

Regular quantizations and covering maps ¹

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Abstract

Let $\tilde{M} \rightarrow M$ be a holomorphic (unbranched) covering map between two compact complex manifolds, with $b_2(\tilde{M}) = 1$. We prove that if \tilde{M} and M both admit regular Kähler forms $\tilde{\omega}$ and ω respectively then, up to homotheties, $(\tilde{M}, \tilde{\omega})$ and (M, ω) are biholomorphically isometric.

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1 Introduction

In the quantum mechanics terminology introduced by Kostant and Souriau (see e.g. [1]), a geometric quantization of a compact n -dimensional Kähler manifold (M, ω) is a pair (L, h) , where L is a holomorphic line bundle over M , called the *quantum line bundle*, and h is an Hermitian metric on L whose Ricci curvature equals ω . For any non negative integer m , consider L^m the m -th tensor power of L . The Hermitian metric h induces, in a natural way, an Hermitian metric h_m on L^m whose Ricci curvature equals $m\omega$. Therefore (L^m, h_m) is a geometric quantization of the Kähler manifold $(M, m\omega)$.

Denote by $H^0(M, L^m)$ the (finite dimensional) space of global holomorphic sections of L^m and let $d_m + 1$ be its complex dimension. Fix a L^2 -orthonormal basis $(s_0^m, \dots, s_{d_m}^m)$ of $H^0(M, L^m)$ with respect to the L^2 -scalar product $\langle \cdot, \cdot \rangle_{h_m}$ induced by h_m and by the Liouville volume form $\frac{\omega^n}{n!}$. Define a holomorphic map of M into the a complex projective space $\mathbb{C}P^{d_m}$

$$\varphi_m : M \rightarrow \mathbb{C}P^{d_m}, [s_0^m, \dots, s_{d_m}^m]. \quad (1)$$

By Kodaira's theorem for m sufficiently large, the map φ_m is an embedding. Let $g_{FS}^{d_m}$ be the standard Fubini–Study metric on $\mathbb{C}P^{d_m}$. This restricts

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to a Kähler metric $g_m = \varphi_m^*(g_{FS}^{d_m})$ on M . The difference between $\frac{g_m}{m}$ and the Kähler metric g associated to ω can be measured by the function

$$T_{m\omega}(x) = \sum_{j=0}^{d_m} \|s_j^m(x)\|_{h_m}^2, \quad (2)$$

where $\|\cdot\|_{h_m}^2$ is the L^2 -norm associated to $\langle \cdot, \cdot \rangle_{h_m}$. Indeed, it is can be seen that for all non-negative integer m

$$\frac{\omega_m}{m} = \omega + \frac{i}{2m} \partial \bar{\partial} \log T_{m\omega}(x). \quad (3)$$

The functions $T_{m\omega}(x)$ (one for each m) play a prominent role in various area of physics and mathematics (see e.g. [1] and [2] and the references therein).

Of central importance for the understanding of this function is the work of Steve Zelditch [16], who, generalizing a Theorem of Gang Tian [14] proves:

Theorem 1.1 (*Zelditch*) *There is a complete asymptotic expansion*

$$T_{m\omega}(x) = \sum_{j=0}^{d_m} \|s_j^m(x)\|_{h_m}^2 = a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \cdots \quad (4)$$

for certain smooth coefficients $a_j(x)$ with $a_0 = 1$.

Later, Ziquin Lu, by using Tian's peak section method, proves:

Theorem 1.2 (*Lu*) *Each coefficients $a_j(x)$, given by the asymptotic expansion (4) is a polynomial of the curvature and its covariant derivatives at x of the metric g . Such a polynomials can be found by finitely many steps of algebraic operations. Furthermore $a_1(x) = \frac{1}{2}\rho$, where ρ is the scalar curvature of the polarized metric g .*

(See also [9] and [10] for the computations of the coefficients a_j 's through Calabi's diastasis function).

