A Laplace integral, the T–Y–Z expansion and Berezin's transform on a Kähler manifold

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Abstract

Let M be a n-dimensional complex manifold endowed with a C^∞ Kähler metric g. We show that a certain Laplace-type integral $\mathcal{L}_m(x)$, when x varies in a sufficiently small open set $U \subset M$, has an asymptotic expansion $\mathcal{L}_m(x) = \frac{1}{m^n}\sum_{r\geq 0} m^{-r} C_r(f)(x)$, where $C_r : C^\infty(U) \to C^\infty(U)$ are smooth differential operators depending on the curvature of g and its covariant derivatives. As a consequence we furnish a different proof of Lu's theorem by computing the lower order terms of Tian–Yau–Zelditch expansion in terms of the operator C_j. Finally, we compute the differential operators Q_j of the expansion $\mathrm{Ber}_m(f) = \sum_{r\geq 0} m^{-r} Q_r(f)$ of Berezin's transform in terms of the operators C_j.

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1 Introduction

Let M be a n-dimensional complex manifold endowed with a Kähler metric g and let ω be the corresponding Kähler form. Let Φ be a Kähler potential for the metric g, namely a real valued function Φ defined on an open set $U \subset M$ satisfying

$$\omega = \frac{i}{2\pi}\partial\bar{\partial}\Phi. \tag{1}$$

If $g = \sum_{j\bar{k}}^{n} g_{j\bar{k}} dz_j d\bar{z}_k$ is the local expression of the metric g then the previous equation is equivalent to

$$g_{j\bar{k}} = \frac{1}{\pi}\frac{\partial^2\Phi}{\partial z_j \partial \bar{z}_k}. \tag{2}$$

Let $\tilde{\Phi}$ be an *almost analytic extension* of the potential Φ from the diagonal of $U \times U$ (see e.g. [1]). This means that $\tilde{\Phi}(x, y)$ is a smooth $\mathbb{C}$-valued function on $U \times U$ such that $\bar{\partial}_x\tilde{\Phi}$ and $\partial_y\tilde{\Phi}$ vanish to infinite order for $x = y$.

More precisely, if $(z_1, \dots z_n)$ and $(w_1, \dots w_n)$ denote complex coordinates in $U \times \{0\}$ and $\{0\} \times U$ respectively then $\frac{\partial \tilde{\Phi}}{\partial \bar{z}_j}$ $\frac{\partial \tilde{\Phi}}{\partial w_j}$, $j = 1, \dots, n$ vanish together with all their partial derivatives of all orders at $x = y$.

Observe that $\tilde{\Phi}$ is defined up to the sum of a smooth function on $U \times U$ vanishing at infinite order for $x = y$. Therefore, by choosing $\frac{1}{2}(\tilde{\Phi}(x, y) + \tilde{\Phi}(y, x))$ instead of $\tilde{\Phi}(x, y)$, we can assume that

$$\tilde{\Phi}(x, y) = \overline{\tilde{\Phi}(y, x)}. \tag{3}$$

Following [6] we define a smooth function $D_{\tilde{\Phi}}$ on $U \times U$ as follows:

$$D_{\tilde{\Phi}}(x, y) = \tilde{\Phi}(x, \bar{x}) + \tilde{\Phi}(y, \bar{y}) - \tilde{\Phi}(x, \bar{y}) - \tilde{\Phi}(y, \bar{x}).$$

Since $D_{\tilde{\Phi}}(x, x) = \tilde{\Phi}(x)$ we have that $D_{\tilde{\Phi}}(x, x) = 0$. Moreover, it follows by (3), that $D_{\tilde{\Phi}}$ is real valued.

Take $U \subset M$ as above. For all $x \in U$, the positive definiteness of the matrix (2) implies that the function

$$D_{\tilde{\Phi}}(x, \cdot) = \tilde{\Phi}(x, x) + \tilde{\Phi}(\cdot, \cdot) - \tilde{\Phi}(x, \cdot) - \tilde{\Phi}(\cdot, x)$$

has a local minimum at x. Diminishing U, if necessary, we can assume that $D_{\tilde{\Phi}}(x, y)$ is a globally defined function on $U \times U$, $D_{\tilde{\Phi}}(x, y) \geq 0$ and $D_{\tilde{\Phi}}(x, y) = 0$ iff $x = y$.

