Bergman and balanced metrics on complex manifolds

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Abstract

In this paper we find sufficient conditions for a Bergman Einstein metric on a complex manifold to be balanced in terms of its Bochner's coordinates.

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1 Introduction

Let M be a complex manifold and let L be its canonical bundle. Under suitable assumptions on L, one can define different Kähler metrics on M. One of these is the well-known Bergman metric on M, denoted here by g_B. Other canonical Kähler metrics on M, which depend on the choice of an Hermitian metric h on L and can be defined for any holomorphic line bundle (not necessarily the canonical one), are the *balanced metrics* introduced by Donaldson [7] in the compact case and by Arezzo and Loi [2] in the noncompact case.

Balanced metrics are important for the theories of quantization of Kähler manifolds (see e.g. [1] and [2]) and for the stability of complex line bundle. They are also deeply related to Kähler–Einstein metrics ([19], [20], [21] and [22]) and to the existence and uniqueness of extremal and constant scalar curvature metrics ([7], [12] and [13]).

The existence and the uniqueness of balanced metrics has been study by Donaldson [7] in the compact case and when the group $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ (the group of biholomorphisms of M which lift to the line bundles L modulo the $\mathbb{C}^*$ action) is discrete. The author with C. Arezzo [2] extend Donaldson's results by dropping the hypothesis on the group $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$.

The study of balanced metrics in the noncompact case is a very interesting area of research. Even in the one-dimensional case many questions on the existence and uniqueness of balanced metrics are still open. For example, it is not known if there exists a complete balanced metric on $\mathbb{C}$ different from the Euclidean one. It is worth mentioning that in [14] it is proven that among rotation-invariant metrics on $\mathbb{C}^n$ the only balanced one is the Euclidean one. (Some results on balanced metrics on locally Hermitian symmetric spaces of noncompact type can be found in [8] and [11]).

In this paper we study the link between the Bergman and balanced metrics on a complex manifold M. Our main result is Theorem 4.1 where we give sufficient conditions for a Bergman Einstein metric to be balanced in terms of Bochner's coordinates. As a consequence we get Corollary 4.2 where we prove that *if the Bergman metric g_B on a bounded domain $D \subset \mathbb{C}^n$ is Einstein with Einstein constant $\lambda = -2$ and the complex coordinates $(Z_1, \dots Z_n)$ of $\mathbb{C}^n$ restrict to Bochner's coordinates in a neighborhood of a point $p \in D$ then g_B is balanced.*

A few comments on our assumptions are in order. As far as the Einstein condition is concerned, it is worth pointing out that all the known examples of Bergman Einstein metrics are homogeneous and their Einstein constant equals -2 (see [9]). Actually Yau raised the question on the classification of Bergman Einstein metrics on strictly pseudoconvex domains and S–Y Cheng conjectured that if the Bergman metric on a strictly pseudoconvex domain is Einstein, then the domain is biholomorphic to the ball (see [17]). About the condition on the Bochner's coordinates, we remark that it is always satisfied in the case of Hermitian symmetric spaces of noncompact type (see Example 3.3 below).

The proof of Theorem 4.1 is based on the the local expression of the volume form of a Kähler-Einstein metric with respect to the Bochner coordinates around a point $p \in M$ (see Lemma 3.4). Similar techniques are used in [3] to study the link between Kähler-Einstein metrics, Bochner's coordinates and the sign of the first Chern class of a compact complex manifold M.

The paper is organized as follows. In the next section we give the definition of Bergman and balanced metrics on a complex manifold M and we describe their main properties. In Section 3, after recalling the definition of Calabi's diastasis function and Bochner's coordinates on a Kähler manifold manifold (M, g), we prove Lemma 3.4. Finally, in Section 4, we prove Theorem 4.1 and its Corollary 4.2.

