Calabi's Diastasis Function for Hermitian Symmetric Spaces

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Abstract

In this paper we study the Calabi diastasis function of Hermitian symmetric spaces. This allows us to prove that if a complete Hermitian locally symmetric space (M, g) admits a Kähler immersion into a globally symmetric space (S, G) then it is globally symmetric and the immersion is injective. Moreover, if (S, G) is symmetric of a specified type (Euclidean, noncompact, compact), then (M, g) is of the same type. We also give a characterization of Hermitian globally symmetric spaces in terms of their diastasis function. Finally, we apply our analysis to study the balanced metrics, introduced by Donaldson, in the case of locally Hermitian symmetric spaces.

Keywords: Kähler metrics; diastasis function; symmetric space; complex space form; balanced metrics.

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1 Introduction

Let M be a complex manifold endowed with a real analytic Kähler metric g. Then in a neighborhood of every point $p \in M$, one can introduce a very special Kähler potential D^g_p for the metric g, which Calabi [3] christened *diastasis*. Recall that a Kähler potential is an analytic function Φ defined in a neighborhood of a point p such that $\omega = \frac{i}{2}\bar{\partial}\partial\Phi$, where ω is the Kähler form associated to g. In a complex coordinate system (z) around p one has:

$$g_{\alpha\bar{\beta}} = 2g(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}) = \frac{\partial^2\Phi}{\partial z_\alpha \partial \bar{z}_\beta}.$$

A Kähler potential is not unique: it is defined up to an addition with the real part of a holomorphic function. By duplicating the variables z and $\bar{z}$ a potential Φ can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighborhood U of the diagonal containing $(p, \bar{p}) \in M \times \bar{M}$ (here $\bar{M}$

denotes the manifold conjugated to M). The *diastasis function* is the Kähler potential D^g_p around p defined by

$$D^g_p(q) = \tilde{\Phi}(q, \bar{q}) + \tilde{\Phi}(p, \bar{p}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p}).$$

Observe the $D^g_p(q)$ is symmetric in p and q and $D^g_p(p) = 0$.

The diastasis function is the key tool for studying Kähler immersions (i.e. holomorphic and isometric immersions) of a Kähler manifold into another Kähler manifold as expressed by the following lemma.

Lemma 1.1 *(Calabi [3]) Let (M, g) be a Kähler manifold which admits a Kähler immersion $f : (M, g) \to (S, G)$ into a real analytic Kähler manifold (S, G). Then the metric g is real analytic. Let $D^g_p : U \to \mathbb{R}$ and $D^G_{f(p)} : V \to \mathbb{R}$ be the diastasis functions of (M, g) and (S, G) around p and $f(p)$ respectively. Then $f^{-1}(D^G_{f(p)}) = D^g_p$ on $f^{-1}(V) \cap U$.*

In [3] Calabi studied Kähler immersions $(M, g) \to (S, G)$ when the ambient space (S, G) is a finite or infinite dimensional complex space form. He gives necessary and sufficient conditions for (M, g) to admit a Kähler immersion into (S, G) in terms of the diastasis function D^g_p. There are three types of complex space forms according to the sign of their constant holomorphic sectional curvature:

(i) the complex Euclidean space $\mathbb{C}^N$, $N \leq \infty$ with the flat metric denoted by G_0. (Here $\mathbb{C}^\infty$ denotes the Hilbert space consisting of sequences w_j, $j = 1 \ldots, w_j \in \mathbb{C}$ such that $\sum_{j=1}^{+\infty} |w_j|^2 < +\infty$). The diastasis function $D^{G_0}_z : \mathbb{C}^N \to \mathbb{R}$ of G_0 around $p \in \mathbb{C}^N$ is given by: $D^{G_0}_p(q) = \sum_{j=1}^N |p_j - q_j|^2$.

(ii) the complex projective space $\mathbb{C}P^N_b$, $N \leq \infty$ and $b > 0$, with the Fubini–Study metric G_{FS} of holomorphic sectional curvature $4b$. If $p = [1, 0 \ldots, 0]$ then in the affine coordinates the diastasis around p is $D^{G_{FS}}_p(z) = \frac{1}{b} \log(1 + b \sum_{j=1}^n |z_j|^2)$. One can show that for all $p \in \mathbb{C}P^N_b$ the diastasis function $D^{G_{FS}}_p$ around p is globally defined except in the cut locus H_p of p where it blows up. Moreover $e^{-D^{G_{FS}}_p}$ is globally defined and smooth on $\mathbb{C}P^N_b$, $e^{-D^{G_{FS}}_p(q)} \leq 1$ and $e^{-D^{G_{FS}}_p(q)} = 1$ implies $p = q$ (see [3] or [1] for details).

