A Laplace integral on a Kähler manifold and Calabi's diastasis function

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Abstract

In this paper we give a different proof of Engliš's result [9] about the asymptotic expansion of a Laplace integral on a a real analytic Kähler manifold (M, g) by using the link between the metric g and the associated Calabi's diastasis function D. We also make explicit the connection between the coefficients of Engliš' expansion and Gray's invariants [10].

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1 Introduction

Let M be an n-dimensional complex manifold endowed with a real analytic Kähler metric g. Consider Calabi's diastasis function $D(x, y)$ defined on $U \times U \subset M \times M$, where U is a suitable open subset of M (see Section 3.3 below for details). In [2] Berezin was able to establish a quantization procedure for a very special class of real analytic Kähler manifolds: the bounded symmetric domains with the Bergman metric and the flat space $\mathbb{C}^n$. One of the key ingredient used by Berezin was the behaviour of the Laplace integral

$$L_\alpha(x) = \int_U f(y) e^{-\alpha D(x,y)} \frac{\omega^n(y)}{n!}$$

as α goes to infinity, where f is a smooth function on U and $D(x, y)$ is Calabi's diastasis function. More precisely, he proved that for any real analytic metric the following holds true:

$$\left(\frac{\alpha}{\pi}\right)^n L_\alpha(x) = f(x) + (\Delta f(x) - \frac{1}{2} f(x)\rho(x))\alpha^{-1} + o(\alpha^{-1}),\ \alpha \to +\infty,$$

where ρ denotes the scalar curvature of the metric g and Δ the associated Laplacian operator. Berezin's ideas and techniques were developed and generalized by many mathematicians and physicists. In the present paper we are particularly interested in the work of Engliš [7], [8], [9]. In [8] it is proven that Berezin's quantization procedure can be carried out for strongly pseudoconvex domains with real analytic boundary (see also [7] where one can find a detailed description of Berezin's work). In [9] Engliš proved that the Laplace integral $L_\alpha(x)$ above, admits an asymptotic expansion

$$L_\alpha(x) \sim \left(\frac{\pi}{\alpha}\right)^n \sum_{r\geq 0} \alpha^{-r} C_r(f)(x),$$

where $C_r : C^\infty(U) \to C^\infty(U)$ are smooth differential operators which depend on the curvature of the metric g and its covariant derivatives. In particular he computed the first three coefficients explicitly (see Theorem 2.1 and formulae (6) below).
Disregarding the applications to the theory of quantization, it is interesting to understand what implications has the previous asymptotic expansion to the geometry of the Kähler manifold (M, g).
The main result of this paper is Theorem 4.1 where we compute the asymptotic expansion of $L_\alpha(x)$ by making an explicit connection between this expansion and the Gray's invariants of the volume of small geodesics balls. The proof of our theorem is based on Proposition 3.3 where we prove that for any point $x \in M$ there exists a neighbourhood of the zero section V_1 and a smooth embedding

$$\nu_x : T_xM \cap V_1 \to T_xM$$

such that

$$D(x, \exp_x(\nu_x(v))) = g_x(v, v), \ (x, v) \in V_1. \tag{1}$$

The techniques used in the present paper to prove Proposition 3.3 and Theorem 4.1, are generalization of those of Cahen, Gutt and Rawnsley in the context of quantization of Kähler manifolds (see [3] and [4] and also Remark 3.4 below).

The paper is organized as follows. In Section 2 we describe Engliš's work and we state his main result Theorem 2.1. In Section 3 we prove the link between the diastasis function and the metric expressed by equation (1) above (see Proposition 3.3). Section 4 is dedicated to the computation of the expansion of $L_\alpha(x)$ (Theorem 4.1). Finally, in Section 5 we show the link between this expansion and the volume of small geodesics balls.