Observe that Φ is defined up to the sum with a real part of a holomorphic function and its extension $\tilde{\Phi}$ is defined up to the sum with a smooth function vanishing at infinite order on the diagonal of $U \times U$. In the definition of the function $D_{\tilde{\Phi}}$ the first ambiguity drops out while the second one is anavoidable (this justify the notation).

Remark 1.1 In the case the metric g is real analytic, the extension $\tilde{\Phi}(x, y)$ is uniquely defined and it is holomorphic in x and anti-holomorphic in y. In this case the function $D = D_{\tilde{\Phi}}$ is well-defined and it is called Calabi's diastasis function (see [2]).

Let f be a C^∞ function on U and $m > 0$. Consider the Laplace integral

$$\mathcal{L}_m(x) = \int_U f(y) e^{-m D_{\tilde{\Phi}}(x,y)} \frac{\omega^n}{n!}(y), \tag{4}$$

where $\frac{\omega^n}{n!}$ is the Riemannian volume form on M, induced by the metric g. The first result of this paper is Theorem 2.1 where we prove that this integral

admits an asymptotic expansion with smooth differential operators which depend only on the curvature of the metric g (and its covariant derivatives) and not on the particular extension $\tilde{\Phi}$ of the potential Φ chosen. This is an extension of Theorem 3 in [4] (cfr. also Theorem 3 in [8]) where the case of real analytic metrics is treated. The second result of this paper is Theorem 3.5 which provides a computation of the coefficients a_j of the T–Y–Z expansion in terms of the operator C_j. In particular we compute explicitly $a_j, j \leq 3$ hence giving an alternative to Lu's computation which is based on Tian's peak section method and Ruan's work on K-coordinates. Finally, in Theorem 4.2 below, we compute the differential operators Q_j of the expansion $\mathrm{Ber}_m(f) = \sum_{r\geq 0} m^{-r} Q_r(f)$ of Berezin's transform in terms of the operators C_j. This is a generalization, to compact Kähler manifolds, of a result due to Engliš in [4] valid for pseudoconvex domains in $\mathbb{C}^n$ equipped with a real analytic Kähler metric g possessing a global potential Φ.
The paper is organized as follows. In the next section we prove Theorem 2.1 by adapting Engliš argument, valid for the real analytic case, to the C^∞ case. In Section 3 we recall the definition of Kempf's distortion function, the Tian–Yau–Zelditch expansion and we prove Theorem 3.5. In Section 4 after recalling the defintion of Berezin's transform and the known results about its expansion, we prove Theorem 4.2. For reader's convenience in the appendix below we collect some notations and formulae used throughout this paper.

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2 The asymptotic expansion of $\mathcal{L}_m(x)$

This section is dedicated to the proof of the following theorem

Theorem 2.1 *If the integral (4) exists for some $m = m_0$ then it also exists for all $m > m_0$ and as $m \to +\infty$ it has an asymptotic expansion*

$$\mathcal{L}_m(x) \sim \frac{1}{m^n} \sum_{r \geq 0} m^{-r} C_r(f)(x), \tag{5}$$

where $C_r : C^\infty(U) \to C^\infty(U)$ are smooth differential operators which can be described explicitly. In particular

$$
\begin{cases}
C_0 = id \\
C_1(f) = \Delta f - \frac{1}{2} f\rho \\
C_2(f) = \frac{1}{2}\Delta\Delta f - \frac{1}{2}L_{Ric}(f) - \frac{\rho}{2}\Delta f - \frac{1}{2}(\langle D^{'}\rho, D^{'}f\rangle + \langle D^{'}f, D^{'}\rho\rangle) \\
-f(\frac{1}{3}\Delta\rho - \frac{1}{8}\rho^2 - \frac{1}{6}|\,Ric\,|^2 + \frac{1}{24}|R|^2) \\
C_3(1) = \frac{1}{4}|D^{'}\rho|^2 - \frac{1}{8}\Delta\Delta\rho + \frac{\rho}{6}\Delta\rho + \frac{3}{8}L_{Ric}(\rho) - \frac{1}{48}\rho^3 \\
+\frac{1}{4}|D^{'}Ric|^2 - \frac{\rho}{12}|Ric|^2 - \frac{1}{24}|D^{'}R|^2 + \frac{\rho}{48}|R|^2 + \frac{1}{4}R(Ric, Ric) \\
-\frac{1}{12}\sigma_1(R) + \frac{1}{24}\sigma_2(R) + \frac{1}{6}\sigma_3(Ric),
\end{cases}
\tag{6}
$$

