BALANCED METRICS ON HARTOGS DOMAINS

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Abstract. An n-dimensional strictly pseudoconvex Hartogs domain D_F can be equipped with a natural Kähler metric g_F. In this paper we prove that if $m_0 g_F$ is balanced for a given positive integer m_0 then $m_0 > n$ and (D_F, g_F) is holomorphically isometric to an open subset of the n-dimensional complex hyperbolic space.

1. Introduction

Let M be a complex manifold endowed with a Kähler metric g and let ω be the Kähler form associated to g, i.e. $\omega(\cdot,\cdot) = g(\cdot, J\cdot)$. Assume that the metric g can be described by a strictly plurisubharmonic real valued function $\Phi : M \to \mathbb{R}$, called a *Kähler potential* for g, i.e. $\omega = \frac{i}{2}\partial\bar\partial\Phi$.

A Kähler potential is not unique, in fact it is defined up to an addition with the real part of a holomorphic function on M. Let $\mathcal{H}_\Phi$ be the weighted Hilbert space of square integrable holomorphic functions on (M, g), with weight $e^{-\Phi}$, namely

$$\mathcal{H}_\Phi = \left\{ f \in \mathrm{Hol}(M) \mid \int_M e^{-\Phi}|f|^2 \frac{\omega^n}{n!} < \infty \right\}, \tag{1}$$

where $\frac{\omega^n}{n!} = \det(\partial\bar\partial\Phi)\frac{\omega_0^n}{n!}$ is the volume form associated to ω and $\omega_0 = \frac{i}{2}\sum_{\alpha=0}^{n-1} dz_\alpha \wedge d\bar z_\alpha$ is the standard Kähler form on $\mathbb{C}^n$. If $\mathcal{H}_\Phi \neq \{0\}$ we can pick an orthonormal basis $\{f_j\}$ and define its reproducing kernel by

$$K_\Phi(z,z) = \sum_{j=0}^{\infty} |f_j(z)|^2.$$

Consider the function

$$\varepsilon_g(z) = e^{-\Phi(z)} K_\Phi(z,z). \tag{2}$$

As suggested by the notation ε_g depends only on the metric g and not on the choice of the Kähler potential Φ. In fact, if $\Phi' = \Phi - \mathrm{Re}(\varphi)$, for some holomorphic function φ, is another potential for ω, we have $e^{-\Phi'} = e^{-\Phi}|e^{\varphi}|^2$.

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Furthermore, since φ is holomorphic and $\partial\bar\partial\Phi' = \partial\bar\partial\Phi$, e^{φ} is an isomorphism between $\mathcal{H}_\Phi$ and $\mathcal{H}_{\Phi'}$, and thus we can write $K_{\Phi'}(z,z) = |e^{\varphi}|^2 K_\Phi(z,z)$, where $K_\Phi(z,z)$ (resp. $K_{\Phi'}(z,z)$) is the reproducing kernel of $\mathcal{H}_\Phi$ (resp. $\mathcal{H}_{\Phi'}$). It follows that $e^{-\Phi(z)}K_\Phi(z,z) = e^{-\Phi'(z)}K_{\Phi'}(z,z)$, as claimed.

In the literature the function ε_g was first introduced under the name of *η-function* by J. Rawnsley in [17], later renamed as *ε-function* in [3], and it is also appear under the name of *distortion function* for the study of Abelian varieties by J. R. Kempf [14] and S. Ji [13], and for complex projective varieties by S. Zhang [18]. It also plays a fundamental role in the geometric quantization of a Kähler manifold and in the Tian-Yau-Zelditch asymptotic expansion (see [11] and references therein).

