The Tian–Yau–Zelditch asymptotic expansion for real analytic Kähler metrics

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Abstract

Let M be a compact Kähler manifold endowed with a real analytic and polarized Kähler metric g and let $T_{m\omega}(x)$ be the corresponding Kempf's distortion function. In this paper we compute the coefficients of Tian–Yau–Zelditch asymptotic expansion of $T_{m\omega}(x)$ using quantization's tecniques alternative to Lu's computations in [10].

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1 Introduction

Let M be a projective algebraic manifold, namely a compact complex manifold which admits a holomorphic embedding into some complex projective space $\mathbb{C}P^N$. The hyperplane bundle on $\mathbb{C}P^N$ restricts to an ample line bundle L on M, which is called a *polarization* on M. A Kähler metric g on M is *polarized* with respect to L if the corresponding Kähler form ω represents the first Chern class $c_1(L)$ of L and this happens if and only if ω is an integral form. Given any polarized Kähler metric g on M, one can find a Hermitian metric h on L with its Ricci curvature form $\mathrm{Ric}(h) = \omega$. Here $\mathrm{Ric}(h)$ is the 2-form on M defined by the equation:

$$\mathrm{Ric}(h) = -\frac{i}{2\pi}\partial\bar{\partial}\log h(\sigma(x), \sigma(x)), \tag{1}$$

for a trivializing holomorphic section $\sigma : U \subset M \to L \setminus \{0\}$ of L.

For a positive integer m, the line bundle $L^m = L^{\otimes m}$, the m-th tensor power of L, is a polarization for the Kähler metric mg and the Hermitian metric h satisfying $\mathrm{Ric}(h) = \omega$ induces, in a natural way an Hermitian metric h_m on L^m such that $\mathrm{Ric}(h_m) = m\omega$. Denote by $H^0(L^m)$ the space of holomorphic sections of L^m and let $\underline{s} = (s_0, \ldots, s_{d_m})$ be an orthonormal basis with respect to the L^2-product induced by h_m, namely

$$\langle s_j, s_k \rangle_{h_m} = \int_M h_m(s_j(x), s_k(x)) \frac{\omega^n}{n!} = \delta_{jk}. \tag{2}$$

Let

$$i_m(\underline{s}) : M \to \mathbb{P}(H^0(L^m)^*) \cong \mathbb{C}P^{d_m}, \tag{3}$$

be the Kodaira emebedding in the basis $\underline{s}$.

Let G_{FS} be the standard Fubini–Study metric on $\mathbb{C}P^{d_m}$, namely the metric whose associated Kähler form is given by

$$\Omega_{FS} = \frac{i}{2\pi} \partial\bar{\partial} \log \sum_{j=0}^{d_m} |z_j|^2 \tag{4}$$

for a homogeneous coordinate system $[z_0, \ldots, z_{d_m}]$ in $\mathbb{C}P^{d_m}$. This restricts to a Kähler metric $g_m = i_m(\underline{s}^m)^*(G_{FS})$ on M which polarized with respect to L^m and its associated Kähler form

$$\omega_m = i_m(\underline{s})^*(\Omega_{FS}) \tag{5}$$

is cohomologous to $m\omega$. In [12] Tian christened the set of normalized metrics $\frac{g_m}{m}$ as the *Bergman* metrics on M with respect to L and solved a conjecture posed by Yau by proving that the sequence $\frac{g_m}{m}$ C^2-converges to the polarized metric g. This result was further generalized by Ruan [11] who proved the C^∞-convergence. As already observed by Tian, the difference between $\frac{g_m}{m}$ and the metric g can be measured by the so called *G. Kempf's distortion function* [9]

$$T_{m\omega}(x) = \sum_{j=0}^{d_m} h_m(s_j(x), s_j(x)). \tag{6}$$

Indeed, it is easily seen that for all non-negative integers m,

$$\frac{\omega_m}{m} = \omega + \frac{i}{2\pi m} \partial\bar{\partial} \log T_{m\omega}, \tag{7}$$

During the last decade, many authors have studied the asymptotic behavior of the function $T_{m\omega}(x)$ when $m \to \infty$.
Zelditch [13] generalized Tian's Theorem by proving:

Theorem 1.1 *(Zelditch) There is a complete asymptotic expansion*

$$T_{m\omega}(x) = a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \cdots \tag{8}$$

for certain smooth coefficients $a_j(x)$ with $a_0 = 1$. More precisely the expression holds in C^∞ in that, for any $r, k \geq 0$

$$||T_{m\omega}(x) - \sum_{j=0}^{k} a_j(x)m^{n-j}||_{C^r} \leq C_{k,r} m^{n-k-1}, \tag{9}$$

where $C_{k,r}$ are constant depending on k, r and on the Kähler form ω.