where, for $f, g \in C^{\infty}(U)$, we have the following notations:

$$
L_{Ric}(f) = \sum_{i,j,p,q=1}^{n} g^{i\bar{q}} g^{p\bar{j}} Ric_{p\bar{q}} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}, \tag{7}
$$

$$
\langle D^{'}f, D^{'}g\rangle = \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial \bar{z}_j}, \tag{8}
$$

$$
|D^{'}f|^2 = \langle D^{'}f, D^{'}f\rangle. \tag{9}
$$

(For the other notations see Section 5 in [9] or Appendix below)

In order to prove 2.1 we need the following lemma whose proof can be found in [4].

Lemma 2.2 *Let U be a neighbourhood of a point $x \in \mathbb{R}^{2n}$ and f a complex valued and S a real-valued C^{∞} functions on $\overline{U}$ such that $S(y) < S(x), \forall y \in \overline{U}, x \neq y$. Assume that the integral*

$$
\mathcal{L}(\lambda) = \int_U f(y) e^{\lambda S(y)} dy
$$

exists for some $\lambda = \lambda_0$. Then it exists for all $\lambda > \lambda_0$ and as $\lambda \to +\infty$ it admits an asymptotic expansion

$$
\mathcal{L}(\lambda) = \frac{e^{\lambda S(x)}(2\pi)^n}{\lambda^n [Hess\, S(x)]^{\frac{1}{2}}} \sum_{r=0}^{+\infty} \left(\sum_{k=r}^{3r} \frac{1}{k!(k-r)!2^k} L_S^k (S(x,y)^{k-r} f(y))_{|y=x} \right) \lambda^{-r}, \tag{10}
$$

where $Hess\, S(x)$ is the determinant of the matrix

$$
A = - \left[\frac{\partial^2 S(x)}{\partial x_j \partial x_k} \right]_{j,k=1}^{2n},
$$

L_S is the (constant-coefficient) differential operator

$$L_S = \sum_{j,k=1}^{2n} (A^{-1})_{jk} \frac{\partial^2}{\partial y_j \partial y_k},$$

and

$$S(x,y) = S(y) - S(x) + \frac{1}{2}\langle A(y-x), y-x\rangle.$$

Proof of Theorem 2.1 If one identifies $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in the usual way and pass from the basis $\{\mathrm{Re}\, z_k, \mathrm{Im}\, z_k\}$ to $\{z_k, \bar{z}_k\}$ one obtains that $L_S = 2\sum_{j,k=1}^{n}(B^{-1})_{jk}\frac{\partial}{\partial \bar{z}_j}\frac{\partial}{\partial z_k}$, where B^{-1} is the inverse of the matrix

$$B = -\left[\frac{\partial^2 S(x)}{\partial z_i \partial \bar{z}_j}\right]_{i,j=1}^{n}$$

and the Hessian equals

$$\mathrm{Hess}\, S(x) = 4^n |\det B|^2.$$

By applying the previous lemma to $S(y) = -D_{\tilde{\Phi}}(x,y)$ where $D_{\tilde{\Phi}}$ is as above, replacing f by $f\det(g_{i\bar{j}})$, by taking into account that $\frac{\omega}{n!}(y) = \det(g_{i\bar{j}})\frac{dz}{\pi^n}$ (where $dz = \frac{i^n}{2^n} dz_1 \wedge d\bar{z}_1 \wedge \dots dz_1 \wedge d\bar{z}_1$ is the Lebesgue measure) and, finally, by changing λ with m one gets

