KÄHLER IMMERSIONS OF HOMOGENEOUS KÄHLER MANIFOLDS INTO COMPLEX SPACE FORMS

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Abstract. In this paper we study the homogeneous Kähler manifolds (h.K.m.) which can be Kähler immersed into finite or infinite dimensional complex space forms. On one hand we completely classify the h.K.m. which can be Kähler immersed into a finite or infinite dimensional complex Euclidean or hyperbolic space. Moreover, we extend known results about Kähler immersions into the finite dimensional complex projective space to the infinite dimensional setting.

1. Introduction and statements of the main results

In this paper we address the following problem: *classify all homogeneous Kähler manifolds (*h.K.m. *for short) which admit a Kähler immersion into a given finite or infinite dimensional complex space form.*

A Kähler immersion $f:(M,g)\to(S,g_S)$ from a Kähler manifold (M,g) into a complex space form (S,g_S) is a holomorphic map such that $f^*g_S=g$ (here g and g_S denote the Kähler metrics on M and S respectively).

Recall that there are three types, up to homotheties, of complex space forms (S,g_S) according to the sign of their constant holomorphic sectional curvature:

- the complex Euclidean space $\mathbb{C}^N$, $N\leq\infty$, with the flat metric denoted by g_0. Here $\mathbb{C}^\infty$ is the complex Hilbert space $\ell^2(\mathbb{C})$ consisting of sequences $z_j, j=1\ldots, z_j\in\mathbb{C}$ such that $\sum_{j=1}^{+\infty}|z_j|^2<+\infty$.
- the complex hyperbolic space $\mathbb{C}H^N$, $N\leq\infty$, namely the unit ball in $\mathbb{C}^N$ ($\sum_{j=1}^{N}|z_j|^2<1$) endowed with the hyperbolic metric g_{hyp} of holomorphic sectional curvature being -4, whose associated Kähler form ω_{hyp} is given by:

$$\omega_{hyp}=-\frac{i}{2}\partial\bar\partial\log(1-\sum_{j=1}^{N}|z_j|^2). \tag{1}$$

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- the complex projective space $\mathbb{C}P^N, N \leq \infty$, with the Fubini–Study metric g_{FS} of holomorphic sectional curvature being 4. If ω_{FS} denotes the Kähler form associated to g_{FS} then, in homogeneous coordinates $[Z_0, \ldots, Z_N]$, $\omega_{FS} = \frac{i}{2}\partial\bar{\partial}\log\sum_{j=0}^{N}|Z_j|^2$.

Notation. When we speak about the Kähler manifold $\mathbb{C}^N$ (resp. $\mathbb{C}H^N$ or $\mathbb{C}P^N$) without mentioning the Kähler metric we will always mean $\mathbb{C}^N$ (resp. $\mathbb{C}H^N$ or $\mathbb{C}P^N$) equipped with the metric g_0 (resp. g_{hyp}, g_{FS}).

Note that, once that a Kähler immersion into a complex space form (S, g_S) is given, then all other Kähler immersions can be obtained by composing it with a unitary transformation of (S, g_S). This is due to the following celebrated rigidity theorem due to E. Calabi [Ca53] which will be of constant use throughout this paper.

Theorem (Calabi's rigidity theorem) *Let $f : (M, g) \to (S, g_S)$ and $\tilde{f} : (M, g) \to (S, g_S)$ be two Kähler immersions into the same complex space form (S, g_S). Then there exists a unitary transformation U of (S, g_S) such that $f = U \circ \tilde{f}$.*

1.1. **Immersions in $\mathbb{C}^N$ and $\mathbb{C}H^N$.** In the following two theorems we give a complete solution of our problem when the ambient space is $\mathbb{C}^N$ or $\mathbb{C}H^N$, $N \leq \infty$. In order to state our result note that the map $f_n : \mathbb{C}H^n \to l^2(\mathbb{C})$ given by:

$$z = (z_1, \ldots z_n) \overset{f_n}{\mapsto} (\ldots, \sqrt{\frac{(|j|-1)!}{j!}} z_1^{j_1} \cdots z_n^{j_n}, \ldots) \tag{2}$$

is a Kähler immersion of $\mathbb{C}H^n$ into $l^2(\mathbb{C})$, i.e. $f_n^* g_0 = g_{hyp}$, (see [Ca53]), where $|j| = j_1 + \cdots + j_n$ and $j! = j_1! \cdots j_n!$.