The expansion (8) is called the *Tian–Yau–Zelditch expansion.*

Observe that Catlin [6] indipendently, proved the previous theorem, using Szegö Kernel's tecniques developed in [3].

In [10] Lu proves:

Theorem 1.2 *(Lu) Each coefficients $a_j(x)$, given by the asymptotic expansion (8) is a polynomial of the curvature and its covariant derivatives at x of the metric g. These polynomials can be found by finitely many steps of algebraic operations. In particular*

$$\begin{cases} a_0 = 1 \\ a_1(x) = \frac{1}{2}\rho \\ a_2(x) = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|\,Ric\,|^2| + 3\rho^2) \\ a_3(x) = \frac{1}{8}\Delta\Delta\rho + \frac{1}{24}\,div\,div(R, Ric) - \frac{1}{6}\,div\,div(\rho\,Ric) \\ +\frac{1}{48}\Delta(|R|^2 - 4|\,Ric\,|^2| + 8\rho^2) + \frac{1}{48}\rho(\rho^2 - 4|\,Ric\,|^2 + |R|^2) \\ +\frac{1}{24}(\sigma_3(Ric) - Ric(R,R) - R(Ric, Ric)). \end{cases} \tag{10}$$

We refer the reader to Section 5 in [10] or to formulae in the Appendix of the present article for the definitions of the terms in the expression of a_2 and a_3.

In this paper we investigate the link between T–Y–Z expansion and Engliš' expansion [7] of the Laplace integral

$$\mathcal{L}_m(x) = \int_U f(y)e^{-mD(x,y)}\frac{\omega^n}{n!}(y) \sim \frac{1}{m^n}\sum_{r\geq 0} m^{-r}C_r(f)(x),$$

where U is an open subset of M, f is a smooth function on U, $D(x, y)$ denotes the Calabi's diastasis function and C_j are smooth differential operators on U

(see next section for details). Our main result is Theorem 3.1 where we prove that, for real analytic Kähler metrics, the functions a_j can be calculated in terms of the operators C_j.

The main tools for the proof of Theorem 3.1 come from the teory of quantization of compact Kähler manifolds developed by Karabegov and Schlichenmaier in [8], which in turn is based on the theory of Szegö Kernel on the unit circle bundle L^* over M.

Therefore, our Theorem 3.1 provides a computation of the functions a_j based on the theory of Szegö Kernel alternative to the Lu's proof of Theorem 1.2 which is complex-geometric being based on Tian's peak section method and Ruan's work on K-coordinates.

The paper is organized as follows. In the next Section we describe Engliš' result about the asymptotic expansion of the Laplace integral using Lu's notation (see Theorem 2.1) and we collect the above mentioned results of Karabegov and Schlichenmaier needed for the proof of Theorem 3.1 (see Lemma 2.3 and Lemma 2.4 below). Section 3 is dedicated to the statement and proof of our main result Theorem 3.1. We conclude the section and the article by reobtaining the functions a_1, a_2 and a_3 given by (10) for the case of real analytic Kähler metrics. For the reader's convenience in the Appendix below we recall Lu's notations and some formulae which will be used in the paper.

2 Engliš' work and Berezin's transform

Let M be a n-dimensional complex manifold endowed with a real analytic Kähler metric g and let ω be the corresponding Kähler form.
Let Φ be a Kähler potential for the metric g, namely a real valued function Φ defined on a open set $U \subset M$ satisfying

$$\omega = \frac{i}{2\pi}\partial\bar{\partial}\Phi. \tag{11}$$

If $g = \sum_{j\bar{k}}^{n} g_{j\bar{k}} dz_j d\bar{z}_k$ is the local expression of the metric g then the previous equation is equivalent to

$$g_{j\bar{k}} = \frac{1}{\pi}\frac{\partial^2\Phi}{\partial z_j \partial \bar{z}_k}. \tag{12}$$

The potential Φ can be complex analytically continued to an open neighbourhood $V \subset U \times \bar{U}$ of the diagonal. Denote this extension by $\Phi(x, \bar{y})$.

