ARTICLE IN PRESS

ELSEVIER

Journal of Geometry and Physics 867 (2002) 1–13

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Quantization of Kähler manifolds and the asymptotic expansion of Tian–Yau–Zelditch

Claudio Arezzo [a], Andrea Loi [b,*]

[a] Università di Parma, Parma, Italy

[b] Dip. di Matematica e Fisica, Università di Sassari, Via Vienna 2, Sassari 07100, Italy

Received 10 September 2001

Abstract

In this paper we study the link between the asymptotic expansion of Tian–Yau–Zelditch [J. Diff. Geom. 32 (1990) 99] and the quantization of compact Kähler manifolds carried out in [J. Geophys. 7 (1990) 45; Trans. Am. Math. Soc. 337 (1993) 73].

MSC: 53C55; 58F06

Subj. Class.: Quantum mechanics

Keywords: Kähler metrics; Bergmann metrics; Quantization; Epsilon function; Diastasis

1. Introduction

Tian [20] solved a conjecture posed by Yau by proving that a polarized Kähler metric g on a compact complex manifold M can be obtained as the limit of Bergmann metrics on M. In that paper Tian introduced, for all non-negative integer m a smooth function on M which we will denote by $T_m(x)$. This function has been extensively studied by several authors and it is strictly related to the stability of Kähler–Einstein metrics. Zelditch [21] showed that $T_m(x)$ admits an asymptotic expansion (in the variable m) and Lu [14] calculates the first three terms of this expansion (see Section 2). The first observation of the present article is that Tian's function is, up to the factor m^n (n is the complex dimension of M) one of the key ingredients in the framework of quantization of Kähler manifolds carried out in [3–7,15,16]. In this context the function is denoted by $\epsilon_m(x)$ and it is called the *Epsilon function*. Cahen, Gutt and Rawnsley [4,5], starting with a geometric quantization of a Kähler manifold (M, ω)

* Corresponding author.
E-mail addresses: claudio.arezzo@unipr.it (C. Arezzo), loi@ssmain.uniss.it (A. Loi).

0393-0440/02/$ – see front matter © 2002 Published by Elsevier Science B.V.
PII: S0393-0440(02)00175-4

introduced by Kostant and Souriau, beautifully generalized Berezin's method [3] to the case of compact Kähler manifolds and, under suitable conditions, they obtain a deformation quantization of (M, ω).

The concept of quantization deformation started with [1,2]. It is defined in terms of a star product which is an associative formal deformation of the usual product of functions and of the Lie algebra structure given by the Poisson bracket $\{,\}$ associated to the symplectic form ω (here ω is the Kähler structure associated to the Kähler metric g). The method to construct a $*$-product for compact Kähler manifolds given by C–G–R involves making a correspondence between operators and functions (their Berezin symbols), transferring the operator composition to the symbols, introducing a suitable parameter into the Berezin composition of symbols, taking the asymptotic expansion in this parameter on a large algebra of functions and then showing that the coefficients of this expansion satisfy the cocycle conditions to define a star product on the smooth functions (see Section 3). In general to define and to find this asymptotic expansion is a difficult task. In [4,5] C–G–R considered a special class of quantization, called *regular*, where the function $\epsilon_m(x)$ is constant for all $m \geq 1$. Examples of Kähler manifolds which admit a regular quantization are the homogeneous and simply connected Kähler manifolds. In [4], the existence of a (convergent) $*$-product for compact coadjoint orbits and in [11] for general compact coadjoint orbits was shown. It is worth to mention that in [17] can be found an analogous result for general Kähler manifolds.

The aim of this paper is two-fold. Firstly, in Theorem 4.3 we observe that an asymptotic expansion given by C–G–R works for the (larger) class of projectively induced Kähler forms on M. Secondly, disregarding the applications to the theory of quantization (actually using some of the results of this theory), we study the function ϵ_m and some of its geometric properties. In particular we address the following natural problems:

1. Classify all Kähler manifolds which admit a regular quantization.
2. Study when the Zelditch's asymptotic expansion of $T_m(x)$ is finite.

Our first result is Theorem 5.3 where we prove that a regular quantization of a compact homogeneous Kähler manifold is necessarily homogeneous and Theorem 5.6 where we describe the link between Zelditch's asymptotic expansion and the expansion given by Theorem 4.3, proving that the knowledge of the expansion of Theorem 4.3 completely determines the one of $T_m(x)$ (and of $\epsilon_m(x)$).

The relevance of this fact lies in that the explicit calculation of $T_m(x)$ is usually very hard while the one of Theorem 4.3 is determined more easily from the knowledge of Calabi's diastasis function. In fact $T_m(x)$ can be computed by knowing all the powers of the quantum line bundle over M and by an explicit calculation of an orthonormal basis for the L^2-metric, while the expansion of Theorem 4.3 requires only the computation (independently of m) of Calabi's function.

