

Stable bundles and the first eigenvalue of the Laplacian

Claudio Arezzo, Alessandro Ghigi, Andrea Loi

Abstract

In this paper we study the first eigenvalue of the Laplacian on a compact manifold using stable bundles and balanced bases. Our main result is the following: let M be a compact Kähler manifold of complex dimension n and E a holomorphic vector bundle of rank r over M . If E is globally generated and its Gieseker point T_E is stable, then for any Kähler metric g on M

$$\lambda_1(M, g) \leq \frac{4\pi h^0(E)}{r(h^0(E) - r)} \cdot \frac{\langle c_1(E) \cup [\omega]^{n-1}, [M] \rangle}{(n-1)! \operatorname{vol}(M, [\omega])},$$

where $\omega = \omega_g$ is the Kähler form associated to g .

By this method we obtain, for example, a sharp upper bound for λ_1 of Kähler metrics on complex Grassmannians.

1 Introduction and statements of the results

The first eigenvalue of the Laplace operator is one of the most natural and studied riemannian invariants. A general question is how the underlying differentiable and topological structure is sensitive of this riemannian invariant. For example, in the case of a compact surface M^2 , it is known that the product $\lambda_1(M, g) \cdot \operatorname{vol}(M, g)$ is bounded above by a function of the genus only [21]. While for general riemannian metrics on higher dimensional manifolds similar results (substituting the genus with any geometric quantity) cannot hold, as proved in [4], it is natural to try to get upper bounds for some restricted natural classes of metrics.

When the underlying compact manifold is Kähler one studies the functional $\lambda_1(M, g) \cdot \operatorname{vol}(M, g)$ within a fixed Kähler class α allowing also complex invariants to appear in the upper bound. Since the volume is constant, this amounts to study the behaviour of λ_1 on the metrics g with $\omega_g \in \alpha$.

By the Rayleigh principle, upper bounds for the first eigenvalue are obtained by constructing functions with zero mean, sensitive to the geometry of the underlying manifold. If the manifold admits a map to a space where one is able to produce abundance of mean zero functions, one can try to import them on the manifold, thus getting an upper bound. This idea has been the core of the work of Hersch [13] and Yang-Yau [21] for Riemann surfaces, using holomorphic maps onto S^2 (see also [22] for further results on Riemann surfaces). This strategy ultimately relies on the possibility of using conformal diffeomorphisms of the two sphere without altering the holomorphicity of the map, in order to cover *all* riemannian metrics on the Riemann surface.

It is clear that this strategy needs a serious change for higher dimensional base manifolds. Such a generalization is possible if one restricts to Kähler metrics and the manifold is immersed in a projective space. Bourguignon, Li and Yau [3] have in fact shown that one can move the algebraic manifold with an automorphism of the projective space in such a way that the pull-back of a certain family of functions on \mathbb{P}^N have zero mean. These functions are in fact the components of the moment map for the action of $SU(N+1)$ on \mathbb{P}^N . It follows that for metrics g with $\omega_g \in \alpha$ the first eigenvalue is bounded above by an invariant depending on the immersion in projective space.

In the light of recent results of Xiaowei Wang [20], we observed that the theorem of Bourguignon, Li and Yau can be rephrased as a suitable stability property (stability of the Gieseker point) of any ample globally generated line bundle. This notion becomes in fact more interesting for higher rank vector bundles and the aim of this paper is precisely to investigate how we can use Gieseker stable vector bundles on Kähler manifolds to improve the upper bounds on λ_1 .

First observe that vector bundles give rise to maps (if globally generated) to Grassmannians. Roughly speaking Wang proved that the Gieseker point of a globally generated vector bundle is stable if and only if the associated map into the Grassmannian can be moved into a “balanced” position. Once this is achieved we can give the sought upper bound:

Theorem 1.1 *Let $E \rightarrow M$ be a holomorphic vector bundle of rank r over a compact Kähler manifold M of complex dimension n . Assume that*

1. *E is globally generated,*
2. *the Gieseker point T_E is stable.*

Then, for any Kähler metric g on M one has the eigenvalue estimate

$$\lambda_1(M, g) \leq \frac{4\pi h^0(E)}{r(h^0(E) - r)} \cdot \frac{\langle c_1(E) \cup [\omega]^{n-1}, [M] \rangle}{(n-1)! \text{vol}(M, [\omega])}, \quad (1)$$

where $\omega = \omega_g$ is the Kähler form associated to g .

In particular, if $\omega_g \in 2\pi c_1(L)$ for some line bundle L over M , then

$$\lambda_1(M, g) \leq \frac{2n h^0(E) \deg E}{r(h^0(E) - r) c_1(L)^n}, \quad (2)$$

where $\deg E = c_1(E) \cdot c_1(L)^{n-1}$.

