# THE BISYMPLECTOMORPHISM GROUP OF A BOUNDED SYMMETRIC DOMAIN

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Abstract. We determine the group of diffeomorphisms of a bounded symmetric domain, which preserve simultaneously the hyperbolic and the flat symplectic form.

## Contents



## Introduction

Let $\Omega$ be an Hermitian bounded symmetric domain in a complex vector space $V$; we always assume that $\Omega$ is given in its circled realization. The domain $\Omega$ is endowed with two natural symplectic forms: the flat form $\omega_0$ and the hyperbolic form $\omega_-$. In a similar way, the ambient vector space $V$ is also endowed with two natural symplectic forms: the Fubini-Study form $\omega_+$ and the flat form $\omega_0$ (see Section 1 for the definition of $\omega_-$, $\omega_0$, $\omega_+$). It has been shown in [1] that there exists a diffeomorphism $F : \Omega \to V$ such that

$$F^*\omega_0 = \omega_-, \quad F^*\omega_+ = \omega_0. \tag{0.1}$$

This map is the same that was used in [4], with the property

$$F^*\omega_+^n = \omega_0^n$$

($n = \dim_{\mathbb{C}} V$), to show that the flat volume of a bounded symmetric domain, with some natural normalization, is equal to the degree of a canonical projective embedding of its compact dual. In the one-dimensional case, where $V = \mathbb{C}$ and $\Omega$ is the unit disc $\Delta$, this map is simply $f : \Delta \to \mathbb{C}$ given by

$$f(z) = \frac{z}{\sqrt{1 - |z|^2}}. \tag{0.2}$$

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Even in this case, it does not seem that the property (0.1) had been noticed before. In the general case, the map $F$ may be defined by

$$F(z) = B(z,z)^{-1/4}z, \tag{0.3}$$

where $B(z,z)$ denotes the Bergman operator of the Jordan triple structure on $V$ associated to $\Omega$; it may also be defined by functional calculus in Hermitian positive Jordan triples. In view of the property (0.1), the map $F$ is called map of (bi)symplectic duality. The part $F^*\omega_0 = \omega_-$ tells that $F$ is a realization of the isomorphism of Mc Duff [3] for the bounded symmetric domain $\Omega$; but the property (0.1), which involves two pairs of symplectic forms, is much stronger. In order to determine all diffeomorphisms $F : \Omega \to V$ verifying (0.1), we determine the group of bisymplectomorphims of $\Omega$, that is, diffeomorphisms $\phi : \Omega \to \Omega$ such that

$$\phi^*\omega_0 = \omega_0, \quad \phi^*\omega_- = \omega_-. \tag{0.4}$$

This group is infinite-dimensional, but is the semi-direct product of the compact group $K$ of linear automorphisms of $\Omega$ with an infinite-dimensional Abelian group of "radial circular diffeomorphisms" (Theorem 4); this is the main result of this article and may be considered as a kind of Schwarz lemma.

The plan of this article is as follows. In Section 1, we recall known facts about Jordan triple systems associated to bounded complex symmetric domains (see mainly [2]); the only result we could not find in the literature is Proposition 1, which describes the tangent space of the manifold of frames (the "Fürstenberg-Satake boundary" of $\Omega$) in terms of Peirce decomposition in Jordan triples. In Section 2, we compute the symplectic forms $\omega_0$ and $\omega_-$ using spectral decomposition in Jordan triples, which is the appropriate generalization of polar coordinates. From this, we derive in Section 3 a simple proof of the property (0.1), different from the proof given in [1]. Section 4 is devoted to the study and characterization of bisymplectomorphisms.

## 1. Hermitian positive Jordan triples

Let $\Omega$ be a bounded symmetric domain in a finite dimensional complex vector space $V$. *We will always consider such a domain in its (unique up to linear isomorphism) circled realization.* Consider the associated Jordan triple system $(V, \{\ ,\ ,\ \})$. For basic facts about Hermitian positive Jordan triples and their correspondence with complex symmetric domains, see [2], [4]. We recall hereunder those which will be used here.

1.1. **Definitions and notations.** Consider the operators on the Jordan triple $V$ defined by

$$D(x,y)z = \{x,y,z\}, \tag{1.1}$$

$$Q(x,z)y = \{x,y,z\}, \tag{1.2}$$

$$Q(x,x) = 2Q(x), \tag{1.3}$$

$$B(x,y) = \mathrm{id}_V - D(x,y) + Q(x)Q(y). \tag{1.4}$$

The operators $D(x,y)$ and $B(x,y)$ are $\mathbb{C}$-linear, the operator $Q(x)$ is $\mathbb{C}$-antilinear. The hermitian form

$$(u \mid v) = \operatorname{tr} D(u,v) \tag{1.5}$$

is a Hermitian scalar product on $V$; with respect to this product, $D(x,x)$ and $B(x,x)$ are self-adjoint.

For $z \in V$, the *odd powers* $z^{(2p+1)}$ of $z$ in the Jordan triple system $V$ are defined by

$$z^{(1)} = z, \qquad z^{(2p+1)} = Q(z) z^{(2p-1)}. \tag{1.6}$$

An element $e \in V$ is called *tripotent* if $e \neq 0$ and $e^{(3)} = e$. Two tripotents $e_1, e_2$ are called *(strongly) orthogonal* if $D(e_1, e_2) = 0$. A tripotent element is called *minimal*, or *primitive*, if it is not the sum of two orthogonal tripotents. A tripotent element $e$ is called *maximal* if there is no tripotent orthogonal to $e$.

1.2. **Spectral decomposition.** Each element $z \in V$ has a unique *spectral decomposition*

$$z = \lambda_1 e_1 + \cdots + \lambda_s e_s \qquad (\lambda_1 > \cdots > \lambda_s > 0), \tag{1.7}$$

where $(e_1, \ldots, e_s)$ is a sequence of pairwise orthogonal tripotents. The integer $s = \operatorname{rk} z$ is called the *rank* of $z$. Let $V_z^+$ be the $\mathbb{R}$-subspace of $V$ generated by the odd powers $z, \ldots, z^{(2p+1)}, \ldots$ and $V_z = V_z^+ \oplus \mathrm{i}\, V_z^+$ the $\mathbb{C}$-subspace generated by the odd powers of $z$. Then $\operatorname{rk} z = \dim_{\mathbb{R}} V_z^+$ and $(e_1, \ldots, e_s)$ is an $\mathbb{R}$-basis of $V_z^+$. The *rank* of $V$ is $r = \operatorname{rk} V = \max\{\operatorname{rk} z \mid z \in V\}$; elements $z$ such that $\operatorname{rk} z = \operatorname{rk} V$ are called *regular*. If $z \in V$ is regular, with spectral decomposition

$$z = \lambda_1 e_1 + \cdots + \lambda_r e_r \qquad (\lambda_1 > \cdots > \lambda_r > 0), \tag{1.8}$$

then $(e_1, \ldots, e_r)$ is a *(Jordan) frame of* $V$, that is, a maximal sequence of pairwise orthogonal minimal tripotents.

1.3. **Peirce decompositions.** Let $e \in V$ be a tripotent. Then the eigenvalues of $D(e,e)$ are contained in $\{0,1,2\}$. Define the *Peirce subspaces* of $e$ as

$$V_i(e) = \{z \in V \mid D(e,e)z = iz\} \qquad (i \in \{0,1,2\}). \tag{1.9}$$

The decomposition

$$V = V_0(e) \oplus V_1(e) \oplus V_2(e) \tag{1.10}$$

is called the *Peirce decomposition* of $V$ w.r. to $e$. A tripotent $e$ is maximal if $V_0(e) = 0$, minimal if $V_2(e) = \mathbb{C}e$. The Peirce subspaces compose according to the law

$$\{V_i(e), V_j(e), V_k(e)\} \subset V_{i-j+k}(e), \tag{1.11}$$

where $V_m(e) = 0$ if $m \notin \{0,1,2\}$; in particular, Peirce subspaces are Jordan subsystems of $V$. The $\mathbb{C}$-antilinear operator $Q(e)$ is 0 on $V_0(e) \oplus V_1(e)$; its restriction to $V_2(e)$ is involutive. Let

$$V_2^+(e) = \{v \in V \mid D(e,e)v = 2v,\ Q(e)v = v\}. \tag{1.12}$$

Then the decomposition of $V_2(e)$ into (real) eigenspaces of $Q(e)$ is

$$V_2(e) = V_2^+(e) \oplus \mathrm{i}\, V_2^+(e). \tag{1.13}$$

Let $\mathbf{e} = (e_1, \ldots, e_s)$ be a sequence of pairwise orthogonal tripotents. Then the operators $D(e_j, e_j)$, $1 \leq j \leq s$ commute and have the common eigenspaces

$$V_{jj}(\mathbf{e}) = V_2(e_j) \qquad (1 \leq j \leq s),$$
$$V_{jk}(\mathbf{e}) = V_1(e_j) \cap V_1(e_k) \qquad (1 \leq j < k \leq s),$$

$$V_{0j}(\mathbf{e}) = V_1(e_j) \cap \bigcap_{k \neq j} V_0(e_k) \qquad (1 \leq j \leq s), \tag{1.14}$$
$$V_{00}(\mathbf{e}) = \bigcap_k V_0(e_k)$$