2 The work of Engliš

Let M be an n-dimensional complex manifold endowed with a real analytic Kähler metric g and let ω be the corresponding Kähler form. Let Φ be a Kähler potential for the metric g, namely a real valued function Φ defined on a open set $U \subset M$ satisfying

$$\omega = \frac{i}{2}\partial\bar{\partial}\Phi. \tag{2}$$

If $g = \sum_{j\bar{k}}^{n} g_{j\bar{k}} dz_j d\bar{z}_k$ is the local expression for the metric g then the previous equation is equivalent to

$$g_{j\bar{k}} = \frac{\partial^2\Phi}{\partial z_j \partial \bar{z}_k}. \tag{3}$$

The potential Φ can be complex analytically continued to an open neighbourhood $W \subset U \times U$ of the diagonal. Denote this extension by $\Phi(x, \bar{y})$. It is holomorphic in x and anti-holomorphic in y and, with this notation, $\Phi(x) = \Phi(x, x)$. Observe also that $\overline{\Phi(x, \bar{y})} = \Phi(\bar{x}, y)$. Consider the real valued function

$$D(x, y) = \Phi(x, \bar{x}) + \Phi(y, \bar{y}) - \Phi(x, \bar{y}) - \Phi(y, \bar{x})$$

on W. It is easily seen that the function $D(x, y)$ is independent from the potential chosen which is defined up to the sum with the real part of a holomorphic function. Calabi [5] christened the function $D(x, y)$ the *diastasis function*. We refer to [5] for details and further results on the diastasis function.

For all $x \in U$ (U as above), the positive definiteness of the matrix (3) implies that the function

$$D(x, \cdot) = \Phi(x, \bar{x}) + \Phi(\cdot, \bar{\cdot}) - \Phi(x, \bar{\cdot}) - \Phi(\cdot, \bar{x})$$

has a local minimum at x. Shrinking U, if necessary, we can assume that $D(x, y)$ is a globally defined on $U \times U$, $D(x, y) \geq 0$ and $D(x, y) = 0$ iff $x = y$. Let f be a C^∞-function on U and $\alpha > 0$. Consider the Laplace integral

$$L_\alpha(x) = \int_U f(y) e^{-\alpha D(x,y)} \frac{\omega^n}{n!}(y), \tag{4}$$

Before stating Engliš' main result about this integral (Theorem 2.1 below), we fix our notations and conventions.

The curvature tensor is defined as

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l},\ i,j,k,l = 1,\ldots,n$$

The Ricci curvature is

$$Ric_{i\bar{j}} = -\sum_{k,l=1}^{n} g^{k\bar{l}} R_{i\bar{j}k\bar{l}},\ i,j = 1,\ldots,n$$

and the scalar curvature is the trace of the Ricci curvature

$$\rho = -\sum_{i,j=1}^{n} g^{i\bar{j}} Ric_{i\bar{j}}.$$

The Laplace operator, denoted by Δ, is given by

$$\Delta f = \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}.$$

Finally, we set

$$|R|^2 = \sum_{i,j,k,l=1}^{n} |R_{i\bar{j}k\bar{l}}|^2,\quad |Ric|^2 = \sum_{i,j=1}^{n} |Ric_{i\bar{j}}|^2.$$

We are now in the position to state Engliš's result.

Theorem 2.1 *(Engliš) If the integral (4) exists for some $\alpha = \alpha_0$ then it also exists for all $\alpha > \alpha_0$ and as $\alpha \to +\infty$ it has an asymptotic expansion*

$$L_\alpha(x) \sim \left(\frac{\pi}{\alpha}\right)^n \sum_{r \geq 0} \alpha^{-r} C_r(f)(x), \tag{5}$$

where $C_r : C^\infty(U) \to C^\infty(U)$ are smooth differential operators which can be described explicitly. In particular

$$\begin{cases} C_0 = id \\ C_1(f) = \Delta f - \frac{1}{2} f \rho \\ C_2(f) = \frac{1}{2}\Delta\Delta f - \frac{1}{2} L_{Ric}(f) - \frac{\rho}{2}\Delta f - \frac{1}{2}(\langle D^{'}\rho, D^{'}f\rangle + \langle D^{'}f, D^{'}\rho\rangle) \\ -f(\frac{1}{3}\Delta\rho - \frac{1}{8}\rho^2 - \frac{1}{6}|\,Ric\,|^2 + \frac{1}{24}|R|^2), \end{cases} \tag{6}$$

where, for $f, g \in C^\infty(U)$, we have the following notations:

$$L_{Ric}(f) = \sum_{i,j,p,q=1}^{n} g^{i\bar{q}} g^{p\bar{j}} Ric_{p\bar{q}} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}, \tag{7}$$

$$\langle D^{'} f, D^{'} g\rangle = \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial \bar{z}_j}, \tag{8}$$

$$|D^{'} f|^2 = \langle D^{'} f, D^{'} f\rangle. \tag{9}$$

Proof: For the proof of the first part we refer to Theorem 3 in [9] where the operators C_j are denoted by R_j. The expression for the operators C_1, C_2 above can be deduced from the expression for R_1, R_2 in Section 4 of [9] by translating Engliš notations into ours and by taking into account that the Ricci curvature considered by Engliš has opposite sign to the one we are considering in the present article. □