If E is a line bundle this result reduces to the following generalized version of Bourguignon-Li-Yau's estimate:

Theorem 1.2 (*Bourguignon-Li-Yau*) *Let g be a Kähler metric on a Kähler manifold M and let E be a globally generated line bundle on M with $N = h^0(E) = \dim H^0(E)$. Let $\varphi_{\mathbf{t}} : M \rightarrow \mathbb{C}P^{N-1}$ be the Kodaira map in a basis $\mathbf{t} = (t_1, \dots, t_N)$ of $H^0(E)$. Then*

$$\lambda_1(M, g) \leq \frac{4nN}{N-1} d. \quad (3)$$

Here d is the so-called holomorphic immersion degree defined by

$$d = \frac{\int_M \varphi_{\mathbf{t}}^*(\sigma) \wedge \omega^{n-1}}{\int_M \omega^n},$$

$\omega = \omega_g$ and σ is half of the Fubini-Study form

$$\omega_{FS} = i\partial\bar{\partial} \log(|z_0|^2 + \dots + |z_{n-1}|^2)$$

on $\mathbb{C}P^{N-1}$ (and hence $[\sigma] = \pi c_1(\mathcal{O}(1))$). If moreover $\omega \in 2\pi c_1(L)$ as above, then $d = \frac{\deg E}{2 c_1(L)^n}$.

If $M = \mathbb{P}^n$ and L is the hyperplane bundle the estimate given by Theorem 1.2 is sharp, since the bound is realized by the Fubini-Study metrics.

It is then natural to suspect that our result gives a sharp estimate for the Grassmannian manifolds, which are now the natural target manifolds. This is indeed the case:

Theorem 1.3 *For any Kähler form ω_g on $M = \mathbb{G}(r, N)$ in the class $2\pi c_1(M)$ one has:*

$$\lambda_1(M, g) \leq 2.$$

It is easy to see that this bound cannot be achieved using line bundles and the theorem of Bourguignon, Li and Yau. Moreover the value $\lambda_1 = 2$ is indeed achieved by the symmetric Kähler-Einstein metric on $\mathbb{G}(r, N)$. Thus the symmetric metric is a maximum point of the functional λ_1 restricted to the set of Kähler metrics with fixed volume. This should be compared with a result of El Soufi and Ilias according to which the symmetric metric is a critical point (in suitable sense) for λ_1 on the set of *all* Riemannian metrics with fixed volume (see Remark 1 at p. 96 of [7]).

As mentioned above, our results are obtained by using balanced maps into Grassmannians, hence implicitly using en passant a special type of metrics on M (those obtained as pull back of the symmetric metric via these embeddings). The main point regarding these metrics is their abundance, since, as we prove adapting Wang's work, they are sensitive only of the stability of the Gieseker point of the vector bundle. This allows to prove general estimates for *all* Kähler metrics on M . On the other hand a stronger notion of "balanced" metric has been deeply studied in the last few years for line bundles (see e.g. [6], [1], [14] and references therein). While in general more rare (they in fact measure a non vacuous stability property of line bundles), it would be very interesting to know whether these metrics, when they exist, can give stronger informations on the spectrum of the laplacian. We leave this, and other question discussed in the last section, for future research.

Acknowledgements: We wish to thank Gian Pietro Pirola for many enlightening discussions concerning various aspects of this work. The first author wishes to thank also Kieran O'Grady for many helpful discussions about stability constructions. This work has been done while the authors were enjoying the peace, beauty and silence of Marina di Ragusa and Torre delle Stelle.

2 The set up and the proofs

2.1 Grassmannians

If W is a complex vector space we denote by $\mathbb{G}(r, W)$ the Grassmannian of r -dimensional *subspaces* of W . When $W = \mathbb{C}^N$ we will write $\mathbb{G}(r, N)$ for $\mathbb{G}(r, \mathbb{C}^N)$. We denote by $U = U_{r, N} \rightarrow \mathbb{G}(r, N)$ the *universal subbundle*. It is the subbundle of the trivial bundle $\mathbb{G}(r, N) \times \mathbb{C}^N$ whose fibre over a point $x \in \mathbb{G}(r, N)$ is simply the subspace $U_x = x$ represented by x . If a_1, \dots, a_r is a basis of U_x , consider the $N \times r$ matrix $A = (a_1, \dots, a_r)$, whose columns are

the vectors a_α . This means that if $a_\alpha = (a_{1\alpha}, \dots, a_{N\alpha})$ are the components of the vector a_α , then $A = (a_{i\alpha})$. (We let the Greek indices run over $1, \dots, r$ and the Latin ones over $1, \dots, N$.) We say that $A(x)$ is a *Stiefel matrix* for the point x or that $A(x)$ are *Stiefel coordinates* for x (see [10]). If a'_1, \dots, a'_r is another basis of U_x , there is a nonsingular matrix $C = (c_{\alpha\beta}) \in \text{GL}(r, \mathbb{C})$, such that $a'_\beta = c_{\alpha\beta} a_\alpha$. This means that $A' = AC$. Therefore the Stiefel coordinates are defined only up to right multiplication by a nonsingular $r \times r$ matrix. This simply reflects the fact that $\mathbb{G}(r, N) = M^*(N, r, \mathbb{C}) / \text{GL}(r, \mathbb{C})$, where $M^*(N, r, \mathbb{C})$ denotes the set of $N \times r$ matrices of maximal rank. In terms of Stiefel coordinates the standard action of $\text{GL}(N, \mathbb{C})$ on $\mathbb{G}(r, N)$ reads as follows: let x be a point in $\mathbb{G}(r, N)$, P an element of $\text{GL}(N, \mathbb{C})$ and A a Stiefel matrix for x ; then PA is a Stiefel matrix for Px .

