Partially Isometric Immersions and Free Maps

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Abstract

In this paper we investigate the existence of "partially" isometric immersions. These are maps $f : M \to \mathbb{R}^q$ which, for a given Riemannian manifold M, are isometries on some sub-bundle $\mathcal{H} \subset TM$. The concept of free maps, which is essential in the Nash–Gromov theory of isometric immersions, is replaced here by that of $\mathcal{H}$–free maps, i.e. maps whose restriction to $\mathcal{H}$ is free. We prove, under suitable conditions on the dimension q of the Euclidean space, that $\mathcal{H}$–free maps are generic and we provide, for the smallest possible value of q, explicit expressions for $\mathcal{H}$–free maps in the following three settings: 1–dimensional distributions in $\mathbb{R}^2$, Lagrangian distributions of completely integrable systems, Hamiltonian distributions of a particular kind of Poisson Bracket.

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1 Introduction and motivation

The present paper deals with the solvability of certain "large" systems of partial differential equations (PDEs) that appear naturally when one considers the problem of inducing a quadratic form on a given distribution, where the term *large* refers to the fact that these systems contain much more unknowns than equations. Our work is partly inspired by earlier works ([D'A93, DL03, DL07]), all of them related, in different guises, to the general program of inducing geometric structures developed, among many methods for solving the isometric immersion problem, by M. Gromov in his seminal book [Gro86].

To trace the origin of this theory, we must go back to the work of Nash who showed [Nas56] that every compact C^∞ Riemannian manifold can be isometrically immersed in $\mathbb{R}^q$ with $q \geq \frac{1}{2}n(3n+11)$. This fundamental result has been

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substantially improved and refined by several authors [Gr70A, Gro86, Gun90]. In particular, the hard analytical part of Nash's original proof (based on a powerful technique generalizing the classical implicit function theorem) was taken up by Gromov who stated a *Nash-type implicit function* theorem for a certain class of differential operators called *infinitesimally invertible operators* (for the exact statement, see [Gro86], p. 117). This is a result which, besides its importance in itself (it serves as a basis to develop the whole of M. Gromov's method based on h-prinviple), represents the starting point to afford the study of many problems about the possibility of inducing arbitrary geometric structures.

In this paper, M will always denote a C^∞ smooth manifold of dimension m with metric $g = g_{\alpha\beta}dx^\alpha dx^\beta$, $\alpha, \beta = 1, \ldots, m$. Let $f : M \to \mathbb{R}^q$ be a C^∞ immersion $f = (f^1, \cdots, f^q)$ and let $g_{can} = \delta_{ij}dx^i dx^j$ be the canonical Euclidean metric on $\mathbb{R}^q$. The map f induces on M a metric tensor given in local coordinates by:

$$f^* g_{can} = \delta_{ij}\frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}dx^\alpha dx^\beta. \tag{1}$$

Here and throughout the paper we write the above pull-back $f^* g_{can}$ as $\mathcal{D}(f)$, where

$$\mathcal{D} : C^\infty(M, \mathbb{R}^q) \to \Gamma$$

denotes the *metric-inducing operator*, viewed as a map between the space $C^\infty(M, \mathbb{R}^q)$ of smooth maps $f : M \to \mathbb{R}^q$ and the space Γ of smooth quadratic differential forms g on M. These function spaces will always be considered equipped with the C^∞-fine (also called Whitney) topology. Recall that, if the manifold M is compact, the Whitney topology coincides with the ordinary C^∞ topology.

Placed in this setting, the isometric immersion problem [Nas56] asks for a solution f to the (inducing) equation

$$\mathcal{D}(f) = g \tag{2}$$

which, in local coordinates, requires to solve, for a given positive-definite symmetric matrix of functions $g_{\alpha\beta} = g_{\alpha\beta}(x)$ on M, the system

$$\frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}\delta_{ij} = g_{\alpha\beta},\ i, j = 1, \ldots, q. \tag{3}$$

which consists of $m(m+1)/2$ equations in the q unknowns f^i. One of the main steps in the original proof of Nash's isometric immersion theorem amounts to inverting algebraically the linearized equations corresponding to the system (3). It turns out that the same idea applies to many other instances of nonlinear PDEs of geometric nature, which, following the general approach indicated in [Gro86] can be locally solved after an appropriate infinite dimensional implicit function theorem is established to pass from (solutions of) the linearized system to (solutions of) the non-linear system. Once this first step is done, the efficency of Gromov's method allows to derive specific (local) statements as direct corollaries of the generalized theorem just by specializing, to the pertinent differential operator, the basic properties of infinitesimally invertible operators and the consequent analytical results (compare [Gr72]).

The formulation of Gromov's generalized implicit function theorem we need for our application is stated below (see [Gr70A]):

Theorem IFT. *Let $\mathcal{D} : C^\infty(M, \mathbb{R}^q) \to \Gamma$ be a C^∞ differential operator which is infinitesimally invertible over some open subset $U \subset C^\infty(M, \mathbb{R}^q)$. Then the restriction of $\mathcal{D}$ to U is an open map.*

The formal definition of infinitesimally invertible operator is omitted here as it requires elaborate preliminaries (see [Gro86], p.115-116). For our purposes it is enough to say that an operator is infinitesimally invertible if its linearization is invertible.

In the case of the metric inducing operator, the class of those maps to which Theorem IFT is applicable consists of the maps called *free* by Nash. Recall that an immersion $f : M \to \mathbb{R}^q$ is said to be *free* if for all $x \in M$ the $m + s_m$ vectors $\{\partial_\alpha f^i(x), \partial_{\alpha\beta} f^i(x)\}$ are linearly independent (clearly a necessary condition for the existence of free maps $f : M \to \mathbb{R}^q$ is that $q \geq m + s_m$).

Theorem N1. ([Nas56] and [Gro86], p.116) *The metric inducing operator $\mathcal{D}$: $C^\infty(M, \mathbb{R}^q) \to \Gamma$ is infinitesimally invertible over the set of free maps $M \to \mathbb{R}^q$.*

The next result, which follows by a straightforward application of Thom's transversality theorem is also due to Nash ([Nas56]) (compare [Gro86], p.33).

Theorem N2. *A generic map $M \to \mathbb{R}^q$ is free for $q \geq 2m + s_m$, where $s_m = \frac{m(m+1)}{2}$.*

(The above means that free maps are open and dense among all C^2-maps for $q \geq 2m + s_m$).

Let us turn now to the subject matter of the present paper, where we are given a k-dimensional distribution $\mathcal{H}$ on a m-dimensional smooth manifold M. Denote by $\Gamma_\mathcal{H}$ the space of quadratic differential forms on $\mathcal{H}$ and by $G_\mathcal{H} \subset \Gamma_\mathcal{H}$ the (open) sub-set of the positive ones, i.e. the set of smooth Riemannian metrics on $\mathcal{H}$.

We are concerned with the following question. Assume that there exists a smooth map $f_0 : M \to \mathbb{R}^q$ which induces on M a metric $g_0 \in G_\mathcal{H}$ (from the canonical Euclidean metric g_{can} on $\mathbb{R}^q$). This means that:

$$f_0^* g_{can}|_\mathcal{H} = g_0 \,, \tag{4}$$

which can be written as

$$\mathcal{D}_\mathcal{H}(f_0) = g_0,$$

where $\mathcal{D}_\mathcal{H} : C^\infty(M, \mathbb{R}^q) \to \Gamma_\mathcal{H}$ denotes the partial differential operator defined by:

$$f \mapsto \mathcal{D}_\mathcal{H}(f) = f^* g_{can}|_\mathcal{H}.$$

We shall be interested in seeing whether the same is true for any other $g \in G_{\mathcal{H}}$ close enough to g_0, namely if the inducing equation

$$\mathcal{D}_{\mathcal{H}}(f) = g \tag{5}$$

is solvable in a neighbourhood of g_0. In different words, we want to find an open neighbourhood $U_{g_0} \subset G_{\mathcal{H}}$ of g_0 which is contained in the image of $\mathcal{D}_{\mathcal{H}}$.