3 The diastasis and the exponential map

In this section we find a very natural and nice link between the diastasis and the exponential map of a real analytic Kähler manifold that in the author's opinion deserves further study. This is expressed by Proposition 3.3 (see also equations (12) and (13) below). In order to prove it we need two lemmata. In the first one (Lemma 3.1 below), we show that the Hessian of Calabi's diastasis function $D(x, y)$, with respect to its second variable evaluated at the point $x = y$ equals twice the metric at the point x. The second one (Lemma 3.2 below) is a generalization of Morse's lemma for a smooth function defined on an open neighbourhood of the zero section.

Lemma 3.1 *Let M be a complex manifold endowed with a real analytic Kähler metric g. Then, for every $x \in M$, we have:*

$$(Hess_2 D)_{|x=y} = 2g_x \tag{10}$$

where $Hess_2 D$ denotes the Hessian of $D(x, y)$ with respect to its second variable y.

Proof: Choose a system $(z_1, \dots, z_n)$ of Bochner's coordinates centered in x (see e.g. Section 2 in [5]). Thus, we can write

$$D(z(x), \bar{z}(y)) = \Phi(z(y), \bar{z}(y)) = \sum_{j=1}^{n} |z_j|^2 + \psi(z, \bar{z}),$$

where $\psi(z, \bar{z})$ is a power series in $(z, \bar{z})$ with no term of degree ≤ 2 in either the variables z or $\bar{z}$. Since, in these coordinates $g_{i\bar{j}}(x) = \delta_{ij}$, $\frac{\partial g_{i\bar{j}}(x)}{\partial z_k} = 0$ and $\mathrm{Hess}\,\psi_{|(z,\bar{z})=(0,0)} = 1$ equation (10) follows immediately. $\Box$

Lemma 3.2 *Let M be a complex manifold endowed with a real analytic Kähler metric g. Let $V \subset TM$ be a neighbourhood of the zero section and $f : V \subset TM \to \mathbb{R}$ be a smooth function on V which admits the points of the zero section as non-degenerate critical zeros, namely:*

- $f(0_x) = 0$;
- $(D_2 f)_{0_x} = 0$;
- $(\mathit{Hess}_2\, f)_{0_x}$ *is a non-degenerate bilinear form on* $T_{0_x}V$,

where D_2 and Hess_2 denote respectively the differentiation and the Hessian with respect to the vertical direction and $0_x = (x, 0), 0 \in T_xM$ is any element of the zero section. Then there exist an open neighbourhood $V_1 \subset V$ of the zero section and a diffeomorphism $\nu : V_1 \to \nu(V_1) \subset TM$ such that $\nu(V_1) \subset V$ and:

- $p(\nu(X)) = p(X),\ X \in V_1$;
- $(f \circ \nu)(X) = \frac{1}{2}(\mathit{Hess}_2\, f)_{0_x}(X, X)$,

and the differential of ν at the zero section is the identity.

Proof: The proof can be easily obtained by using a version of the Morse lemma on page 5 of Combet [6] which one has to adapt to our case. $\Box$

We denote by $\exp_x(v)$ the exponential map at $x \in M$ and $v \in T_xM$. Let $V \subset TM$ be an open subset, containing the zero section, where the exponential map is defined for all $x \in M$ and $v \in T_xM$. The differential of the exponential map at the zero section is the identity so the map $\alpha : V \to M \times M$ given by

$$X \mapsto (p(X), \exp_{p(x)} X),\ X \in V$$

where $p : TM \to M$ is the projection in the tangent bundle, is a diffeomorphism near the zero-section.
We can now state and prove the main result of this section.