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2 Bergman and balanced metrics

Let M be a n-dimensional complex manifold and L be its canonical bundle. Let $U \subset M$ be a sufficiently small open neighborhood of a point $p \in M$ endowed with local coordinates $(z_1, \dots, z_n)$. If α is a holomorphic section of L then there exists a complex-valued function $f_\alpha : U \to \mathbb{C}$ such that

$$\alpha(z) = f_\alpha(z)\ dz_1 \wedge \ldots \wedge dz_n,\ \forall z \in U. \tag{1}$$

Let $(\alpha_0, \dots, \alpha_N)$ $(N \leq \infty)$ be an orthonormal basis for the complex Hilbert space $(\mathcal{F}, (\cdot, \cdot))$ of all holomorphic n-forms α bounded with respect to

$$(\alpha, \alpha) = \frac{i^n}{2^n} \int_M \alpha \wedge \bar{\alpha}.$$

Let K^* be the smooth function on U given by

$$K^*(z, \bar{z}) = \sum_{j=0}^{N} f_{\alpha_j}(z) \bar{f}_{\alpha_j}(z).$$

The expression $\partial\bar{\partial} \log K^*$ does not depend on the coordinates and, therefore, $\omega_B = \frac{i}{2}\partial\bar{\partial} \log K^*$ is a globally defined real 2-form on M. In the sequel we will suppose ω_B is non-degenerate hence Kähler. For example this happens for the bounded domains in $\mathbb{C}^n$ or for compact manifolds with ample canonical bundle (see [9] for details). The *Bergman metric* g_B is the metric associated to the Kähler form ω_B. An alternative description of g_B is the following. Consider the holomorphic map

$$\varphi : M \to \mathbb{C}^{N+1}, x \mapsto (f_{\alpha_0}(x), \dots, f_{\alpha_N}(x)), \tag{2}$$

where $f_{\alpha_j} : U \to \mathbb{C}$ is the local expression of the orthonormal basis α_j. This map gives rise to a holomorphic map (denoted with the same symbol φ)

$$\varphi : M \to \mathbb{C}P^N, x \mapsto [\alpha_0(x), \dots, \alpha_N(x)] \tag{3}$$

whose local expression on U is given by (2). One can easily verify that the Bergman metric g_B is equal to the pull-back via φ of the Fubini–Study metric G_{FS}^N on $\mathbb{C}P^N$, namely $g_B = \varphi^*(G_{FS}^N)$ (see [9] for details).

We now define the concept of *balanced metrics* on M (with respect to the canonical bundle L). Let g be any Kähler metric on M polarized with respect to L, i.e. $c_1(L) = [\omega]$, where ω is the Kähler form associated to g.

Let h be an Hermitian metric on L such that its Ricci curvature $\mathrm{Ric}(h) = \omega$. Here $\mathrm{Ric}(h)$ is the two form on M whose local expression is given by

$$\mathrm{Ric}(h) = -\frac{i}{2}\partial\bar{\partial}\log h(\sigma(x), \sigma(x)), \tag{4}$$

for a trivializing holomorphic section $\sigma : U \to L$. In the quantum mechanics terminology L is called the *prequantum line bundle* and the pair (L, h) is called a *geometric prequantization* of the Kähler manifold (M, ω) (see e.g. [1]).

Consider the space $\mathcal{H}_h$ consisting of global holomorphic sections s of L (i.e. holomorphic n-forms on M) which are bounded with respect to

$$\langle s, s\rangle_h = \int_M h(s(x), s(x))\frac{\omega^n}{n!}.$$

One can show that $\mathcal{H}_h$ is a separable complex Hilbert space.

Let $x \in M$ and $q \in L \setminus \{0\}$ a fixed point of the fiber over x. If one evaluates $s \in \mathcal{H}_h$ at x, one gets a multiple $\delta_q(s)$ of q, i.e. $s(x) = \delta_q(s)q$. The map $\delta_q : \mathcal{H}_h \to \mathbb{C}$ is a continuous linear functional [6]. Hence from Riesz's theorem, there exists a unique $e_q \in \mathcal{H}_h$ such that $\delta_q(s) = \langle s, e_q\rangle_h, \forall s \in \mathcal{H}_h$, i.e.