The paper is organized as follows. In Section 2 we recall the Tian's construction for polarized Kähler metrics, we give the definition of the function $T_m(x)$ and we describe its Zelditch's asymptotic expansion. In Section 3 we recall the definition of the function $\epsilon_m(x)$ in the context of quantization of Kähler manifolds, its link with $T_m(x)$ and the C–G–R construction. In Section 4 we prove Theorem 4.3. Finally, Section 5 is dedicated to problems 1 and 2; our main results are Theorems 5.3, 5.5 and 5.6.

2. The work of Tian, Zelditch and Lu

Let M be a projective algebraic manifold, namely a compact complex manifold which admits a holomorphic embedding into some complex projective space $\mathbb{C}P^N$. The hyperplane bundle on $\mathbb{C}P^N$ restricts to an ample line bundle L on M, which is called a *polarization* on M. A Kähler metric g on M is *polarized* with respect to L if the corresponding Kähler form ω represents the first Chern class $c_1(L)$ of L and this happens if and only if ω is an integral form. Given any polarized Kähler metric g on M, one can find a hermitian metric h on L with its Ricci curvature form $\mathrm{Ric}(h) = \omega$. Here $\mathrm{Ric}(h)$ is the 2-form on M defined by the equation:

$$\mathrm{Ric}(h) = -\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\log h(\sigma(x), \sigma(x)), \tag{1}$$

for a trivializing holomorphic section $\sigma : U \subset M \to L \setminus \{0\}$ of L.

For each positive integer m, we denote by $L^m = L^{\otimes m}$ the mth tensor power of L. It is a polarization of the Kähler metric mg and the hermitian metric h induces a natural hermitian metric h_m on L^m such that $\mathrm{Ric}(h_m) = m\omega$. Denote by $H^0(M, L^m)$ the space of global holomorphic sections of L^m. It is in a natural way a (finite dimensional) hermitian space with respect to the norm

$$\|s\|_{h^m} = \langle s, s\rangle_{h_m} = \int_M h_m(s(x), s(x))\frac{\omega^n}{n!}(x) \quad \forall s \in H^0(M, L^m). \tag{2}$$

For sufficiently large m we can define a holomorphic embedding of M into a complex projective space as follows. Let $(s_0^m, \ldots, s_{d_m-1}^m)$, be a orthonormal basis for $H^0(M, L^m)$ and let $\sigma : U \to L$ be a trivializing holomorphic section on the open set $U \subset M$. Define the map

$$\varphi_\sigma : U \to \mathbb{C}^{d_m} \setminus \{0\} : x \mapsto \left(\frac{s_0(x)}{\sigma(x)}, \ldots, \frac{s_{d_m-1}(x)}{\sigma(x)}\right). \tag{3}$$

If $\tau : V \to L$ is another holomorphic trivialization then there exists a non-vanishing holomorphic function f on $U \cap V$ such that $\sigma(x) = f(x)\tau(x)$. Therefore, one can define a holomorphic map

$$\varphi_m : M \to \mathbb{C}P^{d_m-1}, \tag{4}$$

whose local expression in the open set U is given by (3). Since L is ample, by Kodaira's theorem for m sufficiently large, the map φ_m is an embedding.

Let g_{FS} be the standard Fubini–Study metric on $\mathbb{C}P^{d_m-1}$, namely the metric whose associated Kähler form is given by

$$\omega_{\mathrm{FS}} = \frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\log\sum_{j=0}^{d_m-1}|z_j|^2, \tag{5}$$

for a homogeneous coordinate system $[z_0, \ldots, z_{d_m-1}]$ in $\mathbb{C}P^{d_m-1}$. This restricts to a Kähler metric $g_m = \varphi_m^* g_{\mathrm{FS}}$ on M. Its associated Kähler form $\omega_m = \varphi_m^*\omega_{\mathrm{FS}}$ is cohomologous to $m\omega$ and is polarized with respect to L^m. Tian [20] christened the set of normalized metrics

g_m/m as the *Bergmann* metrics on M with respect to L and solves a conjecture posed by Yau by proving that the sequence g_m/m C^2-converges to the polarized metric g. This result was further generalized by Ruan [18] who proved the C^∞-convergence. As already observed by Tian, the difference between g_m/m and the metric g can be measured by the function

$$T_m(x) = \sum_{j=0}^{d_m-1} \|s_j^m(x)\|_{h_m}^2. \tag{6}$$

Indeed, it is easily seen that for all non-negative integer m

$$\frac{\omega_m}{m} = \omega + \frac{\mathrm{i}}{2\pi m}\partial\bar{\partial}\log T_m(x). \tag{7}$$

Tian's Theorem was generalized by Zelditch [21], who has used the theory of Szegö Kernel on the unit circle bundle L^* over M, which proves the following theorem.

