A Symplectic version of Nash C^1-Isometric Embedding Theorem

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Abstract

In this paper the Nash C^1-embedding theorem for Riemannian manifolds is extended to symplectic manifolds.

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1 Preliminaries and the problem setting

Our basic object in this paper is a symplectic manifold $W = (W, \Omega)$ i.e. a smooth $2n$-manifold W on which a closed non degenerate 2-form is given. We assume that on this W is defined an *adapted* Riemannnian metric G, where adapted means that there exists an almost complex structure $J : TW \to TW$ such that $\Omega(X, Y) = G(X, JY)$ and $G(JX, JY) = G(X, Y)$ for all pairs of vector fields X, Y on W. In other words, the pair (G, Ω) defines an *almost Kähler structure* on W (if the almost complex structure J is integrable then one calls it a Kähler structure). This implies the existence of a Hermitian metric H on W such that G equals the real part and Ω is the imaginary part of H, i.e.

$$H = G + \sqrt{-1}\Omega$$

(compare, e.g. [10]).

Next, let V be a smooth manifold endowed with a symplectic structure ω and a Riemannian metric g. We consider the following

Isometric embedding problem:

Let (W, Ω, G) and (V, ω, g) as above. We look for a map $f : V \to W$ such that

$$f^*(\Omega) = \omega \tag{1}$$

and

$$f^*(G) = g. \tag{2}$$

The equation (1) was studied in [7] where the complete solution in the form of an h-principle is found. Namely, one has the following *h-principle* for symplectic embeddings (see [5] and [7]).

Theorem 1.1 *(Gromov) Let (V, ω) and (W, Ω) be two symplectic manifolds. Suppose that an embedding $f_0 : V \to W$ satisfies the following two conditions:*

(i) the cohomology classes of the forms ω and $f_0^*(\Omega)$ coincide;

(ii) the differential $Df_0 : T(V) \to T(W)$ is homotopic through injective homomorphisms to a symplectic homomorphism, $\Delta : T(V) \to T(W)$.

Then in the following two cases f_0 is isotopic to a symplectic embedding $f : V \to W$.

(a) $\dim W \geq \dim V + 4$;

(b) $\dim W > \dim V$ and V is open.

In the case when symplectic *immersions* are considered this h-principle holds with the inequality (a) replaced by $\dim W \geq \dim V + 2$ and the inequality (b) replaced by $\dim W = \dim V$.

On the other hand, as far as we are concerned with the isometric embedding problem expressed by (1)+(2) above, it does not make difference to distinguish between immersions and embeddings as the dimension of the manifold W is big enough for both (compare condition (ii) in the main Theorem 1.4).

The equation (2) constitutes the classical isometric embedding problem of Riemannian geometry solved by Nash in his famous papers of 1954 and 1956. In the first paper (see [9]) he proved his celebrated C^1-isometric embedding theorem which is stated as follows:

Theorem 1.2 *If a compact Riemannian manifold* (V, g) *admits a smooth immersion (or embedding)* $f_0 : V \to \mathbb{R}^q$ *then there also exists an isometric* C^1*-immersion (respectively,* C^1*-embedding)* $f : (V, g) \to \mathbb{R}^q$*, provided* $\mathit{dim}\, V \leq q - 2$*. In particular* (V, g) *can be isometrically* C^1*-embedded into* $\mathbb{R}^q$ *for* $q = 2\, \mathit{dim}\, V$*.*

Theorem 1.2 was improved a year later (1955) by Kuiper [8] who modified Nash's tecnique to make it work also for $q = \dim V + 1$. (Clearly this is impossible, in general, for $\dim V = q$).

Now, let us briefly recall the basic structure of the Nash's C^1-isometric theory for the *Riemannian manifolds*. One starts with a general smooth (not at all isometric) immersion f_0 of a Riemannian manifold (V, g) into $\mathbb{R}^q$. If V is compact, then by an obvious scaling one can make such $f_0 : V \to \mathbb{R}^q$ *strictly short* (or *g-short*). This means that the Riemannian metric g_0 induced by f_0 from $\mathbb{R}^q$ is strictly smaller than g, i.e. the difference $g - g_0$ is positive definite on V. The key idea of the C^1-immersion theory of Nash and Kuiper (compare [8], [9] and see also Sect. 6 in this paper) is as follows. One "stretches" a given strictly short immersion f_0 to an isometric C^1-immersion f_1, i.e. such that the metric g_1 induced by f_1 equals g. Obviously, to perform the complete construction one needs to have at disposal the starting immersion $f_0 : V \to \mathbb{R}^q$. For this, one invokes the classical result of Whitney claiming that such an f_0 always exists for $q \geq 2 \dim V$. In fact, a generic C^∞-map $f : V \to \mathbb{R}^q$ is an immersion. There is also another possibility offered by the Smale–Hirsch immersion theory, which provides smooth immersions $V \to \mathbb{R}^q$ for a given $q \geq \dim V$ provided the manifold V satisfies the necessary topological restrictions. For example, every parallelizable (e.g. contractible) manifold V can be smoothly immersed in $\mathbb{R}^q$ with $q = \dim V + 1$ according to Hirsch's theorem (which was not known to Nash and Kuiper as it was proven in 1959).

Next, we also recall that a contact version of Nash's C^1-isometric embedding Theorem 1.2 was proven in [3], where a stretching construction in the same style as Nash was carried out for the case of contact manifolds.

