Structural stability and catastrophes

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Abstract: We show that, in a pure exchange smooth economy, a redistribution of endowments involving singular economies can be supported by a unique and continuous path of supporting equilibrium price vectors if this redistribution is the projection of a path on the equilibrium manifold transversal to the set of critical equilibria.

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1 Introduction

It is well-known that continuity of supporting equilibrium price changes characterizes local perturbations of (regular) economies [1, 4]. This is a stability property with evident implications in comparative statics and dynamics. In the sequel, we will refer to this property as *smooth selection property* (SSP). In a recent paper [9] have investigated whether this property holds if endowments are generically redistributed across consumers. By using standard properties of covering spaces, [9] show that SSP can be extended to non-local redistributions of regular endowments (see also Theorem 2.4 in Section 2). More precisely, there exists a unique continuous path of equilibrium prices which support a redistribution of regular economies. The key ingredient to prove this result, a property of *covering spaces* known as *arc lifting property* (ALP) (see Proposition 2.1 in Section 2), cannot be used if the endowments redistributed are singular economies. In this case the existence of such a path becomes an issue.

In this paper we explore the connection between structural stability and singular economies. A redistribution policy which encounters catastrophes cannot be generically supported by a continuous equilibrium price path. Using the geomeric construction by [8], we show in Theorem 3.1 under what conditions it is still possible to get a (unique) continuous path of equilibrium prices vectors if endowments are changed according to a redistribution which encounters catastrophes. In our construction we rely on the existence of *minimal* paths, i.e., paths which minimize the distance between regular equilibria, where the length between two regular equilibria is defined as the number of intersection points of all the paths connecting them with the set of critical equilibria (see [8]). The idea is that one needs to construct a path which intersects the set of critical equilibria transversally in a finite number of points and to project it onto the space of endowments. The structure of the paper is the following. Section 2 is devoted to the illustration of the economic model and some mathematical results. In Section 3 we prove our main result, Theorem 3.1.

2 Preliminaries

We refer to [9] and references therein for some mathematical results on covering spaces and lifting properties. For reader's convenience we recall here some definitions and properties.

Let $\tilde{X}$ and X be two (not necessarily connected) topological spaces. A continuous map $p : \tilde{X} \to X$ is called a *covering map* if it satisfies the following conditions:

(a) p is surjective;

(b) each $x \in X$ has an open neighbourhood U such that $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$, each of which is mapped by p homeomorphically onto U.

The neighbourhood U is said to be *well-covered* for p and the set $p^{-1}(x)$ is called the *fiber* of x. Let $p : \tilde{X} \to X$ be a continuous map (not necessarily a covering) and let Y be a topological space. A *lift* of a continuous map $f : Y \to X$ is a map $\tilde{f} : Y \to \tilde{X}$ such that $p\tilde{f} = f$. We recall that an arc on X is a map $\alpha : I \to X$, where $I = [0, 1]$. The points $\alpha(0)$ and $\alpha(1)$ are called the starting and final points of α. The following proposition holds true.

Proposition 2.1 (ALP: arc lifting property) *Given a covering space $p : \tilde{X} \to X$, let $\alpha : I \to X$ be an arc with starting point x_0 and let $\tilde{x}_0$ be any point in the fiber of x_0. There exists a unique lift $\tilde{\alpha} : I \to \tilde{X}$ of α with starting point $\tilde{x}_0$.*

In the previous proposition the existence of a lift relies on properties of covering maps. Once a lift is given its uniqueness depends only on the fact that p is a local diffeomorphism as expressed by the following proposition which will be of constant use in the proof of Theorem 3.1. For completeness, we include a proof of this standard result.