Theorem 1. *Let (M, g) be a n-dimensional h.K.m..*

(a) *If (M, g) can be Kähler immersed into $\mathbb{C}^N$, $N < \infty$, then $(M, g) = \mathbb{C}^n$;*

(b) *if (M, g) can be Kähler immersed into $\ell^2(\mathbb{C})$, then (M, g) equals*

$$\mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_r}_{\lambda_r},$$

where $k + n_1 + \cdots + n_r = n$, λ_j, $j = 1, \ldots, r$ are positive real numbers and $\mathbb{C}H^{n_j}_{\lambda_j} = (\mathbb{C}H^{n_j}, \lambda_j g_{hyp})$, $j = 1, \ldots, r$ (hence $\mathbb{C}H^n_1 = \mathbb{C}H^n$).

Moreover, in case (a) (resp. case (b)) the immersion is given, up to a unitary transformation of $\mathbb{C}^N$ (resp. $\ell^2(\mathbb{C})$), by the linear inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^N$ (resp. by $(f_0, f_1, \ldots, f_r)$, where f_0 the linear inclusion $\mathbb{C}^k \hookrightarrow \ell^2(\mathbb{C})$ and each $f_j : \mathbb{C}H^{n_j} \to \ell^2(\mathbb{C})$ is $\sqrt{\lambda_j}$ times the map (2)).

Theorem 2. *Let (M, g) be a n-dimensional h.K.m.. Then if (M, g) can be Kähler immersed into $\mathbb{C}H^N$, $N \leq \infty$, then $(M, g) = \mathbb{C}H^n$ and the immersion* *is given, up to a unitary transformation of* $\mathbb{C}H^N$*, by the linear inclusion* $\mathbb{C}H^n \hookrightarrow \mathbb{C}H^N$

Remark 1. Since a Kähler immersion is minimal, an alternative proof of (1) in Theorem 1 when $N < \infty$ follows by the work of A. J. Di Scala [DS02].

Remark 2. Assertion (2) in Theorem 1 is a generalization to arbitrary h.K.m. of Theorem 3.3 in [DL07] where the first and the third authors proved that a bounded symmetric domain which can be Kähler immersed into $\ell^2(\mathbb{C})$ has necessarily rank one. Actually, the method of the present paper, when applied to bounded symmetric domains, provides us with an alternative and more elegant proof of this result (cfr. Remark 5 below).

1.2. **Immersion in $\mathbb{C}P^N$.** There exists a large class (cfr. Conjecture 1 below) of h.K.m. which can be Kähler immersed into $\mathbb{C}P^N$. In this paper a Kähler metric g on a complex manifold M will be called *projectively induced* if there exists an immersion $f : M \to \mathbb{C}P^N$, $N \leq \infty$, such that $f^* g_{FS} = g$. An obvious necessary condition for g to be projectively induced is that its associated Kähler form ω is integral i.e. it represents the first Chern class $c_1(L)$ in $H^2(M, \mathbb{Z})$ of a holomorphic line bundle $L \to M$. Indeed L can be taken as the pull-back of the hyperplane line bundle on $\mathbb{C}P^N$ whose first Chern class is given by ω_{FS}. Notice that if ω is an exact form (e.g. when M is contractible) then ω is obviously integral since its second cohomology class vanishes.

Other (less obvious) conditions are expressed by the following theorem and its corollary which represent our first result about projectively induced Kähler metrics.

Theorem 3. *Assume that a h.K.m.* (M, g) *admits a Kähler immersion* $f : M \to \mathbb{C}P^N$*,* $N \leq \infty$*. Then* M *is simply-connected and* f *is injective.*

Corollary 3. *Let* (M, g) *be a complete and locally h.K.m.. Assume that* $f : (M, g) \to \mathbb{C}P^N$*,* $N \leq \infty$*, is a Kähler immersion. Then* (M, g) *is a h.K.m..*

When the dimension of the ambient space is finite, i.e. $(S, g_S) = \mathbb{C}P^N$, $N < \infty$, M is forced to be compact and a proof of Theorem 3 is well-known by the work of M. Takeuchi [TA78]. In this case he also provides a complete classification of all compact h.K.m. which can be Kähler immersed into $\mathbb{C}P^N$ by making use of the representation theory of semisimple Lie groups. Viceversa, it is not hard to see that if a *compact* Kähler manifold can be Kähler immersed into $\mathbb{C}P^\infty$ then it can also be Kähler immersed into $\mathbb{C}P^N$ with $N < \infty$.

We believe that, up to homotheties, *any* simply-connected h.K.m. such that its associated Kähler form is integral can be Kähler immersed into $\mathbb{C}P^N$, with $N \leq \infty$. This is expressed by the following conjecture.

