Riemannian Geometry of Hartogs domains [*]

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Abstract

Let $D_F = \{(z_0, z) \in \mathbb{C}^n \mid |z_0|^2 < b, \|z\|^2 < F(|z_0|^2)\}$ be a strongly pseudoconvex Hartogs domain endowed with the Kähler metric g_F associated to the Kähler form $\omega_F = -\frac{i}{2} \partial \overline{\partial} \log\left(F(|z_0|^2) - \|z\|^2\right)$.

This paper contains several results on the Riemannian geometry of these domains. These are summarized in Theorem 1.1, Theorem 1.2 and Theorem 1.3. In the first one we prove that if D_F admits a non special geodesic (see definition below) through the origin whose trace is a straight line then D_F is holomorphically isometric to an open subset of the complex hyperbolic space. In the second theorem we prove that all the geodesics through the origin of D_F do not self-intersect, we find necessary and sufficient conditions on F for D_F to be geodesically complete and we prove that D_F is locally irreducible as a Riemannian manifold. Finally, in Theorem 1.3, we compare the Bergman metric g_B and the metric g_F in a bounded Hartogs domain and we prove that if g_B is a multiple of g_F, namely $g_B = \lambda g_F$, for some $\lambda \in \mathbb{R}^+$, then D_F is holomorphically isometric to an open subset of the complex hyperbolic space.

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1 Introduction

Let $b \in \mathbb{R}^+ \cup \{+\infty\}$ and let $F : [0, b) \to (0, +\infty)$ be a non increasing smooth function on $[0, b)$. The n-dimensional *Hartogs domain* $D_F \subset \mathbb{C}^n$ associated to the function F is defined by

$$D_F = \{(z_0, z) \in \mathbb{C}^n \mid |z_0|^2 < b, \|z\|^2 < F(|z_0|^2)\}, \tag{1}$$

where $z = (z_1, \ldots, z_{n-1})$ and $\|z\|^2 = |z_1|^2 + \cdots + |z_{n-1}|^2$.

Under the assumption that D_F is strongly pseudoconvex one can prove (see Proposition 2.1 in [11]) that the natural $(1,1)$-form on D_F given by

$$\omega_F = -\frac{i}{2}\partial\bar{\partial}\log\left(F(|z_0|^2) - \|z\|^2\right) \qquad (2)$$

is a Kähler form on D_F and this is equivalent to the requirement that F satisfies the condition

$$\left(\frac{xF'}{F}\right)' < 0, \quad x \in [0, b), \qquad (3)$$

where the derivatives are taken with respect to the variable $x = |z_0|^2$. Notice that (3) (and hence the strongly pseudoconvexity of D_F) turns out to be equivalent to the strongly pseudoconvexity of the boundary of D_F at all $z = (z_0, z_1, \ldots, z_{n-1})$ with $|z_0|^2 < x_0$ (see Proposition 2.1 in [11] for a proof). Denote by g_F the Kähler metric associated to the Kähler form ω_F. Throughout all this paper we assume that D_F is equipped with this Kähler metric. Notice that when $F(x) = 1 - x, 0 \leq x < 1$, D_F equals the n-dimensional complex hyperbolic space $\mathbb{C}H^n$, namely the unit ball B^n in $\mathbb{C}^n$ equipped with the hyperbolic metric $g_{hyp} = g_F$. Hartogs domains are interesting both from the mathematical and the physical point of view (see for example [4], [9] and [11] for the study of some Riemannian properties of g_F and the Berezin quantization of (D_F, g_F), [1] and [10] for the construction of global symplectic coordinates on these domains and [12] for the construction of Kähler-Einstein metrics on Hartogs type domains on symmetric spaces).