Proposition 2.2 (uniqueness of lifts for local homeomorphisms) *Let $p : \tilde{X} \to X$ be a surjective local homeomorphism, Y a connected topological space, $y_0 \in Y$, $\tilde{x}_0 \in \tilde{X}$ and $f : Y \to X$ a map such that $f(y_0) = x_0 = p(\tilde{x}_0)$. A lift $\tilde{f} : Y \to \tilde{X}$ of f such that $\tilde{f}(y_0) = \tilde{x}_0$ is unique.*

Proof: Assume there exists another lift $\tilde{f}'$ of f such that $\tilde{f}(y_0) = \tilde{f}'(y_0) = \tilde{x}_0$ and consider the set $Y' = \{y \in Y | \tilde{f}(y) = \tilde{f}'(y)\}$. We are going to prove that Y' is open and closed in Y and hence, since Y is connected, $Y' = Y$. In order to prove that Y' is open choose a point $y \in Y'$ and let U be an open neighborhood of $\tilde{f}(y)$ such that $p_{|U} : U \to p(U)$ is a homeomorphism. The open set $V = \tilde{f}^{-1}(U) \cap \tilde{f}'^{-1}(U)$ contains y. Let z be any point in V. Observe that $\tilde{f}(z), \tilde{f}'(z) \in U$. By the injectivity of $p_{|U}$, $p(\tilde{f}(z)) = p(\tilde{f}'(z))$ implies $\tilde{f}(z) = \tilde{f}'(z)$ and, hence, $V \subset Y'$. This shows that Y' is open. Define now $\bar{Y} = Y \setminus Y'$ and let $w \in \bar{Y}$. Observe that $p(\tilde{f}(w)) = p(\tilde{f}'(w)) = f(w)$. Let U' and U'' be two disjoint open neighborhoods of $\tilde{f}(w)$ and $\tilde{f}'(w)$, respectively, homeomorphic to an open neighborhood Z of $f(w)$. Consider the open set $\bar{V} = \tilde{f}^{-1}(U') \cup \tilde{f}'^{-1}(U'')$ and choose any point $\bar{z} \in \bar{V}$. Since $f(z) \in Z$ we have that $\tilde{f}(z) \in U'$ and $\tilde{f}'(z) \in U''$, which implies that $w \in \bar{Y}$. This shows that Y' is closed. $\square$

As far as the economic setting is concerned, we consider a smooth pure exchange economy with fixed total resources (see [2]). Let m and l be, respectively, the (finite) number of agents and commodities. Let $S = \{p \in \mathbb{R}^l \mid p_i \geq 0, i = 1, 2, \ldots, l-1, p_l = 1\}$ be the set of prices normalized by the numeraire convention. Let $r \in \mathbb{R}^l$ be the vector of fixed total resources and denote by $\Omega(r)$ the set of endowments with fixed total resources, i.e., $\Omega(r) = \{\omega \in \mathbb{R}^{lm} | \sum_{i=1}^m \omega_i = r\}$. Define the equilibrium manifold, denoted by $E(r)$, the set of pairs of prices and endowments such that aggregate net demand is zero, i.e.,

$$E(r) = \{(p, \omega) \in S \times \Omega(r) \mid \sum_{i=1}^{m} f_i(p, p \cdot \omega_i) = r\},$$

where $f_i(p, w_i)$, denotes consumer i's demand.

The set $E(r)$ is globally diffeomorphic to $\mathbb{R}^{l(m-1)}$ (see [2, Ch. 5]). Let $\pi : E(r) \to \Omega(r)$ be the *natural projection*, i.e. the restriction to $E(r)$ of the projection $S \times \Omega(r) \to \Omega(r)$, such that $(p, \omega) \mapsto \omega$. The map π is smooth, proper and surjective. One can define the set of *critical equilibria*, denoted by $E_c(r)$, as the pairs $(p, \omega) \in E(r)$ such that the derivative of π is not onto [1]. The set $E_c(r)$ is a closed subset of measure zero of the equilibrium manifold $E(r)$ [3]. The set of *singular economies*, denoted by Σ, is the image via π of the set $E_c(r)$. The set Σ is a closed (by properness of π) and a measure zero set in $\Omega(r)$ (by Sard's theorem). Let us define the regular economies $\mathcal{R} = \Omega(r) \setminus \Sigma$ as the regular values of the map π. We state as a theorem the following important result due to Balasko.