Definition. *Let g be a Kähler metric on a complex manifold M such that $\omega = \frac{i}{2}\partial\bar\partial\Phi$. The metric g is* balanced *if the function ε_g is a positive constant.*

Remark 1. The definition of balanced metrics was originally given by S. Donaldson [7] in the case of a compact polarized Kähler manifold (M, g) and generalized in [2] (see also [4], [10], [9]). In the compact case the potential Φ will certainly not exist globally and the only holomorphic functions on M are the constants. Nevertheless, since g is polarized there exists an hermitian line bundle $(L, h) \to M$ such that $\mathrm{Ric}(h) = \omega$. One can then enodwed the space of global holomorphic sections of L, denoted by $H^0(L)$, with the scalar product

$$\langle s, t\rangle_h = \int_M h(s(x), t(x))\frac{\omega^n}{n!} < \infty, s, t \in H^0(L).$$

If $H^0(L) \neq \{0\}$ one can set

$$\varepsilon_g(x) = \sum_{j=0}^{N} h(s_j(x), s_j(x)),$$

where $\{s_0, \dots, s_N\}$, $N+1 = \dim H^0(L)$, is an orthonormal basis of $(H^0(L), \langle,\rangle_h)$ and define the metric g *balanced* if ε_g is a positive constant.

In this paper we study the balanced condition for a particular class of strictly pseudoconvex domains D_F of $\mathbb{C}^n$, called *Hartogs domains* (see next section or [8]), equipped with a Kähler metric g_F depending on a real valued function F. Our main result is Theorem 7 below where we prove that if the metric $m_0 g_F$ of a n-dimensional Hartogs domain D_F is balanced for a given $m_0 > n$, then (D_F, g_F) is holomorphically isometric to an open subset of the n-dimensional complex hyperbolic space. The paper contains another section with the description of the Hartogs domains and the proof of Theorem 7.

2. Statement and proof of the main result

Let $x_0 \in \mathbb{R}^+ \cup \{+\infty\}$ and let $F : [0, x_0) \to (0, +\infty)$ be a decreasing continuous function, smooth on $(0, x_0)$. The Hartogs domain $D_F \subset \mathbb{C}^n$ associated to the function F is defined by

$$D_F = \{(z_0, z_1, \dots, z_{n-1}) \in \mathbb{C}^n \mid |z_0|^2 < x_0,\ ||z||^2 < F(|z_0|^2)\},$$

where $||z||^2 = |z_1|^2 + \dots + |z_{n-1}|^2$. We shall assume that the natural $(1,1)$-form on D_F given by

$$\omega_F = \frac{i}{2}\partial\overline{\partial}\log\left(\frac{1}{F(|z_0|^2) - ||z||^2}\right), \tag{3}$$

is a Kähler form on D_F. The following proposition gives some conditions on D_F equivalent to this assumption:

Proposition 2 ([16])**.** *Let D_F be a Hartogs domain in $\mathbb{C}^n$. Then the following conditions are equivalent:*

(i) *the $(1,1)$-form ω_F given by (3) is a Kähler form;*
(ii) *the function $-\frac{xF'(x)}{F(x)}$ is strictly increasing, namely $-\left(\frac{xF'(x)}{F(x)}\right)' > 0$ for every $x \in [0, x_0)$;*
(iii) *the boundary of D_F is strongly pseudoconvex at all $z = (z_0, z_1, \dots, z_{n-1})$ with $|z_0|^2 < x_0$.*

The Kähler metric g_F associated to the Kähler form ω_F is the metric we will be dealing with in the present paper. It follows by (3) that a Kähler potential for this metric is given by

$$\Phi_F = -\log\left(F(|z_0|^2) - ||z||^2\right).$$

Example 1. When $F(x) = 1 - x, 0 \le x < 1$,

$$D_F = \mathbb{C}\mathrm{H}^n = \{(z_0, z_1, \dots, z_{n-1}) \mid |z_0|^2 + \|z\|^2 < 1)\},$$

the n-dimensional complex hyperbolic space $\mathbb{C}\mathrm{H}^n$ and g_F is the hyperbolic metric, i.e. $g_F = g_{hyp}$. A Kähler potential for g_{hyp} is given by $\Phi_{hyp} = -\log(1 - \sum_{\alpha=0}^{n-1}|z_\alpha|^2)$, and the associated volume form reads

$$\frac{\omega_{hyp}^n}{n!} = \left(1 - \sum_{\alpha=0}^{n-1}|z_\alpha|^2\right)^{-(n+1)}\frac{\omega_0^n}{n!}.$$

