^4)} = \dfrac{\chi(X)}{\chi(S^4)} = \dfrac{\chi(X)}{2}$, since $\chi(S^4) = 2$. $\square$

In the odd-dimensional case, it is more difficult to compute the possible volumes of a compact hyperbolic manifold : for instance, an important open problem is the

[1] A *geodesic simplex* is a simplex whose boundary is made of pieces of totally geodesic hypersurfaces.

Question 2.6 *What is the smallest volume of an hyperbolic 3 -manifold ?*

Another important problem is to compute explicitely the supremum T_n of the volumes of all geodesic n-simplices on the real hyperbolic space-form. We have first to find what geodesic n-simplices have maximal volume. Let us, in general, denote by $\widetilde{X}$ the universal covering of X, endowed with the pulled-back metric. In the above situation, $\widetilde{X}$ is the real hyperbolic space $\mathbb{H}^n$, which may be regarded as the unit ball B^n endowed with the metric $\widetilde{g}_0$ defined (at the point $x \in B^n$) by

$$\widetilde{g}_0 = \frac{4}{(1-|x|^2)^2}\, g_E$$

where g_E is the euclidean metric. We may thus compactify B^n by adding the "ideal boundary" $S^{n-1} = \partial\widetilde{X}$, and endowing $B^n \cup S^{n-1}$ with the obvious topology of the closed unit ball. We call *ideal simplices* those geodesic n-simplices of $\widetilde{X}$ all of whose vertices lie on $\partial\widetilde{X}$. Such a simplex will be called *regular* when every permutation of its vertices may be achieved by an isometry of the hyperbolic space (a regular ideal simplex is a limit of simplices all of whose 1-dimensional edges have same length). We then have the

Theorem 2.7 (U. Haagerup and H. J. Munkholm, [H-M]) *Ideal regular simplices have maximal volume among all geodesic n-simplices (and thus their volume is equal to T_n).*

In order to make explicit the equality of the Theorem 2.5, it remains to compute the volume of the ideal regular simplices:
- In dimension 2, as all ideal triangles have zero angles, the Gauss-Bonnet formula (with boundary) implies that all ideal simplices have volume equal to $\pm\pi$.
- In dimension 3, an explicit formula is known (see for example J. Milnor, [Mil]).
- In any dimension $n \geq 4$, the problem is still open.

2.4 A sketch of the proof of the Gromov-Thurston Theorem 2.5:

2.4.1 First step : $\mathrm{Vol}(X) \leq T_n$ Simpl $\mathrm{Vol}(X)$:

This proof is attributed to W. Thurston (see [Gr 1]).
As X is hyperbolic (i. e. with constant curvature -1), any simplex σ on X is homotopic to a geodesic simplex $\bar{\sigma}$ which has the same vertices: just lift the simplex σ as a simplex $\widetilde{\sigma}$ of the total space $\widetilde{X}$ of the (Riemannian) universal covering $\pi : \widetilde{X} \to X$. As the total space $\widetilde{X}$ of this universal covering is the real hyperbolic space $\mathbb{H}^n$, we can construct the geodesic simplex $\bar{\widetilde{\sigma}}$ of $\widetilde{X} = \mathbb{H}^n$ which has the same vertices as $\widetilde{\sigma}$, and push it down on X : we then obtain the "straightened simplex" $\bar{\sigma} = \pi \circ \bar{\widetilde{\sigma}}$.
This process $\sigma \mapsto \bar{\sigma}$ is called "*straightening simplices*". W. Thurston proved that one can straighten a (real) simplicial decomposition $\sum_{i\in I} \lambda_i\, \sigma_i$ of the fundamental class $[X]$ (by straightening simultaneously all the simplices σ_i) in order that the result $\sum_{i\in I} \lambda_i \bar{\sigma}_i$ is still a (real) simplicial decomposition of the same homology class $[X]$. Let ω_0 be the volume form on X associated to the hyperbolic metric, we get

$$\mathrm{Vol}(X) = \langle \omega_0, [X] \rangle = \langle \omega_0, \left[\sum_{i\in I} \lambda_i \,.\, \bar{\sigma}_i \right] \rangle = \sum_{i\in I} \lambda_i \ \mathrm{Vol}(\bar{\sigma}_i) \leq T_n \,.\, \sum_{i\in I} |\lambda_i|.$$

By taking the infimum in the right hand side of this last inequality, we obtain :

$$\mathrm{Vol}(X) \leq T_n \ \|[X]\| = T_n \ \mathrm{Simpl\ Vol}(X) \quad , \tag{1}$$

which is the required inequality.

2.4.2 Second step: $\mathrm{Vol}(X) \geq T_n$ Simpl $\mathrm{Vol}(X)$ (M. Gromov [Gr 1]):

- **In dimension 2**, any compact hyperbolic 2-manifold (with Euler characteristic χ) may be triangulated by $2(|\chi| + 1)$ geodesic triangles : in fact, such a manifold may be obtained from an hyperbolic $k-$gon (with $k = 2|\chi| + 4$) by gluing the edges together, and such a $k-$gon is triangulated by $(k-2)$ triangles.
Let $\pi : X_p \to X$ be a $p-$sheeted covering of X. As $\chi(X_p) = p \,.\, \chi(X)$, X_p is triangulated by $m(p) = 2\,(\,p\,.\,|\chi(X)| + 1\,)$ triangles $\sigma_1\,,\,\ldots\,,\,\sigma_{m(p)}$. We thus get
$$[X] = \frac{1}{p}\ \pi_*([X_p]) = \sum_{i=1}^{m(p)} \frac{1}{p}\ (\pi \circ \sigma_i) \quad ,$$
and deduce that $\|[X]\|$ is bounded from above by $\dfrac{m(p)}{p}$, which goes to $2|\chi(X)|$ when p goes to $+\infty$, and thus $\|[X]\| \leq 2\ |\chi(X)|$. Then the Gauss-Bonnet formula gives that
$$\mathrm{Vol}(X) = \int_X (-K_{gX}) dv_{gX} = 2\pi|\chi(X)| \geq T_2 \|[X]\|$$
because, as we have seen after the Theorem 2.7, $T_2 = \pi$.
Notice that this (and the previous inequality) proves that, in dimension 2, one has Simpl $\mathrm{Vol}(X) = 2|\chi(X)|$.

- **How M. Gromov generalizes this proof in higher dimensions ?**
Roughly speaking, the idea of M. Gromov ([Gr 1], for more explanations see [Mun]) is to admit chains which are limits of linear combinations of simplices, i.e. chains whose coefficients are measures. Thus, instead of writing a chain $c = \sum_i \lambda_i\,.\sigma_i$, we shall write it $c = \int_I \lambda(i)\,.\,\sigma_i\ d\mu(i)$, where I may be a continuous set of parameters and where μ is a positive measure on this set. One may analogously define
$$\|c\|_1 = \int_I |\lambda(i)|\ d\mu(i).$$
In the case where X is a hyperbolic manifold, assumed to be compact (resp. with finite volume), $\pi_1(X)$ acts on $\widetilde{X} = H^n$ by deck-transformations, which are isometries. We thus consider the set of parameters $I = Isom(\mathbb{H}^n)/\pi_1(X)$, which is compact (resp. with finite volume), and the measure μ on I, whose pull-back by the quotient map is the Haar measure of $Isom(\mathbb{H}^n)$. We shall denote by $g \mapsto \mathrm{sign}(g)$ the function which assigns, to each $g \in Isom(\mathbb{H}^n)$, the number $+1$ or -1 whether g preserves or changes the orientation.
Let σ_0 be a fixed regular ideal simplex of $\mathbb{H}^n$ then, for any $g \in Isom(\mathbb{H}^n)$, the simplex $g \circ \sigma_0$ is also a regular ideal simplex whose projection $\pi \circ g \circ \sigma_0$ (by the

covering map $\pi : \widetilde{X} \to X$) only depends on the image $\widehat{g}$ of g in $Isom(\mathbb{H}^n)/\pi_1(X)$ by the quotient map. We thus define a chain c by

$$c = \int_{Isom(\mathbb{H}^n)/\pi_1(X)} \mathrm{sign}(g) \,.\, (\pi \circ g \circ \sigma_0)\ d\mu(\widehat{g}) \tag{2}$$

which gives

$$\|c\|_1 = \mathrm{Vol}\,[Isom(\mathbb{H}^n)/\pi_1(X)] = \mathrm{Vol}(I).$$

As the symmetry, with respect to a totally geodesic hypersurface containing one of the $(n-1)-$dimensional faces of σ_0, changes the sign of g without changing the sign of the corresponding face, it comes that $d(-c) = dc$ and thus that c is a closed chain. Let ω_0 be the canonical volume-form associated to the hyperbolic metric, one gets :

$$\langle \omega_0, [c] \rangle = \mathrm{Vol}(g \circ \sigma_0)\,.\,\mathrm{Vol}(I) = T_n\,\mathrm{Vol}(I).$$

