A note on Kähler-Einstein metrics and Bochner's coordinates

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Abstract

In this paper we prove that if a compact Kähler-Einstein manifold (M, ω) with integral Kähler form satisfies a compatibility condition between the domain of definition of the Bochner coordinates and of the diastasis potential, then $c_1(M) > 0$.

Keywords: Kähler metrics; Kähler-Einstein metrics; diastasis; Bochner's coordinates, canonical coordinates.

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1 Introduction

Let M be a complex manifold and let g be a real analytic Kähler metric on M. Calabi introduced, in a neighborhood of a point $p \in M$, a very special Kähler potential D_p for the metric g, which he christened *diastasis* (see [5]). Recall that a Kähler potential is a analytic function Φ defined in a neighborhood of a point p such that $\omega = \frac{i}{2}\partial\bar{\partial}\Phi$, where ω is the Kähler form associated to g. In a complex coordinate system (z) around p

$$g_{\alpha\beta} = 2g(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}) = \frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta}.$$

A Kähler potential is not unique: it is defined up to the sum with the real part of a holomorphic function. By duplicating the variables z and $\bar{z}$ a potential Φ can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighborhood U of the diagonal containing $(p, \bar{p}) \in M \times \bar{M}$ (here $\bar{M}$ denotes the manifold conjugated of M). The *diastasis function* is the Kähler potential D_p around p defined by

$$D_p(q) = \tilde{\Phi}(q, \bar{q}) + \tilde{\Phi}(p, \bar{p}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p}).$$

Among all the potentials the diastasis is characterized by the fact that in every coordinates system (z) centered in p

$$D_p(z, \bar{z}) = \sum_{|j|,|k| \geq 0} a_{jk} z^j \bar{z}^k,$$

with $a_{j0} = a_{0j} = 0$ for all multi-indices j. The following theorem shows the importance of the diastasis in the context of holomorphic maps between Kähler manifolds.

Theorem 1.1 (Calabi) *Let $\varphi : (M, g) \to (P, G)$ be a holomorphic and isometric embedding between Kähler manifolds and suppose that G is real analytic. Then g is real analytic and for every point $p \in M$*

$$\varphi(D_p) = D_{\varphi(p)},$$

where D_p (resp. $D_{\varphi(p)}$) is the diastasis of g relative to p (resp. of G relative to $\varphi(p)$).

We now introduce the concept the concept of Bochner's coordinates (cfr. [4], [5], [8], [9], [12], [13]). Given a real analytic Kähler metric g on M and a point $p \in M$, one can always find local (complex) coordinates in a neighborhood of p such that

$$D_p(z, \bar{z}) = |z|^2 + \sum_{|j|,|k| \geq 2} b_{jk} z^j \bar{z}^k,$$

where D_p is the diastasis relative to p. These coordinates, uniquely defined up to a unitary transformation, are called *the Bochner coordinates* with respect to the point p.

As in the case of the diastasis these coordinates has a good behavior regarding holomorphic maps between Kähler manifolds. More precisely, the following holds true:

Theorem 1.2 (**Calabi**) *Let $\varphi : (M, g) \to (P, G)$ be a holomorphic and isometric embedding between Kähler manifolds and suppose that G is real* *analytic. If $(z_1, \dots, z_n)$ is a system of Bochner's coordinates in a neighborhood U of $p \in M$ then there exists a system of Bochner's coordinates $(Z_1, \dots, Z_N)$ with respect to $\varphi(p)$ such that*

$$Z_1|_{\varphi(U)} = z_1, \dots, Z_n|_{\varphi(U)} = z_n. \tag{1}$$

Given a point $p \in M$ we denote by $M_p \subset M$ the maximal domain of definition of the diastasis D_p. The main result of this paper is the following:

Theorem 1.3 *Let (M, g) be a compact Kähler-Einstein manifold such that its associated Kähler form ω is integral. Suppose that the following condition is satisfied:*

Condition A: there exists a point p in M such that the Bochner coordinates $(z_1, \dots, z_n)$ around p extend to holomorphic functions $(f_1, \dots, f_n)$ on a connected open set $V \subseteq M_p$ having infinite euclidean volume, namely:

$$\int_V \frac{i^n}{2^n} df_1 \wedge d\bar{f}_1 \wedge \ldots \wedge df_n \wedge d\bar{f}_n = +\infty. \tag{2}$$

Then $c_1(M) > 0$.

Remark 1.4 Observe that in Theorem 1.3 it makes sense to consider the Bochner coordinates around p since g is Kähler-Einstein and hence real analytic (see e.g. [3]).