(iii) the complex hyperbolic space $\mathbb{C}H^N_b$ $N \leq \infty$ and $b < 0$, namely the domain $B \subset \mathbb{C}^N$ given by $B = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^N | \sum_{j=1}^N |z_j|^2 < -\frac{1}{b}\}$. endowed with the hyperbolic metric G_{hyp} of constant holomorphic sectional curvature $4b$. The globally defined diastasis $D^{G_{hyp}}_p$ around $p = (0, 0, \ldots 0)$ is given by: $D^{G_{hyp}}_p(z) = \frac{1}{b} \log(1 + b \sum_{j=1}^n |z_j|^2)$.

This paper is dedicated to the study of the diastasis function of Hermitian symmetric spaces. In Section 2 we show that for Hermitian globally symmetric spaces (M, g) the diastasis function D^g_p around any point $p \in M$ behaves like the diastasis of the complex projective space, namely it is globally defined on M, except in a measure zero subset. Moreover, $e^{-D^g_p}$ is globally defined and smooth on M, $e^{-D^g_p(q)} = 1$ implies $p = q$. We call a Kähler form with these properties *projective-like* (see definition in Section 3 below). These properties are the main ingredients used in Section 3 to study the Kähler immersions of a locally Hermitian symmetric space into *almost projective-like* Kähler manifolds, namely Kähler manifolds (S, G) where $e^{-D^g_x}$ is globally defined for all $x \in S$. The main results of this section are Theorem 3.3 and Theorem 3.7 and their corollaries where we prove that *if there exists a Kähler immersion of a complete Hermitian locally symmetric space* (M, g) *into an almost projective-like Kähler manifold* (S, G) *then* M *is forced to be simply connected and the immersion to be injective.* Moreover, *if* (S, G) *is globally symmetric and of a specified type (Euclidean, noncompact, compact), then* (M, g) *is of the same type.* This generalizes a Theorem of Takeuchi (see Theorem 3.8), where the ambient space is the finite dimensional complex projective space and a Theorem of Nakagawa and Takagi (see Theorem 3.9) where the ambient space is the finite dimensional Euclidean or complex hyperbolic space. We also give a characterization of Hermitian globally symmetric spaces in terms of their diastasis function (see Corollary 3.11). Finally in Section 4 we apply the results obtained in Section 3 to the study of balanced metrics, defined by Donaldson in [5], on Hermitian symmetric spaces or more generally projective-like Kähler manifolds. The main result in this direction is Theorem 4.4 (which is a generalization of a recent result of Engliš [6]) where we prove that *a metric* g *on a complex manifold* M *such that its lift* $\tilde{g}$ *to the universal cover* $\tilde{M}$ *is projective-like cannot be balanced unless* M *is simply-connected.* This raises the question of studying balanced metrics g on a complex manifold M such that their lift to the universal cover $\tilde{M}$ is still balanced. To this regard we have Theorem 4.5. This theorem, combined with a uniqueness result on cohomologous balanced metrics due the author and C. Arezzo [2], implies (see Corollary 4.6) that *any balanced metric* g_1 *cohomologous to the Kähler metric* g *of a compact Hermitian locally symmetric space* (M, g) *(with* $\tilde{M}$ *compact) cannot admit a balanced lift unless* (M, g) *is globally symmetric.*

2 The diastasis function of a symmetric space

We describe the diastasis function for Hermitian (globally) symmetric spaces. (We refer to Sections 2, 3 and 4 of [6] for the basic material we need about these spaces.)
Every Hermitian symmetric space (S, g) is (up to biholomorphisms) the Cartesian product of irreducible ones. Irreducible Hermitian symmetric spaces are of three types: Euclidean, noncompact and compact.