In [7] Donaldson, using Zelditch's expansion, studies the case when a_1 is constant, namely the case of constant scalar curvature metrics. His main result in this direction is the following:

Theorem 1.3 (*Donaldson*) *Let (L, h) be a geometric quantization of a compact Kähler manifold (M, ω) . Assume that $\frac{Aut(M, L)}{\mathbb{C}^*}$ is discrete. Then there exists at most one Kähler form ω on M with $c_1(L) = [\omega]$ having constant scalar curvature.*

(Here $\frac{\text{Aut}(M,L)}{\mathbb{C}^*}$ denotes the group biholomorphisms of M which lift to holomorphic bundles maps $L \rightarrow L$ modulo the trivial automorphism group \mathbb{C}^*).

In [13] Lu and Tian, inspired by Donaldson's work, address the problem of studying the set of Kähler metrics (in a fixed cohomology class) with prescribed a_j 's. In [6] Chen and Tian beautifully generalize Theorem 1.3 to the class of extremal Kähler metrics (notice that constant curvature Kähler metrics are automatically extremal), without any assumption either on the automorphism group of M or on the integrality of the Kähler form ω . More precisely they prove the following:

Theorem 1.4 (*Chen–Tian*) *Let M be a compact complex manifold and let ω_1 and ω_2 be two extremal Kähler metrics on M in the same Kähler class. Then there is a holomorphic transformation $\sigma : M \rightarrow M$ such that $\sigma^*(\omega_2) = \omega_1$.*

In this paper we continue the study (carried out by the author in [1], [2] and in [11] for the noncompact case) of the set of *regular* Kähler forms on M , namely those forms such that the functions $T_{m\omega}(x)$ are constants, for m sufficiently large. (In the terminology introduced for the first time by [3] and [4] a geometric quantization (L, h) of a Kähler manifold (M, ω) is said to be *regular* if ω is a regular form). In particular we address the following question: *how many regular Kähler forms exist?* It is not hard to see that the homogeneous Kähler forms on a simply connected and homogeneous complex manifold are regular (see Theorem 5.1 in [1]) and we conjecture that the existence of a regular Kähler form ω on a complete complex manifold M implies that (M, ω) is simply connected and homogeneous. It is worth noticing that, in view of Zelditch's result, if ω is regular the smooth functions a_j 's of the expansion of $T_{m\omega}$ are constants for all j . Moreover, observe also that (4) is not an equality and therefore the constancy of the a_j 's do not necessarily imply that ω is regular. Indeed, the flat torus admits a geometric quantization such that all the a_j 's vanish but the functions $T_{m\omega}$ are not constants (see Remark 5.2 in [1], for details).

The aim of this paper is to investigate the link between regular quantizations and covering maps. Our main result is the following theorem which can be considered as a step towards the classification of regular Kähler forms:

Theorem 1.5 *Let $\pi : \tilde{M} \rightarrow M$ be a holomorphic covering map between compact complex manifolds. Assume that $b_2(\tilde{M}) = 1$ and both \tilde{M} and M*

admit regular forms $\tilde{\omega}$ and ω respectively. Then there exists $\lambda \in \mathbb{R}^+$ and a biholomorphic isometry $F : (\tilde{M}, \lambda\tilde{\omega}) \rightarrow (M, \omega)$, i.e. $F^*(\omega) = \lambda\tilde{\omega}$.

The next section is devoted to the proof of Theorem 1.5. After recalling some results about Calabi's diastasis function we prove Proposition 2.3, which together with Theorem 1.4 is the key ingredient in the proof Theorem 1.5.

Observe that, when \tilde{M} is simply-connected, the proof of Theorem 1.5 can be obtained more directly without using Proposition 2.3 (see the end of the paper).

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2 Proof of the main results

Here we briefly recall some results about Calabi's diastasis function referring the reader to [5] and [12] for details and further results.