Conjecture 1: *Let (M, g) be a simply-connected h.K.m. such that its associated Kähler form ω is integral. Then there exists $\lambda_0 \in \mathbb{R}^+$ such that $\lambda_0 g$ is projectively induced.*

The integrality of ω in the conjecture is important since there exist simply-connected h.K.m. (M, ω) such that $\lambda\omega$ is not integral for any $\lambda \in \mathbb{R}^+$ (take, for example, $(M, g) = (\mathbb{C}P^1, g_{FS}) \times (\mathbb{C}P^1, \sqrt{2} g_{FS})$). Observe also that there exist simply-connected (even contractible) h.K.m. (M, g) such that ω is an integral form but g is not projectively induced. In order to describe such an example we recall the following result (see Theorem 2 in [LZ09]).

Theorem A. Let g_B be the Bergman metric of an irreducible Hermitian symmetric space of noncompact type Ω. Then λg_B is projectively induced if and only $\lambda\gamma$ belongs to $W(\Omega) \setminus \{0\}$, where γ denotes the genus of Ω and $W(\Omega)$ its Wallach set.

It turns out (see Corollary 4.4 p. 27 in [AR95] and references therein) that $W(\Omega)$ consists only of real numbers and depends on two of the domain's invariants, denoted by a (strictly positive natural number) and r (the rank of Ω). More precisely we have

$$W(\Omega) = \left\{0, \frac{a}{2}, 2\frac{a}{2}, \dots, (r-1)\frac{a}{2}\right\} \cup \left((r-1)\frac{a}{2}, \infty\right). \tag{3}$$

The set $W_d = \left\{0, \frac{a}{2}, 2\frac{a}{2}, \dots, (r-1)\frac{a}{2}\right\}$ and the interval $W_c = \left((r-1)\frac{a}{2}, \infty\right)$ are called respectively the *discrete* and *continuous* part of the Wallach set of the domain Ω. Observe that when $r = 1$, namely Ω is the complex hyperbolic space $\mathbb{C}H^n$, then $g_B = (n+1) g_{hyp}$. In this case (and only in this case) $W_d = \{0\}$ and $W_c = (0, \infty)$. If $\mathrm{rank}(\Omega) = r \geq 2$ and $0 < \lambda < \frac{a}{2\gamma}$ it follows by Theorem A that λg_B is not projectively induced and its associated Kähler form $\lambda\omega_B$ is integral (since Ω is contractible). This provides us with the desired example.

Notice also that from Theorem A it follows that the only irreducible bounded symmetric domain where λg_B is projectively induced for all $\lambda > 0$ is the complex hyperbolic space. In the following theorem, which represents our last result, we generalize this fact to any homogeneous bounded domain (h.b.d. for short). This will be a key ingredient in the proof of Theorem 1.

Theorem 4. *Let (Ω, g) be a n-dimensional h.b.d.. The metric λg is projectively induced for all $\lambda > 0$ if and only if*

$$(\Omega, g) = \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_r}_{\lambda_r}, \tag{4}$$

where $n_1 + \cdots + n_r = n$, λ_j, $j = 1, \dots, r$ are positive real numbers and $\mathbb{C}H^{n_j}_{\lambda_j} = (\mathbb{C}H^{n_j}, \lambda_j g_{hyp})$, $j = 1, \dots, r$.

The paper contains another section dedicated to the proofs of our main results.

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2. Proof of the main results

The basic ingredient for the proof of our results is the following solution due to J. Dorfmeister and K. Nakajima [DN88] of the fundamental conjecture on h.K.m..

Theorem FC *A h.K.m. (M, g) is the total space of a holomorphic fiber bundle over a h.b.d. Ω in which the fiber $\mathcal{F} = \mathcal{E} \times \mathcal{C}$ is (with the induced Kähler metric) the Kähler product of a flat homogeneous Kähler manifold $\mathcal{E}$ and a compact simply-connected homogeneous Kähler manifold $\mathcal{C}$.*

In order to prove Theorem 1 recall that complete connected totally geodesic submanifolds of $\mathbb{R}^n$ are affine subspaces $p + \mathbb{W}$, where $p \in \mathbb{R}^n$ and $\mathbb{W} \subset \mathbb{R}^n$ is a vector subspace. We need the following result from [AD03] which we include here for completeness.

