Symplectic maps of complex domains into complex space forms [1]

Andrea Loi
Dipartimento di Matematica e Informatica – Università di Cagliari – Italy
e-mail address: loi@unica.it

and

Fabio Zuddas
Dipartimento di Matematica e Informatica – Università di Cagliari – Italy
e-mail address: fzuddas@unica.it

Abstract

Let $M \subset \mathbb{C}^n$ be a complex domain of $\mathbb{C}^n$ endowed with a rotation invariant Kähler form $\omega_\Phi = \frac{i}{2}\partial\bar{\partial}\Phi$. In this paper we describe sufficient conditions on the Kähler potential Φ for (M, ω_Φ) to admit a symplectic embedding (explicitely described in terms of Φ) into a complex space form of the same dimension of M. In particular we also provide conditions on Φ for (M, ω_Φ) to admit global symplectic coordinates. As an application of our results we prove that each of the Ricci flat (but not flat) Kähler forms on $\mathbb{C}^2$ constructed by LeBrun in [15] admits explicitely computable global symplectic coordinates.

Keywords: Kähler metrics; diastasis function; complex space form; symplectic coordinates; Darboux theorem.

Subj.Class: 53C55, 58C25, 53D05, 58F06.

1 Introduction and statements of the main results

Let (M, ω) and (S, Ω) be two symplectic manifolds of dimension $2n$ and $2N$, $n \leq N$, respectively. Then, one has the following natural and fundamental question.

Question 1. *Under which conditions there exists a symplectic embedding $\Psi : (M, \omega) \to (S, \Omega)$, namely a smooth embedding $\Psi : M \to S$ satisfying $\Psi^*(\Omega) = \omega$?*

[1] During the preparation of this article the authors were supported by the M.I.U.R. Project "Geometric Properties of Real and Complex Manifolds".

Theorems A, B, C and D below give a topological answer to the previous question when Ω is the Kähler form of a N-dimensional complex space form S, namely (S, Ω) is either the complex Euclidean space $(\mathbb{C}^N, \omega_0)$, the complex hyperbolic space $(\mathbb{C}H^N, \omega_{hyp})$ or the complex projective space $(\mathbb{C}P^N, \omega_{FS})$ (see below for the definition of the symplectic (Kähler) forms ω_0, ω_{hyp} and ω_{FS}). Indeed these theorems are consequences of Gromov's h-principle [12] (see also Chapter 12 in [9] for a beautiful description of Gromov's work).

Theorem A (Gromov [12], see also [10]) *Let (M, ω) be a contractible symplectic manifold. Then there exist a non-negative integer N and a symplectic embedding $\Psi : (M, \omega) \to (\mathbb{C}^N, \omega_0)$, where $\omega_0 = \sum_{j=1}^N dx_j \wedge dy_j$ denotes the standard symplectic form on $\mathbb{C}^N = \mathbb{R}^{2N}$.*

This was further generalized by Popov as follows.

Theorem B (Popov [20]) *Let (M, ω) be a symplectic manifold. Assume ω is exact, namely $\omega = d\alpha$, for a 1-form α. Then there exist a non-negative integer N and a symplectic embedding $\Psi : (M, \omega) \to (\mathbb{C}^N, \omega_0)$.*

Observe that the complex hyperbolic space $(\mathbb{C}H^N, \omega_{hyp})$, namely the unit ball $\mathbb{C}H^N = \{z = (z_1, \dots, z_N) \in \mathbb{C}^N | \ \sum_{j=1}^N |z_j|^2 < 1\}$ in $\mathbb{C}^N$ endowed with the hyperbolic form $\omega_{hyp} = -\frac{i}{2}\partial\bar{\partial}\log(1 - \sum_{j=1}^N |z_j|^2)$ is globally symplectomorphic to $(\mathbb{C}^N, \omega_0)$ (see (20) in Lemma 2.2 below) hence Theorem B immediately implies

Theorem C *Let (M, ω) be a symplectic manifold. Assume ω is exact. Then there exist a non-negative integer N and a symplectic embedding $\Psi : (M, \omega) \to (\mathbb{C}H^N, \omega_{hyp})$.*

The following theorem, further generalized by Popov [20] to the noncompact case, deals with the complex projective $\mathbb{C}P^N$, equipped with the Fubini–Study form ω_{FS}. Recall that if $Z_0, \dots, Z_N$ denote the homogeneous coordinates on $\mathbb{C}P^N$, then, in the affine chart $Z_0 \neq 0$ endowed with coordinates $z_j = \frac{Z_j}{Z_0}, j = 1, \dots, N$, the Fubini-Study form reads as

$$\omega_{FS} = \frac{i}{2}\partial\bar{\partial}\log(1 + \sum_{j=1}^N |z_j|^2).$$

Theorem D (Gromov [10], see also Tischler [22]) *Let (M, ω) be a compact symplectic manifold such that ω is integral. Then there exist a non-negative integer N and a symplectic embedding $\Psi : (M, \omega) \to (\mathbb{C}P^N, \omega_{FS})$.*

At this point a natural problem is that to find the smallest dimension of the complex space form where a given symplectic manifold (M, ω) can be symplectically embedded. In particular one can study the case of equidimensional symplectic maps, as expressed by the following interesting question.

Question 2. *Given a $2n$-dimensional symplectic manifold (M, ω) under which conditions there exists a symplectic embedding Ψ of (M, ω) into $(\mathbb{C}^n, \omega_0)$ or $(\mathbb{C}P^n, \omega_{FS})$?*

Notice that locally there are not obstructions to the existence of such Ψ. Indeed, by a well-known theorem of Darboux for every point $p \in M$ there exist a neighbourhood U of p and an embedding $\Psi : U \to \mathbb{R}^{2n} = \mathbb{C}^n$ such that $\Psi^*(\omega_0) = \omega$. In order to get a local embedding into $(\mathbb{C}P^n, \omega_{FS})$ we can assume (by shrinking U if necesssary) that $\Psi(U) \subset \mathbb{C}H^n$. Therefore $f \circ \Psi : U \to (\mathbb{C}^n, \omega_{FS}) \subset (\mathbb{C}P^n, \omega_{FS})$, with f given by Lemma 2.2 below, is the desired embedding satisfying $(f \circ \Psi)^*(\omega_{FS}) = \Psi^*(\omega_0) = \omega$. Observe also that Darboux's theorem is a special case of the following

Theorem E (Gromov [13]) *A $2n$-dimensional symplectic manifold (M, ω) admits a symplectic immersion into $(\mathbb{C}^n, \omega_0)$ if and only if the following three conditions are satisfied: a) M is open, b) the form ω is exact, c) the tangent bundle (TM, ω) is a trivial $Sp(2n)$-bundle. (Observe that a), b), c) are satisfied if M is contractible).*

It is worth pointing out that the previous theorem is not of any help in order to attack Question 2 due to the existence of exotic symplectic structures on $\mathbb{R}^{2n}$ (cfr. [11]). (We refer the reader to [1] for an explicit construction of a 4-dimensional symplectic manifold diffeomorphic to $\mathbb{R}^4$ which cannot be symplectically embedded in $(\mathbb{R}^4, \omega_0)$).

In the case when our symplectic manifold (M, ω) is a Kähler manifold, with associated Kähler metric g, one can try to impose Riemannian or holomorphic conditions to answer the previous question. From the Riemannian point of view the only complete and known result (to the authors' knowledge) is the following global version of Darboux's theorem.