In this paper we are interested in the Riemannian geometry of Hartogs domains. In particular we study the geodesics, the completeness and local irreducibility of such domains with respect to the metric g_F. We denote by $\mathcal{G}$ the set of those geodesics passing through the origin whose traces are straight lines of $\mathbb{C}^n$ intersected with D_F. Since the isometry group of D_F contains $U(1) \times U(n-1)$, it is easily seen that the set $\mathcal{S}$ of geodesics of D_F passing through the origin and contained in the plane $z_0 = 0$ or in the complex line $z_1 = \cdots = z_{n-1} = 0$ is included in $\mathcal{G}$. A geodesic $\ell \in \mathcal{G}$ will be called a *special geodesic* if it belongs to $\mathcal{S}$.

Our first result is the following interesting characterization of the complex hyperbolic space amongst Hartogs domains.

Theorem 1.1 *Let (D_F, g_F) be a Hartogs domain. If there exists $\ell \in \mathcal{G}$ such that $\ell \notin \mathcal{S}$, then D_F is holomorphically isometric to an open subset of $\mathbb{C}H^n$.*

In other words the previous theorem asserts that if there exists one non special geodesic ℓ through the origin of D_F whose trace is a straight

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line, then $D_F \subset \mathbb{C}H^n$ (and hence, in this case, $\mathcal{G}$ coincide with the set of all the geodesic through the origin).

Our second result is the following:

Theorem 1.2 *Let (D_F, g_F) be a Hartogs domain. Then the following facts hold true.*

(i) *All the geodesics through the origin of D_F do not self-intersect;*

(ii) *D_F is geodesically complete with respect to the Kähler metric g_F if and only if*

$$\int_0^{\sqrt{b}} \sqrt{-\left(\frac{xF'}{F}\right)'}|_{x=u^2} \ du = +\infty, \qquad (4)$$

where we define $\sqrt{b} = +\infty$ for $b = +\infty$;

(iii) *(D_F, g_F) is locally irreducible around any of its points.*

The first part of the previous theorem should be compared with the main result of D'Atri and Zhao [2] asserting that in a bounded homogeneous domain equipped with its Bergman metric all the geodesics do not intersect. (Notice that the homogeneous assumption for an Hartogs domain implies that D_F is holomorphically equivalent to B^n, the unit ball in $\mathbb{C}^n$ (see e.g. Theorem 6.11 in [7]).

Other properties of the geodesics of the Bergman metric can be found in [5] and [6]. In [5] Fefferman deeply studied the geodesics of the Bergman metric at the boundary points of a bounded domain D while [6] is concerned with the existence of a closed geodesic in any non trivial homotopy class of a (non simply-connected) bounded domain. Regarding the completeness of the Bergman metric on a bounded domain, the reader is referred to the classical paper of S. Kobayashi [8].

By the previous discussion it is natural to compare the Bergman metric g_B and the metric g_F on a *bounded* Hartogs domain. Our third and last result is the following:

Theorem 1.3 *Let D_F be a bounded Hartogs domain. Assume that g_B is a multiple of g_F, namely $g_B = \lambda g_F$, for some $\lambda \in \mathbb{R}^+$. Then g_F is Kähler–Einstein and therefore D_F is holomorphically isometric to an open subset of $\mathbb{C}H^n$.*

The first part of the proof of the previous theorem is an adaptation of the proof of the following (unpublished) proposition communicated by Miroslav Engliš to the second author and which deals with the more general class of *generalized* Hartogs domains.

Proposition 1.4 *(Engliš) Let*

$$\widetilde{\Omega} = \{(z, w) \in \Omega \times \mathbb{C}^k : \|w\|^2 < F(z)\}$$

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be a bounded and simply-connected generalized Hartogs domain, where Ω is a pseudoconvex domain in $\mathbb{C}^n$ and $-\log F$ is a smooth strictly-PSH function on Ω. Let g_B be the Bergman metric and let g_F be the Kähler metric on $\widetilde{\Omega}$ whose Kähler potential is $-\log(F(z) - \|w\|^2)$. If $g_B = \lambda g_F$, for some $\lambda \in \mathbb{R}^+$, then g_F is Kähler-Einstein.

The next section is dedicated to the proof of our theorems.