The main result proven in [3] can be stated as follows:

Theorem 1.3 *Let* V *and* W *be odd-dimensional manifolds with contact structures* $S \subset T(V)$ *and* $T \subset T(W)$ *and suppose that* T *is represented as* $\ker \beta$ *for some* 1*-form* β *on* W*. Let* g *and* G *be positive definite quadratic forms defined on* S *and* T *respectively. Assume, moreover, that* G *is adapted with respect to* $\Omega = d\beta_{|T}$*.*

If $\dim W \geq max(2\dim V+3, 3\dim V-2)$ *then there exists a* C^1*-embedding* $f: V \to W$ *which is contact and isometric.*

(Here the term contact means that the differential $Df : T(V) \to T(W)$ sends S to T while isometric means that Df is an isometric homomorphism $(S, g) \to (T, G)$).

Now, we turn to the subject matter of the present paper where we want to study the C^1-solutions of (1)+(2) by applying a symplectic version of Nash C^1-method. That is, we consider two symplectic manifolds (V, ω) and (W, Ω) carrying, besides the symplectic structures, Riemannian metrics g on V and G on W respectively, where we assume the metric G to be adapted.

The main results proved in this paper are the following Theorem 1.4 and its Corollary 1.5.

Theorem 1.4 *Let* $f_0 : V \to W$ *be a symplectic embedding, i.e.* $f_0^*(\Omega) = \omega$.
Suppose that the following two conditions are satisfied:

(i) the map f_0 *is* g*-short, i.e.* f_0 *is such that* $g - f_0^*(G) > 0$*;*

(ii) $\dim W \geq max(2\dim V + 4, 3\dim V)$.

Then there exists a C^1*-embedding* $f : V \to W$*, arbitrarly* C^0*-close to* f_0*, which satisfies both* $f^*(G) = g$ *and* $f^*(\Omega) = \omega$.

We shall prove this theorem in Section 6. Infact, in order to prove it we first need to show that a similar result holds for a special class of maps, called *regular*. This is shown under the hypothesis that $\dim W \geq 2\dim V + 4$ (the proof is incorporated into the proof of the above Theorem 1.4). The notion of *regularity* for maps $V \to W$ is crucial to adapt to the symplectic framework the Nash C^1-construction. Since it is a rather technical notion we discuss it separately in Section 4. It is worth to mention that in [3] (see Sect. 2) a similar notion of regularity was used to adjust to the contact framework the Nash's C^1-extension scheme elaborated in [9].

As a corollary of Theorem 1.4 we get the following

Corollary 1.5 *Let* $f_0 : V \to W$ *be an embedding which satisfies the following two conditions:*

(i) the cohomology classes of the forms ω *and* $f_0^*(\Omega)$ *coincide;*

(ii) the differential $Df_0 : T(V) \to T(W)$ *is homotopic through injective homomorphisms to a symplectic homomorphism.*

Then, for all sufficiently large g, there exists a C^1-embedding $f : V \to W$ satisfying (1) and (2) provided $dim W \geq max(2\, dim V + 4, 3\, dim V)$, where "large" refers to the natural partial order of quadratic forms on V.

Proof: According to Theorem 1.1 there exists a map $f_0 : V \to W$ satisfying equation (1) and if g is large, then, by definition of the partial order for metrics the difference $g - f_0^*(G)$ is positive definite and the Corollary follows by applying Theorem 1.4. $\square$

2 Examples and Remarks

Example 2.1 Let $W = (W, \Omega, G)$ be the $2N$-dimensional Euclidean space $\mathbb{R}^{2N}$ endowed with the flat metric $G_{can} = \sum_{j=1}^{N} dx_j^2 + dy_j^2$ and the standard symplectic form $\Omega_{can} = \sum_{j=1}^{N} dx_j \wedge dy_j$ and assume (V, ω) to be a $2n$-dimensional contractible symplectic manifold. Then, it is not difficult to see that any embedding $V \to \mathbb{R}^{2N}$ satisfies the conditions (i) and (ii) in Theorem 1.1. Moreover, by Corollary 1.5, we see that for $N \geq (2n+2, 3n)$, a solution to the isometric embedding problem expressed by (1) and (2) is given by a C^1-map $f : V \to W$ provided g is large.

Remark 2.2 The "largeness" condition referred to g in Corollary 1.5 is easy to understand in the particular case when one considers the standard embedding $f_0 : V \to W$ where, $(V, \omega) = (\mathbb{R}^{2n}, \sum_{j=1}^{n} dx_j \wedge dy_j)$ and $(W, \Omega) = (\mathbb{R}^{2N}, \sum_{j=1}^{N} dx_j \wedge dy_j)$. Here, in order to have a "large enough" metric g it is sufficient to assume $g > g_0$ where g_0 is the Euclidean metric.

Example 2.3 Let $W = \mathbb{C}P^N$ the the N-dimensional complex projective space endowed with the Fubini–Study metric denoted by G_{FS} and the corresponding Kähler form Ω_{FS}. Assume that the symplectic manifold V is compact and the form ω is integral, namely its cohomology class $[\omega_g]_{dR}$ in the de Rham group, is in the image of the natural map $H^2(M, \mathbb{Z}) \hookrightarrow H^2(M, \mathbb{C})$. Then, the existence of a smooth symplectic embedding $f_0 : (V, \omega) \to (\mathbb{C}P^N, \Omega_{FS})$ follows, for N suffciently large, by a result shown in [11] (see also [7] p. 335). Observe also that Theorem 1.4 implies the existence of a C^1-embedding $f : V \to \mathbb{C}P^N$ such that $f^*(\Omega_{FS}) = \omega$ and $f^*(G_{FS}) = g$ for $N \geq \max(2 \dim V + 4, 3 \dim V)$, provided the map f_0 is large with respect to the metric g fixed on V.