Let M be a complex manifold endowed with a real analytic Kähler metric g . Then in a neighborhood of every point $p \in M$, one can introduce a very special Kähler potential D_p^g for the metric g , which Calabi [5] christened *diastasis*. Recall that a Kähler potential is an analytic function Φ defined in a neighborhood of a point p such that $\omega = \frac{i}{2}\bar{\partial}\partial\Phi$, where ω is the Kähler form associated to g . A Kähler potential is not unique: it is defined up to an addition with the real part of a holomorphic function. By duplicating the variables z and \bar{z} a potential Φ can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighborhood U of the diagonal containing $(p, \bar{p}) \in M \times \bar{M}$ (here \bar{M} denotes the manifold conjugated to M). The *diastasis function* is the Kähler potential D_p^g around p defined by

$$D_p^g(q) = \tilde{\Phi}(q, \bar{q}) + \tilde{\Phi}(p, \bar{p}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p}).$$

Observe that the diastasis does not depend on the potential chosen, $D_p^g(q)$ is symmetric in p and q and $D_p^g(p) = 0$.

The diastasis function is the key tool for studying holomorphic and isometric immersions of a Kähler manifold into another Kähler manifold as expressed by the following lemma.

Lemma 2.1 (Calabi [5]) *Let (M, g) be a Kähler manifold which admits a holomorphic and isometric immersion $\varphi : (M, g) \rightarrow (S, G)$ into a real*

analytic Kähler manifold (S, G) . Then the metric g is real analytic. Let $D_p^g : U \rightarrow \mathbb{R}$ and $D_{\varphi(p)}^G : V \rightarrow \mathbb{R}$ be the diastasis functions of (M, g) and (S, G) around p and $\varphi(p)$ respectively. Then $\varphi^{-1}(D_{\varphi(p)}^G) = D_p^g$ on $\varphi^{-1}(V) \cap U$.

When (S, G) is the N -dimensional complex projective space $S = \mathbb{CP}^N$ equipped with the Fubini–Study metric $G = g_{FS}^N$, one can show that for all $p \in \mathbb{CP}^N$ the diastasis function $D_p^{g_{FS}^N}$ around p is globally defined except in the cut locus H_p of p where it blows up. On the other hand, $e^{-D_p^{g_{FS}^N}}$ is globally defined and smooth on \mathbb{CP}^N , $e^{-D_p^{g_{FS}^N}(q)} \leq 1$ and $e^{-D_p^{g_{FS}^N}(q)} = 1$ implies $p = q$ (see [5] or [1] for details).

We say that a Kähler metric g on a compact complex manifold M is *projectively induced* if there exist a non negative integer N and a holomorphic embedding $\varphi : M \rightarrow \mathbb{CP}^N$, such that $\varphi^*(g_{FS}^N) = g$.

By Lemma 2.1 one gets the following

Lemma 2.2 *Let g be a projectively induced Kähler metric on a compact complex manifold M . Then, $e^{-D_p^g}$ is globally defined on M , $e^{-D_p^g(q)} \leq 1$ and $e^{-D_p^g(q)} = 1$ implies $p = q$.*

Proposition 2.3 *Let g be a projectively induced Kähler metric on a compact complex manifold M . If there exist $\lambda \in \mathbb{R}^+$ and a holomorphic immersion $\varphi : M \rightarrow \mathbb{CP}^N$ such that $\varphi^*(g_{FS}^N) = \lambda g$ then φ is injective and hence also λg is projectively induced.*

Proof: Let $p, q \in M$ such that $\varphi(p) = \varphi(q)$. Then, by Lemma 2.1

$$e^{-D_p^{\lambda g}(q)} = e^{-D_{\varphi(p)}^{g_{FS}^N}(\varphi(q))} = 1.$$

On the other hand, $e^{-D_p^{\lambda g}(q)} = e^{-\lambda D_p^g(q)} = (e^{-D_p^g(q)})^\lambda = 1$ obviously implies $e^{-D_p^g(q)} = 1$. Therefore, $p = q$ by Lemma 2.2. \square