An Hermitian symmmetric space of *Euclidean type* is biholomorphically isometric to $(\mathbb{C}^n, g)$, where $g = cg_0$ and g_0 is the standard Euclidean metric. Therefore a (global) potential for g is given by $\Phi(z) = c\|z\|^2$ and the diastasis function $D^g_z : \mathbb{C}^n \to \mathbb{R}$ around z is given by

$$D^g_z(w) = c\|z - w\|^2. \quad (1)$$

An irreducible Hermitian symmetric space of *noncompact type* is biholomorphically isometric to (D, g), where D is a bounded symmetric domain in $\mathbb{C}^n$ which can be chosen circular (i.e. $z \in D, \theta \in \mathbb{R}$ imply $e^{i\theta}z \in D$) and convex and g is, up to a positive constant, the Bergman metric g_D on D, namely $g = cg_D, c > 0$. Therefore a (globally defined) potential for the metric g is given by $\Phi(z, z) = c \log K_D(z, z)$, where $K_D(z, z)$ is the Bergman kernel function on D. The Kähler form associated to g is $\omega = c\omega_D = c\frac{i}{2}\partial\bar{\partial}\log K_D$. Observe that the circularity of D implies that $K_D(z, 0) = K_D(0, z) = \frac{1}{\mathrm{vol}(D)}$, where $\mathrm{vol}(D)$ is the Euclidean volume. Therefore the diastasis function $D^g_0 : D \to \mathbb{R}$ around 0 is given by:

$$D^g_0(z) = cD^{g_D}_0(z) = c \log[\mathrm{vol}(D)K_D(z, z)]. \quad (2)$$

In the next Lemma we prove that $D^g_0(z) = 0$ implies $z = 0$.

Lemma 2.1 *Let (D, g) be an irreducible Hermitian symmetric space of noncompact type. Then for all $p \in D$ the diastasis function D^g_p is globally defined on D, $D^g_p(q) \geq 0$ for all $p, q \in M$ and $D^g_p(q) = 0$ implies $p = q$.*

Proof: Let $\mathcal{H}_m$ be the Hilbert space consisting of holomorphic functions f on D such that $\int_D \frac{|f|^2}{K_D^{mc}} dz < +\infty$, where dz is the Lebesgue measure on D. Let f^m_j be an orthonormal basis for $\mathcal{H}_m$. It is not hard to see that $\frac{\sum_{j=0}^{+\infty} |f^m_j(z)|^2}{K_D^{mc}(z,z)}$ is invariant under the action of the group of isometric biholomorphisms of (D, g). Since this group acts transitively on D we have that

$$\sum_{j=0}^{+\infty} |f_j^m(z)|^2 = c_m K_D^{mc}(z, z), c_m > 0. \tag{3}$$

It can be shown that, for m sufficiently large, $\mathcal{H}_m$ contains all polynomials and that that $1, z_1, \ldots, z_n$ are L^2-orthogonal. Thus, without loss of generality, we can assume that there exist $\lambda_k \in \mathbb{C}^*, k = 0, \ldots n$ such that $f_0^m = \lambda_0$ and $f_k^m = \lambda_k z_k, k = 1, \ldots n$. Consider now the holomorphic map

$$\Phi_m : D \to \mathbb{C}^\infty : (z_1, \ldots, z_n) \mapsto (\lambda_0, \lambda_1 z_1, \ldots, \lambda_n z_n, f_{n+1}^m, \ldots), \tag{4}$$

where $\mathbb{C}^\infty$ denotes the Hilbert space consisting of sequences $\underline{w} = w_j, j = 1 \ldots, w_j \in \mathbb{C}$ such that $\sum_{j=1}^{+\infty} |w_j|^2 < +\infty$. This space can be seen as the affine chart $Z_0 \neq 0$ of the infinite dimensional complex projective space $\mathbb{C}P^\infty$. Therefore the Fubini-Study metric G_{FS}^∞ of $\mathbb{C}P^\infty$ (see (ii) in the introduction) restricts to a Kähler metric

$$g_\infty = \frac{i}{2} \partial\bar{\partial} \log(1 + \sum_{j=1}^{+\infty} |w_j|^2) \tag{5}$$