Consider $m\, g_{hyp}$, for a positive integer m, and let $\mathcal{H}_{m\Phi_{hyp}}$ be the weighted Hilbert space of square integrable holomorphic functions on $(\mathbb{C}\mathrm{H}^n, m\, g_{hyp})$, with weight $e^{-m\Phi_{hyp}} = \left(1 - \sum_{\alpha=0}^{n-1}|z_\alpha|^2\right)^m$, namely

$$\mathcal{H}_{m\Phi_{hyp}} = \left\{\varphi \in \mathrm{Hol}(\mathbb{C}\mathrm{H}^n) \mid \int_{\mathbb{C}\mathrm{H}^n}\left(1 - \sum_{\alpha=0}^{n-1}|z_\alpha|^2\right)^{m-(n+1)}|\varphi|^2\frac{\omega_0^n}{n!} < \infty\right\}.$$

If $m \leq n$, then it is not hard to see that $\mathcal{H}_{m\Phi_{hyp}} = \{0\}$. On the other hand, for $m > n$, an orthonormal basis for $\mathcal{H}_{m\Phi_{hyp}}$ is given by

$$\left\{ \dots, \frac{\sqrt{(m+j-1)!}}{\sqrt{\pi^n}\sqrt{j_1! \cdots j_{n-1}!(m-n-1)!}} z_0^{j_0} \cdots z_{n-1}^{j_{n-1}}, \dots \right\}.$$

where $j = j_0 + \cdots + j_{n-1}$. In fact, since the metric depends only on the squared module of the variables, it is easy to see that the monomials $z_0^{j_0} \cdots z_{n-1}^{j_{n-1}}$ are a complete orthogonal system for $\mathcal{H}_{m\Phi_{hyp}}$. Further, the following computation

$$\int_{\mathbb{C}\mathrm{H}^n} |z_0^{j_0} \cdots z_{n-1}^{j_{n-1}}|^2 \left(1 - \sum_{\alpha=0}^{n-1} |z_\alpha|^2\right)^{m-(n+1)} \frac{i^n}{2^n} dz_0 \wedge d\bar{z}_0 \wedge \cdots \wedge dz_{n-1} \wedge d\bar{z}_{n-1}$$

$$= \pi^n \int_0^1 \cdots \int_0^{1-r_1-\cdots-r_{n-1}} r_0^{j_0} \cdots r_{n-1}^{j_{n-1}} \left(1 - \sum_{\alpha=0}^{n-1} r_\alpha^{j_\alpha}\right)^{m-(n+1)} dr_0 \cdots dr_{n-1}$$

$$= \pi^n \frac{j_0! \cdots j_{n-1}!(m-n-1)!}{(m+j-1)!},$$

justifies the choice of the normalization constants. The reproducing kernel for $\mathcal{H}_{m\Phi_{hyp}}$ is then given by

$$K_{m\Phi_{hyp}}(z, z) = \frac{(m-1)\cdots(m-n)}{\pi^n (1 - \sum_{j=0}^{n-1} |z_j|^2)^m},$$

and thus

$$\varepsilon_{m g_{hyp}}(z) = \frac{(m-1)\cdots(m-n)}{\pi^n}.$$

In this example we have that the metric $m\, g_{hyp}$ is balanced iff $m > n$. In the geometric quantization framework introduced in [3] the Kähler forms satisfying this property play a fundamental role for the quantization by deformation of the Kähler manifold (M, g). In our setting one says that a Kähler manifold (M, g) admits a *regular quantization* if the functions

$$\varepsilon_{mg}(z) = e^{-m\Phi(z)} K_{m\Phi}(z, z)$$

are positive constants (depending on m) for all sufficiently large positive integers.

Regarding regular quantizations we have the following lemma which will be an important ingredient in the proof of our main result, Theorem 7.

Lemma 3. *Let g be a Kähler metric on a complex manifold M. If (M, g) admits a regular quantization then the scalar curvature of the metric g is constant.*

Proof. See Theorem 5.3 in [1] for the compact case and Theorem 4.1 in [15] for the noncompact one. □

Hartogs domains have been considered in [8] and [15] in the framework of quantization of Kähler manifolds. In [5] is studied the existence of global symplectic coordinates on (D_F, ω_F) and [6] deals with the Riemannian geometry of (D_F, g_F). In [15] (see also [16]) these domains are studied from the scalar curvature viewpoint. The main results obtained in [16] and in [6] are summarized in the following two lemmata needed in the proof of Theorem 7 and its Corollary 8.