The above equation (5) is easily seen to be equivalent to a system of PDEs which is, for any metric $\tilde{g}$ on M, a subsystem of (3). This is the reason for calling *partial isometry* between $(\mathcal{H}, g)$ and $(T\mathbb{R}^q, g_{can})$ any map $f : M \to \mathbb{R}^q$ which is a solution of (5). It can be useful to notice that, since the map f is not necessarily an immersion, unlike in the case of isometries the quadratic form $\mathcal{D}(f)$ is not required to be positive-definite on the whole TM.

To answer the above question we have to study the linearization of the operator $\mathcal{D}_{\mathcal{H}}$. This is done in Section 2, where we introduce the notion of $\mathcal{H}$-free map $f : M \to \mathbb{R}^q$ and prove, for the operator $\mathcal{D}_{\mathcal{H}}$, the analogous of Theorems N1 and N2. This is Theorem 2.5. As a corollary (see Theorem 2.6) we obtain that the restriction of $\mathcal{D}_{\mathcal{H}}$ to the set of $\mathcal{H}$-free maps is an open map.

As we shall see from the very definition of $\mathcal{H}$-freedom, the condition $q \geq k+s_k$ is necessary (but not sufficient) for the existence of $\mathcal{H}$-free maps $f : M \to \mathbb{R}^q$. It is then natural to look for explicit expressions of $\mathcal{H}$-free maps when $q = k + s_k$. Notice that very few natural examples of free maps are known so far (see [Gro86], p.12). In Section 3 we show explicit examples of $\mathcal{H}$-free maps when the distribution $\mathcal{H}$ is one of the following: a one-dimensional distribution on $\mathbb{R}^2$; the Lagrangian distribution of a complete integrable system; the Hamiltonian distribution of a certain kind of Poisson bracket.

We end this introduction by stating the main results contained in this paper. Their proofs are postponed to Section 3.

Theorem 1.1. *Let $\mathcal{H} \subset T\mathbb{R}^2$ be a one-dimensional distribution of finite type on $\mathbb{R}^2$. Then there exists a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F = \psi \circ f$ is an $\mathcal{H}$-free map for any free map $\psi : \mathbb{R} \to \mathbb{R}^2$.*

Theorem 1.2. *Let (M^{2n}, Ω) be a symplectic manifold admitting a completely integrable system $\{I_1, \cdots, I_n\}$, $\mathcal{H} \subset TM$ the n-dimensional Lagrangian distribution $\mathcal{H} = \cap_{i=1}^n \ker dI_i$ and $\mathcal{F}$ the corresponding Lagrangian foliation. Assume that the Hamiltonian vector fields associated to the I_i are all complete and that every leaf of $\mathcal{F}$ has no compact component. Then it is possible to find n smooth real valued functions f^i, $i = 1, \ldots, n$, on M such that the map $F : M \to \mathbb{R}^{n+s_n}$, $s_n = \frac{n(n+1)}{2}$ defined by*

$$F(x) = (\psi_1(f^1(x)), \cdots, \psi_n(f^n(x)), f^1(x)f^2(x), \cdots, f^{n-1}(x)f^n(x))$$

is $\mathcal{H}$-free for any choice of n free maps $\psi_i : \mathbb{R} \to \mathbb{R}^2$.

Theorem 1.3. *Let M be an n-dimensional oriented Riemannian manifold, $H = \{h_1, \cdots, h_{n-2}\}$ a set of $n-2$ functions functionally independent at every* *point and $\{\cdot,\cdot\}_H$ the corresponding Riemann-Poisson bracket. If $h \in C^\infty(M)$ is functionally independent from all the h_i and $\mathcal{H}$ is the corresponding Hamiltonian 1-dimensional distribution, then there exists a (possibly multivalued) smooth function $f : M \to \mathbb{R}$ such the smooth map $F : M \to \mathbb{R}^2$ given by $F(x) = \psi(f(x))$ is $\mathcal{H}$-free for every free map $\psi : A \to \mathbb{R}^2$ where $A = \mathbb{R}$ if f is single-valued or $A = \mathbb{S}^1$ if f is multivalued.*

2 $\mathcal{H}$-free maps and the linearization of the operator $\mathcal{D}_\mathcal{H}$

Let $\mathcal{H}$ be a k-dimensional distribution on M, i.e. a vector subbundle of TM. Fix local coordinates (x^α) on some chart $U \subset M$, $\alpha = 1, \cdots, m$, and let $\{\xi_a\}$, $a = 1, \cdots, k$, be a local trivialization for $\mathcal{H}$. Let $\{\theta^a, \omega^A\}$, $A = 1, \cdots, m-k$, be a dual base for the whole T^*M such that $i_{\xi_b}\theta^a = \delta^a_b$ and $i_{\xi_b}\omega^A = 0$. Then

$$\mathcal{H}|_U = \bigcap_{A=1}^{m-k} \ker \omega^A.$$

and the gradient of the components of a smooth map $f = (f^1, \ldots, f^q) : M \to \mathbb{R}^q$ writes

$$df^i = u^i_A \omega^A \oplus v^i_a \theta^a \ , \ \ i = 1, \cdots, q$$

where $v^i_a = i_{\xi_a} df^i = L_{\xi_a} f^i$ (the u^i_A play no role in what follows). Then, in local terms, the restriction to $\mathcal{H} \subset TM$ of $f^* g_{can} = \delta_{ij} df^i \otimes df^j$ is given by

$$f^* g_{can}|_\mathcal{H} = \delta_{ij} L_{\xi_a} f^i L_{\xi_b} f^j \ \theta^a \otimes \theta^b \tag{6}$$

and the equation $\mathcal{D}_\mathcal{H}(f) = g$ writes locally as

$$\delta_{ij} L_{\xi_a} f^i L_{\xi_b} f^j = g_{ab}, \tag{7}$$

where $g_{ab} = g(\xi_a, \xi_b)$, $a, b = 1, \ldots, k$.

Let $\mathcal{H}$ be a distribution on M. We say that $f \in C^\infty(M, \mathbb{R}^q)$ is an $\mathcal{H}$-*immersion* if the restriction to $\mathcal{H}$ of the tangent map $Tf : TM \to T\mathbb{R}^q$ is injective.

Example 0. Take $M = A \times B$, where A and B are smooth manifolds. Consider the two natural projections π_A and π_B on A and B and the corresponding two canonical distributions $\mathcal{H}_A = \ker T\pi_B = TA \oplus \{0\}$ and $\mathcal{H}_B = \ker T\pi_A = \{0\} \oplus TB$. A map $f \in C^\infty(M, \mathbb{R}^q)$ is a $\mathcal{H}_A$-immersion iff $f(\cdot, b) : A \to \mathbb{R}^q$ is an immersion for every $b \in B$. Similarly for $\mathcal{H}_B$.

Example 1. For any fiber bundle (M, N, π, F) it is defined the canonical distribution of vertical vectors $V = \ker T\pi \subset TM$. A smooth map $f : M \to \mathbb{R}^q$ is a V-immersion iff on every trivialization $U \times F$ of M the map $f(u, \cdot) : F \to \mathbb{R}^q$ is an immersion for every $u \in U$. Let now A be a linear connection on M and let H be the horizontal distribution with respect to A; then a map $f : M \to \mathbb{R}^q$ is a H-immersion iff the covariant derivatives $\{\nabla_\mu f^i\}$ are linearly independent on every point of M.