Proposition 3.3 *Let M be a complex manifold endowed with a real analytic Kähler metric g. Let V be an open neighboorhood of the zero section of the tangent bundle $p : TM \to M$, such that the map $\alpha : V \to M \times M$ above is well-defined. Then there exist an open neighbourhood V_1 of the zero section and a smooth embedding $\nu : V_1 \to TM$ with $\nu(V_1) \subset V$ such that:*

$$p(\nu(X)) = p(X), \ \ X \in V_1; \tag{11}$$

$$(D \circ \alpha \circ \nu)(X) = g_{p(X)}(X, X), \ \ X \in V_1, \tag{12}$$

and the differential of ν at the zero section is the identity.

Proof: Take $W \subset TM$ a neighbourhood of the zero section such that $\alpha_{|W} : W \to M \times M$ is a diffeomorphism onto some neighbourhood of the diagonal in $M \times M$ and such that this neighbourhood is contained in $U \times U$, (where the diastasis function D is defined). Then, we can consider the function

$$f = D \circ \alpha : W \to \mathbb{R}.$$

Since the points of the diagonal are critical points for D and the differential of α at the zero section is the identity we have the following equalities:

$$(D \circ \alpha)(0_x) = D(x, \exp_x(0)) = D(x, x) = 0,$$

$$(D_2 f)_{0_x} = 0,$$

$$(Hess_2 f)_{0_x} = 2g_x.$$

This tells us that the points of the zero section are non-degenerate critical zeros for the function f, and we can apply Lemma 3.2. □

Remark 3.4 Observe that Proposition 3.3 is a generalization of Proposition 1 in [3], where the same result is proved for the very special class of Kähler manifolds admitting a regular quantization. We refer to [1] for some geometric properties of these manifolds.

Now we define a smooth function Θ (see formula (14) below) on the open set V_1 given by Proposition 3.3. This will be the main ingredient in the proof of our main result (see Section 4 below).

From (11) it follows that if we write $X = (x, v) \in V_1 \subset TM$ with $v \in T_xM$ then the embedding $\nu : V_1 \to TM$ can be written as

$$(x, v) \mapsto (x, \nu_x(v))$$

where

$$\nu_x : T_xM \cap V_1 \to T_xM \cap \nu(V_1)$$

is smooth diffeomorphism whose differential at the point $0 \in T_xM$ is the identity. Observe also that equation (12) can then be written as:

$$D(x, \exp_x(\nu_x(v))) = g_x(v, v), \ (x, v) \in V_1. \tag{13}$$

Consider the neighbourhood V_1 of the zero section in TM given by Proposition 3.3 and denote by $U_1 = \alpha(V_1) \subset U \times U$ its image under the diffeomorphism

$$\alpha : V_1 \to U_1, \ X \mapsto (x, \exp_x X).$$

Fix a point $x \in U$, and consider the embedding:

$$\exp_x \circ \nu_x : T_xM \cap V_1 \to U = \{x\} \times U.$$

Hence one can define a non-zero smooth function Θ_x on $T_xM \cap V_1$ by:

$$(\exp_x \circ \nu_x)^*(\frac{\omega^n}{n!})(v) = \Theta_x(v)dv, \tag{14}$$

where dv is the standard Lebsgue measure on T_xM. By varying $x \in U$ we then get a smooth function $\Theta(x, v) = \Theta_x(v)$ on V_1.

4 The main result

In this section we prove Theorem 4.1, where we obtain a different expansion of the Laplace operator $L_\alpha(x)$ in terms of differential operators depending on the function Θ defined at the end of the previous section. Observe that Theorem 4.1 and its proof are an extension of Proposition 2 in [3] where the same result is obtained for the case of Kähler manifolds (M, g) which admit a regular quantization (cfr. Remark 3.4 above).

Theorem 4.1 *Let M be a complex manifold endowed with a real analytic Kähler metric g. Then the Laplace integral (4), namely*

$$L_\alpha(x) = \int_U f(y)e^{-\alpha D(x,y)}\frac{\omega^n}{n!}(y)$$

admits an asymptotic expansion

$$\mathcal{L}_\alpha(x) \sim \left(\frac{\pi}{\alpha}\right)^n \sum_{r\geq 0} \alpha^{-r} C_r(f)(x). \tag{15}$$

for smooth operators $C_r : C^\infty(U) \to C^\infty(U)$. Moreover,

$$C_r(f)(x) = \frac{(r+n-1)!}{2\pi^n(2r)!}\int_{S_xM}(D_v^{2r}(\tilde{f}\Theta))(x,0)dv,\ r = 0, 1, \dots \tag{16}$$

and C_0 is the identity operator. Here

$$\tilde{f}(x, v) = f(\exp_x(\nu_x(v))),$$

$\Theta(x, v) = \Theta_x(v)$ is given by (14) and D_v^p denote the p-th directional derivative with respect to v.