2 Proof of the main results

The following lemma is the main tool in the proofs of Theorem 1.1 and Theorem 1.2.

Lemma 2.1 Let (D_F, g_F) be a Hartogs domain. Let $M \subset D_F$ be the real (plane) surface given by:

$$M = D_F \cap \{Im(z_0) = Im(z_1) = 0, z_j = 0, j = 2, \ldots, n-1\}, \quad (5)$$

and denote by g the metric induced on M by g_F. Then (M, g) is totally geodesic, has constant Gaussian curvature equal to $-\frac{1}{2}$ and is geodesically complete if and only if condition (4) above is satisfied.

Proof: The surface M is the fixed point set of the isometry of D_F given by $(z_0, z_1, z_2, \ldots, z_{n-1}) \mapsto (\bar{z}_0, \bar{z}_1, -z_2, \ldots, -z_{n-1})$ and hence it is totally geodesic in D_F. By setting $u = Re(z_0)$ and $v = Re(z_1)$, this surface can be described as

$$M = \{(u, v) \in \mathbb{R}^2 | \ v^2 < F(u^2), u^2 < b\}. \quad (6)$$

Furthermore, it is not difficult to see that the metric g induced by g_F on M is given by

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \frac{2}{(F - v^2)^2} \begin{pmatrix} C & -F'uv \\ -F'uv & F \end{pmatrix}, \quad (7)$$

where $C = F'^2 \cdot u^2 - (F' + F'' \cdot u^2)(F - v^2)$ and F and its derivatives are evaluated at u^2. By a straightforward, but long computation, one can verify that the Gaussian curvature of g equals $-1/2$. Hence (M, g) is isometric to an open subset, say U, of $\mathbb{R}H^2(-\frac{1}{2})$, namely the unit disk $\{(x, y) \mid x^2 + y^2 < 1\}$ in $\mathbb{R}^2$ endowed with the Beltrami-Klein metric

$$g_{BK} = \frac{2}{(1 - x^2 - y^2)^2} \left[(1 - y^2)dx^2 + 2xydxdy + (1 - x^2)dy^2 \right]. \quad (8)$$

An isometry between (M, g) and U can be described explicitly. Indeed, let $\psi : (-\sqrt{b}, \sqrt{b}) \to \mathbb{R}$ be the strictly increasing real valued function defined by

$$\psi(u) = \int_0^u \sqrt{-\left(\frac{xF'}{F}\right)'}|_{x=s^2} ds.$$

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Then, it is not hard to see that the map

$$\Psi : M \to \mathbb{R}H^2(-\frac{1}{2}), (u,v) \mapsto \left(\text{Tanh}(\psi(u)), \frac{v}{\text{Cosh}(\psi(u))\sqrt{F(u^2)}} \right)$$

is an injective local diffeomorphism satisfying $\Psi^*(g_{BK}) = g$. Therefore, the completeness of M is equivalent to $\Psi(M) = \mathbb{R}H^2$, which is easily seen to be equivalent to condition (4), and we are done. $\qquad\square$

Remark 2.2 The fact that the surface M in Lemma 2.1 is totally geodesic and that the isometry group of D_F contains $U(1) \times U(n-1)$ implies the existence of an isometry of D_F, fixing the origin and taking any given geodesic passing through the origin of D_F to a geodesic lying in M. This will be a key point in the proofs of both Theorem 1.1 and Theorem 1.2.