Remark 2.4 Observe that, in the case when also the domain manifold (V, g, ω) is almost Kähler then it cannot exist a smooth embedding $f : V \to W$ satisfying condition (i) in Theorem 1.4. An explanation of this, on the linear algebra level, is as follows. Let F be a vector space endowed with a scalar product G and a symplectic form Ω related by the equality $\Omega(X, Y) = G(X, JY), \forall X, Y \in F$ for an almost complex structure J on F. Then it cannot exist a symplectic linear map $f : F \to F$ satisfying $G - f^*(G) > 0$. Indeed, the fact that f is symplectic implies that its eigenvalues are in pair $\lambda, \frac{1}{\lambda}$ while the condition $G - f^*(G) > 0$ would imply that the eigenvalues of f are all strictly less than 1.

Remark 2.5 In the case (W, Ω, G) and (V, ω, g) are Kähler manifolds with integrable almost complex structures J and j respectively, then for a C^1-map $f : V \to W$ satisfying equations (1) and (2) above we have: $Df \circ j = J \circ Df$ and hence the map f is necessarily holomorphic.

Remark 2.6 The above remarks suggest that the condition (i) in Theorem 1.4 is necessary both for the compact and non-compact case. To see this take, for example, the unit disk $D \subset \mathbb{C}$ endowed with the hyperbolic metric $g_{hyp} = \frac{dx^2+dy^2}{(1-x^2-y^2)^2}$ and the corresponding Kähler form $\omega_{hyp} = \frac{dx\wedge dy}{(1-x^2-y^2)^2}$. Thus, for N sufficiently large (compare Example 2.1) there exists a symplectic embedding $f_0 : (D, \omega_{hyp}) \to (\mathbb{R}^{2N}, \Omega_{can})$. Hence the pairs $(V, \omega) = (D, \omega_{hyp})$ and $(W, \Omega) = (\mathbb{C}^N, \Omega_{can})$ together with the map f_0 satisfy all the hypothesis of Theorem 1.4. Furthermore, it cannot exist a C^1-map $f : D \to \mathbb{C}^N$ such that $f^*(\Omega_{can}) = \omega_{hyp}$ and $f^*(G_{can}) = g_{hyp}$. Indeed, by Remark 2.5, such a map would be holomorphic and according to a theorem due to Calabi [2], it cannot exist a holomorphic and isometric map from (D, g_{hyp}) to $(\mathbb{C}^N, G_{can})$ for any natural number N.

Now, let us briefly describe an example in the compact case. Take $V = T^n = \mathbb{R}^{2n}/\Lambda$, the n complex dimensional torus, where Λ is the standard $2n$-lattice on $\mathbb{R}^{2n}$. We equipe T^n with the flat metric $G = G_{can}$ and the Kähler form $\Omega = \Omega_{can}$ coming from $\mathbb{R}^{2n}$ which we still denote by G_{can} and Ω_{can}. Since the form Ω_{can} is integral there exists, for N sufficiently large, a symplectic embedding $f_0 : (T^n, \Omega_{can}) \to (\mathbb{C}P^N, \Omega_{FS})$, (compare Example 2.3). On the other hand, again by a result of Calabi [2] it cannot exists a holomorphic and isometric map from (T^n, G_{can}) to $(\mathbb{C}P^N, G_{FS})$ for any natural number N.

3 Normal symplectic extension

The proof of Theorem 1.4 is close in spirit to the argument used by Nash in [9]. There, one basic ingredient is the construction of a *normal extension* (see Sect.6) of a given (Riemannian) immersion which we need here for the case of symplectic immersions. Here we start with a given map $f_0 : V = (V, g, \omega) \to W = (W, G, \Omega)$ and a ball $B \subset V$ (i.e. a contractible compact set in V) and we want to construct a map $\tilde{f}_0 : B \times \mathbb{R}^2 \to W$, which we call *a normal symplectic extension of* f_0 *at* $B \subset V$, satisfying the following four properties:

(i) $\tilde{f}_0$ restricted to $B \times \{(0,0)\}$ equals f_0 restricted to B;

(ii) the two vector fields $\partial_1 = \frac{\partial \tilde{f}_0}{\partial x}|_{B\times\{(0,0)\}}$ and $\partial_2 = \frac{\partial \tilde{f}_0}{\partial y}|_{B\times\{(0,0)\}}$ are G-orthogonal to B, mutually G-orthogonal and moreover their norm with respect to G is one (here x and y are the natural coordinates in $\mathbb{R}^2$);

(iii) the above vector fields ∂_1 and ∂_2 are Ω-orthogonal to B and mutually Ω-orthogonal;

(iv) the induced form $\tilde{f}_0^*(\Omega)$ on $B \times \mathbb{R}^2$ equals $p^*(\omega)$, where $p : B \times \mathbb{R}^2 \to B$ is the standard projection (and where, recall, $\omega = f_0^*(\Omega)$).

Remark 3.1 Observe that the two conditions (ii) and (iii) together can be equivalently expressed in terms of the Hermitian metric $H = G + \sqrt{-1}\Omega$ as follows: ∂_1 and ∂_2 are H-orthogonal to B, mutually H-orthogonal and of unit norm with respect to G.