Proof: We can assume that there exists a constant $C > 0$ such that:

$$\left|\int_U f(y)\frac{\omega^n(y)}{n!}\right| \leq C, \tag{17}$$

where $\frac{\omega^n}{n!}$ is the Riemannian volume form on U, induced by the metric g. Shrinking V_1 and U_1, if necessary, one may assume that Θ is defined on $\overline{V}_1$ and hence bounded as well as all its derivatives for x in a compact subset of U. Choose an open neighbourhood U_2 of the diagonal in $U\times U$, with $\overline{U_2} \subset U_1$ and define $V_2 = \alpha^{-1}(U_2)$. Let $\chi : U \times U \to [0, 1]$ be a smooth function such that $\chi_{|U_2} = 1$ and $\operatorname{supp}\chi \subset U_1$. Set $\eta = \max_{x,y\notin U_2} e^{-D(x,y)}$. Then $\eta < 1$ and $e^{-D(x,y)} \leq \eta$ on $U \setminus U_2$. Let $U_{i,x} = \{y \in M|(x, y) \in U_i\},\ i = 1, 2$ and $\chi_x(y) = \chi(x, y)$. The function χ_x is equal to 1 on $U_{2,x}$ and has compact support in $U_{1,x}$. One can write $L_\alpha(x)$ as the sum of three integrals:

$$\begin{aligned} L_\alpha(x) = \int_{U_{1,x}} f(y)\chi_x(y)e^{-\alpha D(x,y)}\tfrac{\omega^n(y)}{n!} &+ \int_{U_{1,x}\setminus U_{2,x}} f(y)(1-\chi_x(y))e^{-\alpha D(x,y)}\tfrac{\omega^n(y)}{n!} \\ &+ \int_{U\setminus U_{1,x}} f(y)e^{-\alpha D(x,y)}\tfrac{\omega^n(y)}{n!} \end{aligned}.$$

The absolute values of the last two integrals are less or equal to $C\eta^\alpha$, where C is a constant given by (17). Therefore

$$|L_\alpha(x) - \int_{U_{1,x}} f(y)\chi_x(y)e^{-\alpha D(x,y)}\frac{\omega^n(y)}{n!}| \le 2C\eta^\alpha.$$

Thus this difference is exponentially small for each x. By formula (14), the remaining integral may be computed in the tangent space T_xM as a Gauss integral, namely:

$$\int_{U_{1,x}} f(y)\chi(x,y)e^{-\alpha D(x,y)}\frac{\omega^n(y)}{n!} = \int_{V_{1,x}} f(\exp_x \nu_x(v))\chi_x(\exp_x \nu_x(v))e^{-\alpha g_x(v,v)}\Theta(x,v)dv.$$

Consider the function on TM defined by

$$G(x,v) = \begin{cases} \chi_x(\exp_x \nu_x(v))f(\exp_x \nu_x(v))\Theta(x,v), & \text{if} \quad (x,v) \in V_1 \\ 0 & \text{if} \quad (x,v) \notin V_1 \end{cases}$$

It is smooth and compactly supported for x in a compact set and

$$\begin{aligned} \textstyle\int_{U_{1,x}} f(y)\chi(x,y)e^{-\alpha D(x,y)}\frac{\omega^n(y)}{n!} &= \textstyle\int_{T_xM} G(x,v)e^{-\alpha g_x(v,v)}dv \\ &= \textstyle\int_0^{+\infty} e^{-\alpha r^2}r^{2n-1}dr \int_{S_xM} G(x,rv)dv, \end{aligned}$$

where $r(v)^2 = g_x(v,v)$ and S_xM is the unit sphere in T_xM and where we are using the notation dv for both the volume measure on T_xM and the surface measure on S_xM. Now use Taylor's formula with integral remainder for $G(x,rv)$

$$G(x,rv) = \sum_{p=0}^{2N} \frac{r^p}{p!}(D_v^pG)(x,0) + R_{2N}(x,rv)$$

where

$$R_{2N}(x,rv) = r^{2N+1}\int_0^1 \frac{(1-s)^{2N}}{(2N)!}(D_v^{2N+1}G)(x,rsv)ds.$$