Proposition 2.1. *Let $f \in C^\infty(M, \mathbb{R}^q)$. The quadratic form $\mathcal{D}_{\mathcal{H}}(f)$ is positive-definite iff f is a $\mathcal{H}$-immersion.*

Proof. Let $\{\xi_a\}$ be a local trivialization for $\mathcal{H}$. Then

$$Tf(\xi_a) = \partial_\alpha f^i \partial_i \otimes dx^\alpha(\xi_a^\beta \partial_\beta) = \xi_a^\alpha \partial_\alpha f^i \partial_i = (L_{\xi_a} f^i)\partial_i$$

and the proposition follows. □

Proposition 2.2. *Let $\mathcal{H} \subset TM$ be a k-dimensional distribution. If $q \geq m + k$ the set of $\mathcal{H}$-immersions is open and dense in $C^\infty(M, \mathbb{R}^q)$.*

Proof. A map $f : M \to \mathbb{R}^q$ is a $\mathcal{H}$-immersion iff the $k \times q$ matrix $(L_{\xi_a} f^i)$ has rank k at every point. Denote by $D : M \to M_{k,q}(\mathbb{R})$ the corresponding map with $D = (L_{\xi_a} f^i)$. The image $D(M)$ and the set of non maximal rank matrices whose codimension in $M_{k,q}(\mathbb{R})$ is $q - k + 1$ [Arn71] do not intersect if $m < q - k + 1$. □

Let us consider now a smooth 1-parameter deformation g_ϵ of the metric g on M such that $g_0 = g$ and assume that there exists a corresponding smooth 1-parameter deformation f_ϵ such that $f_0 = f$. It follows by (7) that

$$\delta_{ij} L_{\xi_a} {f_\epsilon}^i L_{\xi_b} {f_\epsilon}^j = g_{\epsilon,ab} \ ,$$

where $g_{\epsilon,ab} = g_\epsilon(\xi_a, \xi_b)$. Differentiate with respect to ϵ and set

$$\delta f^i = \left.\frac{df_\epsilon^i}{d\epsilon}\right|_{\epsilon=0} \ , \quad \delta g_{ab} = \left.\frac{dg_{\epsilon,ab}}{d\epsilon}\right|_{\epsilon=0}$$

thus obtainining the system of $k(k+1)/2$ PDEs:

$$\delta_{ij}\big(L_{\xi_a} f^i(x)\delta[L_{\xi_b} f^j(x)] + \delta[L_{\xi_a} f^i(x)]L_{\xi_b} f^j(x)\big) = \delta g_{ab}(x).$$

Since

$$L_{\xi_a} f^i \delta[L_{\xi_b} f^j] = L_{\xi_b}[L_{\xi_a} f^i \delta f^j] - L_{\xi_b} L_{\xi_a} f^i \delta f^j$$

by defining $\psi_a(x) = \delta_{ij} L_{\xi_a} f^i(x) \delta f^j(x)$ one gets the following equivalent algebraic system in the q unknown δf^j:

$$\begin{cases} \delta_{ij} \ L_{\xi_a} f^i \delta f^j & = \psi_a \\ \delta_{ij} \ (L_{\xi_a} L_{\xi_b} f^i + L_{\xi_b} L_{\xi_a} f^i)\delta f^j & = L_{\xi_a}\psi_b + L_{\xi_b}\psi_a - \delta g_{ab} \end{cases} \tag{8}$$

where the ψ_a are arbitrary functions.

A sufficient condition for this system to be solvable is that the matrix

$$D_{\xi_1,\cdots,\xi_k,f} = \begin{pmatrix} L_{\xi_1} f^1 & \cdots & L_{\xi_1} f^q \\ \vdots & \vdots & \vdots \\ L_{\xi_k} f^1 & \cdots & L_{\xi_k} f^q \\ L^2_{\xi_1} f^1 & \cdots & L^2_{\xi_1} f^q \\ L_{\xi_1} L_{\xi_2} f^1 + L_{\xi_2} L_{\xi_1} f^1 & \cdots & L_{\xi_1} L_{\xi_2} f^q + L_{\xi_2} L_{\xi_1} f^q \\ \vdots & \vdots & \vdots \\ L^2_{\xi_k} f^1 & \cdots & L^2_{\xi_k} f^q \end{pmatrix} \tag{9}$$

has maximal rank. Equivalently, the vectors

$$L_{\xi_a} f^i,\ \{L_{\xi_a}, L_{\xi_b}\}f = \ L_{\xi_a} L_{\xi_b} f^i + L_{\xi_b} L_{\xi_a} f^i$$

must be linearly independent. Notice that we need to assume $q \geq 2$, since the matrix (9) has at least two lines.

Definition. *Let $\mathcal{H} \subset TM$ be a k-dimensional distribution on a smooth manifold M. We say that a smooth map $f : M \to \mathbb{R}^q$ is* $\mathcal{H}$-free at $x \in M$ *if (in a neighbourhood U of x) there exists a trivialization $\{\xi_a\}$ of $\mathcal{H}$ such that*

$$\mathrm{rank} D_{\xi_1,\cdots,\xi_k,f} = k + s_k,\ \ s_k = \frac{k(k+1)}{2}\ .$$

Notice that $\mathcal{H}$-free maps only can exist for $q \geq k+s_k$. Moreover every $\mathcal{H}$-free map is an $\mathcal{H}$-immersion so that the quadratic differential form $\mathcal{D}_{\mathcal{H}}(f)$ induced on $\mathcal{H} \subset TM$ by a $\mathcal{H}$-free map $f : M \to \mathbb{R}^q$ is always positive-definite (i.e. $\mathcal{D}_{\mathcal{H}}(f)$ is a Riemannian metric on $\mathcal{H}$).

We do not deal here with the question of inducing a given metric on a given $\mathcal{H}$. We shall return to this problem in another paper where the notion of partial isometry will be considered in a more general context (see [Gr70B]).

Next proposition shows that the above definition is well posed.

Proposition 2.3. *The rank of the matrix $D_{\xi_1,\cdots,\xi_k,f}$ given by (9) does not depend on the particular choice of the trivialization of $\mathcal{H}$.*

Proof. Take another trivialization $\{\zeta_a\}$ of $\mathcal{H}$ in the same neighbourhood $U(x)$ of x. Then $\zeta_a(x) = \lambda_a^b(x)\xi_b(x)$ for some local section $\lambda_a^b(x)$ of the frame bundle over $\mathcal{H}$ and

$$L_{\zeta_a} f = \lambda_a^b L_{\xi_b} f\ ,\ \ L_{\zeta_a} L_{\zeta_b} f = \lambda_a^c L_{\xi_c} \lambda_b^d L_{\xi_d} f + \lambda_a^c \lambda_b^d L_{\xi_c} L_{\xi_d} f\ .$$

Clearly $\mathrm{rank}(L_{\zeta_a} f^i) = \mathrm{rank}(L_{\xi_a} f^i)$. Hence

$$\mathrm{rank}\begin{pmatrix} L_{\zeta_a} f \\ \{L_{\zeta_a}, L_{\zeta_b}\}f \end{pmatrix} = \mathrm{rank}\begin{pmatrix} L_{\xi_a} f \\ (\lambda_a^c L_{\xi_c} \lambda_b^d + \lambda_b^c L_{\xi_c} \lambda_z^d) L_{\xi_d} f + \lambda_a^c \lambda_b^d \{L_{\xi_a}, L_{\xi_b}\} f \end{pmatrix}$$

$$= \mathrm{rank}\begin{pmatrix} L_{\xi_a} f \\ \lambda_a^c \lambda_b^d \{L_{\xi_a}, L_{\xi_b}\} f \end{pmatrix} = \mathrm{rank}\begin{pmatrix} L_{\xi_a} f \\ \{L_{\xi_a}, L_{\xi_b}\} f \end{pmatrix}$$

□

Example 2. It is known (see [Kap40]) that every one-dimensional distribution $\mathcal{H}$ on the plane $\mathbb{R}^2$ is orientable and then it is the kernel of a regular [1] 1-form ω. The metric induced on $\mathcal{H} = \ker \omega$ by a map $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (\alpha(x,y), \beta(x,y))$, is, by Eq. (7),

$$\mathcal{D}_{\mathcal{H}}(f) = [(L_\xi \alpha)^2 + (L_\xi \beta)^2](*\omega)^2$$

[1] Throughout this paper we call a vector field or a k-form *regular* if they are different from zero at every point of M.

where $*$ is the Euclidean Hodge operator. Then f is a $\mathcal{H}$-immersion iff $L_\xi\alpha$ and $L_\xi\beta$ do not vanish simultaneously at any point. Here the matrix $D_{\xi,f}$ is given by the 2×2 matrix

$$D_{\xi,f} = \begin{pmatrix} L_\xi\alpha & L_\xi\beta \\ L^2_\xi\alpha & L^2_\xi\beta \end{pmatrix} .$$

Therefore f is $\mathcal{H}$-free iff there is a regular section ξ of $\mathcal{H}$ such that

$$L_\xi\alpha\, L^2_\xi\beta - L_\xi\beta\, L^2_\xi\alpha > 0$$

on the whole plane $\mathbb{R}^2$.