The previous theorem can be considered as a generalization of a Theorem of Hulin [8] asserting that a Kähler-Einstein submanifold of the complex projective space has necessarily positive scalar curvature. Indeed, in the case of a projectively induced Kähler-Einstein metric g condition A is achieved by strongly using Theorem 1.1 and Theorem 1.2 with $P = \mathbb{C}P^N$ the N-dimensional complex projective space and with $G = G_{FS}$ the Fubini–Study metric on P (see [8] for details). One of the key point of Hulin's proof is the fact that the Diastasis D_p of the metric g with respect to the point p is globally defined and non-negative in $M \setminus H$ where H is the restriction to M of an hyperplane of $\mathbb{C}P^N$. This is no longer true for general Kähler metrics. Hence, in order to make work Hulin's proof in our case, we need a similar result when the metric g is not necessarily projectively induced. This is done in Proposition 2.1, by using Tian's result on polarized metrics.

We finally want to point out that without the condition of the infinity of the euclidean volume given by formula (2), Theorem 1.3 does not hold. Indeed let $M = \mathbb{C}^n/\mathbb{Z}^{2n}$ be the n-dimensional complex torus with the flat metric $g = \frac{i}{2}\sum_{j=1}^{n} |dz_j|^2$ coming from $\mathbb{C}^n$. The associated Kähler form ω is clearly integral. Let p be the origin in $\mathbb{C}^n$ and let V be a fundamental domain containing p. The diastasis D_p of g with respect to p reads as $D_p(z, \bar{z}) = \sum_{j=1}^{n} |z_j|^2$ and the euclidean coordinates $(z_1, \dots, z_n)$ are in fact Bochner's coordinates around p. Observe that, in this case, V is the maximal domain of extension of these coordinates and also that V is the maximal domain of extension of D_p, namely $V = M_p$. Moreover V is an open connected subset of M (even dense on M) with finite euclidean volume. This shows that without condition A Theorem 1.3 does not hold since the first Chern class of M is zero. The same argument can be applied to any Riemann surface Σ_g of genus $g \geq 2$ endowed with the hyperbolic metric.

This leads to the following problem, which we believe to be very interesting on its own sake:

Problem: *Let M be a complex manifold endowed with a real analytic Kähler metric g and let $(z_1, \dots, z_n)$ be the Bochner coordinates of the metric g around any point p in M. Suppose that $c_1(M) > 0$. Is it true that $(z_1, \dots, z_n)$ extend to holomorphic functions $(f_1, \dots, f_n)$ satisfying condition (2)?*

2 The diastasis of a polarized metric and the proof of Theorem 1.3

In this section we describe the diastasis of a polarized Kähler metric. We refer the reader to [13] for details and further results about polarized Kähler metrics.

Let M be a projective algebraic manifold, namely a compact complex manifold which admits a holomorphic embedding in some complex projective space $\mathbb{C}P^N$. The hyperplane line bundle of $\mathbb{C}P^N$ restricts to a holomorphic line bundle L on M which is called a *polarization* on M. A Kähler metric g (not necessarily Einstein) on M is *polarized* with respect to L if its associated Kähler form ω represents the first Chern class $c_1(L)$ of L in $H^2(M, \mathbb{Z})$. Given a polarized Kähler metric g on M one can find a Hermitian metric h on L with its Ricci curvature $\mathrm{Ric}(h) = \omega$ (see Lemma 1.1 in [13]).

For each positive integer k, we denote by $L^{\otimes k}$ the k-th tensor power of L. It is a polarization of the Kähler metric kg and the Hermitian metric h induces a natural Hermitian metric h^k on $L^{\otimes k}$ such that $\mathrm{Ric}(h^k) = k\omega$.

Denote by $H^0(M, L^{\otimes k})$ the space of global holomorphic sections of $L^{\otimes k}$.

It is in a natural way a complex Hilbert space with respect to the norm

$$\|s\|_{h^k} = \langle s, s\rangle_{h^k} = \int_N h^k(s(x), s(x))\frac{\omega^n(x)}{n!} < \infty,$$

for $s \in H^0(M, L^{\otimes k})$.

By a well-known Theorem of Kodaira (see e.g. [7]), for all k sufficiently large, one can define a holomorphic embedding of M into a complex projective space as follows. Let $(s_0, \dots, s_{d_k})$, be a orthonormal basis for $H^0(M, L^{\otimes k})$ and let $\sigma : U \to L^{\otimes k}$ be a trivialising holomorphic section on the open set $U \subset M$. Define the map

$$\varphi_\sigma : U \to \mathbb{C}^{d_k+1} \setminus \{0\}, x \mapsto \left(\frac{s_0(x)}{\sigma(x)}, \dots, \frac{s_{d_k}(x)}{\sigma(x)}\right). \tag{3}$$

If $\tau : V \to L$ is another holomorphic trivialisation then there exists a non-vanishing holomorphic function f on $U \cap V$ such that $\sigma(x) = f(x)\tau(x)$. Therefore one can define a holomorphic embedding

$$\varphi_k : M \to \mathbb{C}P^{d_k}, \tag{4}$$

whose local expression in the open set U is given by (3).