$$\mathcal{L}_m(x) = \mathcal{L}(\lambda) = \sum_{r\geq 0}\left(\sum_{k=r}^{3r}\frac{1}{k!(k-r)!}L^k(f\det(g_{i\bar{j}})S_{\tilde{\Phi}}^{k-r})_{|y=x}\right) m^{-r},$$

where L is the (constant-coefficients) differential operator

$$L\phi(y) = \sum_{i\bar{j}=1}^{n} g^{i\bar{j}}(x)\frac{\partial\phi}{\partial z_i}\frac{\partial\phi}{\partial z_j},$$

with $g^{i\bar{j}}$ the inverse of the matrix $(g_{i\bar{j}})$ and

$$S_{\tilde{\Phi}}(x,y) = -D_{\tilde{\Phi}}(x,y) + \sum_{i\bar{j}=1}^{n} g_{i\bar{j}}(y)(y_i - x_i)(\bar{y}_j - \bar{x}_j).$$

Observe that if $\hat{\Phi}$ is another almost analytic extension of the potential Φ then $L^k(S_{\tilde{\Phi}})_{|y=x} = L^k(S_{\hat{\Phi}})_{|y=x}$ and therefore the expansion of $\mathcal{L}_m(x)$ depends only on the Kähler metric g. By setting

$$C_r(f)(x) = \sum_{k=r}^{3r} \frac{1}{k!(k-r)!} L^k(f \det(g_{i\bar{j}}) S^{k-r})_{|y=x} \tag{11}$$

one gets the first part of the theorem. Finally, in Section 4 of [4] one can find the computation of C_1, C_2 and C_3 by using formula (11) above. (Observe that in [4] the operators C_j are denoted by R_j and the Ricci curvature of the metric g has opppsite sign to the one we are considering here, (see also [8])). □

3 The Tian–Yau–Zelditch expansion

Let M be a compact complex manifold endowed with a Kähler metric g and associated Kähler form ω and suppose that g is polarized with respect to a holomorphic line bundle L, i.e. the Kähler form ω represents the first Chern class $c_1(L)$. Let h be a Hermitian metric on L with its Ricci curvature form $\mathrm{Ric}(h) = \omega$. For a positive integer m, the line bundle $L^m = L^{\otimes m}$, the m-th tensor power of L, is a polarization for the Kähler metric mg and the Hermitian metric h satisfying $\mathrm{Ric}(h) = \omega$ induces, in a natural way an Hermitian metric h_m on L^m such that $\mathrm{Ric}(h_m) = m\omega$. Denote by $H^0(L^m)$ the space of holomorphic sections of L^m and let $(s_0, \ldots, s_{d_m})$ be an orthonormal basis with respect to the L^2-product induced by h_m.

The *G. Kempf's distortion function* [7] is defined by

$$T_{m\omega}(x) = \sum_{j=0}^{d_m} h_m(s_j(x), s_j(x)). \tag{12}$$

During the last fifteen years, many authors have studied this function and its asymptotic expansion when $m \to \infty$.
Zelditch [12] generalized Tian's Theorem [11] by proving:

Theorem 3.1 *(Zelditch) There is a complete asymptotic expansion*

$$T_{m\omega}(x) = a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \cdots \tag{13}$$

for certain smooth coefficients $a_j(x)$ with $a_0 = 1$. More precisely the expression holds in C^∞ in that, for any $r, k \geq 0$

$$||T_{m\omega}(x) - \sum_{j=0}^{k} a_j(x) m^{n-j}||_{C^r} \leq C_{k,r} m^{n-k-1}, \tag{14}$$

where $C_{k,r}$ are constant depending on k, r and on the Kähler form ω.