Theorem 2.3 (Balasko [2]) *The map $\pi_{|_{\pi^{-1}(R)}} : \pi^{-1}(R) \to R$ is a finite covering.*

According to this theorem, see [2, p. 94], smooth local changes of the parameter ω imply smooth changes of the corresponding equilibrium price vectors, namely there exists a supporting equilibrium price vector sufficiently close to the initial one (a property known as smooth selection property, SSP).
SSP can be extended to arbitrary changes of regular economies, represented by a continuous map $\gamma : [0, 1] \to \Omega(r)$, where $\omega_0 = \gamma(0)$ and $\omega_1 = \gamma(1)$ (the map γ can be thought as a redistribution policy). If one writes, using standard vector notation to denote the aggregate excess demand function, the equilibrium condition as $z(p(t), \gamma(t)) = 0$, $t \in [0, 1]$, [9] have showed that $p(t)$ is locally unique and it is changing continuously while the parameter $\gamma(t) \in \Omega(r)$ is varying.

Theorem 2.4 (Loi and Matta [9]) *Let $\gamma : I \to \mathcal{R}$ be a regular policy connecting $\omega_0 = \gamma(0)$ and $\omega_1 = \gamma(1)$ and let p_0 be the supporting equilibrium price vector associated with ω_0. Then there exists a unique lift $\tilde{\gamma} : I \to \pi^{-1}(\mathcal{R})$ of γ.*

3 Main result

Consider now a redistribution $\sigma : [0,1] \to \Omega(r)$ such that $\sigma(t) \in \Sigma$ for some $t \in (0,1)$. In this case the existence of a unique continuous equilibrium price change supporting $\sigma(t)$ becomes an issue since ALP cannot be applied (the map $\pi_{|_{\pi^{-1}(\Sigma)}} : \pi^{-1}(\Sigma) \to \Sigma$ is not a covering).
We address this problem using a different strategy: assuming that an equilibrium path exists, under what conditions is it unique? Let ω $(\omega') \in \mathcal{R}$ be the initial (final) allocation and let p be he supporting equilibrium price vector of ω. We construct a *minimal* path $\tilde{\gamma}(t)$ on $E(r)$, i.e., a path which connects two regular equilibria $(p,\omega) = x$ and $(p',\omega') = y$ and which minimizes the number of intersection points with the set of critical equilibria $E_c(r)$. The existence of such a path has been showed by [8]. The redistribution $\gamma : [0,1] \to \Omega(r)$ is found by projecting $\tilde{\gamma}$ onto the space of economies, i.e. $\gamma(t) = \pi(\tilde{\gamma}(t))$. Theorem 3.1 shows under what conditions this policy admits a (unique) lift.

Theorem 3.1 *Let $\tilde{\gamma} : I \to E(r)$ be a minimal arc connecting two reqular equilibria x and y, where $x, y \in \pi^{-1}(\mathcal{R})$. Then $\tilde{\gamma}$ is uniquely determined by its projection $\gamma = \pi(\tilde{\gamma})$ and by a finite number of its points.*