As $\langle \omega_0, [X] \rangle = \mathrm{Vol}(X)$, this implies that

$$[X] = \frac{\mathrm{Vol}(X)}{T_n\ \mathrm{Vol}(I)}\ [c]\ \ , \tag{3}$$

and thus that

$$\|[X]\| \leq \frac{\mathrm{Vol}(X)}{T_n.\mathrm{Vol}(I)}.\|c\|_1 = \frac{\mathrm{Vol}(X)}{T_n}\ \ . \tag{4}$$

This inequality, together with the inequality (1), proves that $|[X]| = \dfrac{\mathrm{Vol}(X)}{T_n}$. $\square$

Notice that this last equality implies that the first inequality of (4) is in fact an equality and we obtain the following computation of the norm of the chain c in terms of the simplicial volume:

$$\mathrm{Simpl\ Vol}(X) = \frac{\mathrm{Vol}(X)}{T_n.\mathrm{Vol}(I)}\ \|c\|_1\ \ . \tag{5}$$

2.5 A first application : Gromov's proof of Mostow's rigidity theorem :

The above ideas led M. Gromov to a new proof of the

Theorem 2.8 *Let X, Y be two $n-$dimensional hyperbolic manifolds ($n \geq 3$), which are both compact (or noncompact with same (finite) volume), then any homotopy-equivalence $f : Y \to X$ is homotopic to an isometry.*

Some rough ideas for the proof (see [Mun]) : Any homotopy equivalence $f : Y \to X$ may be lifted to a quasi-isometry $\widetilde{f} : \widetilde{Y} \to \widetilde{X}$, which may be continuously extended to an homeomorphism $\bar{f} : \partial\widetilde{Y} \to \partial\widetilde{X}$.
Let σ be any ideal simplex of $\widetilde{Y}$, with vertices $\theta_0, \ldots, \theta_n$; we shall denote by $\bar{f}(\sigma)$ the ideal simplex of $\widetilde{X}$ whose vertices are $\bar{f}(\theta_0), \ldots, \bar{f}(\theta_n)$. Let I and μ be as in the

previous proof, and $\lambda_0 = \dfrac{\mathrm{Vol}(Y)}{T_n \,\mathrm{Vol}(I)}$. We have seen in the previous proof that $[Y]$ may be represented by the chain

$$c = \lambda_0 \int_I \mathrm{sign}(g) \,.\, (\pi \circ g \circ \sigma_0) \; d\mu(\hat{g}) \quad ;$$

this implies that $[X] = f_*[Y]$ is represented by the chain f_*c, which may be straightened as the (homotopic) chain

$$\lambda_0 \int_I \mathrm{sign}(g) \,.\, (\pi[\bar{f}(g \circ \sigma_0)]) \; d\mu(\hat{g}) \quad .$$

Computing $\langle \omega_0, [X] \rangle = \langle \omega_0, f_*c \rangle$ by two different ways, we obtain :

$$\mathrm{Vol}(X) \leq \lambda_0 \int_I \mathrm{Vol}[\bar{f}(g \circ \sigma_0)] d\mu(\hat{g}) \leq \lambda_0 \, T_n \, \mathrm{Vol}(I) = \mathrm{Vol}(Y) = T_n \, \|[Y]\| = T_n \, \|[X]\| \, , \quad (6)$$

where the last two equalities come from the Theorem 2.5, from the Lemma 2.3 (used in both senses), and from the fact that f is a homotopy equivalence.
By the Theorem 2.5, all inequalities of (6) are in fact equalities, this implies that $\mathrm{Vol}(X) = \mathrm{Vol}(Y)$ and that, for almost every g, the ideal simplex $\bar{f}(g \circ \sigma_0)$ has maximal volume T_n, and thus is also regular. This implies that every regular ideal simplex is sent onto a regular ideal simplex by $\bar{f}$, and thus that $\bar{f}$ is the trace on the ideal boundaries of an isometry from $\mathbb{H}^n = \widetilde{Y}$ onto $\mathbb{H}^n = \widetilde{X}$. $\square$

2.6 A lower bound for the minimal volume of a compact hyperbolic manifold :

Let us consider any manifold X which admits a hyperbolic metric (denoted by g_0). Theorems 2.1 and 2.5 imply the

Theorem 2.9 (M. Gromov, [Gr 1]) .– $\mathrm{MinVol}(X) \geq \dfrac{C_n}{T_n} \mathrm{Vol}(X, g_0)$, *where the constants C_n and T_n are defined in Theorems 2.1 and 2.5 .*

Let us remind the two objections that we have made in the section 1.4 about the Gauss-Bonnet-Allendœrfer-Chern-Weil approach. M. Gromov's theorem 2.9 answers quite conveniently to the objection (2) of the section 1.4, for it provides a non trivial lower bound for the Minimal Volume of every compact hyperbolic manifolds in any dimension (in the odd dimensional as in the even dimensional case). On the contrary, it does not answer the objection (1) of the section 1.4 because it is not sharp (for $\dfrac{C_n}{T_n} << 1$).
This led M. Gromov ([Gr 1]) making the following conjectures.

2.7 M. Gromov's conjectures about Minimal Volume :

Except in dimension 2 (cf.section 1) or when it vanishes, one could never compute the exact value of the minimal volume of a given manifold. That is the reason why M. Gromov was naturally led to ask the following :

Question 2.10 *For a given manifold X, what is the exact value of* $\mathrm{MinVol}(X)$ *?*

The answer to this question in dimension 2 (see section 1.2) is that the minimal volume of a surface which admits a hyperbolic metric is the area of this hyperbolic metric (this metric is not unique, but all hyperbolic metrics on the same surface have the same area). As the possibility of a generalization of this result in dimension n ($n \geq 3$) is suggested by the theorem 2.9, in the case where X admits a *hyperbolic metric* g_0 (i. e. a metric with constant sectional curvature $K_{g_0} \equiv -1$), M. Gromov asked the following more precise

Conjecture 2.11 *If some manifold* X *admits a hyperbolic metric (denoted by* g_0*) then* $\mathrm{MinVol}(X) = \mathrm{Vol}(X, g_0)$.

It is clear that, if the answer to the conjecture 2.11 is positive, it solves the question 2.10 in the hyperbolic case.
These two conjectures are explicitely asked by M. Gromov in [Gr 1].

The conjecture 2.11 says that the functional $g \mapsto \mathrm{Vol}(g)$, defined on the set of the metrics g on M which satisfy $-1 \leq K_g \leq 1$, attains its minimum at the point g_0. It is thus a natural question to ask if this minimum is unique, this leads to the

Conjecture 2.12 *If some compact manifold* X *(of dimension* $n \geq 3$*) admits a metric* g_0 *with constant sectional curvature* $K_{g_0} \equiv -1$*, any metric on* X *for which the minimal volume is attained is isometric to* g_0.

The answer to the conjecture 2.12 would be negative in dimension 2, for the minimal volume is attained for any metric lying in the Teichmüller space of hyperbolic metrics, and it is well-known that these metrics are not isometric to a fixed one (here denoted by g_0). The problem is quite different in higher dimensions because, by Mostow's rigidity theorem 2.8, when X has dimension $n \geq 3$ and admits a metric g_0 with constant sectional curvature $K_{g_0} \equiv -1$, this metric is unique (up to isometries). By the way, let us notice that Mostow's rigidity theorem would be an immediate consequence of a positive answer to the conjectures 2.11 and 2.12.

3 Our approach :

3.1 Classical Kählerian Schwarz lemmas :

The original Schwarz Lemma may be rewritten in the language of the hyperbolic geometry, that is to say on the ball B^2 endowed with the hyperbolic metric $g_0 = \dfrac{4}{(1 - \|x\|^2)^2}\, g_E$, where g_E is the canonical euclidean metric.

Theorem 3.1 (Schwarz Lemma, revisited by Pick).– *Any holomorphic map* $f : B^2 \to B^2$ *is a contracting map from* (B^2, g_0) *into* (B^2, g_0).

In other words, let $\mathrm{Jac}_g f(x)$ be the determinant of the (real) differential of f at the point $x \in B^2$, computed with respect to the metric g, the classical Schwarz Lemma says that $\mathrm{Jac}_{g_E} f(0) \leq 1$ and Pick remarked that this implies that $\mathrm{Jac}_{g_0} f(x) \leq 1$ at any point $x \in B^2$.

Considering holomorphic maps between compact Kählerian manifolds of any dimensions, there have been many extensions of the above Schwarz lemma (due in particular to L. Ahlfors, S. T. Yau, N. Mok, and others...). We shall choose the following one, which may be found for instance in [Mok] (see [B-C-G 3] appendix A for a complete proof).