(some of these subspaces may be 0). The decomposition

$$V = \bigoplus_{0 \leq j \leq k \leq s} V_{jk}(\mathbf{e}) \tag{1.15}$$

is called the *simultaneous Peirce decomposition* of $V$ w.r.t. to $\mathbf{e}$. If

$$z = \lambda_1 e_1 + \cdots + \lambda_s e_s \qquad (\lambda_j \in \mathbb{C}),$$
$$e = e_1 + \cdots + e_s$$

and $v \in V_{jk}(\mathbf{e})$, then

$$D(z,z)v = \left( |\lambda_j|^2 + |\lambda_k|^2 \right) v \tag{1.16}$$
$$Q(z)v = \lambda_j \lambda_k Q(e)v, \tag{1.17}$$
$$Q(z)Q(z) = |\lambda_j \lambda_k|^2 \, v, \tag{1.18}$$
$$B(z,z)v = \left(1 - |\lambda_j|^2\right) \left(1 - |\lambda_k|^2\right) v, \tag{1.19}$$
$$B(z,-z)v = \left(1 + |\lambda_j|^2\right) \left(1 + |\lambda_k|^2\right) v, \tag{1.20}$$

where $\lambda_0 = 0$. So the $V_{jk}(\mathbf{e})$'s are eigenspaces for all the operators $D(z,z)$, $B(z,z)$, $B(z,-z)$, $z = \lambda_1 e_1 + \cdots + \lambda_s e_s$.

1.4. **Hermitian metrics and symplectic forms.** Let $V$ be a Hermitian positive Jordan triple and let $\Omega$ be the associated Hermitian bounded symmetric domain. Let

$$h_0(z)(u,v) = (u \mid v) = \operatorname{tr} D(u,v) \tag{1.21}$$

be the flat Hermitian metric and let

$$\omega_0(z) = \frac{\mathrm{i}}{2} \partial \overline{\partial} \, (z \mid z) \, ,$$
$$\omega_0(z)(u,v) = \frac{\mathrm{i}}{2} \left( (u \mid v) - (v \mid u) \right)$$

be the associated *flat symplectic form*. If $\Omega$ is endowed with the volume form $\omega_0^n$ ($n = \dim_{\mathbb{C}} V$), the Bergman kernel of $\Omega$ is

$$K(x,y) = \frac{C}{\det B(x,y)},$$

with $C = \left( \int_\Omega \omega_0^n \right)^{-1}$. The *Bergman metric* of $\Omega$ is

$$h_-(z)(u,v) = \partial_u \overline{\partial}_v \ln K(z,z) = -\partial_u \overline{\partial}_v \ln \det B(z,z).$$

It satisfies the relation

$$h_-(z)(u,v) = h_0 \left( B(z,z)^{-1} u, v \right). \tag{1.22}$$

In view of this relation, $B(z,z)$ is called the *Bergman operator* at $z \in \Omega$.

The *hyperbolic symplectic form* of $\Omega$, associated to the Bergman metric, is defined by

$$\omega_-(z) = -\frac{\mathrm{i}}{2}\partial\overline{\partial}\ln\det B(z,z). \tag{1.23}$$

From (1.22), it results that the forms $\omega_0$ and $\omega_-$ are related with the Bergman operator by

$$\omega_-(z)(u,v) = \omega_0(B(z,z)^{-1}u,v), \tag{1.24}$$

for $z \in \Omega$ and $u, v \in T_z\Omega$.

The (generalized) *Fubini-Study* metric on $V$ is defined by

$$h_+(z)(u,v) = \partial_u\overline{\partial}_v \ln\det B(z,-z).$$

The associated Kähler form is

$$\omega_+(z) = \frac{\mathrm{i}}{2}\partial\overline{\partial}\ln\det B(z,-z).$$

It is related to the flat form by

$$\omega_+(z)(u,v) = \omega_0(B(z,-z)^{-1}u,v). \tag{1.25}$$

1.5. **Polar coordinates.** Let $M$ be the set ot tripotents elements of the positive Jordan triple $V$. Then $M$ is a compact submanifold of $V$ (with connected components of different dimensions). At $e \in M$, the tangent space $T_eM$ and the normal space $N_eM$ to $M$ are

$$T_eM = \mathrm{i}\,V_2^+(e) \oplus V_1(e), \tag{1.26}$$

$$N_eM = V_0(e) \oplus V_2^+(e) \tag{1.27}$$

(see [2], Theorem 5.6).

The *height* $k$ of a tripotent element $e$ is the maximal length of a decomposition $e = e_1 + \cdots + e_k$ into a sum of pairwise orthogonal (minimal) tripotents. Minimal tripotents have height 1, maximal tripotents have height $r = \operatorname{rk} V$. Denote by $M_k$ the set of tripotents of height $k$. *If $V$ is simple (that is, if $\Omega$ is irreducible)*, the submanifolds $M_k$ are the connected components of $M$.

The set $\mathcal{F}$ of frames (also called Fürstenberg-Satake boundary of $\Omega$):

$$\mathcal{F} = \{(e_1,\ldots,e_r) \mid e_j \in M_1,\ e_j \perp e_k\ (1 \le j < k \le r)\}, \tag{1.28}$$

(where $e_j \perp e_k$ means orthogonality of tripotents: $D(e_j,e_k) = 0$ or equivalently $\{e_j,e_j,e_k\} = 0$) is a submanifold of $V^r$. The following proposition provides a description of the tangent space of $\mathcal{F}$.

**Proposition 1.** *Let $\mathbf{e} = (e_1,\ldots,e_r) \in \mathcal{F} \subset V^r$ and $e = e_1 + \cdots + e_r$. Then $(v_1,\ldots,v_r) \in T_{\mathbf{e}}\mathcal{F}$ if and only if*

$$v_j = \mathrm{i}\,\alpha_j e_j + v_{j0} + \sum_{\substack{1\le k\le r\\ k\ne j}} v_{jk} \qquad (1 \le j \le r), \tag{1.29}$$

*where $\alpha_j \in \mathbb{R}$, $v_{j0} \in V_{0j}(\mathbf{e})$, $v_{jk} \in V_{jk}(\mathbf{e}) = V_{kj}(\mathbf{e})$ and*

$$Q(e)v_{jk} = -v_{kj} \qquad (1 \le j < k \le r). \tag{1.30}$$

*Proof.* Let $(v_1, \dots, v_r) \in T_{\mathbf{e}}\mathcal{F}$. As $e_j$ are minimal tripotents, we have

$$v_j = T_{e_j} M_1 = \mathrm{i}\,\mathbb{R}e_j \oplus V_1(e_j) = \mathrm{i}\,\mathbb{R}e_j \oplus V_{0j}(\mathbf{e}) \oplus \bigoplus_{\substack{1\le k\le r\\ k\ne j}} V_{jk}(\mathbf{e}),$$

which shows that $v_j$ has the form (1.29).