Lemma 4. *Let (D_F, g_F) be a n-dimensional Hartogs domain. Assume that its scalar curvatures is constant. Then (D_F, g_F) is holomorphically isometric to an open subset of the complex hyperbolic space $(\mathbb{C}\mathrm{H}^n, g_{hyp})$.*

Lemma 5. *A Hartogs domain (D_F, g_F) is geodesically complete if and only if*

$$\int_0^{\sqrt{x_0}} \sqrt{-\left(\frac{xF'}{F}\right)'}|_{x=u^2}\, du = +\infty, \tag{4}$$

where we define $\sqrt{x_0} = +\infty$ for $x_0 = +\infty$.

For the proof of Theorem 7 we need another result, Lemma 6 below, which is a straightforward generalization to dimension n of Propositions 3.12 and 3.14 proven by M. Engliš in [8]. In order to state it we set

$$c_k(F^m) = \int_0^{x_0} t^k F(t)^m G(t) dt, \tag{5}$$

where

$$G(t) = -\left(\frac{tF'}{F}\right)', \tag{6}$$

(notice that $G(t) > 0$ by (ii) in Proposition 2) and assume that there exists a real number γ such that for all positive integers m

$$\sum_{k=0}^{\infty} \frac{t^k}{c_k(F^m)} = (m - 1 + \gamma) F(t)^{-m}. \tag{7}$$

Many examples of Hartogs domains satisfy this condition (see [8, pp. 450-454]). Such domains admit a quantization by deformation (see [8] for details) and so they are also interesting from the physical point of view.

Let us also write the volume element corresponding to the metric ω_F by

$$\frac{\omega_F^n}{n!} = \frac{F^2(|z_0|^2)}{(F(|z_0|^2) - ||z||^2)^{n+1}} G(|z_0|^2) \frac{\omega_0^n}{n!}. \tag{8}$$

Lemma 6. *Let (D_F, g_F) be an Hartogs domain and let $\mathcal{H}_{m\Phi_F}$ be the corresponding weighted Hilbert space given by (1). Assume that condition (7) is satisfied for all positive integers m. Then $\mathcal{H}_{m\Phi_F} \neq \{0\}$ iff $m > n$ and its reproducing kernel is given by*

$$K_{m\Phi_F}(z, z) = \frac{(m-2)\cdots(m-n)}{\pi^n (F(|z_0|^2) - ||z||^2)^m} \left[m - 1 + (1 - w)\gamma\right],$$

where $w=\frac{\|z\|^2}{F(|z_0|^2)}$ *and* γ *is the real number appearing in (7).*

Proof. It is not hard to verify that the monomials $z_0^{j_0}z_1^{j_1}\cdots z_{n-1}^{j_{n-1}}$ are a complete orthogonal system for $\mathcal{H}_{m\Phi_F}$, for $m>n$. Hence, the well-known formula for reproducing kernels gives for the Hilbert space $\mathcal{H}_{m\Phi_F}$

$$K_{m\Phi_F}(z,z)=\sum_{j_0,\dots,j_{n-1}}\frac{|z_0|^{2j_0}\cdots|z_{n-1}|^{2j_{n-1}}}{\|z_0^{j_0}\cdots z_{n-1}^{j_{n-1}}\|_m^2},\tag{9}$$

where

$$\|z_0^{j_0}\cdots z_{n-1}^{j_{n-1}}\|_m^2=\int_{D_F}\left(F(|z_0|^2)-||z||^2\right)^m\prod_{k=0}^{n-1}|z_k|^{2j_k}\frac{\omega_F^n}{n!}.$$

By formula (8) the right hand side is equal to

$$\int_{D_F}\left(F(|z_0|^2)-||z||^2\right)^{m-n-1}\prod_{k=0}^{n-1}|z_k|^{2j_k}F^2(|z_0|^2)G(|z_0|^2)\frac{\omega_0^n}{n!},$$

which passing to polar coordinates reads

$$\pi^n\int_0^{x_0{}^{1/2}}\int_0^{F(r_0^2)^{1/2}}\cdots\int_0^{(F(r_0^2)-\sum_{i=2}^{n-1}r_i^2)^{1/2}}\left(F(r_0^2)-r^2\right)^{m-n-1}\prod_{k=0}^{n-1}r_k^{2j_k}F^2(r_0^2)G(r_0^2)2^n dr dr_0,$$