Theorem F (McDuff [19]) *Let (M, g) be a simply-connected and complete n-dimensional Kähler manifold of non-positive sectional curvature. Then* *there exists a diffeomorphism* $\Psi : M \to \mathbb{R}^{2n}$ *such that* $\Psi^*(\omega_0) = \omega$.

(See also [4], [5], [6] and [8] for further properties of McDuff's symplectomorphism).

The aim of this paper is to give an answer to Question 2 in terms of the Kähler potential of the Kähler metric of complex domains (open and connected) $M \subset \mathbb{C}^n$ equipped with a Kähler form ω which admits a rotation invariant Kähler potential. More precisely, throughout this paper we assume that there exists a Kähler potential for ω, namely a smooth function $\Phi : M \to \mathbb{R}$ such that $\omega = \frac{i}{2}\partial\bar{\partial}\Phi$, depending only on $|z_1|^2, \dots, |z_n|^2$, where $z_1, \dots, z_n$ are the standard complex coordinates on $\mathbb{C}^n$. Therefore, there exists a smooth function $\tilde{\Phi} : \tilde{M} \to \mathbb{R}$, defined on the open subset $\tilde{M} \subset \mathbb{R}^n$ given by

$$\tilde{M} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n |\ x_j = |z_j|^2, z = (z_1, \dots z_n) \in M\} \qquad (1)$$

such that

$$\Phi(z_1, \dots, z_n) = \tilde{\Phi}(x_1, \dots, x_n),\ x_j = |z_j|^2,\ j = 1, \dots, n.$$

We set $\omega := \omega_\Phi$ and call ω_Φ a *rotation invariant* symplectic (Kähler) form with *associated* function $\tilde{\Phi}$. It is worth pointing out that many interesting examples of Kähler forms on complex domains are rotation invariant (even radial, namely depending only on $r = |z_1|^2 + \cdots + |z_n|^2$), since they often arise from solutions of ordinary differential equations on the variable r (see Example 3.3 below and also [3] in the case of extremal metrics).

Our first result is Theorem 1.1 below where we describe explicit conditions in terms of the potential Φ for the existence of an explicit symplectic embedding of a rotation invariant domain (M, ω_Φ) into a given complex space form (S, ω_Ξ) of the same dimension. In particular we find conditions on Φ for the existence of global symplectic coordinates of (M, ω_Φ).

Theorem 1.1 *Let* $M \subseteq \mathbb{C}^n$ *be a complex domain such that condition*

$$M \cap \{z_j = 0\} \neq \emptyset,\ j = 1, \dots, n \qquad (2)$$

is satisfied [2] *and let* $\omega_\Phi = \frac{i}{2}\partial\bar{\partial}\Phi$ *be a rotation invariant Kähler form on* M *with associated function* $\tilde{\Phi} : \tilde{M} \to \mathbb{R}$. *Then*

[2] Obviously (2) is satisfied if $0 \in M$, but there are other interesting cases, see Examples 3.2 and 3.3 below, where this condition is fulfilled.

(i) there exists a uniquely determined special [3] *symplectic immersion*

$$\Psi_0 : (M, \omega_\Phi) \to (\mathbb{C}^n, \omega_0)$$

(resp. $\Psi_{hyp} : (M, \omega_\Phi) \to (\mathbb{C}H^n, \omega_{hyp})$*) if and only if,*

$$\frac{\partial \tilde{\Phi}}{\partial x_k} \geq 0, \quad k = 1, \dots, n. \tag{3}$$

(ii) there exists a uniquely determined special symplectic immersion

$$\Psi_{FS} : (M, \omega_\Phi) \to (\mathbb{C}^n, \omega_{FS}),$$

if and only if

$$\frac{\partial \tilde{\Phi}}{\partial x_k} \geq 0, \quad k = 1, \dots, n \text{ and } \sum_{j=1}^{n} \frac{\partial \tilde{\Phi}}{\partial x_j} x_j < 1, \tag{4}$$

where we are looking at $\mathbb{C}^n \overset{i}{\hookrightarrow} \mathbb{C}P^n$ *as the affine chart* $Z_0 \neq 0$ *in* $\mathbb{C}P^n$ *endowed with the restriction of the Fubini–Study form* ω_{FS}*.*

Moreover, assume that $0 \in M$*. If (3) (resp.(4)) is satisfied then* Ψ_0 *(resp.* Ψ_{FS}*) is a global symplectomorphism (and hence* $i \circ \Psi_{FS} : M \to \mathbb{C}P^n$ *is a symplectic embedding) if and only if*

$$\frac{\partial \tilde{\Phi}}{\partial x_k} > 0, \quad k = 1, \dots, n \tag{5}$$

and

$$\lim_{x \to \partial M} \sum_{j=1}^{n} \frac{\partial \tilde{\Phi}}{\partial x_j} x_j = +\infty \quad (\textit{resp. } \lim_{x \to \partial M} \sum_{j=1}^{n} \frac{\partial \tilde{\Phi}}{\partial x_j} x_j = 1). \text{ [4]} \tag{6}$$

[3] See (7) in the next section for the definition of special map between complex domains.

[4] For a rotation invariant continuous map $F : M \to \mathbb{R}$ we write

$$\lim_{x \to \partial M} \tilde{F}(x) = l \in \mathbb{R} \cup \{\infty\}, \ x = (x_1, \dots, x_n),$$

if, for $\|x\| \to +\infty$ or $z \to z_0 \in \partial M$, we have $\|\tilde{F}(x)\| \to l$, where ∂M denotes the boundary of $M \subset \mathbb{C}^n$ and $\tilde{F} : \tilde{M} \to \mathbb{R}$, $\tilde{M}$ given by (1), is the continuous map such that

$$F(z_1, \dots, z_n) = \tilde{F}(x_1, \dots, x_n), \ x_j = |z_j|^2.$$

Observe that the maps Ψ_0, Ψ_{hyp} and Ψ_{FS} can be described explicitely (see (23), (24) and (25) below). This is a rare phenomenon. In fact the proofs of Theorems A, B, C, D and E above are existential and the explicit form of the symplectic embedding or symplectomorphism into a given complex space form is, in general, very hard to find.

Theorem 1.1 is an extension and a generalization of the results obtained by the first author and Fabrizio Cuccu in [7] for complete Reinhardt domains in $\mathbb{C}^2$. Actually, all the results obtained there become a straightforward corollary of our Theorem 1.1 (see Example 3.1 in Section 3).

Our second result is Theorem 1.2 below where we describe geometric conditions on Φ, related to Calabi's work on Kähler immersions, which implies the existence of a special symplectic immersion of (M, ω_Φ) in $(\mathbb{R}^{2n}, \omega_0)$, $n = \dim_{\mathbb{C}} M$ (and in particular the existence of global symplectic coordinates of (M, ω_Φ)).