$$s(x) = \langle s, e_q\rangle_h. \tag{5}$$

It follows that

$$e_{cq} = \overline{c}^{-1}e_q,\ \forall c \in \mathbb{C}^*.$$

The holomorphic section $e_q \in \mathcal{H}_h$ is called the *coherent state* relative to the point q. Thus, one can define a smooth function on M

$$\epsilon_{(L,h)}(x) = h(q, q)\|e_q\|^2_h, \tag{6}$$

where $q \in L \setminus \{0\}$ is any point on the fiber of x. If $s_l,\ l = 0, \ldots r$, $r \leq \infty$ is a orthonormal basis for $(\mathcal{H}_h, \langle\cdot,\cdot\rangle_h)$ then one can easily verify that $\epsilon_{(L,h)}(x) = \sum_{l=0}^{N} h(s_l(x), s_l(x))$.

We are now in the position to recall the definition of *balanced metrics* in the special case of the canonical bundle (cfr. [7] and [2] for the general definition in the compact and in the noncompact case).

Definition 2.1 *A Kähler metric g on M is balanced if the following conditions are satisfied:*

1. *g is polarized with respect to the canonical bundle L i.e. there exists an Hermitian structure h on L such that $Ric(h) = \omega$;*

2. *the function $\epsilon_{(L,h)}$ is constant.*

Under the hypothesis that for each point $x \in M$ there exists $s \in \mathcal{H}_h$ non-vanishing at x, one can give an alternative defintion of balanced metric as follows. Consider the holomorphic map of M into the complex projective space $\mathbb{C}P^r$:

$$\varphi_{(L,h)} : M \to \mathbb{C}P^r : x \mapsto [s_0(x), \ldots, s_r(x)], \tag{7}$$

where s_l, $l = 0, \ldots r$, $r \leq \infty$ is a orthonormal basis for $(\mathcal{H}_h, \langle \cdot, \cdot \rangle_h)$.

The map $\varphi_{(L,h)}$ was introduced by Rawnsley [15] in the context of quantization of Kähler manifolds and it is called the *coherent states map*. It is not difficult to see that

$$\varphi^*_{(L,h)}(\Omega^r_{FS}) = \omega + \frac{i}{2}\partial\bar{\partial} \log \epsilon_{(L,h)}, \tag{8}$$

where Ω^r_{FS} is the Fubini–Study form on $\mathbb{C}P^r$. Therefore, if g is a balanced metric, then g is projectively induced via the coherent states map. In the case the manifold M is compact also the contrary holds. Indeed, if $\partial\bar{\partial} \log \epsilon_{(L,h)} = 0$ then $\epsilon_{(L,h)}$ is harmonic and hence constant.

3 Bochner's coordinates and Einstein metrics

Let M be a complex manifold and let g be a real analytic Kähler metric on M. Calabi introduced, in a neighborhood of a point $p \in M$, a very special Kähler potential D^g_p for the metric g, which he christened *diastasis* (see [5]). Recall that a Kähler potential is an analytic function Φ defined in a neighborhood of a point p such that $\omega = \frac{i}{2}\partial\bar{\partial}\Phi$, where ω is the Kähler form associated to g. In a complex coordinate system (z) around p

$$g_{\alpha\beta} = 2g(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}) = \frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta}.$$

A Kähler potential is not unique: it is defined up to the sum with the real part of a holomorphic function. By duplicating the variables z and $\bar{z}$ a potential Φ can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighborhood U of the diagonal containing $(p, \bar{p}) \in M \times \bar{M}$ (here $\bar{M}$

denotes the manifold conjugated of M). The *diastasis function* is the Kähler potential D_p^g around p defined by

$$D_p^g(q) = \tilde{\Phi}(q, \bar{q}) + \tilde{\Phi}(p, \bar{p}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p}).$$

Among all the potentials the diastasis is characterized by the fact that in every coordinates system (z) centered in p

$$D_p^g(z, \bar{z}) = \sum_{|j|,|k| \geq 0} a_{jk} z^j \bar{z}^k,$$

with $a_{j0} = a_{0j} = 0$ for all multi-indices j.

The following proposition shows the importance of the diastasis in the context of holomorphic maps between Kähler manifolds.