Remark 2.3 All the n-dimensional Hartogs domain D_F contains the complex totally geodesic surface

$$\mathcal{B} = \{z_j = 0, \ j = 1, \dots, n-1\} \cap D_F, \qquad (9)$$

which in the literature of complex analysis is called the *base* of the Hartogs domain D_F. In view of the previous lemma, it is natural to consider the Hartogs domains where the Gaussian curvature of $\mathcal{B}$ is constant, , say equal to K_0,. It is not hard to see that such domains can be classified as follows:

(a) if $K_0 = 0$ then $F(t) = ce^{-kt}, c, k > 0, \ t \in [0, +\infty)$, (complex analysts often refer to these domains as *Spring domains*);

(b) If $K_0 > 0$ then $F(t) = (c_1 + c_2 t)^{-\frac{2}{K_0}}$, with $c_1 > 0, c_2 > 0,$ $t \in [0, +\infty)$;

(c) If $K_0 < 0$ then $F(t) = (c_1 + c_2 t)^{-\frac{2}{K_0}}$, with $c_1 > 0, c_2 < 0,$ $t \in [0, -\frac{c_1}{c_2})$.

Notice that in the case (b), the corresponding Hartogs domain D_F cannot be geodesically complete. In fact in this case also its base $\mathcal{B}$ would be complete and hence biholomorphic to $\mathbb{C}P^1$, yielding the contradiction $\mathcal{B} \cong \mathbb{C}P^1 \subset D_F \subset \mathbb{C}^n$ (cfr. Example 2.6 at the end of the paper).

Proof of Theorem 1.1.
Let ℓ be a geodesic as in the statement of the theorem. Since $\ell \notin \mathcal{S}$, by Remark 2.2, we can assume $\ell \subset M$ and that

$$\ell = \{v = ku, k \neq 0\} \cap M,$$

where u and v are the parameters introduced in the proof of Lemma 2.1. Hence ℓ can be parametrized as $t \mapsto (u(t), v(t) = ku(t))$, where t varies in a real interval containing the origin and the following geodesic equations have to be satisfied

$$u'' + \Gamma^1_{11} u'^2 + 2\Gamma^1_{12} u'v' + \Gamma^1_{22} v'^2 = 0, \quad v'' + \Gamma^2_{11} u'^2 + 2\Gamma^2_{12} u'v' + \Gamma^2_{22} v'^2 = 0,$$

namely

$$u'' + \Gamma^1_{11} u'^2 + 2k\Gamma^1_{12} u'^2 + k^2 \Gamma^1_{22} u'^2 = 0 \tag{10}$$

$$ku'' + \Gamma^2_{11} u'^2 + 2k\Gamma^2_{12} u'^2 + k^2 \Gamma^2_{22} u'^2 = 0, \tag{11}$$

where $\Gamma^i_{jk}, i, j, k = 1, 2$ are the Christoffel symbols (see e.g. [3]). A straightforward computation gives :

$$\Gamma^1_{11} = \frac{1}{2D}\left(g_{22}\frac{\partial g_{11}}{\partial u} - g_{12}\left(2\frac{\partial g_{12}}{\partial u} - \frac{\partial g_{11}}{\partial v}\right)\right) =$$

$$= \frac{-4u}{D(v^2 - F)^4}[u^2(2F'^2 + v^2 F'') - F(v^2 - F)(2F'' + u^2 F''') - FF'(2F' + 3u^2 F'')], \tag{12}$$

$$\Gamma^2_{11} = \frac{1}{2D}\left(-g_{12}\frac{\partial g_{11}}{\partial u} + g_{11}\left(2\frac{\partial g_{12}}{\partial u} - \frac{\partial g_{11}}{\partial v}\right)\right) =$$

$$= \frac{4u^2 v}{D(v^2 - F)^3}[-u^2 F''^2 + F'(F'' + u^2 F''')], \tag{13}$$

$$\Gamma^1_{12} = \frac{1}{2D}\left(g_{22}\frac{\partial g_{11}}{\partial v} - g_{12}\frac{\partial g_{22}}{\partial u}\right) =$$

$$= \frac{-4v}{D(v^2 - F)^4}[-u^2 F'^2 + F(F' + u^2 F'')], \tag{14}$$

$$\Gamma^2_{12} = \frac{1}{2D}\left(g_{11}\frac{\partial g_{22}}{\partial u} - g_{12}\frac{\partial g_{11}}{\partial v}\right) =$$