Theorem 2.1 (Zelditch). *There is a complete asymptotic expansion*

$$T_m(x) = \sum_{j=0}^{d_m-1} \|s_j^m(x)\|_{h_m}^2 = a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \cdots, \tag{8}$$

for certain smooth coefficients $a_j(x)$ with $a_0 = 1$. More precisely, for any m

$$\|T_m(x) - \sum_{j<R} a_j(x)m^{n-j}\|_{C^k} \le C_{R,k}m^{n-R},$$

where $C_{R,k}$ depends on R, k and the manifold M.

Recently Lu [14], by using Tian's peak section method, proved the following theorem.

Theorem 2.2 (Lu). *Each coefficients $a_j(x)$, given by the asymptotic expansion (8) is a polynomial of the curvature and its covariant derivatives at x of the metric g. Such a polynomials can be found by finitely many steps of algebraic operations. Furthermore $a_1(x) = \rho/2$, where ρ is the scalar curvature of the polarized metric g.*

3. The work of Cahen, Gutt and Rawnsley

In the quantum mechanics terminology a couple (L, h) such that $\mathrm{Ric}(h) = \omega$ is called a *geometric quantization* of the Kähler manifold (M, ω) and L is called a *quantum line bundle*. For any non-negative integer m, (L^m, h_m) is a geometric quantization of the Kähler manifold $(M, m\omega)$. Let $x \in M$ and $q \in L^m \setminus \{0\}$ a fixed point of the fibre over x. If one evaluates $s \in H^0(M, L^m)$ at x, one gets a multiple $\delta_q(s)$ of q, i.e. $s(x) = \delta_q(s)q$. The map $\delta_q : H^0(M, L^m) \to \mathbb{C}$ is a continuous linear functional [4] hence by Riesz's theorem, there exists a unique $e_q^m \in H^0(M, L^m)$ such that $\delta_q(s) = \langle s, e_q^m\rangle_{h_m}$ $\forall s \in H^0(M, L^m)$, i.e.

$$s(x) = \langle s, e_q^m\rangle_{h_m} q. \tag{9}$$

It follows, by (9), that

$$e^m_{cq} = \bar{c}^{-1} e^m_q \quad \forall c \in \mathbb{C}^*.$$

The holomorphic section e^m_q is called the *coherent state* relative to the point q. Thus, for all $m \geq 1$, one can define a smooth function on M

$$\epsilon_m(x) = \frac{1}{m^n} h_m(q, q) \| e^m_q \|^2_{h_m}, \tag{10}$$

where $q \in L^m \setminus \{0\}$ is any point on the fibre of x. Now it is easily seen that

$$m^n \epsilon_m(x) = h_m(q, q) \| e^m_q \|^2_{h_m} = \sum_{j=0}^{d_m - 1} h_m(s^m_j(x), s^m_j(x)) = T_m(x), \tag{11}$$

where $T_m(x)$ is defined by (6) and where s^m_j is an orthonormal basis for $H^0(M, L^m)$. This gives the claimed link between the function $\epsilon_m(x)$ and $T_m(x)$.

Remark 3.1. The factor $1/m^n$ in the definition (10) does not appear in the definition of $\epsilon_m(x)$ given in [4,5] since they consider the L^2-product

$$\int_M h_m(s(x), s(x)) \frac{(m\omega)^n}{n!}(x), \tag{12}$$

which equals m^n times the L^2-product $\| \cdot \|^2_{h_m}$ given by (2).

Let $A : H^0(L, M) \to H^0(L, M)$ be a linear operator. The symbol of A is the real analytic function on M defined by

$$\hat{A}(x) = \frac{\langle A e_q, e_q \rangle_h}{\langle e_q, e_q \rangle_h}, \quad q \in L_x \setminus \{0\},$$

where L_x denotes the fibre of x. The function $\hat{A}$ has an analytic continuation to an open neighbourhood of the diagonal $M \times \bar{M}$ given by

$$\hat{A}(x, \bar{y}) = \frac{\langle A e_q, e_{q'} \rangle_h}{\langle e_q, e_{q'} \rangle_h}, \quad q \in L_x \setminus \{0\}, \quad q' \in L_y \setminus \{0\}.$$

We denote by $\hat{E}(L)$ the space of symbols of bounded linear operators. The composition of operators on $H^0(M, L)$ gives rise to a product for the corresponding symbols, which is associative. It is given by the integral formula

$$(\hat{A} * \hat{B})(x) = \int_M \hat{A}(x, \bar{y}) \hat{B}(y, \bar{x}) \psi(x, y) \epsilon(y) \frac{\omega^n}{n!}(y), \tag{13}$$

where

$$\psi(x, y) = \frac{|\langle e_q, e_{q'} \rangle_h|^2}{\| e_q \|^2_h \| e_{q'} \|^2_h},$$

is the so called 2-point function globally defined function on $M \times M$. The quantization is *regular* if ϵ_m is constant for all $m \geq 1$. The proof of the following proposition can be found in [4].