In [9] Lu proves:

Theorem 3.2 *(Lu) Each coefficients $a_j(x)$, given by the asymptotic expansion (13) is a polynomial of the curvature and its covariant derivatives at x of the metric g. These polynomials can be found by finitely many steps of algebraic operations. In particular*

$$\begin{cases} a_0 = 1 \\ a_1(x) = \frac{1}{2}\rho \\ a_2(x) = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|\,Ric\,|^2| + 3\rho^2) \\ a_3(x) = \frac{1}{8}\Delta\Delta\rho + \frac{1}{24}\, div\, div(R, Ric) - \frac{1}{6}\, div\, div(\rho\; Ric) \\ +\frac{1}{48}\Delta(|R|^2 - 4|\,Ric\,|^2| + 8\rho^2) + \frac{1}{48}\rho(\rho^2 - 4|\,Ric\,|^2 + |R|^2) \\ +\frac{1}{24}(\sigma_3(Ric) - Ric(R, R) - R(Ric, Ric)). \end{cases} \tag{15}$$

We refer the reader to Section 5 in [9] or to formulae in the Appendix below for the definitions of the terms in the expression of a_2 and a_3.

The expansion (13) is called the *Tian–Yau–Zelditch expansion.*

Recently, Karabegov and Schlichenmaier (see Theorem 5.6 in [6]) prove the following generalization of Zelditch's expansion (see also [3]).

Theorem 3.3 *(Karabegov and Schlichenmaier) Let M be a compact complex manifold endowed with a polarized Kähler metric g. Let $T_{m\omega}(x, y)$ denote an almost analytic extension of the Kempf's distortion function $T_{m\omega}(x)$ to an open neighbourhood, say $U \times U$, of the diagonal. Then, for U sufficiently small, $T_{m\omega}(x, y)$ admits an asymptotic expansion (as $m \to +\infty$) of the form*

$$T_{m\omega}(x, y) \sim \sum_{j \geq 0} a_j(x, y) m^{n-j} \tag{16}$$

such that for any compact set $K \subset U \times U$ and for any non-negative integer k one has:

$$\sup_{x,y \in K} e^{\frac{-m}{2} D_{\tilde{\Phi}}(x,y)} |T_{m\omega}(x, y) - \sum_{j=0}^{k} a_j(x, y) m^{n-j}| = O(m^{n-k-1}), \tag{17}$$

where $D_{\tilde{\Phi}} : U \times U \to \mathbb{R}$ is as in the previous section and $a_r(x, y)$ is an almost analytic extension of $a_r(x)$ to the diagonal; in particular $a_0(x, y) = 1$.

In Theorem 3.5 below we prove that the coefficients a_j can be computed in terms of the differential operator C_j coming from expansion (5). We need the following Lemma which is a straightforward extension of formulae (21) and (22) in [8] to the smooth case.

Lemma 3.4 *For m sufficiently large the real valued function*

$$\psi_m(x, y) = \frac{e^{-mD_{\tilde{\Phi}}(x,y)}|T_{m\omega}(x, y)|^2}{T_{m\omega}(x)T_{m\omega}(y)}. \tag{18}$$

is globally defined on $M \times M$ and

$$\int_M \psi_m(x, y)T_{m\omega}(y)\frac{\omega^n(y)}{n!} = 1. \tag{19}$$

Theorem 3.5 *Let M be a n-dimensional complex manifold endowed with a polarized Kähler metric g. Then Zelditch's functions a_j given by (13) can be computed in terms of the operators C_j of the expansion (5). Moreover, we can recover the lower terms a_1, a_2 and a_3 of Zelditch's expansion given by (15).*

Proof: Fix a point $x \in M$ and a neighbourhood U of x where we can apply Theorem 3.3 and such that $D_{\tilde{\Phi}}(x, y)$ is defined on $U \times U$. Thus, on $U \times U$,

$$e^{-mD_{\tilde{\Phi}}(x,y)}|T_{m\omega}(x, y)|^2 = m^{2n}e^{-mD_{\tilde{\Phi}}(x,y)}(1 + \sum_{j=1}^{+\infty} \tilde{a}_j(x, y)m^{-j}), \text{ as } m \to +\infty \tag{20}$$

uniformly on compact subset $K \subset U \times U$. Here the $\tilde{a}_j(x, y)$'s are smooth functions on $U \times U$.