Theorem 1.2 *Let $M \subseteq \mathbb{C}^n$ be a complex domain such that $0 \in M$ endowed with a rotation invariant Kähler form ω_Φ. Assume that there exists a Kähler (i.e. a holomorphic and isometric) immersion of (M, g_Φ) into some finite or infinite dimensional complex space form, where g_Φ is the metric associated to ω_Φ. Then, (5) is satisfied and hence there exists a special symplectic immersion Ψ_0 of (M, ω_Φ) into $(\mathbb{C}^n, \omega_0)$, which is a global symplectomorphism if and only if $\lim_{x \to \partial M} \sum_{j=1}^n \frac{\partial \tilde{\Phi}}{\partial x_j} x_j = +\infty$. If $\sum_{j=1}^n \frac{\partial \tilde{\Phi}}{\partial x_j} x_j < 1$ then there exists a symplectic immersion Ψ_{FS} of (M, ω_Φ) into $(\mathbb{C}P^n, \omega_{FS})$ which is an embedding if and only if $\lim_{x \to \partial M} \sum_{j=1}^n \frac{\partial \tilde{\Phi}}{\partial x_j} x_j = 1$.*

The paper is organized as follows. In the next section we prove Theorem 1.1 and Theorem 1.2. The later will follow by an application of Calabi's results, which will be briefly recalled in that section. Finally, in Section 3 we apply Theorem 1.1 to some important cases. In particular we recover the results proved in [7] and we prove that each of the Ricci flat (but not flat) Kähler forms on $\mathbb{C}^2$ constructed by LeBrun in [15] admits explicitely computable global symplectic coordinates. Observe that this last result cannot be obtained by Theorem F above (see Remark 3.6 below).

2 Proof of the main results

The following general lemma, used in the proof of our main results Theorem 1.1 and Theorem 1.2, describes the structure of a special symplectic immersion between two complex domains $M \subset \mathbb{C}^n$ and $S \subset \mathbb{C}^n$ endowed with rotation invariant Kähler forms ω_Φ and ω_Ξ respectively. In all the paper we consider smooth maps from M into S of the form

$$\Psi : M \to S, z \mapsto (\Psi_1(z) = \tilde{\psi}_1(x)z_1, \ldots, \Psi_n(z) = \tilde{\psi}_n(x)z_n), \tag{7}$$

$z = (z_1, ..., z_n)$, $x = (x_1, \ldots, x_n)$, $x_j = |z_j|^2$ for some real functions $\tilde{\psi}_j : \tilde{M} \to \mathbb{R}$, $j = 1, \ldots, n$, where $\tilde{M} \subset \mathbb{R}^n$ is given by (1). A smooth map like (7) will be called a *special* map.

Lemma 2.1 *Let $M \subseteq \mathbb{C}^n$ and $S \subseteq \mathbb{C}^n$ be complex domains as above. A special map $\Psi : M \to S$, $z \mapsto (\Psi_1(z), \ldots, \Psi_n(z))$, is symplectic, namely $\Psi^*(\omega_\Xi) = \omega_\Phi$, if and only if there exist constants $c_k \in \mathbb{R}$ such that the following equalities hold on $\tilde{M}$:*

$$\tilde{\psi}_k^2 \frac{\partial \tilde{\Xi}}{\partial x_k}(\Psi) = \frac{\partial \tilde{\Phi}}{\partial x_k} + \frac{c_k}{x_k}, \quad k = 1, \ldots, n, \tag{8}$$

where $\tilde{\Phi}$ (resp. $\tilde{\Xi}$) is the function associated to ω_Φ (resp. ω_Ξ), and

$$\frac{\partial \tilde{\Xi}}{\partial x_k}(\Psi) = \frac{\partial \tilde{\Xi}}{\partial x_k}(\tilde{\psi}_1^2 x_1, \ldots, \tilde{\psi}_n^2 x_n), \quad k = 1, \ldots, n.$$

Proof: From

$$\omega_\Xi = \frac{i}{2} \sum_{i,j=1}^{n} \left(\frac{\partial^2 \tilde{\Xi}}{\partial x_i \partial x_j} \bar{z}_j z_i + \frac{\partial \tilde{\Xi}}{\partial x_i} \delta_{ij} \right)_{x_1=|z_1|^2, \ldots, x_n=|z_n|^2} dz_j \wedge d\bar{z}_i$$

one gets

$$\Psi^*(\omega_\Xi) = \frac{i}{2} \sum_{i,j=1}^{n} \left(\frac{\partial^2 \tilde{\Xi}}{\partial x_i \partial x_j}(\Psi) \Psi_i \bar{\Psi}_j + \frac{\partial \tilde{\Xi}}{\partial x_j}(\Psi) \delta_{ij} \right)_{x_1=|z_1|^2, \ldots, x_n=|z_n|^2} d\Psi_j \wedge d\bar{\Psi}_i,$$

where

$$\frac{\partial^2 \tilde{\Xi}}{\partial x_i \partial x_j}(\Psi) = \frac{\partial^2 \tilde{\Xi}}{\partial x_i \partial x_j}(\tilde{\psi}_1^2 x_1, \ldots, \tilde{\psi}_n^2 x_n).$$

If one denotes by

$$\Psi^*(\omega_\Xi) = \Psi^*(\omega_\Xi)_{(2,0)} + \Psi^*(\omega_\Xi)_{(1,1)} + \Psi^*(\omega_\Xi)_{(0,2)}$$

the decomposition of $\Psi^*(\omega_\Xi)$ into addenda of type $(2,0), (1,1)$ and $(0,2)$ one has:

$$\Psi^*(\omega_\Xi)_{(2,0)} = \frac{i}{2} \sum_{i,j,k,l=1}^{n} \left(\frac{\partial^2 \tilde{\Xi}}{\partial x_i \partial x_j}(\Psi)\Psi_i \bar{\Psi}_j + \frac{\partial \tilde{\Xi}}{\partial x_j}(\Psi)\delta_{ij} \right) \frac{\partial \Psi_j}{\partial z_k} \frac{\partial \bar{\Psi}_i}{\partial z_l} dz_k \wedge dz_l \tag{9}$$

$$\Psi^*(\omega_\Xi)_{(1,1)} = \frac{i}{2} \sum_{i,j,k,l=1}^{n} \left(\frac{\partial^2 \tilde{\Xi}}{\partial x_i \partial x_j}(\Psi)\Psi_i \bar{\Psi}_j + \frac{\partial \tilde{\Xi}}{\partial x_j}(\Psi)\delta_{ij} \right) \left(\frac{\partial \Psi_j}{\partial z_k} \frac{\partial \bar{\Psi}_i}{\partial \bar{z}_l} - \frac{\partial \Psi_j}{\partial \bar{z}_l} \frac{\partial \bar{\Psi}_i}{\partial z_k} \right) dz_k \wedge d\bar{z}_l \tag{10}$$

$$\Psi^*(\omega_\Xi)_{(0,2)} = \frac{i}{2} \sum_{i,j,k,l=1}^{n} \left(\frac{\partial^2 \tilde{\Xi}}{\partial x_i \partial x_j}(\Psi)\Psi_i \bar{\Psi}_j + \frac{\partial \tilde{\Xi}}{\partial x_j}(\Psi)\delta_{ij} \right) \frac{\partial \Psi_j}{\partial \bar{z}_k} \frac{\partial \bar{\Psi}_i}{\partial \bar{z}_l}\, d\bar{z}_k \wedge d\bar{z}_l. \tag{11}$$