Proposition 3.2 .– *Let X, Y be compact Kählerian manifolds of the same dimension. If* $\mathrm{Ricci}_{g_Y} \geq -C^2 \geq \mathrm{Ricci}_{g_X}$ *, then any holomorphic map $F : Y \to X$ satisfies : $\forall y \in Y$, $|\mathrm{Jac}\, F|\,(y) \leq 1$. Moreover, if $|\mathrm{Jac}\, F|\,(y) = 1$ at some point y , then d_yF is isometric.*

Let us recall that Ricci_g is *the Ricci curvature tensor of the metric g* , and that *the assumption* $\mathrm{Ricci}_g \geq -C^2$ *means that* $\mathrm{Ricci}_g(u, u) \geq -C^2 \,.\, g(u, u)$ for any tangent vector u .
Here we have denoted by $|\mathrm{Jac}\, F|\,(y)$ the "absolute Jacobian of F" at the point y , i. e. the square root of the determinant of the matrix $\left(g_X(d_yF\,(e_i)\,,\, d_yF\,(e_j)\right)_{1\leq i,j\leq \dim(Y)}$, where $\{e_1\,,\, e_2\,,\,\ldots\,,\, e_n\}$ is any g_Y−orthonormal basis of T_yY . This definition gives the absolute value of the usual Jacobian determinant when $\dim(Y) = \dim(X)$, but notice that this definition also works in the case where $\dim(Y) < \dim(X)$.

Let us remark that, when the sectional curvature of X is negative, there is at most one holomorphic map : $Y \to X$ in each homotopy class ([Ha]). When such a map exists, it realizes (in some sense) the "best possible choice" (in the homotopy class) for a map F if one wants F to contract the volumes. So a natural question is the following one : when the manifolds Y, X are not any more assumed to be complex, or when the homotopy class does not contain any holomorphic map, what is the "best possible choice" for F ?

3.2 A real Schwarz lemma :

Let us denote by $\mathrm{Vol}_n(B)$ the Riemannian Hausdorff measure of some subset B of a Riemannian manifold (M, g) . If $\dim(M) = n$, and if B is a measurable subset of M , then $\mathrm{Vol}_n(B)$ is the usual Riemannian measure of B . If $\dim(M) > n$, and if B is a measurable subset of some n−dimensional submanifold of M , then $\mathrm{Vol}_n(B)$ is the usual Riemannian n−dimensional submanifold measure of B .

Theorem 3.3 ([B-C-G 1], improved in [B-C-G 2] and [B-C-G 3]) .– *Let Y^n , X^m be (real) complete Riemannian manifolds satisfying $3 \leq \dim(Y) \leq \dim(X)$, let us assume that there exists some constant $C \neq 0$ such that $K_{g_X} \leq -C^2$ and that $Ricci_{g_Y} \geq -(n-1)\, C^2 .\, g_Y$. Then any continuous map $f : Y \to X$ may be deformed to a family of C^1 maps F_ε $(\varepsilon \to 0_+)$ such that $\mathrm{Vol}_n[F_\varepsilon(A)] \leq (1+\varepsilon)\ \mathrm{Vol}(A)$ for any domain A (with smooth boundary) in Y . Moreover:*

(i) If Y, X are compact with the same dimension, then $\mathrm{Vol}(Y) \geq |degf|\ \mathrm{Vol}(X)$ and if $\mathrm{Vol}(Y) = |degf|\ \mathrm{Vol}(X)$, then Y, X have constant sectional curvature (equal to $-C^2$), and we may choose the family of maps $(F_\varepsilon)_{\varepsilon>0}$ such that they converge, when $\varepsilon \to 0$, to a Riemannian covering F (an isometry when $|\deg f| = 1$).

(ii) If Y, X are compact, homotopically equivalent, of the same dimension,and if $K_{g_Y} < 0$, then any homotopy equivalence f may be deformed to a smooth map F such that $\mathrm{Vol}_n[F(A)] \leq \mathrm{Vol}(A)$ for any open subset A in Y , the equality beeing attained iff F is an isometry on A.

Remarks 3.4 (cf [B-C-G 3]) : The property (ii) of the theorem 3.3 remains valid when $\dim(Y) < \dim(X)$ and when X is not compact (however, we have then to assume that $\pi_1(X)$ acts on the total space $\widetilde{X}$ of the universal covering $\widetilde{X} \to X$ in a "convex-cocompact" way, i. e. we have to assume that X may be retracted to a compact submanifold with convex boundary).
In this case, we still can deform any homotopy equivalence $f : Y \to X$ to some map F such that $|\mathrm{Jac}F| \leq 1$ and such that $|\mathrm{Jac}F| \equiv 1$ iff F is isometric and moreover totally geodesic.

Let us also notice that the maps F_ε and F of the Theorem 3.3 and of the remark 3.4 are explicitely constructed (see [B-C-G 1], [B-C-G 2], [B-C-G 3] and the sections 4.1, 4.5 and 4.6 of the present paper). Thus, applied to the case where Y and X are both hyperbolic, this theorem also gives a construction of the isometry which was only proved to exist in the Mostow's rigidity Theorem (Theorem 2.8).

Before giving the ideas of the proof of the theorem 3.3, let us first settle some applications:

3.3 Applications to minimal (and maximal) volume problems :

The following corollary answers the conjectures 2.11, 2.12 (and the conjecture 2.10 in the case of a manifold which admits a hyperbolic metric).

Corollary 3.5 ([B-C-G 1]), improved in [B-C-G 3]) .– *Let X be a compact manifold with dimension $n \geq 3$. If X admits a hyperbolic metric g_0 (i.e. $K_{g_0} \equiv -1$), then*

(i) $\mathrm{MinVol}(X) = \mathrm{Vol}(g_0) = \mathrm{MaxVol}(X)$.

(ii) A metric g on X (such that $|K_g| \leq 1$) realizes the minimal (resp. the maximal) volume iff it is isometric to g_0.

(iii) For any other manifold Y^n and any map $f : Y^n \to X^n$, one has : $\mathrm{MinVol}(Y) \geq |\deg f|\ \mathrm{MinVol}(X)$.

Proof of (i) and (ii) : Any metric g on X satisfying the assumption $|K_g| \leq 1$ obviously verifies $Ricci_g \geq -(n-1)$. As $K_{g_0} \equiv -1$, by the theorem 3.3, this implies, for every ε, the existence of a map $F_\varepsilon : (X, g) \to (X, g_0)$, homotopic to id_X, such that $|\mathrm{Jac}F_\varepsilon| \leq 1 + \varepsilon$. We thus get :

$$(1+\varepsilon)\ \mathrm{Vol}(g) \geq \int_X |\mathrm{Jac}F_\varepsilon|\ dv_g \geq |\deg F_\varepsilon|\ \mathrm{Vol}(g_0) = \mathrm{Vol}(g_0) \quad .$$

This proves the first equality of (i).
Moreover, if $\mathrm{Vol}(g) = Vol(g_0)$, the equality case in the theorem 3.3 (i) proves that $(F_\varepsilon)_{\varepsilon>0}$ converges, when $\varepsilon \to 0$, to a Riemannian covering of degree 1, i.e. an isometry. This proves (ii).
On the contrary, if $K_g \leq -1$, by the theorem 3.3 (ii), there exists a map $F : (X, g_0) \to (X, g)$, homotopic to id_X, such that $|\mathrm{Jac}F| \leq 1$. We thus get :

$$\mathrm{Vol}(g_0) \geq \int_X |\mathrm{Jac}F| dv_{g_0} \geq |degF|.\mathrm{Vol}(g) = \mathrm{Vol}(g)$$

This proves the second equality of (i)... □

Remarks 3.6 : Improving the arguments of the corollary 3.5, A.Sambusetti ([Sam 3]), theorem 5.1) proved a sharp version of M. Gromov's theorem 2.1 for any manifold Y^n when there exists some continuous map f from Y^n to an hyperbolic manifold X^n such that the induced representation $f_* : \pi_1(Y) \to \pi_1(X)$ is an isomorphism (or, more generally, such that its kernel has subexponential growth). In fact, he proves that, in these cases, $\mathrm{MinVol}(Y) \geq T_n$ Simpl $\mathrm{Vol}(Y)$ (let us recall that the equality is attained for real hyperbolic manifolds by the theorems 2.5 and 3.5 (i)).

3.4 Applications to Einstein manifolds :

On a given manifold, *an* **Einstein metric** *is a Riemannian metric whose Ricci curvature tensor is proportional to the metric* (and thus is constant on the unit tangent bundle). In dimensions 2 and 3, every Einstein metric has constant sectional curvature, so the fundamental problem of describing the *whole* moduli space of Einstein metrics on a given manifold only starts at dimension 4.

Notice that, since the solution by Aubin and Yau of the Calabi's conjecture, we know quite well what is the moduli space of Kähler-Einstein metrics on a given complex manifold (at least when the Ricci curvature is nonpositive).
Here we are interested with the moduli space of all Riemannian Einstein metrics, the moduli space of Kähler-Einstein metrics beeing a relatively small subset of this moduli space.

3.4.1 Obstructions to the existence of Einstein metrics:

When the dimension (denoted by n) is greater than 5, one does not know any obstruction to the existence of Einstein metrics: for instance, there is no counter-example to the following.