Proposition 3.1 (Calabi) *Let $\varphi : (M, g) \to (P, G)$ be a holomorphic and isometric embedding between Kähler manifolds and suppose that G is real analytic. Then g is real analytic and for every point $p \in M$*

$$\varphi(D_p^g) = D_{\varphi(p)}^G,$$

where D_p^g (resp. $D_{\varphi(p)}^G$) is the diastasis of g relative to p (resp. of G relative to $\varphi(p)$).

Given a real analytic Kähler metric g on M and a point $p \in M$, one can always find local (complex) coordinates in a neighborhood of p such that

$$D_p^g(z, \bar{z}) = |z|^2 + \sum_{|j|,|k| \geq 2} b_{jk} z^j \bar{z}^k,$$

where D_p^g is the diastasis relative to p. These coordinates, uniquely defined up to a unitary transformation, are called *the Bochner coordinates* for g around p (cfr. [4], [5], [16], [18]).

Example 3.2 Let $\mathbb{C}P^N$ ($N \leq \infty$) be the complex projective space endowed with the Fubini–Study metric G_{FS}^N. Let $p = [1, 0, \ldots, 0]$. The Bochner's coordinates for G_{FS}^N around p are given by the affine coordinates $(u_1, \ldots, u_N)$ on $U_0 = \{Z_0 \neq 0\}$ where $u_j = \frac{Z_j}{Z_0}, j = 1, \ldots N$. The diastasis around p is given by

$$D_p^{G_{FS}^N}(u, \bar{u}) = \log(1 + \sum_{j=1}^N |u_j|^2).$$

Example 3.3 Let $B \subset \mathbb{C}^n$ be the unit ball endowed with its Bergman metric g_B whose associated Kähler form is given by $-\frac{i}{2}\partial\bar{\partial}\log(1-\sum_{j=1}^n |z_j|^2)$. Take the origin $p = (0, \ldots, 0)$. The Bochner coordinates for g_B around p are given by the global coordinates on $\mathbb{C}^n$ restricted to B and the diastasis is given by

$$D_p^g(z, \bar{z}) = -\log(1 - \sum_{j=1}^n |z_j|^2).$$

More generally let D be an irreducible Hermitian symmetric space of noncompact type endowed with its Bergman metric g_B. One can always assume that D is a bounded symmetric domain in $\mathbb{C}^n$ which can be chosen in such a way that $p = (0, 0, \ldots 0)$ belongs to D and D is circular with respect to D. Also in this case, the (global) Bochner coordinates for g_B around $p = (0, 0, \ldots 0)$ are given by the global coordinates on $\mathbb{C}^n$ restricted to D.

We now relate the Bochner coordinates to the Einstein condition.

Lemma 3.4 *Let M be a complex manifold endowed with a Kähler-Einstein metric g with Einstein constant λ. Given a point $p \in M$, there exist Bocnher's coordinates $(z_1, \ldots, z_n)$ for g in a neighbohood U of p such that*

$$\frac{\omega^n}{n!} = e^{-\frac{\lambda}{2}D_p^g} d\mu(z) \ ,$$

where D_p^g is the diastasis around p and $d\mu(z) = \frac{i^n}{2^n}dz$, $dz = dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n$.

Proof: Choose Bochner's coordinates $(z_1, \ldots, z_n)$ for g in a neighborhood U of the point p where D_p^g is defined (this is possible since the metric g is real analytic being Einstein). Since g is Einstein with (constant) scalar curvature s one has: $\rho_\omega = \lambda\omega$, where λ is the Einstein constant, i.e., $\lambda = \frac{s}{2n}$ and ρ_ω is the *Ricci form.* If $\omega = \frac{i}{2}\sum_{j=1}^n g_{j\bar{k}} dz_j \wedge d\bar{z}_{\bar{k}}$ then $\rho_\omega = -i\partial\bar{\partial}\log\det g_{j\bar{k}}$ is the local expression of its *Ricci form.* Thus the volume form of (M, g) reads on U as:

$$\frac{\omega^n}{n!} = e^{-\frac{\lambda}{2}D_p^g + F + \bar{F}} d\mu(z) \ , \tag{9}$$

where F is a holomorphic function on U and D_p^g is the diastasis around p.