(Here and below, with a slight abuse of notation, we are omitting the fact that all the previous expressions have to be evaluated at $x_1 = |z_1|^2, \dots, x_n = |z_n|^2$.) Since $\Psi_j(z) = \tilde{\psi}_j(|z_1|^2, ..., |z_n|^2)z_j$, one has:

$$\frac{\partial \Psi_i}{\partial z_k} = \frac{\partial \tilde{\psi}_i}{\partial x_k} z_i \bar{z}_k + \tilde{\psi}_i \delta_{ik},\ \ \frac{\partial \Psi_i}{\partial \bar{z}_k} = \frac{\partial \tilde{\psi}_i}{\partial x_k} z_k z_i \tag{12}$$

and

$$\frac{\partial \bar{\Psi}_i}{\partial \bar{z}_k} = \frac{\partial \tilde{\psi}_i}{\partial x_k} z_k \bar{z}_i + \tilde{\psi}_i \delta_{ik},\ \ \frac{\partial \bar{\Psi}_i}{\partial z_k} = \frac{\partial \tilde{\psi}_i}{\partial x_k} \bar{z}_k \bar{z}_i, \tag{13}$$

By inserting (12) and (13) into (9) and (10) after a long, but straightforward computation, one obtains:

$$\Psi^*(\omega_\Xi)_{(2,0)} = \frac{i}{2} \sum_{k,l=1}^{n} \frac{A_{kl}}{2} \bar{z}_k \bar{z}_l\, dz_k \wedge dz_l \tag{14}$$

and

$$\Psi^*(\omega_\Xi)_{(1,1)} = \frac{i}{2}\sum_{k,l=1}^{n}\left[\left(\frac{A_{kl}+A_{lk}}{2} + \frac{\partial^2\tilde{\Xi}}{\partial x_k\partial x_l}(\Psi)\tilde{\psi}_k^2\tilde{\psi}_l^2\right)\bar{z}_k z_l + \frac{\partial\tilde{\Xi}}{\partial x_k}(\Psi)\delta_{kl}\tilde{\psi}_k^2\right] dz_k\wedge d\bar{z}_l, \tag{15}$$

where

$$A_{kl} = \frac{\partial\tilde{\Xi}}{\partial x_k}(\Psi)\frac{\partial\tilde{\psi}_k^2}{\partial x_l} + \tilde{\psi}_k^2\sum_{j=1}^{n}\frac{\partial^2\tilde{\Xi}}{\partial x_j\partial x_k}(\Psi)\frac{\partial\tilde{\psi}_j^2}{\partial x_l}|z_j|^2. \tag{16}$$

Now, we assume that

$$\Psi^*(\omega_\Xi) = \omega_\Phi = \frac{i}{2}\sum_{k,l=1}^{n}\left(\frac{\partial^2\tilde{\Phi}}{\partial x_k\partial x_l}\bar{z}_k z_l + \frac{\partial\tilde{\Phi}}{\partial x_l}\delta_{lk}\right)_{x_1=|z_1|^2,\dots,x_n=|z_n|^2} dz_k\wedge d\bar{z}_l.$$

Then the terms $\Psi^*(\omega_\Xi)_{(2,0)}$ and $\Psi^*(\omega_\Xi)_{(0,2)}$ are equal to zero. This is equivalent to the fact that (16) is symmetric in k,l.

Hence, by setting $\Gamma_l = \tilde{\psi}_l^2\,\frac{\partial\tilde{\Xi}}{\partial x_l}(\Psi),\ l=1,\dots,n$ equation (15) becomes

$$\Psi^*(\omega_\Xi)_{(1,1)} = \frac{i}{2}\sum_{k,l=1}^{n}\left[\left(A_{kl} + \frac{\partial^2\tilde{\Xi}}{\partial x_k\partial x_l}(\Psi)\tilde{\psi}_k^2\tilde{\psi}_l^2\right)\bar{z}_k z_l + \frac{\partial\tilde{\Xi}}{\partial x_k}(\Psi)\delta_{kl}\tilde{\psi}_k^2\right] dz_k\wedge d\bar{z}_l =$$

$$= \frac{i}{2}\sum_{k,l=1}^{n}\left(\frac{\partial\Gamma_l}{\partial x_k}\bar{z}_k z_l + \Gamma_k\delta_{kl}\right) dz_k\wedge d\bar{z}_l. \tag{17}$$

So, $\Psi^*(\omega_\Xi) = \omega_\Phi$ implies

$$\frac{i}{2}\sum_{k,l=1}^{n}\left(\frac{\partial\Gamma_l}{\partial x_k}\bar{z}_k z_l + \Gamma_k\delta_{lk}\right) dz_k\wedge d\bar{z}_l = \frac{i}{2}\sum_{k,l=1}^{n}\left(\frac{\partial^2\tilde{\Phi}}{\partial x_k\partial x_l}\bar{z}_k z_l + \frac{\partial\tilde{\Phi}}{\partial x_l}\delta_{kl}\right) dz_k\wedge d\bar{z}_l.$$

In this equality, we distinguish the cases $l\neq k$ and $l=k$ and get respectively

$$\frac{\partial\Gamma_l}{\partial x_k} = \frac{\partial^2\tilde{\Phi}}{\partial x_k\partial x_l}\quad (k\neq l)$$

and

$$\frac{\partial \Gamma_k}{\partial x_k} x_k + \Gamma_k = \frac{\partial^2 \tilde{\Phi}}{\partial x_k^2} x_k + \frac{\partial \tilde{\Phi}}{\partial x_k}.$$

By defining $A_k = \Gamma_k - \frac{\partial \tilde{\Phi}}{\partial x_k}$, these equations become respectively

$$\frac{\partial A_k}{\partial x_l} = 0 \quad (l \neq k)$$

and

$$\frac{\partial A_k}{\partial x_k}\ x_k = -A_k.$$

The first equation implies that A_k does not depend on x_l and so by the second one we have

$$A_k = \Gamma_k - \frac{\partial \tilde{\Phi}}{\partial x_k} = \frac{c_k}{x_k}, \tag{18}$$

for some constant $c_k \in \mathbb{R}$, i.e.

$$\Gamma_k = \tilde{\psi}_k^2 \frac{\partial \tilde{\Xi}}{\partial x_k}(\Psi) = \frac{\partial \tilde{\Phi}}{\partial x_k} + \frac{c_k}{x_k}, \quad k = 1, \ldots, n,$$

namely (8).

In order to prove the converse of Lemma 2.1, notice that by differentiating (8) with respect to l one gets:

$$\frac{\partial^2 \tilde{\Phi}}{\partial x_k \partial x_l} - \frac{c_k}{x_k^2}\delta_{kl} = A_{kl} + \frac{\partial^2 \tilde{\Xi}}{\partial x_k \partial x_l}\tilde{\psi}_k^2 \tilde{\psi}_l^2$$

with A_{kl} given by (16). By $\frac{\partial^2 \tilde{\Phi}}{\partial x_k \partial x_l} = \frac{\partial^2 \tilde{\Phi}}{\partial x_l \partial x_k}$ and $\frac{\partial^2 \tilde{\Xi}}{\partial x_k \partial x_l}\tilde{\psi}_k^2 \tilde{\psi}_l^2 = \frac{\partial^2 \tilde{\Xi}}{\partial x_l \partial x_k}\tilde{\psi}_l^2 \tilde{\psi}_k^2$ one gets $A_{kl} = A_{lk}$. Then, by (14), the addenda of type (2,0) (and (0,2)) in $\Psi^*(\omega_\Xi)$ vanish. Moreover, by (16) and (17), it follows that $\Psi^*(\omega_\Xi) = \omega_\Phi$. $\Box$

In the proof of Theorem 1.1 we also need the following lemma whose proof follows by Lemma 2.1, or by a direct computation.