Let now e_1, \dots, e_N be the standard basis of \mathbb{C}^N . For I a multiindex of length r put

$$U_I = \{x \in \mathbb{G}(r, N) : U_x + \text{span}(e_i : i \notin I) = \mathbb{C}^N\}.$$

For simplicity of notation assume $I = (0, \dots, 0)$. If $x \in U_0$ let A be some Stiefel coordinates of x . Then

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

where $A_1 \in \text{GL}(r, \mathbb{C})$ and A_2 is an $(N-r) \times r$ matrix. The affine coordinates of x in the chart U_0 are then given by the matrix $Z = A_1^{-1} A_2$. In other words the affine coordinates are the entries of the matrix Z such that $(I_r \ Z^t)^t$ be a Stiefel coordinate of x . Of course affine coordinates are honest holomorphic coordinates for the complex manifold $\mathbb{G}(r, N)$.

If $x \in \mathbb{G}(r, N)$ and a_1, \dots, a_r is a basis of U_x , then $a_1 \wedge \dots \wedge a_r$ is a nonzero element of $\Lambda^r \mathbb{C}^N$ defined up to multiplication by a nonzero scalar. Therefore it represents a well-defined point in $\mathbb{P}(\Lambda^r \mathbb{C}^N)$. The corresponding map is an embedding of $\mathbb{G}(r, N)$ in this projective space, and it is called the *Plücker embedding*. Observe that if we let $\mathcal{O}(1)$ be the hyperplane bundle on $\mathbb{P}(\Lambda^r \mathbb{C}^N)$ then $\det U^* = \mathcal{O}_{\mathbb{G}(r, N)}(1)$ and $K_{\mathbb{G}(r, N)} = \mathcal{O}_{\mathbb{G}(r, N)}(-N)$.

The constant metric on the fibres of $\mathbb{G}(r, N) \times \mathbb{C}^N$ induces a Hermitian metric on the subbundle U . Let $H_{\mathbb{G}}$ and $h_{\mathbb{G}} = \det H_{\mathbb{G}}$ be the induced metric on U^* and $\det U^* = \mathcal{O}_{\mathbb{G}(r, N)}(1)$ respectively. Put $\omega_{\mathbb{G}} = iR(h_{\mathbb{G}})$. Let $A = (a_{i\alpha})$ be a Stiefel matrix for $x \in \mathbb{G}(r, N)$ and let a_α be the columns of A . Then

$$\|a_1 \wedge \dots \wedge a_r\|_{h_{\mathbb{G}}}^2 = \det(A^* A),$$

where $h_{\mathbb{G}}^*$ is the induced metric on $\det U$. Assume for simplicity of notation that $x \in U_0$, and let Z be the affine coordinates of x . Then we can choose $A = (I_r \ Z^t)^t$. The corresponding basis of U_x is $\{e_\alpha + z_\alpha\}$, where z_α is the α -th column of Z . Then $s = (e_1 + z_1) \wedge \cdots \wedge (e_r + z_r)$ is a nonzero section of $\Lambda^r U$ over U_0 and $\|s\|^2 = \det(I_r + Z^*Z)$. Therefore

$$\omega_{\mathbb{G}} = i\partial\bar{\partial} \log \det(I_r + Z^*Z).$$

At the point $x = (I_r \ 0)^t$ the expression of $\omega_{\mathbb{G}}$ in the affine coordinates $Z = (z_{p\alpha})$ is simply

$$\omega_{\mathbb{G}}(x) = i \sum_{p=1}^{N-r} \sum_{\alpha=1}^r dz_{p\alpha} \wedge d\bar{z}_{p\alpha}. \quad (4)$$

When $r = 1$, $\omega_{\mathbb{G}}$ is just the Fubini-Study metric. When $r > 1$ it is the pull-back of the Fubini-Study metric on $\mathbb{P}^{\binom{N}{r}-1}$ via the Plücker embedding.

The standard action of $\mathrm{SU}(N)$ on $\mathbb{G}(r, N)$ is holomorphic and preserves $\omega_{\mathbb{G}}$. Its moment map $\mu_{\mathbb{G}} : \mathbb{G}(r, N) \rightarrow \mathfrak{su}(N)$ is given by

$$\mu_{\mathbb{G}}(x) = i \left(A(A^*A)^{-1}A^* - \frac{r}{N}I_N \right). \quad (5)$$

Here A are Stiefel coordinates of x and we identify $\mathfrak{su}(N)^*$ with $\mathfrak{su}(N)$ by means of the Killing scalar product $\langle X, Y \rangle = \mathrm{tr} X^*Y = -\mathrm{tr} XY$. The normalization is chosen so that

$$\int_{\mathbb{G}(r, N)} \mu_{\mathbb{G}} \mathrm{vol}_{\mathbb{G}} = 0. \quad (6)$$

We use the sign convention so that

$$d\langle \mu, v \rangle = -i\xi_v \omega_{\mathbb{G}} \quad (7)$$

where $v \in \mathfrak{su}(N)$ and ξ_v is the fundamental vector field of the action on $\mathbb{G}(r, N)$.