$$= \frac{4uF'}{D(v^2 - F)^4}[-u^2 F'^2 + F(F' + u^2 F'')], \tag{15}$$

$$\Gamma^1_{22} = \frac{1}{2D}\left(-g_{12}\frac{\partial g_{22}}{\partial v} + g_{22}\left(2\frac{\partial g_{12}}{\partial v} - \frac{\partial g_{22}}{\partial u}\right)\right) = 0, \tag{16}$$

$$\Gamma^2_{22} = \frac{1}{2D}\left(g_{11}\frac{\partial g_{22}}{\partial v} - g_{12}\left(2\frac{\partial g_{12}}{\partial v} - \frac{\partial g_{22}}{\partial u}\right)\right) =$$

$$= \frac{-8v}{D(v^2 - F)^4}[-u^2 F'^2 + F(F' + u^2 F'')], \tag{17}$$

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where
$$D = g_{11}g_{22} - g_{12}^2 = 4\frac{CF - F'^2 u^2 v^2}{(F - v^2)^4}.$$

By solving (10) with respect to u'' and substituting into (11) we get

$$u'^2[\Gamma_{11}^2 + k(2\Gamma_{12}^2 - \Gamma_{11}^1) + k^2(\Gamma_{22}^2 - 2\Gamma_{12}^1) - k^3\Gamma_{22}^1] = 0 \qquad (18)$$

Since $u' \neq 0$ we get

$$\Gamma_{11}^2 + k(2\Gamma_{12}^2 - \Gamma_{11}^1) + k^2(\Gamma_{22}^2 - 2\Gamma_{12}^1) - k^3\Gamma_{22}^1 = 0 \qquad (19)$$

(where $\Gamma_{ij}^k = \Gamma_{ij}^k(u, ku)$).

By using (12) - (17), after a very long but straightforward calculation, the previous equation becomes

$$\frac{8ku\left(u^4 F''^2 + F(2F'' + u^2 F''') - F'(2u^2 F'' + u^4 F''')\right)}{D(k^2 u^2 - F)^3} = 0, \qquad (20)$$

which, by setting $u^2 = t, 0 \leq t < b$, is equivalent to the following ODE

$$t^2 F''^2 + F(2F'' + tF''') - F'(2tF'' + t^2 F''') = 0. \qquad (21)$$

Notice that for $t \neq 0$ this equation can be written as

$$t^2 F''^2 + \left(\frac{F}{t} - F'\right)(t^2 F'')' = 0. \qquad (22)$$

By setting $G = t^2 F''$ equation (22) becomes

$$G' = -\frac{F''t}{F - F't}G \qquad (23)$$

(notice that $F - F't > 0$ for all $0 < t < b$ since F is not decreasing) and hence

$$G(t) = c\, e^{\int \frac{-F''t}{F - F't}dt} = c\,(F - F't), \qquad (24)$$

for some $c \in \mathbb{R}$. For $t \to 0$ we get $cF(0) = 0$, i.e. $c = 0$. Therefore $G = t^2 F'' = 0$, which implies $F(t) = c_1 - c_2 t$ for some $c_1, c_2 > 0$. Then the map

$$\phi : D_F \to \mathbb{C}H^n, \ (z_0, z_1, \ldots, z_{n-1}) \mapsto \left(\frac{z_0}{\sqrt{c_1/c_2}}, \frac{z_1}{\sqrt{c_1}}, \ldots, \frac{z_{n-1}}{\sqrt{c_1}}\right)$$

is a holomorphic isometry of D_F into an open subset of $\mathbb{C}H^n$ and this concludes the proof of Theorem 1.1. $\qquad\square$

Remark 2.4 In the very definition of a Hartogs domain D_F we have assumed (cfr. the introduction) that F is non increasing in the interval $[0, b)$. The statement of Theorem 1.1 holds true also without this assumption. Indeed, it follows by condition (3) that $F'(t) < 0$ in a suitable interval $0 \leq t < \epsilon < b$, for some ϵ, and the proof works also in this case.