We show below a few concrete examples of $\mathcal{H}$-free maps $f : M \to \mathbb{R}^q$ for $q = k + s_k$ and either $\dim\mathcal{H} = 1$ or $\operatorname{codim}\mathcal{H} = 1$.

Example 3. Take a regular vector field $\xi \in \mathfrak{X}(\mathbb{R}^m)$ and $\mathcal{H} = \operatorname{span}\{\xi\}$. Assume that ξ has a component always different from zero, e.g. $\xi^1 = 1$. Then $L_\xi x^1 = 1$ and a direct calculation shows that the function $F(x) = \psi(x^1)$ is $\mathcal{H}$-free for every free map $\psi : \mathbb{R} \to \mathbb{R}^2$ (for example, one can take $\psi(t) = (t, e^t)$ or $\psi(t) = (\sin t, \cos t)$).

Example 4. Consider a Riemannian manifold (M, g) and a regular vector field ξ on M which is the gradient of some function f, i.e. $\xi^\alpha = g^{\alpha\beta}\partial_\beta f$. Then $L_\xi f = \|\xi\|^2 > 0$ and, as in the previous example, $F(x) = \psi(f(x))$ is $\mathcal{H}$-free (for $\mathcal{H} = \operatorname{span}\{\xi\}$) for any free map $\psi : \mathbb{R} \to \mathbb{R}^2$.

Example 5. Take the two commuting vector fields $\xi_1 = (\cos y, -\sin y, 0)$ and $\xi_2 = (0, 0, 1)$ and consider the (integrable) distribution $\mathcal{H} = \operatorname{span}\{\xi_1, \xi_2\} \subset T\mathbb{R}^3$. The leaves of this $\mathcal{H}$ are the direct product of the level sets $f(x, y) = e^x \sin(y)$ with the z axis and, the space of leaves being not Hausdorff, this foliation is not topologically equivalent to the trivial one of $\mathbb{R}^3$. A direct computation gives that $L_{\xi_1}(e^x \cos y) = e^{2x} > 0$ and $L_{\xi_2} z = 1 > 0$ so that the map $f : \mathbb{R}^3 \to \mathbb{R}^5$ defined by

$$f(x, y, z) = (\psi(e^x \cos y), \varphi(z), ze^x \cos y))$$

is $\mathcal{H}$-free for every pair of free maps $\psi, \varphi : \mathbb{R} \to \mathbb{R}^2$.

Example 6. Let $\omega = y\,dx - dz$ be the canonical contact structure on $\mathbb{R}^3$ and $\mathcal{H} = \ker\omega \subset T\mathbb{R}^3$ the corresponding (non-integrable) codimension-1 distribution on $\mathbb{R}^3$. Equivalently, $\mathcal{H} = \operatorname{span}\{\xi_1, \xi_2\}$ for $\xi_1 = (0, 1, 0)$ and $\xi_2 = (1, 0, -y)$. The fields ξ_1 and ξ_2 are both transversal to the level sets of the function $f : \mathbb{R}^3 \to \mathbb{R}^5$ defined by

$$f(x, y, z) = (y, x, e^y, e^x, z)$$

Here the matrix (9) writes:

$$D_{\xi_1,\xi_2,f} = \begin{pmatrix} 1 & 0 & e^y & 0 & 0 \\ 0 & 1 & 0 & e^x & -y\cos y \\ 0 & 0 & e^y & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & e^x & 0 \end{pmatrix}$$

and f is $\mathcal{H}$-free as $\det D_{\xi_1,\xi_2,f} = e^{x+y} > 0$.

In the following proposition we state a characterization of $\mathcal{H}$-freedom which, in all respects (including the argument for its proof), is similar to that of freedom for maps $f : M \to \mathbb{R}^q$ (compare [Gr70A], p.45).

Proposition 2.4. *Let $\mathcal{H}$ be a distribution on a smooth manifold M, let $S = S^2\mathcal{H}^*$ be the set of its symmetric $(0,2)$ tensors and $N = (Tf(\mathcal{H}))^\perp$ the normal bundle to $Tf(\mathcal{H})$ in $\mathbb{R}^q$ with respect to g_{can}. Then a $\mathcal{H}$-immersion $f : M \to \mathbb{R}^q$ is $\mathcal{H}$-free iff the "Wintergarten map" $\nu : N \to S$ defined by*

$$\nu_x(n_x) = \{L_{\xi_a}, L_{\xi_b}\} f^i(x)\delta_{ij} n^j_x \, \theta^a \otimes \theta^b$$

is surjective.

The next proposition is the analogous of Theorems N1 and N2 for partial isometries:

Theorem 2.5. *Let $\mathcal{H} \subset TM$ be a k-distribution on M, $\dim M = m$. The operator $\mathcal{D}_\mathcal{H}$ is infinitesimally invertible on the set of $\mathcal{H}$-free maps $f : M \to \mathbb{R}^q$. Moreover, if $q \geq m + k + s_k$, a generic map $M \to \mathbb{R}^q$ is $\mathcal{H}$-free.*

Proof. The infinitesimally invertibility of $\mathcal{D}_\mathcal{H}$ follows directly from the definition of $\mathcal{H}$-freedom. Observe that a map $f : M \to \mathbb{R}^q$ is $\mathcal{H}$-free when the image of the map

$$D_{\xi_1,\cdots,\xi_k,f} : M \to M_{s_k,q}(\mathbb{R})$$

is contained in the set of matrices of maximal rank. In particular a map is *not* $\mathcal{H}$-free when the image of $D_{\xi_1,\cdots,\xi_k,f}$ intersects the set $\mathcal{N}_{s_k,q}$ of matrices of non-maximal rank, whose codimension is $c = q + 1 - s_k$ [Arn71]. For a generic f the image $D_{\xi_1,\cdots,\xi_k,f}(M)$ and $\mathcal{N}_{s_k,q}$ are transversal and therefore they do not have points in common when $\dim D_{\xi_1,\cdots,\xi_k,f}(M) < \operatorname{codim} \mathcal{N}_{s_k,q}$, namely for $m < c$. Hence a generic map f is $\mathcal{H}$-free for $q > m - 1 + s_k$. □

By combining Theorem 2.5 and Theorem IFT we get as a corollary:

Theorem 2.6. *The restriction of $\mathcal{D}_\mathcal{H}$ to the subset of $\mathcal{H}$-free maps of $C^\infty(M, \mathbb{R}^q)$ is an open map.*

3 Construction of $\mathcal{H}$-free maps for $q = k + s_k$

The purpose of this section is to give proofs of theorems 1.1, 1.2 and 1.3. For all three, the argument of the proof stems from a result obtained by J.L. Weiner in an article [Wei88] where he studies the problem of existence of smooth first integrals for a direction field on the plane. Here we give three different variants (quoted below as Lemma 3.1, Lemma 3.3 and Lemma 3.4) of Weiner's result which, in the original formulation, reads as follows:

Lemma W. *Let h be a smooth function on $\mathbb{R}^2$ without critical points and let $\mathcal{H} = \ker dh \subset T\mathbb{R}^2$. Then there exists a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ whose level sets are transverse to $\mathcal{H}$ at every point.*

3.1 One-dimensional distributions on $\mathbb{R}^2$

Here is our first generalization of the Lemma W above where the hypothesis $\mathcal{H} = \ker dh$ is weakened.