Proposition 3.2. *Let (L, h) be a regular quantization of a Kähler manifold (M, ω) then the following facts hold true*:

(i) $\hat{E}(L^m) \subset \hat{E}(L^{m+1})$;
(ii) $\cup_m \hat{E}(L^m)$ *is dense on* $C^0(M)$.

From the nesting property one sees that if $\hat{A}, \hat{B}$ belongs to $\hat{E}(L^l)$, $m \geq l$ one may define

$$(\hat{A} *_m \hat{B})(x) = m^n \int_M \hat{A}(x, \bar{y})\hat{B}(y, \bar{x})\psi_m(x, y)\epsilon_m(y)\frac{\omega^n}{n!}(y). \tag{14}$$

Here

$$\psi_m(x, y) = \frac{|\langle e_q^m, e_{q'}^m\rangle_{h_m}|^2}{\|e_q^m\|_{h_m}^2 \|e_{q'}^m\|_{h_m}^2},$$

is the 2-point function for L^m. If the quantization is regular then one can prove that $\psi_m(x, y) = \psi^m(x, y)$. Observe also that

$$1 = 1 *_m 1 = m^n \int_M \psi_m(x, y)\epsilon_m(y)\frac{\omega^n}{n!}(y). \tag{15}$$

In [5], it can be found the proof of the following theorem.

Theorem 3.3. *Let (L, h) be a regular quantization of a Kähler manifold (M, ω). Then the following facts hold true*:

(i) *For any $f \in \hat{E}(L^l)$ the integral*

$$F_m(x) = m^n \int_M f(x, \bar{y})\psi^m(x, y)\frac{\omega^n}{n!}(y), \quad m \geq l, \tag{16}$$

admits an asymptotic expansion (*as m goes to infinity*)

$$F_m(x) \sim m^{-r}C_r(f)(x), \tag{17}$$

where the C_r's are smooth differential operators of order $2r$ depending only on the geometry of M. Moreover, the leading term is given by $C_0(f) = f$.

(ii) *The $*_m$-product given by (14) admits an asymptotic expansion for m tending to infinity*

$$\hat{A} *_m \hat{B} \sim m^{-r}C_r(\hat{A}, \hat{B}),$$

where C_r are smooth bidifferential operators defined by the geometry alone. Furthermore

$$C_0(\hat{A}, \hat{B})(x) = \hat{A}(x)\hat{B}(x), \tag{18}$$

$$(C_1(\hat{A}, \hat{B}) - C_1(\hat{B}, \hat{A}))(x) = \frac{\mathrm{i}}{\pi}\{\hat{A}, \hat{B}\}(x), \tag{19}$$

where $\{,\}$ is the Poisson bracket of functions on M associated to ω.

In the case of flag manifolds or generalized flags manifolds the asymptotic expansion defined above defines an associative star product (see [5,11]).

4. An asymptotic expansion for projectively induced Kähler metrics

We start by defining the diastasis function. We refer to [8] for details and further results. A potential for the Kähler form ω is a real valued function Φ defined on a open set $U \subset M$ satisfying

$$\omega = \frac{\mathrm{i}}{4\pi}\bar{\partial}\partial\Phi.$$

A potential is not unique: it is defined up to the sum with the real part of a holomorphic function. Therefore, since ω is real analytic a potential Φ can be complex analytically continued to an open neighbourhood $V \subset U \times \bar{U}$ of the diagonal. Denote this extension by $\Phi(x, \bar{y})$. It is holomorphic in x and antiholomorphic in y and, with this notation, $\Phi(x) = \Phi(x, \bar{x})$.

Calabi's diastasis function $D : V \to \mathbb{R}$, is defined by

$$D(x, y) = \Phi(x, \bar{x}) + \Phi(y, \bar{y}) - \Phi(x, \bar{y}) - \Phi(y, \bar{x}).$$

It is real valued since $\overline{\Phi(x, \bar{y})} = \Phi(\bar{x}, y)$ and it is independent from the potential chosen.

Example 4.1. In the case of $M = \mathbb{C}P^N$ endowed with the Fubini–Study form ω_{FS} the diastasis can be written in terms of the coordinates in $\mathbb{C}^{N+1}$ as

$$D_{\mathrm{FS}}(\pi(z), \pi(w)) = 2\log\frac{\|z\|^2\|w\|^2}{|\langle z, w\rangle|^2},$$

where $\pi : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{C}P^N$ is the canonical projection and where we are denoting by $\langle\cdot, \cdot\rangle$ the standard hermitian metric on $\mathbb{C}^{N+1}$. In particular $D > 0$ unless $\pi(z) = \pi(w)$ where $D = 0$. Observe also that the function $\mathrm{e}^{-D_{\mathrm{FS}}(\pi(z),\pi(w))/2} = |\langle z, w\rangle|^2/\|z\|^2\|w\|^2$ is globally defined on $\mathbb{C}P^N \times \mathbb{C}P^N$ and vanishes on the diagonal.