on $\mathbb{C}^\infty$. It follows by (3) that $\Phi_m^*(g_\infty) = mcg$ and therefore the map (4) is a injective Kähler immersion. Denote by $D_{\underline{0}}^{g_\infty} : \mathbb{C}^\infty \to \mathbb{R}$ the diastasis function for the metric g_∞ around $\underline{0}$ ($\underline{0}$ is the sequence with all zero entries). Fix a point $p \in D$. Without loss of generality we can assume, by the aid of a unitary transformation in $\mathbb{C}^\infty$, that $\Phi_m(p) = \underline{0}$. By Lemma 1.1 we then get that $D_p^{mcg} = \Phi_m^{-1}(D_{\underline{0}}^{g_\infty})$. Since Φ_m is injective and since the equality $D_{\underline{0}}^{g_\infty}(\underline{w}) = \log(1 + \sum_{j=1}^{+\infty} |w_j|^2) = 0$ implies $\underline{w} = \underline{0}$, it follows that $D_p^{mcg}(q) \geq 0$ for all $p, q \in M$ and $D_p^{mcg}(q) \geq 0$ implies $p = q$. The equality $D_p^g = \frac{1}{mc} D_p^{mcg}$ concludes the proof of the Lemma. □

Remark 2.2 The fact that a bounded domain endowed with its Bergman metric admits a Kähler immersion φ into the infinite dimensional complex projective space is well-known and was first pointed out by Kobayashi [8]. This is not sufficient to deduce that its diastasis function satisfies the properties stated in Lemma 2.1 since we do not know if the Kähler immersion φ is injective.

Finally, we consider irreducible Hermitian symmetric spaces of *compact type* (C, g). It is well known that C is a compact homogeneous simply-connected Kähler-Einstein manifold with strictly positive scalar curvature.

Therefore, if ω denotes the Kähler form associated to g, there exists $c > 0$ such that $c\omega$ represents the first Chern class of C and hence it is an integral Kähler form. The integrality of $c\omega$ implies that there exists a holomorphic line bundle L on C such that $c_1(L) = [c\omega]$. Let m be a non-negative integer and let $H^0(L^m)$ be the space of global holomorphic sections of L^m, the m-th tensor power of L. Let $s_0, \dots, s_{d_m}$ be an orthonormal basis of $H^0(L^m)$. Since C is simply-connected and homogeneous one can find a positive integer m such that the Kodaira map

$$K_m : C \to \mathbb{C}P^{d_m} : x \mapsto [s_0(x), \dots, s_{d_m}(x)] \tag{6}$$

is a Kähler embedding, i.e. $K_m^*(G_{FS}) = mcg$, where G_{FS} denotes the Fubini–Study metric on $\mathbb{C}P^{d_m}$. By Lemma 1.1, by the structure of the diastasis function for the Fubini–Study metric (see (ii) in the introduction) and by the fact that $D_p^g = \frac{1}{mc} D_p^{mcg}$, we get:

Lemma 2.3 *Let (C, g) be a Hermitian globally symmetric space of compact type and let p be a point of C. Then the diastasis function D_p^g around p is non-negative and globally defined except in a set of measure zero Y_p where it blows up. Furthermore, $e^{-D_p^g}$ is globally defined on C, $e^{-D_p^g(q)} \leq 1$ for all $p, q \in C$ and $e^{-D_p^g(q)} = 1$ implies $p = q$.*

Remark 2.4 One can give an explicit formula for the diastasis function for a Hermitian symmetric space of compact type analogous to formula (2). More precisely, one has that for all $p \in C$, $C \setminus Y_p$ is biholomorphic to $\mathbb{C}^n$ (n being the complex dimension of M). Let $D \subset \mathbb{C}^n$ be the Hermitian symmetric domain of noncompact type dual to C. Under the previous biholomorphism, p can be made to correspond to the origin 0 of D and the diastasis function $D_p^g = D_0^g : D \to \mathbb{R}$ is given by:

$$D_0^g(z) = -c \log[\mathrm{vol}(D) K_D(z, -z)],$$

where c is a positive constant and K_D is the Bergman Kernel function on D.

Remark 2.5 In the general case of a Kähler immersion of a Kähler manifold (M, g) into $\mathbb{C}P^N$, the set Y_p, where the distasis D_p^g blows up, is called the polar variety with respect to p (see [3]). In [14] it is proved that the polar variety with respect to a point p of a Hermitian globally symmetric space of compact type equals the cut locus of p with respect to the metric g.

3 Kähler immersions of Hermitian spaces

In this section we study Kähler immersions of Hermitian locally symmetric spaces into Kähler manifolds which admit a diastasis similar to that of the projective space. Due to the structure of the diastasis function for the complex projective space (see (ii) in the introduction), we give the following:

Definition 3.1 Let S be a complex manifold. We say that a real analytic Kähler metric G on S is *almost projective-like* if for all points $x \in S$ the function $e^{-D^G_x}$ is globally defined on S. If, moreover, $e^{-D^G_x(y)} = 1$ implies $x = y$ then the Kähler metric G is said to be *projective-like*. We say that a real analytic Kähler manifold (S, G) is (almost) projective-like if the Kähler form G is (almost) projective-like.