The Fubini–Study metric on $\mathbb{C}P^{d_k}$, which we denote by G_k, restricts to a Kähler metric $g_k = \varphi_k^* G_k$ on M which is polarized with respect to $L^{\otimes k}$. In [13] Tian called the set of normalized metrics $\frac{1}{k}g_k$ the *Bergmann* metrics on M with respect to L and he proves that the sequence $\frac{1}{k}g_k$ converges to the original Kähler metric g in the C^2-topology (see Theorem A in [13]). This theorem was further generalizes to the C^∞ case by Ruan [12] (see also [11] and [17]).

The mentioned results have found different important applications such as the existence problem of K–E metrics, K-stability and CM-stability ([14], [15] and [16]), the existence of constant scalar curvature metrics (see [6]) and the quantization of Kähler manifolds (see [1], [2] and [10]).

We now show that they also imply interesting conditions on the diastasis of any polarized Kähler metric g.

Proposition 2.1 *Let M be a compact complex manifold endowed with a real analytic and polarized Kähler metric g and let p be any point of M. Then there exists a measure zero set $H \subset M$ such that D_p (the diastasis of the metric g around p) is non-negative on $M_p \setminus (M_p \cap H)$, where M_p denotes the maximal domain of definition of D_p.*

Proof: Fix a natural number k. Let us denote by D_p^k the diastasis of the metric $g_k = \varphi_k^* G_k$ with respect to the point p. Without loss of generality we can choose the above map φ_k in such a way that $\varphi_k(p) = [1, 0, \dots, 0] \in \mathbb{C}P^{d_k}$.

Since the diastasis of $(\mathbb{C}P^{d_k}, G_k)$ with respect to $[1, 0, \dots, 0]$ is globally defined on $\mathbb{C}P^{d_k} \setminus Y_k$, where $Y_k = \mathbb{C}P^{d_k - 1}$ is the hyperplane $Z_0 = 0$ in $\mathbb{C}P^{d_k}$, it follows by Proposition 1.1 that D_p^k is globally defined on $M \setminus H_k$, where $H_k = \varphi_k^{-1}(Y_k)$.

By the very definition of the diastasis in any coordinate system $(z_1, \dots, z_n)$ centered in p we have: $D_p^k = \sum_{|i|,|j| \geq 0} a_{ij}(k) z^i \bar{z}^j$ where $a_{ij}(k)$ is a matrix depending on k such that $a_{j0}(k) = a_{0j}(k) = 0$. This is equivalent to

$$\frac{\partial D_p^k}{\partial z^l}(p) = \frac{\partial D_p^k}{\partial \bar{z}^m}(p) = 0, \tag{5}$$

for any multi-indices l, m and for any k. Consider the sequence of functions D_p^k globally defined on $M \setminus H$ where $H = \cup_k U_k$. Ruan's proof of the above mentioned theorem is actually based on the stronger statement that the sequence $\frac{D_p^k}{k}$ C^∞-converges to a Kähler potential Φ for the metric g, whenever the D_p^k's and Φ are defined (see p. 600 in [12]). More precisely, if U_p denote the maximal domain of definition of the potential Φ, then the sequence $\frac{D_p^k}{k}$ C^∞-converges to Φ in $U_p \setminus (U_p \cap H)$. Since, for all k, D_p^k is non-negative on $M \setminus H$, being the restriction of the diastasis of the Fubini–Study metric on $\mathbb{C}P^{d_k}$, the potential Φ is non-negative on $U_p \setminus (U_p \cap H)$. Moreover property (5) goes to the limit, namely

$$\frac{\partial \Phi}{\partial z^l}(p) = \frac{\partial \Phi}{\partial \bar{z}^m}(p) = 0,$$

for any multi-indices l, m. Since $(z_1, \dots, z_n)$ is an arbitrary coordinate system centered in p it follows, again by the definition of the diastasis, that $\Phi = D_p$. Hence, $U_p = M_p$ and so D_p is non-negative on $M_p \setminus (M_p \cap H)$ as desired. □

