Inducing connections on $SU(2)$-bundles

Giuseppina D'Ambra
Dipartimento di Matematica, Via Ospedale 72 – Università di Cagliari – Italy
e-mail address: dambra@unica.it

and

Andrea Loi
Dipartimento di Matematica, Via Ospedale 72 – Università di Cagliari – Italy
e-mail address: loi@unica.it

Abstract

Given $SU(2)$-bundles $X \to V$ and $Y \to W$ with connections Γ on X and ∇ on Y and Riemannian metrics g on V and h on W, we study immersions $V \to W$ inducing both the given bundle with connection and the given metric. Our study relies on Nash–Gromov implicit function theorem. We show, for the case when $Y \to W = \mathbb{H}P^q$ is the Hopf bundle, that if $4q \geq \frac{n(n+1)}{2} + 5n$, $n = \dim V$, then there exists a non-empty open subset in the space of C^∞-pairs (g, Γ) on V which can be induced from (h, ∇) on $\mathbb{H}P^q$.

Keywords: Connections, Nash's implicit function theorem.
Subj.Class: 53C05,58C15.

1 Introduction and preliminaries

In this paper we deal with the following problem. Let (V, g) and (W, h) be C^∞-Riemannian manifolds. We consider two C^∞-smooth $SU(2)$-bundles, $X \to V$ and $Y \to W$ with connection Γ on X and ∇ on Y respectively. We look for C^∞ $SU(2)$-bundle maps $\tilde{f} : X \to Y$ such that:

i) the map $f : V \to W$ underlying $\tilde{f}$ is isometric, i.e. $f^*(h) = g$;

ii) $\tilde{f} : X \to Y$ is a connection preserving map, i.e. $\tilde{f}^*(\nabla) = \Gamma$.

Set $G = (g, \Gamma)$ and $H = (h, \nabla)$. Then i) and ii) can be expressed by

$$\tilde{f}^*(H) = G,$$

which is equivalent to the system of equations

$$f^*(h) = g \qquad \tilde{f}^*(\nabla) = \Gamma. \tag{1}$$

Observe that, in (1) above the lifted map $\tilde{f} : X \to Y$ is uniquely determined by $f : V \to W$ up to the $SU(2)$-action on X as the automorphism group of (X, Γ) over V is given by this action.

The above problem can be also interpreted in the following way. Let the pair $H = (h, \nabla)$ be fixed on Y. Then, to every morphism $\tilde{f}$, we can associate the induced structure $G = \tilde{f}^*(H) = (f^*(h), \tilde{f}^*(\nabla))$ on X. That is, we get a map (operator)

$$\tilde{f} \mapsto G = \mathcal{D}_H(\tilde{f}) \stackrel{\text{def}}{=} \tilde{f}^*(H) \tag{2}$$

from the function space, call it $\{\tilde{f}\}$, of C^∞ $SU(2)$-bundle morphisms $\tilde{f} : X \to Y$, to the function space $\{G\}$ of pairs $G = (g, \Gamma)$ on X. (See Section 2 for further description of the problem in this setting).

It is well-known (see, e.g., [13]) that the Yang–Mills theory admits an interpretation in terms of connections and curvatures on principal bundles with the structure groups $U(1)$ and $SU(2)$ respectively and, infact, a major inspiration for the study of the pairs (g, Γ) comes from this theory where one could hope to get some new insight for the construction of Yang–Mills fields (which involves both metric and connection) by inducing both from some universal object (Grassmannian). For example, the $SU(2)$-instantons over S^4 can be induced by certain maps $f : S^4 \to \mathbb{H}P^q$ (see e.g., [1], [6]).

For the case of $S^1 \cong U(1)$-bundles, the problem (1) has been studied by the first author in a previous paper [5] in the following formulation: given circle bundles $X \to V$ and $Y \to W$ with connections and Riemannian metrics on V and W one wants an immersion $V \to W$ inducing both the given bundle and connection and the given metric. Related to this, the main result (quoted there as Theorem 0.4.A) is as follows: *If* $2q \geq \frac{n(n+1)}{2} + 3n, n = \dim V$*, then there exists a non-empty open subset in the space of pairs* (g, Γ) *on* V *which can be induced from* (h, ∇) *on* $W = \mathbb{C}P^q$. The reader may notice that this kind of result can be interpreted as a (local) variant of the celebrated Nash's isometric immersion theorem [10], which says that every Riemannian manifold can be isometrically embedded in some $\mathbb{R}^N$, for $N \geq$ some functions of the dimension. Indeed, the above statement is weaker than Nash's: there is an obvious obstruction stemming from the invariance of the norm of the curvature (see Section 0.5 in [5] for details), which prevents a simple general theorem like Nash's .

The principal result in this paper (see Theorem 1.1 below) consists of the S^3-analog ($S^3 \cong SU(2)$) of the above S^1-statement for the case where the inducing bundle Y is the Hopf bundle $Y \to W = \mathbb{H}P^q$. Our study of maps inducing (g, Γ) is based on Nash's implicit function theorem for *infinitesimally invertible* differential operators. This technique, introduced and developed by Gromov in [7], requires certain genericity assumptions on the partial differential equations expressing the inducing relations $f^*(h) = g$, $\tilde{f}^*(\nabla) = \Gamma$ which, as was already observed in [5] for the S^1-case, are rather degenerate. Having in mind the Yang–Mills motivation, it is useful to examine this degeneration phenomenon, which is similar to that which produces overdetermined systems, also for $SU(2)$-bundles. Unfortunately, the results we have so far are such that our approach does not seem to bring anything new in the Yang–Mills theory. But, still the question raised is of interest from the point of view of the general program of inducing geometric structures [7].

The paper is organized as follows: in this section, after setting up the problem to be studied, we introduce some notations and preliminaries. In Section 2, we derive the linearization formula for the differential operator $\mathcal{D}_H$ which corresponds to the geometric structure under study. In Section 3 we discuss the notion of regularity for $SU(2)$-(metric, connection) inducing maps which is relevant to define the class of maps where Gromov's method is applicable. The paper ends up with Section 4, where we illustrate by means of examples the obstructions for given pairs (metric, connection) to be realized as induced from $SU(2)$-bundle maps.