For later use we also recall also the following identity:

$$\omega_{\mathbb{G}} = -i \sum_{j,k=1}^N d\mu_{jk} \wedge d\mu_{kj}. \quad (8)$$

where μ_{jk} are the entries of $\mu_{\mathbb{G}}$. Just as for (4) and (5), it is enough to prove this formula at one point, thanks to the equivariance properties. A simple computation in affine coordinates shows that it holds at the origin of the chart.

2.2 Kempf-Ness theorem

Let G be complex reductive group, $K \subset G$ a maximal compact subgroup, W a linear representation of G and $\langle \cdot, \cdot \rangle$ a K -invariant Hermitian product on W . For $v \in W$ the function $\rho_v(g) = \log \|g^{-1}v\|$ is K -invariant and descends to a convex function ν_v on the symmetric space $X = G/K$. The moment map $\mu : \mathbb{P}(W) \rightarrow \mathfrak{k}$ for the action of K is given by $\mu([v]) = (d\rho_v)_e = (d\nu_v)_e$ where we consider $(d\nu_v)_e$ as an element of $(\sqrt{-1}\mathfrak{k})^* = \mathfrak{k}$.

A point $x = [v] \in \mathbb{P}(W)$ is said to be *semistable* if $0 \notin \overline{G \cdot v}$, and it is called *stable* if the orbit $G \cdot v$ is closed and the stabiliser G_v is finite.

Theorem 2.1 (*Kempf-Ness*) *A point $x = [v] \in \mathbb{P}(W)$ is semistable if and only if the function ν_v is bounded below. It is stable if and only if ν_v is proper if and only if μ has a unique zero on $G \cdot x$.*

2.3 Vector bundles, stability and balanced bases

Let E be a holomorphic vector bundle of rank r over a compact complex manifold M . Let $V = H^0(E)$ be the space of global holomorphic sections of E . Assuming that E is globally generated, for each $x \in M$ the subspace $V_x \subset V$ of sections vanishing at x is an $(N-r)$ -dimensional subspace. Denote by $\text{Ann}(V_x)$ its annihilator, that is

$$\text{Ann}(V_x) = \{\lambda \in V^* : \lambda \equiv 0 \text{ on } V_x\}.$$

Then one can define the so-called *Kodaira map*

$$\varphi_V : M \rightarrow \mathbb{G}(r, V^*), \quad x \mapsto \text{Ann}(V_x).$$

A choice of a basis s_1, \dots, s_N of V identifies $\mathbb{G}(r, V^*)$ with $\mathbb{G}(r, N) = \mathbb{G}(r, \mathbb{C}^N)$. We denote by

$$\varphi_{\mathbf{s}} : M \rightarrow \mathbb{G}(r, N)$$

the map φ_V written in the basis $\mathbf{s} = (s_1, \dots, s_N)$. If $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_r)$ is a local frame for E on a trivializing open set $U_{\boldsymbol{\sigma}} \subset M$ one has:

$$s_j = \sum_{\alpha=1}^r a_{j\alpha} \sigma_{\alpha}, \quad j = 1, \dots, N. \quad (9)$$

If $A(x)$ denotes the $N \times r$ matrix with complex entries $a_{j\alpha}(x)$ then $A : U_{\boldsymbol{\sigma}} \rightarrow M^*(N, r, \mathbb{C})$ is a local expression of $\varphi_{\mathbf{s}}$ in the Stiefel coordinates, that

is $A(x)$ is a Stiefel matrix for the point $\varphi_{\mathbf{s}}(x)$. By construction we have the commutative diagram

$$\begin{array}{ccc} E = \varphi_{\mathbf{s}}^* U_{r,N}^* & \longrightarrow & U_{r,N}^* \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi_{\mathbf{s}}} & \mathbb{G}(r, N). \end{array}$$

Denote by $k_{\mathbf{s}}$ the pull-back of $H_{\mathbb{G}}$ via the map $\varphi_{\mathbf{s}}$, namely

$$k_{\mathbf{s}} = \varphi_{\mathbf{s}}^*(H_{\mathbb{G}}). \quad (10)$$

Let $\omega = \omega_g$ be a fixed Kähler form on M . A basis $\mathbf{s} = (s_1, \dots, s_N)$ of V is called ω -balanced iff

$$\langle s_j, s_k \rangle_{k_{\mathbf{s}}, \omega} := \int_M k_{\mathbf{s}}(s_j, s_k) \frac{\omega^n}{n!} = \frac{r \operatorname{vol}(M, g)}{N} \delta_{jk}. \quad (11)$$

In other words a basis \mathbf{s} of V is ω -balanced iff, up to the product with the positive constant $(r/N) \operatorname{vol}(M, g)$, it is an orthonormal basis of V with respect to the L^2 -product $\langle \cdot, \cdot \rangle_{k_{\mathbf{s}}, \omega}$ defined using the metric $k_{\mathbf{s}}$ and the volume $\omega^n/n!$.