Conjecture 3.7 .– *Every compact manifold of dimension $n \geq 5$ admits at least one Einstein metric.*

In dimension 4, the following obstructions to the existence of Einstein metrics on a given manifold Y were known :

Obstructions 3.8 *to the existence of Einstein metrics on a given compact $4-$dimensional manifold Y :*

(i) *If $\chi(Y) < 0$, then Y does not admit any Einstein metric* (M. Berger, [Bes 2]).

(ii) *If $\chi(Y) - \frac{3}{2}|\tau(Y)| < 0$ (where $\tau(Y)$ is the signature of Y), then Y does not admit any Einstein metric* (J. Thorpe, [Bes 2] p 210).

(iii) *If $\chi(Y) < \dfrac{1}{2592\,\pi^2}$ $\mathrm{Simpl\,Vol}\,(Y)$, then Y does not admit any Einstein metric* (M. Gromov, [Gr 1], see also [Bes 2] Theorem 6.47).

The obstructions 3.8 (i) and 3.8 (ii) derive from the Allendœrfer-Chern-Weil formulas (see the section 1.3). In fact, the space of the quadrilinear forms (on a $4-$dimensional euclidean space V) which satisfy the same algebraic properties as a curvature tensor (i.e.

the space of symmetric bilinear forms on $\wedge^2(V)$ which satisfy the first Bianchi identity) splits as the direct sum of 4 subspaces which are irreducibly invariant under the action of SO(V). When $V = T_yY$ (endowed with the scalar product g at the point y), let W_g^+, W_g^-, Z_g and U_g be the components of the Riemannian curvature tensor R_g with respect to this decomposition. The components W_g^+ and W_g^- both vanish iff the metric g is locally conformally flat and only differ by the fact that the composition by the Hodge operator $*$ acts as $+id$ or $-id$ on each of them. The component Z_g corresponds to the trace-free part of the Ricci curvature tensor and vanishes iff g is Einstein. On the contrary, the component U_g is the canonical scalar product on $\wedge^2(V)$ associated to g, multiplied by the scalar curvature (see for instance [Bes 2], theorem 1.126, and [Bes 3] for more explanations).
The Allendœrfer-Chern-Weil theory (see section 1.3, [Bes 2] p. 161 and [Bes 3]) gives the following formulas for the Euler characteristic $\chi(Y)$ and for the signature $\tau(Y)$ of Y:

$$8\pi^2\ \chi(Y) = \int_Y (\|W_g^+\|^2 + \|W_g^-\|^2 - \|Z_g\|^2 + \|U_g\|^2)\ dv_g,$$

$$12\pi^2\ \tau(Y) = \int_Y (\|W_g^+\|^2 - \|W_g^-\|^2)\ dv_g.$$

As a consequence, any Einstein metric g on Y is such that $\|U_g\|^2$ is constant and satisfies:

$$8\pi^2\ \chi(Y) = \int_Y (\|W_g^+\|^2 + \|W_g^-\|^2 + \|U_g\|^2)\ dv_g \geq \|U_g\|^2\ \mathrm{Vol}(Y,g)\quad , \tag{7}$$

$$8\pi^2 \left(\chi(Y) \pm \frac{3}{2}\ \tau(Y)\right) = \int_Y (2\,\|W_g^\pm\|^2 + \|U_g\|^2)\ dv_g \geq \|U_g\|^2\ \mathrm{Vol}(Y,g)\quad ; \tag{8}$$

moreover any metric g_0 with constant sectional curvature satisfies $W_{g_0}^+ = W_{g_0}^- = 0$ (because g_0 is then locally conformally flat), and both inequalities (7) and (8) are equalities in this case.
The obstructions 3.8 (i) and 3.8 (ii) immediately follows from the fact that all the terms of the inequalities (7) and (8) are nonnegative when g is Einstein.
On the other hand, the Theorem 2.1 bounds the volume from below in terms of the simplicial volume when one assumes the sectional curvature to be bounded. In fact, M. Gromov proved a result which is stronger than the Theorem 2.1: he proved that $\mathrm{Vol}(M,g) \geq C_n\ \mathrm{Simpl\,Vol}(M)$ for any metric g satisfying $\mathrm{Ricci}_g \geq -(n-1)\,g$ ([Gr 1]) p. 12, [Bes 2] result 6.46). For an Einstein manifold (Y,g), this new version of the Theorem 2.1 writes

$$|\mathrm{scal}_g|^{n/2}\ \mathrm{Vol}(Y,g) \geq A_n\ \mathrm{Simpl\,Vol}(Y)\quad ,$$

where scal_g is the (constant) scalar curvature of the metric g and where A_n is a universal constant. In dimension 4, comparing with the inequality (7), this implies the existence of a universal constant B such that

$$\chi(Y) \geq B\ |scal_g|^2\ \mathrm{Vol}(Y,g) \geq B\,A_4\ \mathrm{Simpl\,Vol}(Y)\quad .$$

One ends the proof of the obstruction 3.9 by estimating the constants B and A_4 ([Gr 1], [Bes 2] result 6.46). $\square$

Looking at J. Thorpe's Theorem 3.8 (ii), one might conjecture that any manifold which satisfies $\chi(Y) - \frac{3}{2}|\tau(Y)| > 0$ admits an Einstein metric. M. Gromov's Theorem 3.8 (iii) already gave some counter-examples (see for example [Bes 2] example 6.48); A. Sambusetti gave a systematic answer to this question by proving the

Proposition 3.9 (A.Sambusetti, [Sam 1], [Sam 2]) .– *For any* $4-$*dimensional manifold* Z *, there exists an infinity of (non pairwise homeomorphic)* $4-$*dimensional manifolds* Y_i *such that each* Y_i *has the same signature and the same Euler characteristic as* Z*, and does not admit any Einstein metric.*

The manifolds Y_i are obtained by gluing to any hyperbolic compact manifold X (such that $\chi(X) > \chi(Z)$) enough copies of $\mathbb{C}P^2$ (with the direct or reverse orientation) and enough copies of $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{S}^2 \times \mathbb{T}^2$, in order to obtain the right signature and the right Euler characteristic (let us notice that $\tau(X) = 0$). As there exists a map $Y_i \to X$ of degree 1 , the non-existence of Einstein metrics on Y_i is a direct consequence of the

Proposition 3.10 (A. Sambusetti, [Sam 1], [Sam 2]) .– *Let* Y *be a compact* $4-$*dimensional manifold. Let us assume that there exists a hyperbolic* $4-$*dimensional manifold* X *and a continuous map* $f : Y \to X$ *which satisfies* $|\deg f|\ (\chi(X) - \frac{3}{2}|\tau(X)|) > \chi(Y) - \frac{3}{2}\ |\tau(Y)|$*, then* Y *does not admit any Einstein metric.*

Example : As an illustration of this proposition, let us consider the connected sum $X\sharp X$ of two copies of a compact hyperbolic $4-$manifold. By cellular decomposition, one gets $\chi(X\sharp X) = 2\,\chi(X) - 2 < 2\,\chi(X)$, and there exists an obvious map $f : X\sharp X \to X$ of degree 2 . The proposition 3.10 then proves that $X\sharp X$ does not admit any Einstein metric.

Further obstructions to the existence of Einstein metrics on some 4-manifolds (obtained by blow-up from complex surfaces, cf. [LeB 2]) were given by C. LeBrun : for example, if one glues sufficiently many copies of $\mathbb{C}P^2$ (with the reverse orientation) to a compact complex hyperbolic manifold, one obtains manifolds which do not admit any Einstein metric.
For a complete survey about Einstein manifolds, see [Bes 2].

3.4.2 Is a given Einstein metric unique ?

The solution by Aubin and Yau of the Calabi's conjecture provides us with large moduli spaces of Kähler Einstein metrics on given complex manifolds (like, for example, the $K3-$spaces of (real) dimension 4). Thus uniqueness is not true in general.
On the contrary, when a manifold X is known to admit a very canonical Einstein structure, i. e. a (rank one) locally symmetric one, the main problem is the

Conjecture 3.11 .– *On a manifold* X *which admits a locally symmetric metric with negative curvature, this metric is (modulo homotheties) the only Einstein metric.*

This would be a very strong version of the Mostow's rigidity Theorem 2.8. In fact, in Mostow's rigidity theorem, one assumes the sectional curvature to be constant (equal to -1), thus every possible candidate is a quotient of the hyperbolic space-form $\mathbb{H}^n$ (endowed with its canonical hyperbolic metric) by some discrete subgroup Γ of $Isom(\mathbb{H}^n)$. So all possible candidates are locally isometric and the problem is then to decide if two

isomorphic subgroups Γ and Γ' of $Isom(\mathbb{H}^n)$ give two globally isometric quotient-spaces $\mathbb{H}^n/\Gamma$ and $\mathbb{H}^n/\Gamma'$ (assumed to be manifolds), in which case the isomorphism $\rho : \Gamma \mapsto \Gamma'$ extends to an isomorphism $\bar{\rho} : Isom(\mathbb{H}^n) \to Isom(\mathbb{H}^n)$ given by $\bar{\rho}(g) = \widetilde{F} \circ g \circ \widetilde{F}^{-1}$, where $\widetilde{F} : \mathbb{H}^n \to \mathbb{H}^n$ is the lift of the isometry $F : \mathbb{H}^n/\Gamma \to \mathbb{H}^n/\Gamma'$.
On the contrary, two Einstein metrics (with the same constant Ricci curvature, equal to $-(n-1)$ for example) are generally *not locally isometric* : for example, the real and complex hyperbolic spaces are both Einstein with constant Ricci curvature equal to $-(n-1)$ (if we rescale the complex hyperbolic metric), but they are not locally isometric. Moreover, the possible local models for Einstein manifolds are much more numerous, and generally not locally symmetric. There thus must be some global arguments proving that, even if local a priori considerations say that all these models have to be taken into account, only one of this models remains valid at the end for global reasons.
This difficulty is certainly the reason why the first (partial) answers to the above conjecture 3.11 where given in 1994 by two different methods (one uses the real Schwarz lemma, the other the Seiberg-Witten invariants). That is the