Following [5] we now describe the link between the diastasis function and the Epsilon function. First since the function $\epsilon_m(x)$ is real analytic we can take its analytic extension $\epsilon_m(x, \bar{y})$ to a neighbourhood of the diagonal holomorphic in the first variable and antiholomorphic in the second. One can prove the following (see formula (1.11), Section 1 of [5])

$$\mathrm{e}^{-mD(x,y)/2}|\epsilon_m(x, \bar{y})|^2 = \epsilon_m(x)\epsilon_m(y)\psi_m(x, y), \tag{20}$$

where

$$\psi_m(x, y) = \frac{|\langle e_q^m, e_{q'}^m\rangle_{h_m}|^2}{\|e_q^m\|_h^2\|e_{q'}^m\|_{h_m}^2},$$

is the 2-point function for L^m (cfr. previous section).

Observe that since the right hand side of (20) is globally defined on $M \times M$ then the function $\mathrm{e}^{-mD(x,y)/2}|\epsilon_m(x, \bar{y})|^2$ is a well-defined function on $M \times M$ even if the single functions $\mathrm{e}^{-mD(x,y)/2}$ and $|\epsilon_m(x, \bar{y})|^2$ are a priori defined only in a neighbourhood of the diagonal.

From now on we will assume that the metric g (resp. the associated Kähler form ω) is *projectively induced*, namely there exists a holomorphic embedding $\varphi : M \to \mathbb{C}P^N$ such that $\varphi^* g_{\mathrm{FS}} = g$ (equivalently $\varphi^* \omega_{\mathrm{FS}} = \omega$). Observe that this condition is automatically satisfied when (M, ω) admits a regular quantization. Indeed by formula (7) with $m = 1$ one gets $\omega = \omega_1 = \varphi_1^* \omega_{\mathrm{FS}}$. As a result due to Calabi [8] the Kähler form ω is then real analytic and its diastasis function D is obtained by the restriction, via the map φ, of the diastasis function D_{FS} on $\mathbb{C}P^N$, namely $\varphi^* D_{\mathrm{FS}} = D$. Since the map φ is an embedding it follows by the previous example that the diastasis function $D(x, y)$ vanishes if and only if $x = y$ and moreover the function $\mathrm{e}^{-D(x,y)/2}$ is globally defined on $M \times M$. This function admits the point of the diagonal as critical points. In fact at these points it has its maximum value, namely 1 and $\mathrm{e}^{-D(x,y)/2} = 1$ if and only if $x = y$.

Suppose now that (L, h) is a quantization of the Kähler manifold (M, ω). In [5] the function $\mathrm{e}^{-D(x,y)/2}$ is shown to be equal to the so called *characteristic function* denoted there by $\psi_L(x, y)$. Furthermore in Proposition 4, Section 1 of [5] is proved that, if $\epsilon_m(x)$ in constant for all $m \geq 1$, then the Hessian of ψ_L (considered as a function of its second argument only) is given by

$$\mathrm{Hess}_2 \psi_L = -2\pi g. \tag{21}$$

Observe that formula (21) includes the missing factor 2 in formula 1.16 of [5]. Nevertheless a careful reading of [5] shows that the proof of (21) is based only on the fact that the metric g is projectively induced, being $\epsilon_m(x)$ constant and hence it is still valid for the (larger) class of projectively induced Kähler metrics.

This also implies that Proposition 2 in Section 2 of [5] generalizes for projectively induced Kähler metrics.

Proposition 4.2. *Let (M, g) be a compact Kähler manifold and let ω be its associated Kähler form. Let V be an open neighbourhood of the zero section of the tangent bundle $p : TM \to M$, such that the map $\alpha : V \to M \times M$, $X \mapsto (p(X), \exp_{p(x)} X)$ is well-defined. Suppose that the metric g is projectively induced and let ψ_L be the characteristic function on $M \times M$. Then there exists an open neighbourhood W of the zero section and a smooth embedding $\mu : W \to TM$ such that*:

$$(-\log \psi_L \circ \alpha \circ \mu)(X) = \pi g_{p(X)}(X, X).$$

By the previous proposition and by a slightly modification of the proof of (i) of Theorem 3.3 one gets an asymptotic expansion for projectively induced Kähler metrics.