It remains to show that $F + \bar{F} = 0$. To prove this, we first observe that

$$\frac{\omega^n}{n!} = \det(\frac{\partial^2 D_p^g}{\partial z_\alpha \partial \bar{z}_\beta}) d\mu(z). \tag{10}$$

Secondly, by the very definition of Bochner's coordinates, it is easy to check that the expansion of $\log\det(\frac{\partial^2 D^g_p}{\partial z_\alpha \partial \bar{z}_\beta})$ in the $(z, \bar{z})$-coordinates contains only mixed terms (i.e. of the form $z^j \bar{z}^k, j \neq 0, k \neq 0$). On the other hand, by formulae (9) and (10) one has:

$$-\frac{\lambda}{2} D^g_p + F + \bar{F} = \log\det(\frac{\partial^2 D^g_p}{\partial z_\alpha \partial \bar{z}_\beta})$$

and again, by the defintion of the Bochner coordinates, this forces $F + \bar{F}$ to be zero. $\square$

4 Statement and proof of the main result

In Section 2 we have seen that given an n-dimensional complex manifold M, under suitable conditions on the canonical bundle, one can define two different Hilbert spaces $(\mathcal{F}, (\cdot, \cdot))$ and $(\mathcal{H}_h, \langle \cdot, \cdot \rangle_h)$ (the latter depending on an Hermitian metric h on L). Starting from these spaces we have defined the Bergman and the balanced metrics on M. In the following proposition and theorem we relate these spaces and we find sufficient conditions for the Bergman metric to be balanced.

Theorem 4.1 *Let M be a n-dimensional complex manifold. Suppose that the following conditions are satisfied:*

(i) the Bergman metric g_B on M can be defined and it is Kähler-Einstein with Einstein constant equal -2;

(ii) there exists a point $p \in M$ and $\alpha \in \mathcal{F}$ such that the local expression of α in the Bochner coordinates $(z_1, \ldots z_n)$ for g_B in a neighborhood U of p is given by $\alpha = dz_1 \wedge \cdots \wedge dz_n$.

Then there exists an Hermitian metric k on L such that $Ric(k) = \omega_B$ and $\epsilon_{(L,k)}(x) = 1$ and hence g_B is balanced.

Proof: Let $\varphi : M \to \mathbb{C}P^N, \varphi(x) = [\alpha_0(x), \ldots, \alpha_N(x)]$ be the holomorphic map given by (3). Without loss of generality, one can assume that the point p and the form α given by (ii) satisfy $\varphi(p) = [1, 0, \ldots, 0]$ and $\alpha = \alpha_0$. Furthermore, we can assume, by shrinking U if necessary, that D^g_p is defined on U and $\varphi(U) \subset U_0$, where U_0 is the open subset of $\mathbb{C}P^N$ given by $[Z_0, \ldots, Z_N]$ with $Z_0 \neq 0$. The local expression of the map φ in the Bochner's coordinates $(z_1, \dots z_n)$ for g_B around p and $(u_1 = \frac{Z_1}{Z_0}, \dots, u_N = \frac{Z_N}{Z_0})$ for G_{FS}^N around $\varphi(p)$ (cfr. Example 3.2 above) is then given by:

$$\varphi : U \to U_0 : z = (z_1, \dots z_n) \mapsto (f_{\alpha_1}(z), \dots, f_{\alpha_N}(z)),$$

where $f_{\alpha_j}(z) = \frac{\alpha_j(z)}{\alpha_0(z)}, j = 1, \dots, n$. This is equivalent to:

$$\alpha_j(z) = f_{\alpha_j}(z)\alpha_0(z) = f_{\alpha_j}(z) dz_1 \wedge \dots \wedge dz_n, j = 1, \dots, n$$

since, by hypothesis, $\alpha = \alpha_0 = dz_1 \wedge \dots \wedge dz_n$ on U.