Consider the foliation $\mathcal{F}$ of the integral leaves of $\mathcal{H}$, that is $\mathcal{H} = T\mathcal{F}$, and let $\pi : \mathbb{R}^2 \to \mathcal{F}$ be the canonical projection which associates to every point of $\mathbb{R}^2$ the leaf of $\mathcal{F}$ passing through it. A leaf s of $\mathcal{F}$ is said a *separatrix* if there is another leaf s' such that every saturated (with respect to π) neighbourhood of s has non-empty intersection with every saturated neighbourhood of s' (e.g. see Fig. 1). We call leaves s and s' in such a relation *inseparable leaves* of $\mathcal{F}$ and denote by I_s the set of all leaves of $\mathcal{F}$ which are inseparable from s. Finally, we say that a foliation $\mathcal{F}$ is of *finite type* if the set I_s is finite for every separatrix s of $\mathcal{F}$.

Lemma 3.1. *Let $\mathcal{H}$ be any 1-dimensional distribution on $\mathbb{R}^2$ of finite type. The following three (equivalent) properties hold true:*

1. *there exists a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ whose level sets are transverse to $\mathcal{H}$ at every point;*

2. *for any regular 1-form ω such that $\mathcal{H} = \ker \omega$ there exists a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $*(\omega \wedge df) > 0$;*

3. *for any regular section ξ of $\mathcal{H}$ there exists a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $L_\xi f > 0$.*

Proof. Let $\mathcal{H}^\perp$ be the distribution orthogonal to $\mathcal{H}$ with respect to the Euclidean metric on $\mathbb{R}^2$ and let $\mathcal{F}$ and $\mathcal{G}$ be the foliations such that $T\mathcal{F} = \mathcal{H}$ and $T\mathcal{G} = \mathcal{H}^\perp$. By construction, $\mathcal{F}$ and $\mathcal{G}$ are transversal at every point. Moreover, they can be oriented in such a way that the induced orientation on $\mathcal{H}_p \oplus \mathcal{H}_p^\perp$ coincides with the orientation of $T_p\mathbb{R}^2$ at every point $p \in \mathbb{R}^2$.

Consider any point $p \in \mathbb{R}^2$ and let h_p and g_p be the leaves of $\mathcal{F}$ and $\mathcal{G}$ passing through p. The set $U_p = \pi^{-1}(g_p)$ is a tubular neighbourhood of h_p. Let ξ be the unitary (with respect to the Euclidean norm) section of $\mathcal{H}$ pointing in the positive direction, Φ^t_ξ its flow and $\phi : \mathbb{R} \to g_p$ a chart on g_p such that $\phi(0) = p$

and compatible with the orientation of $\mathcal{G}$. It is easy to verify that the map $\psi_p : \mathbb{R}^2 \to U_p$, defined as $\psi_p(x,t) = \Phi^t_\xi(\phi(x))$, is a chart for U_p compatible with the orientation of $\mathbb{R}^2$ and that the leaves h_p and g_p correspond respectively, in this coordinate system, to the coordinate axes $x = 0$ and $t = 0$.

Let now ω be a 1-form such that $\mathcal{H} = \ker\omega$ (cfr. Example 2) oriented positively with respect to $\mathcal{G}$, i.e. such that $\omega(\zeta) > 0$ if ζ is a regular vector field tangent to $\mathcal{G}$ and oriented positively. In the chart ψ_p the vector field $\xi \in \ker\omega$ equals ∂_t and so $\psi_p^*\omega = \alpha(x,t)dx$ (i.e. ω has no dt component) with $\alpha(x,t) > 0$.

Next, we define on U_p the smooth function $f_p : U_p \to \mathbb{R}$ represented in coordinates by $f_{\psi_p}(x,t) := \psi_p^* f_p(x,t) = l(t)$, where l is any real smooth function equal to 0 for $t \leq -1$, to 1 for $t \geq 1$ and strictly increasing for $|t| < 1$. Clearly, the support of f_p is the set $V_p = \psi_p(\mathbb{R} \times [-1,1])$. We have

$$\psi_p^*(\omega \wedge df_p)(x,t) = \alpha(x,t)\partial_t f_{\psi_p}(x,t)\, dx \wedge dt$$

so that $\psi_p^*(*(\omega\wedge df_p))(x,t) = \alpha(x,t) f_{\psi_p}(x,t) *(dx \wedge dt)$ is strictly positive inside $\mathbb{R}\times(-1,1)$ and constant elsewhere. In particular, df_p has compact support equal to V_p.

Furthermore, every such function f_p can be extended from U_p to the whole $\mathbb{R}^2$. Indeed, the leaf h_p divides $\mathbb{R}^2$ in two disjoint components and the value of f_p on h_p is strictly in between 0 and 1. By setting $f_p = 0$ on all those points outside V_p which belong to the component where f_p takes the value 0 and $f_p = 1$ elsewhere, we get the desired extension $f_p : \mathbb{R}^2 \to \mathbb{R}$.

Next, let us (arbitrarily) fix, for every separatrix $s_k \in \mathcal{F}$, a leaf $g_k \in \mathcal{G}$ so that $g_k \cap s_k \neq \emptyset$ (which implies $g_k \cap s_{k'} = \emptyset$ for $k \neq k'$). Consider, inside every $U_k := U_{p_k}$, the locally finite covering consisting of the sets $C_{k,i} = \psi_k(\mathbb{R}\times(i-1,i+1))$, $i \in \mathbb{Z}$, where $\psi_k := \psi_{p_k}$. The set $C = \{C_{k,i}, k \in \mathbb{N}, i \in \mathbb{Z}\}$ obviously covers the whole $\mathbb{R}^2$ and, since the foliation $\mathcal{F}$ is of finite type, for every $k \in \mathbb{N}$ there is a finite number of $k' \in \mathbb{N}$ such that $U_k \cap U_{k'} \neq \emptyset$. Therefore, C is locally finite. We denote by $p_{k,i}$ the points $\psi_k(0,i)$ and set $f_{k,i} = f_{p_{k,i}}$. The series

$$\sum_{k=1}^{\infty}\sum_{i=-\infty}^{\infty} 2^{-N(k,i)} f_{k,i}$$

(here $N : \mathbb{N}\times\mathbb{Z} \to \mathbb{N}$ indicates any bijection) converges pointwise to a smooth function f. Indeed the $f_{k,i}$ are uniformly bounded and the series $\sum_{k=1}^{\infty}\sum_{i=-\infty}^{\infty} D_\alpha f_{k,i}$ of the derivatives of all positive orders is a finite sum since all $D_\alpha f_{k,i}$ have support equal to $C_{k,i}$.