Lemma 4. *Let G be a connected Lie subgroup of isometries of the Euclidean space $\mathbb{R}^n$. Let $G.p = p+\mathbb{V}$ and $G.q = q+\mathbb{W}$ be two totally geodesic G-orbits. Then $\mathbb{V} = \mathbb{W}$, i.e. $G.p$ and $G.q$ are parallel affine subspaces of $\mathbb{R}^n$.*

Proof. We can assume that $p = 0 \in \mathbb{R}^n$ and that p, q are the points that realize the distance between both orbits $G \cdot p$, $G \cdot q$, i.e. $\mathrm{dist}(p, q) = \mathrm{dist}(G \cdot p, G \cdot q)$. Let $\gamma(t) = tq$ be the geodesic that realizes the distance between q and $\mathbb{V}$. So the vector q is perpendicular to any G-orbit $G_t = G \cdot \gamma(t)$ $t \in \mathbb{R}$. Let $X = x^*$ be any Killing vector field of G and $\mathrm{Exp}(tX)$ its associated one-parameter group of isometries. Define $h : I \times \mathbb{R} \to \mathbb{R}^n$ by $h_s(t) := \mathrm{Exp}(sX) \cdot \gamma(t)$. Note that $X(h_s(t)) = \frac{\partial h}{\partial s}$ and that, for a fixed s, $h_s(t)$ is a geodesic.

Let A_t be the shape operator at the point $\gamma(t)$ of the orbit $G\cdot\gamma(t)$ in the direction of $\dot\gamma(t)$. Define $f(t) := -\langle A_t(X(\gamma(t))), X(\gamma(t))\rangle = \langle \frac{D}{\partial s}\frac{\partial h}{\partial t}, X(h_s(t))\rangle\mid_{s=0}$. We have

$$\begin{aligned}\frac{d}{dt}f(t) &= \langle \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial h}{\partial t}, X(h_s(t))\rangle \mid_{s=0} + \langle \frac{D}{\partial s}\frac{\partial h}{\partial t}, \frac{D}{\partial t}X(h_s(t))\rangle \mid_{s=0} \\ &= \langle \frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial h}{\partial t}, X(h_s(t))\rangle \mid_{s=0} + \langle \frac{D}{\partial t}\frac{\partial h}{\partial s}, \frac{D}{\partial t}X(h_s(t))\rangle \mid_{s=0} \\ &= \|\nabla_{\dot\gamma(t)}(X(\gamma(t)))\|^2.\end{aligned}$$

Since $f(0) = 0$ because $G \cdot p$ is totally geodesic, we get

$$f(1) = -\langle A_t(X(q)), X(q)\rangle \geq 0\,.$$

Hence A_1 is negative definite and since $G \cdot q$ is totally geodesic, any Killing vector field X is parallel along $\gamma(t)$. We can write $\mathrm{Exp}(sX)\cdot p = e^{s\overline{X}}(p-c) + c+sd$, where $\overline{X}$ is the projection of X into $\mathfrak{so}_n$, $d \in \ker(\overline{X})$ and $c \in \ker(\overline{X})^\perp$.

Then a Killing vector field X is parallel along $\gamma(t)$ if and only if $q \in \ker(\overline{X})$. Thus $\mathrm{Exp}(sX) \cdot q = q + \mathrm{Exp}(sX) \cdot p$ which implies that

$$\mathbb{V} = T_p(G \cdot p) \subset T_q(G \cdot q) = \mathbb{W}\,.$$

Reversing the role of $\mathbb{V}$ and $\mathbb{W}$ the same argument yields $\mathbb{W} \subset \mathbb{V}$. This completes the proof of the lemma. $\square$

Proof of Theorem 1. Assume that there exists a Kähler immersion $f : M \to \mathbb{C}^N$. By Theorem FC and by the fact that a h.b.d. is contractible we get that $M = \mathbb{C}^k \times \Omega$ as a complex manifold since, by the maximum principle, the fiber $\mathcal{F}$ cannot contain a compact manifold. Let $M = G/K$ be the homogeneous realization of M (so the metric g is G-invariant). It follows again by Theorem FC that there exists $L \subset G$ such that the L-orbits are the fibers of the fibration $\pi : M = G/K \to \Omega = G/L$. Let F_p, F_q be the fibers over $p, q \in \Omega$. We claim that $f(F_p)$ and $f(F_q)$ are parallel affine subspaces of $\mathbb{C}^N$. Indeed, by Calabi's rigidity $f(F_p)$ and $f(F_q)$ are affine subspaces of $\mathbb{C}^N$ since both F_p and F_q are flat Kähler manifolds of $\mathbb{C}^n$. Moreover, Calabi rigidity theorem implies the existence of a morphism of groups $\rho : G \to Iso_{\mathbb{C}}(\mathbb{C}^N) = \mathrm{U}(\mathbb{C}^N) \ltimes \mathbb{C}^N$ such that $f(g \cdot x) = \rho(g) f(x)$ for all $g \in G, x \in M$. Let $W_{p,q}$ be the affine subspace generated by $f(F_p)$ and $f(F_q)$. Since both $f(F_p)$ and $f(F_q)$ are $\rho(L)$-invariant it follows that $W_{p,q}$ is also $\rho(L)$-invariant. Indeed, for any $g \in L$ the isometry $\rho(g)$ is an affine map and so must preserve the affine space generated by $f(F_p)$ and $f(F_q)$. Observe that $W_{p,q}$ is a finite dimensional complex Euclidean space, $\rho(L)$ acts on $W_{p,q}$ and $f(F_p)$ and $f(F_q)$ are two complex totally geodesic orbits in $W_{p,q}$. Then, by Lemma 4, we get that $f(F_p)$ and $f(F_q)$ are parallel affine subspaces of $W_{p,q}$ and hence of $\mathbb{C}^N$. Since $p, q \in \Omega$ are two arbitrary points it follows that $f(M)$ is a Kähler product. Thus $M = \mathbb{C}^k \times \Omega$ is a Kähler product of homogeneous Kähler manifolds. Using again the fact M can be Kähler immersed into $\mathbb{C}^N$ it follows that the h.b.d. Ω can be Kähler immersed into $\mathbb{C}^N$. If one denotes by φ this immersion and by g_Ω the homogeneous Kähler metric of Ω, it follows that the map $\sqrt{\lambda}\varphi$ is a Kähler immersion of $(\Omega, \lambda g_\Omega)$ into $\mathbb{C}^N$. Therefore, by Theorem 14 in [Bo47], λg_Ω is projectively induced for all $\lambda > 0$ and Theorem 4 yields