Lemma 2.2 *The map $f : \mathbb{C}H^n \to \mathbb{C}^n$ given by*

$$(z_1, \ldots, z_n) \mapsto \left(\frac{z_1}{\sqrt{1 - \sum_{i=1}^n |z_i|^2}}, \ldots, \frac{z_n}{\sqrt{1 - \sum_{i=1}^n |z_i|^2}} \right) \tag{19}$$

is a special global diffeomorphism satisfying

$$f^*(\omega_0) = \omega_{hyp} \tag{20}$$

and

$$f^*(\omega_{FS}) = \omega_0, \tag{21}$$

where, in the second equation, we are looking at $\mathbb{C}^n \stackrel{i}{\hookrightarrow} \mathbb{C}P^n$ *as the affine chart* $Z_0 \neq 0$ *in* $\mathbb{C}P^n$ *endowed with the restriction of the Fubini–Study form* ω_{FS} *and where* ω_0 *denotes the restriction of the flat form of* $\mathbb{C}^n$ *to* $\mathbb{C}H^n \subset \mathbb{C}^n$.

We are now in the position to prove our first result.

Proof of Theorem 1.1 First of all observe that under assumption (2), the c_k's appearing in the statement of Lemma 2.1 are forced to be zero. So, the existence of a special symplectic immersion $\Psi : M \to S$ is equivalent to

$$\tilde{\psi}_k^2 \frac{\partial \tilde{\Xi}}{\partial x_k}(\Psi) = \frac{\partial \tilde{\Phi}}{\partial x_k},\ k = 1, \ldots, n. \tag{22}$$

If we further assume $(S = \mathbb{C}^n, \omega_\Xi = \omega_0)$, namely $\tilde{\Xi} = \sum_{j=1}^n x_j$, then condition (8) reduces to

$$\tilde{\psi}_k^2 = \frac{\partial \tilde{\Phi}}{\partial x_k},\ k = 1, \ldots, n,$$

and hence (3) follows by Lemma 2.1. Further Ψ_0 is given by:

$$\Psi_0(z) = \left(\sqrt{\frac{\partial \tilde{\Phi}}{\partial x_1}}\ z_1, \cdots, \sqrt{\frac{\partial \tilde{\Phi}}{\partial x_n}}\ z_n \right)_{x_i = |z_i|^2} \tag{23}$$

In order to prove (i) when $(S = \mathbb{C}H^n, \omega_\Xi = \omega_{hyp})$ observe that since the composition of two special maps is a special map it follows by (20) that the existence of a special symplectic map $\Psi : (M, \omega_\Phi) \to (\mathbb{C}H^n, \omega_{hyp})$ gives rise to a special symplectic map $f \circ \Psi : (M, \omega_\Phi) \to (\mathbb{C}^n, \omega_0)$. The later is uniquely determined by the previous case, i.e. $\Psi_0 = f \circ \Psi$. So $\Psi_{hyp} := \Psi = f^{-1} \circ \Psi_0$ and since the inverse of f is given by

$$f^{-1} : \mathbb{C}^n \to \mathbb{C}H^n, z \mapsto \left(\frac{z_1}{\sqrt{1 + \sum_{i=1}^n |z_i|^2}}, \ldots, \frac{z_n}{\sqrt{1 + \sum_{i=1}^n |z_i|^2}} \right),$$

one obtains:

$$\Psi_{hyp}(z) = \left(\sqrt{\frac{\frac{\partial \tilde{\Phi}}{\partial x_1}}{1 + \sum_{k=1}^n \frac{\partial \tilde{\Phi}}{\partial x_k} x_k}}\ z_1, \dots, \sqrt{\frac{\frac{\partial \tilde{\Phi}}{\partial x_n}}{1 + \sum_{k=1}^n \frac{\partial \tilde{\Phi}}{\partial x_k} x_k}}\ z_n \right)_{x_i = |z_i|^2} . \tag{24}$$

In order to prove (ii), notice that by (21) a special symplectic map $\Psi : (M, \omega_\Phi) \to (\mathbb{C}^n, \omega_{FS})$ is uniquely determined by the special symplectic map $\Psi_0 = f^{-1} \circ \Psi : (M, \omega_\Phi) \to (\mathbb{C}H^n, \omega_0) \subset (\mathbb{C}^n, \omega_0)$ and therefore (4) is a straightforward consequence of the previous case (i). Furthermore Ψ_{FS} is given by

$$\Psi_{FS}(z) = \left(\sqrt{\frac{\frac{\partial \tilde{\Phi}}{\partial x_1}}{1 - \sum_{k=1}^n \frac{\partial \tilde{\Phi}}{\partial x_k} x_k}}\ z_1, \dots, \sqrt{\frac{\frac{\partial \tilde{\Phi}}{\partial x_n}}{1 - \sum_{k=1}^n \frac{\partial \tilde{\Phi}}{\partial x_k} x_k}}\ z_n \right)_{x_i = |z_i|^2} \tag{25}$$

Finally, notice that conditions (5) and (6) for the special map (23) (resp. (25)) are equivalent to $\Psi_0^{-1}(\{0\}) = \{0\}$ (resp. $\Psi_{FS}^{-1}(\{0\}) = \{0\}$) and to the properness of Ψ_0 (resp. Ψ_{FS}). Hence, the fact that under these conditions Ψ_0 (resp. Ψ_{FS}) is a global diffeomorphism follows by standard topological arguments. $\Box$

Remark 2.3 Observe that, by Theorem 1.1, if (M, ω_Φ) admits a special symplectic immersion into $(\mathbb{C}^n, \omega_{FS})$, then it admits a special symplectic immersion in $(\mathbb{C}^n, \omega_0)$ (or $(\mathbb{C}H^n, \omega_{hyp})$). The converse is false even if one restricts to an arbitrary small open set $U \subseteq M$ endowed with the restriction of ω_Φ (see Remark 3.4 below).

In order to prove our second result (Theorem 1.2) we briefly recall Calabi's work on Kähler immersions and his fundamental Theorem 2.9. We refer the reader to [2] for details and further results (see also [17] and [18]).

Calabi's work In his seminal paper Calabi [2] gave a complete answer to the problem of the existence and uniqueness of Kähler immersions of a Kähler manifold (M, g) into a finite or infinite dimensional complex space form. Calabi's first observation was that if such Kähler immersion exists then the metric g is forced to be real analytic being the pull-back via a holomorphic map of the real analytic metric of a complex space form. Then in a neighborhood of every point $p \in M$, one can introduce a very special Kähler potential D_p^g for the metric g, which Calabi christened *diastasis*.