The orthogonality conditions in a frame are

$$\{e_j, e_j, e_k\} = 0 \qquad (1 \le j < k \le r).$$

Differentiating these conditions yields

$$\{v_j, e_j, e_k\} + \{e_j, v_j, e_k\} + \{e_j, e_j, v_k\} = 0 \qquad (1 \le j < k \le r).$$

As $D(e_j, e_k) = 0$, this condition is reduced to

$$Q(e_j, e_k)v_j + D(e_j, e_j)v_k = 0 \qquad (1 \le j < k \le r). \tag{1.31}$$

Let

$$v_j = \mathrm{i}\,\alpha_j e_j + v_{j0} + \sum_{\substack{1\le m\le r\\ m\ne j}} v_{jm} \qquad (1 \le j \le r).$$

Then

$$D(e_j, e_j)v_k = v_{kj}, \quad Q(e_j, e_k)e_j = 0,$$
$$Q(e_j, e_k) = Q(e_j + e_k) - Q(e_j) - Q(e_k),$$

and we get from (1.17)

$$Q(e_j + e_k)v_{jm} = \delta_k^m Q(e)v_{jm}, \quad Q(e_j)v_{jm} = 0, \quad Q(e_k)v_{jm} = 0,$$
$$Q(e_j, e_k)v_j = Q(e)v_{jk}.$$

This shows that the conditions (1.31) are equivalent to (1.30). □

Comparing the description of $T_{\mathbf{e}}\mathcal{F}$ with the simultaneous Peirce decomposition of $V$ w.r.t. to $\mathbf{e}$, it is easily checked that $T_{\mathbf{e}}\mathcal{F}$ is a real vector space of dimension $2n - r$, where $n = \dim_{\mathbb{C}} V$. This implies that the map

$$\begin{aligned} \{\lambda_1 > \dots > \lambda_r > 0\} \times \mathcal{F} &\to V_{\mathrm{reg}} \\ ((\lambda_1, \dots, \lambda_r), (e_1, \dots, e_r)) &\mapsto \sum \lambda_j e_j \end{aligned} \tag{1.32}$$

is a diffeomorphism onto the set $V_{\mathrm{reg}}$ of regular elements of $V$; its restriction

$$\{1 > \lambda_1 > \dots > \lambda_r > 0\} \times \mathcal{F} \to \Omega_{\mathrm{reg}}$$

is a diffeomorphism onto the set $\Omega_{\mathrm{reg}}$ of regular elements of $\Omega$. This map plays the same role as polar coordinates in rank one.

1.6. **Functional calculus.** Using the spectral decomposition, it is possible to associate to an *odd* function $f : (-1, 1) \to \mathbb{C}$ (resp. $f : \mathbb{R} \to \mathbb{C}$) a "radial" map $F : \Omega \to V$ (resp. $F : V \to V$) in the following way. Let $z \in V$ and let

$$z = \lambda_1 e_1 + \dots + \lambda_k e_k, \quad \lambda_1 > \dots > \lambda_k > 0$$

be the spectral decomposition of $z$. Define the map $F = \widetilde{f}$ associated to $f$ by

$$F(z) = f(\lambda_1)e_1 + \dots + f(\lambda_k)e_k. \tag{1.33}$$

Since $f$ is odd, it follows from properties of tripotents that, for

$$z = \lambda_1 e_1 + \dots + \lambda_r e_r,$$

where $\mathbf{e} = (e_1, \ldots, e_r)$ is a frame and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, we have

$$F(z) = f(\lambda_1)e_1 + \cdots + f(\lambda_r)e_r. \tag{1.34}$$

If $f$ is continuous, then $F$ is continuous.

If

$$f(t) = \sum_{k=0}^{N} a_k t^{2k+1}$$

is a polynomial, then $F$ is the map defined by

$$F(z) = \sum_{k=0}^{N} a_k z^{(2k+1)} \qquad (z \in V). \tag{1.35}$$

If $f$ is analytic, then $F$ is real-analytic; if $f$ is given near 0 by

$$f(t) = \sum_{k=0}^{\infty} a_k t^{2k+1},$$

then $F$ has the Taylor expansion near $0 \in V$:

$$F(z) = \sum_{k=0}^{\infty} a_k z^{(2k+1)}. \tag{1.36}$$

If $f$ is $C^\infty$, then $F$ is also $C^\infty$.

## 2. Symplectic forms in polar coordinates

2.1. We compute the flat symplectic form $\omega_0$ in the "polar coordinates"

$$\begin{aligned} &\{\lambda_1 > \cdots > \lambda_r > 0\} \times \mathcal{F} \to V_{\mathrm{reg}} \\ &((\lambda_1, \ldots, \lambda_r), (e_1, \ldots, e_r)) \mapsto \sum \lambda_j e_j. \end{aligned}$$

If $(z_1, \ldots, z_n)$ $(n = \dim V)$ are orthonormal coordinates for the Hermitian product $(u \mid v)$, then

$$\omega_0 = \mathrm{i} \sum_{m=1}^{n} \mathrm{d}\, z_m \wedge \mathrm{d}\, \overline{z}_m.$$

From

$$z = \sum_{j=1}^{r} \lambda_j e_j,$$

we have

$$\begin{aligned} \mathrm{d}\, z_m \wedge \mathrm{d}\, \overline{z}_m = &\sum_{j,k=1}^{r} \lambda_j \lambda_k \,\mathrm{d}\, e_{jm} \wedge \mathrm{d}\, \overline{e}_{km} + \sum_{j,k=1}^{r} e_{jm} \overline{e}_{km} \,\mathrm{d}\, \lambda_j \wedge \mathrm{d}\, \lambda_k \\ &+ \sum_{j,k=1}^{r} \lambda_k \,\mathrm{d}\, \lambda_j \wedge \left(e_{jm} \,\mathrm{d}\, \overline{e}_{km} - \overline{e}_{km} \,\mathrm{d}\, e_{jm}\right). \end{aligned}$$

Using $(e_j \mid e_k) = \delta_{jk}$, we get

$$\omega_0 = \sum_{j,k=1}^{r} \lambda_j \lambda_k \omega_{jk} + 2 \sum_{j,k=1}^{r} \lambda_k \,\mathrm{d}\, \lambda_j \wedge \eta_{jk}, \tag{2.1}$$

where

$$\eta_{jk} = \frac{\mathrm{i}}{2} \sum_{m=1}^{n} \left( e_{jm} \,\mathrm{d}\, \overline{e}_{km} - \overline{e}_{km} \,\mathrm{d}\, e_{jm} \right) \Big|_{\mathcal{F}} = \mathrm{i} \sum_{m=1}^{n} e_{jm} \,\mathrm{d}\, \overline{e}_{km} \Big|_{\mathcal{F}}, \tag{2.2}$$

$$\omega_{jk} = \mathrm{i} \sum_{m=1}^{n} \mathrm{d}\, e_{jm} \wedge \mathrm{d}\, \overline{e}_{km} \Big|_{\mathcal{F}} = \mathrm{d}\, \eta_{jk}. \tag{2.3}$$

We compute $\eta_{jk}$ and $\omega_{jk}$ using the description of $T_{\mathbf{e}}\mathcal{F}$ given in Proposition 1. Let $v, w \in T_{\mathbf{e}}\mathcal{F}$ with $v = (v_1, \dots, v_r)$, $w = (w_1, \dots, w_r)$,

$$\begin{aligned}
v_j &= \mathrm{i}\,\alpha_j e_j + v_{j0} + \sum_{\substack{1 \le m \le r \\ m \ne j}} v_{jm} \qquad (1 \le j \le r), \\
w_j &= \mathrm{i}\,\beta_j e_j + w_{j0} + \sum_{\substack{1 \le m \le r \\ m \ne j}} w_{jm} \qquad (1 \le j \le r), \\
&\alpha_j \in \mathbb{R}, \quad v_{j0} \in V_{0j}(\mathbf{e}), \quad v_{jk} \in V_{jk}(\mathbf{e}), \\
&\beta_j \in \mathbb{R}, \quad w_{j0} \in V_{0j}(\mathbf{e}), \quad w_{jk} \in V_{jk}(\mathbf{e}), \\
&Q(e) v_{jk} = -v_{kj}, \quad Q(e) w_{jk} = -w_{kj} \qquad (1 \le j < k \le r).
\end{aligned}$$

Then, as the Peirce subspaces are orthogonal w.r. to $(\ |\ )$, we deduce from (2.2)-(2.3)

$$\begin{aligned}
\eta_{jj}(\mathbf{e})(v) &= \mathrm{i}\,(e_j \mid v_j) = \alpha_j, \qquad (2.4)\\
\eta_{jk}(\mathbf{e})(v) &= \mathrm{i}\,(e_j \mid v_k) = 0 \qquad (j \ne k), \\
\omega_{jk} &= \mathrm{d}\,\eta_{jk} = 0 \qquad (j \ne k)
\end{aligned}$$

and

$$\begin{aligned}
\omega_{jj}(\mathbf{e})(v, w) &= \frac{\mathrm{i}}{2} \left( (v_j \mid w_j) - (w_j \mid v_j) \right) \\
&= \langle v_{j0} \mid w_{j0} \rangle + \sum_{\substack{1 \le m \le r \\ m \ne j}} \langle v_{jm} \mid w_{jm} \rangle, \qquad (2.5)
\end{aligned}$$

where $\langle\ |\ \rangle$ denotes the symplectic product

$$\langle x \mid y \rangle = \frac{\mathrm{i}}{2} \left( (x \mid y) - (y \mid x) \right). \tag{2.6}$$

Finally, we have

$$\omega_0 = \sum_{j=1}^{r} \lambda_j^2 \omega_{jj} + 2 \sum_{j=1}^{r} \lambda_j \,\mathrm{d}\, \lambda_j \wedge \eta_{jj}, \tag{2.7}$$

with $\omega_{jj}$ and $\eta_{jj}$ given by (2.5), (2.4). The expression (2.5) shows that the $\omega_{jj}$ $(1 \le j \le r)$ are linearly independent at each point $\mathbf{e} \in \mathcal{F}$.