One can not construct a normal symplectic extension for an arbitrary symplectic embedding $f_0 : V \to W$. However, this can be done if we impose some additional *regularity* assumptions on our map f_0. Infact, we shall see that for a map $f_0 : V \to W$ which is regular in the sense of definition 4.3.1 below the following holds true (see Sect. 5 for the proof):

Lemma 3.2 (normal symplectic extension) *Let* $f_0 : V \to W$, *be a regular symplectic embedding and let* $B \subset V$ *be a ball in* V. *If* $dim W \geq 2\, dim V + 4$ *then there exists a normal symplectic extension of* f_0 *at* B.

Remark 3.3 Let $\tilde{f}_0 : B \times \mathbb{R}^2 \to W$ be a normal symplectic extension of a symplectic map $f_0 : V \to W$ at B and let $s : V \to V \times \mathbb{R}^2$ be an arbitrary section with $\mathrm{supp}(s) \subset B$. Define a smooth map $f : V \to W$ by posing: $f_{|B} = \tilde{f}_0 \circ s_{|B}$ and $f = f_0$ outside B. Now, this f is well-defined by property (i) in the above definition. Moreover, it follows by property (iii) that f is symplectic, i.e. $f^*(\Omega) = \omega$.

4 Regular maps

As it was already pointed out in Section 1, in order to construct a normal symplectic extension we need to start with a *regular* map $f_0 : V \to W$. The regularity conditions used in this paper are expressed by the definitions 4.1.1 and 4.3.1 below.

4.1 Regular spaces

Let $F = (F, G, \Omega)$ be a linear space endowed with a symmetric quadratic form G and an antisymmetric form Ω.

Definition 4.1.1 *A subspace $E \subset F = (F, G, \Omega)$ is called (G, Ω)- regular if the intersection of the orthogonal complements $E^G \cap E^\Omega$ has the correct codimension in F, namely $2\, dim E$.*

Here E^G (resp. E^Ω) denotes the G-orthogonal complement (resp. Ω-symplectic complement) of E. We shall call the intersection $E^G \cap E^\Omega$ the (G, Ω)-orthogonal complement of E.

Another definition of (G, Ω)-regularity, which is useful for a clearer understanding of this concept, is as follows.

Definition 4.1.2 *A subspace $E \subset F$ is (G, Ω)-regular if the following two equivalent conditions are satisfied:*

(i) the linear system in $x \in F$,

$$G(x, e_i) = 0, \quad \Omega(x, e_i) = 0, i = 1, \ldots, m \tag{3}$$

where the vectors e_i form a basis in E, is non singular;

(ii) the $2m$ covectors $x \to G(x, e_i)$ and $x \to \Omega(x, e_i), i = 1, \ldots, m$ are linearly independent.

Next, we list a few observations which are straightforward after the previous definitions 4.1.1 and 4.1.2.

(a) The space F_1 of solutions of (3) clearly equals the (G,Ω)-orthogonal complement $E^G \cap E^\Omega$;

(b) If the quadratic form G is non-singular (e.g. is positive definite) then $\operatorname{codim} E^G = \dim E$ and the same is true for non-singular Ω. However our (G,Ω)-regularity condition is stronger than the joint regularity of G and Ω;

(c) If a subspace $E \subset F$ is regular, then every subspace $E' \subset E$ is regular.

Observe that if the space (F, G, Ω) admits a complex structure $J : F \to F$ and a Hermitian form H compatible with J such that its real part equals G and its imaginary equals Ω, then in our framework this is equivalent to the metric G being adapted (compare with the preliminaries in Section 1). It is also clear that J, if exists, is uniquely determined by G and Ω via the condition $\Omega(X, Y) = G(X, JY)$ as the forms Ω and G are non-singular (here H is the usual Hermitian metric of the complex vector space (F, J)).

There is one more useful characterization of regularity, related to the notion of *totally real subspaces.*

Recall, that a $\mathbb{R}$-linear subspace of $E \subset \mathbb{C}^q$ is said *totally real* if one of the following four equivalent conditions is satisfied:

(i) $E \cap \sqrt{-1}E = \{0\}$;

(ii) the real dimension of $\operatorname{Span}_{\mathbb{R}}(E, \sqrt{-1}E)$ equals $2 \dim E$;

(iii) if $e_1, \ldots, e_m$ is a real basis in E then the vectors $e_1, \ldots, e_m$ are $\mathbb{C}$-independent in $\mathbb{C}^q$;

(iv) if $e_1, \ldots, e_m$ is a real basis on E then the vectors $e_1, \ldots, e_m, \sqrt{-1}e_1, \ldots, \sqrt{-1}e_m$ are $\mathbb{R}$-independent in E.

From now on, we assume that $F = (F, G, \Omega)$ is a Hermitian space, with the implied complex structure denoted by J and with corresponding Hermitian form $H = G + \sqrt{-1}\Omega$. Such F will be obviously isomorphic to some $\mathbb{C}^q$ with $q = \frac{1}{2} \dim_{\mathbb{R}} F$ and with $J = \sqrt{-1}$. We also observe that the the H-orthogonality and the orthogonality with respect to G "plus" Ω expressed by (3) are equivalent. Hence, we may state the following:

Lemma 4.1.3 *Let $E \subset (F, G, \Omega)$ be a linear subspace.*

Then the space F_1 of solutions of system (3) equals the Hermitian orthogonal complement E^H. In particular F_1 is complex linear, i.e. $JF_1 = F_1$.

Corollary 4.1.4 *A subspace $E \subset F$ is regular iff it is totally real.*

We end our discussion on the algebraic meaning of regularity by adding two more properties which are satisfied by regular subspaces of a Hermitian space. These are expressed by the following Lemma 4.1.5 and its Corollary.