It is holomorphic in x and anti-holomorphic in y and, with this notation, $\Phi(x) = \Phi(x, \bar{x})$. Observe also that $\overline{\Phi(x, \bar{y})} = \Phi(\bar{x}, y)$. Consider the real valued function

$$D(x, y) = \Phi(x, \bar{x}) + \Phi(y, \bar{y}) - \Phi(x, \bar{y}) - \Phi(y, \bar{x})$$

on V. It is easily seen that the function $D(x, y)$ is independent from the potential chosen which is defined up to the sum with the real part of a holomorphic function. Calabi [5] christened the function $D(x, y)$ the *diastasis function.* Take $U \subset M$ as above. For all $x \in U$, the positive definiteness of the matrix (12) implies that the function

$$D(x, \cdot) = \Phi(x, \bar{x}) + \Phi(\cdot, \bar{\cdot}) - \Phi(x, \bar{\cdot}) - \Phi(\cdot, \bar{x})$$

has a local minimum at x. Diminishing U, if necessary, we can assume that $D(x, y)$ is a globally defined function on $U \times U$, $D(x, y) \geq 0$ and $D(x, y) = 0$ iff $x = y$. Now, let f be a C^∞ function on U and $m > 0$. Consider the Laplace integral

$$\mathcal{L}_m(x) = \int_U f(y) e^{-mD(x,y)} \frac{\omega^n}{n!}(y), \tag{13}$$

where $\frac{\omega^n}{n!}$ is the Riemannian volume form on M, induced by the metric g.

The following result is the crucial step in the proof of our main result.

Theorem 2.1 *(Engliš) If the integral (13) exists for some $m = m_0$ then it also exists for all $m > m_0$ and as $m \to +\infty$ it has an asymptotic expansion*

$$\mathcal{L}_m(x) \sim \frac{1}{m^n} \sum_{r \geq 0} m^{-r} C_r(f)(x), \tag{14}$$

where $C_r : C^\infty(U) \to C^\infty(U)$ are smooth differential operator which can be described explicitly. In particular

$$\begin{cases} C_0 = id \\ C_1(f) = \Delta f - \frac{1}{2} f \rho \\ C_2(f) = \frac{1}{2}\Delta\Delta f - \frac{1}{2} L_{Ric}(f) - \frac{\rho}{2}\Delta f - \frac{1}{2}(\langle D^{'}\rho, D^{'}f\rangle + \langle D^{'}f, D^{'}\rho\rangle) \\ -f(\frac{1}{3}\Delta\rho - \frac{1}{8}\rho^2 - \frac{1}{6}|\, Ric\,|^2 + \frac{1}{24}|R|^2) \\ c_3 = C_3(1) = \frac{1}{4}|D^{'}\rho|^2 - \frac{1}{8}\Delta\Delta\rho + \frac{\rho}{6}\Delta\rho + \frac{3}{8} L_{Ric}(\rho) - \frac{1}{48}\rho^3 \\ +\frac{1}{4}|D^{'}Ric|^2 - \frac{\rho}{12}|Ric|^2 - \frac{1}{24}|D^{'}R|^2 + \frac{\rho}{48}|R|^2 + \frac{1}{4}R(Ric, Ric) \\ -\frac{1}{12}\sigma_1(R) + \frac{1}{24}\sigma_2(R) + \frac{1}{6}\sigma_3(Ric), \end{cases} \tag{15}$$

where, for $f, g \in C^\infty(U)$, we have the following notations:

$$L_{Ric}(f) = \sum_{i,j,p,q=1}^{n} g^{i\bar{q}} g^{p\bar{j}} Ric_{p\bar{q}} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}, \tag{16}$$

$$\langle D^{'} f, D^{'} g \rangle = \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial \bar{z}_j}, \tag{17}$$