As

$$\begin{aligned}
2 \left( Q(z)x \mid y \right) &= \left( D(z, x) z \mid y \right) = \left( z \mid D(x, z) y \right) \\
&= \left( z \mid D(y, z) x \right) = \left( D(z, y) z \mid x \right) = 2 \left( Q(z) y \mid x \right),
\end{aligned}$$

the $Q$ operator satisfies

$$\left( Q(z)x \mid y \right) = \left( Q(z) y \mid x \right), \tag{2.8}$$

$$\langle Q(z) x \mid y \rangle = - \langle x \mid Q(z) y \rangle \tag{2.9}$$

for all $z, x, y \in V$.

For $v, w \in T_{\mathbf{e}}\mathcal{F}$, using

$$Q(e)v_{jk} = -v_{kj}, \quad Q(e)w_{jk} = -w_{kj} \qquad (1 \leq j \neq k \leq r),$$

we have then

$$\langle v_{jk} \mid w_{jk} \rangle = \langle Q(e)v_{kj} \mid Q(e)w_{kj} \rangle = -\left\langle v_{kj} \mid Q(e)^2 w_{kj} \right\rangle,$$

that is,

$$\langle v_{jk} \mid w_{jk} \rangle = -\langle v_{kj} \mid w_{kj} \rangle, \tag{2.10}$$

as $w_{kj} \in V_2(e)$ and $Q(e)$ is involutive on $V_2(e)$. In view of (2.10), the flat symplectic form $\omega_0$ in polar coordinates may be rewritten

$$\omega_0 = \sum_{j=1}^{r} \lambda_j^2 \theta_{j0} + \sum_{\substack{j,k \\ 1 \leq j < k \leq r}} \left(\lambda_j^2 - \lambda_k^2\right) \theta_{jk} + 2 \sum_{j=1}^{r} \lambda_j \, \mathrm{d}\, \lambda_j \wedge \eta_{jj}, \tag{2.11}$$

where the $\eta_{jj}$'s are defined by (2.4), and the $\theta_{j0}$'s and $\theta_{jk}$'s by

$$\theta_{j0}(\mathbf{e})(v, w) = \langle v_{j0} \mid w_{j0} \rangle, \qquad (1 \leq j \leq r), \tag{2.12}$$

$$\theta_{jk}(\mathbf{e})(v, w) = \langle v_{jk} \mid w_{jk} \rangle \qquad (1 \leq j < k \leq r). \tag{2.13}$$

for $v, w \in T_{\mathbf{e}}\mathcal{F}$. Note that these forms are (pull-backs of) forms on the manifold of frames $\mathcal{F}$ and that the $\theta_{j0}$'s are 0 when the domain is of tube type.

2.2. We compute now the hyperbolic form $\omega_-$ and the Fubini-Study form $\omega_+$ in polar coordinates.

Using

$$B(z, z)^{-1} v_{j0} = \left(1 - \lambda_j^2\right)^{-1} v_{j0} \qquad (1 \leq j \leq r),$$

$$B(z, z)^{-1} v_{jk} = \left(1 - \lambda_j^2\right)^{-1} \left(1 - \lambda_k^2\right)^{-1} v_{jk} \qquad (1 \leq j < k \leq r),$$

$$B(z, z)^{-1} e_j = \left(1 - \lambda_j^2\right)^{-2} e_j \qquad (1 \leq j \leq r)$$

in (2.11), (2.12), (2.13), we obtain

$$\eta_{jj}(\mathbf{e})(B(z, z)^{-1} v) = \left(1 - \lambda_j^2\right)^{-2} \eta_{jj}(\mathbf{e})(v), \tag{2.14}$$

$$\theta_{j0}(\mathbf{e})(B(z, z)^{-1} v, w) = \left(1 - \lambda_j^2\right)^{-1} \theta_{j0}(\mathbf{e})(v, w), \tag{2.15}$$

$$\theta_{jk}(\mathbf{e})(B(z, z)^{-1} v, w) = \left(1 - \lambda_j^2\right)^{-1} \left(1 - \lambda_k^2\right)^{-1} \theta_{jk}(\mathbf{e})(v, w), \tag{2.16}$$

for $z = \lambda_1 e_1 + \cdots + \lambda_r e_r$ and $v, w \in T_{\mathbf{e}}\mathcal{F}$. From (1.24):

$$\omega_-(z)(v, w) = \omega_0(B(z, z)^{-1} v, w)$$

and the expression of $\omega_0$ in polar coordinates, we have

$$\omega_- = \sum_{j=1}^{r} \frac{\lambda_j^2}{1 - \lambda_j^2} \theta_{j0} + \sum_{\substack{j,k \\ 1 \leq j < k \leq r}} \frac{\lambda_j^2 - \lambda_k^2}{\left(1 - \lambda_j^2\right)\left(1 - \lambda_k^2\right)} \theta_{jk} + 2 \sum_{j=1}^{r} \frac{\lambda_j \, \mathrm{d}\, \lambda_j}{\left(1 - \lambda_j^2\right)^2} \wedge \eta_{jj}. \tag{2.17}$$

In the same way, the Fubini-Study symplectic form on $V$ is

$$\omega_+ = \sum_{j=1}^{r} \frac{\lambda_j^2}{1+\lambda_j^2}\theta_{j0} + \sum_{\substack{j,k\\ 1\leq j<k\leq r}} \frac{\lambda_j^2+\lambda_k^2}{\left(1+\lambda_j^2\right)\left(1+\lambda_k^2\right)}\theta_{jk} + 2\sum_{j=1}^{r} \frac{\lambda_j \,\mathrm{d}\,\lambda_j}{\left(1+\lambda_j^2\right)^2}\wedge \eta_{jj}. \tag{2.18}$$

## 3. Symplectic duality

Consider the real analytic maps $f = ]-1,1[ \to \mathbb{R}$ and $g : \mathbb{R} \to ]-1,1[$, inverse of each other, defined by

$$f(t) = \frac{t}{\sqrt{1-t^2}} \qquad (-1<t<1), \tag{3.1}$$

$$g(t) = \frac{t}{\sqrt{1+t^2}} \qquad (t\in\mathbb{R}). \tag{3.2}$$

By the functional calculus described in Subsection 1.6, we associate to these maps the real analytic diffeomorphisms, also inverse of each other

$$F = \widehat{f} : \Omega \to V,$$
$$G = \widehat{g} : V \to \Omega,$$

where $\Omega$ is the bounded symmetric domain associated to the Jordan triple $V$. If $\mathbf{e} = (e_1,\ldots,e_r)$ is a frame and $z = \sum_{j=1}^{r} \lambda_j e_j$, then

$$F(z) = \sum_{j=1}^{r} \frac{\lambda_j}{\sqrt{1-\lambda_j^2}} e_j \qquad (z\in\Omega), \tag{3.3}$$

$$G(z) = \sum_{j=1}^{r} \frac{\lambda_j}{\sqrt{1+\lambda_j^2}} e_j \qquad (z\in V). \tag{3.4}$$

Using (1.16)-(1.20), the maps $F$ and $G$ may also be defined by

$$F(z) = B(z,z)^{-1/4}z = \left(\mathrm{id}_V - \frac{1}{2}D(z,z)\right)^{-1/2} z \qquad (z\in\Omega), \tag{3.5}$$

$$G(z) = B(z,-z)^{-1/4}z = \left(\mathrm{id}_V - \frac{1}{2}D(z,-z)\right)^{-1/2} z \qquad (z\in V). \tag{3.6}$$

The following theorem is the main result of [1]. We give here a different and simpler proof, using the expression of the symplectic forms $\omega_0, \omega_-, \omega_+$ in generalized polar coordinates.