Proof: Let $C = E_c(r) \cap \tilde{\gamma}(I)$. Since $\tilde{\gamma}$ is a minimal path either $C = \emptyset$ or C is a finite number of points. If $C = \emptyset$ then the conclusion follows by Proposition 2.2 applied to the local diffeomorphism $\pi : E \setminus E_c(r) \to \pi(E \setminus E_c(r))$. In this case one does not need to fix any point. If C is nonempty set $C = \{c_1, c_2, \ldots, c_k\}$. Then there exist $0 < t_1 \leq \ldots \leq t_k < 1$ such that $c_i = \tilde{\gamma}(t_i)$, $i = 1, \ldots, k$. Choose $\xi_i = \tilde{\gamma}(s_i)$, with $i = 1, \ldots, k-1$, with $t_j < s_j < t_{j+1}$, $j = 1, \ldots, k-1$ such that $\tilde{\gamma}(s_j) \in E \setminus E_c(r)$. Consider the following subarcs of $\tilde{\gamma}$: $\tilde{\gamma}_x^{c_1}, \tilde{\gamma}_{c_1}^{\xi_1}, \tilde{\gamma}_{\xi_1}^{c_2}, \ldots, \tilde{\gamma}_{\xi_{k-1}}^{c_k}, \tilde{\gamma}_{c_k}^{y}$ connecting x with c_1, c_1 with ξ_1,..., ξ_{k-1} with c_k, and c_k with y. By applying again Proposition 2.2 to the local diffeomorphism $\pi : E \setminus E_c(r) \to \pi(E \setminus E_c(r))$ it follows that $\tilde{\gamma}_x^{c_1} \setminus \{c_1\}, \tilde{\gamma}_{c_1}^{\xi_1} \setminus \{c_1\}, \tilde{\gamma}_{\xi_1}^{c_2} \setminus \{c_2\}, \ldots, \tilde{\gamma}_{\xi_{k-1}}^{c_k} \setminus \{c_k\}, \tilde{\gamma}_{c_k}^{y} \setminus \{c_k\}$ are the unique lifts of $\pi(\tilde{\gamma}_x^{c_1}) \setminus \{\pi(c_1)\}, \pi(\tilde{\gamma}_{c_1}^{\xi_1}) \setminus \{\pi(c_1)\}, \pi(\tilde{\gamma}_{\xi_1}^{c_2}) \setminus \{\pi(c_2)\}, \ldots, \pi(\tilde{\gamma}_{\pi(\xi_{k-1})}^{c_k}) \setminus \{\pi(c_k)\}, \pi(\tilde{\gamma}_{\pi(c_k}^{\pi(y)}) \setminus \{\pi(c_k)\}$ passing through the points $\{\xi_1, \ldots, \xi_{k-1}\}$, respectively. Then, by a continuity argument, $\tilde{\gamma}$ is the unique lift of $\gamma = \pi(\tilde{\gamma})$ passing through the finite set of points $C \cup \{\xi_1, \ldots, \xi_{k-1}\}$. $\square$

Remark 3.2 In order to get uniqueness, it is necessary to fix as many supporting price vectors as many connected components crossed by the path on $E(r)$. This is due to the potential discontinuity which may arise when $\gamma(t)$ crosses singular economies. If it is reasonable to assume that social planner's intervention to fix prices is costly, a minimal path has also another economic meaning since it allows to minimize the number of supporting price vectors to be fixed.

Remark 3.3 This construction can be extended to a smooth setting and, hence, potentially extended to construct an algorithm to find an optimal redistribution policy by applying the result by [7], where the path is a minimal geodesic according to a Riemannian metric constructed on $E(r)$. In [7] it is shown that there exists a Riemannian metric on the equilibrium manifold $E(r)$ such that a minimal geodesic connecting two (sufficiently close) regular equilibria intersects the set of critical equilibria in a finite number of points. Roughly speaking, the idea is the following. Given an initial and final economy ω and ω', the social planner wants to redistribute endowments across consumers to move the economy toward the target ω'. One constructs a Riemannian metric on $E(r)$ which embodies the objectives of the redistribution policy. This metric can be regarded as an algorithm to calculate the optimal path (geodesic) $\tilde{\gamma} : [0,1] \to E(r)$ connecting two regular equilibria $x = (p, \omega)$ and $y = (p', \omega')$. The redistribution policy is then $\pi(\tilde{\gamma}(t)) = \gamma(t)$, which represents the optimal choice among the infinite policies joining ω and ω'. Under this perspective, Theorems 2.4 and 3.1 can provide foundations to apply the tools developed by [7, 8] to construct an optimal policy. Finally, observe that this policy, which is optimal from the perspective of the equilibrium manifold, can appear quite counterintuitive in the space of the endowments (for example, its self-intersections can be homeomorphic to intervals).

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