$$|D^{'} f|^2 = \langle D^{'} f, D^{'} f \rangle. \tag{18}$$

(For the other notations see Section 5 *in [10] or Appendix below.)*

Proof: For the proof of the first part we refer to Theorem 3 in [7] where the operators C_j are denoted by R_j. The expression of the operators C_1, C_2 and of the function c_3 above can be deduced by the expression of R_1, R_2 and of the function $r_3 = R_3(1)$ in Section 4 of [7] by translating Engliš's notations in our notations and by taking into account that the Ricci curvature considered by Engliš has opposite signs to the one we are considering in the present article. While to pass from R_1, R_2 to C_1, C_2 is straightforward, in order to obtain the function $c_3 = C_3(1)$ from the function $r_3 = R_3(1)$ one needs more work by going through Engliš' computation at the end of Section 4 in [7]. □

Remark 2.2 Observe that the factor π^n which appears in the expansion (2.21) given by Engliš in [7] is missing in (14) above since Engliš uses the expression $\omega = \frac{i}{2}\partial\bar{\partial}\Phi$ which differs from (11) by the factor $\frac{1}{\pi}$.

We conclude this section by describing some results of the theory of the quantization of Kähler manifolds due to Karabegov and Schlichenmaier [8] which we need in the proof of our main Theorem 3.1. These are summarized in the following Lemma 2.3 and Lemma 2.4. The proof of these lemmata follows by Theorem 5.6 and Proposition 5.1 in [8] by specializing them to the case of real analytic Kähler metrics. Observe that in [8] the functions $T_{m\omega}(x)$ and $\psi_m(x, y)T_{m\omega}(y)$ (see formula (21) below) are denoted by $u_m(x)$ and $v_m(x, y)$ respectively.

Lemma 2.3 *Let M be a complex manifold endowed with a polarized and real analytic Kähler metric g. Let $T_{m\omega}(x, \bar{y})$ denote the holomorphic extension of the Kempf's distortion function $T_{m\omega}(x)$ to an open neighbourhood, say $U \times U$ of the diagonal. Then, for U sufficiently small, $T_{m\omega}(x, \bar{y})$ admits an asymptotic expansion as $m \to +\infty$ of the form:*

$$T_{m\omega}(x, \bar{y}) \sim \sum_{j \geq 0} a_j(x, \bar{y}) m^{n-j} \tag{19}$$

such that for any compact set $K \subset U \times U$ and for any non-negative integer k one has:

$$\sup_{x,y \in K} |T_{m\omega}(x, \bar{y}) - \sum_{j=0}^{k} a_j(x, \bar{y}) m^{n-j}| = O(m^{n-k-1}), \tag{20}$$

where $a_r(x, \bar{y})$ is the holomorphic extension of $a_r(x)$; in particular $a_0(x, \bar{y}) = 1$.

Before stating Lemma 2.4 assume that the open set U given by Lemma 2.3 is such that Calabi's diastasis function $D(x, y)$ is defined on $U \times U$. Then one can define the real valued function

$$\psi_m(x, y) = \frac{e^{-mD(x,y)} |T_{m\omega}(x, \bar{y})|^2}{T_{m\omega}(x) T_{m\omega}(y)} \tag{21}$$

which turns out to be globally defined on $M \times M$ (cfr. [1] and [4]).

Lemma 2.4 *In the same hypothesis of Lemma 2.3*

$$\int_M \psi_m(x, y) T_{m\omega}(y) \frac{\omega^n(y)}{n!} = 1. \tag{22}$$

Moreover, for any neighbourhood U of a point $x \in M$ and for every smooth function f on M,

$$\int_{M \setminus U} \psi_m(x, y) T_{m\omega}(y) f(y) \frac{\omega^n(y)}{n!} = O(m^{-k}), \ \forall k \geq 1. \tag{23}$$

Remark 2.5 Lemma 2.3 can be considered a generalization of T–Y–Z expansion (8). It is worth to mention that this generalization is somehow implicit in the above mentioned paper of Catlin [6].

In the quantum mechanics terminology the integral

$$I^{(m)}(f(x)) = \int_M \psi_m(x, y) T_{m\omega}(y) f(y) \frac{\omega^n(y)}{n!}$$

is called the *Berezin transform* of the the smooth function f (see [7] and [8] for further details). Therefore formula (22) is telling us that the Berezin transform of the constant function 1 equals 1 while formula (23) expresses the fact that the expansion of $I^{(m)}(f(x))$ as $m \to +\infty$ depends only on the germ of the function f at the point x.