Proof of Theorem 1.2

Let $\ell \subset D_F$ be a geodesic passing through the origin. By Remark 2.2 we can assume $\ell \subset M$. On the other hand by Lemma 2.1, (M, g) is isometric to an open subset of $\mathbb{R}H^2(-\frac{1}{2})$ where it is well-known that all the geodesics do not self intersect. This proves (i) of Theorem 1.2.

Notice that, again by Remark 2.2 and by Hopf–Rinow's theorem the completeness of g_F is equivalent to that of g, which by Lemma 2.1 is equivalent to (4) and we this proves (ii).

In order to prove (iii), assume by contradiction that D_F is locally reducible around some point, say $p \in D_F$. Since the group $U(1) \times U(n-1)$ acts by isometries on (D_F, g_F) we can assume that $p \in M$ where M is given by (5). So $p = (u, v, 0, \cdots, 0)$ and we can assume that both u, v are (real numbers) different from zero (indeed if one of them is zero, say $p_1 = 0$, then D_F is locally reducible around the point $p' = (p'_1, p_2, 0, \ldots 0)$ with p'_1 sufficiently close to zero). Therefore there exists an neighborhood $D \subset D_F$ of $p \in D_F$ such that (D, g_F) splits as a Riemannian product i.e. $D = A \times B$, where A and B are Kähler manifolds. So the Lie algebra $\mathfrak{g}$ of Killing vector fields of D also splits into two (or more) factors. Since $\mathfrak{u}(1) \times \mathfrak{u}(n-1) \subset \mathfrak{g}$ it follows that $\mathfrak{g}$ has at most two factors. Moreover since $p = (u, v, 0, \cdots, 0)$ with $u, v \neq 0$ we can recover the tangent space to the Riemannian factors A and B. Thus, the factor A is an open subset $A \subset \mathbb{C}$, with $u \in A$, and B is an open subset $B \subset \mathbb{C}^{n-1}$, with $(v, 0, \cdots, 0) \in B$. In particular such Riemannian factors must be orthogonal w.r. to g_F. Then the coefficient g_{12} of the metric g on M induced by g_F has to be zero for u and v different from zero. On the other hand, by (7) above, $g_{12} = -F'uv \neq 0$, a contradiction. This concludes the proof of (iii). $\square$

With the aid of (ii) in Theorem 1.2 we now study the completeness of two specific Hartogs domains.

Example 2.5 If $F(t) = ce^{-kt}$, $c, k > 0$, $t \in [0, +\infty)$ then condition (4) is easily seen to be satisfied, so we get that the the Spring domains are complete (cfr. (a) of Remark 2.3).

Example 2.6 If $F(t) = \frac{1}{(c_1 + c_2 t)^p}$ $(p \in \mathbb{N}^+)$, $t \in [0, +\infty)$, then

$$\int_0^{\sqrt{b}} \sqrt{-\left(\frac{xF'}{F}\right)'}\Big|_{x=u^2} \, du = \int_0^{+\infty} \frac{\sqrt{c_1 c_2 p}}{c_1 + c_2 u^2} du = \frac{\pi}{2}\sqrt{p} < \infty$$

which proves that, for such F, the domain D_F is not complete (cfr. (b) of Remark 2.3).

We now prove the last result of this paper.

Proof of Theorem 1.3

Recall that the Bergman metric g_B on D_F is, by definition, the one given by the Kähler potential $\log \widetilde{K}(z_0, z; z_0, z)$, where $\widetilde{K}(z_0, z; z_0', z')$ is the Bergman kernel of D_F. Let

$$\tilde{F}(z_0, z) := F(|z_0|^2) - \|z\|^2. \tag{25}$$

Note that this is a local defining function (positively signed) for D_F at any boundary point (z_0, z) with $z \in \mathcal{B}$, and such boundary points are strictly pseudoconvex. The hypothesis of the theorem and the fact that D_F is contractible means that