1.A. The Hopf bundle

Let us consider the problem of inducing (g, Γ) stated in the beginning. A case of special interest in our context is when the base manifold W is the quaternionic projective space $\mathbb{H}P^q$ and $Y = S^{4q+3} \to W$ is the Hopf bundle. Recall that, on this Y, the canonical connection ∇ is defined by the (horizontal hyperplane) subbundle $S \subset TY$ consisting of the vectors normal to the Hopf sphere $S^3 \cong SU(2)$. The relevant metric h on $W = \mathbb{H}P^q$ is the $\mathrm{Sp}(q+1)$-invariant metric corresponding to twice the usual spherical metric on S^{4q+3} restricted to the horizontal distribution S (see, e.g., [3]). Notice, that the bundle Y is n-universal [12]. Morever, if $\dim W \geq 2 \dim X$ one can choose the bundle inducing morphism $\tilde{f} : X \to Y$ such that it induces the connection as well (see: [4], [11]).

1.B. The main Theorem

In the theorem below we refer to the respective fine (Whitney) C^∞-topologies in the function spaces $\{\tilde{f}\}$ and $\{G\}$. (We remind to the reader that if the manifold V is compact the fine C^∞-topology coincides with the ordinary

C^∞-topology).
We now assume that $W = \mathbb{H}P^q$. The aim of the present paper is to prove the following local result:

Theorem 1.1 *If* $4q \geq \frac{n(n+1)}{2}+5n$, $n = dim V$ *then there exists a* C^∞*-vector bundle map* $\tilde{f}_0 : X \to Y$ *and a* C^∞*-neighborhood* $\tilde{U}$ *of the induced structure* $\tilde{f}_0^*(H) \in G$ *such that every* $G \in \tilde{U}$ *is induced by some* C^∞*-morphism* $\tilde{f} : X \to Y$.

Remarks 1.2 (i) An alternative formulation of Theorem 1.1 is as follows: *if* $4q \geq \frac{n(n+1)}{2} + 5n$, *then there exists a nonempty open set* $\tilde{U} \subset \{G\}$ *such that* $\tilde{U} \subset \mathcal{D}_H(\{\tilde{f}\})$. Roughly speaking, this can be expressed by saying that the differential system $\tilde{f}^*(H) = G$ has a solution for "many" right hand sides.
(ii) Analytically speaking, to solve the problem (1) amounts to solving $\frac{n(n+1)}{2}$ equations, given by the components of the metric tensor, together with $3n$ equations coming from the connection which is, locally, an $su(2)$-valued 1-form. The number of unknown functions equals $4q + 3$, corresponding to $4q$ coordinates in $\mathbb{H}P^q$ accompanied by 3 extra degrees of freedom corresponding to the gauge transformation. Therefore, there is *no* solution for a generic pair (g, Γ), if $4q + 3 < \frac{n(n+1)}{2} + 3n$ (see Section 4 for further discussion).
(iii) For the intermediate range of $\dim W$, i.e. for $\frac{n(n+1)}{2} + 3n - 3 \leq 4q < \frac{n(n+1)}{2} + 5n$, the situation is rather not clear. If one takes into account Gromov's general theorem on the infinitesimal invertibility of generic underdetermined differential operators (see Section 2.3.8 in [7]) it can be conjectured that our Theorem 1.1 is true for $4q \geq \frac{n(n+1)}{2} + 3n$.

2 The operator $\mathcal{D}_H$ and its linearization L_H

Our study of $SU(2)$-connection inducing maps follows the same approach and uses the same terminology as in [7], to where the reader is referred for a general discussion on induced geometric structures developed in the context of Nash's immersion theory. The key tool is a Nash-type implicit function theorem proved by Gromov for a special class of differential operators (see Section 2.3.1 in [7] for its various formulations and refinements). A general criterion for the validity of the Nash-Gromov implicit function theorem is the *infinitesimal invertibility* of the relevant differential operator and infact what we shall do in the following $2A - 2D$ is to work it out explicitely for the differential operator $\mathcal{D}_H$ under scrutiny. This $\mathcal{D}_H$ is the operator (see (2) above) which assigns to each smooth bundle morphism $\tilde{f} : X \to Y$ the induced structure $G = \tilde{f}^*(H)$ on X for the fixed pair $H = (\nabla, h)$ on Y. More precisely, the operator $\mathcal{D}_H : \{\tilde{f}\} \to \{G = (g, \Gamma)\}$ is a differential operator betweeen spaces of sections (considered with the fine C^∞-topology) of certain fibrations over V described below. Observe, that Riemannian metrics g on V are C^∞-smooth sections $g : V \to S^2(V)$ where $S^2(V)$ denotes the symmetric square of the cotangent bundle of V, while connections on X are viewed as C^∞-sections of the bundle $E \to V$ with fiber $E_v \subset E$ for $v \in V$ which can be described as follows. We denote by X_v^1 the space of linear maps $T_vV \to TX$ which project to the identity $Id : T_vV \to T_vV$ by the differential (of the projection map) of the fibration $X \to V$. The group $SU(2) \cong S^3$ naturally acts on this X_v^1 and the corresponding fiber E_v, equals X_v^1/S^3. This allows us to interpret our pair of structures (metric, connection) $= (g, \Gamma)$ as sections $V \to S^2(V) \oplus E$.

We can also interpret the $SU(2)$-bundle morphisms $X \to Y$ as sections of a bundle $\tilde{\mathcal{F}}$ over V. To describe this $\tilde{\mathcal{F}}$, we choose a section $s : V \to X$ and see that the fiber $\tilde{\mathcal{F}}_v$ over $v \in V$ consists of all maps of the sphere $s(v) \cong S^3$ (that is the fiber $X_v \subset X$) into Y such that $s(v)$ goes onto some sphere $s(w) \cong S^3 = Y_w$ for a different $w \in W$, by an S^3-equivariant map.

Thus $\tilde{\mathcal{F}}_v$ can be identified (non-canonically, because of the freedom in the choice of the section s) with the manifold $\tilde{Y} = \{w = f(v), \tilde{f}(s(v))\}$.

2.A. The linearization of the operator $\mathcal{D}_H$

Here we construct the linearization of the operator $\mathcal{D}_H$. In easy terms, this linearization, denoted by L_H, is the differential of $\mathcal{D}_H$ at $\tilde{f} \in \{\tilde{f}\}$ and so it is a linear operator from the tangent space to the space $\{\tilde{f}\}$ of smooth maps $X \to Y$, say $T_{\tilde{f}}\{\tilde{f}\}$, to $T_G\{G\}$.