Finally, we have

$$*(\omega \wedge df) = \sum_{k=0}^{\infty}\sum_{i=-\infty}^{\infty} 2^{-N(k,i)}[*(\omega \wedge df_{k,i})] > 0$$

since every point of $\mathbb{R}^2$ belongs to some $C_{k,i}$, where $*(\omega \wedge df_{k,i}) > 0$. □

Proof of Theorem 1.1. Let ξ be a regular section of the distribution $\mathcal{H} \subset T\mathbb{R}^2$, $\mathcal{H} = \ker\omega$ such that $i_\xi(*\omega) = 1$ and let $U = L_\xi^{-1}\left(C^\infty_+(\mathbb{R}^2)\right)$ denote the set of

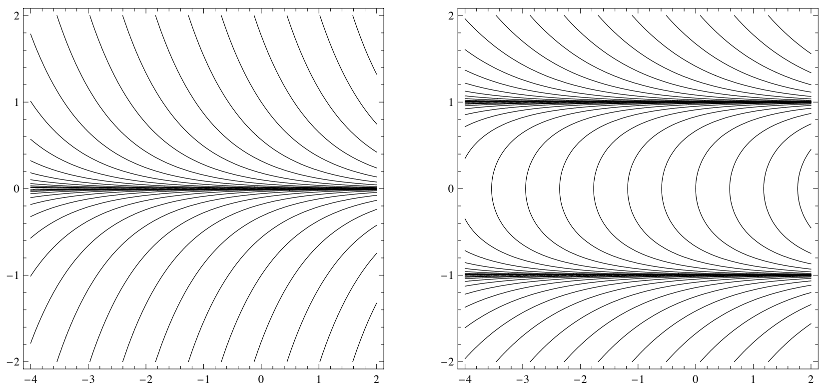

Figure 1: Level sets of the 1-st degree quasi-polynomial regular function $f(x,y)=ye^x$ (left). A transversal 2-nd degree quasi-polynomial function $g(x,y)=(y^2-1)e^x$ (right). The foliation given by the level sets of f admits a global transversal (e.g., any vertical straight line). The foliation given by the level sets of g does not, since it admits two separatrices ($y=\pm 1$) delimiting a stripe filled by leaves of the kind $x=\sec y$.

all smooth real valued maps f on $\mathbb{R}^2$ such that $L_\xi f>0$ (this set is non-empty by the previous lemma). We want to show that, for every free map $\psi:\mathbb{R}\to\mathbb{R}^2$, the map

$$F(x,y)=\psi(f(x,y))$$

is $\mathcal{H}$-free. Take $\psi(t)=(a(t),b(t))$. We must verify that the matrix

$$\begin{pmatrix} L_\xi[a(f)] & L_\xi[b(f)] \\ L_\xi^2[a(f)] & L_\xi^2[b(f)] \end{pmatrix}$$

has rank 2 (cfr. Example 2).
A direct computation shows that

$$\det D_{\xi,F}=\begin{vmatrix} L_\xi[a(f)] & L_\xi[b(f)] \\ L_\xi^2[a(f)] & L_\xi^2[b(f)] \end{vmatrix}=$$

$$=\begin{vmatrix} a'(f)L_\xi f & b'(f)L_\xi f \\ a'(f)L_\xi^2 f+a''(f)[L_\xi f]^2 & b'(f)L_\xi^2 f+b''(f)[L_\xi f]^2 \end{vmatrix}=$$

$$=[a'(f)b''(f)-b'(f)a''(f)][L_\xi f]^3\neq 0$$

which, by hypothesis, is never zero. In fact we assumed $L_\xi f>0$ and the first factor cannot vanish since ψ is free. $\square$

Example 7. Consider the distribution $\mathcal{H}\subset\mathbb{R}^2$ defined by

$$\mathcal{H}=\operatorname{span}\{\xi=2y\partial_x+(1-y^2)\partial_y\}\,.$$

This $\mathcal{H}$ is of finite type since it has only a pair of separatrices, the straight lines $y = \pm 1$. Indeed $\mathcal{H}$ is the tangent space to the (Hamiltonian) foliation $\mathcal{F}$ of the level sets of $f(x,y) = (y^2-1)e^x$ (a direct computation shows that $L_\xi((y^2-1)e^x) = 0)$. Moreover, $L_\xi(ye^x) = (1+y^2)e^x > 0$, i.e. the foliation of the level sets of the function $g(x,y) = (1+y^2)e^x$ is transverse to $\mathcal{F}$ at every point (cfr. Fig. 1). Then, by Theorem 1.1, $F(x,y) = \psi(g(x,y))$ is $\mathcal{H}$-free for every free map $\psi : \mathbb{R} \to \mathbb{R}^2$. For example, $F(x,y) = (ye^x, e^{ye^x})$ is $\mathcal{H}$-free (cfr. Example 3).

3.2 The case of completely integrable systems

We can prove another generalization (Lemma 3.3 below) of Weiner's Lemma in terms of completely integrable systems.

Let (M^{2n}, Ω) be a connected symplectic manifold. Since the symplectic 2-form is non-degenerate it sets up a linear isomorphism between vector fields ξ and 1-forms ω on M through the relation $i_\xi\Omega = \omega$. Moreover, every real valued function $f : M \to \mathbb{R}$ determines a unique vector field ξ_f called *Hamiltonian vector field* with the *Hamiltonian* f by requiring that for every vector field η on M the identity $df(\eta) = \omega(\xi_f, \xi_\eta)$ must hold. To the given symplectic structure Ω we can associate, in a natural way, the *Poisson bracket* via the formula $\{f,g\} = \Omega(\xi_f, \xi_\eta)$ which turns the algebra $C^\infty(M)$ of smooth functions on M into a Poisson algebra. Assume that (M^{2n}, Ω) admits a regular completely integrable system. This means that there exists a maximal set of functionally independent Poisson commuting functions $\{I_i\}$, i.e., such that $dI_1 \wedge \cdots \wedge dI_n \neq 0$ at every point of M and that the Poisson subalgebra generated by the I_i in $C^\infty(M)$ is abelian. Consider the distribution $\mathcal{H} = \ker dI_1 \cap \cdots \cap \ker dI_n$ and the corresponding Lagrangian foliation $\mathcal{F}$ (so that $\mathcal{H} = T\mathcal{F}$). Then the following theorem, classically known as *Arnold–Liouville theorem* holds true (see [Arn78] or [AM78] for details).

Theorem 3.2 (Arnold–Liouville)**.** *Let $\mathcal{F}$ the Lagrangian foliation defined above. If every Hamiltonian vector field ξ_{I_i} is complete then every leaf of $\mathcal{F}$ is diffeomorphic to $\mathbb{T}^r \times \mathbb{R}^{n-r}$ and has a saturated neighbourhood U (with respect to the projection onto the space of leaves $\mathcal{F}$) symplectomorphic to the product manifold $D \times (\mathbb{T}^r \times \mathbb{R}^{n-r})$, where $D \subset \mathbb{R}^n$ is open, endowed with the coordinates (I_i, φ^j) and with the canonical symplectic form $\Omega_0 = dI_i \wedge d\varphi^i$.*

This statement means, in particular, that the commutation relations between the special coordinates (I_i, φ^j) are given by the well-known

$$\{I_i, I_j\} = 0\,, \quad \{\varphi^i, \varphi^j\} = 0\,, \quad \{I_i, \varphi^j\} = \delta_i^j.$$

The (I_i, φ^j) are usually called "action-angle" coordinates.

When $M = \mathbb{R}^2$ every Hamiltonian system (represented by a single Hamiltonian) is, trivially, a completely integrable system. In particular, Lemma W can be restated as follows:

Let $\{I\}$ be a regular completely integrable system on the symplectic manifold $(\mathbb{R}^2, \Omega_0 = dx \wedge dy)$. Then there exists a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\Omega_0(\xi_I, \xi_f) > 0$ for all points of $\mathbb{R}^2$.

We come now to prove, based on what we have seen above, the following Lemma:

Lemma 3.3. *Let $\{I_1, \cdots, I_n\}$ be a regular completely integrable system on (M^{2n}, Ω) and suppose that all the Hamiltonian vector fields ξ_{I_i} are complete. Then there exist n smooth functions $\{f^1, \cdots, f^n\}$ (possibly multi-valued) such that*

$$\{I_i, f^i\} > 0\,, \quad \{I_i, f^j\} = 0\,,\; j \neq i \tag{10}$$

on the whole manifold M^{2n}.