3 Proof of main result

In this section we state and prove the main result of this article.

Theorem 3.1 *Let M be a n-dimensional complex manifold endowed with a polarized and real analytic Kähler metric g. Then Zelditch's functions a_j given by (8) can be computed in terms of the operators C_j of the expansion (14) (see formulae (29) and (30) below).*

Proof: Fix a point $x \in M$ and a neighbourhood U of x where one can apply Lemma 2.3 and such that the diastasis $D(x, y)$ is defined on $U \times U$. Thus, on $U \times U$, $|T_{m\omega}(x, \bar{y})|^2$ admits an asymptotic expansion

$$|T_{m\omega}(x, \bar{y})|^2 = m^{2n}(1 + \sum_{j=1}^{+\infty} \tilde{a}_j(x, y)m^{-j}), \text{ as } m \to +\infty \tag{24}$$

uniformly on compact subset $K \subset U \times U$. Here the $\tilde{a}_j(x, y)$'s are smooth real valued functions on $U \times U$.

In particular, we have:

$$\begin{array}{l} \tilde{a}_1(x, y) = a_1(x, \bar{y}) + a_1(y, \bar{x}) \\ \tilde{a}_2(x, y) = |a_1(x, \bar{y})|^2 + a_2(x, \bar{y}) + a_2(y, \bar{x}) \\ \tilde{a}_3(x, y) = a_1(x, \bar{y})a_2(y, \bar{x}) + a_1(y, \bar{x})a_2(x, \bar{y}) + a_3(x, \bar{y}) + a_3(y, \bar{x}). \end{array} \tag{25}$$

By formulae (21) and (22) one can write

$$T_{m\omega}(x) = T_{m\omega}(x) \int_{M \setminus U} \psi_m(x, y)T_{m\omega}(y)\frac{\omega^n}{n!}(y) + \int_U e^{-mD(x,y)}|T_{m\omega}(x, \bar{y})|^2 \frac{\omega^n}{n!}(y). \tag{26}$$

Zelditch's expansion (8) of $T_{m\omega}(x)$ together with (23) above (with $f = 1$) imply that,

$$\lim_{m\to\infty} m^k T_{m\omega}(x) \int_{M \setminus U} \psi_m(x, y)T_{m\omega}(y)\frac{\omega^n}{n!}(y) = 0,$$

for all non-negative integers $k \geq 1$. Therefore, by inserting (24) in (26) and by taking into account Engliš' expansion (14), we obtain

$$1+\sum_{r=1}^{k} a_r(x)m^{-r} = 1+\sum_{r=1}^{k} c_r(x)m^{-r} + \sum_{r+j=1,\ r\geq 0,\ j\geq 1}^{k} m^{-r-j}C_r(\tilde{a}_j(x, y))_{|y=x} + R_k(x, m), \tag{27}$$

with $\lim_{m\to\infty} m^k R_k(x,m) = 0$.

Here we are denoting by $c_j(x)$ the function obtained by applying the operators C_j to the constant function 1, namely

$$C_j(1)(x) = c_j(x), \tag{28}$$

and by $C_r(\tilde{a}_j(x,y))_{|y=x}$ we mean to apply the operator C_r to the y-variable of the function $\tilde{a}_j(x,y))$ and to set $x = y$.
Therefore, by formula (27) one gets

$$a_1(x) = c_1(x) + C_0(\tilde{a}_1(x,y))_{|y=x} = c_1(x) + \tilde{a}_1(x,x),$$

which implies

$$a_1(x) = -c_1(x). \tag{29}$$

Finally, for any integer $k \geq 2$, due to the fact that C_0 is the identity operator, one obtains:

$$\begin{array}{rcl} a_k(x) & = & c_k(x) + \sum_{r+j=k,\ r\geq 0,\ j\geq 1} C_r(\tilde{a}_j(x,y))_{|y=x} \\ & = & c_k(x) + \tilde{a}_k(x,x) + \sum_{r+j=k,\ r\geq 1,\ j\geq 1} C_r(\tilde{a}_j(x,y))_{|y=x}. \end{array} \tag{30}$$

□

We conclude this article by reobtaining the functions $a_1(x)$, $a_2(x)$ and $a_3(x)$ given by Lu (see formulae (10) above). The fact that $a_1(x) = \frac{1}{2}\rho$ follows immediately by (29) and the second of (15) (with $f = 1$).