If $\sigma_1, \dots, \sigma_r$ is a local frame for E and s_j are given by (9) then

$$\left(k_{\mathbf{s}}(s_j(x), s_k(x)) \right)_{j,k} = A(AA^*)^{-1} A^*$$

that is

$$\mu_{jk}(\varphi_{\mathbf{s}}(x)) = k_{\mathbf{s}}(s_j(x), s_k(x)) - \frac{r}{N} \delta_{jk}.$$

Therefore the basis \mathbf{s} is ω -balanced if and only if

$$\int_M \mu_{\mathbb{G}} \circ \varphi_{\mathbf{s}} \frac{\omega^n}{n!} = 0. \quad (12)$$

An important theorem of Xiaowei Wang (conjectured by Donaldson in the case of Riemann surfaces) relates ω -balanced metrics to stable bundles when $\omega \in 2\pi c_1(L)$. In order to state it (see Theorem 2.2 below) we recall some definitions. Let (M, L) be a polarised projective manifold. This means that M is a Kähler manifold and L is an ample line bundle over M . Given a holomorphic vector bundle E over M of rank r , define the *degree* $\deg E$ and the *slope* $\mu(E)$ of E (with respect to the polarisation L) by the formulas

$$\deg E = c_1(E) \cdot c_1(L)^{n-1} \quad \mu(E) = \frac{\deg E}{r}. \quad (13)$$

Set $E(m) = E \otimes L^m$ and $p_E(m) = (1/r) \cdot \chi(E(m))$, where χ denotes the Euler characteristic of a sheaf. The vector bundle E is said to be *Gieseker stable* if for any coherent subsheaf $F \subset E$, and m sufficiently large (depending on F) one has the inequality $p_F(m) < p_E(m)$. On the other hand E is said to be *Mumford-Takemoto stable* (or simply *Mumford stable* or *slope stable*) if for any coherent subsheaf F of E one has $\mu(F) < \mu(E)$. Since $\mu(E)$ is the leading coefficient of p_E , Mumford stability is a stronger condition than Gieseker stability (see e.g. [8], Chapter 4). In order to study the Gieseker stability of a globally generated bundle E , Gieseker [11] considered the linear map

$$T_E : \Lambda^r H^0(E) \longrightarrow H^0(\det E), (s_1, \dots, s_r) \mapsto s_1 \wedge \dots \wedge s_r. \quad (14)$$

This map, regarded as a point in $\mathbb{P}(\text{Hom}(\Lambda^r H^0(E), H^0(\det E)))$ is called the *Gieseker point* of E . On this projective space there is a natural action of $\text{SL}(V)$, $V = H^0(E)$, and so it makes sense to speak about the stability of T_E . Observe that the stability of the Gieseker point does not involve the choice of an ample line bundle L .

In the case where X is a projective surface, Gieseker showed that a vector bundle on (X, L) is Gieseker stable if and only if $T_{E(m)}$ is stable for sufficiently large m (see Theorem 0.7 in [11]). Wang, on the other hand, used the Gieseker point to prove the following theorem (see Theorem 1.1 in [20]).

Theorem 2.2 (Wang) *Let E be a holomorphic vector bundle on a polarised projective manifold (M, L) and let $\omega \in 2\pi c_1(L)$ be a Kähler form. Then E is Gieseker stable iff there is a m_0 such that for all $m \geq m_0$ $E(m)$ admits an ω -balanced basis.*

The result we actually need is the following slightly different version of this theorem, where the polarization does not play any role.

Lemma 2.3 *Let E be a holomorphic vector bundle of rank r over a compact Kähler manifold M , and let ω be a Kähler form on M . If E is globally generated and the Gieseker point T_E is stable, then $V = H^0(E)$ admits an ω -balanced basis.*

Sketch of the proof. Set $W = H^0(\det E)$ and $\mathbb{W} = \text{Hom}(\Lambda^r V, W)$. Fix an arbitrary Hermitian metric h on E and consider the L^2 -scalar product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{h, \omega}$ on V built from ω and h . Let $\mathbf{s} = \{s_1, \dots, s_N\}$ be an orthonormal basis with respect to this product and $\varphi = \varphi_{\mathbf{s}} : M \rightarrow \mathbb{G}(r, N)$

the corresponding map to the Grassmannian. On the line bundle $\det E$ consider the metric $k_{\mathbf{s}}$ (see (10)) and let $\langle \cdot, \cdot \rangle_{\mathbb{W}}$ and $\|\cdot\|_{\mathbb{W}}$ be respectively the Hermitian inner product and the norm gotten on \mathbb{W} using $\langle \cdot, \cdot \rangle$ on V and the L^2 -metric $\langle \cdot, \cdot \rangle_{L^2}$ on W . Since \mathbf{s} is an orthonormal basis this means that for $\alpha \in \mathbb{W}$

$$\|\alpha\|_{\mathbb{W}}^2 = \sum_I \|\alpha(s_{i_1}, \dots, s_{i_r})\|_{L^2}^2 \quad (15)$$

the sum being taken over all r -indices $I = (i_1, \dots, i_r)$ such that $i_1 < \dots < i_r$. An application of the Kempf-Ness theorem 2.1 ensures that the function $\|g^{-1} \cdot T_E\|_{\mathbb{W}} = \exp(\nu(g))$ admits a minimum on $X = \mathrm{SL}(V)/\mathrm{SU}(V)$ and is proper on geodesics transversal to $\exp(i\mathfrak{k})$, where \mathfrak{k} is the Lie algebra of the stabilizer of T_E inside $\mathrm{SU}(V)$. Here ν is the Kempf-Ness function based at the point T_E for the action of $\mathrm{SL}(V)$ on $\mathbb{P}(\mathbb{W})$. By (15)