In particular, we have:

$$\begin{aligned} \tilde{a}_1(x, y) &= a_1(x, y) + a_1(y, x) \\ \tilde{a}_2(x, y) &= a_1(x, y)a_1(y, x) + a_2(x, y) + a_2(y, x) \\ \tilde{a}_3(x, y) &= a_1(x, y)a_2(y, x) + a_1(y, x)a_2(x, y) + a_3(x, y) + a_3(y, x). \end{aligned} \tag{21}$$

By formulae (18) and (19), one can write

$$T_{m\omega}(x) = T_{m\omega}(x)\int_{M\setminus U} \psi_m(x, y)T_{m\omega}(y)\frac{\omega^n}{n!}(y) + \int_U e^{-mD_{\tilde{\Phi}}(x,y)}|T_{m\omega}(x, \bar{y})|^2\frac{\omega^n}{n!}(y). \tag{22}$$

Moreover, one can prove (see e.g. Proposition 5.1. in [6]) that for any neighbourhood U of a point $x \in M$ and for every smooth function f on M,

$$\int_{M \setminus U} \psi_m(x,y) T_{m\omega}(y) f(y) \frac{\omega^n(y)}{n!} = O(m^{-k}), \ \forall k \geq 1. \tag{23}$$

Therefore, Zelditch's expansion (13) of $T_{m\omega}(x)$ together with (23) above (with $f = 1$) imply that,

$$\lim_{m \to \infty} m^k T_{m\omega}(x) \int_{M \setminus U} \psi_m(x,y) T_{m\omega}(y) \frac{\omega^n}{n!}(y) = 0,$$

for all non-negative integers $k \geq 1$. By inserting (20) in (22) and by taking into account expansion (5) we obtain

$$1 + \sum_{r=1}^{k} a_r(x) m^{-r} = 1 + \sum_{r=1}^{k} c_r(x) m^{-r} + \sum_{r+j=1, \ r \geq 0, \ j \geq 1}^{k} m^{-r-j} C_r(\tilde{a}_j(x,y))_{|y=x} + R_k(x,m), \tag{24}$$

with $\lim_{m \to \infty} m^k R_k(x,m) = 0$.

Here we are denoting by $c_j(x)$ the function obtained by applying the operators C_j to the constant function 1, namely

$$C_j(1)(x) = c_j(x), \tag{25}$$

and by $C_r(\tilde{a}_j(x,y))_{|y=x}$ we mean to apply the operator C_r to the y-variable of the function $\tilde{a}_j(x,y))$ and to set $y = x$.
Therefore, by formula (24), one gets

$$a_1(x) = c_1(x) + C_0(\tilde{a}_1(x,y))_{|y=x} = c_1(x) + \tilde{a}_1(x,x),$$

which implies

$$a_1(x) = -c_1(x). \tag{26}$$

Further, for any integer $k \geq 2$, due to the fact that C_0 is the identity operator, one obtains:

$$\begin{aligned} a_k(x) &= c_k(x) + \textstyle\sum_{r+j=k, \ r \geq 0, \ j \geq 1} C_r(\tilde{a}_j(x,y))_{|y=x} \\ &= c_k(x) + \tilde{a}_k(x,x) + \textstyle\sum_{r+j=k, \ r \geq 1, \ j \geq 1} C_r(\tilde{a}_j(x,y))_{|y=x}. \end{aligned} \tag{27}$$

The fact that $a_1(x) = \frac{1}{2}\rho$ follows by (26) and the second of (6) (with $f = 1$). Finally, the expression of a_2 and a_3, can be obtained by (27), by formally carrying out the same computations at page 261 in [8] and by taking into account that $\bar{\partial}_x a_k(x,y)$ and $\partial_y a_k(x,y)$ vanish to infinite order for $y = x$. $\square$