Theorem 4.3. *Let (M, g) be a compact Kähler manifold and suppose g is projectively induced. Let $f(x, y)$ be a function defined in a neighbourhood of the diagonal in $M \times M$ such that $\mathrm{e}^{-mD(x,y)/2} f(x, y)$ is globally defined and smooth on $M \times M$ for m sufficiently large. Then the integral*

$$F_m(x) = m^n \int_M \mathrm{e}^{-mD(x,y)/2} f(x, y) \frac{\omega^n}{n!}(y), \tag{22}$$

admits an asymptotic expansion (as m goes to infinity)

$$F_m(x) \sim \sum_{r \geq 0} m^{-r} C_r(f)(x), \tag{23}$$

where $f(x)$ is defined as $f(x, x)$ and where the C_r's are smooth differential operators of order $2r$ depending only on the geometry of M. Moreover, the leading term is given by $C_0(f) = f$.

In the case $f = 1$ we will denote by $b_j(x)$ the function of the previous expansion, namely

$$C_j(1)(x) = b_j(x). \tag{24}$$

Remark 4.4. The further step to obtain a quantization deformation for projectively induced Kähler forms could be to generalize the asymptotic expansion given by (ii) in Theorem 3.3 for this class of Kähler forms.

5. Geometric properties of the Epsilon function

Let (L, h) be a geometric quantization of a Kähler manifold (M, ω). In this last section we attack problems 1 and 2 posed in Section 1. We start with problem 1 and so we try to understand what kind of properties are enjoyed by the Kähler forms which admit a regular quantization. First of all as we have already pointed out such a Kähler forms are projectively induced. Secondly, a large class of these forms is given by the following (see [4] for a proof).

Theorem 5.1. *A quantization (L, h) of a homogeneous and simply connected compact Kähler manifold (M, g) is regular.*

Recall that a Kähler manifold (M, ω) is homogeneous if the group $\mathrm{Aut}(M) \cap \mathrm{Isom}(M, g)$ acts transitively on M, where $\mathrm{Aut}(M)$ denotes the group of holomorphic diffeomorphisms of M and $\mathrm{Isom}(M, g)$ the isometry group of (M, g) (g denotes as usual the Kähler metric associated to ω).

Remark 5.2. Note that the condition of simply connectedness in Theorem 5.1 cannot be relaxed. In fact the n-dimensional complex torus $M = \mathbb{C}^n/\mathbb{Z}^{2n}$ endowed with the flat Kähler form ω is a homogeneous Kähler manifold. On the other hand the flat metric cannot be projectively induced (see Lemma 22 in [19] for a proof) and hence in particular any quantization of (M, ω) cannot be regular (see also [13] for the calculation of the Epsilon function in this case).

In view of Theorem 5.1 the following question naturally arises: *Is it true that a Kähler manifold (M, ω) which admits a regular quantization is necessarily homogeneous*? We give a partial answer to this question in the following theorem.

Theorem 5.3. *Let (M, ω) be a compact homogeneous and simply connected Kähler manifold. Let $\tilde{\omega}$ be a Kähler form on M cohomologous to ω which admits a regular quantization. Then there exists $f \in \mathrm{Aut}(M)$ such that $f^*\tilde{\omega} = \omega$ and hence $(M, \tilde{\omega})$ is homogeneous.*

Proof. By Theorem 5.1 (M, ω) admits a regular quantization and hence the function ϵ_m is constant for all $m \geq 1$. This implies that all the coefficients of the asymptotic expansion (8) of $T_m(x) = m^n \epsilon_m(x)$ are constant. In particular, by Theorem 2.2 the scalar curvature of the metric g (the metric whose associated Kähler form is ω) is constant. For the same reason the scalar curvature of the metric $\tilde{g}$ associated to $\tilde{\omega}$ is constant. Therefore, by applying Theorem B in [10] one can find $f \in \mathrm{Aut}(M)$ such that $f^*\tilde{\omega} = \omega$. □

Corollary 5.4. *Let $\tilde{\omega}$ be a Kähler form on $\mathbb{C}P^N$ and suppose that $(\mathbb{C}P^N, \tilde{\omega})$ admits a regular quantization. Then there exists a natural number k and $f \in \mathrm{PGL}(N+1, \mathbb{C})$) (the projective linear group) such that $f^*\tilde{\omega} = k\omega_{\mathrm{FS}}$ where ω_{FS} is the Fubini–Study Kähler form.*

Proof. Since the first betti number of $\mathbb{C}P^N$ is 1 there exists a natural number k such that $\tilde{\omega}$ is cohomologous to $k\omega_{\mathrm{FS}}$ and thus by Theorem 5.3 there exists $f \in \mathrm{PGL}(N+1, \mathbb{C}) = \mathrm{Aut}(\mathbb{C}P^N)$ satisfying $f^*\tilde{\omega} = k\omega_{\mathrm{FS}}$. □

Another case when the answer to the above question is affirmative is when M is a complete intersection submanifold of $\mathbb{C}P^N$.

