THE DIASTATIC EXPONENTIAL OF A SYMMETRIC SPACE

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Abstract. Let (M, g) be a real analytic Kähler manifold. We say that a smooth map $\mathrm{Exp}_p : W \to M$ from a neighbourhood W of the origin of T_pM into M is a *diastatic exponential* at p if it satisfies

$$\left(d\,\mathrm{Exp}_p\right)_0 = \mathrm{id}_{T_pM},$$

$$D_p\left(\mathrm{Exp}_p(v)\right) = g_p(v, v)\,,\ \forall v \in W,$$

where D_p is Calabi's diastasis function at p (the usual exponential $\exp_p$ obviously satisfied these equations when D_p is replaced by the square of the geodesics distance from p). In this paper we prove that for every point p of an Hermitian symmetric space of noncompact type M there exists a globally defined diastatic exponential centered in p which is a diffeomorphism and it is uniquely determined by its restriction to polydisks. An analogous result holds true in an open dense neighbourhood of every point of M^*, the compact dual of M . We also provide a geometric interpretation of the symplectic duality map (recently introduced in [5]) in terms of diastatic exponentials. As a byproduct of our analysis we show that the symplectic duality map pulls back the reproducing kernel of M^* to the reproducing kernel of M.

Introduction and statements of the main results

Let M be a n-dimensional complex manifold endowed with a real analytic Kähler metric g. For a fixed point $p \in M$ let $D_p : U \to \mathbb{R}$ be the *Calabi diastasis function*, defined in the following way. Recall that a Kähler potential is an analytic function Φ defined in a neighborhood of a point p such that $\omega = \frac{i}{2}\partial\bar{\partial}\Phi$, where ω is the Kähler form associated to g. By duplicating the variables z and $\bar{z}$ a potential Φ can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighborhood U of the diagonal containing $(p, \bar{p}) \in M \times \bar{M}$ (here $\bar{M}$ denotes the manifold conjugated to M). The diastasis function is the Kähler potential D_p around p defined by

$$D_p(q) = \tilde{\Phi}(q, \bar{q}) + \tilde{\Phi}(p, \bar{p}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p})\,.$$

If $d_p : \exp_p(V) \subset M \to \mathbb{R}$ denotes the geodesic distance from p then one has:

$$D_p(q) = d_p(q)^2 + O\left(d_p(q)^4\right)$$

and $D_p = d_p^2$ if and only if g is the flat metric. We refer the reader to the seminal paper of Calabi [3] for more details and further results on the diastasis function (see also [8], [9] and [4]).

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In [9] it is proven that there exists an open neighbourhood S of the zero section of the tangent bundle TM of M and a smooth embedding $\nu : S \to TM$ such that $p \circ \nu = p$, where $p : TM \to M$ is the natural projection, satisfying the following conditions: if one writes

$$\nu\left(p, v\right) = \left(p, \nu_p\left(v\right)\right), \ (p, v) \in S$$

then the diffeomorphism

$$\nu_p : T_pM \cap S \to T_pM \cap \nu\left(S\right)$$

satisfies

$$\left(d\nu_p\right)_0 = \mathrm{id}_{T_pM}$$

$$D_p\left(\exp_p\left(\nu_p\left(v\right)\right)\right) = g_p\left(v, v\right), \ \forall v \in T_pM \cap S,$$

where $\exp_p : V \subset T_pM \to M$ denotes the exponential map at p (V is a suitable neighbourhood of the origin of T_pM where the restriction of $\exp_p$ is a diffeomorphism). Thus, the smooth map

$$\mathrm{Exp}_p := \exp_p \circ \nu_p : T_pM \cap S \to M$$

satisfies

$$\left(d\,\mathrm{Exp}_p\right)_0 = \mathrm{id}_{T_pM} \tag{1}$$

$$D_p\left(\mathrm{Exp}_p\left(v\right)\right) = g_p\left(v, v\right), \ \forall v \in W. \tag{2}$$

In analogy with the exponential at p (which satisfies $d_p\left(\exp_p\left(v\right)\right) = \sqrt{g_p\left(v, v\right)}$, $\forall v \in V$) any smooth map $\mathrm{Exp}_p : W \to M$ from a neighbourhood W of the origin of T_pM into M satisfying (1) and (2) will be called a *diastatic exponential* at p.

It is worth pointing out (see [2] for a proof) that $\exp_p$ is holomorphic if and only if the metric g is flat and it is not hard to see that the same assertion holds true for a diastatic exponential Exp_p.

In this paper we study the diastatic exponentials for the Hermitian symmetric spaces of noncompact type (HSSNT) and their compact duals. The following examples deal with the rank one case and it will be our prototypes for the general case.