$$\exp(\nu(g)) = \sum_I \|g^{-1}T_E(s_I)\|_{L^2}^2.$$

So for any $g \in \mathrm{SL}(V)$ there is some multi-index I_g such that

$$\|g^{-1}T_E(s_{I_g})\|_{L^2}^2 \geq \frac{\exp(\nu(g))}{K}$$

where $K = \binom{N}{r}$. If we set $\varepsilon(g) = \sqrt{K \exp(-\nu(g))}$, then

$$\|\varepsilon(g)(g^{-1}T_E)(s_{I_g})\|_{L^2}^2 \geq 1$$

so Lemma 3.6 in [20] implies that there is a $C_1 \in \mathbb{R}$ such that for any $g \in \mathrm{SL}(V)$

$$\int_M \log \|\varepsilon(g^{-1}T_E)(s_{I_g})\|_{k_{\mathbf{s}}}^2 \frac{\omega^n}{n!} \geq C.$$

Therefore we get the inequality

$$L(g) := \int_M \left(\sum_I \|\varepsilon(g^{-1}T_E)(s_I)\|_{k_{\mathbf{s}}}^2 \right) \frac{\omega^n}{n!} \geq \nu(g) + C_2.$$

It follows that the function L is proper on $\mathrm{SL}(V)/\mathrm{SU}(V)$ and it must attain its minimum at some point g . Then gs_1, \dots, gs_N is the desired ω -balanced basis. This concludes the proof.

Q.E.D.

2.4 The proofs

Proof of Theorem 1.1. Let r be the rank of E and $N = h^0(E)$. Hypotheses 1. and 2. and Lemma 2.3 yield the existence of an ω -balanced basis $\mathbf{s} = (s_1, \dots, s_N)$ of $V = H^0(E)$. Denote by $\varphi = \varphi_{\mathbf{s}}$ the holomorphic map $\varphi : M \rightarrow \mathbb{G}(r, N)$ obtained using the sections $\mathbf{s} = (s_1, \dots, s_N)$. Let $F : M \rightarrow \mathfrak{su}(N)$ be the matrix function $F(x) = \mu_{\mathbb{G}}(\varphi(x))$ and let f_{jk} be the entries of F . The balanced condition, as rephrased in (12), says that

$$\int_M f_{jk} \frac{\omega^n}{n!} = 0.$$

In fact the only use of the balanced metric is to provide us with these test functions. Using the Rayleigh principle we get

$$\lambda_1(M, g) \leq \frac{\int_M |\nabla f_{jk}|^2 \frac{\omega^n}{n!}}{\int_M |f_{jk}|^2 \frac{\omega^n}{n!}}.$$

Thus

$$\lambda_1(M, g) \cdot \left(\sum_{j,k=1}^N \int_M |f_{jk}|^2 \frac{\omega^n}{n!} \right) \leq \sum_{j,k=1}^N \int_M |\nabla f_{jk}|^2 \frac{\omega^n}{n!}. \quad (16)$$

We claim that

$$\sum_{j,k=1}^N \int_M |f_{jk}|^2 \frac{\omega^n}{n!} = \frac{r(N-r)}{N} \text{vol}(M, g) \quad (17)$$

and

$$\sum_{j,k=1}^N \int_M |\nabla f_{jk}|^2 \frac{\omega^n}{n!} = \frac{4\pi}{(n-1)!} \langle c_1(E) \cup [\omega]^{n-1}, [M] \rangle. \quad (18)$$

To prove (17) observe that

$$\sum_{j,k=1}^N |f_{jk}(x)|^2 = \|F(x)\|^2 = \|\mu_{\mathbb{G}}(\varphi(x))\|^2.$$

Since the moment map is $\text{SU}(N)$ -equivariant its norm is constant on the Grassmannian. Calculating at the point x_0 with Stiefel matrix $(I_r \ 0)^t$ we get

$$\sum_{j,k=1}^N |f_{jk}(x)|^2 = \|F(x_0)\|^2 = \frac{r(N-r)}{N}.$$

From this (17) follows immediately.

In order to prove (18) observe that for any $f \in C^\infty(M, \mathbb{C})$ we have

$$|\nabla f|^2 \omega^n = n(i\partial f \wedge \bar{\partial} \bar{f} + i\partial \bar{f} \wedge \bar{\partial} f) \wedge \omega^{n-1}. \quad (19)$$

Therefore

$$\begin{aligned} |\nabla f_{jk}|^2 \frac{\omega^n}{n!} &= \frac{1}{(n-1)!} (i\partial f_{jk} \wedge \bar{\partial} \bar{f}_{jk} + i\partial \bar{f}_{jk} \wedge \bar{\partial} f_{jk}) \wedge \omega^{n-1} = \\ &= -\frac{i}{(n-1)!} \varphi^* (\partial \mu_{jk} \wedge \bar{\partial} \bar{\mu}_{kj} + \partial \mu_{kj} \wedge \bar{\partial} \bar{\mu}_{jk}) \wedge \omega^{n-1}. \end{aligned}$$

Using (8) we get

$$\sum_{j,k=1}^N |\nabla f_{jk}|^2 \frac{\omega^n}{n!} = \frac{2}{(n-1)!} \varphi^* (\omega_{\mathbb{G}}) \wedge \omega^{n-1}.$$

To get (18) it is enough to recall that $[\varphi^*(\omega_{\mathbb{G}})] = 2\pi c_1(E)$.