Observe, that, due to the above splitting $S^2(V) \oplus E$, one can decompose the operator $\mathcal{D}_H$ into the sum of two operators

$$\mathcal{D}_H = \mathcal{D}_h \oplus \mathcal{D}_\nabla : \{\tilde{f}\} \to \{G = (g, \Gamma)\}$$

where, for a given morphism $\tilde{f} : X \to Y$ with underlying map $f : V \to W$, one has:

$$\mathcal{D}_h(\tilde{f}) = \mathcal{D}_h(f) = f^*(h) = g$$

and

$$\mathcal{D}_\nabla(\tilde{f}) = \tilde{f}^*(\nabla) = \Gamma.$$

One should note at this point that the study of $\mathcal{D}_H$ (as well as the solvability of the equation $\mathcal{D}_H(\tilde{f}) = G$) cannot be reduced to separate problems for $\mathcal{D}_h$ and $\mathcal{D}_\nabla$ as they depend on the same argument $\tilde{f}$. But, on the other hand, all we need in this paper is to compute the linearization formula of $\mathcal{D}_H$ so that our considerations will be purely local and we can proceed by analyzing $\mathcal{D}_h$ and $\mathcal{D}_\nabla$ separately.

2.B. The linearization of the operator $\mathcal{D}_h$

Our first operator $\mathcal{D}_h$ only depends on f and, in a neighborhood of $v \in V$ equipped with local coordinates $u_1, \dots u_n$, it can be expressed by

$$\mathcal{D}_h(f) = \{g_{ij} = h(\partial_i f, \partial_j f)\}, j = 1, \dots, n,$$

where $\partial_i f = df(\frac{\partial}{\partial u_i})$, $i = 1, \dots, n$, denote the images of the vector fields $\frac{\partial}{\partial u_i}$ on V under the differential of f and where g_{ij} are the components of the metric $g = f^*(h)$ in our local coordinates.

The linearization of the operator $\mathcal{D}_h$ at the morphism f is the linear operator

$$L_h : \Gamma(f^*(TW)) \to S^2(V),$$

assigning to each vector field ∂ on W along $f(V)$ a quadratic form g on V. (Here $\Gamma(f^*(TW))$ denote the space of smooth section of the bundle $f^*(TW) \to V$). We take a smooth 1-parametric family of maps $f_t : V \to W$, $t \in [0, 1]$ such that $f_0 = f$ and $\frac{df_t}{dt}_{|t=0} = \partial$ for a given $\partial : V \to f^*(TW)$ and set $\partial_i = \frac{\partial}{\partial u_i}$ and $\bar{\partial}_i = \partial_i f$. Then (compare [5]) the expression for $L_h(\partial) = \frac{d}{dt}\mathcal{D}_h(f_t)_{t=0}$ in local coordinates $u_1, \dots, u_n$ is as follows:

$$\partial \mapsto h(\nabla^h_{\bar{\partial}_i}\partial, \bar{\partial}_j) + h(\bar{\partial}_i, \nabla^h_{\bar{\partial}_j}\partial), \tag{3}$$

where ∇^h denotes the Levi–Civita connection for the metric h.

2.C. The linearization of the operator $\mathcal{D}_\nabla$

To describe our second operator $\mathcal{D}_\nabla$ we also act locally. We may assume that the bundles $X \to V$ and $Y \to W$ are trivial and after fixing two trivializing sections $\alpha : V \to X$ and $\beta : W \to Y$, we interpret our bundle maps $\tilde{f} : X \to Y$ as pairs (f, φ) where $f : V \to W$ is the underlying map of $\tilde{f}$ and $\varphi : V \to SU(2)$ moves the given section α to the $\tilde{f}$ pull-back of the section β.

The connection ∇ on Y can be represented by a $su(2)$-valued 1-form on Y, say γ_β, for $\gamma_\beta = \nabla - \nabla_\beta$, where ∇_β denotes the trivial connection for which the section β is parallel. We claim that the inducing connection relation $\tilde{f}^*(\nabla) = \Gamma$ can be written as

$$\Gamma = \mathcal{D}_\nabla(\tilde{f}) = \tilde{f}^*(\nabla) = \varphi^{-1} f^*(\gamma_\beta)\varphi + \varphi^*(\Theta) + \nabla_\alpha, \tag{4}$$

where $f^*(\gamma_\beta)$ is the induced $su(2)$-valued 1-form on V, Θ is the canonical $su(2)$-valued 1-form on $SU(2)$ and ∇_α is the trivial connection on X associated to the section $\alpha : V \to X$. To prove (4), we first assume that $\tilde{f}$ sends α to β. Then $\varphi = id$ and $\varphi^*(\Theta) = 0$. In this case, we have

$$\tilde{f}^*(\nabla_\beta) = \nabla_\alpha \quad \text{(as } \alpha \text{ goes to } \beta)$$

and

$$f^*(\gamma_\beta) = \tilde{f}^*(\nabla) - \tilde{f}^*(\nabla_\beta) \quad (\text{since } \gamma_\beta = \nabla - \nabla_\beta).$$

Hence

$$\tilde{f}^*(\nabla) = f^*(\gamma_\beta) + \nabla_\alpha$$

which is exactly (4) for $\varphi = id$.

Now, a general $\tilde{f}$ is obtained from the special one (where $\varphi = id$) by composing it with φ thought as a gauge transformation on X. The effect of this on the $su(2)$-valued 1-form $f^*(\gamma_\beta)$ on V is

$$f^*(\gamma_\beta) \mapsto \varphi^{-1} f^*(\gamma_\beta)\varphi + \varphi^*(\Theta),$$

(see e.g. [9]), and thus the general case (4) follows from the special one.

The linearization L_∇ of the operator $\mathcal{D}_\nabla$ at a morphism $\tilde{f}$ is the linear operator

$$L_\nabla : \Gamma(\tilde{f}^*(TY)) \to \Omega^1(V, su(2))$$

assigning to each vector field $\tilde{\partial}$ on Y along $\tilde{f}(X)$ a vector tangent to the space of connections on X. We identify this space with the space of $su(2)$-valued 1-forms on V noted by $\Omega^1(V, su(2))$. A field $\tilde{\partial}$ on $\Gamma(\tilde{f}^*(TY))$ is given by a pair $\tilde{\partial} = (\partial, \tilde{\partial}^v)$, where ∂ and $\tilde{\partial}^v$ denote the ∇-horizontal and ∇-vertical components of $\tilde{\partial}$ respectively. By applying (4) to a family of bundle maps $\tilde{f}_t = (f_t, \varphi_t), t \in [0,1]$, with $\tilde{f}_{t|_{t=0}} = (f, \varphi)$, $\partial = \frac{d}{dt} f_{t|_{t=0}}$ and $\tilde{\partial}^v = \frac{d}{dt}\varphi_{t|_{t=0}}$ and differentiating at $t = 0$, we have

$$\frac{d}{dt}\mathcal{D}_\nabla(\tilde{f}_t)_{|_{t=0}} = L_\nabla(\tilde{\partial} = (\partial, \tilde{\partial}^v))(\tau) = \varphi^{-1}\Omega(\partial, df(\tau))\varphi + R(\tau). \tag{5}$$