4 The expansion of Berezin's transform

In the quantum mechanics terminology a couple (L, h) such that $\mathrm{Ric}(h) = \omega$ is called a *geometric quantization* of the Kähler manifold (M, ω) and L is called a *quantum line bundle*. Therefore, a polarized Kähler manifold (M, g) as in the previous section gives rise to a geometric quantization of (M, ω) and viceversa. In this context the integral operator

$$\mathrm{Ber}_m(f) = \int_M \psi_m(x, y) T_{m\omega}(y) f(y) \frac{\omega^n(y)}{n!}(y),$$

which comes in the proof of Theorem 3.5 is called the *Berezin transform* of the the smooth function f. Formula (19) means that the Berezin transform of the constant function 1 equals 1 while formula (23) expresses the fact that the expansion of $\mathrm{Ber}_m(f)$ as $m \to +\infty$ depends only on the germ of the function f at the point x. More generally one can prove that $\mathrm{Ber}_m(f)$ admits an asymptotic expansion

$$\mathrm{Ber}_m(f) = \sum_{r \geq 0} m^{-r} Q_r(f), \tag{28}$$

with

$$Q_0(f) = f, Q_1(f) = \Delta f \tag{29}$$

(see e.g. [10] also for the construction of a deformation quantization of (M, ω)).

In [4] Engliš proves the following Theorem valid on a quite general class of noncompact domains in $\mathbb{C}^n$ (see also [5]) for a characterization of symmetric spaces in terms of its Berezin's transform).

Theorem 4.1 *(Engliš) Let $\Omega \subset \mathbb{C}^n$ be a strongly pseudoconvex domain with real analytic boundary endowed with its (real analytic) Bergman metric. Then the Berezin's transform $Ber_m(f)$ of a smooth function f on Ω admits an asymptotic expansion (as $m \to +\infty$)*

$$Ber_m(f) = \sum_{r \geq 0} m^{-r} Q_r(f),$$

where $Q_0(f) = f$, $Q_1(f) = \Delta f$, $Q_2(f) = \Delta\Delta f - \frac{1}{2} L_{Ric}(f)$ and $L_{Ric}(f)$ is the operator (7) appearing in Theorem 2.1.

The following theorem extends the previous theorem to compact complex manifolds endowed with a C^∞ Kähler metric.

Theorem 4.2 *Let M be a compact complex manifold equipped with a polarized Kähler metric g. Then the operaotor Q_j of the expansion (28) can be written in terms of the differential operators C_j of Theorem 2.1 (see formula (33) below). Moreover,*

$$Q_2(f) = \Delta\Delta f - \frac{1}{2} L_{Ric}(f) \tag{30}$$

for any smooth function f on M.

Proof: In a similar way as in the proof of Theorem 3.5 by taking U (the neighbourhood of x) sufficiently small we can write

$$\begin{aligned}
\mathrm{Ber}_m(f) &= \int_M \psi_m(x,y) T_{m\omega}(y) f(y) \tfrac{\omega^n(y)}{n!}(y) \\
&= \int_{M\setminus U} \psi_m(x,y) T_{m\omega}(y) f(y) \tfrac{\omega^n}{n!}(y) + \int_U e^{-mD_{\tilde\Phi}(x,y)} \tfrac{|T_{m\omega}(x,\bar y)|^2}{T_{m\omega}(x)} f(y) \tfrac{\omega^n}{n!}(y).
\end{aligned} \tag{31}$$

□

Let

$$(T_{m\omega}(x))^{-1} = 1 + \sum_{s\geq 1} m^{-s} d_s(x) \tag{32}$$

be the asymptotic expansion of $(T_{m\omega}(x))^{-1}$. By inserting (32) in (31) and by taking into account expansion (5) and (23) we obtain that $\mathrm{Ber}_m(f)(x)$ is equal to

$$\begin{aligned}
f(x) &+ \textstyle\sum_{r=1}^k m^{-r} C_r(f)(x) + \sum_{t+j=1}^k m^{-t-j} C_t(\tilde f(y) a_j(x,y))_{|y=x} \\
&+ f(x) \textstyle\sum_{s=1}^k m^{-s} d_s(x) + \sum_{r+s=2}^k m^{-r-s} d_s(x) C_r(f) \\
&+ \textstyle\sum_{t+s+j=2}^k m^{-t-s-j} d_s(x) C_t(\tilde f(y) a_j(x,y))_{|y=x} + S_k(x,m),
\end{aligned} \tag{33}$$

with $j\geq 1, r\geq 1, s\geq 1, t\geq 0$ and $\lim_{m\to\infty} m^k S_k(x,m) = 0$.