Lemma 4.1.5 *Let $E \subset F = (F, G, \Omega)$ be a regular subspace and let $\tau_1 \in F$ be a non-zero vector H-orthogonal to E. Then the linear space $Span(E, \tau_1)$ is regular.*

Corollary 4.1.6 *Let τ_1 and τ_2 be two independent vectors which are H-orthogonal to $E \subset F$ and also mutually H- orthogonal. Then the space $Span(E, \tau_1, \tau_2)$ is regular provided the subspace E was regular.*

Remark 4.1.7 Notice that for the existence of two such vectors τ_1 and τ_2 as in the Corollary above one needs to suppose $\dim E^H = \dim(E^G \cap E^\Omega) \geq 4$.

4.2 On the space of non-regular homomorphisms

Let (F, G, Ω) be a Hermitian vector space of complex dimension n and let $E \subset F$ be a vector subspace of F of real dimension m. We denote by Σ the subset of $\mathrm{Hom}(E, F)$ consisting of *non-regular* homomorphisms $E \to F$, namely of those homomorphisms which are either non-injective or have non-regular image. We have the following:

Lemma 4.2.1 *The subset $\Sigma \subset Hom(E, F)$ is a stratified complex manifold of real codimension $c = 2(n - m + 1)$.*

Proof: Let $(e_1, \dots, e_m)$ be the canonical basis in $\mathbb{R}^m$. By Corollary 4.1.4 and by the definition of totally real subspace we see that Σ can be identified with the subset of $\mathrm{Hom}(\mathbb{R}^m, \mathbb{C}^n)$ which take $(e_1, \dots, e_m)$ to m-linearly $\mathbb{C}$-dependent vectors in $\mathbb{C}^n$. Then $\Sigma = \cup_{i=0}^{m-1} \Sigma_i$, where each $\Sigma_i \subset \mathrm{Hom}(\mathbb{R}^m, \mathbb{C}^n)$ consists of $\mathbb{R}$-linear maps which take $(e_1, \dots, e_i)$ to i-linearly $\mathbb{C}$-independent vectors in $\mathbb{C}^n$ and $(e_1, \dots, e_{i+1})$ to $i+1$-linearly $\mathbb{C}$-dependent vectors in $\mathbb{C}^n$. It is easy to see that Σ_i is a complex submanifold of $M(n, m)$ (the space of $n \times m$ matrices with complex entries) and its complex codimension is equal to $(n - i)(m - i)$ (cf., e.g., [1]). Then $\Sigma \subset M(n, m)$ is a stratified complex manifold and its complex codimension (= the complex codimension of Σ_{m-1}) is given by $(n - (m - 1))(m - (m - 1)) = n - m + 1$. $\square$

Remark 4.2.2 Our conventions concerning the dimension and the codimension of stratified sets are those usually accepted (see, e.g. 1.3.2 in [7]).

4.3 Regular maps

Coming to a non-linear situation, let us now extend to the case of maps between smooth manifolds the definition of regularity given in subsection 4.1 at the linear algebra level. Let (V, ω) be a symplectic manifold and let $W = (W, G, \Omega)$ be an almost Kähler manifold as in Section 1.

Definition 4.3.1 *A symplectic immersion $f : V \to W$ is called (G, Ω)-regular if, for all $v \in V$, the (image) subspace $Df_v(T_v(V)) \subset T_w(W)$ of $T_v(V)$, $w = f(v)$ is (G_w, Ω_w)-regular.*

Proposition 4.3.2 *Assume that $dim W \geq 3\, dim V$. Then generic maps $V \to W$ are $(G, \Omega) - regular$.*

Proof: The standard idea in the proof of this proposition is to interpret non-(G, Ω)-regularity as a singularity in the space $\mathrm{J}^1(V, W)$ of 1-jets of our maps $V \to W$, so that one can use *Thom's transversality theorem.* Recall that $\mathrm{J}^1(V, W)$ forms a bundle over $V \times W$ whose fibers, usually denoted by $J^1_{v,w}$, consist of linear maps $L : T_v(V) \to T_w(W)$. The jet J^1_f of a given map $f : V \to W$ at a point v, is given by the differential in v of f, i.e. $J^1_f(v) = df_v : T_v(V) \to T_{f(v)}W \in J^1_{v,f(v)}$. Denote by $\Sigma_{v,w}$ the space $\Sigma_{v,w} \subset J^1_{v,w}$ consisting of the jets of non-(G, Ω)-regular maps. By Lemma 4.2.1, $\Sigma_{v,w} \subset J^1_{v,w}$ is a stratified manifold of real codimension $\dim W - 2\dim V + 2$.

The set $\Sigma = \cup_{(v,w) \in V \times W} \Sigma_{v,w} \subset \mathrm{J}^1(V, W)$ fibers over $V \times W$ and hence it is a stratified manifold in $\mathrm{J}^1(V, W)$ of codimension $\dim W - 2\dim V + 2$. By the very definition of Σ it follows that a map $f : V \to W$ is non-(G, Ω)-regular iff $J^1_f(V) \subset \mathrm{J}^1(V, W)$ does not meet Σ. By (the special case) of Thom's transversality theorem (see , e.g. [7] Corollary D', p. 33) generic maps do have the property $J^1_f(V) \cap \Sigma = \emptyset$ iff $\dim W - 2\dim V + 2 \geq \dim V + 1$ which is equivalent to $\dim W \geq 3\dim V$ being V and W even dimensional manifolds. $\square$

Corollary 4.3.3 *If $dim W \geq 3\, dim V$ every symplectic embedding $f_0 : (V, \omega) \to (W, \Omega)$ can be C^∞-approximated by a regular symplectic embedding $f_0^{reg} : (V, \omega) \to (W, \Omega)$.*

Proof: By Proposition 4.3.2 f_0 admits an approximation by a regular map f_1 and since we work in the C^∞-fine topology we may assume that there exists an homotopy $\varphi_t : V \to W$ between $\varphi_0 = f_0$ and $\varphi_1 = f_1$ such that all φ_t are uniformly close to f_0 in the C^∞-fine topology.