Now substitute (17) and (18) in (16). Recalling that $\int_M \omega^n = n! \operatorname{vol}(M, g)$, one immediately gets (1).

Q.E.D.

Proof of Theorem 1.3.

The proof will follow applying Theorem 1.1 to the bundle $E = U^*$ and $L = -K$ where $U = U_{r,N}$ and K are respectively the universal subbundle and the canonical bundle on $M = \mathbb{G}(r, N)$. Observe that if $H^0(U^*) = V$ then $H^0(\det U^*) = \Lambda^r V$. Therefore, it is easily seen that the Gieseker point T_{U^*} is simply the identity map Id of $\Lambda^r V$, the action of $a \in \operatorname{SL}(V)$ on V is just the pull-back and the action of $\operatorname{SL}(V)$ on $\operatorname{Hom}(\Lambda^r V, \Lambda^r V)$ is given by

$$(a \cdot \Phi)(s_1 \wedge \cdots \wedge s_r) = \Phi(as_1 \wedge \cdots \wedge as_r) \quad (20)$$

where $a \in \operatorname{SL}(V)$, $s_j \in V$ and $\Phi \in \operatorname{Hom}(\Lambda^r V, \Lambda^r V)$.

To apply Theorem 1.1 we need to check that $T_{U^*} = Id$ is stable, i.e. its stabiliser is finite and its $\operatorname{SL}(V)$ -orbit is closed.

If $a \cdot I = I$ then $as_1 \wedge \cdots \wedge as_r = s_1 \wedge \cdots \wedge s_r$ for any $s_1 \wedge \cdots \wedge s_r \in \Lambda^r V$. It follows that $a = I$. Therefore the stabiliser of $T_{U^*} = I$ is trivial.

If the orbit of $\operatorname{SL}(V)$ through I were not closed, by the Hilbert-Mumford criterion (see e.g. Theorem 4.2 in [2]) there would be a non-trivial algebraic one-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow \operatorname{SL}(V)$ such that

$$\lim_{t \rightarrow 0} \lambda(t) \cdot I = T_\infty \quad (21)$$

for some $T_\infty \in \text{Hom}(\Lambda^r V, \Lambda^r V)$. Let s_1, \dots, s_N be a basis of V such that $\lambda(t)s_j = t^{m_j}s_j$. Since λ is a 1-parameter subgroup in $\text{SL}(V)$, $m_1 + \dots + m_N = 0$. Assume $m_1 \geq m_2 \geq \dots \geq m_N$. As λ is non-trivial, we have $m_1 > 0 > m_N$. We claim that $m_1 + \dots + m_r > 0$. In fact, assume that $m_j \geq 0$ for $j \leq s$ and $m_j < 0$ for $j > s$. If $s \geq r$, the sum $m_1 + \dots + m_r$ is clearly positive, since the first term is positive and the others are nonnegative. If instead $s < r$, then $m_{r+1} + \dots + m_N < 0$ since all terms are negative. Therefore

$$m_1 + \dots + m_r = -(m_{r+1} + \dots + m_N) > 0. \quad (22)$$

Then we have indeed $m_1 + \dots + m_r > 0$. But then

$$T_\infty(s_1, \dots, s_r) = \lim_{t \rightarrow 0} (\lambda(t) \cdot I)(s_1, \dots, s_r) = \quad (23)$$

$$= \lim_{t \rightarrow 0} \lambda(t)s_1 \wedge \dots \wedge \lambda(t)s_r = \lim_{t \rightarrow 0} t^{m_1 + \dots + m_r} s_1 \wedge \dots \wedge s_r. \quad (24)$$

This is impossible since the right hand side diverges. Therefore the orbit is closed, and T_{U^*} is stable. The assumptions of Theorem 1.1 are therefore satisfied. Therefore (2) with $\omega_g \in 2\pi c_1(M) = 2\pi c_1(-K)$ and $N_r = r(N - r) = \dim M$ yields the estimate

$$\begin{aligned} \lambda_1(M, g) &\leq 2 \frac{N_r h^0(U^*)}{r(h^0(U^*) - r)} \frac{\deg(U^*)}{c_1(-K)^{N_r}} = 2 \frac{N_r N}{N_r} \frac{c_1(\mathcal{O}(1)) \cdot c_1(-K)^{N_r-1}}{c_1(-K)^{N_r}} \\ &= 2 \frac{c_1(-K) \cdot c_1(-K)^{N_r-1}}{c_1(-K)^{N_r}} = 2, \end{aligned}$$

where the second equality follows from $K = \mathcal{O}(-N)$.

Q.E.D.

3 Final remarks

Let us indicate some lines of future research which we feel are worth pursuing in light of our result.