Therefore, by Proposition 3.1, the diastasis $D_p^{g_B} = \varphi^*(D_{\varphi(p)}^{G_{FS}^N})$ of the Bergman metric $g_B = \varphi^*(G_{FS}^N)$ on the open set U is equal to

$$D_p^{g_B}(z) = \log(1 + \sum_{j=1}^{N} |f_{\alpha_j}(z)|^2).$$

Since, by assumption, g_B is Einstein, we can apply Lemma 3.4 above (with $\lambda = -2$) and get

$$\frac{1}{1 + \sum_{j=1}^{N} |f_{\alpha_j}(z)|^2} \frac{\omega_B^n(z)}{n!} = d\mu(z), \tag{11}$$

Consider the Hermitian metric k on L whose local expression in the Bochner's coordinates $z = (z_1, \dots, z_n)$ is given by:

$$k(\beta(z), \beta(z)) = \frac{|f_\beta(z)|^2}{1 + \sum_{j=1}^{N} |f_{\alpha_j}(z)|^2}, \tag{12}$$

where β is a holomorphic n-form on M and f_β is the holomorphic function on U such that $\beta = f_\beta \alpha_0$. It is easily seen that the pair (L, k) is a geometric prequantization of (M, g_B), i.e. $\mathrm{Ric}(k) = \omega_B$. Observe that the $2n$-form $\frac{i^n}{2^n}\beta \wedge \bar{\beta}$ on the open set U equipped with Bochner's coordinates $(z_1, \dots, z_n)$ reads as:

$$\frac{i^n}{2^n}(\beta \wedge \bar{\beta})(z) = |f_\beta(z)|^2 d\mu(z). \tag{13}$$

Formulae (11), (12) and (13) tell us that, for all holomorphic n-form β of M, the two complex-valued $2n$-forms on M given by $k(\beta, \beta)\frac{\omega_B^n}{n!}$ and $\frac{i^n}{2^n}\beta \wedge \bar{\beta}$ respectively, coincide on the open set U. Since they are real analytic they must agree on all of M. This implies that the Hilbert spaces $\mathcal{F}$ and $\mathcal{H}_k$ defined in the previous section coincide and $(\alpha_0, \dots, \alpha_N)$ is an orthonormal basis with respect to $\langle\cdot,\cdot\rangle_k$. Thus $\epsilon_{(L,k)}(x) = \sum_{j=0}^{N} k(\alpha_j(x), \alpha_j(x)) = 1$, which concludes the proof. $\square$

Consider now a bounded domain $D \subset \mathbb{C}^n$, namely an open and connected subset of $\mathbb{C}^n$ such that

$$\int_D d\mu(Z) < \infty,$$

where $d\mu(Z) = \frac{i^n}{2^n} dZ$, $dZ = dZ_1 \wedge d\bar{Z}_1 \wedge \ldots \wedge dZ_n \wedge d\bar{Z}_n$ is the Lebesgue measure with respect to the Euclidean coordinates $(Z_1, \ldots, Z_n)$ of $\mathbb{C}^n$.

In this case the Bergman metric g_B on D can be defined (see [9] for details). Let $\alpha = dZ_1 \wedge \ldots \wedge dZ_n$. Since D is a bounded domain, α belongs to $\mathcal{F}$, i.e., $\frac{i^n}{2^n}\int_D \alpha \wedge \alpha$ is finite. Assume now that there exist a point $p \in D$ and an open neighborhood U of p where the restriction of $(Z_1, \ldots Z_n)$ defines Bochner's coordinates $(z_1, \ldots z_n)$ around p (cfr. Example 3.3). This implies that the form α equals $dz_1 \wedge \ldots \wedge dz_n$ on U and therefore condition (ii) in Theorem 4.1 is satisfied. This leads to the following corollary.

Corollary 4.2 *Let $D \subset \mathbb{C}^n$ be a bounded domain and let g_B its Bergman metric. Suppose that:*

(i) g_B is Einstein with Einstein constant -2;

(ii) the Eucliden coordinates $(Z_1, \ldots, Z_n)$ of $\mathbb{C}^n$ restrict to Bochner coordinates for g_B in a neighborhood of a point $p \in D$.

Then g_B is balanced.

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