Proof. We follow closely the original argument in [Wei88]. By Arnold-Liouville Theorem, every leaf $l \in \mathcal{F}$ has a saturated neighbourhood $U_l \simeq \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{T}^{n-k}$ with coordinates (I_i, φ^j_l) such that U_l is defined by the inequalities $\alpha^l_i \leq I_i \leq \beta^l_i$ and

$$\{I_i, \varphi^j_l\} = \delta^j_i .$$

We renormalize the action coordinates I_i (which, by hypothesis, are global) by $J^l_i = \mu^l_i(I_i - \nu^l_i)$ so that U_l is characterized as the connected component of $|J^l_i| < 2$ containing l. Now, let $b : \mathbb{R} \to \mathbb{R}$ be any bump function with support equal to $(-1, 1)$ and let $l : \mathbb{R} \to \mathbb{R}$ be any smooth non-decreasing function which is equal to 0 on $(-\infty, -1]$, to 1 on $[1, \infty)$ and strictly increasing between 0 and 1 on $(-1, 1)$.

The functions defined on U_l as

$$f^j_l = b(J^l_1) \cdots b(J^l_n) l(\varphi^j_l)$$

can be trivially extended to the whole M by setting $f^j_l = 0$ outside U_l. Note that their differentials

$$df^i_l = \sum_{i=1}^{\infty} \mu^l_i b'(J^l_i) dI_i + \sum_{i=1}^{\infty} b(J^l_n) l'(\varphi^i_l) d\varphi^i_l$$

have (modulo the span of the dI_i) compact support

$$V_l = \{p \in U_l \,|\, |J^l_i(p)| < 1\,,\; |\varphi^j_l(p)| < 1\} .$$

Moreover, we have that

$$\{I_i, f^i_l\} = b(J^l_1) \cdots b(J^l_n) l'(\varphi^i_l) > 0\,, \quad \{I_i, f^j_l\} = 0\,, \quad j \neq i$$

inside V_l while all Poisson brackets are identically zero outside V_l.

We extract from the covering $\{U_l\}$ a countable subcovering $\{U_{l_k}\}$. and show that, by a convenient choice of the coefficients a_k, the series

$$f^i = \sum_{k \in \mathbb{N}} a_k f^i_{l_k}$$

can be made convergent. In fact the $f^i_{l_k}$ are uniformly bounded so that, by taking $a_k = 2^{-k}$, the series can be made uniformly convergent. Next, let us fix any n-dimensional distribution $\mathcal{H}'$ transverse to $\mathcal{F}$ and consider on M the Riemannian metric $g = \sum_{i=1}^n (dI_i)^2 + g'$ (where g' is any metric on $\mathcal{H}'$). Denote by $\|D^{(j)} f^i_{l_k}\|$ the norm (associated to the metric g) of the derivatives of order j of $f^i_{l_k}$. This is seen as a map with domain M and range the symmetric product bundle (of order j) $S^j M$ based on M. We thus get that, for every value of k, there is some finite constant M'_k such that, outside V_{l_k}, $\|D^{(j)} f^i_{l_k}\| \leq M'_{k,i}$ for $1 \leq j \leq k$. Since V_{l_k} has compact closure, there exists another constant $M''_{k,i}$ such that $\|D^{(j)} f^i_{l_k}\| \leq M'_{k,i}$ within V_{l_k}. This means that $\|M^{-1}_{k,i} D^{(j)} f^i_{l_k}\| \leq 1$ for $M_{k,i} = \min\{1, M'_{k,i}, M''_{k,i}\}$. Therefore, if we take $a_k = 2^{-k} M^{-1}_{k,i}$, the series $\sum_{k\in\mathbb{N}} a_k D^{(j)} f^i_{l_k}$ uniformly converges for each $j \in \mathbb{N}$.

Then the f^i are smooth and one has

$$\{I_i, f^i\} > 0\,,\ \ \{I_i, f^j\} = 0\,,\ \ j \neq i.$$

Finally, Arnold–Liouville's Theorem tells that the neighbourhoods U_l are all symplectomorphic to $\mathbb{R}^n \times (\mathbb{T}^r \times \mathbb{R}^{n-r})$ for some r between 0 and n and, for $r > 0$ the leaves have compact components. Observe that, on these components, the df^j are well-defined closed 1-forms. Nevertheless, these forms may be non-exact, due to the non-triviality of the first cohomology group of the leaves. Consequently, in this case the functions f^j may be multivalued, namely well-defined only on some covering of M. $\square$

Proof of Theorem 1.2 We can apply Lemma 3.3 to see that there exist n smooth functions f^i satisfying (10) which, since the leaves of the foliation $\mathcal{F}$ have no compact component, are all single-valued. We consider n free maps $\{\psi_1, \cdots, \psi_n\}$ from $\mathbb{R}$ to $\mathbb{R}^2$ and prove that the map $F : M \to \mathbb{R}^{n+s_n}$ defined as

$$F(x) = (\psi_1(f^1(x)), \cdots, \psi_n(f^n(x)), f^1(x)f^2(x), \cdots, f^{n-1}(x)f^n(x))$$

is $\mathcal{H}$-free.

Let $\psi_i(t) = (a_i(t), b_i(t))$ and set $D\psi_i = a'_i b''_i - a''_i b'_i$. The square matrix $D_{\xi_1,\cdots,\xi_n,F}$ (see (9) above) is given, up to a permutation of its rows, by

$$\left(\begin{array}{c|c|c|c|c|c}
A_1 & * & * & * & * & * \\ \hline
0 & \ddots & * & * & * & * \\ \hline
0 & 0 & A_n & * & * & * \\ \hline
0 & 0 & 0 & 2g_1 g_2 & * & * \\ \hline
0 & 0 & 0 & 0 & \ddots & * \\ \hline
0 & 0 & 0 & 0 & 0 & 2g_{n-1} g_n
\end{array}\right)$$

where $g_i = L_{\xi_{I_i}} f^i$,

$$A_i = \begin{pmatrix} a'_i(f^i) g_i & b'_i(f^i) g_i \\ a'_i(f^i) L^2_{\xi_{I_i}} f^i + a''_i(f^i) g_i^2 & b'_i(f^i) L^2_{\xi_{I_i}} f^i + b''_i(f^i) g_i^2 \end{pmatrix}$$

and the stars represent terms which do not contribute to the determinant.

Since $\det A_i = g_i^3 D\psi_i$ and the blocks below the diagonal are identically zero, the determinant of $D_{\xi_1,\cdots,\xi_n,F}$ equals

$$2^{s_n}\Pi_{k=1}^{n}(g_i^{n+2}D\psi_k)$$

which differs from zero at every point because, by construction, $g_i > 0$ and, by hypothesis, $D\psi_i \neq 0$. Hence F is a $\mathcal{H}$-free map. □

Remark 1. Clearly the map F defined in the proof above is modeled after the canonical free map $G : \mathbb{R}^n \to \mathbb{R}^{n+s_n}$ given by

$$G(x^1,\ldots,x^n) = (x^1,\ldots,x^n,(x^1)^2,x^1x^2,\cdots,(x^n)^2)\ .$$

So far it is not known (see [Gro86], p.9) whether, for $n \geq 2$, there exist free maps from $\mathbb{T}^n$ to $\mathbb{R}^{n+s_n}$. It is for this reason that in Theorem 1.2 we require that the leaves of the foliation $\mathcal{F}$ have no compact component. On the other hand, it is an easy matter to check that the map $G : \mathbb{T}^n \to \mathbb{R}^{n+s_n+s_{n-1}}$ defined by

$$G(\theta^1,\ldots,\theta^n) = (\mathrm{cs}\,(\theta^1),\ldots,\mathrm{cs}\,(\theta^n),\mathrm{cs}\,(\theta^1+\theta^2),\ldots,\mathrm{cs}\,(\theta^{n-1}+\theta^n))\ ,$$

where $\mathrm{cs}\,\theta = (\cos\theta,\sin\theta)$, is free.

