Report for "RIGIDITY PROPERTIES OF HOLOMORPHIC ISOMETRIES INTO HOMOGENEOUS KÄHLER MANIFOLDS"

The holomorphic isometry between complex manifolds have been a classical problem in complex geometry. Calabi systematically studied the existence, rigidity of holomorphic isometries. Umehara later uses Calabi's method to characterize the Kähler-Einstein submanifolds in the complex space forms and he also proved that two complex space forms with different curvature signs cannot share a common Kähler submanifold. Later, Di Scala and Loi called two complex manifolds that share a common Kähler submanifold relatives. Combining the results of Di Scala-Loi and Huang-Yuan, it is known that two Hermitian symmetric spaces of different types are not relatives. More recently, Loi and Mossa extend the relativity problem to homogeneous manifolds and proved that the Euclidean space and the homogeneous bounded domain are not relatives. On the other hand, Loi and Mossa also proved the triviality for Kähler-Ricci solitons that are holomorphically isometrically embedded in the indefinite complex space forms and the homogeneous bounded domains. Namely, the Kähler-Ricci soliton must be Kähler-Einstein.

In this paper, the author characterizes the holomorphic isometric immersion from the Kähler-Ricci solitons into the special flag manifolds. In particular, one main theorem of the paper shows that such Kähler-Ricci solitons must be the Kähler-Einstein metrics. The other main theorem concerns the non-relativity for any two objects among the complex Euclidean space, the bounded homogeneous domain and the special flag manifold. The main contribution of the paper is to extend the existing non-relativity results and the triviality of the KRS to the special flag manifold. These extensions are interesting. Thus I recommend the paper be published in the Proceedings of AMS with the comments below taken into consideration.

Some comments are as follows:

- Page 2, line 7, the statement "most of the work have been done by Mok" does not truly describe the state of the art of this direction. Mok for sure has done the fundamental works in this direction, while many other authors such as Chan, Ng, Upmeier, Wang, Xiao, Yuan, G. Zhang, Y. Zhang also have major contribution. To name a few deep results:
  - Mok, Ngaiming; Ng, Sui Chung. Germs of measure-preserving holomorphic maps from bounded symmetric domains to their Cartesian products. J. Reine Angew. Math. 669 (2012), 47–73.
  - Yuan, Yuan; Zhang, Yuan. Rigidity for local holomorphic isometric embeddings from B^n into $B^{N_1} \times \cdots \times B^{N_m}$ up to conformal factors. J. Differential Geom. 90 (2012), no. 2, 329–349.
  - Upmeier, Harald; Wang, Kai; Zhang, Genkai. Holomorphic isometries from the unit ball into symmetric domains. Int. Math. Res. Not. IMRN 2019, no. 1, 55–89.
  - Xiao, Ming; Yuan, Yuan. Holomorphic maps from the complex unit ball to type IV classical domains. J. Math. Pures Appl. (9) 133 (2020), 139–166.
  - Chan, Shan Tai; Mok, Ngaiming. Asymptotic total geodesy of local holomorphic curves exiting a bounded symmetric domain and applications to a uniformization problem for algebraic subsets. J. Differential Geom. 120 (2022), no. 1, 1–49.
  - Xiao, Ming. Holomorphic isometric maps from the complex unit ball to reducible bounded symmetric domains. J. Reine Angew. Math. 789 (2022), 187–209.
- It would be more friendly to the reader if the definition of the generalized flag manifold of classical type is given.
- Page 3, it is better to reformulate the statement Theorem 1.1 as follows. "Let (M, g) be a KRS. If (M, g) admits a holomorphic isometry into the Kähler product .... , then g is KE".

- Page 3, line -7: ”immediately”, line -6: ”a Kähler manifold”, line -1: ”a non-trivial KRS”. The author should read the paper thoroughly to correct all obvious typos like these.
- Page 7, ”$\partial_w^{\log}\Psi$” reads ”$\partial_w \log \Psi$”.