**Theorem 1.** *(Symplectic duality)*

$$F^*\omega_0 = \omega_-, \quad F^*\omega_+ = \omega_0, \tag{3.7}$$

$$G^*\omega_0 = \omega_+, \quad G^*\omega_- = \omega_0. \tag{3.8}$$

*Proof.* In polar coordinates, the map $F$ is written

$$((\lambda_1,\ldots,\lambda_r),\mathbf{e}) \mapsto ((\mu_1,\ldots,\mu_r),\mathbf{e})$$

with

$$\mu_j = \frac{\lambda_j}{\sqrt{1-\lambda_j^2}}. \tag{3.9}$$

As, by (2.11),

$$\omega_0 = \sum_{j=1}^r \lambda_j^2 \theta_{j0} + \sum_{\substack{j,k \\ 1\le j<k\le r}} \left(\lambda_j^2 - \lambda_k^2\right)\theta_{jk} + 2\sum_{j=1}^r \lambda_j \,\mathrm{d}\,\lambda_j \wedge \eta_{jj},$$

we obtain, using (3.9),

$$F^*\omega_0 = \sum_{j=1}^r \frac{\lambda_j^2}{1-\lambda_j^2}\theta_{j0} + \sum_{\substack{j,k \\ 1\le j<k\le r}} \left(\frac{\lambda_j^2}{1-\lambda_j^2} - \frac{\lambda_k^2}{1-\lambda_k^2}\right)\theta_{jk} + \\ + 2\sum_{j=1}^r \frac{\lambda_j \,\mathrm{d}\,\lambda_j}{\left(1-\lambda_j^2\right)^2} \wedge \eta_{jj},$$

which, compared to (2.17), gives $F^*\omega_0 = \omega_-$ on the open dense subset $\Omega_{\mathrm{reg}}$ of regular elements, and by continuity on all of $\Omega$.

The relation $G^*\omega_0 = \omega_+$ is proved along the same lines. The relations $F^*\omega_+ = \omega_0$ and $G^*\omega_- = \omega_0$ follow, as $F$ and $G$ are inverse of each other. □

In view of this theorem, the map $F$ (or the map $G = F^{-1}$) is called the *duality map*.

*Example* 1. (Type $I_{1,1}$) Here $V = \mathbb{C}$, $\Omega$ is the unit disc,

$$\omega_0 = \frac{\mathrm{i}}{2}\,\mathrm{d}\,z \wedge \mathrm{d}\,\overline{z}, \quad \omega_- = \frac{\mathrm{i}}{2}\frac{\mathrm{d}\,z\wedge \mathrm{d}\,\overline{z}}{(1-z\overline{z})^2}, \quad \omega_+ = \frac{\mathrm{i}}{2}\frac{\mathrm{d}\,z\wedge \mathrm{d}\,\overline{z}}{(1+z\overline{z})^2}.$$

The duality map is

$$F(z) = \frac{z}{\sqrt{1-z\overline{z}}}.$$

*Example* 2. (Type $I_{1,n}$) Here $V = \mathbb{C}^n$ with the Hermitian norm $\|z\|^2 = \sum z_j \overline{z}_j$, $\Omega$ is the unit Hermitian ball,

$$\omega_0 = \frac{\mathrm{i}}{2}\sum \mathrm{d}\,z_j \wedge \mathrm{d}\,\overline{z}_j, \quad \omega_- = \frac{\omega_0}{\left(1-\|z\|^2\right)^{n+1}}, \quad \omega_+ = \frac{\omega_0}{\left(1+\|z\|^2\right)^{n+1}}.$$

The duality map is

$$F(z) = \frac{z}{\sqrt{1-\|z\|^2}}.$$

## 4. The bisymplectomorphism group

4.1. In this section we study to which extent a diffeomorphism $F : \Omega \to V$ satisfying the property (3.7) is unique. If two diffeomorphisms $F_1, F_2 : \Omega \to V$ satisfy (3.7), then $F_2 = F_1 \circ f$, where $f : \Omega \to \Omega$ preserves $\omega_0$ and $\omega_-$. This leads us to the following definition.

**Definition 1.** *A* bisymplectomorphism of $\Omega$ *is a diffeomorphism* $f : \Omega \to \Omega$ *which satisfies*

$$f^*\omega_0 = \omega_0, \tag{4.1}$$

$$f^*\omega_- = \omega_-. \tag{4.2}$$

Clearly, bisymplectomorphisms of $\Omega$ form a group, which will be denoted by $\mathcal{B}(\Omega)$ and called the *bisymplectomorphism group* of $\Omega$.

From (1.24), we derive a characterization of $\mathcal{B}(\Omega)$ in terms of the Bergman operator:

**Proposition 2.** *Let $\Omega$ be a bounded symmetric domain. Then a diffeomorphism $f \in \mathrm{Diff}(\Omega)$ is a bisymplectomorphism if and only if it satisfies*

$$f^*\omega_0 = \omega_0, \tag{4.3}$$

$$B\left(f(z), f(z)\right) \circ \mathrm{d}\, f(z) = \mathrm{d}\, f(z) \circ B(z,z) \quad (z \in \Omega). \tag{4.4}$$

Note that the second condition implies that the tangent map $\mathrm{d}\, f(z)$ maps invariant subspaces of $B_z = B(z,z)$ to invariant subspaces of $B_{f(z)}$, and that $B_{f(z)}$ has the same eigenvalues as $B_z$.

*Proof.* Let $z \in \Omega$, $u, v \in T_z\Omega$. We have from (1.24) and (4.1)

$$\begin{aligned} \omega_-(z)(u,v) &= \omega_0(B(z,z)^{-1}u, sv) \\ &= \omega_0(\mathrm{d}\, f(z) B(z,z)^{-1}u, \mathrm{d}\, f(z)v), \\ \left(f^*\omega_-\right)(z)(u,v) &= \omega_-(f(z))(\mathrm{d}\, f(z)u, \mathrm{d}\, f(z)v) \\ &= \omega_0\left(B(f(z), f(z))^{-1}\, \mathrm{d}\, f(z)u, \mathrm{d}\, f(z)v\right), \end{aligned}$$

so that, assuming (4.1), the condition (4.2) is equivalent to

$$\omega_0(\mathrm{d}\, f(z)B(z,z)^{-1}u, \mathrm{d}\, f(z)v) = \omega_0\left(B(f(z), f(z))^{-1}\, \mathrm{d}\, f(z)u, \mathrm{d}\, f(z)v\right)$$

for all $u, v$. As $\omega_0$ is non singular and $\mathrm{d}\, f(z)$ is bijective, this is equivalent to

$$\mathrm{d}\, f(z) \circ B(z,z)^{-1} = B(f(z), f(z))^{-1} \circ \mathrm{d}\, f(z),$$

that is, to (4.4). □

4.2. Here we study diffeomorphisms of $\Omega$ satisfying the condition (4.4). Recall that $B_z = B(z,z) : V \to V$ is a $\mathbb{C}$-linear operator, self-adjoint w.r. to the Hermitian metric $h_0$, positive if $z \in \Omega$. Let $r$ denote the *rank* of $\Omega$ and $V$.

For $z \in \Omega$, consider the spectral decomposition

$$z = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_s e_s, \quad 1 > \lambda_1 > \lambda_2 > \cdots > \lambda_s > 0, \tag{4.5}$$

where $s = \operatorname{rk} z \leq r = \operatorname{rk} V$. An element is called *regular* if $\operatorname{rk} z = \operatorname{rk} V$; for regular elements, which form an open dense subset of $\Omega$, the decomposition (4.5) is the decomposition using generalized polar coordinates.