By (30), the third of (25) and the second of (15) one obtains:

$$a_2(x) = -c_2(x) - a_1^2(x) - C_1(a_1(x,\bar{y}) + a_1(y,\bar{x}))_{|y=x} = -c_2(x) - a_1^2(x) + \rho(x)a_1(x).$$

By substituting $a_1(x) = \frac{1}{2}\rho$ and $c_2(x) = -\frac{1}{3}\Delta\rho + \frac{1}{8}\rho^2 + \frac{1}{6}|\operatorname{Ric}|^2 - \frac{1}{24}|R|^2$ (the latter obtained by the third of (15) with $f = 1$), one gets:

$$a_2(x) = \frac{1}{3}\Delta\rho + \frac{1}{24}|R|^2 - \frac{1}{6}|\operatorname{Ric}|^2 + \frac{1}{8}\rho^2. \tag{31}$$

Finally, (30) with $k = 3$ gives:

$$a_3(x) = -c_3(x) - 2a_1(x)a_2(x) - C_1(\tilde{a}_2(x,y))_{|y=x} - C_2(\tilde{a}_1(x,y))_{|y=x}. \tag{32}$$

By (15) and (25), one has:

$$\begin{array}{rcl} C_1(\tilde{a}_2(x,y))_{|y=x} & = & C_1(|a_1(x,\bar{y})|^2 + a_2(x,\bar{y}) + a_2(y,\bar{x})) \\ & = & -\frac{\rho}{2}(a_1(x)^2 + 2a_2(x)) = -\frac{\rho}{2}(\frac{\rho^2}{4} + 2a_2(x)) \end{array} \tag{33}$$

$$
\begin{aligned}
C_2(\tilde{a}_1(x,y))_{|y=x} = & - \tfrac{1}{2}(\langle D^{'}\rho, D^{'}\tilde{a}_1\rangle + \langle D^{'}\tilde{a}_1, D^{'}\rho\rangle)_{|y=x} \\
& - 2a_1(x)(\tfrac{1}{3}\Delta\rho - \tfrac{1}{8}\rho^2 - \tfrac{1}{6}|\operatorname{Ric}|^2 + \tfrac{1}{24}|R|^2) \\
& = -\tfrac{1}{4}|D^{'}\rho|^2 - \rho((\tfrac{1}{3}\Delta\rho - \tfrac{1}{8}\rho^2 - \tfrac{1}{6}|\operatorname{Ric}|^2 + \tfrac{1}{24}|R|^2)),
\end{aligned} \tag{34}
$$

(The equality $(\langle D^{'}\rho, D^{'}\tilde{a}_1\rangle + \langle D^{'}\tilde{a}_1, D^{'}\rho\rangle)_{|y=x} = \frac{1}{2}|D^{'}\rho|^2$ follows by:

$$
\frac{\partial\tilde{a}_1}{\partial z_j}_{|y=x} = \frac{1}{2}\frac{\partial a_1}{\partial z_j} = \frac{1}{4}\frac{\partial\rho}{\partial z_j}, \quad \frac{\partial\tilde{a}_1}{\partial\bar{z}_j}_{|y=x} = \frac{1}{2}\frac{\partial a_1}{\partial\bar{z}_j} = \frac{1}{4}\frac{\partial\rho}{\partial\bar{z}_j}.)
$$

Thus, by inserting (33), (34), (31) and c_3 (given by (15)) into (32) one gets:

$$
\begin{aligned}
& a_3(x) = \tfrac{1}{8}\Delta\Delta\rho + \tfrac{1}{48}\rho(\rho^2 - 4|Ric|^2 + |R|^2) + \tfrac{\rho}{6}\Delta\rho \\
& -\tfrac{3}{8}L_{Ric}(\rho) - \tfrac{1}{4}|D^{'}Ric|^2 + \tfrac{1}{24}|D^{'}R|^2 - \tfrac{1}{4}R(Ric, Ric) \\
& +\tfrac{1}{12}\sigma_1(R) - \tfrac{1}{24}\sigma_2(R) - \tfrac{1}{6}\sigma_3(Ric)
\end{aligned}
$$

which, by using the formulae in the Appendix below, can be seen to be the same as the function $a_3(x)$ given by Lu in formula (10) above.