The construction goes as follows. Take a a real-analytic Kähler potential Φ around the point p (it exists since g is real analytic). By duplicating the variables z and $\bar{z}$ Φ can be complex analytically continued to a function $\hat{\Phi}$ defined in a neighborhood U of the diagonal containing $(p, \bar{p}) \in M \times \bar{M}$ (here $\bar{M}$ denotes the manifold conjugated to M). The *diastasis function* is the Kähler potential D^g_p around p defined by

$$D^g_p(q) = \hat{\Phi}(q, \bar{q}) + \hat{\Phi}(p, \bar{p}) - \hat{\Phi}(p, \bar{q}) - \hat{\Phi}(q, \bar{p}).$$

Example 2.4 Let g_0 be the Euclidean metric on $\mathbb{C}^N$, $N \leq \infty$, namely the metric whose associated Kähler form is given by $\omega_0 = \frac{i}{2}\sum_{j=1}^{N} dz_j \wedge d\bar{z}_j$. Here $\mathbb{C}^\infty$ is the complex Hilbert space $l^2(\mathbb{C})$ consisting of sequences $(z_j)_{j\geq 1}$, $z_j \in \mathbb{C}$ such that $\sum_{j=1}^{+\infty} |z_j|^2 < +\infty$. The diastasis function $D^{g_0}_0 : \mathbb{C}^N \to \mathbb{R}$ around the origin $0 \in \mathbb{C}^N$ is given by

$$D^{g_0}_0(z) = \sum_{j=1}^{N} |z_j|^2. \tag{26}$$

Example 2.5 Let $(Z_0, Z_1, \ldots, Z_N)$ be the homogeneous coordinates in the complex projective space in $\mathbb{C}P^N$, $N \leq \infty$, endowed with the Fubini–Study metric g_{FS}. Let $p = [1, 0, \ldots, 0]$. In the affine chart $U_0 = \{Z_0 \neq 0\}$ endowed with coordinates $(z_1, \ldots, z_N), z_j = \frac{Z_j}{Z_0}$ the diastasis around p reads as:

$$D^{g_{FS}}_p(z) = \log(1 + \sum_{j=1}^{N} |z_j|^2). \tag{27}$$

Example 2.6 Let $\mathbb{C}H^N = \{z \in \mathbb{C}^N | \ \sum_{j=1}^{N} |z_j|^2 < 1\} \subset \mathbb{C}^N, N \leq \infty$ be the complex hyperbolic space endowed with the hyperbolic metric g_{hyp}. Then the diastasis around the origin is given by:

$$D^{g_{hyp}}_0(z) = -\log(1 - \sum_{j=1}^{N} |z_j|^2). \tag{28}$$

A very useful characterization of the diastasis (see below) can be obtained as follows. Let (z) be a system of complex coordinates in a neighbourhood of p where D^g_p is defined. Consider its power series development:

$$D^g_p(z) = \sum_{j,k\geq 0} a_{jk}(g) z^{m_j} \bar{z}^{m_k}, \tag{29}$$

where we are using the following convention: we arrange every n-tuple of nonnegative integers as the sequence

$$m_j = (m_{1,j}, m_{2,j}, \dots, m_{n,j})_{j=0,1,\dots}$$

such that $m_0 = (0, \dots, 0)$, $|m_j| \leq |m_{j+1}|$, with $|m_j| = \sum_{\alpha=1}^{n} m_{\alpha,j}$ and $z^{m_j} = \prod_{\alpha=1}^{n} (z_\alpha)^{m_{\alpha,j}}$. Further, we order all the m_j's with the same $|m_j|$ using the lexicographic order.

Characterization of the diastasis: *Among all the potentials the diastasis is characterized by the fact that in every coordinates system (z) centered in p the coefficients $a_{jk}(g)$ of the expansion (29) satisfy $a_{j0}(g) = a_{0j}(g) = 0$ for every nonnegative integer j.*

Definition 2.7 *A Kähler immersion φ of (M, g) into a complex space form (S, G) is said to be full if $\varphi(M)$ is not contained in a proper complex totally geodesic submanifold of (S, G).*

Definition 2.8 *Let g be a real analytic Kähler metric on a complex manifold M. The metric g is said to be resolvable of rank N if the $\infty \times \infty$ matrix $a_{jk}(g)$ given by (29) is positive semidefinite and of rank N. Consider the function $e^{D_p^g} - 1$ (resp. $1 - e^{-D_p^g}$) and its power series development:*

$$e^{D_p^g} - 1 = \sum_{j,k \geq 0} b_{jk}(g) z^{m_j} \bar{z}^{m_k}. \tag{30}$$

(resp.

$$1 - e^{-D_p^g} = \sum_{j,k \geq 0} c_{jk}(g) z^{m_j} \bar{z}^{m_k}.) \tag{31}$$

The metric g is said to be 1-resolvable (resp. -1-resolvable) of rank N at p if the $\infty \times \infty$ matrix $b_{jk}(g)$ (resp. $c_{jk}(g)$) is positive semidefinite and of rank N.

We are now in the position to state Calabi's fundamental theorem and to prove our Theorem 1.2.

Theorem 2.9 (Calabi) *Let M be a complex manifold endowed with a real analytic Kähler metric g. A neighbourhood of a point p admits a (full) Kähler immersion into $(\mathbb{C}^N, g_0)$ if and only if g is resolvable of rank at most (exactly) N at p. A neighbourhood of a point p admits a (full) Kähler immersion into $(\mathbb{C}P^N, g_{FS})$ (resp. $(\mathbb{C}H^N, g_{hyp})$) if and only if g is 1-resolvable (resp. -1-resolvable) of rank at most (exactly) N at p.*

Proof of Theorem 1.2 Without loss of generality we can assume $\Phi(0)=0$. Then, it follows by the characterization of the diastasis function that Φ is indeed the (globally defined) diastasis function for the Kähler metric g_Φ (associated to ω_Φ) around the origin, namely $\Phi = D_0^{g_\Phi}$. Since $\Phi = D_0^{g_\Phi}$ is rotation invariant, namely it depends only on $|z_1|^2, \dots, |z_n|^2$, the matrices $a_{jk}(g)$, $b_{jk}(g)$ and $c_{jk}(g)$ above are diagonal, i.e.

$$a_{jk}(g) = a_j\delta_{jk},\ b_{jk}(g) = b_j\delta_{jk},\ c_{jk}(g) = c_j\delta_{jk},\ a_j, b_j, c_j \in \mathbb{R}. \tag{32}$$

Therefore, by Calabi's Theorem 2.9 if (M, g_Φ) admits a (full) Kähler immersion into $(\mathbb{C}^N, g_0)$ (resp. $(\mathbb{C}P^N, g_{FS})$ or $(\mathbb{C}H^N, g_{hyp})$) then all the $a_j's$ (resp. the $b_j's$ or the $c_j's$) are greater or equal than 0 and at most (exactly) N of them are positive. Moreover, it follows by the fact that the metric g_Φ is positive definite (at $0 \in M$) that the coefficients a_k (resp. b_k or c_k), $k = 1, \dots, n$, are strictly greater than zero . Hence, by using (29) (resp. (30) or (31)) with $p = 0$ and $g = g_\Phi$ we get $\frac{\partial\tilde\Phi}{\partial x_k}(x) = a_k + P_0(x)$ (resp. $\frac{\partial\tilde\Phi}{\partial x_k}(x) = \frac{b_k + P_+(x)}{1+\sum_j b_j x^{m_j}}$ or $\frac{\partial\tilde\Phi}{\partial x_k}(x) = \frac{c_k + P_-(x)}{1-\sum_j c_j x^{m_j}}$) where P_0 (resp. P_+ or P_-) is a polynomial with non-negative coefficients in the variables $x = (x_1, \dots, x_n)$, $x_j = |z_j|^2$. Hence condition (3) above is satisfied. The last two assertions of Theorem 1.2 are immediately consequences of Theorem 1.1. □