$$(M, g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_r}_{\lambda_r},$$

where $k + n_1 + \cdots + n_r = n$ and λ_j, $j = 1, \ldots, r$ are positive real numbers. If the dimension N of the ambient space $\mathbb{C}^N$ is finite then $M = \mathbb{C}^n$ since there cannot exist a Kähler immersion of $(\mathbb{C}H^{n_j}, \lambda_j g_{hyp})$ into $\mathbb{C}^N$, $N < \infty$ (see [Ca53]) and this proves (a). The last part of Theorem 1 is a consequence of Calabi's rigidity theorem together with Lemma 3.1 in [DL07] which asserts that a Kähler map $f : M \times M' \to \mathbb{C}^N$, $N \leq \infty$, from a product $M \times M'$ of two Kähler manifolds is a product, i.e. $f(p, q) = (f_1(p), f_2(q))$ where $f_1 : M \to \mathbb{C}^N$ and $f_2 : M' \to \mathbb{C}^N$ are Kähler maps. $\square$

Remark 5. As we have already pointed, Theorem 4, which is an important step in the proof of the Theorem 1, is a straightforward consequence of Theorem A above when the h.K.m. is a bounded symmetric domain. Therefore the last part of Theorem 1 provides an alternative proof of Theorem 3.3 in [DL07] without the use of Calabi's diastasis function (cfr. Remark 2).

In order to prove Theorem 2 we need the following lemma.

Lemma 6. *If a Kähler manifold (M, g) can be Kähler immersed into $\mathbb{C}H^N$, $N \leq \infty$, then it can also be Kähler immersed into $\ell^2(\mathbb{C})$.*

Proof. Let f be the Kähler immersion of (M, g) into $\mathbb{C}H^N$. If $N < \infty$ then the map $f_n \circ f : (M, g) \to \ell^2(\mathbb{C})$, where f_n is given by (2), is a Kähler immersion. If $N = \infty$, it follows by (1) in the introduction that $\Phi = -\log(1-\sum_{j=1}^{\infty} |\phi_j|^2) = \sum_{k=1}^{\infty}(\sum_{j=1}^{\infty} |\phi_j|^2)^k$ is a Kähler potential for the metric g, i.e. $\frac{i}{2}\partial\bar{\partial}\Phi = \omega$, where ω is the Kähler form associated to the metric g and the ϕ_j's are the components of f. Hence $\Phi = \sum_{j=1}^{\infty} |h_j|^2$ for suitable holomorphic functions h_j, $j = 1, 2, \dots$ on M and the map $h = (\dots, h_j, \dots) : (M, g) \to \ell^2(\mathbb{C})$ is the desired Kähler immersion. □

Proof of Theorem 2. If a h.K.m. (M, g) can be Kähler immersed into $\mathbb{C}H^N$, $N \leq \infty$, then, by Lemma 6 it can also be Kähler immersed into $\ell^2(\mathbb{C})$. By Theorem 1, (M, g) is then a Kähler product of complex space forms, namely

$$(M, g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_r}_{\lambda_r}.$$

Then the conclusion follows by the fact that $\mathbb{C}^k$ cannot be Kähler immersed into $\mathbb{C}H^N$ for all $N \leq \infty$ (see [Ca53]), by Calabi's rigidity theorem and by Theorem 2.11 in [AD03] which shows that there are not Kähler maps from a product $M \times M'$ of Kähler manifolds into $\mathbb{C}H^N$, $N \leq \infty$, (the proof in [AD03] is given for $N < \infty$ but it extends without any substantial change to the infinite dimensional case). □