4 Appendix

In this section we recall Lu's notations and formulae used in this paper.

Let M be a n-dimensional complex manifold endowed with a Kähler metric g whose local expression is $g = \sum_{j\bar{k}}^{n} g_{j\bar{k}} dz_j d\bar{z}_k$.
The curvature tensor is defined as

$$
R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial\bar{z}_l} - \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k}\frac{\partial g_{p\bar{j}}}{\partial\bar{z}_l}, \ i,j,k,l = 1,\ldots,n
$$

The Ricci curvature is

$$
Ric_{i\bar{j}} = -\sum_{k,l=1}^{n} g^{k\bar{l}} R_{i\bar{j}k\bar{l}}, \ i,j = 1,\ldots,n
$$

and the scalar curvature is the trace of the Ricci curvature

$$
\rho = -\sum_{i,j=1}^{n} g^{i\bar{j}} Ric_{i\bar{j}}.
$$

The Laplace operator, denoted by Δ, is given by

$$
\Delta f = \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial^2 f}{\partial z_i \partial\bar{z}_j}.
$$

The following formulae can be found in Section 5 of [10] (cfr. also Proposition 5.2 there):

$$
\begin{aligned}
&|R|^2 = \sum_{i,j,k,l=1}^n |R_{i\bar{j}k\bar{l}}|^2 \\
&|Ric|^2 = \sum_{i,j=1}^n |Ric_{i\bar{j}}|^2 \\
&|D^{'}\rho|^2 = \sum_{i=1}^n |\tfrac{\partial\rho}{\partial z_i}|^2 \\
&|D^{'}Ric|^2 = \sum_{i,j,k=1}^n |Ric_{i\bar{j},k}|^2 \\
&|D^{'}R|^2 = \sum_{i,j,k,l,p=1}^n |R_{i\bar{j}k\bar{l},p}|^2 \\
&\mathrm{div}\,\mathrm{div}(\rho Ric) = 2|D^{'}\rho|^2 + \sum_{i,j=1}^n Ric_{i\bar{j}} \tfrac{\partial^2\rho}{\partial\bar{z}_j\partial z_i} + \rho\Delta\rho \\
&\mathrm{div}\,\mathrm{div}(R, Ric) = -\sum_{i,j=1}^n Ric_{i\bar{j}} \tfrac{\partial^2\rho}{\partial\bar{z}_j\partial z_i} - 2|D^{'}Ric|^2 \\
&+\sum_{i,j,k,l=1}^n R_{j\bar{i}l\bar{k}} R_{i\bar{j},k\bar{l}} - R(Ric, Ric) - \sigma_3(Ric) \\
&R(Ric, Ric) = \sum_{i,j,k,l=1}^n R_{i\bar{j}k\bar{l}} Ric_{j\bar{i}} Ric_{l\bar{k}} \\
&Ric(R, R) = \sum_{i,j,k,l,p,q=1}^n Ric_{i\bar{j}} R_{j\bar{k}p\bar{q}} R_{k\bar{i}q\bar{p}} \\
&\sigma_1(R) = \sum_{i,j,k,l,p,q=1}^n R_{i\bar{j}k\bar{l}} R_{l\bar{k}p\bar{q}} R_{q\bar{p}j\bar{i}} \\
&\sigma_2(R) = \sum_{i,j,k,l,p,q=1}^n R_{i\bar{j}k\bar{l}} R_{p\bar{i}q\bar{k}} R_{j\bar{p}l\bar{q}} \\
&\sigma_3(Ric) = \sum_{i,j,k=1}^n Ric_{i\bar{j}} Ric_{j\bar{k}} Ric_{k\bar{i}},
\end{aligned}
$$

where " ,p" represents the covariant derivative in the direction $\frac{\partial}{\partial z_p}$.

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