3 Applications and further results

Example 3.1 (cfr. [7]) Let $x_0 \in \mathbb{R}^+ \cup \{+\infty\}$ and let $F : [0, x_0) \to (0, +\infty)$ be a non-increasing smooth function. Consider the domain

$$D_F = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 < x_0,\ |z_2|^2 < F(|z_1|^2)\}$$

endowed with the 2-form $\omega_F = \frac{i}{2}\partial\bar\partial \log \frac{1}{F(|z_1|^2) - |z_2|^2}$. If the function $A(x) = -\frac{xF'(x)}{F(x)}$ satisfies $A'(x) > 0$ for every $x \in [0, x_0)$, then ω_F is a Kähler form on D_F and (D_F, ω_F) is called the *complete Reinhardt domain* associated with F. Notice that ω_F is rotation invariant with associated real function $\tilde F(x_1, x_2) = \log \frac{1}{F(x_1) - x_2}$. We now apply Theorem 1.1 to (D_F, ω_F). We have

$$\frac{\partial \tilde F}{\partial x_1} = -\frac{F'(x_1)}{F(x_1) - x_2} > 0, \quad \frac{\partial \tilde F}{\partial x_2} = \frac{1}{F(x_1) - x_2} > 0,\ x_j = |z_j|^2,\ j = 1, 2.$$

So, by Theorem 1.1, (D_F, ω_F) admits a special symplectic immersion in $(\mathbb{C}^2, \omega_0)$ (and in $(\mathbb{C}H^2, \omega_{hyp})$). Moreover, this immersion is a global symplectomorphism only when

$$\frac{\partial \tilde{F}}{\partial x_1} x_1 + \frac{\partial \tilde{F}}{\partial x_2} x_2 = \frac{x_2 - F'(x_1)x_1}{F(x_1) - x_2}$$

tends to infinity on the boundary of D_F. For example, let $F : [0, +\infty) \to \mathbb{R}^+$ given by $F(x) = \frac{c}{c+x}$, $c > 0$ (resp. $F(x) = \frac{1}{(1+x)^p}$, $p \in \mathbb{N}^+$). Then $\sum_{i=1}^2 \frac{\partial \tilde{F}}{\partial x_i} x_i = \frac{x_2(c+x_1)^2 + cx_1}{c(c+x_1) - x_2(c+x_1)^2}$ (resp. $\sum_{i=1}^2 \frac{\partial \tilde{F}}{\partial x_i} x_i = \frac{x_2 + px_1(1+x_1)^{-p-1}}{(1+x_1)^{-p} - x_2}$) does not tend to infinity, for $t \to \infty$, along the curve $x_1 = t$, $x_2 = \frac{\varepsilon c}{c+t}$, for any $\varepsilon \in (0, 1)$ (resp. does not tend to infinity, for $t \to \infty$, along the curve $x_1 = t$, $x_2 = \varepsilon(1 + t)^{-p}$, for any $\varepsilon \in (0, 1)$). On the other hand, one verifies by a straight calculation that, if $F : [0, +\infty) \to \mathbb{R}^+$ is given by $F(x) = e^{-x}$ (resp. $F : [0, 1) \to \mathbb{R}^+$, $F(x) = (1 - x)^p$, $p > 0$), then $\sum_{i=1}^2 \frac{\partial \tilde{F}}{\partial x_i} x_i = \frac{x_2 + e^{-x_1} x_1}{e^{-x_1} - x_2}$ (resp. $\sum_{i=1}^2 \frac{\partial \tilde{F}}{\partial x_i} x_i = \frac{x_2 + px_1(1-x_1)^{p-1}}{(1-x_1)^p - x_2}$) tends to infinity on the boundary of D_F. We then recover the conclusions of Examples 3.3, 3.4, 3.5, 3.6 in [7].

Example 3.2 Let us endow $\mathbb{C}^2 \setminus \{0\}$ with the rotation invariant Kähler form $\omega_\Phi = \frac{i}{2}\partial\bar{\partial}\Phi$ with associated real function

$$\tilde{\Phi}(x_1, x_2) = a \log(x_1 + x_2) + b(x_1 + x_2) + c, \;\; a, b, c > 0.$$

The metric g_Φ associated to ω_Φ is used in [21] (see also [16]) for the construction of Kähler metrics of constant scalar curvature on bundles on $\mathbb{C}P^{n-1}$.

Since $\frac{\partial \tilde{\Phi}}{\partial x_i} = b + \frac{a}{x_1 + x_2} > 0$, by Theorem 1.1 there exists a special symplectic immersion of $(\mathbb{C}^2 \setminus \{0\}, \omega_\Phi)$ in $(\mathbb{C}^2, \omega_0)$ (or in $(\mathbb{C}H^2, \omega_{hyp})$).

Example 3.3 Let us endow $\mathbb{C}^2 \setminus \{0\}$ with the metric $\omega_\Phi = \frac{i}{2}\partial\bar{\partial}\Phi$, where

$$\tilde{\Phi} = \sqrt{r^4 + 1} + 2 \log r - \log(\sqrt{r^4 + 1} + 1), \;\; r = \sqrt{|z_1|^2 + |z_2|^2}.$$

The metric g_Φ is used in [14] for the construction of the Eguchi–Hanson metric. A straight calculation shows that

$$\frac{\partial \tilde{\Phi}}{\partial x_i} = \frac{\partial \tilde{\Phi}}{\partial r} \frac{\partial r}{\partial x_i} = \left[\frac{4r^3}{2\sqrt{r^4 + 1}} \left(1 - \frac{1}{\sqrt{r^4 + 1} + 1} \right) + \frac{2}{r} \right] \frac{1}{2r} > 0,$$

so by Theorem 1.1 there exists a special symplectic immersion of $(\mathbb{C}^2 \setminus \{0\}, \omega_\Phi)$ in $(\mathbb{C}^2, \omega_0)$ (or in $(\mathbb{C}H^2, \omega_{hyp})$).

Remark 3.4 Notice that in the previous Example 3.3 one has

$$\frac{\partial\tilde{\Phi}}{\partial x_1}x_1+\frac{\partial\tilde{\Phi}}{\partial x_2}x_2=\frac{r^4}{\sqrt{r^4+1}}\left(1-\frac{1}{\sqrt{r^4+1}+1}\right)+1>1,$$

so again by Theorem 1.1 it does not exist a special symplectic immersion of $(\mathbb{C}^2\setminus\{0\},\omega_\Phi)$ in $(\mathbb{C}^2,\omega_{FS})$. Moreover, such an immersion does not exist for any arbitrarily small $U\subseteq\mathbb{C}^2\setminus\{0\}$ endowed with the restriction of ω_Φ (cfr. Remark 2.3 above).

Example 3.5 In [15] Claude LeBrun constructed the following family of Kähler forms on $\mathbb{C}^2$ defined by $\omega_m=\frac{i}{2}\partial\bar{\partial}\Phi_m$, where

$$\Phi_m(u,v)=u^2+v^2+m(u^4+v^4),\ m\geq 0$$

and u and v are implicitly defined by

$$|z_1|=e^{m(u^2-v^2)}u,\ |z_2|=e^{m(v^2-u^2)}v.$$

For $m=0$ one gets the flat metric, while for $m>0$ each of the metrics of this family represents the first example of complete Ricci flat (non-flat) metric on $\mathbb{C}^2$ having the same volume form of the flat metric ω_0, namely $\omega_m\wedge\omega_m=\omega_0\wedge\omega_0$. Moreover, for $m>0$, these metrics are isometric (up to dilation and rescaling) to the Taub-NUT metric.