Observe that the Kähler metric of a finite or infinite dimensional complex spase form is projective-like. This is true also for Hermitian globally symmetric space as expressed by the following proposition.

Proposition 3.2 *An Hermitian globally symmetric space (M, g) is projective-like.*

Proof: Write $(M, g) = (M_1, g_1) \times \ldots \times (M_r, g_r)$ as a product of irreducible globally symmetric spaces (M_j, g_j) and observe that

$$e^{-D^g_p(q)} = e^{-D^{g_1}_{p_1}(q_1)} \cdots e^{-D^{g_r}_{p_r}(q_r)}, \tag{7}$$

for all $p = (p_1, \ldots p_r)$ and $q = (q_1, \ldots q_r)$ in M. Suppose that $e^{-D^g_p(q)} = 1$. Since, by (1), Lemmata 2.1 and 2.3, $e^{-D^{g_j}_{p_j}(q_j)} \leq 1$ for all j it follows that $e^{-D^{g_j}_{p_j}(q_j)} = 1$ for all j. Again by formula (1), Lemmata 2.1 and 2.3 we know that an irreducible Hermitian globally symmetric space is projective-like and hence $p_j = q_j$, i.e. $p = q$. □

The following theorem is an easy consequence of the definitions and of the property of Calabi's diastasis function.

Theorem 3.3 *Let (S, G) be an almost projective-like Kähler manifold and let (M, g) be a Kähler manifold such that its universal cover $(\tilde{M}, \tilde{g})$ is projective-like. If (M, g) admits a Kähler immersion f into (S, G), then (M, g) is simply connected and the map f is injective.*

Proof: Let $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$ be the universal covering map and consider the Kähler immersion $\tilde{f} = f \circ \pi : (\tilde{M}, \tilde{g}) \to (S, G)$. We claim that $\tilde{f}$ is injective. Indeed, let p and q be two points in $\tilde{M}$ such that $\tilde{f}(p) = \tilde{f}(q) = x$. It follows by Lemma 1.1 that $e^{-D^g_p(q)} = e^{-D^G_x(x)} = 1$ (the previous equality makes sense since, by hypothesis, $e^{-D^G_x}$ is globally defined on S). Since, by assumption, $(\tilde{M}, \tilde{g})$ is projective-like it follows that $p = q$. Therefore π injective and, since it is a covering map, it is forced to be a biholomorphism.

□

Consider now a complete Hermitian locally symmetric space (M, g). It is a classical fact that its universal cover $(\tilde{M}, \tilde{g})$ is an Hermitian globally symmetric space (see e.g. [7]). Therefore, by taking into account Proposition 3.2 and the previous theorem, we get:

Corollary 3.4 *If a complete Hermitian locally symmetric space admits a Kähler immersion into an almost projective-like Kähler manifold (S, G) then (M, g) is globally symmetric and the immersion is injective.*

By taking $f = id_M$ (the identity map of M) in the previous corollary we get the following:

Corollary 3.5 *Let (M, g) be a complete Hermitian locally symmmetric space. If g is almost projective-like, then (M, g) is globally symmetric. In particular, any complete, almost projective-like and flat Kähler manifold is simply-connected.*

Observe that the metric g of a Hermitian locally symmetric space has constant scalar curvature. In the compact case the previous corollaries hold true for any constant scalar curvature Kähler metric cohomologous to g. More precisely we have the following:

Theorem 3.6 *Let (M, g) be a Hermitian locally symmetric space such that its universal cover $\tilde{M}$ is compact (and hence M is compact). Let g_1 be a constant scalar curvature Kähler metric cohomologous to g (namely the Kähler forms associated to g and g_1 are cohomologous). If (M, g_1) admits a Kähler immersion f into an almost projective-like Kähler manifold (S, G) then (M, g) is globally symmetric and the map f is an embedding. In particular if (M, g_1) is almost projective-like then (M, g) is globally symmetric.*

Proof: As before consider the the universal covering map $\pi : \tilde{M} \to M$ and let $\tilde{g}_1$ be the constant scalar curvature Kähler metric on $\tilde{M}$ given by $\tilde{g}_1 =$