where $r^2=r_1^2+\cdots+r_{n-1}^2$, $dr=dr_1\cdots dr_{n-1}$. Making now the substitution $r_i^2=t_i$ and using again the short notation $t=t_1+\cdots+t_{n-1}$, $dt=dt_1\cdots dt_{n-1}$, we get

$$\pi^n\int_0^{x_0}\int_0^{F(t_0)}\cdots\int_0^{F(t_0)-\sum_{i=2}^{n-1}t_i}(F(t_0)-t)^{m-n-1}\prod_{k=0}^{n-1}t_k^{j_k}F^2(t_0)G(t_0)dt dt_0,$$

which substituting $t_k=w_kF(t_0)$ for $k=1,\dots,n-1$, becomes

$$\pi^n\int_0^{x_0}t_0^{j_0}F(t_0)^{m+j}G(t_0)dt_0\int_0^1\cdots\int_0^{1-\sum_{i=2}^{n-1}w_i}(1-w)^{m-n-1}\prod_{k=1}^{n-1}w_k^{j_k}dw,$$

where again $w=w_1+\cdots+w_{n-1}$, $dw=dw_1\cdots dw_{n-1}$. If $m\leq n$ the last integral diverges, so we can assume $m>n$. Therefore,

$$\|z_0^{j_0}\cdots z_{n-1}^{j_{n-1}}\|_m^2=\pi^n\frac{j_1!\cdots j_{n-1}!(m-n-1)!}{(m+j-2)!}c_{j_0}(F^{m+j}),\tag{10}$$

where $j=j_1+\cdots+j_{n-1}$ and $c_{j_0}(F^{m+j})$ is defined by (5). Thus

$$K_{m\Phi_F}(z,z)=\sum_{j_0,\dots,j_{n-1}}|z_0|^{2j_0}\cdots|z_{n-1}|^{2j_{n-1}}\frac{(m+j-2)!}{\pi^n j_1!\cdots j_{n-1}!(m-n-1)!}\left(c_{j_0}(F^{m+j})\right)^{-1}.$$

By (7) we can carry out the summation over j_0, getting

$$
\begin{aligned}
K_{m\Phi_F}(z,z) &= \sum_{j_1\cdots,j_{n-1}} |z_1|^{2j_1}\cdots|z_{n-1}|^{2j_{n-1}}\frac{(m+j-2)!(m+j-1+\gamma)}{\pi^n j_1!\cdots j_{n-1}!(m-n-1)!}F^{-m-j}(|z_0|^2)\\
&= \sum_{j_1\cdots,j_{n-1}} \frac{|z_1|^{2j_1}}{F^{j_1}(|z_0|^2)}\cdots\frac{|z_{n-1}|^{2j_{n-1}}}{F^{j_{n-1}}(|z_0|^2)}\frac{(m+j-2)!(m+j-1+\gamma)}{\pi^n j_1!\cdots j_{n-1}!(m-n-1)!}F^{-m}(|z_0|^2)\\
&= \sum_{j_1\cdots,j_{n-1}} w_1^{j_1}\cdots w_{n-1}^{j_{n-1}}\frac{(m+j-2)!(m+j-1+\gamma)}{\pi^n j_1!\cdots j_{n-1}!(m-n-1)!}F^{-m}(|z_0|^2)\\
&=\frac{(m-2)\cdots(m-n)}{\pi^n}\sum_{j_1\cdots,j_{n-1}}\frac{w_1^{j_1}}{j_1!}\cdots\frac{w_{n-1}^{j_{n-1}}}{j_{n-1}!}\left[\binom{m+j-1}{m-1}(m-1)+\right.\\
&\qquad\qquad\left.+\binom{m+j-2}{m-2}\gamma\right]F^{-m}(|z_0|^2)\\
&=\frac{(m-2)\cdots(m-n)}{\pi^n}\left[\frac{m-1}{(1-w)^m}+\frac{\gamma}{(1-w)^{m-1}}\right]F^{-m}(|z_0|^2)\\
&=\frac{(m-2)\cdots(m-n)}{\pi^n}\left[\frac{m-1}{(F(|z_0|^2)-||z||^2)^m}+\frac{(1-w)\gamma}{(F(|z_0|^2)-||z||^2)^m}\right]\\
&=\frac{(m-2)\cdots(m-n)}{\pi^n(F(|z_0|^2)-||z||^2)^m}\left[m-1+(1-w)\gamma\right].
\end{aligned}
$$

□

We can now state and prove our main result, which characterizes the hyperbolic space among Hartogs domains in terms of a balanced condition.