Here τ is the tangent vector field on V on which the $su(2)$-valued 1-form $L_\nabla(\tilde{\partial})$ is evaluated and Ω is the $su(2)$-valued 2-form on W representing the curvature of the connection ∇. The last term R is an $su(2)$-valued 1-form on V which, evaluated in τ, is given by:

$$R(\tau) = \frac{d}{dt}(\varphi_t^{-1} f^*(\gamma_\beta)\varphi_t + \varphi_t^*(\Theta))_{|t=0}(\tau).$$

In local coordinates $u_1, \dots, u_n$ (5) reads as:

$$\tilde{\partial} \mapsto \varphi^{-1}\Omega(\partial, \bar{\partial}_i)\varphi + R(\partial_i), \tag{6}$$

where, as before, $\partial_i = \frac{\partial}{\partial u_i}$ and $\bar{\partial}_i = \partial_i f$.

Remark 2.1 The $su(2)$-valued 1-form R on V can be written explicitely in terms of $\tilde{\partial}^v$. All we need here is to observe that if $\tilde{\partial}^v = \frac{d\varphi_t}{dt}_{|t=0} = 0$ then $R = 0$. To see a global expression of (5) the reader may compare the above linearization formula (5) with formula (2) at pag. 71 of [4]. (There, the computation is done in a slightly different setting).

2.D. The inversion of the operator $L_H = (L_h, L_\nabla)$
To invert (locally) the operator D_H, we invert its linearization $L_H = (L_h, L_\nabla)$. This amounts to solving the equation

$$L_H(\tilde{\partial}) = (L_h(\partial), L_\nabla(\tilde{\partial})) = H^{'} = (g^{'}, \Gamma^{'}), \tag{7}$$

where the right-hand side $H^{'} = (g^{'}, \Gamma^{'})$ consists of an arbitrary quadratic form $g^{'}$ on V and an arbitrary $su(2)$-valued 1-form $\Gamma^{'}$ on V respectively. In view of (3) and (6), we express (7) by the following system of p.d.e. in the unknowns ∂ and $\tilde{\partial}^v$:

$$h(\nabla^h_{\bar{\partial}_i}\partial, \bar{\partial}_j) + h(\bar{\partial}_i, \nabla^h_{\bar{\partial}_j}\partial) = g^{'}_{ij}, \qquad \varphi^{-1}\Omega(\partial, \bar{\partial}_i)\varphi + R(\partial_i) = \Gamma^{'}_i, \tag{8}$$

where $g^{'}_{ij}, i, j = 1, \dots n$ are functions on V and $\Gamma^{'}_i, i = 1, \dots n$ are $su(2)$-functions on V, representing, in the local coordinates u_i, the components of $g^{'}$ and $\Gamma^{'}$ respectively.

Next, we impose two additional conditions for the field $\tilde{\partial}$ (see [4] and [10]), namely

$$h(\partial, \bar{\partial}_i) = 0 \tag{9}$$

and

$$\tilde{\partial}^v = 0 \tag{10}$$

(the latter condition means that we seek a solution of (7) among horizontal fields $\tilde{\partial}$). Finally, we differentiate covariantly (9) and alternate the index i and j. As a result of all this the system (3) together with the extra-condition (9) become equivalent to:

$$h(\nabla^h_{\bar{\partial}_i}\bar{\partial}_j, \partial) = -\frac{1}{2}g^{'}_{ij} \qquad h(\partial, \bar{\partial}_i) = 0. \tag{11}$$

Finally, we deduce from (8) that the construction of an infinitesimal inverse for the operator $\mathcal{D}_H$ reduces to find the solution ∂ of the system

$$h(\partial, \bar{\partial}_i) = 0,\ h(\nabla^h_{\bar{\partial}_i}\bar{\partial}_j, \partial) = g^{'}_{ij},\ \tilde{\partial}^v = 0,\ \Omega(\partial, \bar{\partial}_i) = \Gamma^{'}_i, \tag{12}$$

which is an *algebraic* system in the unknown field ∂ and where $g^{'}_{ij} : V \to \mathbb{R}$ and $\Gamma^{'}_i : V \to su(2)$ are arbitrarly chosen functions. To get the last equation (12) we use the fact that the $su(2)$-valued 1-form R vanishes for $\tilde{\partial}^v = 0$ (see Remark 2.1 above). Notice, that every solution of (12) also gives a solution of the original linearized system (7) with the extra conditions $\tilde{\partial}^v = 0$ and $h(\partial, \bar{\partial}_i) = 0$.

3 Regular maps and the proof of Theorem 1.1

The previous discussion prepares us to see how our linearization formula and the consequent infinitesimal invertibility of the differential operator $\mathcal{D}_H$ can be used for obtaining our desired result (Theorem 1.1) essentially claiming that the operator $\mathcal{D}_H$ relating to each map $\tilde{f} : X \to Y$ the induced structure $\tilde{f}^*(H)$ is an open map on a certain subset in the space of maps. These are maps which satisfy a certain regularity condition, called "(h, Ω)" regularity, defined in 3.2 below. Our proof parallels that of Theorem 0.4.A in [5] and infact both follow the same pattern (see: [7], [10]) of the case of Riemannian isometric immersions [10]. There, the relevant regularity condition is *freedom* of the involved map $f : V \to W$, i.e. linear independence of the $n + \frac{n+1}{2}$ vectors of the first and second partial derivatives of f.

3.A. Regular maps
Let $f : V \to W$ be a smooth map. We denote by $T^1_f(v_0) \subset T^2_f(v_0) \subset T_{w_0}W$, $w_0 = f(v_0)$ the first and the second osculating space respectively of the map f at the given point $v_0 \in V$. Namely, $T^1_f(v_0) = Df_{v_0}(T_{v_0}V)$ and $T^2_f(v_0) \subset T_{w_0}W$ is the subspace spanned by $T^1_f(v_0)$ and by the second covariant derivatives $\nabla^h_{\bar{\partial}_i}\bar{\partial}_j, 1 \le i, j \le n$ at w_0 with respect to some (fixed)

local coordinates $u_1, \dots, u_n$ around v_0. (Here we use, as before, the notation $\bar{\partial}_i = \partial_i f$).