Let

$$V = \bigoplus_{0 \leq i \leq j \leq s} V_{ij}$$

be the simultaneous Peirce decomposition relative to $(e_1, \ldots, e_s)$. Note that some subspaces $V_{ij}$ may be 0. The operator $B(z,z)$ may only have the eigenvalues

$$(1-\lambda_i^2)^2 \quad (1 \leq i \leq s), \quad (1-\lambda_i^2)(1-\lambda_j^2) \quad (1 \leq i < j \leq s), \tag{4.6}$$

$$(1-\lambda_i^2) \quad (1 \leq i \leq s), \quad 1,$$

which occur respectively on the subspaces

$$V_{ii}, \quad V_{ij}, \quad V_{0i}, \quad V_{00}. \tag{4.7}$$

The relation (4.4) then implies that $B(z,z)$ and $B(f(z),f(z))$ have the same eigenvalues with the same multiplicities. Moreover, *if all eigenvalues in the list (4.6) occurring for non-zero $V_{ij}$ are different*, the non-zero subspaces of the list (4.7) are the eigenspaces of $B(z,z)$ and are mapped by $\mathrm{d}\, f(z)$ to the corresponding eigenspaces of $B(f(z),f(z))$.

**Definition 2.** *Let $\Omega$ be an irreducible bounded symmetric domain and denote by $(V,\{\ ,\ ,\ \})$ be the corresponding Jordan triple. An element $z \in V$ is called* super-regular *if $z$ is regular and if all eigenvalues in the list (4.6), occurring on a non-zero $V_{ij}$, are different.*

If $V$ is of tube type or of rank 1, any regular element $z \in V$ is super-regular. If $V$ is not of tube type, an element $z \in V$ is super-regular if it is regular and if its spectral values satisfy

$$1-\lambda_i^2 \neq \left(1-\lambda_j^2\right)^2 \tag{4.8}$$

for all $(i,j)$, $i<j$. Clearly, super-regular elements form an open dense subset of $V$.

Let $z \in \Omega$ be an element of *rank one*. Then the spectral decomposition of $z$ is

$$z = \lambda_1 e_1, \quad 0 < \lambda_1 < 1,$$

where $e_1$ is a tripotent; the associated Peirce decomposition is

$$V_{11} = V_2(e_1), \quad V_{01} = V_1(e_1), \quad V_{00} = V_0(e_1)$$

and the eigenvalues of $B(z,z)$ on these subspaces are respectively

$$\left(1-\lambda_1^2\right)^2, \quad 1-\lambda_1^2, \quad 1.$$

It then follows that

$$f(z) = \lambda_1 \varepsilon_1,$$

where $\varepsilon_1$ is a tripotent such that $V_2(\varepsilon_1) = \mathrm{d}\, f(z) V_2(e_1)$, which means that $e_1$ and $\varepsilon_1$ have the same height $\dim V_2(e_1) = \dim V_2(\varepsilon_1)$. In particular, if $e_1$ is minimal (resp. maximal), then $\varepsilon_1$ is minimal (resp. maximal).

Let $V_z$ be the $\mathbb{C}$-subspace generated by the odd powers $z, \ldots, z^{(2p+1)}, \ldots$ Then $\dim_{\mathbb{C}} V_z = \operatorname{rk} z \leq r$, and $\dim_{\mathbb{C}} V_z = r$ if and only if $z$ is a regular element. Denote

$$P_z = V_z \cap \Omega.$$

For $z \in \Omega$, $z \neq 0$ with spectral decomposition $z = \sum_{j=1}^k \alpha_j e_j$ ($\alpha_1 > \cdots > \alpha_k > 0$; $k = \operatorname{rk}_V x$), $P_z$ is the $k$-polydisc

$$P_z = \left\{ u = \sum u_j e_j \mid |u_j| < 1 \right\}.$$

Note that if $u \in P_z$ and $\operatorname{rk}_V u = \operatorname{rk}_V z$, then $V_u = V_z$ and $P_u = P_z$.

**Lemma 3.** *Let $f : \Omega \to \Omega$ be a diffeomorphism of $\Omega$ such that*

$$B\left(f(z), f(z)\right) \circ \mathrm{d}\, f(z) = \mathrm{d}\, f(z) \circ B(z,z)$$

*for all $z \in \Omega$. Then for all $z \in \Omega$,*

$$\mathrm{d}\, f(z) V_z = V_{f(z)}. \tag{4.9}$$

*Proof.* Let $z$ be a super-regular element and let

$$z = \alpha_1 e_1 + \cdots + \alpha_r e_r$$

be the spectral decomposition of $z$ in $V$; then $V_z := \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_r$ is the sum of the eigenspaces of $B(z,z)$ relative to the eigenvalues

$$\left(1-\alpha_j^2\right)^2 \quad (1 \leq j \leq r),$$

so that it follows from (4.4) that

$$\mathrm{d}\, f(z) V_z = V_{f(z)}.$$

By continuity, (4.9) holds for all $z \in \Omega$. □

**Proposition 4.** *Let $f : \Omega \to \Omega$ be a diffeomorphism of $\Omega$ such that*

$$B\left(f(z), f(z)\right) \circ \mathrm{d}\, f(z) = \mathrm{d}\, f(z) \circ B(z,z)$$

*for all $z \in \Omega$. Then for any element $z \in \Omega$, $z \neq 0$, we have*

$$f\left(P_z\right) = P_{f(z)}.$$

*Proof.* Let $s = \mathrm{rk}_V\, z$. We already know that $\mathrm{d}\, f(z) V_z = V_{f(z)}$, which implies that $\mathrm{rk}_V\, f(z) = s$. So there exist continuous functions $\beta_j$ such that

$$\mathrm{d}\, f(z) u = \sum_{j=1}^{s} \beta_j(u,z) \left(f(z)\right)^{(2j+1)} \quad (z \in \Omega,\ u \in P_z).$$

Consider a $C^1$ path $\eta : [0,1] \to P_z$ from $\eta(0) = z$ to $\eta(1) = w$; let $g(t) = f(\eta(t))$. Then $g$ satisfies the differential equation

$$\begin{aligned} g'(t) &= \sum_{j=1}^{s} \beta_j(\eta'(t), \eta(t)) \left(g(t)\right)^{(2j+1)}, \\ g(0) &= f(z). \end{aligned} \tag{4.10}$$

Let

$$f(z) = \sum_{k=1}^{s} \alpha_k \varepsilon_k$$

be the spectral decomposition of $f(z)$. Let $h : [0,1] \to \mathbb{C}^s$ be the solution of the differential system

$$\begin{aligned} h_k'(t) &= \sum_{j=1}^{s} \beta_j(\eta'(t), \eta(t)) \left(h_k(t)\right)^{2j+1)} \quad (1 \leq k \leq s), \\ h_k(0) &= \alpha_k. \end{aligned}$$

Then the solution of (4.10) is

$$g(t) = \sum_{k=1}^{s} h_k(t) \varepsilon_k,$$

which shows that $g(1) = f(w)$ belongs to $P_{f(z)}$. □

Let $K$ denote the group of linear automorphisms of $\Omega$.

**Proposition 5.** *Let $f : \Omega \to \Omega$ be a diffeomorphism of $\Omega$ such that*

$$B\left(f(z), f(z)\right) \circ \mathrm{d}\, f(z) = \mathrm{d}\, f(z) \circ B(z, z)$$

*for all $z \in \Omega$. Then any $K$-orbit is globally invariant by $f$.*

*Proof.* Assume first that $z = \lambda_1 e_1 + \cdots + \lambda_r e_r$ is super-regular. Then $f(z) = \mu_1 \varepsilon_1 + \cdots + \mu_r \varepsilon_r$, and equality between the eigenvalues of $B(z, z)$ and $B(f(z), f(z))$ implies $\lambda_j = \mu_j$, hence $f(z) \in Kz$ and $f(Kz) = Kz$. By continuity, this also holds for any $z \in \Omega$. □

4.3. We now go back to bisysmplectomorphisms of $\Omega$.

**Proposition 6.** *Let $\Omega$ be a bounded circled symmetric domain and denote by $K$ the group of linear automorphisms of $\Omega$. For each bisymplectomorphism $f \in \mathcal{B}(\Omega)$, we have $\mathrm{d}\, f(0) \in K$.*

As $K \subset \mathcal{B}(\Omega)$, it will be sufficient to study the subgroup

$$\mathcal{B}_0(\Omega) = \{f \in \mathcal{B}(\Omega) \mid \mathrm{d}\, f(0) = \mathrm{id}_V\}. \tag{4.11}$$