Proof of Theorem 3. Theorem FC and the fact that a h.b.d. is contractible imply that M is a *complex* product $\Omega \times \mathcal{F}$, where $\mathcal{F} = \mathcal{E} \times \mathcal{C}$ is a Kähler product of a flat Kähler manifold $\mathcal{E}$ Kähler embedded into (M, g) and a simply-connected h.K.m. $\mathcal{C}$. We claim that $\mathcal{E}$ is simply-connected and hence $M = \Omega \times \mathcal{E} \times \mathcal{C}$ is simply-connected. In order to prove our claim notice that $\mathcal{E}$ is the Kähler product $\mathbb{C}^k \times T_1 \times \cdots \times T_s$, where T_j are flat complex tori. So one needs to show that each T_j reduces to a point. If, by a contradiction, the dimension of one of this tori, say T_{j_0} is not zero, then by composing the Kähler immersion of T_{j_0} in (M, g) with the immersion $f : M \to \mathbb{C}P^N$ we would get a Kähler immersion of T_{j_0} into $\mathbb{C}P^N$ in contrast with a well-known result of Calabi [Ca53] (see also Lemma 2.2 in [TA78]). In order to prove that f is injective we first observe that, by Calabi's rigidity theorem, $f(M)$ is still a h.K.m.. Then, by the first part of the theorem, $f(M) \subset \mathbb{C}P^N$ is simply-connected. Moreover, since M is complete and $f : M \to f(M)$ is a local isometry, it is a covering map (see, e.g., Lemma 3.3 p. 150 in [DC92]) and hence injective.□

Proof of Corollary 3. Let $\pi : \tilde{M} \to M$ be the universal covering map. Then $(\tilde{M}, \tilde{g})$ is a h.K.m. and, by Theorem 3, $f \circ \pi : \tilde{M} \to \mathbb{C}P^n$ is injective. Therefore π is injective, and since it is a covering map, it defines a holomorphic isometry between $(\tilde{M}, \tilde{g})$ and (M, g). □

Proof of Theorem 4. First we find a global potential of the homogeneous Kähler metric g on the domain Ω following Dorfmeister [D85]. By [D85, Theorem 2 (c)], there exists a split solvable Lie subgroup $S \subset \mathrm{Aut}(\Omega, g)$ acting simply transitively on the domain Ω. Taking a reference point $z_0 \in \Omega$, we have a diffeomorphism $S \ni s \overset{\sim}{\mapsto} s \cdot z_0 \in \Omega$, and by the differentiation, we get the linear isomorphism $\mathfrak{s} := \mathrm{Lie}(S) \ni X \overset{\sim}{\mapsto} X \cdot z_0 \in T_{z_0}\Omega \equiv \mathbb{C}^n$. Then the evaluation of the Kähler form ω on $T_{z_o}\Omega$ is given by $\omega(X \cdot z_o, Y \cdot z_0) = \beta([X, Y])$ $(X, Y \in \mathfrak{s})$ with a certain linear form $\beta \in \mathfrak{s}^*$. Let $j : \mathfrak{s} \to \mathfrak{s}$ be the linear map defined in such a way that $(jX) \cdot z_0 = \sqrt{-1}(X \cdot z_0)$ for $X \in \mathfrak{s}$. We have $\Re g(X \cdot z_0, Y \cdot z_0) = \beta([jX, Y])$ for $X, Y \in \mathfrak{s}$, and the right-hand side defines a positive inner product on $\mathfrak{s}$. Let $\mathfrak{a}$ be the orthogonal complement of $[\mathfrak{s}, \mathfrak{s}]$ in $\mathfrak{s}$ with respect to the inner product. Then $\mathfrak{a}$ is a commutative Cartan subalgebra of $\mathfrak{s}$. Define $\gamma \in \mathfrak{a}^*$ by $\gamma(C) := -4\beta(jC)$ $(C \in \mathfrak{a})$, and we extended γ to $\mathfrak{s} = \mathfrak{a} \oplus [\mathfrak{s}, \mathfrak{s}]$ by the zero-extension. Keeping the diffeomorphism between S and Ω in mind, we define a positive smooth function Ψ on Ω by