Remark 3.1 The spaces $T^1_f(v_0)$ and $T^2_f(v_0)$ defined above are independent on the choice of the coordinates around $v_0 \in V$. Observe that the dimension of $T^2_f(v_0)$ can vary between 0 and $\min(4q, n+s)$, for $s = \frac{1}{2}n(n+1)$.

Let Ω denote the curvature of the canonical connection ∇ of the Hopf bundle Y over $W = \mathbb{H}P^q$. Given a (trivializing) open set $U \subset W$ we can express locally Ω as an $su(2)$-valued 2-form. Namely,

$$\Omega_w : T_w W \otimes T_w W \to su(2),$$

for all $w \in U$. This Ω_w obviously depends on the chosen trivialization. Hence it is well defined up to the adjoint action of $SU(2)$ on its Lie algebra su(2).

Definition 3.2 *A map $f : V \to W$ is called (h, Ω)-regular if for all $v \in V$ the pair $T^1_f(v)$ and $T^2_f(v)$ obey the following conditions:*

(i) $dim T^1_f(v) = n = dim V$ and $dim T^2_f(v) = \frac{1}{2}n(n+3) = n+s$

(ii) for some (and hence for every) basis $\tau_1, \dots \tau_n, \tau_{n+1}, \dots \tau_{n+s}$ in $T^2_f(v)$ such that $\tau_1, \dots \tau_n$ form a basis in $T^1_f(v)$ the equations

$$h_w(\tau_i, \partial) = a_i, \ i = 1, \dots n+s$$

$$\Omega_w(\tau_i, \partial) = b_i, \ i = 1, \dots n,$$

are solvable in $\partial \in T_w W$, $w = f(v)$, for arbitrarly $a_i \in \mathbb{R}$ and $b_i \in su(2)$.

Remarks 3.3 a) The above definition does not depend on the trivializing open set U chosen to define Ω_w.

b) Observe, that condition (ii) can be equivalently expressed by the following: for some (and hence for every) basis $\tau_1, \dots \tau_n, \tau_{n+1}, \dots \tau_{n+s}$ in $T^2_f(v)$ such that $\tau_1, \dots \tau_n$ form a basis in $T^1_f(v)$ the homogeneous system

$$h_w(\tau_i, \partial) = 0, \ i = 1, \dots n+s$$

$$\Omega_w(\tau_i, \partial) = 0, \ i = 1, \dots n,$$

(in the unknowns $\partial \in T_w W$) is non-singular. Namely, the dimension of the space of solutions equals $4q - (4n+s)$.

c) The reader may notice that the above number $4q-(4n+s)$ is positive for $4q=\dim W \geq s+5n$, which is the condition which appears in Theorem 1.1.

The following Proposition in now straightforward:

Proposition 3.4 *Let $f: V \to W$ be a (h,Ω)-regular map. Then the linear operator L_H is invertible over all of V by some differential operator $M_{\tilde{f}}$, i.e. $L_H \circ M_{\tilde{f}} = id$.*

Proof: It follows from the discussion in 2.D. that what we need is to find a solution ∂ of the system of equations (12). Since the map $f: V \to W$ is (h,Ω)-regular, it follows that the solution of (12) form an affine bundle over V of rank $4q - \frac{n(n+1)}{2} - 4n$. Now, every affine bundle admits a section over V. To choose it in a canonical way, one may use any fixed auxiliary Riemannian metric on W (e.g., we can use h) and then take as canonical solution say, ∂_{can}, the solution ∂ of (12) which has the minimal norm with respect to this metric at every point $w=f(v) \in W$ (see, e.g., [7], [8], [10]). Finally, we define the inverse $M_{\tilde{f}}$ of L_H by

$$M_{\tilde{f}}(g^{'}, \Gamma^{'}) = (\partial_{can}, 0)$$

where 0 corresponds to the choice $\tilde{\partial}^v = 0$. □

To make sure that the results we get are non-empty, we show the following:

Proposition 3.5 *For $4q \geq s+5n$ generic maps $f: V \to W$ are (h,Ω)-regular.*

Proof: We shall interpret *non* (h,Ω)-regularity as a singularity in the space $J^2(V,W)$ of 2-jets of our maps $V \to W$, so that we can use an argument based on *Thom's transversality theorem.* Recall that $J^2(V,W)$ form a bundle over $V \times W$ whose fibers are denoted by $J^2_{v,w}$. If we fix local coordinates $u_1, \ldots, u_n$ around $v \in V$, then $J^2_{v,w}$ can be identified with the vector space of linear maps $T_vV \oplus S^2(T_vV) \to T_wW$. The jet J^2_f of a given map $f: V \to W$ at the point v is given by the first and second covariant derivatives

$$J^2_f(v) = (\bar{\partial}_i, \nabla^h_{\bar{\partial}_i} \bar{\partial}_j), \ i,j = 1, \ldots, n.$$

Here $S^2(T_vV)$ denotes the symmetric square of T_vV and one should notice that the identification $J^2_{v,w} = \mathrm{Hom}(T_vV \oplus S^2(T_vV) \to T_wW)$ depends on the local coordinates since $\nabla^h_{\bar{\partial}_i}\bar{\partial}_j$ is not invariantly defined. We also notice that the *non*-(h,Ω)-regularity at $v \in V$ depends on $J^2_f(v)$ and hence we can define the subspace $\Sigma_{v,w} \subset J^2_{v,w}$ consisting of the jets of *non* (h,Ω)-regular maps. The manifold $\Sigma_{v,w}$ is a stratified manifold of real codimension $c = 4q - s - 4n + 1$. This is shown later (see 3.C). Here, we use this fact to show that the set $\Sigma = \cup_{(v,w)\in V\times W}\Sigma_{v,w} \subset \mathrm{J}^2(V,W)$, which fibers over $V \times W$, is a stratified manifold of codimension $4q - s - 4n + 1$. Now, by the very definition of Σ, it follows that a map $f : V \to W$ is *non* (h,Ω)-regular iff $J^2_f(V) \subset \mathrm{J}^2(V,W)$ does not meet Σ. Finally, (the special case) of Thom's transversality theorem (see , e.g. [7] Corollary D', p. 33) tells us that generic maps do have the property $J^2_f(V) \cap \Sigma = \emptyset$ iff $4q - s - 4n + 1 \geq n + 1$ or equivalently $4q \geq s + 5n$. $\square$

Conclusion of the proof of Theorem 1.1 Take any morphism $\tilde{f} : X \to Y$ with underlying map $f : V \to W$. The existence of such morphism is guaranteed by the fact that the Hopf bundle $Y = S^{4q+3}$ is q-universal hence n-universal being $q > n = \dim V$ (see [12]). By Proposition 3.5 above we can choose an (h,Ω)-regular map $f_0 : V \to W$ which is C^∞-close to f. Such f_0 admits a lifting $\tilde{f}_0 : X \to Y$. Indeed the natural map between function spaces $\{\tilde{f}\} \to \{f\}$ (where $\{\tilde{f}\}$ is the space of C^∞-bundle morphisms $X \to Y$ and $\{f\}$ is the space of C^∞-maps $V \to W$) is a Serre fibration and so if the map f is covered by $\tilde{f}$ the same is true for the map f_0 which is C^∞-close (and hence homotopic) to f.