$\pi^*(g_1)$. By Theorem B in [10] two cohomologous constant scalar curvature metrics on a compact homogeneous space have to be equivalent, hence there exists a biholomorphism $F : \tilde{M} \to \tilde{M}$ such that $F^*(\tilde{g}_1) = \tilde{g}$, where $\tilde{g} = \pi^*(g)$. Thus the metric $\tilde{g}_1$ is projective-like and the conclusion follows by applying Theorem 3.3 to the Kähler immersion $f : (M, g_1) \to (S, G)$. $\square$

In the case the ambient space (S, G) is globally symmetric we get the following.

Theorem 3.7 *Let (S, G) be a globally Hermitian symmetric space. If a complete Hermitian locally symmetric space (M, g) admits a Kähler immersion f into (S, G), then (M, g) is globally symmetric and f is injective. If, moreover, (S, G) is a globally symmetric space of a specified type, then M is of the same type.*

Proof: Since (S, G) is projective-like by Proposition 3.2 we can apply Corollary 3.4 to prove the first part. In order to prove the second part we start from the case when (S, G) is of Euclidean type. It follows immediately that in the decomposition of (M, g) into irreducible factors it cannot appear compact factors. Moreover, Theorem 3.9 below implies that in this decomposition there cannot exist a noncompact factor and hence M is of Euclidean type. Suppose now that (S, G) is of compact type and suppose, by contradiction, that (M, g) is of Euclidean or noncompact type and it admits a Kähler immersion f into (S, G). In the Euclidean case, by composing f with the map $K_m : (S, G) \to (\mathbb{C}P^{d_m})$ given by (6) in the previous section, we get (for sufficiently large m) a Kähler immersion $K_m \circ f : (\mathbb{C}^n, mcg_0) \to (\mathbb{C}P^{d_m}, G_{FS})$. Since $(\mathbb{C}^n, mcg_0)$ is flat, this is forbidden by a well-known result of Calabi (see [3] or Lemma 2.2 in [14]). The case of a Kähler immersion $f : (D, g) \to (S, G)$ of a symmetric space of noncompact type (D, g) into a symmetric space of compact type (S, G) is excluded by the following argument. As before, by composing f with the map K_m we get a Kähler immersion $K_m \circ f : (D, mg) \to (\mathbb{C}P^{d_m}, G_{FS})$. On the other hand one can construct a Kähler immersion Φ_m of (D, mg) into $\mathbb{C}P^\infty$ as in the proof of Lemma 2.1. This map is built with an orthonormal basis of holomorphic functions on D and hence it is full, i.e. $\Phi_m(D)$ is not contained in $\mathbb{C}P^N$ with $N < \infty$. Observe now, that the Kähler immersions $K_m \circ f$ and Φ_m contradict the rigidity Theorem of Calabi [3] since they cannot be joined by a unitary transformation of $\mathbb{C}P^\infty$. Finally, we need to show that $(M, g) = (\mathbb{C}^n, cg_0)$ *cannot be immersed into a Hermitian symmetric space of noncompact type (S, G)* (the case (M, g) compact into (S, G) noncompact is trivially excluded). This is proved in a unpublished work of Antonio J. Di Scala. His proof goes as follows. Let $f : (\mathbb{C}^n, cg_0) \to (S, G)$ be such an immersion. Therefore there exist a open set $U \subset \mathbb{C}^n$ with the flat metric which can be Kähler embedded into (S, G). By Proposition 9.2 p. 176 in [9] (by taking f as the inclusion) one has:

$$R^S(X, JX, JY, Y) = R^M(X, JX, JY, Y) + 2\|\alpha(X, Y)\|^2,$$

where R^S and R^M denote the curvature tensors of S and M respectively, X and Y are two vectors fields on M, J is the almost complex structure on S (and M) and α is the second fundamental form. Since $R^S(X, JX, JY, Y)$ (i.e. the *holomoprhic bisectional curvature*) of an Hermitian symmetric space of noncompact type is non positive, we get that such f is totally geodesic, i.e. $\alpha = 0$. But totally geodesic submanifolds of symmetric spaces are characterized by Lie triple system, namely subspace $V_p \subset T_pM$ invariant by the curvature tensor (see [7]). Being this an algebraic characterization we can think this "flat" and "complex " Lie triple V_p in the complex dual of S. So, we get a complex flat submanifold of an Hermitian symmetric space of compact type which is excluded by Calabi's theorem as before. $\square$