Theorem 5.5. *Let (M, ω) be a compact manifold which admits a regular quantization. Suppose that $\varphi_1(M)$ is a complete intersection submanifold of $\mathbb{C}P^{d_1-1}$, where the map φ_1 is the embedding given by formula (4) with $m = 1$. Then (M, ω) is either a quadric or a totally geodesics projective space.*

Proof. As in the proof of Theorem 5.3 we deduce that the scalar curvature of g is constant. Being $\varphi_1(M)$ a complete intersection and being g projectively induced we can apply Kobayashi's theorem [12] to conclude that M is infact a quadric or a totally geodesics projective space. □

We now consider problem 2 and then we suppose that the Zelditch's expansion of $T_m(x)$ is finite. This means that there exists a natural number p such that

$$\epsilon_m(x) = \frac{T_m(x)}{m^n} = 1 + \sum_{j=1}^{p} \frac{a_j(x)}{m^j}. \tag{25}$$

This condition is obviously satisfied in the case of a regular quantization (with $p = n$). Indeed, when ϵ_m is constant it follows by (11) that

$$\epsilon_m = \frac{\dim H^0(M, L^m)}{m^n \,\mathrm{vol}(M)},$$

and $\dim H^0(M, L^m)$ is a polynomial of degree $\dim M$ in m by Riemann–Roch–Hirzebruch's formula (see [9]). It is also easy to give examples when the asymptotic expansion of $T_m(x)$ cannot be finite. For example, consider the complex torus endowed with the flat metric. If the Zelditch's expansion of $T_m(x)$ were finite then all a_j would be constant since they depend on the curvature of g (by Theorem 2.2). Thus the Epsilon function would be constant which is impossible by Remark 5.2.

In the case of finite asymptotic expansion we get by (25) that, in a suitable neighbourhood of the diagonal

$$\epsilon_m(x,\bar{y}) = 1 + \sum_{j=1}^{p} \frac{a_j(x,\bar{y})}{m^j},$$

where $a_j(x,\bar{y})$ are the analytic extensions of the functions $a_j(x)$, $j = 1,\ldots,p$. Consequently

$$|\epsilon_m(x,\bar{y})|^2 = 1 + \sum_{j=1}^{2p} \frac{\tilde{a}_j(x,y)}{m^j}, \tag{26}$$

where

$$\tilde{a}_1(x,y) = a_1(x,\bar{y}) + a_1(\bar{x},y), \qquad \tilde{a}_2(x,y) = |a_1(x,\bar{y})|^2 + a_2(x,\bar{y}) + a_2(\bar{x},y),$$

and so on. If ω is projectively induced then $\mathrm{e}^{-mD(x,y)/2}$ is globally defined on $M \times M$. On the other hand $\mathrm{e}^{-mD(x,y)/2}|\epsilon_m(x,\bar{y})|^2$ is globally defined on $M \times M$ for m sufficiently large (cfr. Section 4) and then $\mathrm{e}^{-mD(x,y)/2}\tilde{a}_j(x,y)$ are globally defined on $M \times M$ for all $j = 1,\ldots,2p$. Therefore

$$\begin{aligned} &m^n \int_M \mathrm{e}^{-mD(x,y)/2}|\epsilon_m(x,\bar{y})|^2 \frac{\omega^n}{n!}(y) \\ &\quad = m^n \int_M \mathrm{e}^{-mD(x,y)/2}\frac{\omega^n}{n!}(y) + \sum_{j=1}^{2p} \frac{m^n}{m^j} \int_M \mathrm{e}^{-mD(x,y)/2}\tilde{a}_j(x,y)\frac{\omega^n}{n!}(y). \end{aligned}$$

We can now apply Theorem 4.3 to the above two addenda and get

$$\begin{aligned} \epsilon_m(x) &= m^n \int_M \mathrm{e}^{-mD(x,y)/2}|\epsilon_m(x,\bar{y})|^2 \frac{\omega^n}{n!}(y) \\ &\sim 1 + \sum_{r\geq 1} \frac{b_r(x)}{m^r} + \sum_{j=1}^{2p}\sum_{r\geq 0} \frac{C_r(\tilde{a}_j(x,x))}{m^{r+j}}. \end{aligned}$$

The first equality follows by formulae (15) and (20) and the b_j's are defined by (24). Hence by taking $p = 1$ and developing up to order 2 in r

$$\begin{aligned} \epsilon_m(x) &= 1 + \frac{a_1(x)}{m} + \frac{a_2(x)}{m^2} + R(m,x) \\ &= 1 + \frac{b_1(x) + C_0(\tilde{a}_1(x,x))}{m} + \frac{b_2(x) + C_1(2a_1(x)) + C_0(2a_2(x) + a_1^2(x))}{m^2} \\ &\quad + S(m,x), \end{aligned}$$

where $\lim_{m\to\infty} m^2 R(m,x) = \lim_{m\to\infty} m^2 S(m,x) = 0$.