Finally, we know by Proposition 3.4 that the linearization of the operator $\mathcal{D}_H$ at $\tilde{f}_0$ admits an inverse (or, using the terminology in [7], that the operator $\mathcal{D}_H$ is infinitesimally invertible at $\tilde{f}_0$). This allows us to apply Nash's implicit function theorem to deduce that $\mathcal{D}_H$ is an open operator from a neighborhood of $\tilde{f}_0$ to a neighborhood $\tilde{U}$ of $\mathcal{D}_H(\tilde{f}_0) = \tilde{f}^*_0(H)$ and therefore all the structures G in $\tilde{U}$ are inducible from H.

3.B. More on the Hopf bundle

Let us turn to the canonical Hopf bundle $Y = S^{4q+3}$ over $W = \mathbb{H}P^q$ equipped with the canonical connection ∇ and the canonical metric h (compare 1.A.). Consider the chart $U_0 = \{r = [r_0, \ldots, r_q] | r_0 \neq 0\} \subset W$ with the *affine quaternionic* coordinates

$$q = (q_1 = \frac{r_1}{r_0}, \ldots, q_q = \frac{r_q}{r_0}). \tag{13}$$

Before proving Lemma 3.6 and its Corollary 3.7 needed to complete the proof of Proposition 3.5, we wish to describe some relationship between the curvature Ω of the connection ∇ and the metric h.

To do this, we trivialize (in the chart U_0) the fibration $p : Y \to W$ by

$$\psi : U_0 \times SU(2) \to p^{-1}(U_0) \subset Y : ([r], g) \mapsto (\frac{g}{1+|q|^2}, \frac{q \cdot g}{1+|q|^2}).$$

Thus, the connection ∇ is given by the following $su(2)$-valued 1-form:

$$\nabla_{|U_0} = \frac{1}{2}\frac{\bar{q} \cdot dq - d\bar{q} \cdot q}{1+|q|^2}, \tag{14}$$

where we use the quaternion differential

$$dq = (dq_1, \ldots, dq_q)$$

and its conjugate

$$d\bar{q} = (d\bar{q}_1, \ldots, d\bar{q}_q).$$

Explicitely,

$$dq_\alpha = dx_\alpha + dx_{\alpha+1} i + dx_{\alpha+2} j + dx_{\alpha+3} k$$

and

$$d\bar{q}_\alpha = dx_\alpha - dx_{\alpha+1} i - dx_{\alpha+2} j - dx_{\alpha+3} k$$

for $q_\alpha = x_\alpha + x_{\alpha+1} i + x_{\alpha+2} j + x_{\alpha+3} k,\ \alpha = 1, \ldots q$.

The curvature Ω of the connection ∇ is given (in the same chart U_0) by the $su(2)$-valued 2-form (compare, e.g. [1], pp. $21-22$):

$$\Omega_{|U_0} = \frac{d\bar{q} \wedge dq}{(1+|q|^2)^2} \tag{15}$$

where, as in (14) above, the Lie algebra $su(2)$ is identified with the space of purely imaginary quaternions.

The manifold W being a quaternionic manifold, in U_0 there exist three almost complex structures I, J, K, obeying the quaternionic axioms, namely $IJ = -JI = K$ and such that at the point $w_0 = [1, 0, \ldots, 0] \in U_0 \subset W$ they correspond to the multiplication (on the right) of the unitary quaternions i, j and k on $T_{w_0}W = \mathbb{H}^q = \mathbb{R}^{4q}$ (see, e.g. [3] p. 410). Specifically, let $x = (x_1, \ldots, x_{4q}), x_j \in \mathbb{R}$, $x \in T_{w_0}W = \mathbb{R}^{4q}$. Then the multiplication by i, j and k are given respectively by:

$$xi = (\ldots, -x_{4\alpha-2}, x_{4\alpha-3}, x_{4\alpha}, -x_{4\alpha-1}, \ldots), \alpha = 1, \ldots q \tag{16}$$

$$xj = (\ldots, -x_{4\alpha-1}, -x_{4\alpha}, x_{4\alpha-3}, x_{4\alpha-2}, \ldots), \alpha = 1, \ldots q \tag{17}$$

$$xk = (\ldots, -x_{4\alpha}, x_{4\alpha-1}, -x_{4\alpha-2}, x_{4\alpha-3}, \ldots), \alpha = 1, \ldots q. \tag{18}$$

3.C. The codimension of $\Sigma_{v,w}$

To complete the proof of Proposition (3.5) it remains to show that the manifold $\Sigma_{v,w}$ has codimension $c = 4q - s - 4n + 1$. This is stated by Corollary 3.7 below.

Consider now the $su(2)$-valued 2-form ω on U_0 defined by

$$\omega(X, Y) = h(XI, Y)i + h(XJ, Y)j + h(XK, Y)k, \tag{19}$$

for all vectors fields X, Y on U_0.