Theorem 3.12 ([B-C-G 1]) .– *Let X be a compact $4-$dimensional manifold which admits a real hyperbolic metric, then this metric is (modulo homotheties) the only Einstein metric on X .*

Theorem 3.13 (C. LeBrun, [LeB 1]) .– *Let X be a compact $4-$dimensional manifold which admits a complex hyperbolic metric, then this metric is (modulo homotheties) the only Einstein metric on X .*

Remarks 3.14 *:* A. Sambusetti noticed that the theorem 3.12 may be generalized in the following way: *For any compact Einstein $4-$dimensional manifold (Y, g) , if there exists some hyperbolic $4-$dimensional manifold (X, g_0) and some $d \in \mathbb{N}^*$ such that $(\chi(X) - \frac{3}{2}|\tau(X)|) = \dfrac{1}{d}\left(\chi(Y) - \frac{3}{2}\,|\tau(Y)|\right)$ and if there exists some continuous map $f : Y \to X$ of degree $\pm d$, then (Y, g) is hyperbolic. Moreover f is homotopic to a Riemannian covering (an isometry if f has degree 1).*

This also gives a complete characterization of the equality case in the inequality of the Theorem 3.10.

Proofs of theorems 3.12 and 3.10 and of the remark 3.14 .– Let (Y, g) be the Einstein manifold and (X, g_0) the real hyperbolic one (in the theorem 3.12, we have $Y = X$). We may always assume the existence of a map $f : Y \to X$ of non zero degree (in the theorem 3.12, $f = id_X$ has degree 1). As the assumptions imply that the Ricci curvature cannot be nonnegative, one may assume (after rescaling) that $Ricci_g = -(n-1)g$. The real Schwarz lemma (Theorem 3.3 (i)) then implies that

$$\mathrm{Vol}(Y, g) \geq |\deg f|\ \mathrm{Vol}(X, g_0). \tag{9}$$

On the other hand, as $|U_g|^2 = |U_{g_0}|^2$ is a constant (here denoted by C^2), the inequality (8) and its equality case give :

$$\chi(Y) - \frac{3}{2}\,|\tau(Y)| \geq \frac{C^2}{8\pi^2}\,\mathrm{Vol}(Y, g) \quad \text{and} \quad \chi(X) - \frac{3}{2}\,|\tau(X)| = \frac{C^2}{8\pi^2}\,\mathrm{Vol}(X, g_0) \ \ ,$$

and implies that $\chi(Y) - \dfrac{3}{2}\,|\tau(Y)| \geq |\deg f|\,\left(\chi(X) - \dfrac{3}{2}\,|\tau(X)|\right)$, in contradiction with the assumptions of the Proposition 3.10. Thus, under the assumptions of the Proposition 3.10, the non-existence of any Einstein metric on Y is proved.
In the remark 3.14 (resp. in the Theorem 3.12), the last inequality is an equality. This implies that the inequality (9) is also an equality and thus, applying the equality-case of the Theorem 3.3 (i), that f is homotopic to a Riemannian covering (resp. to an isometry) $F : (Y, g) \to (X, g_0)$.
This ends the proofs of the remark 3.14 (resp. of the Theorem 3.12). $\square$

4 Sketch of the proof of the real Schwarz lemma: (see [B-C-G 1], [B-C-G 2], [B-C-G 3] for a complete proof)

We shall first prove the theorem 3.3 (ii).
Rescaling the metrics g_Y and g_X, we notice that, in order to prove the theorem 3.3 in the general case, it is sufficient to prove it in the case where the constant $-C^2$ which bounds the curvature of Y and X is equal to -1. In order to simplify the proof, we shall moreover suppose that $K_{g_X} = -1$ (the proof in the general case $K_{g_X} \leq -1$ is given in [B-C-G 3]).

4.1 Reduction of the problem:

We have already seen how to compactify the Riemannian universal covering $(\widetilde{X}, \widetilde{g}_0)$ of a compact hyperbolic manifold (X, g_0) by adding the ideal boundary $\partial\widetilde{X} = S^{n-1}$ (see section 2.3.2). It is classical that this construction generalizes to the Riemannian universal covering $(\widetilde{Y}, \widetilde{g}_Y)$ of any negatively curved compact Riemannian manifold (Y, g_Y), the ideal boundary $\partial\widetilde{Y}$ beeing defined as the set of geodesics of $(\widetilde{Y}, \widetilde{g}_Y)$, quotiented by the equivalence relation which identifies two geodesics c_1 and c_2 of $(\widetilde{Y}, \widetilde{g}_Y)$ iff $d_{\widetilde{Y}}(\,c_1(t)\,,\,c_2(t)\,)$ is bounded when $t \to +\infty$ (*where* $d_{\widetilde{Y}}$ *denotes the Riemannian distance in* $(\widetilde{Y}, \widetilde{g}_Y)$); the corresponding point of the ideal boundary is denoted by $\theta = c_1(+\infty) = c_2(+\infty)$. Choosing an origin $\tilde{y}_0$ in $\widetilde{Y}$, one may identify the unit sphere $U_{\tilde{y}_0}$ of $T_{\tilde{y}_0}\widetilde{Y}$ with $\partial\widetilde{Y}$ by the map $v \mapsto c_v(+\infty)$, where c_v is the geodesic of $(\widetilde{Y}, \widetilde{g}_Y)$ such that $\dot{c}_v(0) = v$.
The topology of $\partial\widetilde{Y}$ is defined when deciding that this map is a homeomorphism. Rescaling the time-parameter of the geodesic c_v, one may also identify $\widetilde{Y}$ with the unit ball $B_{\tilde{y}_0}$ of $T_{\tilde{y}_0}\widetilde{Y}$: this gives the (topological) compactification of $\widetilde{Y}$, i. e. $\widetilde{Y} \cup \partial\widetilde{Y} = B_{\tilde{y}_0} \cup U_{\tilde{y}_0}$.
Let us also notice that the action $\tilde{y} \mapsto \gamma\cdot\tilde{y}$ of $\Gamma = \pi_1(Y)$ on $\widetilde{Y}$ (by deck-transformations) induces an action $c \mapsto \gamma \circ c$ on geodesics, an thus an action on $\partial\widetilde{Y}$ that we shall still denote by $\theta \mapsto \gamma \cdot \theta$.
As we have already seen in the proof of the Theorem 2.8, any homotopy equivalence $f : Y \to X$ may be lifted to a quasi-isometry $\widetilde{f} : \widetilde{Y} \to \widetilde{X}$, which may be continuously extended to a homeomorphism $\bar{f} : \partial\widetilde{Y} \to \partial\widetilde{X}$ (see for instance [B-P]). Moreover, if $\rho = [f]$ is the induced representation $\pi_1(Y) \to \pi_1(X)$, for any $\gamma \in \pi_1(Y)$, one has the following "*equivariance properties*" :

$$\widetilde{f} \circ \gamma = \rho(\gamma) \circ \widetilde{f} \qquad , \qquad \bar{f} \circ \gamma = \rho(\gamma) \circ \bar{f} \quad . \tag{10}$$

For any topological space Z, we shall denote by $\mathcal{M}(Z)$ the space of positive finite Borel measures on Z. Denoting by $\bar{f} * \mu$ the push-forward of a measure $\mu \in \mathcal{M}(\partial\widetilde{Y})$, one defines a map $\bar{f}_* : \begin{cases} \mathcal{M}(\partial\widetilde{Y}) \to \mathcal{M}(\partial\widetilde{X}) \\ \quad \mu \quad \mapsto \quad \bar{f} * \mu \end{cases}$.

By the equivariance properties (10), this map is equivariant, i. e. $\bar{f}_* \circ \gamma_* = \rho(\gamma)_* \circ \bar{f}_*$. In order to construct the map $F : Y \to X$, it is thus sufficient to construct an equivariant map $\widetilde{F} : \widetilde{Y} \to \widetilde{X}$ that we shall define by

$$\widetilde{F}(\tilde{y}) = \mathrm{bar}\,(\bar{f} * \mu_{\tilde{y}}), \tag{11}$$

where bar is the canonical "*barycentre map*" : $\mathcal{M}(\partial\widetilde{X}) \to \widetilde{X}$, introduced by H. Furstenberg ([Fu], see also [D-E]), whose definition will be recalled in section 4.4, and where $\tilde{y} \mapsto \mu_{\tilde{y}}$ is the canonical map : $\widetilde{Y} \to \mathcal{M}(\partial\widetilde{Y})$ called the "*Patterson-Sullivan measures*", whose definition will be recalled in section 4.3.