*Proof.* Let $(e_1, \ldots, e_r)$ be a frame of $V$ and let $V = \bigoplus V_{ij}$ be the associated simultaneous Peirce decomposition. Consider a regular element

$$z = \alpha_1 e_1 + \cdots + \alpha_r e_r,$$

$1 > \alpha_1 > \cdots > \alpha_r > 0$. For $f \in \mathcal{B}(\Omega)$ and $z$ super-regular, we have $f(z) = \alpha_1 \varepsilon_1 + \cdots + \alpha_r \varepsilon_r$, where $(\varepsilon_1, \ldots, \varepsilon_r)$ is a frame of $V$, which may depend on $(\alpha_1, \ldots, \alpha_r)$. As $V_{jj} = \mathbb{C} e_j$ is the eigenspace of $B(z, z)$ for the eigenvalue $\left(1 - \alpha_j^2\right)^2$, we deduce from (4.4) that

$$\mathrm{d}\, f(z)\,(e_j) \in \mathbb{C}\varepsilon_j.$$

By continuity, there exists a frame $(e'_1, \ldots, e'_r)$ such that

$$\mathrm{d}\, f(0)\,(e_j) \in \mathbb{C} e'_j \qquad (1 \le j \le r).$$

Multiplying $e'_j$ by a suitable complex of modulus 1, we may assume that

$$\mathrm{d}\, f(0)\,(e_j) = \lambda'_j e'_j, \quad \lambda'_j > 0 \quad (1 \le j \le r). \tag{4.12}$$

We have also

$$\mathrm{d}\, f(0)(e_j) = \lim_{t \to 0+} \frac{f(te_j)}{t}$$

and $f(te_j)$ is a multiple of a minimal tripotent, with spectral norm $\|f(te_j)\| \le t\,\|e_j\|$, hence $\|\mathrm{d}\, f(0)(e_j)\| \le \|e_j\|$ and $\lambda'_j \le 1$. The same argument applied to $f^{-1}$ gives $\lambda'_j = 1$. So the image of a frame $(e_1, \ldots, e_r)$ under $\mathrm{d}\, f(0)$ is a frame $(e'_1, \ldots, e'_r)$ and the Peirce spaces $V_{jk}$ relative to $(e_1, \ldots, e_r)$ are mapped by $\mathrm{d}\, f(0)$ onto the corresponding Peirce spaces $V'_{jk}$ relative to $(e'_1, \ldots, e'_r)$. In particular, $\mathrm{d}\, f(0)$ is an $\mathbb{R}$-linear map from $\mathbb{C} e_j$ onto $\mathbb{C} e'_j$. We have $\mathrm{d}\, f(0)(\mathrm{i}\, e_j) = \beta e'_j$, with $|\beta| = 1$. As

$$\omega_0(u, v) = \frac{\mathrm{i}}{2}\left(\operatorname{tr} D(u, v) - \operatorname{tr} D(v, u)\right),$$

we have

$$\omega_0(e_j, \mathrm{i}\, e_j) = \operatorname{tr} D(e_j, e_j),$$
$$\omega_0\left(\mathrm{d}\, f(0)\,(e_j), \mathrm{d}\, f(0)\,(\mathrm{i}\, e_j)\right) = \operatorname{Im} \beta \operatorname{tr} D(e'_j, e'_j).$$

From $\operatorname{tr} D(e'_j, e'_j) = \operatorname{tr} D(e_j, e_j)$ and $f^*\omega_0 = \omega_0$, we get $\operatorname{Im}\beta = 1$, $\beta = \mathrm{i}$, which means that $\mathrm{d}\, f(0)$ is $\mathbb{C}$-linear on $\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_r$. Finally, $\mathrm{d}\, f(0)$ is $\mathbb{C}$-linear on $V$, maps minimal tripotents to minimal tripotents and $\Omega$ to $\Omega$. □

4.4. **The unit disc.** Let $\Delta$ be the unit disc of $\mathbb{C}$. The associated triple product is $\{u, v, w\} = 2u\overline{v}w$. The Bergman operator is $B(z, z)w = (1 - |z|^2)^2 w$. The Bergman metric is

$$h_z(u, v) = \frac{2u\overline{v}}{(1 - |z|^2)^2}.$$

The two symplectic forms are

$$\omega_0 = \mathrm{i}\,\mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}, \tag{4.13}$$

$$\omega_- = \frac{\omega_0}{(1 - |z|^2)^2}. \tag{4.14}$$

Denote by $S^1$ the unit circle in $\mathbb{C}$ and consider the "polar coordinates" diffeomorphism

$$\begin{aligned} \Theta : (0, 1) \times S^1 &\to \Delta \setminus \{0\}, \\ (r, \zeta) &\mapsto r\zeta. \end{aligned}$$

Then we have

$$\Theta^*\omega_0 = 2r\,\mathrm{d}\, r \wedge \frac{\mathrm{d}\,\zeta}{\mathrm{i}\,\zeta}.$$

The following theorem characterizes the elements of $\mathcal{B}(\Delta)$.

**Theorem 2.** *The elements $f \in \mathcal{B}(\Delta)$ are the maps defined by*

$$f(z) = u\left(|z|^2\right) z \qquad (z \in \Delta),$$

*where $u$ is a smooth function $u : [0, 1) \to S^1 \simeq U(1)$.*

In other words, the restriction of $f$ to a circle of radius $r$ $(0 < r < 1)$ is the rotation $u\left(r^2\right)$.

*Proof.* From (4.14), we see that a diffeomorphism $f : \Delta \to \Delta$ is a bisymplectomorphism if and only if $f$ preserves $\omega_0$ and $|f(z)| = |z|$ for all $z \in \Delta$.

If $|f(z)| = |z|$ for all $z \in \Delta$, the map $F = \Theta^{-1} \circ f \circ \Theta$ may be written

$$(r, \zeta) \xrightarrow{F} (r, Z(r, \zeta))$$

for some smooth function $Z : (0, 1) \times S^1 \to S^1$. We have

$$F^*\left(\Theta^*\omega_0\right) = 2r\,\mathrm{d}\, r \wedge \frac{\mathrm{d}\, Z}{\mathrm{i}\, Z} = 2r\,\mathrm{d}\, r \wedge \frac{\mathrm{d}_\zeta\, Z}{\mathrm{i}\, Z}.$$

If $f$ preserves $\omega_0$, then $F$ preserves $\Theta^*\omega_0$, which implies

$$\frac{\mathrm{d}_\zeta\, Z}{Z} = \frac{\mathrm{d}\,\zeta}{\zeta}. \tag{4.15}$$

For $r$ fixed, let $u_r(\zeta) = Z(r, \zeta)$ ; the condition (4.15) is then equivalent to

$$\frac{\mathrm{d}\, u_r}{u_r} = \frac{\mathrm{d}\,\zeta}{\zeta} \tag{4.16}$$

for all $r \in (0, 1)$. The condition (4.16) is in turn equivalent to

$$u_r(\zeta) = v(r)\zeta, \tag{4.17}$$

with $v(r) \in S^1$. The function $v$, which is given by

$$v(r) = \frac{Z(r,\zeta)}{\zeta},$$

is smooth on $(0,1)$ and

$$f(r\zeta) = rv(r)\zeta,$$

or

$$f(z) = v\left(|z|\right) z \tag{4.18}$$

for $z \in \Delta$, $z \neq 0$; on the other hand, $f(0) = 0$. Let $g$ be the restriction of $f$ to $(-1,1)$; from (4.18), we see that $g$ is odd. For $x \neq 0$, we have

$$g(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\, s} g(sx)\, \mathrm{d}\, s = x \int_0^1 g'(sx)\, \mathrm{d}\, s,$$

which shows that

$$v(r) = \int_0^1 g'(sr)\, \mathrm{d}\, s$$

extends to a smooth even function $v : (-1,1) \to S^1$. Let $u : [0,1) \to S^1$ be defined by

$$u(r) = v\left(\sqrt{r}\right).$$

It follows then from Whitney's theorem [5] that $u$ is smooth at 0.