Lemma 3.6 *Let Ω be the curvature of the canonical connection ∇ of Y. There exists $g \in SU(2)$ such that $\Omega_{w_0} = g^{-1}\omega_{w_0}g$, where ω_{w_0} is the value of the form (19) at $w_0 = [1, 0 \ldots, 0]$.*

Proof: In view of (15) above it is enough to verify that

$$\Omega_{w_0} = d\bar{q} \wedge dq = \omega_{w_0}.$$

We have:

$$d\bar{q} \wedge dq = 2(\sum_{\alpha=1}^{q} \omega_{1\alpha} i + \sum_{\alpha=1}^{q} \omega_{2\alpha} j + \sum_{\alpha=1}^{q} \omega_{3\alpha} k),$$

where

$$\omega_{1\alpha} = dx_{4\alpha-3} \wedge dx_{4\alpha-2} - dx_{4\alpha-1} \wedge dx_{4\alpha}$$

$$\omega_{2\alpha} = dx_{4\alpha-3} \wedge dx_{4\alpha-1} - dx_{4\alpha-2} \wedge dx_{4\alpha}$$

$$\omega_{3\alpha} = dx_{4\alpha-3} \wedge dx_{4\alpha} - dx_{4\alpha-2} \wedge dx_{4\alpha-1},$$

for $\alpha = 1, \ldots q$. If $x = (x_1, \ldots, x_{4q})$ and $y = (y_1, \ldots, y_{4q})$ are vectors in $T_{w_0}W = \mathbb{R}^{4q}$ it remains to show that:

$$2 \sum_{\alpha=1}^{q} \omega_{1\alpha}(x, y) = h_{w_0}(xi, y)$$

$$2 \sum_{\alpha=1}^{q} \omega_{2\alpha}(x, y) = h_{w_0}(xj, y)$$

$$2\sum_{\alpha=1}^{q}\omega_{3\alpha}(x,y) = h_{w_0}(xk,y).$$

Appealing to the definition of h (see 1.A.) it is easy to see that h_{w_0} is twice the standard metric on $T_{w_0}W \cong \mathbb{R}^{4q}$. That is,

$$h_{w_0} = 2dq\cdot d\bar{q} = 2(dx_1^2+\cdots+dx_{4q}^2) \tag{20}$$

and the previous three identities follow by a straightforward computation involving (16), (17) and (18). □

Corollary 3.7 *The set $\Sigma_{v,w}\subset J^2_{v,w}$ is a stratified manifold of real codimension $c = 4q-s-4n+1$.*

Proof: First, observe that the metric h on W and the curvature Ω of the connection ∇ are $Sp(q+1)$-invariant so that we can assume that $f(v)=w=w_0=[1,0,\dots,0]$. Next, we identify the space $J^2_{v,w}$ with $\mathrm{Hom}(\mathbb{R}^{n+s},\mathbb{H}^q)$. and denote by $\tau_1,\dots,\tau_n,\tau_{n+1},\dots,\tau_{n+s}$ the images of the standard basis of $\mathbb{R}^{n+s}$ in $\mathbb{H}^q$ and by $\langle\cdot,\cdot\rangle$ the standard metric on $T_{w_0}W=\mathbb{H}^q\cong\mathbb{R}^{4q}$. which, by formula (20), equals $\frac{h_{w_0}}{2}$. It follows that the (h,Ω)-regularity of the map f at v is equivalent to the solvability of the following system of linear equations:

$$\langle\tau_i,\partial\rangle = (\cdot)_i,\ i=1,\dots n+s$$

$$\Omega_{w_0}(\tau_r,\partial) = (\cdot)_r,\ r=1,\dots n,$$

with $(\cdot)_i$ and $(\cdot)_r$ arbitrarly chosen elements in $\mathbb{R}$ and $su(2)$ respectively.

On the other hand, by Lemma (3.6) one has

$$\frac{1}{2}\Omega_{w_0}(\tau_r,\partial) = \langle\tau_r i,\partial\rangle i + \langle\tau_r j,\partial\rangle j + \langle\tau_r k,\partial\rangle k,\ r=1,\dots n.$$

Thus, the (h,Ω)-regularity of the map f at v amounts to the solvability of the following system of linear equations:

$$\langle\tau_i,\partial\rangle = (\cdot)_i,\ i=1,\dots n+s$$

$$\langle\tau_r i,\partial\rangle = (\cdot)_r,\ r=1,\dots n$$

$$\langle\tau_r j,\partial\rangle = (\cdot)_r,\ r=1,\dots n$$

$$\langle\tau_r k,\partial\rangle = (\cdot)_r,\ r=1,\dots n,$$

and we derive, by elementary linear algebra, that (h,Ω)-regularity for maps $f:V\to W$ is expressed in terms of the following property of the vectors $\tau_i, i=1,\dots,n+s$.

- The vectors $\tau_1, \ldots, \tau_n$ are linearly independent over the field of quaternion $\mathbb{H}$ and the vectors $\tau_{n+1}, \ldots, \tau_{n+s}$ are $\mathbb{R}$-independent modulo $\mathrm{Span}_{\mathbb{H}}(\tau_1, \ldots, \tau_n)$.

To conclude the proof we only have to show that the subset $\Sigma_0 \subset \mathbb{H}^{qn} \oplus \mathbb{H}^{qs}$ consisting of $(n+s)$-tuples $\tau_1, \ldots, \tau_n, \tau_{n+1}, \ldots, \tau_{n+s}$ in $\mathbb{H}^q$ such that either $\tau_1, \ldots, \tau_n$ are $\mathbb{H}$-linearly dependent or $\tau_{n+1}, \ldots, \tau_{n+s}$ are $\mathbb{R}$-dependent modulo $\mathrm{Span}_{\mathbb{H}}(\tau_1, \ldots, \tau_n)$ has real codimension $4q - s - 4n + 1$.
For the proof of this we refer to Sublemma $2.1.G'$ p. 795 in [5], since it is completely analogous. One just needs to formally reiterate (extending it to the quaternions) the same argument (there valid for $\mathbb{C}$). □

4 Obstructions for inducing pairs (metric, connection).

Bearing in mind the p.d.e. system of equations (1), let us start with a formal discussion applicable to all systems of p.d.e.. By counting the number of equations and the unknown functions (see Remark 1.2 (ii)) one sees that, for a generic G, there is no solution in the present case if $4q + 3 < \frac{n(n+1)}{2} + 3n$, i.e. for $q < \frac{1}{4}\frac{n(n+1)}{2} + 3n - 3$ as the system (1) is overdetermined. On the other hand, a natural question in the situation at hand is of course, the following: is it possible to prove that the system (1) is solvable for all $G \in \{G\}$? An affirmative answer is obtained if one considers only one of the two equations in (1) as they are separately solvable for q sufficiently large (see: [5], [7], [10]). However, the system (1) is not always solvable even for large q. This may be illustrated by the aid of the following examples.

4.A. Choose an open set $U \subset V$ where the curvature Ω_Γ of the connection Γ is given by an $su(2)$-valued 2-form on U and still denote, with a slight abuse of notation, this form by Ω_Γ. We can associate to Ω_Γ its maximal norm on U, noted by $N(\Omega_\Gamma, g)$. This is the supremum of $\|\Omega_{\Gamma_p}(x, y)\|$ over all g-orthonormal pairs of tangent vectors $x, y \in T_pU, p \in U$, where $\| \cdot \| : su(2) \cong \mathbb{R}^3 \to \mathbb{R}$ refers to the standard euclidean norm on $\mathbb{R}^3$.