If M is a Fano manifold and g is a Kähler-Einstein metric, then $\lambda_1(M, g) \geq 2$ and equality holds if and only if M admits nonzero holomorphic vector fields. This follows from work of Futaki, see [9], p.40ff. Therefore Theorem 1.1 could be used to rule out the existence of Kähler-Einstein metrics on Fano manifolds. In fact if M is an n -dimensional Fano manifold and E a globally generated rank r vector bundle over M such that

$$\frac{n h^0(E) c_1(E) \cdot c_1(-K)^{n-1}}{r(h^0(E) - r) c_1(-K)^n} < 1, \quad (25)$$

then either the Gieseker point T_E is not stable or M does not admit a Kähler-Einstein metric. If M does not have any nontrivial holomorphic vector field, the equality in (25) is enough to get the conclusion.

We do not know any example of a Fano manifold with a *line* bundle E for which (25) holds. We believe such examples, if any, would be quite interesting in view of the connection with Kähler-Einstein metrics. If $\text{Pic}(M) = \mathbb{Z}$ one can rule out the existence of such line bundles using a classical result of Kobayashi and Ochiai, according to which the index of a Fano manifold cannot exceed $n + 1$.

We believe the extension to higher rank vector bundles should on the contrary forbid some Fano manifold to have a Kähler-Einstein metric.

If M is a surface of genus g , Yang and Yau proved that

$$\lambda_1(M, g) \cdot \text{vol}(M, g) \leq 8\pi \left[\frac{g+3}{2} \right]. \quad (26)$$

Optimal estimates are only known for $g = 0$ or $g = 1$ and for $g = 1$ the estimate (26) is not sharp, see [17]. It is therefore natural to tackle this problem with the help of Theorem 1.1. Unfortunately it is not easy to construct bundles that improve (26). To get the best possible estimate, one has to minimize the ratio

$$\frac{h^0(E)}{r(h^0(E) - r)} \cdot \deg E$$

among globally generated rank r vector bundles with stable Gieseker point. For rank $r = 1$ the best possible choice is $h^0(E) = 2$ and

$$\deg E = \left[\frac{g+3}{2} \right].$$

For higher rank the existence of globally generated stable bundles of given rank and degree with fixed h^0 is still unanswered (see [18]). But it seems very hard to improve the estimate using vector bundles of higher rank.

References

- [1] C. Arezzo and A. Loi, *Moment maps, scalar curvature and quantization of Kähler manifolds*, Comm. Math. Phys. 243 (2004), 543–559.
- [2] D. Birkes, *Orbits of linear algebraic groups*, Ann. of Math. 93(2) (1971), 459–475.

- [3] J.P. Bourguignon, P. Li, S. T. Yau, *Upper bound for the first eigenvalue of algebraic submanifolds*, Comment. Math. Helvetici 69 (1994), 199-207.
- [4] B. Colbois and J. Dodziuk, *Riemannian metrics with large λ_1* , Proc. Amer. Math. Soc., 122 (1994), 905-906.
- [5] I. Dolgachev, *Lectures on invariant theory*, Cambridge Univ. Press, Cambridge, 2003.
- [6] S. Donaldson, *Scalar curvature and projective embeddings*, Journ. Diff. Geom., 59 (2001), 479-522.
- [7] A. El Soufi and S. Ilias, *Riemannian manifolds admitting isometric immersions by their first eigenfunctions*, Pacific J. Math. 195(1) (2000), 91-99.
- [8] Robert Friedman, *Algebraic surfaces and holomorphic vector bundles*, Universitext. Springer-Verlag, New York, 1998.
- [9] Akito Futaki, *Kähler-Einstein metrics and integral invariants*. Springer-Verlag, Berlin, 1988.
- [10] I. M. Gel'fand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*. Birkhäuser, Boston, 1994.
- [11] D. Gieseker, *On the moduli of vector bundles on an algebraic surface*. Ann. of Math. 106(1) (1977), 45-60.
- [12] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley and Sons Inc. 1978.
- [13] J. Hersch, *Quatre propriétés isopérimétriques de membranes sphérique homogènes*, C. R. Acad. Paris 270 (1970), 1645-1648.
- [14] H. Luo, *Geometric criterion for Mumford-Gieseker stability of polarized manifolds*. Journ. Diff. Geom. 49 (1998), 577-599.
- [15] Shigeru Mukai, *An introduction to invariants and moduli*, Cambridge Univ. Press, Cambridge, 2003.
- [16] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, third edition, 1994.
- [17] N. Nadirashvili, *Berger's isoperimetric problem and minimal immersions of surfaces*, *Geom. Fun. Anal.*, 6(5) (1996), 877-898.
- [18] M. Teixidor i Bigas, *Brill-Noether theory for stable vector bundles*, *Duke Math. J.*, 62 (2), 1991, 385-400.
- [19] X. Wang, *Canonical metrics and stability of vector bundles over a projective manifold*, Ph. D. thesis 2002.
- [20] X. Wang, *Balance point and stability of vector bundles over a projective manifold*, Math. Res. Lett. 9 (2002), 393-411.
- [21] P. C. Yang and S. T. Yau, *Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds*, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 7(1) (1980), 55-63.

- [22] Shing-Tung Yau. An application of eigenvalue estimate to algebraic curves defined by congruence subgroups. *Math. Res. Lett.*, 3(2):167–172, 1996.

Università di Parma,
E-mail: `claudio.arezzo@unipr.it`

Università di Milano Bicocca,
E-mail: `alessandro.ghigi@unimib.it`

Università di Cagliari,
E-mail: `loi@unica.it`