$$\log \widetilde{K}(z_0, z) = -\lambda \log \widetilde{F}(z_0, z) + 2 \operatorname{Re} G(z_0, z)$$

for some holomorphic function G on D_F; here and below we will write just $\widetilde{K}(z_0, z)$ for $\widetilde{K}(z_0, z; z_0, z)$. By rotational symmetry of $\widetilde{K}$ and F, the pluriharmonic function $2 \operatorname{Re} G$ must depend only on $|z_0|^2$ and $\|z\|^2$, hence must be a positive constant, say μ. Thus

$$\widetilde{K}(z_0, z) = \frac{\mu}{\widetilde{F}(z_0, z)^\lambda}. \tag{26}$$

On the other hand, by Fefferman's formula [5] for the boundary singularity of the Bergman kernel,

$$\widetilde{K}(z_0, z) = \frac{a(z_0, z)}{\widetilde{F}(z_0, z)^{n+1}} + b(z_0, z) \log \widetilde{F}(z_0, z), \qquad (z_0, z) \in D_F, \tag{27}$$

where $a, b \in C^\infty(\mathcal{B} \times \mathbb{C}^{n-1})$ and

$$a(z_0, z) = \frac{n!}{\pi^n} J[\tilde{F}](z_0, z), \tag{28}$$

for $z_0 \in \mathcal{B}$ and $\|z\|^2 = F(|z_0|^2)$ and where $J[\widetilde{F}]$ is the Monge-Ampere determinant

$$J[\widetilde{F}] = (-1)^n \det \begin{bmatrix} \widetilde{F} & \frac{\partial \widetilde{F}}{\partial z_0} & \partial_z \widetilde{F} \\ \frac{\partial \widetilde{F}}{\partial \overline{z}_0} & \frac{\partial^2 \widetilde{F}}{\partial z_0 \partial \overline{z}_0} & \partial_z \left(\frac{\partial \widetilde{F}}{\partial \overline{z}_0}\right) \\ \partial_{\overline{z}} \widetilde{F} & \partial_{\overline{z}} \left(\frac{\partial \widetilde{F}}{\partial \overline{z}_0}\right) & \partial_{\overline{z}} \partial_z \widetilde{F} \end{bmatrix}.$$

A direct computation gives

$$J[\widetilde{F}] = -F^2 \frac{\partial^2 \log F}{\partial z_0 \partial \overline{z}_0}. \tag{29}$$

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(which depends only on $|z_0|^2$). By comparing (26) with (27) one gets:

$$\mu = \frac{a(z_0, z)\widetilde{F}(z_0, z)^\lambda}{\widetilde{F}(z_0, z)^{n+1}} + b(z_0, z)\widetilde{F}(z_0, z)^\lambda \log \widetilde{F}(z_0, z), \qquad (z_0, z) \in D_F,$$

which evaluated at $\|z\|^2 = F(|z_0|^2)$, forces $\lambda = n + 1$. Further, by (28) and (29), the last expression gives

$$-F^2 \frac{\partial^2 \log F}{\partial z_0 \bar{\partial} z_0} = c,$$

for all $z_0 \in \mathcal{B}$ and $\|z\|^2 = F(|z_0|^2)$, where c is the negative constant given by $c = -\frac{\mu \pi^n}{n!}$ (notice that the condition $\|z\|^2 = F(|z_0|^2)$ is superfluous, since nothing there depends on z). Feeding this back into formula (29) one gets $J[F](z_0, z) = c$ for all $(z_0, z) \in D_F$, i.e. g_F is Kähler-Einstein.

Let us recall now Lemma 3.1. of [11].

Lemma 2.7 *Let (D_F, g_F) be an n-dimensional Hartogs domain. Assume that one of its generalized scalar curvatures is constant. Then (D_F, g_F) is holomorphically isometric to an open subset of the n-dimensional complex hyperbolic space.*

Since the scalar curvature is one of the generalized scalar curvatures the proof of Theorem 1.3 is complete. $\square$

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