Assume the map $f : V \to W$ can be lifted to $\tilde{f} : X \to Y$ in such a way that $\tilde{f}^*(\nabla) = \Gamma$ and that $f(U)$ is contained in some $U_0 \subset W$ (we may shrink U if necessary). (Here U_0 is a trivializing open set for the bundle Y). Let denote by Ω the $su(2)$-valued 2-form on U_0 representing the curvature of the connection ∇. Then we have $f^*(\Omega) = \Omega_\Gamma$.

Next, we assume the map f to be isometric or, more generally, distance decreasing. This implies that f also decreases the curvature, that is:

$$N(\Omega_\Gamma, g) \leq N(\Omega, h), \tag{21}$$

where $N(\Omega, h)$ stands for the supremum norm of Ω on the open set $f(U)$. We observe that the inequality (21) provides a strong local restriction for the solvability of the system (1) regardless of dimensions. For example, if the connection Γ given on (V, g) has non-vanishing curvature Ω_Γ, then there exists a positive constant ϵ such that

$$N(\Omega_\Gamma, \epsilon g) = \frac{1}{\epsilon} N(\Omega_\Gamma, g) > N(\Omega, h).$$

Therefore, it follows by (21) that the pair $(\epsilon g, \Gamma)$ cannot be induced from (h, ∇) by any map $f : V \to W$.

4.B. The above obstruction in 4.A. can be globalized as follows. We fix a loop γ in W and consider the parallel transport τ along γ with respect to the connection ∇ on Y. This τ is seen as an element of $SU(2)$ which can be naturally embedded in $SO(4)$. Hence, it makes sense to consider the difference $\tau - I$, where I denotes the identity matrix in $SO(4)$. Its norm is:

$$\|\tau - I\| = \sup_{\|u\|=1} \|\tau(u) - u\|, \; u \in \mathbb{R}^4.$$

Let us estimate the difference $\tau - I$ in terms of the norm of the curvature Ω of the connection ∇. To do this, one may assume that γ is homotopic to its base point $m \in W$ through an homotopy $H : [0,1] \times [0,1] \to W$, and show, by elementary calculation, that

$$\|\tau - I\| \leq \|\Omega\| A_h(D), \tag{22}$$

where $\mathrm{A}_h(D)$ denotes the area of $D = H([0,1] \times [0,1])$ with respect to the metric h and

$$\|\Omega\| = \sup_{u,v} \frac{\|\Omega(u^*, v^*)\|}{\|u \wedge v\|},$$

where u, v are vectors in $T_m W$ and u^*, v^* denote their horizontal lifts.

Now, denote by K_h the sectional curvature of (W, h). The manifold W being compact, there exist two constants a e b such that

$$-b^2 \leq K_h \leq a^2.$$

Starting from this inequality, one can easily show that if the length of γ, denoted by $L_h(\gamma)$, is small enough, namely if

$$L_h(\gamma) < k_1$$

then the homotopy H can be chosen in such a way that the following holds:

$$A_h(D) \leq k_2[L_h(\gamma)]^2, \tag{23}$$

where k_1 and k_2 are constants which only depend on a, b and on the injectivity radius of (W, h).

Let us look at the above in the simplest case where the base manifold V of the bundle X is the unit circle (i.e. take $V = S^1$) with some non-trivial connection Γ (which necessarily has curvature $\Omega_\Gamma = 0$). We assume that there exists a smooth map $f : S^1 \to W$ admitting a lift $\tilde{f} : X \to Y$ such that $\tilde{f}^*(\nabla) = \Gamma$. Thus the parallel transport τ on $\gamma = f(S^1)$ induces the parallel transport ρ on S^1. If g is a metric on S^1 satisfying:

$$L_g(S^1) < \min(k_1, \sqrt{\frac{\|\rho - I\|}{k_2\|\Omega\|}})$$

then, such a g cannot be induced by the map $f : S^1 \to W$ as, in that case, one should have $\mathrm{L}_h(\gamma) = \mathrm{L}_g(S^1)$ which contradicts the above inequalities (22) and (23).

4.C. Let (V, g) be a Riemannian manifold and let $X = V \times SU(2)$ be the trivial bundle based on V with a non trivial connection Γ. Also in this case we have zero curvature Ω_Γ and, as before in 4.B., by making the metric g smaller (namely, by taking ϵg for small ϵ) we obtain an obstruction for the inducibility of the pair $(\epsilon g, \Gamma)$. Indeed, we can apply the previous argument in 4.B. to a non-trivial loop $S^1 \subset V$ with the induced structures (trivial bundle, metric and connection).

References

[1] M. F. Atiyah, *Geometry of Yang–Mills Fields*, Lezioni Fermiane, Scuola Normale Superiore, Pisa 1979.

[2] V. Arnold, A. Varchenko, and S. Goussein–Zadé, *Singularités des applications différentiable I*, Mir, Moscow (1986).

[3] A. Besse, *Einstein manifolds*, Springer-Verlag, 1987.

[4] G. D'Ambra, *Constructions of Connections inducing maps between principal bundles, part I*, Trans. of AMS vol. 338 n.2 (1993), 783-797.

[5] G. D'Ambra, *Induced Connections on S^1-bundles over Riemannian Manifolds*, Trans. of AMS vol. 338 n.2 (1993), 783-797.

[6] S. K. Donaldson, *The Yang–Mills equations on the Euclidean space*, Perspectives in Mathematics (1984), 93-109.

[7] M. Gromov, *Partial Differential Relations*, Springer-Verlag (1986).

[8] M. Gromov and V. Rokhlin, *Embeddings and immersions in Riemannian geometry*, Uspekhi Mat. Nauk. 25 (1970) n.5, 3-62.

[9] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry I*, John Wiley and Sons Inc. (1967).

[10] J. Nash, *The embedding problem for Riemannian manifolds*, Ann. of Math. 63 (2) (1956), 20-63.

[11] N. Narasimhan and S. Ramanan, *Existence of universal connections*, Amer. J. Math. 83 (1961), 563-572.

[12] N. Steenrod, *The Topology of fibre bundles*, Princeton Univ. Press, Princeton, N.J., (1970).

[13] A. Trautman, *Solutions of the Maxwell and Yang–Mills equations associated with Hopf fiberings*, Internat. J. of Theoret. Phys. (1977).