Then we get

$$b_1(x) + C_0(\tilde{a}_1(x,x)) = b_1(x) + 2a_1(x) = a_1(x),$$

namely

$$a_1(x) + b_1(x) = 0, \tag{27}$$

and

$$b_2(x) + C_1(2a_1) + C_0(2a_2(x) + a_1^2(x)) = a_2(x),$$

and by (27)

$$a_2(x) = -b_1^2(x) - b_2(x) + 2C_1(b_1).$$

Next, we do the same for $p = 2$ and so on. Therefore, one can recursively calculate all functions $a_j(x)$ and hence the function $\epsilon_m(x)$. We have then proved the following theorem.

Theorem 5.6. *Let (L, h) be a quantization of a compact Kähler manifold (M, ω) with projectively induced Kähler form ω. Suppose that Zelditch's asymptotic expansion of the function $T_m(x)$ is finite. Then the function $\epsilon_m(x)$ can be obtained by the knowledge of the $b_j(x)$'s and of the operators C_j's applied to the b_k's.*

Remark 5.7. Observe that, as we have already noticed, in the case of a regular quantization the Kähler form ω is automatically projectively induced and the asymptotic expansion of the function Epsilon is finite. In this case the proof of Theorem 5.6 is immediate (cfr. [5]).

Corollary 5.8. *In the same hypothesis of Theorem 5.6 suppose further that the b_j's are constant. Then the quantization is regular.*

Proof. It follows by the very definition of the C_j's that $C_j(b_k) = b_k C_j(1) = b_j b_k$, if the b_k's are constant. Then, by Theorem 5.6, ϵ_m is determined only by the b_j's and hence it is constant. □

References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, Quantum mechanics as a deformation of classical mechanics, Lett. Math. Phys. 1 (1977) 521–530.

[2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, Deformation theory and quantization I, Ann. Phys. 111 (1978) 61–110.

[3] F.A. Berezin, Quantization, Math. USSR Izvestija 8 (1974) 1109–1165.

[4] M. Cahen, S. Gutt, J.H. Rawnsley, Quantization of Kähler manifolds I: geometric interpretation of Berezin's quantization, J. Geophys. 7 (1990) 45–62.

[5] M. Cahen, S. Gutt, J.H. Rawnsley, Quantization of Kähler manifolds II, Trans. Am. Math. Soc. 337 (1993) 73–98.

[6] M. Cahen, S. Gutt, J.H. Rawnsley, Quantization of Kähler manifolds III, Lett. Math. Phys. 30 (1994) 291–305.

[7] M. Cahen, S. Gutt, J.H. Rawnsley, Quantization of Kähler manifolds IV, Lett. Math. Phys. 34 (1995) 159–168.

[8] E. Calabi, Isometric imbeddings of complex manifolds, Ann. Math. 58 (1953) 1–23.

[9] P. Griffiths, J. Harris, Principles of Algebraic Geometry, Wiley, New York, 1978.

[10] M. Itoh, On Kaehler metrics on a compact homogeneous complex manifold, Tôhoku Math. J. 28 (1976) 1–5.

[11] A. Karabegov, Berezin's quantization on flag manifolds and spherical modules, Trans. Am. Math. Soc. 350 (1998) 1467–1479.

[12] S. Kobayashi, Hypersurfaces of complex projective space with constant scalar curvature, J. Diff. Geom. 1 (1967) 369–370.
[13] A. Loi, The function epsilon for complex tori and Riemann surfaces, Bull. Belg. Math. Soc. Simon Stevin 7 (2) (2000) 229–236.
[14] Z. Lu, On the lower terms of the asymptotic expansion of Tian–Yau–Zelditch, Am. J. Math. 122 (2000) 235–273.
[15] C. Moreno, P. Ortega-Navarro, $*$-Products on $D^1(C)$, S^2 and related spectral analysis, Lett. Math. Phys. 7 (1983) 181–193.
[16] C. Moreno, Star-products on some Kähler manifolds, Lett. Math. Phys. 11 (1986) 361–372.
[17] N. Reshetikhin, L. Takhtajan, Deformation quantization of Kähler manifolds, Am. Math. Soc. Trans. Ser. 2 201 (2000) 257–276.
[18] W.D. Ruan, Canonical coordinates and Bergmann metrics, Commun. Anal. Geom. 6 (1998) 589–631.
[19] M. Takeuchi, Homogeneous Kähler manifolds in complex projective space, Jpn. J. Math. 4 (1978) 171–219.
[20] G. Tian, On a set of polarized Kähler metrics on algebraic manifolds, J. Diff. Geom. 32 (1990) 99–130.
[21] S. Zelditch, Szegö Kernel and a theorem of Tian, J. Diff. Geom. 32 (1990) 99–130.