Observe that the previous Theorem is a generalization of the following theorems (see [14] and Section 3 in [13])

Theorem 3.8 *(Takeuchi) Let $f : (M, g) \to \mathbb{C}P_b^N$ be a Kähler immersion of a complete Hermitian locally symmetric space into the finite dimensional complex projective space $\mathbb{C}P_b^N$. Then (M, g) is a globally symmetric space of compact type and f is an embedding.*

Theorem 3.9 *(Nakagawa and Takagi) A Hermitian locally symmetric space (M, g), which admits a Kähler immersion into $\mathbb{C}H_b^N$ or $\mathbb{C}^N$ (N finite), is totally geodesic.*

Remark 3.10 Theorem 3.9 (and hence our Theorem 3.7) cannot be generalized when the ambient space is an infinite dimensional complex space form. Indeed, one can contruct Kähler immersions of a flat or hyperbolic complex space form into $\mathbb{C}P^\infty$ (see Chapter 5 in [3]). It is also worth mentioning that Theorem 3.9 has been generalized when the manifold (M, g) is Kähler-Einstein or homogeneous (see [20] and [4]).

We conclude this section with a nice characterization of globally symmetric spaces in terms of their diastasis function.

Corollary 3.11 *Let (M, g) be any real analytic complete Kähler manifold. Suppose that the following two conditions are satisifed:*

1. *the metric g is almost projective-like;*
2. *$\Delta_p(F(p,q)) = \Delta_q(F(p,q))$, where $F(p,q) = e^{-D^g_p(q)}$ and Δ_p (resp. Δ_q) denotes the Laplacian with respect to the variable p (resp. to the variable q).*

Then (M, g) is globally symmetric.

Proof: In Section 3 of [6] it is proven that $\Delta_p(F(p,q)) = \Delta_q(F(p,q))$ implies that (M, g) is locally symmetric and so the conclusion follows by Corollary 3.5. □

4 Balanced metrics on Hermitian symmetric spaces

Let M be a complex manifold equipped with a polarized Kähler metric g. This means that the Kähler form ω associated to g is integral. Let L be a holomorphic line bundle on M endowed with a Hermitian metric h such that its Ricci curvature $\mathrm{Ric}(h) = \omega$. Consider the space $H^0_h(L) \subset H^0(L)$ consisting of global holomorphic sections s of L which are bounded with respect to

$$\langle s, s\rangle_h = \|s\|^2_h = \int_M h(s(x), s(x))\frac{\omega^n}{n!}.$$

One can show that $H^0_h(L)$ is a separable complex Hilbert space. Let $x \in M$ and $q \in L\setminus\{0\}$ a fixed point of the fiber over x. If one evaluates $s \in H^0_h(L)$ at x, one gets a multiple $\delta_q(s)$ of q, i.e. $s(x) = \delta_q(s)q$. The map $\delta_q : H^0_h(L) \to \mathbb{C}$ is a continuous linear functional hence from Riesz's theorem, there exists a unique $e_q \in H^0_h(L)$ such that $\delta_q(s) = \langle s, e_q\rangle_h, \forall s \in H^0_h(L)$, i.e.

$$s(x) = \langle s, e_q\rangle_h. \tag{8}$$

It follows, by (8), that

$$e_{cq} = \overline{c}^{-1}e_q,\ \forall c \in \mathbb{C}^*.$$

The holomorphic section $e_q \in H^0_h(L)$ is called the *coherent state* relative to the point q. Thus, one can define a smooth function on M (depending on the Kähler metric g)

$$\epsilon_g(x) = h(q, q)\|e_q\|^2_h, \tag{9}$$

where $q \in L \setminus \{0\}$ is any point on the fiber of x. If $s_j,\ j = 0, \dots N\ N \leq \infty$ is a orthonormal basis for $(H^0_h(L), \langle \cdot, \cdot \rangle_h)$ then one can easily verify that $\epsilon_g(x) = \sum_{j=0}^{N} h(s_j(x), s_j(x))$. We are now in the position to recall the definition of balanced metrics (see [5] for the compact case and [2] for the noncompact case).