Therefore $Q_0(f) = f$. Moreover, since $C_1(f) = \Delta f - \frac{1}{2} f\rho$, $a_1(x) = \frac{\rho(x)}{2}$ and $d_1(x) = -a_1(x)$, one easily gets $Q_1(f) = \Delta f$ in accordance with (29). The expression of $Q_2(f)$ can be obtained by (33) by taking into account (6), (15) and $d_2(x) = a_1^2(x) - a_2(x)$. □

5 Appendix

In this section we recall Lu's notations and formulae used in this paper.

Let M be a n-dimensional complex manifold endowed with a Kähler metric g whose local expression is $g = \sum_{j\bar k}^n g_{j\bar k} dz_j d\bar z_k$.

The curvature tensor is defined as

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l},\ i, j, k, l = 1, \ldots, n$$

The Ricci curvature is

$$Ric_{i\bar{j}} = -\sum_{k,l=1}^{n} g^{k\bar{l}} R_{i\bar{j}k\bar{l}},\ i, j = 1, \ldots, n$$

and the scalar curvature is the trace of the Ricci curvature

$$\rho = -\sum_{i,j=1}^{n} g^{i\bar{j}} Ric_{i\bar{j}}.$$

The Laplace operator, denoted by Δ, is given by

$$\Delta f = \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}.$$

The following formulae can be found in Section 5 of [9] (cfr. also Proposition 5.2 there):

$$\begin{aligned}
&|R|^2 = \textstyle\sum_{i,j,k,l=1}^{n} |R_{i\bar{j}k\bar{l}}|^2 \\
&|Ric|^2 = \textstyle\sum_{i,j=1}^{n} |Ric_{i\bar{j}}|^2 \\
&|D'\rho|^2 = \textstyle\sum_{i=1}^{n} |\frac{\partial \rho}{\partial z_i}|^2 \\
&|D'Ric|^2 = \textstyle\sum_{i,j,k=1}^{n} |Ric_{i\bar{j},k}|^2 \\
&|D'R|^2 = \textstyle\sum_{i,j,k,l,p=1}^{n} |R_{i\bar{j}k\bar{l},p}|^2 \\
&\operatorname{div}\operatorname{div}(\rho Ric) = 2|D'\rho|^2 + \textstyle\sum_{i,j=1}^{n} Ric_{i\bar{j}} \frac{\partial^2 \rho}{\partial \bar{z}_j \partial z_i} + \rho\Delta\rho \\
&\operatorname{div}\operatorname{div}(R, Ric) = -\textstyle\sum_{i,j=1}^{n} Ric_{i\bar{j}} \frac{\partial^2 \rho}{\partial \bar{z}_j \partial z_i} - 2|D'Ric|^2 \\
&+ \textstyle\sum_{i,j,k,l=1}^{n} R_{j\bar{i}l\bar{k}} R_{i\bar{j},k\bar{l}} - R(Ric, Ric) - \sigma_3(Ric) \\
&R(Ric, Ric) = \textstyle\sum_{i,j,k,l=1}^{n} R_{i\bar{j}k\bar{l}} Ric_{j\bar{i}} Ric_{l\bar{k}} \\
&Ric(R, R) = \textstyle\sum_{i,j,k,l,p,q=1}^{n} Ric_{i\bar{j}} R_{j\bar{k}p\bar{q}} R_{k\bar{i}q\bar{p}} \\
&\sigma_1(R) = \textstyle\sum_{i,j,k,l,p,q=1}^{n} R_{i\bar{j}k\bar{l}} R_{l\bar{k}p\bar{q}} R_{q\bar{p}j\bar{i}} \\
&\sigma_2(R) = \textstyle\sum_{i,j,k,l,p,q=1}^{n} R_{i\bar{j}k\bar{l}} R_{p\bar{i}q\bar{k}} R_{j\bar{p}l\bar{q}} \\
&\sigma_3(Ric) = \textstyle\sum_{i,j,k=1}^{n} Ric_{i\bar{j}} Ric_{j\bar{k}} Ric_{k\bar{i}},
\end{aligned}$$

where " ,p" represents the covariant derivative in the direction $\frac{\partial}{\partial z_p}$.

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