Since being symplectic is an open condition, the forms $\omega_t = \varphi_t^*(\Omega)$ are all symplectic forms on V. Moreover, the forms ω_t are all cohomologous and hence we can suppose that for all $t \in [0,1]$ there exists a 1-form α_t such that

$$\omega_t = \omega + d\alpha_t.$$

If V is compact, by Moser's theorem (see, e.g. [10]) there exists a diffeomorphism $\delta : V \to V$, C^∞-close to the identity Id_V such that $\delta^*(\omega_1) = \omega$. Then $f_0^{reg} = f_1 \circ \delta$ is an embedding which is regular, C^∞-close to f_0, and satisfying $f_0^{reg*}(\Omega) = \omega$.

In the non compact case, one writes $V = \cup_{j=1}^{\infty} K_j$ as the union of exhausting compact sets K_j ($K_j \subset \mathrm{Int}\, K_{j+1}$), and considers for fixed j, the family $\beta_{j,t}$ of 1-forms whose value at $x \in V$ is defined by

$$\beta_{j,t}(x) = \alpha_{t\lambda_j(x)+(1-t)\lambda_{j-1}(x)}(x),$$

where λ_j is a smooth function on V such that $\lambda_j(x) = 1$ for all $x \in K_j$ and $\lambda_j(x) = 0$ outside K_{j+1}. Then the result follows by applying the relative Moser's theorem for symplectic forms (as stated in [7], p. 337) to the family of 2-forms $\omega_{j,t} = \omega + d\beta_{j,t}$. Infact, one gets a diffeomorphism $\delta_j : V \to V$, C^∞-close to Id_V, such that $\delta_j^*(\omega_{j,1}) = \omega_{j,0}$ and $\delta_j = \mathrm{Id}_V$ outside $K_j \setminus \mathrm{Int}\, K_{j-1}$. Finally, one defines a sequence of diffeomorphisms $F_j = \delta_j \circ \delta_{j-1} \circ \cdots \delta_1$, which C^∞-fine converges to a diffeomorphism $\delta : V \to V$ C^∞-close to Id_V, and satisfies $\delta^*(\omega_1) = \omega$. $\square$

5 Construction of the normal symplectic extension

Let's begin by recalling the definition of a Hamiltonian vector field on a symplectic manifold (W, Ω). The form Ω being non-degenerate, one can define an isomorphism between the space of vector fields and the space of 1-forms on W, by associating to a given vector field X defined on W the 1-form $i_X\Omega$, where $i_X\Omega(Y) = \Omega(X, Y)$ for all vector fields Y on W. Then, for a given a smooth function f on W we define the *Hamiltonian vector field* X_f associated to f by

$$i_{X_f}\Omega = df.$$

Observe that X_f is uniquely determined by f up to the multiplication by a constant (W being connected). Furthermore it is not hard to verify that, if we denote by $\varphi_t, t \in (-\epsilon, \epsilon)$, the 1-parameter group of diffeomorphisms generated by X_f, then $\varphi_t^*(\Omega) = \Omega$ for all $t \in (-\epsilon, \epsilon)$. (see [10] for details).

We are now in the position to prove the following:

Lemma 5.1 *Let $B \subset W$ be a ball in W and let ∂_0 be a vector field in W along B which is normal to B with respect to the symplectic form Ω. Then ∂_0 extends to a Hamiltonian vector field ∂ on W.*

Proof: Consider the 1-form γ on W along B given by $\gamma = i_{\partial_0}\Omega$. Choose a smooth function H on W such that $H_{|B} = 0$ and $dH_{|T(W)_{|B}} = \gamma$. Then the vector field ∂ on W defined by $i_\partial \Omega = dH$ is the desired Hamiltonian vector field. $\square$

Proof of Lemma 3.2: Without loss of generality we can suppose that $B \subset W$ and f_0 is the inclusion of B in W.

Let $TB^\Omega = \cup_{b\in B}(T_bB)^\Omega$ and $TB^G = \cup_{b\in B}(T_bB)^G$ where $(T_bB)^\Omega$ (resp. $(T_bB)^G$) is the Ω-orthogonal complement of T_bB (resp. G-orthogonal complement). Since $B \subset W$ is a regular submanifold in W, $\Theta = TB^\Omega \cap TB^G$ is a vector bundle on the base B which is trivial as B is contractible. Moreover, by the very definition of regularity (cf. definition 4.1.1) we have $rank\Theta \geq \dim W - 2\dim V \geq 4$. Thus there exist two vectors fields ∂_1 and ∂_2 on W along B which are H-orthogonal to B, mutually H-orthogonal and unitary with respect to G (cf. Remark 4.1.7)

By Lemma 5.1 we can extend the field ∂_1 to a Hamiltonian vector field $\tilde{\partial}_1$ on W which is integrable in a neighborhood of B (as B is compact). Now, consider the one-parameter group of diffeomorphisms generated by $\tilde{\partial}_1$ which we denote by $\varphi_{1,x}$, for $x \in [-\epsilon, \epsilon]$, and take the $\tilde{\partial}_1$-orbit $\varphi_{1,x}(B)$ of B. Let $\bar{\partial}_2$ be the vector field on W along $\varphi_{1,x}(B)$ obtained by transporting the field ∂_2 by the one-parameter group $\varphi_{1,x}$, i.e. $(\bar{\partial}_2)_{\varphi_{1,x}(b)} = (D\varphi_{1,x})_b(\partial_2)_b$.