$$\Psi((\exp X) \cdot z_0) = e^{-\gamma(X)} \ \ (X \in \mathfrak{s}).$$

From the argument in [D85, pp. 302–304], we see that

$$\omega = \frac{i}{2}\partial\bar{\partial} \log \Psi. \tag{5}$$

It is known that there exists a unique kernel function $\tilde{\Psi} : \Omega \times \Omega \to \mathbb{C}$ such that (1) $\tilde{\Psi}(z, z) = \Psi(z)$ for $z \in \Omega$ and (2) $\tilde{\Psi}(z, w)$ is holomorphic in z and anti-holomorphic in w (cf. [I99, Proposition 4.6]). Let us observe that the metric g is projectively induced if and only if $\tilde{\Psi}$ is a reproducing kernel of a Hilbert space of holomorphic functions on Ω. Indeed, if $f : \Omega \to \mathbb{C}P^N$ $(N \leq \infty)$ is a Kähler immersion with $f(z) = [\psi_0(z) : \psi_1(z) : \cdots]$ $(z \in \Omega)$ its homogeneous coordinate expression, then we have $\omega = \frac{i}{2}\partial\bar{\partial} \log \sum_{j=0}^N |\psi_j|^2$. Comparing (5) with it, we see that there exists a holomorphic function ϕ on Ω for which $\Psi = |e^\phi|^2 \sum_{j=0}^N |\psi_j|^2$. By analytic continuation, we obtain $\tilde{\Psi}(z, w) = e^{\phi(z)}\overline{e^{\phi(w)}} \sum_{j=0}^N \psi_j(z)\overline{\psi_j(w)}$ for $z, w \in \Omega$. For any $z_1, \ldots, z_m \in \Omega$

and $c_1, \dots, c_m \in \mathbb{C}$, we have

$$\begin{aligned}\sum_{p,q=1}^{m} c_p \bar{c}_q \tilde{\Psi}(z_p, z_q) &= \sum_{p,q=1}^{m} c_p \bar{c}_q e^{\phi(z_p)} \overline{e^{\phi(z_q)}} \sum_{j=0}^{N} \psi_j(z_p) \overline{\psi_j(z_q)} \\ &= \sum_{j=0}^{N} | \sum_{p=1}^{m} c_p e^{\phi(z_p)} \psi_j(z_p)|^2 \geq 0.\end{aligned}$$

Thus the matrix $(\tilde{\Psi}(z_p, z_q))_{p,q} \in \mathrm{Mat}(m, \mathbb{C})$ is always a positive Hermitian matrix. Therefore $\tilde{\Psi}$ is a reproducing kernel of a Hilbert space (see [Ar50, p. 344]).

On the other hand, if $\tilde{\Psi}$ is a reproducing kernel of a Hilbert space $\mathcal{H} \subset \mathcal{O}(\Omega)$, then by taking an orthonormal basis $\{\psi_j\}_{j=0}^{N}$ of $\mathcal{H}$, we have a Kähler immersion $f : M \ni z \mapsto [\psi_0(z) : \psi_1(z) : \cdots] \in \mathbb{C}P^N$ because we have $\Psi(z) = \tilde{\Psi}(z, z) = \sum_{j=0}^{N} |\psi_j(z)|^2$. Note that there exists no point $a \in \Omega$ such that $\psi_j(a) = 0$ for all $1 \leq j \leq N$ since $\Psi(z) = \sum_{j=0}^{N} |\psi_j(z)|^2$ is always positive.

The condition for $\tilde{\Psi}$ to be a reproducing kernel is described in [I99]. In order to apply the results, we need a fine description of the Lie algebra $\mathfrak{s}$ with j due to Piatetskii-Shapiro [PS69]. Indeed, it is shown in [PS69, Chapter 2] that the correspondence between the h.b.d. Ω and the structure of $(\mathfrak{s}, j)$ is one-to-one up to natural equivalence. For a linear form α on the Cartan algebra $\mathfrak{a}$, we denote by $\mathfrak{s}_\alpha$ the root subspace $\{ X \in \mathfrak{s}\,;\, [C, X] = \alpha(C)X \ (\forall C \in \mathfrak{a}) \}$ of $\mathfrak{s}$. The number $r := \dim \mathfrak{a}$ is nothing but the rank of Ω. Thanks to [PS69, Chapter 2, Section 3], there exists a basis $\{\alpha_1, \dots, \alpha_r\}$ of $\mathfrak{a}^*$ such that $\mathfrak{s} = \mathfrak{s}(0) \oplus \mathfrak{s}(1/2) \oplus \mathfrak{s}(1)$ with

$$\begin{aligned}&\mathfrak{s}(0) = \mathfrak{a} \oplus \sum_{1 \leq k < l \leq r}^{\oplus} \mathfrak{s}_{(\alpha_l - \alpha_k)/2}, \quad \mathfrak{s}(1/2) = \sum_{1 \leq k \leq r}^{\oplus} \mathfrak{s}_{\alpha_k/2}, \\ &\mathfrak{s}(1) = \sum_{1 \leq k \leq r}^{\oplus} \mathfrak{s}_{\alpha_k} \oplus \sum_{1 \leq k < l \leq r}^{\oplus} \mathfrak{s}_{(\alpha_l + \alpha_k)/2}.\end{aligned}$$