Definition 4.1 *A polarized Kähler metric g on M is balanced if the function $\epsilon_g(x)$ is constant and nonzero on M.*

Balanced metrics are important for the theory of quantization of Kähler manifolds (see e.g. [2]), stability of complex line bundle and Kähler-Einstein metrics ([16], [17], [18] and [19]) and existence and uniqueness of extremal and constant scalar curvature metrics ([5], [11] and [12]).

The existence and the uniqueness of balanced metrics has been study by Donaldson [5] in the compact case and when the group $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ (the group of biholomorphisms of M which lift to the line bundle L modulo the $\mathbb{C}^*$ action) is discrete. The author and C. Arezzo [2] extend Donaldoson's results by dropping the hypothesis on the group $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$. The main result in [2] is the following:

Theorem 4.2 *(Arezzo and Loi) Let g_0 and g_1 be two balanced Kähler metrics on a compact complex manifold M such that their corresponding Kähler forms are cohomologous. Then there exists $F \in Aut(M, L)$ such that $F^*(g_1) = g_0$.*

Observe that in the noncompact case the existence and uniqueness of balanced metrics is still an open and very interesting problem.
Here we are interested in balanced metrics on Hermitian locally symmetric spaces. Recently Engliš proves the following theorem (see Section 4 in [6]).

Theorem 4.3 *(Engliš) Let (M, g) be a complete Hermitian locally symmetric space. If g is balanced then (M, g) is globally symmetric.*

Observe that the Kähler metric $\tilde{g} = \pi^*(g)$ on $\tilde{M}$ in the previous theorem is projective-like by Proposition 3.2. Indeed this is the only assumption one needs to prove Theorem 4.3. This is expressed by the following:

Theorem 4.4 *Let (M, g) be a Kähler manifold such that the metric $\tilde{g} = \pi^*(g)$ on $\tilde{M}$ is projective-like. If g is balanced then M is simply connected.*

Proof: Since g is balanced there exists a Kähler immersion $i_g : (M, g) \rightarrow (\mathbb{C}P^N, G_{FS}), N \leq \infty$ built with the aid of an L^2-orthonormal basis of the Hilbert space $H^0_h(L)$ (see [2]). Thus the conclusion follows by applying Theorem 3.3 to the Kähler immersion i_g. $\Box$

Observe that if if $(\tilde{M}, \tilde{g})$ is a globally Hermitian symmetric space and we assume that the metric $\tilde{g}$ is polarized then $\tilde{g}$ is balanced. In fact all the metrics $m\tilde{g}$ are balanced for all non negative integer m and it is conjecturally true that if a complete space $\tilde{M}$ admits a balanced metric $\tilde{g}$ such that $m\tilde{g}$ is balanced for all m, then $(\tilde{M}, \tilde{g})$ is a globally Hermitian symmetric space (see Section 4 in [2]).

The following question naturally arises: *if g is a complete balanced metric on complex manifold M such that also $\tilde{g} = \pi^*(g)$ is balanced can we conclude that M is simply connected?*

The following theorem gives a partial answer to this question.

Theorem 4.5 *Let (M, g) be a Kähler manifold such that its universal cover $\tilde{M}$ is compact. Suppose that the metric $\tilde{g} = \pi^*(g)$ on $\tilde{M}$ is projective-like and balanced. Let g_1 be a metric on M such that its Kähler form is cohomologous to the Kähler form associated to g. If both g_1 and $\tilde{g}_1 = \pi^*(g_1)$ are balanced then M is simply connected.*

Proof: Since g_1 is balanced there exists a Kähler immersion $i_{g_1} : (M, g_1) \rightarrow (\mathbb{C}P^N, G_{FS})$ where $N+1$ is the complex dimension of $H^0(L)$ (see [2]). Since $\tilde{g}$ and $\tilde{g}_1$ are balanced and cohomologous it follows by Theorem 4.2 that there exists a biholomorphism F of $\tilde{M}$ such that $F^*(\tilde{g}_1) = \tilde{g}$. Thus also $\tilde{g}_1$ is projective-like and the conclusion follows by applying Theorem 3.3 to the Kähler immersion i_{g_1}. $\Box$

Corollary 4.6 *Let (M, g) be a Hermitian locally symmetric space such that $\tilde{M}$ is compact. Let g_1 be a metric on M such that its Kähler form is cohomologous to the Kähler form associated to g. If both g_1 and $\tilde{g}_1 = \pi^*(g_1)$ are balanced then (M, g) is globally symmetric.*

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