Since $\varphi_{1,x}$ is symplectic, for every x the vector field $\bar{\partial}_2$ is Ω- orthogonal to $\varphi_{1,x}(B)$. Again, Lemma 5.1 allows us to extend $\bar{\partial}_2$ to a Hamiltonian vector field $\tilde{\partial}_2$ on W. Let $\varphi_{2,y}$, for $y \in [-\epsilon, \epsilon]$, be the corresponding one-parameter group of diffeomorphisms.

Then the map

$$\tilde{f}_0 : B \times (-\epsilon, \epsilon)^2 \cong B \times \mathbb{R}^2 \to W : (b, x, y) \mapsto \varphi_{2,y}(\varphi_{1,x}(b)).$$

is a normal symplectic extension of f_0 at B. Infact the properties (i),(ii) and (iii) of the definition of normal symplectic extension are easily verified, while property (iv) follows by the fact that $\varphi_{1,x}$ and $\varphi_{2,y}$ are symplectomorphisms. $\square$

6 Proof of Theorem 1.4

We first briefly discuss Nash's proof of his C^1-isometric immersion Theorem 1.2 since the basic ingredients of our proof of Theorem 1.4 are similar to those used in [9]. For a detailed treatment of Nash's "stretching and twisting" tecniques we refer the reader to [9], [3] and also to Sect. 3.1.1 in [7].

In proving his theorem Nash starts with a C^∞-immersion $f_0 : V \to \mathbb{R}^q$ which is strictly short with respect to the Riemannian metric g on V. We repeat from Sect. 1 that, for a C^1-map $f_0 : V \to \mathbb{R}^q$, strictly short means that the difference $g - f_0^*(G) > 0$, where G is the Euclidean metric on $\mathbb{R}^q$. Notice, that short immersions $V \to \mathbb{R}^q$ exist whenever V admits an immersion in $\mathbb{R}^q$ ([9], [7]). Starting with a strictly short map f_0, Nash's process consists in performing a sequence of "stretching " and "twisting " operations to finally arrive at the desired isometric immersion. An important tool for the Nash-type construction is the following Lemma that we quote here in a stronger version than the original one used by Nash (see [7], Sect. 3.1.1).

Lemma 6.1 *(Nash decomposition) Let $V = \cup_j U_j$ be an arbitrary locally finite cover of a manifold V by open subsets $U_j \subset V$. Then every C^∞-metric g on V decomposes as the sum $g = \sum_j \delta_j$ where each δ_i is induced by a smooth function $\varphi_i : V \to \mathbb{R}$ from dx^2 in $\mathbb{R}$, i.e. $\delta_i = (d\varphi_i)^2$ and where the support of each φ_i is contained in some subset U_j for $j = j(i)$.*

The key idea for the proof of Theorem 1.4 is to construct a sequence of strictly short immersions $f_j : (V, g) \to (\mathbb{R}^q, G)$ satisfying the following conditions:

$$\| g - f_j^*(G) \|_0 < \frac{2}{3} \| g - f_{j-1}^*(G) \|_0 \tag{4}$$

$$\| f_j - f_{j-1} \|_1 < c(n) \| g - f_{j-1}^*(G) \|_0^{\frac{1}{2}}, \tag{5}$$

where $\| \cdot \|_r$ denotes the norm in the C^r-fine topology and $c(n)$ is a constant depending on $\dim V$. It follows that the sequence $\{f_j\}$ is a Cauchy sequence in the fine C^1-topology and hence it converges to some C^1-function $f : V \to \mathbb{R}^q$ such that $f^*(G) = g$. Therefore the problem essentially reduces to prove the result stated in the following:

Basic Lemma 6.2 *Assume $q \geq dim V + 2$ and let $f_0 : V \to \mathbb{R}^q$ be a strictly short immersion. Then there exists a C^∞ strictly short immersion $f_1 : V \to \mathbb{R}^q$ such that:*

$$\| g - f_1^*(G) \|_0 < \frac{2}{3} \| g - f_0^*(G) \|_0 \tag{6}$$

and

$$\| f_1 - f_0 \|_1 < c(n) \| g - f_0^*(G) \|_0^{\frac{1}{2}} . \tag{7}$$

Sketch of the proof of Lemma 6.2: (see [3] and [9]) The construction of f_1 uses a sequence of successive corrections ("twisting perturbations") applied to the initial map f_0. Let $V = \cup_j B_j$ be an arbitrary locally finite covering of V by balls $B_j \subset V$. Since f_0 is strictly short one can set $g_1 = g - f_0^*(G)$ and decompose the metric g_1 as the sum $g_1 = \sum_j \delta_j$ where each δ_j is as in Lemma 6.1. The basic step of the proof is to fix any $\epsilon > 0$ and perturb the map f_0 inside its ϵ-neighbourhood (with respect to the C^0-topology) to get a C^∞-immersion $f_\epsilon : V \to W$ such that $f_\epsilon^*(G) = f_0^*(G) + (d\varphi_j)^2 + O(\epsilon)$.