The fact that $\widetilde{f}$ and $\widetilde{F}$ obey to the equivariance property with respect to the same representation ρ implies that f and F induce the same representation $[f] = \rho = [F] : \pi_1(Y) \to \pi_1(X)$. All manifolds of negative curvature beeing $K(\pi, 1)$, this proves that f and F are homotopic.

4.2 The Busemann function:

The Busemann function $B^Y : \widetilde{Y} \times \partial\widetilde{Y} \to \mathbb{R}$ is defined by

$$B^Y(\tilde{y}, \theta) = \lim_{t \to +\infty} \left[d_{\widetilde{Y}}(c_\theta(t), \tilde{y}) - d_{\widetilde{Y}}(c_\theta(t), \tilde{y}_0) \right] \tag{12}$$

where $\tilde{y}_0$ is the fixed origin in $\widetilde{Y}$ and where c_θ is the normal geodesic ray such that $c_\theta(0) = \tilde{y}_0$ and $c_\theta(+\infty) = \theta$.

Roughly speaking, $B^Y(\tilde{y}, \theta)$ measures the (rescaled by the choice of the origin) distance from $\tilde{y}$ to θ. Thus the function $\tilde{y} \mapsto B^Y(\tilde{y}, \theta)$ inherits all the properties of the distance function, in particular its gradient has norm equal to 1; moreover the level set of the Busemann function (called *horosphere*) which contain the point $\tilde{y}$ is the limit (when $t \to +\infty$) of the spheres of radius $d(c_\theta(t), \tilde{y})$ centered at the points $c_\theta(t)$.

4.3 The Patterson-Sullivan measures $\tilde{y} \mapsto \mu_{\tilde{y}}$:

Let us denote by $\widetilde{B}(\tilde{y}, R)$ the geodesic balls of $(\widetilde{Y}, \tilde{g}_Y)$ centered at $\tilde{y} \in \widetilde{Y}$ and of radius R. We can then settle the

Definition 4.1 *The entropy* $\mathrm{Ent}(Y)$ *of* (Y, g_Y) *is defined by*

$$\mathrm{Ent}(Y) = \lim_{R \to +\infty} \left(\frac{1}{R} \log[\,\mathrm{Vol}\,\widetilde{B}(\tilde{y}, R)\,] \right).$$

It is classical that this limit exists (when Y is compact) and does not depend on the particular choice of $\tilde{y}$.

Let now μ_0 be a finite measure on $\partial\widetilde{Y}$, *one defines the measure* $\mu_{\tilde{y}}$ on $\partial\widetilde{Y}$ by $\mu_{\tilde{y}} = e^{-\mathrm{Ent}(Y)\, B^Y(\tilde{y}, \cdot)} \mu_0$.

Remarks 4.2 .– Only some very particular choices of the measure μ_0 are convenient: in fact, in order to obtain the equivariance property for $\widetilde{F}$, we want the measures $\mu_{\tilde{y}}$ to satisfy $\mu_{\gamma\cdot\tilde{y}} = \gamma * \mu_{\tilde{y}}$ for any $\gamma \in \pi_1(Y)$ and any $\tilde{y} \in \widetilde{Y}$. This may be done by a geometric construction as follows : let $\mu^c_{\tilde{y}}$ be defined by $\mu^c_{\tilde{y}} = e^{-c\, d_{\widetilde{Y}}(\tilde{y}\,,\,\cdot)}\, dv_{\widetilde{g}_Y}$, where $d_{\widetilde{Y}}$ and $dv_{\widetilde{g}_Y}$ are the Riemannian distance and the Riemannian measure of $(\widetilde{Y}, \widetilde{g}_Y)$; $\mu^c_{\tilde{y}}$ is a measure on $\widetilde{Y}$ which is finite iff $c > \mathrm{Ent}(Y)$. We shall see $\mu^c_{\tilde{y}}$ as a family of measures on the compact set $\widetilde{Y} \cup \partial\widetilde{Y}$ and a classical compactness result says that there exists a decreasing sequence $(c_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty}(c_n) = \mathrm{Ent}(Y)$ and such that the sequence of measures $\dfrac{1}{\mu^{c_n}_{\tilde{y}_0}(\widetilde{Y})}\ \mu^{c_n}_{\tilde{y}}$ weakly converges to a measure $\mu_{\tilde{y}}$, whose support lies in $\partial\widetilde{Y}$.
From the equality $\mu^c_{\tilde{y}} = e^{-c\,[\,d_{\widetilde{Y}}(\tilde{y}\,,\,\cdot)\,-\,d_{\widetilde{Y}}(\tilde{y}_0\,,\,\cdot)]}\,\mu^c_{\tilde{y}_0}$, one easily deduces that the sequence $(c_n)_{n\in\mathbb{N}}$ may be chosen independant from $\tilde{y}$ and that $\mu_{\tilde{y}} = e^{-\mathrm{Ent}(Y)\,B^Y(\tilde{y}\,,\,\cdot)}\,\mu_{\tilde{y}_0}$. The equivariance of $\tilde{y} \mapsto \mu_{\tilde{y}}$ then derives from the equivariance of $\tilde{y} \mapsto \mu^c_{\tilde{y}}$, which is a consequence of the invariance of $d_{\widetilde{Y}}$ and $dv_{\widetilde{g}_Y}$ with respect to the isometries $\gamma \in \pi_1(Y)$. We conclude by making $\mu_0 := \mu_{\tilde{y}_0}$.

4.4 The barycentre map:

Let B^X be the Busemann function of $(\widetilde{X}, \widetilde{g}_X)$; given $\mu \in \mathcal{M}(\partial\widetilde{X})$, the function

$$\mathcal{D}_\mu(\tilde{x}) = \int_{\partial\widetilde{X}} B^X(\tilde{x}, b)\ d\mu(b)$$

may be seen as the mean value of the distance from the point $\tilde{x}$ to $\partial\widetilde{X}$. When $K_{g_X} < 0$, the distance-function $d_{\widetilde{X}}(\cdot\,,\,z)$ is convex and thus $\tilde{x} \mapsto B^X(\tilde{x}, b)$ and $\mathbf{D}_\mu$ are also convex when restricted to any geodesic. Moreover, if the measure μ is sufficiently spread, more precisely if every single point $\tilde{x} \in \partial\widetilde{X}$ satisfies $\mu(\{\tilde{x}\}) < \frac{1}{2}\,\mu(\partial\widetilde{X})$, then $\mathbf{D}_\mu$ is *strictly* convex and goes to $+\infty$ at infinity. Thus $\mathcal{D}_\mu$ attains its minimal value at its unique critical point, denoted by $\mathrm{bar}(\mu)$ and called the "*barycentre of* μ" and characterized as the solution of the implicit equation :

$$(d\mathcal{D}_\mu)_{|_{\mathrm{bar}(\mu)}} = 0 \quad . \tag{13}$$

4.5 Implicit formulas for $\widetilde{F}$ and $d\widetilde{F}$:

Let us define $\mathbb{D} : \widetilde{X} \times \widetilde{Y} \to \mathbf{R}$ by

$$\mathbb{D}(\tilde{x}, \tilde{y}) = \mathcal{D}_{\bar{f}*\mu_{\tilde{y}}}(\tilde{x}) = \int_{\partial\widetilde{Y}} B^X(\tilde{x}, \bar{f}(\theta))\ e^{-\mathrm{Ent}(Y)\,B^Y(\tilde{y}\,,\,\theta)}\ d\mu_0(\theta)$$

and let ∂^1 (resp. ∂^2) denotes the derivative with respect to the first (resp. to the second) parameter in $\widetilde{X}$ (resp. in $\widetilde{Y}$). By the definition (11) and the characterization (13), $\widetilde{F}$ is defined by the implicit equation: $\partial^1\mathbb{D}_{|_{(\widetilde{F}(\tilde{y}),\tilde{y})}} = 0$. By derivation, it comes

$$\partial^1\partial^1\mathbb{D}_{|_{(\widetilde{F}(\tilde{y}),\tilde{y})}}(d\widetilde{F}(u), v) = -\partial^2\partial^1\mathbb{D}_{|_{(\widetilde{F}(\tilde{y}),\tilde{y})}}(u, v)$$

for any $u \in T_{\tilde{y}}\widetilde{Y}$ and any $v \in T_{\widetilde{F}(\tilde{y})}\widetilde{X}$. This writes

$$\int_{\partial\widetilde{Y}} DdB^X_{|(\widetilde{F}(\tilde{y}),\bar{f}(\theta))}\left(d\widetilde{F}(u), v\right) d\mu_{\tilde{y}}(\theta) = \mathrm{Ent}(Y) \int_{\partial\widetilde{Y}} dB^X_{|(\widetilde{F}(\tilde{y}),\bar{f}(\theta))}(v)\; dB^Y_{|(\tilde{y},\theta)}(u)\; d\mu_{\tilde{y}}(\theta)$$

$$\leq \mathrm{Ent}(Y)\;\; \widetilde{g}_X(H_{\tilde{y}}(v), v)^{1/2}\;\; \widetilde{g}_Y(K_{\tilde{y}}(u), u)^{1/2}, \tag{14}$$

where $H_{\tilde{y}}$ (resp. $K_{\tilde{y}}$) is the symmetric endomorphism of $T_{\widetilde{F}(\tilde{y})}\widetilde{X}$ (resp. of $T_{\tilde{y}}\widetilde{Y}$) associated to the quadratic form $v \mapsto \int_{\partial\widetilde{Y}}\left(dB^X_{|(\widetilde{F}(\tilde{y}),\bar{f}(\theta))}(v)\right)^2 d\mu_{\tilde{y}}(\theta)$ (resp. to the quadratic form $u \mapsto \int_{\partial\widetilde{Y}}\left(dB^Y_{|(\tilde{y},\theta)}(u)\right)^2 d\mu_{\tilde{y}}(\theta)$).