Proof of Theorem 1.5 Fix a non-negative integer m_0 sufficiently large so that the functions $T_{m_0\tilde{\omega}}$ and $T_{m_0\omega}$ are constants. and the holomorphic maps $\tilde{\varphi}_{m_0} : \tilde{M} \rightarrow \mathbb{CP}^{\tilde{d}_{m_0}}$ and $\varphi_{m_0} : M \rightarrow \mathbb{CP}^{d_{m_0}}$ given by (1) are embedding. By (3), the Kähler metrics $\tilde{g}_{m_0} = m_0\tilde{g} = \tilde{\varphi}_{m_0}^*(g_{FS}^{\tilde{d}_{m_0}})$ and $g_{m_0} = m_0g = \varphi_{m_0}^*(g_{FS}^{d_{m_0}})$ are projectively induced.

Let \tilde{g} (resp. g) be the Kähler metric on \tilde{M} (resp. M) associated to the regular form $\tilde{\omega}$ (resp. ω). By the regularity assumption, for m sufficiently

large $T_{m\tilde{\omega}}$ and $T_{m\omega}$ are constants and hence all the coefficients of the corresponding Zelditch's expansions (4) must be constants. In particular, by Theorem 1.2 both the metrics \tilde{g} and g must have constant scalar curvature. Consider the Kähler metric $g_\pi = \pi^*(m_0g)$ on \tilde{M} . Since $b_2(\tilde{M}) = 1$ and g_π has constant scalar curvature, by Theorem 1.4 there exist $\lambda \in \mathbb{R}^+$ and a holomorphic transformation $\sigma : \tilde{M} \rightarrow \tilde{M}$ such that $\sigma^*(g_\pi) = \lambda m_0 \tilde{g}$. Thus, the holomorphic map

$$\varphi = \varphi_{m_0} \circ \pi \circ \sigma : \tilde{M} \rightarrow \mathbb{C}P^{d_{m_0}},$$

satisfies $\varphi^*(g_{FS}^{d_{m_0}}) = \lambda m_0 \tilde{g}$. Since $m_0 \tilde{g}$ is projectively induced it follows by Proposition 2.3 that the map φ must be injective. Therefore π is injective, and since it is a covering map, it is forced to be a biholomorphism. Then $F = \pi \circ \sigma : \tilde{M} \rightarrow M$ is the desired biholomorphism satisfying $F^*(\omega) = \lambda \tilde{\omega}$.

Proof of Theorem 1.5 when \tilde{M} is simply-connected Since $b_1(\tilde{M}) = b_2(\tilde{M}) = 1$ the first Chern class $c_1(\tilde{M})$ of \tilde{M} is forced to have a sign. Moreover, since π is an unbranched covering the sign of $c_1(M)$ is the same of that of $c_1(\tilde{M})$. If $c_1(M) > 0$, then by Yau's work on Calabi's conjecture it follows that M is simply connected, hence it remains to deal with the cases $c_1(\tilde{M}) = 0$ or $c_1(\tilde{M}) < 0$. Actually these cases are excluded as follows. Let g_{KE} be the Kähler-Einstein metric on \tilde{M} (with zero, if $c_1(\tilde{M}) = 0$, or negative, if $c_1(\tilde{M}) < 0$, scalar curvature) whose existence is guaranteed again by the solution of Calabi's conjecture. By Theorem 1.4 and by the fact that \tilde{g} has constant scalar curvature there exist λ and a holomorphic transformation of \tilde{M} connecting $\lambda \tilde{g}$ with g_{KE} . Thus, \tilde{g} itself is a Kähler-Einstein metric with negative or zero scalar curvature. Moreover, by (3) and by the regularity of \tilde{g} , there exists a non-negative integer m_0 such that $m_0 \tilde{g}$ is projectively induced. Hence $m_0 \tilde{g}$ is a projectively induced Kähler-Einstein metric on \tilde{M} with zero or negative scalar curvature. This fact contradicts a Theorem of Dominique Hulin [8] which asserts that a projectively induced Kähler-Einstein metric must have positive scalar curvature.

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