A straightforward computation, using the fact that G is compactly supported, shows that

$$|\int_0^{+\infty} e^{-\alpha r^2}r^{2n-1}dr \int_{S_xM} R_{2N}(x,rv)dv| \le \alpha^{-(n+N)}\frac{D}{\sqrt{\alpha}}$$

for some constant D. Observe also that if p is odd

$$\int_{S_xM} (D_v^p G)(x,0)dv = 0$$

since this is the integral of the restriction to the sphere of a homogeneous polynomial of odd degree. Thus

$$\int_0^{+\infty} \sum_{p=0}^{2N} \frac{r^p}{p!} e^{-\alpha r^2} r^{2n-1} dr \int_{S_xM} (D_v^p G)(x,0)dv = \sum_{p=0}^{N} \frac{(p+n-1)!}{2\alpha^{p+n}(2p)!} \int_{S_xM} (D_v^{2p} G)(x,0)dv.$$

Putting these facts together we get:

$$\alpha^N |L_\alpha(x) - \sum_{p=0}^{N} \frac{(p+n-1)!}{2\alpha^{p+n}(2p)!} \int_{S_xM} (D_v^{2p} G)(x,0)dv| \leq 2C\alpha^N \eta^\alpha + \alpha^{-n} \frac{D}{\sqrt{\alpha}}$$

This implies that $L_\alpha(x)$ admits an asymptotic expansion

$$L_\alpha(x) \sim \left(\frac{\pi}{\alpha}\right)^n \sum_{r \geq 0} \alpha^{-r} C_r(f)(x),$$

where

$$C_r(f)(x) = \frac{(r+n-1)!}{2\pi^n (2r)!} \int_{S_xM} (D_v^{2r} G)(x,0)dv, \ r = 0, 1, \ldots .$$

Finally, observe that the derivatives of the function G in the vertical direction for $v = 0$ do not depend on the choice of the cut–off function χ, but depend only on f and Θ. Therefore,

$$C_r(f)(x) = \frac{(r+n-1)!}{2\pi^n (2r)!} \int_{S_xM} (D_v^{2r} (\tilde{f}\Theta))(x,0)dv, \ r = 0, 1, \ldots ,$$

where $\tilde{f}(x,v) = f(x, \exp_x(\nu_x(v)))$. Finally, for $r = 0$

$$C_0(f)(x) = \frac{(n-1)!}{2\pi^n} f(x)\Theta(x,0) \operatorname{vol}(S^{2n-1}) = f(x),$$

since the differential of $\exp_x$ and ν_x at the zero section are equal to the identity and $\operatorname{vol}(S^{2n-1}) = \frac{2\pi^n}{(n-1)!}$. □

5 The link with Gray's invariants and applications

In order to see the link with Gray's work, fix a point $x \in M$, and observe that the function $\Theta(x, v)$ can be written as:

$$\Theta(x, v)dv = \nu_x^*(S(x, v)dv)$$

where

$$S(x, v)dv = \exp_x^*(\frac{\omega^n}{n!})(v). \tag{18}$$

By a result of Gray on the volume of small geodesic balls of an arbitrary $2n$-dimensional Riemannian manifold (M, g) (see Section 3 in [10]) one knows that the function $S(x, v)$ admits the Taylor expansion

$$S(x, v) = 1 - \frac{1}{2}(D_v^2 S)(x, 0) + \frac{1}{4!}(D_v^4 S)(x, 0) + \cdots,$$

where

$$(D_v^2 S)(x, 0) = -\frac{1}{3} Ric(v, v)$$

and

$$\int_{S_x} (D_v^4 S)(x, 0)dv = \frac{\mathrm{vol}(S^{2n-1})}{60n(n+1)}(-3|R|^2 + 8|Ric|^2 + 5\rho^2 - \frac{9}{2}\Delta\rho).$$

With these formulae at hands one can express the coefficients of the asymptotic expansion (16) in terms of Gray's invariants and of the map ν. One can also deduce some properties of the map ν. For example, we can prove the following corollary (whose proof is given after Example 5.2).