As the gradient of $B^X(\,\cdot\,,\bar{f}(\theta))$ at the point $\tilde{x}$ is the unit vector normal to the horosphere of $\widetilde{X}$ centered at the point $\bar{f}(\theta)$ and containing the point $\tilde{x}$, the second fundamental form of this horosphere is equal to the restriction (to the hyperplane $T_{\tilde{x}}$ tangent to the horosphere at the point $\tilde{x}$) of $DdB^X_{|(\tilde{x},\bar{f}(\theta))}(\,\cdot\,,\,\cdot\,)$. When $(\widetilde{X}, \widetilde{g}_X)$ is the real hyperbolic space, the subgroup of the isotropy group of $\tilde{x}$ which fixes the unit normal vector acts irreducibily on $T_{\tilde{x}}$, thus the second fundamental form is diagonal and it implies

$$DdB^X = \widetilde{g}_X - dB^X \otimes dB^X \quad .$$

Plugging this in (14), it gives :

$$\widetilde{g}_X\left((Id - H_{\tilde{y}}) \circ d_{\tilde{y}}\widetilde{F}(u)\,,\, v\right) \leq \mathrm{Ent}(Y)\; \widetilde{g}_X\left(H_{\tilde{y}}(v), v\right)^{1/2}\;\; \widetilde{g}_Y\left(K_{\tilde{y}}(u), u\right)^{1/2} \quad , \tag{15}$$

which induces (by a simple argument of linear algebra) the same inequality on determinants, that is

$$\frac{\det(Id - H_{\tilde{y}})}{(\det H_{\tilde{y}})^{1/2}}\, |\det\,(d_{\tilde{y}}\widetilde{F})| \leq \mathrm{Ent}(Y)^n\; (\det\, K_{\tilde{y}})^{1/2} \leq \mathrm{Ent}(Y)^n \left(\frac{1}{n}\;\mathrm{Trace}\, K_{\tilde{y}}\right)^{n/2} \quad . \tag{16}$$

The fact that $\|dB^Y\| = 1 = \|dB^X\|$ implies that $\mathrm{Trace}\, K_{\tilde{y}} = 1 = \mathrm{Trace}\, H_{\tilde{y}}$; on the other hand, the function $A \mapsto \dfrac{\det\,(I - A)}{(\det\, A)^{1/2}}$ (defined on the set of symmetric positive definite $n \times n$ matrices ($n \geq 3$) whose trace is equal to 1) attains its minimum at the unique point $A_0 = \dfrac{1}{n}\, I$. Plugging this in (16), it gives :

$$|\det\,(d_{\tilde{y}}\widetilde{F})| \leq \left(\frac{\mathrm{Ent}(Y)}{n-1}\right)^n \leq 1 \quad , \tag{17}$$

where the last inequality comes from the comparison theorem of R. L. Bishop and the assumption $\mathrm{Ricci}_{g_Y} \geq -(n-1)\, g_Y$.

When $|\det\,(d_{\tilde{y}}\widetilde{F})| = 1$, then inequalities (16) and (17) are equalities and

$$\det\, K_{\tilde{y}} = \left(\frac{1}{n}\;\mathrm{Trace}\, K_{\tilde{y}}\right)^n \qquad \text{thus} \qquad K_{\tilde{y}} = \frac{1}{n}\, I \quad ;$$

moreover $\dfrac{\det(Id - H_{\tilde y})}{(\det H_{\tilde y})^{1/2}} = \left(\dfrac{1-\frac{1}{n}}{\sqrt{\frac{1}{n}}}\right)^n$, and thus $H_{\tilde y}$ is the point where the function $A \mapsto \dfrac{\det(I-A)}{(\det\, A)^{1/2}}$ (defined on the set of symmetric positive definite $n\times n$ matrices whose trace is equal to 1) attains its minimum; this gives $H_{\tilde y} = A_0 = \frac{1}{n}\, I$. Plugging this estimates of $H_{\tilde y}$ and $K_{\tilde y}$ in (15) and replacing v by $d_{\tilde y}\widetilde F(u)$, we deduce that

$$\|(d_{\tilde y}\widetilde F)(u)\| \le \left(\frac{\mathrm{Ent}(Y)}{n-1}\right)\|u\| \le \|u\| \quad ,$$

and thus that $d_{\tilde y}\widetilde F$ is a contracting linear map whose determinant is equal to 1 , thus it is an isometry. This ends the proof of the Theorem 3.3 (ii). $\square$

4.6 Extensions and generalizations:

The inequality of the theorem 3.3 (i) may be obtained even more easily : considering the family of measures $\mu^c_{\tilde y}$ on $\widetilde Y$ defined in the remark 4.2, we define the map $\widetilde F_c : \widetilde Y \to \widetilde X$ by

$$\widetilde F_c(\tilde y) = \mathrm{Bar}(\widetilde f * \mu^c_{\tilde y})$$

where we have modified the previous notion of Barycentre : in fact, this new Barycentre $\mathrm{Bar}(\mu)$ of a measure μ on $\widetilde X$ is now defined as the unique point where the function

$$\triangle_\mu(\tilde x) = \int_{\widetilde X} d_{\widetilde X}(\tilde x, \tilde z)^2\; d\mu(\tilde z)$$

attains its minimum (we restrict this study to measures μ such that the above integral is finite).
Replacing $B^Y(\,\cdot\,,\theta)$ and $B^X(\,\cdot\,,\bar f(\theta))$ by $d_{\widetilde Y}(\,\cdot\,,z)$ and $d_{\widetilde X}(\,\cdot\,,z)$, the same proof as in the section 4.5 works and, choosing $c = \mathrm{Ent}(Y)(1+\varepsilon)$, it gives :

$$|\det(d_{\tilde y}\widetilde F_c)| \le \left(\frac{c}{n-1}\right)^n \le (1+\varepsilon)^n \quad , \tag{18}$$

because $\mathrm{Ent}(Y) \le n-1$ by the comparison Theorem of R. L. Bishop and thus $c \le (n-1)(1+\varepsilon)$. $\square$

In the theorem 3.3 (at least in the case where the curvature is negative), it is not necessary to assume the existence of some map $f : Y \to X$. In fact, we could notice the

Remarks 4.3 *Given any homomorphism $\rho : \pi_1(Y) \to \pi_1(X)$, we can directly construct the family of maps $F_\varepsilon : Y \to X$, satisfying $|\mathrm{Jac}(F_\varepsilon)| \le 1+\varepsilon$, such that the induced homomorphism $[F_\varepsilon] : \pi_1(Y) \to \pi_1(X)$ is equal to ρ .*

Proof : Let us fix origins $\tilde y_0$ and $\tilde x_0$ in $\widetilde Y$ and $\widetilde X$ respectively, and define $\nu^c_{\tilde y} \in \mathcal M(\widetilde X)$ by

$$\nu^c_{\tilde y} = \sum_{\gamma\in\pi_1(Y)} e^{-c\, d_{\widetilde Y}(\tilde y\,,\,\gamma\cdot\tilde y_0)}\ \delta_{\rho(\gamma)\cdot\tilde x_0}$$

where $\delta_{\tilde z}$ is the Dirac measure at the point $\tilde z \in \widetilde X$. We now define $\widetilde F_c(\tilde y)$ as $\mathrm{Bar}(\nu^c_{\tilde y})$. Then the same proof as above gives

$$|\det(d_{\tilde y}\widetilde F_c)| \le \left(\frac{c}{n-1}\right)^n \le (1+\varepsilon)^n$$

It is moreover easy to verify the equivariance of $\widetilde F_c$, because $\nu^c_{\gamma\cdot\tilde y} = (\rho(\gamma)) * \nu^c_{\tilde y}$, etc. $\square$

We have just seen that the inequality of the Theorem 3.3 (i) is somewhat easier to settle than the inequality of the Theorem 3.3 (ii). On the contrary, the equality case of the Theorem 3.3 (i) is much more difficult to prove than the equality case of Theorem 3.3 (ii): the reason is that, in the Theorem 3.3 (ii), we directly construct the good candidate to be the isometry $F : Y \to X$ while, in the Theorem 3.3 (i), we have to prove that the limit F of the $F_c : Y \to X$ (when c goes to $\mathrm{Ent}(Y)$) exists and is an isometry (see [B-C-G 1] sections 7 and 8 for a proof).