3.3 The case of Poisson systems

Poisson structures are a generalization of symplectic structures having the nice property of existing even in odd-dimensional manifolds. Recall that a Poisson manifold is a pair $(M,\{,\})$ where $\{,\} : C^\infty(M)\times C^\infty(M) \to C^\infty(M)$ is a $\mathbb{R}$-bilinear skew-symmetric derivation satisfying the Jacobi identity. To every smooth function $f \in C^\infty(M)$ it is associated canonically a *Hamiltonian vector field* ξ_f defined by $\xi_f(g) \stackrel{\text{def}}{=} \{f,g\}$.

In particular, when $M = \mathbb{R}^2$, the canonical symplectic form $\Omega_0 = dx\wedge dy$ induces on M a Poisson Bracket $\{f,g\} = \Omega_0(\xi_f,\xi_g)$ which can also be obtained via the Euclidean metric as

$$\{f,g\} = *[df\wedge dg]$$

where $*$ is the Euclidean Hodge operator. Observe that this Poisson bracket does not need a symplectic structure to be defined but rather an orientable Riemannian structure. Furthermore, it can be defined in any dimension n as follows. Let M be an oriented Riemannian manifold of dimension $n \geq 2$, $*$ its Hodge operator and $H = \{h_1,\cdots,h_{n-2}\}$ a set of $n-2$ smooth functions. We set

$$\{f,g\}_H \stackrel{\text{def}}{=} *[dh_1\wedge\cdots\wedge dh_{n-2}\wedge df\wedge dg]$$

and call it *Riemann-Poisson* bracket with respect to H. In particular, the foliation corresponding to a Hamiltonian vector field ξ_h, with $h \in C^\infty(M)$, is given by the intersections of the level sets of the h_i with the level sets of h.

In the general case each function h_i is what, in the language of Poisson's system, is called a Casimir for $\{,\}_H$, meaning that it is an element of the center of $C^\infty(M)$.

Example 8. Let $M = \mathbb{R}^3$ with the Euclidean metric and coordinates (x, y, z) and let $H = \{x\}$. Then the Riemann-Poisson bracket is given by $\{f, g\}_H = \partial_y f\, \partial_z g - \partial_y g\, \partial_z f$. In particular $\xi_y = \partial_z$ and $\xi_z = -\partial_y$ and the coordinate x is a Casimir.

Remark 2. Another example, considered by S.P. Novikov in [Nov82], is given when M is the three-torus $\mathbb{T}^3$ with angular coordinates $(\theta^1, \theta^2, \theta^3)$ and $H = \{h(\theta^i) = B_i\theta^i\}$, $i = 1, 2, 3$, for some constant 1-form $B = B_i d\theta^i$. Then the Riemann-Poisson bracket is given by

$$\{f, g\}_H = \epsilon^{ijk}\partial_i f\, \partial_l g\, B_k \ ,$$

where ϵ^{ijk} is the totally antisymmetric Levi–Civita tensor (for an example of the rich topological structure hidden behind this Riemann-Poisson bracket, the interested reader is referred to [DD09]).

Placed in this setting Weiner's Lemma reads as follows:

Consider the Euclidean plane $\mathbb{R}^2$ endowed with the Riemann-Poisson bracket $\{,\}$ and let $h \in C^\infty(\mathbb{R}^2)$ be a regular Hamiltonian. Then there exists $f \in C^\infty(\mathbb{R}^2)$ such that $\{h, f\} > 0$ on the whole $\mathbb{R}^2$.

Furthermore, Weiner's result in the latter formulation can be extended, under a non-degeneracy condition, to Riemann-Poisson brackets:

Lemma 3.4. *Let M be an oriented Riemannian manifold of dimension $n \geq 2$ and let $H = \{h_1, \cdots, h_{n-2}\}$ be a set of $n-2$ functions functionally independent at every point (i.e. such that $dh_1 \wedge \cdots \wedge dh_{n-2}$ never vanishes). Then, for any $h \in C^\infty(M)$ functionally independent from the h_i, there exists a smooth function (possibly multivalued) $f : M \to \mathbb{R}$ such that the Riemann–Poisson bracket $\{h, f\}_H$ is strictly positive at every point.*

Proof. Set $h_{n-1} = h$. Let $\mathcal{F}$ the 1-dimensional Hamiltonian foliation associated to h, namely the one defined by $dh_1 = \cdots = dh_{n-1} = 0$, and let $\pi : M \to \mathcal{F}$ be the canonical associated projection. At every point $p \in M$ there exists a saturated (with respect to π) neighbourhood $U_p \simeq D \times X$, where $D \simeq \mathbb{R}^{n-1}$ and X is either $\mathbb{R}$ or $\mathbb{S}^1$, defined as the connected component of the set $W_p = \{a_i < h_i < b_i, i = 1, \dots, n-1\}$ which contains p. We renormalize these coordinates by using new ones $\hat{h}_i = \mu_i(h_i - \nu_i)$ so that W_p is defined by $|\hat{h}_i| < 2$, $i = 1, \dots, n-1$.

Let now A_p be the subset of U_p defined by $|\hat{h}_i| < 1$, $i = 1, \dots, n-1$ and take two functions b and l like in the proof of Lemma 3.3.

The real-valued function

$$f_p(h_1, \dots, h_{n-1}, \varphi) = l(\varphi) \times \prod_{i=1,\dots,n-1} b(h_i)$$

is well-defined and smooth in A_p and it can be extended to a smooth function on the whole M by setting it equal to zero outside A_p. Clearly

$$df_p = \omega_{n-1} \oplus \Pi_{i=1}^{n-1} b(h^i) l'(\varphi) d\varphi$$

where $\omega_{n-1} \in \text{span}\{dh_1, \cdots, dh_{n-1}\}$. Then $\{h, f_p\}_H$ everywhere vanishes except within $B_p = \{p' \in A_p | \, |\varphi(p')| < 1\}$, where we have

$$\begin{aligned}\{h, f_p\}_H &= *[dh_1 \wedge \cdots \wedge dh_{n-1} \wedge df_p] \\ &= *[dh_1 \wedge \cdots \wedge dh_{n-1} \wedge \Pi_{i=1}^{n-1} b(h_i) l'(\varphi) d\varphi] \\ &= *[dh_1 \wedge \cdots \wedge dh_{n-1} \wedge d\varphi] \prod_{i=1}^{n-1} b(h_i) l'(\varphi) > 0 \,,\end{aligned}$$

the function $*[dh_1 \wedge \cdots \wedge dh_{n-1} \wedge d\varphi]$ being positive for all points $q \in U_p$ and every $p \in M$ since M is oriented.

Now extract a countable subcovering $\{A_{p_k}\}_{k \in \mathbb{N}}$ from $\{A_p\}$ and let $f_k = f_{p_k}$ be the correponding function on every $A_k := A_{p_k}$. As in Lemma 3.3, the series $\sum_k a_k f_k$ can be made convergent to a smooth function f by choosing a convenient sequence a_k. Then $\{h, f\}_H > 0$ on the whole M since every point p is covered by at least one A_k, so that $\{h, f\}_H \geq \{h, f_k\}_H > 0$. □

Proof of Theorem 1.3 Let ξ_h be the Hamiltonian vector field associated to h through $\{,\}_H$. Then, by Lemma 3.4, there exists a function f (possibly multi-valued) such that $L_\xi f > 0$. Hence, as seen in Theorem 1.1, the smooth map $F : M \to \mathbb{R}^2$ given by $F(x) = \psi(f(x))$ is $\mathcal{H}$-free ($\mathcal{H} = \text{span}\{\xi_h\}$), where $\psi : \mathbb{R} \to \mathbb{R}^2$ (respectively $\psi : \mathbb{S}^1 \to \mathbb{R}^2$) is free if f is single-valued (respectively multi-valued). □

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