Now, with the aid of Theorem 1.1, we prove that for every m the Kähler manifold $(\mathbb{C}^2,\omega_m)$ admits global symplectic coordinates. Set $u^2=U$, $v^2=V$. Then $\tilde{\Phi}_m$ (the function associated to Φ_m) satisfies:

$$\frac{\partial\tilde{\Phi}_m}{\partial x_1}=\frac{\partial\tilde{\Phi}_m}{\partial U}\frac{\partial U}{\partial x_1}+\frac{\partial\tilde{\Phi}_m}{\partial V}\frac{\partial V}{\partial x_1},$$

$$\frac{\partial\tilde{\Phi}_m}{\partial x_2}=\frac{\partial\tilde{\Phi}_m}{\partial U}\frac{\partial U}{\partial x_2}+\frac{\partial\tilde{\Phi}_m}{\partial V}\frac{\partial V}{\partial x_2},$$

where $x_j=|z_j|^2, j=1,2$. In order to calculate $\frac{\partial U}{\partial x_j}$ and $\frac{\partial V}{\partial x_j}, j=1,2$, let us consider the map

$$G:\mathbb{R}^2\to\mathbb{R}^2,\ (U,V)\mapsto(x_1=e^{2m(U-V)}U,\ x_2=e^{2m(V-U)}V)$$

and its Jacobian matrix

$$J_G=\begin{pmatrix}(1+2mU)\,e^{2m(U-V)} & -2mU\,e^{2m(U-V)}\\ -2mV\,e^{2m(V-U)} & (1+2mV)\,e^{2m(V-U)}\end{pmatrix}.$$

We have $detJ_G = 1 + 2m(U+V) \neq 0$, so

$$J_G^{-1} = J_{G^{-1}} = \frac{1}{1+2m(U+V)} \begin{pmatrix} (1+2mV)e^{2m(V-U)} & 2mUe^{2m(U-V)} \\ 2mVe^{2m(V-U)} & (1+2mU)e^{2m(U-V)} \end{pmatrix}.$$

Since $J_{G^{-1}} = \begin{pmatrix} \frac{\partial U}{\partial x_1} & \frac{\partial U}{\partial x_2} \\ \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{pmatrix}$, by a straightforward calculation we get

$$\frac{\partial \tilde{\Phi}_m}{\partial x_1} = (1+2mV)e^{2m(V-U)} > 0, \ \frac{\partial \tilde{\Phi}_m}{\partial x_2} = (1+2mU)e^{2m(U-V)} > 0,$$

and

$$\lim_{\|x\|\to+\infty} (\frac{\partial \tilde{\Phi}_m}{\partial x_1} x_1 + \frac{\partial \tilde{\Phi}_m}{\partial x_2} x_2) = \lim_{\|x\|\to+\infty} (U + V + 4mUV) = +\infty,$$

namely (5) and (6) above respectively . Hence, by Theorem 1.1, the map

$$\Psi_0 : \mathbb{C}^2 \to \mathbb{C}^2, (z_1, z_2) \mapsto \left((1+2mV)^{\frac{1}{2}} e^{m(V-U)} z_1, (1+2mU)^{\frac{1}{2}} e^{m(U-V)} z_2\right)$$

is a special global symplectomorphism from $(\mathbb{C}^2, \omega_m)$ into $(\mathbb{C}^2, \omega_0)$.

Remark 3.6 Notice that for $m > 0$ we cannot apply McDuff's Theorem F in the Introduction in order to get the existence of global symplectic coordinates on $(\mathbb{C}^2, \omega_m)$. Indeed, the sectional curvature of $(\mathbb{C}^2, g_m)$ (where g_m is the Kähler metric associated to ω_m) is positive at some point since g_m is Ricci-flat but not flat.

Remark 3.7 In a forthcoming paper, among other properties, we prove that $(\mathbb{C}^2, g_m)$ cannot admit a Kähler immersion into any complex space form, for all $m > \frac{1}{2}$ (this is achieved by using Calabi's diastasis function). Therefore, the previous example shows that the assumption in Theorem 1.2 that (M, ω_Φ) admits a Kähler immersion into some complex space form is a sufficient but not a necessary condition for the existence of a special symplectic immersion into $(\mathbb{C}^n, \omega_0)$.

References

[1] L. Bates and G. Peschke, *A remarkable symplectic structure*, J. Diff. Geom. (1990), 533-538.

[2] E. Calabi, *Isometric Imbeddings of Complex Manifolds*, Ann. Math. 58 (1953), 1-23.

[3] E. Calabi, *Extremal Kähler metrics*, Seminar on Differential Geometry, pp. 259–290, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.

[4] E. Ciriza, *The local structure of a Liouville vector field*, Amer. J. Math. 115 (1993), 735-747.

[5] E. Ciriza, *On special submanifolds in symplectic geometry*, Diff. Geom. Appl. 3 (1993), 91-99.

[6] E. Ciriza, *Symplectomorphic codimension* 1 *totally geodesic submanifolds*, Diff. Geom. Appl. 5 (1995), 99-104.

[7] F. Cuccu and A. Loi, *Global symplectic coordinates on complex domains*, J. Geom. and Phys. 56 (2006), 247-259.

[8] A. J. Di Scala and A. Loi, *Symplectic duality of symmetric spaces* math.DG/0603141.

[9] Y. Eliashberg and N. Mishachev, *Introduction to the h-principle*, Graduate Studies in Mathematics, 48. American Mathematical Society, Providence, RI, 2002.

[10] M. Gromov, *A topological technique for the construction of differential equations and inequalities*, Proc. ICM. 70, Nice, Vol. 2, Gauthier–Villars, Paris (1971), 221-225.

[11] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985), 307-347.

[12] M. Gromov, *Partial Differential Relations*, Springer-Verlag (1986).

[13] M. Gromov, Soft and hard symplectic geometry. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 81–98, Amer. Math. Soc., Providence, RI, 1987.

[14] D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000.

[15] C. LeBrun, *Complete Ricci-flat Kähler metrics on* $\mathbb{C}^n$ *need not be flat*, Proceedings of Symposia in Pure Mathematics, vol. 52 (1991), Part 2, 297-304.

[16] C. LeBrun, *Counter-examples to the generalized positive action conjecture*, Commun. Math. Phys. 118, 591-596 (1988).

[17] A. Loi, *Holomorphic maps of Hartogs domains into complex space forms*, Riv. Mat. Univ. Parma (7) vol. 1 (2002), 103-113.

[18] A. Loi, *Calabi's diastasis function for Hermitian symmetric spaces*, Diff. Geom. Appl. 24 (2006), 311-319.

[19] D. McDuff, *The symplectic structure of Kähler manifolds of non-positive curvature*, J. Diff. Geometry 28 (1988), 467-475.

[20] V. A. Popov, *Closed 2-forms and embedding theorems for arbitrary symplectic manifolds*, Soviet Math. (Iz. VUZ) 32 (1988), no. 4, 60-66.

[21] S. R. Simanca, *Kähler metrics of constant scalar curvature on bundles over* $\mathbb{C}P_{n-1}$, Math. Ann. 291, 239-246 (1991).

[22] D. Tischler, *Closed two forms and an embedding theorem for symplectic manifolds*, J. Diff. Geom. 12 (1977), 229-235.