Namely, we start by fixing $\epsilon_1 > 0$ and consider a "perturbation of the map f_0 inside B_1" that can be described as follows. We extend the map f_0 to $\tilde{f}_0 : B_1 \times \mathbb{R}^2 \to W$ in such a way that $\tilde{f}_0^*(G) = f_0^*(G) + \sum dx^2 + \sum dy^2$ on $T\mathbb{R}^q_{|V}$, where (x, y) denote the global coordinates in $\mathbb{R}^2$. To get the extension $\tilde{f}_0$ (which is called *normal extension* (compare Sect. 3)) we need to choose two vector fields ∂_1 and ∂_2 on B_1 which are mutually orthogonal and also normal to $T(V)$ (as $q \geq \dim V + 2$ and $B^1 \subset V$ is contractible, such vector fields always exist). Next, we take the smooth map $f_{\epsilon_1} : V \to \mathbb{R}^q$ defined as

$$f_{\epsilon_1} = \tilde{f}_0 \circ s_{1|_{B_1}} \tag{8}$$

and set $f_{\epsilon_1} = f_0$ outside B_1. Here $s_1 : V \to V \times \mathbb{R}^2$ is given by the composition of maps indicated below:

$$s_1 : V \xrightarrow{\Gamma_\phi} V \times \mathbb{R} \xrightarrow{\alpha_{\epsilon_1}} V \times \mathbb{R}^2, \tag{9}$$

where $\Gamma_{\phi_1} : V \to V \times \mathbb{R}$ is the graph of ϕ_1 and $\alpha_{\epsilon_1}(v, t) = (v, \frac{\epsilon_1}{2\pi} \sin \frac{t}{\epsilon_1}, \frac{\epsilon_1}{2\pi} \cos \frac{t}{\epsilon_1})$. We repeat this first step and apply the same construction to the map f_{ϵ_1}. Namely, we fix $\epsilon_2 > 0$ and perturb the map f_{ϵ_1} inside B_2 using ϕ_2 in place of ϕ_1 to obtain a new immersion $f_{\epsilon_2} : V \to W$ and so on in succession . The desired immersion $f_1 : V \to \mathbb{R}^q$ is obtained by constructing a sequence of maps $f_{\epsilon_1}, f_{\epsilon_2}, \ldots$. Since U_j is a locally finite family, each point $v \in V$ has a neighbourhood U_v which intersects only finitely many supp(φ_j)'s. Therefore the sequence $\{f_{\epsilon_j}\}$ is eventually constant on U_v and the $\lim f_{\epsilon_j}$ exists uniformly on it. Moreover one can choose ϵ_j at the j-step in such a way that with all the ϵ_j's small enough this limit can be made C^0-close to f_0. $\square$

To conclude the proof of Theorem 1.4 we need to modify Lemma 6.2 to our context as follows. The initial map $f_0 : (V, \omega, g) \to (W, \Omega, G)$ is assumed to be symplectic and such that:

(a) f_0 is strictly short (i.e. satisfies (i) in Theorem 1.4;

(b) f_0 is (G, Ω)-regular (see Sect. 4);

Moreover the dimension of the range manifold W abides the condition: $\dim W \geq 2 \dim V + 4$.

We decompose the metric $g_1 = g - f_0^*(G)$ as the sum $g_1 = \sum_j \delta_j$ with the δ_j's as in Lemma 6.1.

Next, we fix any $\epsilon_1 > 0$ and consider a "symplectic perturbation of the map f_0 inside B_1", namely we take the smooth map $f_{\epsilon_1} : V \to W$ defined as $f_{\epsilon_1} = \tilde{f}_0 \circ s_{1|_{B_1}}$ and set $f_{\epsilon_1} = f_0$ outside B_1, where the map s_1 is defined as in formula (9) and where the map $\tilde{f}_0 : B_1 \times \mathbb{R}^2 \to W$ is the normal symplectic extension of $f_0 : V \to W$ at B_1 whose existence, with the above assumptions on the initial f_0, follows by Lemma 3.2. In view of Remark 3.3 the map f_{ϵ_1} is well-defined and symplectic. Observe also that, as a consequence of Corollary 4.1.6, the map $\tilde{f}_0$ is (G, Ω)-regular near $B_1 \times \{(0,0)\}$. Hence (see (c) in Sect. 4.1) the map f_{ϵ_1} is (G, Ω)-regular for small ϵ_1. This enables us to apply the Nash-type construction to our initial symplectic embedding f_0, thus obtaining the map $f_1 : V \to W$ which is strictly short, (G, Ω)-regular and symplectic. By iterating this process one then gets a sequence of symplectic embeddings $f_j : V \to W$ which C^1-converges to a C^1-embedding $f : V \to W$ satisfying $f^*(G) = g$ and $f^*(\Omega) = \omega$. This concludes the proof of Theorem 1.4 when the initial map f_0 is (G, Ω)-regular. Assume that the map f_0 is not (G, Ω)-regular and let $\dim W \geq \max(2 \dim V + 4, 3 \dim V)$. We invoke Corollary 4.3.3 so the we can take a symplectic (G, Ω)-regular map $f_0^{reg} : (V, \omega) \to (W, \Omega)$ C^∞-close to f_0 which is strictly short (shortness being an open condition). By applying again the above procedure we get a map $f : V \to W$ satisfying equations $f^*(G) = g$ and $f^*(\Omega) = \omega$. This map f is arbitrarily C^0-close to f_0^{reg} and so it is also arbitrarily C^0-close to f_0 as desired.

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