Theorem 7. *Let (D_F, g_F) be a n-dimensional Hartogs domain. Assume that condition (7) is satisfied for all positive integers m. If $m_0 g_F$ is balanced then $m_0 > n$ and (D_F, g_F) is holomorphically isometric to an open subset of the complex hyperbolic space $(\mathbb{C}\mathrm{H}^n, g_{hyp})$.*

Proof. Since by Lemma 6 $\mathcal{H}_{m\Phi_F} = \{0\}$ for $m_0 \leq n$, we can set $m_0 > n$. Assume that $m_0 g_F$ is balanced, namely $e^{m_0\Phi_F} = c_{m_0}K_{m_0\Phi_F}$, for some positive constant c_{m_0}. Therefore,

$$\left(F(|z_0|^2)-||z||^2\right)^{-m_0} = c_{m_0}K_{m_0\Phi_F}(z,z).$$

By Lemma 6 we get

$$\left(F(|z_0|^2)-||z||^2\right)^{-m_0} = c_{m_0}\frac{(m_0-2)\cdots(m_0-n)}{\pi^n(F(|z_0|^2)-||z||^2)^{m_0}}\left[m_0-1+(1-w)\gamma\right],$$

that is

$$\pi^n = c_{m_0}(m_0-2)\cdots(m_0-n)\left[m_0-1+(1-w)\gamma\right],$$

which yelds $\gamma = 0$, being $(1-w)\gamma$ the only term depending on the variables. Since γ is fixed for all m, it follows that the reproducing kernel of $\mathcal{H}_{m\Phi_F}$, for $m > n$, is given by

$$K_{m\Phi_F}(z,z) = \frac{(m-1)(m-2)\cdots(m-n)}{\pi^n(F(|z_0|^2) - ||z||^2)^m}.$$

By (2), we have

$$\varepsilon_{mg_F}(z) = K_{m\Phi_F}(z,z)\left(F(|z_0|^2) - ||z||^2\right)^m = \frac{(m-1)(m-2)\cdots(m-n)}{\pi^n}.$$

Hence, for all $m > n$, (D_F, g_F) admits a regular quantization. By Lemma 3 and Lemma 4 above, (D_F, g_F) is then holomorphically isometric to an open subset of the complex hyperbolic space. □

Combining Lemma 5 with Theorem 7 one gets:

Corollary 8. *Let (D_F, g_F) be an n-dimensional Hartogs domain. Assume that conditions (4) and (7) are satisfied (the latter for all positive integers m). If for some positive integer m_0, $m_0 g_F$ is a balanced metric then (D_F, g_F) is holomorphically isometric to the complex hyperbolic space $(\mathbb{C}\mathrm{H}^n, g_{hyp})$.*

Remark 9. A balanced metric g on a complex manifold M is projectively induced. Indeed, there exists a holomorphic map $f : M \to \mathbb{C}\mathrm{P}^\infty$, called the *coherent states map* in J. Rawnsley terminology [17], into the infinite dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$ such that $f^* g_{FS} = g$, where g_{FS} denotes the Fubini–Study metric on $\mathbb{C}\mathrm{P}^\infty$ (see [2] for details). Not all projectively induced metrics are balanced. Indeed, there exist n-dimensional Hartogs domains (D_F, g_F), $D_F \neq \mathbb{C}\mathrm{H}^n$, where $m_0 g_F$ is projectively induced for $m_0 > n$. An example is given by the so called *Springer domain* (D_F, g_F) corresponding to the function $F(x) = e^{-x}, x \in [0, +\infty)$ (see [12])). Moreover, it is not hard to verify that this domain satisfies condition (7) in Theorem 7 with $\gamma = 1$ (see also [8]). This shows that the condition that $m_0 g_F$ is balanced in Theorem 7 cannot be replaced by the weaker condition that $m_0 g_F$ is projectively induced.

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