5 Generalization to locally symmetric manifolds: (cf. [B-C-G 1], [B-C-G 2])

We shall now assume that (X, g_X) is a compact n-dimensional locally symmetric manifold with negative curvature, i.e. a compact quotient of the real or complex or quaternionic or Cayley hyperbolic space. The entropy of such a manifold will be denoted by $\mathrm{Ent}(X)$, and is equal to $n+d-2$ (where d is the real dimension of the corresponding real or complex or quaternionic or Cayley field) when the locally symmetric metrics are rescaled in order that the maximum of their sectional curvatures is equal to -1.

5.1 Main Theorem:

The following theorem solves conjectures of A. Katok and M. Gromov about "minimal entropy" :

Theorem 5.1 ([B-C-G 1], [B-C-G 2], [B-C-G 3]) .– *Let (X, g_X) be a compact locally symmetric manifold with negative curvature and (Y, g_Y) be any compact Riemannian manifold such that $\dim X = \dim Y \ge 3$, then any continuous map $f : Y \to X$ may be deformed to a family of C^1 maps F_ε $(\varepsilon \to 0_+)$ such that*

$$\mathrm{Vol}[F_\varepsilon(A)] \le \left(\frac{\mathrm{Ent}(Y)+\varepsilon}{\mathrm{Ent}(X)}\right)^n \mathrm{Vol}(A)$$

for any open set A in Y. In particular, one has

$$(\mathrm{Ent}(Y))^n\ \mathrm{Vol}(Y) \ge |\deg f|\ (\mathrm{Ent}(X))^n\ \mathrm{Vol}(X) \quad .$$

Moreover, if $(\mathrm{Ent}(Y))^n\ \mathrm{Vol}(Y) = |\deg f|\ (\mathrm{Ent}(X))^n\ \mathrm{Vol}(X)$, then Y is also locally symmetric and f is homotopic to a Riemannian covering F (an isometry when $|\deg f| = 1$).

Remarks 5.2 .– As in the theorem 3.3 (ii), when (Y, g_Y) is also negatively curved and when f is a homotopy equivalence, we may construct directly and explicitely the limit $F : Y \to X$ of the family $(F_\varepsilon)_{\varepsilon>0}$ when $\varepsilon \to 0_+$. It satisfies

$$\forall y \in Y \quad |\det(d_y F)| \le \left(\frac{\mathrm{Ent}(Y)}{\mathrm{Ent}(X)}\right)^n \quad , \quad \forall A \subset Y \quad \mathrm{Vol}[F(A)] \le \left(\frac{\mathrm{Ent}(Y)}{\mathrm{Ent}(X)}\right)^n \mathrm{Vol}(A)$$

and F is isometric in the equality case.

Sketch of the proof of the Theorem 5.1 : Let us recall that μ_y^c is a finite measure iff $c > \mathrm{Ent}(Y)$. Then the first inequalities of (17) and (18) and their equality-cases already proved the Theorem 5.1 and the remark 5.2 when (X, g_X) is (locally) real hyperbolic. It easily generalizes to the other cases. For example, let us assume that (X, g_X) is (locally) complex hyperbolic, the proof is exactly the same as it was when (X, g_X) was (locally) real hyperbolic (cf. section 4.5), except for the fact that DdB^X now writes

$$DdB^X = \widetilde{g}_X - dB^X \otimes dB^X + (dB^X \circ J) \otimes (dB^X \circ J) \quad .$$

This modifies the inequalities (15) and (16) but, looking at the inequality (16), the only new thing to prove is that the function $A \mapsto \dfrac{\det(I - A - JAJ)}{(\det A)^{1/2}}$ (still defined on the set of symmetric positive definite $n \times n$ real matrices whose trace is equal to 1) still attains its minimum at the unique point $A_0 = \frac{1}{n} I$. This comes from the Log-concavity of the determinant which reduces the problem to the previous one (see[B-C-G 1], Appendix B).□

5.2 Application to the general Mostow's rigidity theorem :

A corollary is a unified proof of the following rigidity theorem, initially proved by G. D. Mostow:

Theorem 5.3 *Let (X, g_X) and (Y, g_Y) be two compact locally symmetric manifolds with negative curvature such that* $\dim X = \dim Y \ge 3$*, then any homotopy-equivalence $f : Y \to X$ is homotopic to an isometry.*

Proof: Let $g : X \to Y$ be such that $g \circ f \sim id_Y$. By the Theorem 5.1 (or, more directly, by the sharp upper bounds of the determinants of $d_y F$ and $d_{F(y)} G$ given by the remark 5.2), there exists deformations F and G of f and g such that

$$\mathrm{Vol}\,(\,G \circ F(Y)\,) \le \left(\frac{\mathrm{Ent}(X)}{\mathrm{Ent}(Y)}\right)^n \left(\frac{\mathrm{Ent}(Y)}{\mathrm{Ent}(X)}\right)^n \mathrm{Vol}(Y).$$

As the degree of $G \circ F$ is equal to 1, this inequality is an equality and we are in the equality case of the remark 5.2, thus F is an isometry. □

5.3 Application to dynamics:

Two Riemannian manifolds Y and X are said to "*have the same dynamics*" iff there exists a C^1-diffeomorphism ϕ between their unitary tangent bundles UY and UX which exchanges their geodesic flows, i. e. $\phi(\dot c_v(t)) \equiv \dot c_{\phi(v)}(t)$ for any unit vector $v \in UY$ and any $t \in \mathbb{R}$, where c_v is the geodesic such that $\dot c_v(0) = v$. The fundamental question is the following

Conjecture 5.4 (believed to be by E. Hopf) .– *Two compact Riemannian manifolds Y and X which have the same dynamics are isometric.*

In the absence of any additive assumption, this conjecture is false, because their exists non isometric manifolds all of whose geodesics are closed with the same period (see [Bes 1] for more informations).

C. B. Croke and J. P. Otal separately proved this conjecture to be true when Y and X are $2-$dimensional and negatively curved.

C. B. Croke and B. Kleiner proved this conjecture to be true when one of the two manifolds is flat or, more generally, when it admits a parallel section.

A corollary of the theorem 5.1 is the following

Theorem 5.5 [B-C-G 1] .– *The conjecture is true in any dimension n, provided that one of the two manifolds is locally symmetric with negative curvature.*

Proof: The theorem is known in dimension 2, thus let us suppose that $n \geq 3$. Suppose that X is the locally symmetric manifold with negative curvature and that the other manifold (which has the same dynamics as X) is called Y. As UY and UX are homeomorphic and $n \geq 3$, the manifolds Y and X are homotopically equivalent. It is well known that the volume and the entropy are invariants of the dynamics, thus the fact that Y and X have the same dynamics implies that $\mathrm{Ent}(Y) = \mathrm{Ent}(X)$ and that $\mathrm{Vol}(Y) = \mathrm{Vol}(X)$, which means that we are in the equality-case of the Theorem 5.1. The Theorem 5.1 proves that, in the equality-case, Y is isometric to X. $\square$

5.4 Application to the Lichnerowicz's conjecture :

A Riemannian manifold is said to be "*locally harmonic*" when all geodesic spheres of its universal covering have constant mean curvature. It is well known that any locally symmetric manifold of rank one is locally harmonic. A. Lichnerowicz asked the following converse question :

Conjecture 5.6 *Any locally harmonic manifold X is locally symmetric of rank one.*

In the case where the universal covering $\widetilde{X}$, the conjecture has been proved by Z. Szabo ([Sz]), any locally harmonic manifold beeing (in this case) locally symmetric with strictly positive sectional curvature.

In the case where the universal covering $\widetilde{X}$ of X is noncompact, it is known ([Bes 1]) that the geodesics of X have no conjugate points, and the conjecture is not significatively changed when assuming the sectional curvature to be negative.
However, E. Damek and F. Ricci ([D-R]) proved the conjecture to be false (even for negatively curved locally harmonic manifolds), but we may notice that their counter-examples are noncompact harmonic manifolds which do not admit any compact quotient.

Thus the only case where the conjecture remained open was the case where $\widetilde{X}$ is non compact and admits a quotient X which is compact. It is answered by the

Corollary 5.7 [B-C-G 1] .– *Any compact negatively curved locally harmonic manifold is locally symmetric of rank one.*

Proof : It was proved by P. Foulon and F. Labourie ([F-L]) that, under these assumptions, the manifold has the same dynamics as a locally symmetric manifold with negative curvature. We conclude by applying the Theorem 5.5. $\square$

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