Corollary 5.1 *Suppose that for some $x \in M$ the n-form $S(x, v)dv$ given by (18) is invariant under the function ν_x. Then the the scalar curvature of the metric g is zero at x. If, moreover, the metric g is Einstein at x, then also the curvature tensor of g vanishes at x.*

Observe that when ν_x is the identity the hypothesis of the previous corollary is obviously satisfied and we get directly that the metric g is flat at x. This is shown in the following example.

Example 5.2 Consider the n-dimensional complex space $\mathbb{C}^n$ endowed with the flat metric $g = \sum_{j=1}^n |dz_j|^2$ and the corresponding Kähler form $\omega = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j$. A (globally defined) Kähler potential for g is $\Phi(z) =$

$\sum_{j=1}^{n} |z_j|^2$ and its analytic continuation is given by $\Phi(z, \bar{w}) = \sum_{j,k=1}^{n} z_j \bar{w}_k$. Therefore the diastasis reads as:

$$D(z, w) = \sum_{j=1}^{n} |z_j - w_j|^2, \ z, w \in \mathbb{C}^n,$$

namely the square of the distance between the points z and w. For a fixed point $z_0 \in \mathbb{C}^n$ the exponential map

$$\exp_{z_0} : T_{z_0}\mathbb{C}^n = \mathbb{C}^n \to \mathbb{C}^n$$

satisfies

$$\exp_{z_0}(v) = z_0 + v, v \in \mathbb{C}^n.$$

Therefore

$$g_{z_0}(v) = |v|^2 = D(z_0, \exp_{z_0}(v)).$$

Formula (10) implies that $\nu_{z_0} : \mathbb{C}^n \to \mathbb{C}^n$ can be taken to be the identity of $\mathbb{C}^n$. Thus

$$(\exp_{z_0} \circ \nu_{z_0})^* (\frac{\omega^n}{n!})(v) = dv$$

and $\Theta(z, v)$ identically equals to the constant function 1 for all $(z, v) \in \mathbb{C}^n \times \mathbb{C}^n$. Viceversa, by using Bochner's coordinates, one can see that if the map ν_x is the identity at a point x then the curvature tensor of the metric g vanishes identically at the point x.

Proof of Corollary 5.1: Denote by $c_j(x)$ the value of the operators C_j at the constant function one, namely $c_j(x) = C_j(1)(x)$. It follows by our hypothesis that

$$\Theta(x, v) = S(x, v)$$

and by formula (16) with $f = 1$ we get

$$c_0(x) = \frac{(n-1)!}{2\pi^n} S(x, 0) \operatorname{vol}(S^{2n-1}) = 1$$

$$\begin{aligned} c_1(x) = \tfrac{n!}{4\pi^n} \textstyle\int_{S_x M} (D_v^2 S)(x, 0) dv &= -\tfrac{n!}{12\pi^n} \textstyle\int_{S_x M} Ric(v, v) dv \\ &= -\tfrac{n!}{12\pi^n} \tfrac{1}{n} \operatorname{vol}(S^{2n-1}) \rho(x) = -\tfrac{\rho(x)}{6}. \end{aligned}$$

$$c_2(x) = \frac{(n+1)!}{48\pi^n} \int_{S_x M} (D_v^4 S)(x, 0) dv = \frac{1}{1440}(-3|R|^2 + 8|Ric|^2 + 5\rho^2 - \frac{9}{2}\Delta\rho),$$

where we are using the same notations as in Theorem 2.1 and where the functions on the right hand side of the last formula are evaluated at the point x. By comparing these values with those from formulae (6) above (with $f = 1$) we obtain:

$$-\frac{\rho(x)}{2} = -\frac{\rho(x)}{6},$$

which implies that $\rho = 0$ and

$$\frac{1}{6}|\operatorname{Ric}|^2 - \frac{1}{24}|R|^2 = \frac{1}{1440}(-3|R|^2 + 8|Ric|^2).$$

If g is Einstein, then also the Ricci tensor must vanish and thus the curvature tensor is forced to be identically zero. $\square$

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