Conversely, if

$$f(z) = u(|z|^2)z,$$

with $u : [0,1) \to S^1$ smooth, then $f$ satisfies $|f(z)| = |z|$, $f$ is a diffeomorphism with inverse $f^{-1}(w) = u(|w|^2)^{-1}w$; also, $f$ preserves $\omega_0$ on $\Delta \setminus \{0\}$, hence on $\Delta$ by continuity. This implies that $f \in \mathcal{B}(\Delta)$. □

4.5. **The polydisc.** Let $\Delta^r \subset \mathbb{C}^r$ be the product of $r$ unit discs. The Jordan triple product on $V = \mathbb{C}^r$ is just the component-wise product

$$\{x,y,z\} = 2\left(x_1\overline{y_1}z_1, \ldots, x_r\overline{y_r}z_r\right).$$

Let $(e_1, \ldots, e_r)$ denote the canonical basis of $\mathbb{C}^r$. The minimal tripotents of $V$ are the elements $\lambda_j e_j$, $1 \leq j \leq r$, $|\lambda_j| = 1$. Any frame (maximal ordered set of mutually orthogonal tripotents) has the form

$$\left(\lambda_j e_{\sigma(j)}\right)_{1 \leq j \leq r},$$

where $|\lambda_j| = 1$ and $\sigma \in \mathfrak{S}_r$ is a permutation of $\{1, \cdots, r\}$. The corresponding Peirce decomposition is $V = \mathbb{C}e_{\sigma(1)} \oplus \cdots \oplus \mathbb{C}e_{\sigma(r)}$.

**Theorem 3.** *A diffeomorphism $f : \Delta^r \to \Delta^r$ belongs to $\mathcal{B}_0\left(\Delta^r\right)$ if and only if there exist smooth functions $u_j : [0,1) \to S^1$ such that $u_j(0) = 1$ and*

$$f\left(z_1, \ldots, z_r\right) = \sum_{j=1}^{r} u_j\left(|z_j|^2\right) z_j e_j \qquad \left(z_j \in \Delta\right). \tag{4.19}$$

*Proof.* Let $f \in \mathcal{B}_0(\Delta^r)$ be a bisymplectomorphism with $\mathrm{d}\, f(0) = \mathrm{id}_V$. Consider a regular element $z \in \Delta^r$, that is,

$$z = z_1 e_1 + \cdots + z_r e_r,$$

with all $|z_j|$ different. The spaces $\mathbb{C}e_j$ of the corresponding Peirce decomposition are mapped by $\mathrm{d}\, f(z)$ to the spaces $\mathbb{C}e_{\sigma(j)}$ of another Peirce decomposition, for some permutation $\sigma \in \mathfrak{S}_r$. This means that $[\mathrm{d}\, f(z)\,(e_j)] = \left[e_{\sigma(j)}\right]$ for all regular $z \in \Delta^r$, where $[\quad]$ denotes the class in $\mathbb{P}(\mathbb{C}^r)$; by continuity, this is true for all $z \in \Delta^r$. As $\{[e_1], \ldots, [e_r]\}$ is discrete in $\mathbb{P}(\mathbb{C}^r)$ and $\mathrm{d}\, f(0) = \mathrm{id}$, we have $[\mathrm{d}\, f(z)\,(e_j)] = [e_j]$ and

$$\mathrm{d}\, f(z)\,(\mathbb{C}e_j) = \mathbb{C}e_j \tag{4.20}$$

for all $z \in \Delta^r$. This shows that $f\,(z_1, \ldots, z_r) = \sum_{j=1}^r f_j(z) e_j$, with $|f_j(z)| = |z_j|$. From (4.20), we deduce that $f_j$ depends only of $z_j$. Each $f_j$ is then a bisymplectomorphism of the unit disc. According to Theorem 2, there exists a smooth function $u_j : [0,1) \to S^1 \simeq U(1)$ such that

$$f_j(z_j) = u_j\left(|z_j|^2\right)\, z_j \qquad (z_j \in \Delta)\,.$$

Finally, any $f \in \mathcal{B}_0(\Delta^r)$ has the form (4.19).

Conversely, each $f$ of the form (4.19) is easily seen to be a bisymplectomorphism. □

4.6. **The general case.** We assume now that the domain $\Omega$ is *irreducible*, that is, not a product of two bounded symmetric domains. In Theorem 4, we will characterize the bisymplectomorphisms of $\Omega$. This theorem may be considered as a kind of Schwarz lemma.

Let $f \in \mathcal{B}_0(\Omega)$. If $z = \sum \lambda_j e_j$ is a regular element, then $P_z = (\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_r) \cap \Omega$ and $f\,(P_z) = P_z$. For a frame $\mathbf{e} = (e_1, \ldots, e_r)$, let $P(\mathbf{e}) = (\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_r) \cap \Omega$. By the same argument as in the proof of Theorem 3, the restriction of $f$ to $P(\mathbf{e})$ has the form

$$\sum_{j=1}^r \lambda_j e_j \mapsto \sum \lambda_j u_j\left(|\lambda_j|^2\right) e_j,$$

where the $u_j$'s are smooth functions $u_j : [0,1) \to S^1 \simeq U(1)$. In the "polar coordinates" $((\lambda_1, \ldots, \lambda_r), \mathbf{e} = (e_1, \ldots, e_r))$, the map $f$ is then represented by

$$((\lambda_1, \ldots, \lambda_r), \mathbf{e}) \mapsto \left((\lambda_1, \ldots, \lambda_r), \left(u_1\left(\lambda_1^2, \mathbf{e}\right) e_1, \ldots, u_r\left(\lambda_r^2, \mathbf{e}\right) e_r\right)\right). \tag{4.21}$$

Let $w_j\,(\lambda_j, \mathbf{e}) = u_j\left(\lambda_j^2, \mathbf{e}\right)$. We obtain

$$\begin{aligned} f^*\eta_{jj} &= \mathrm{i}\, f^* \sum_{m=1}^n e_{jm}\, \mathrm{d}\, \overline{e}_{jm} = \mathrm{i} \sum_{m=1}^n w_j e_{jm}\left(\overline{w}_j\, \mathrm{d}\, \overline{e}_{jm} + \overline{e}_{jm}\, \mathrm{d}\, \overline{w}_j\right) \\ &= \eta_{jj} - \mathrm{i}\,(e_j \mid e_j)\, \frac{\mathrm{d}\, w_j}{w_j}, \\ f^*\omega_{jj} &= \omega_{jj}. \end{aligned}$$

As $\Omega$ is irreducible, $(e_j \mid e_j)$ has the same value $g$ for all minimal tripotents. Finally

$$f^*\omega_0 = \sum_{j=1}^r \lambda_j^2 \omega_{jj} + 2 \sum_{j=1}^r \lambda_j\, \mathrm{d}\, \lambda_j \wedge \left(\eta_{jj} - \mathrm{i}\, g \frac{\mathrm{d}\, w_j}{w_j}\right)$$

$$= \omega_0 - 2\,\mathrm{i}\,g \sum_{j=1}^{r} \lambda_j \,\mathrm{d}\,\lambda_j \wedge \frac{\mathrm{d}\,w_j}{w_j}.$$

Then $f^*\omega_0 = \omega_0$ implies that $f \in \mathcal{B}_0(\Omega)$, written in the form (4.21), satisfies

$$\sum_{j=1}^{r} \lambda_j \,\mathrm{d}\,\lambda_j \wedge \frac{\mathrm{d}\,w_j}{w_j} = 0. \tag{4.22}$$

As $w_j$ depends only on $\lambda_j$ and $\mathbf{e}$, this implies that $\mathrm{d}_{\mathbf{e}}\,w_j = 0$. As the manifold of frames is connected when the domain $\Omega$ is irreducible, $w_j$ does not depend on $\mathbf{e} \in \mathcal{F}$. As a permutation of a frame is again a frame, we have $w_1 = \cdots = w_r$ and $u_1 = \cdots = u_r$.

Finally, an element $f \in \mathcal{B}_0(\Omega)$ is written in polar coordinates

$$\left((\lambda_1, \ldots, \lambda_r), (e_1, \ldots, e_r)\right) \mapsto \left((\lambda_1, \ldots, \lambda_r), \left(u\left(\lambda_1^2\right) e_1, \ldots, u\left(\lambda_r^2\right) e_r\right)\right), \tag{4.23}$$

where $u$ is a smooth function $u : [0,1) \to S^1 \simeq U(1)$.

**Theorem 4.** *Let $\Omega$ be an irreducible Hermitian bounded circled symmetric domain and let $K$ be the isotropy group of $0$. The analytic (resp. $C^\infty$) bisymplectomorphisms of $\Omega$ are the maps $\phi = f \circ g$, where $g = \mathrm{d}\,\phi(0) \in K$ and $f$ is associated to $v(t) = tu(t^2)$, with $u : [0,1) \to S^1 \simeq U(1)$ analytic (resp. $C^\infty$) and $u(0) = 1$.*

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