Conclusion of the proof of Theorem 1.3

Since ω is supposed to be integral there exists a holomorphic line bundle L on M such that $c_1(L) = \omega$, where $[\omega]$ denotes the de–Rham class of ω. Let $\varphi : M \to \mathbb{C}P^N$ be a holomorphic embedding such that the restriction to M of the hyperplane bundle on $\mathbb{C}P^N$ is given by $L^{\otimes k_0}$ for some k_0 sufficiently large. This implies that $c_1(L^{\otimes k_0}) = k_0 \omega$ and then the metric $k_0 g$ is polarized with respect to $L^{\otimes k_0}$. Moreover condtion A in our Theorem 1.3 remains true for the metric $k_0 g$. Indeed the Bochner coordinates $(z_1^{k_0}, \dots, z_n^{k_0})$ of the metric $k_0 g$ around p are given by $z_j^{k_0} = \frac{z_j}{\sqrt{k_0}}, j = 1, \dots, n$ where $(z_1, \dots, z_n)$ are the Bochner coordinates of the metric g around p. Therefore, without loss of generality, we can assume that g is polarized with respect to the line bundle L.

Let us choose Bochner's coordinates $(z_1, \dots, z_n)$ for g in a neighborhood U of the point p which we take small enough to be contractible.

Since g is Einstein with (constant) scalar curvature s one has: $\rho_\omega = \lambda\omega$, where λ is the Einstein constant, i.e. $\lambda = \frac{s}{2n}$, and ρ_ω is the *Ricci form.* If $\omega = \frac{i}{2}\sum_{j=1}^n g_{j\bar{k}} dz_j \wedge d\bar{z}_{\bar{k}}$ then $\rho_\omega = -i\partial\bar{\partial}\log\det g_{j\bar{k}}$ is the local expression of its *Ricci form.*

Thus the volume form of (M, g) reads on U as:

$$\frac{\omega^n}{n!} = \frac{i^n}{2^n} e^{-\frac{\lambda}{2}D_p + F + \bar{F}} dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n \ , \tag{6}$$

where F is a holomorphic function on U and D_p is the diastasis around p.

We claim that $F + \bar{F} = 0$. To prove this, we first observe that

$$\frac{\omega^n}{n!} = \frac{i^n}{2^n}\det\left(\frac{\partial^2 D_p}{\partial z_\alpha \partial \bar{z}_\beta}\right) dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n.$$

Secondly, by the very definition of Bochner's coordinates it is easy to check that the expansion of $\log\det(\frac{\partial^2 D_p}{\partial z_\alpha \partial \bar{z}_\beta})$ in the $(z, \bar{z})$-coordinates contains only mixed terms (i.e. of the form $z^j \bar{z}^k, j \neq 0, k \neq 0$). On the other hand by formula (6)

$$-\frac{\lambda}{2} D_p + F + \bar{F} = \log\det\left(\frac{\partial^2 D_p}{\partial z_\alpha \partial \bar{z}_\beta}\right).$$

Again by the defintion of the Bochner coordinates this forces $F + \bar{F}$ to be zero, proving our claim.

By hypothesis (condition A) there exist holomorphic functions $(f_1, \dots, f_n)$ on a connected set $V \subseteq M_p$ which extend $(z_1, \dots, z_n)$. Hence, by formula (6) (with $F + \bar{F} = 0$) the n-forms $\frac{\omega^n}{n!}$ and $e^{-\frac{\lambda}{2}D_p} df_1 \wedge d\bar{f}_1 \wedge \dots \wedge df_n \wedge d\bar{f}_n$ globally defined on $V \setminus (V \cap H) \subseteq M_p \setminus (M_p \cap H)$ agree on the open set U. Since they are real analytic they must agree on $V \setminus (V \cap H)$ which is connected since we are deleting from a connected set V a countable union $\cup_k (V \cap H_k)$ of real codimension 2 subsets (cfr. the proof of Proposition 2.1). This means that

$$\frac{\omega^n}{n!} = \frac{i^n}{2^n} e^{-\frac{\lambda}{2}D_p} df_1 \wedge d\bar{f}_1 \wedge \dots \wedge df_n \wedge d\bar{f}_n. \tag{7}$$

Suppose now that the scalar curvature s of g is non-positive. By formula (7) and by the fact that D_p is non-negative on $V \setminus (V \cap H) \subseteq M_p \setminus (M_p \cap H)$ (by Proposition 2.1), we get

$$\int_{V\setminus(V\cap H)} \frac{\omega^n}{n!} \geq \int_{V\setminus(V\cap H)} \frac{i^n}{2^n} df_1 \wedge d\bar{f}_1 \wedge \ldots \wedge df_n \wedge d\bar{f}_n.$$

Since $V \cap H$ has measure zero in V it follows, by hypothesis, that the right hand side of the previous inequality has infinite value. This is the desired contradiction, the volume of (M, g) (and hence that of $(V \setminus (V \cap H), g)$) being finite. $\square$

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