If $\{A_1, \dots, A_r\}$ is the basis of $\mathfrak{a}$ dual to $\{\alpha_1, \dots, \alpha_r\}$, then $\mathfrak{s}_{\alpha_k} = \mathbb{R} j A_k$. Thus $\mathfrak{s}_{\alpha_k}$ $(k = 1, \dots, r)$ is always one dimensional, whereas other root spaces $\mathfrak{s}_{\alpha_k/2}$ and $\mathfrak{s}_{(\alpha_l \pm \alpha_k)/2}$ may be $\{0\}$. Since $\{\alpha_1, \dots, \alpha_r\}$ is a basis of $\mathfrak{a}^*$, the linear form $\gamma \in \mathfrak{a}^*$ is written as $\gamma = \sum_{k=1}^{r} \gamma_k \alpha_k$ with unique $\gamma_1, \dots, \gamma_r \in \mathbb{R}$. Since $jA_k \in \mathfrak{s}_{\alpha_k}$, we have

$$\gamma_k = \gamma(A_k) = -4\beta(jA_k) = -4\beta([A_k, jA_k]) = 4\beta([jA_k, A_k])$$

and the last term equals $4g(A_k \cdot z_0, A_k \cdot z_0)$. Thus we get $\gamma_k > 0$.

For $\epsilon = (\epsilon_1, \dots, \epsilon_r) \in \{0, 1\}^r$, put $q_k(\epsilon) := \sum_{l > k} \epsilon_l \dim \mathfrak{s}_{(\alpha_l - \alpha_k)/2}$ $(k = 1, \dots, r)$. Define

$$\mathfrak{X}(\epsilon) := \left\{ (\sigma_1, \dots, \sigma_r) \in \mathbb{C}^r \,;\, \begin{array}{ll} \sigma_k > q_k(\epsilon)/2 & (\epsilon_k = 1) \\ \sigma_k = q_k(\epsilon)/2 & (\epsilon_k = 0) \end{array} \right\},$$

and $\mathfrak{X} := \bigsqcup_{\epsilon\in\{0,1\}^r} \mathfrak{X}(\epsilon)$. By [I99, Theorem 4.8], $\tilde{\Psi}$ is a reproducing kernel if and only if $\underline{\gamma} := (\gamma_1, \dots, \gamma_r)$ belongs to $\mathfrak{X}$. We denote by $W(g)$ the set of $\lambda > 0$ for which λg is projectively induced. Since the metric λg corresponds to the parameter $\lambda\underline{\gamma}$, we see that λg is projectively induced if and only if $\lambda\underline{\gamma} \in \mathfrak{X}$. Namely we obtain

$$W(g) = \left\{ \lambda > 0 \,;\, \lambda\underline{\gamma} \in \mathfrak{X} \right\},$$

and the right-hand side is considered in [I10]. Put $q_k = \sum_{l>k} \dim \mathfrak{s}_{(\alpha_l-\alpha_k)/2}$ for $k = 1, \dots, r$. Then [I10, Theorem 15] tells us that

$$W(g) \cup \{0\} \subset \left\{ \frac{q_k}{2\gamma_k} \,;\, k = 1, \dots, r \right\} \cup (c_0, +\infty),$$

where $c_0 := \max\left\{ \frac{q_k}{2\gamma_k} \,;\, k = 1, \dots, r \right\}$.

Now assume that λg is projectively induced for all $\lambda > 0$. Then we have $c_0 = 0$, so that $\dim \mathfrak{s}_{(\alpha_l-\alpha_k)/2} = 0$ for all $1 \le k < l \le r$. In this case, we see that $\mathfrak{s}$ is a direct sum of ideals $\mathfrak{s}_k := j\mathfrak{s}_{\alpha_k} \oplus \mathfrak{s}_{\alpha_k/2} \oplus \mathfrak{s}_{\alpha_k}$ $(k = 1, \dots, r)$, which correspond to the hyperbolic spaces $\mathbb{C}H^{n_k}$ with $n_k = 1 + (\dim_{\alpha_k/2})/2$ ([PS69, pp. 52–53]). Therefore the Lie algebra $\mathfrak{s}$ corresponds to the direct product $\mathbb{C}H^{n_1} \times \cdots \times \mathbb{C}H^{n_r}$, which is biholomorphic to Ω because the homogeneous domain Ω also corresponds to $\mathfrak{s}$. Hence (4) holds and Theorem 4 is verified. □

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