Example 1. Let $\mathbb{C}H^n = \{z \in \mathbb{C}^n \mid |z|^2 = |z_1|^2 + \cdots + |z_n|^2 < 1\}$ be the complex hyperbolic space endowed with the hyperbolic metric, namely this metric g^{hyp} whose associated Kähler form is given by $\omega^{\mathrm{hyp}} = -\frac{i}{2}\partial\bar\partial \log\left(1 - |z|^2\right)$. Thus the diastasis function $D_0^{\mathrm{hyp}} : \mathbb{C}H^n \to \mathbb{R}$ and the exponential map $\exp_0^{\mathrm{hyp}} : T_0\mathbb{C}H^n \cong \mathbb{C}^n \to \mathbb{C}H^n$ around the origin $0 \in \mathbb{C}^n$ are given respectively by

$$D_0^{\mathrm{hyp}}\left(z\right) = -\log\left(1 - |z|^2\right)$$

and

$$\exp_0^{\mathrm{hyp}}\left(v\right) = \tanh\left(|v|\right)\frac{v}{|v|}, \quad \exp_0^{\mathrm{hyp}}\left(0\right) = 0.$$

It is then immediate to verify that the map $\mathrm{Exp}_0^{\mathrm{hyp}} : T_0\mathbb{C}H^n \to \mathbb{C}H^n$ given by:

$$\mathrm{Exp}_0^{\mathrm{hyp}}\left(v\right) = \sqrt{1 - e^{-|v|^2}}\,\frac{v}{|v|}, \quad \mathrm{Exp}_0^{\mathrm{hyp}}\left(0\right) = 0, \quad v = (v_1, \dots v_n)$$

satisfies $\left(d\,\mathrm{Exp}_0^{\mathrm{hyp}}\right)_0 = \mathrm{id}_{T_0\mathbb{C}H^n}$ and

$$D_0^{\mathrm{hyp}}\left(\mathrm{Exp}_0^{\mathrm{hyp}}(v)\right) = g_0^{\mathrm{hyp}}(v,v) = |v|^2, \quad \forall v \in T_0\mathbb{C}H^n = \mathbb{C}^n.$$

Hence $\mathrm{Exp}_0^{\mathrm{hyp}}$ is a diastatic exponential at 0. Notice that $\mathrm{Exp}_0^{\mathrm{hyp}}$ is characterized by the fact that it is direction preserving. More precisely, if $F : T_0\mathbb{C}H^n \to \mathbb{C}H^n$ is a diastatic exponential satisfying $F(v) = \lambda(v)\,v$, for some smooth nonnegative function $\lambda : \mathbb{C}^n \to \mathbb{R}$, then $F = \mathrm{Exp}_0^{\mathrm{hyp}}$.

Example 2. Let $P = \left(\mathbb{C}H^1\right)^\ell$ be a polydisk. If z_k, $k = 1,\dots,\ell$, denotes the complex coordinate in each factor of P and $v = (v_1,\dots,v_\ell) \in T_0P \cong \mathbb{C}^\ell$. Then the diastasis $D_0^P : P \to \mathbb{R}$, the exponential map $\exp_0^P : T_0P \to P$ and a diastatic exponential $\mathrm{Exp}_0^P : T_0P \to P$ at the origin are given respectively by:

$$D_0^P(z) = -\sum_{k=1}^{\ell} \log\left(1 - |z_k|^2\right),$$

$$\exp_0^P(v) = \left(\tanh(|v_1|)\frac{v_1}{|v_1|},\dots,\tanh(|v_\ell|)\frac{v_\ell}{|v_\ell|}\right), \quad \exp_0^{\mathrm{hyp}}(0) = 0,$$

$$\mathrm{Exp}_0^P(v) = \left(\sqrt{1-e^{-|v_1|^2}}\frac{v_1}{|v_1|},\dots,\sqrt{1-e^{-|v_\ell|^2}}\frac{v_\ell}{|v_\ell|}\right), \quad \mathrm{Exp}_0^P(0) = 0. \tag{3}$$

Let now M be an HSSNT which we identify with a bounded symmetric domain of $\mathbb{C}^n$ centered at the origin $0 \in \mathbb{C}^n$ equipped with the hyperbolic metric g^{hyp}, namely the Kähler metric whose associated Kähler form (in the irreducible case) is given by

$$\omega^{\mathrm{hyp}} = \frac{i}{2g}\partial\bar\partial \log K_M.$$

Here $K_M(z,\bar z)$ (holomorphic in the first variable and antiholomorphic in the second one) denotes the reproducing kernel of M and g its genus. By using the rotational symmetries of M one can show that the diastasis function at the origin $D_0^{\mathrm{hyp}} : M \to \mathbb{R}$ is globally defined and reads as

$$D_0^{\mathrm{hyp}}(z) = \frac{1}{g}\log K_M(z,\bar z),$$

(see [8] for a proof and further results on Calabi's function for HSSNT). Notice also that, by Hadamard theorem, the exponential map $\exp_0^{\mathrm{hyp}} : T_0M \to M$ is a global diffeomorphism.

The following theorem which is the first result of this paper, contains a description of the diastatic exponential for HSSNT.

Theorem 1. *Let $\left(M, g^{\mathrm{hyp}}\right)$ be an HSSNT. Then there exists a globally defined diastatic exponential $\mathrm{Exp}_0^{\mathrm{hyp}} : T_0M \to M$ which is a diffeomorphism and is uniquely determined by the fact that ${\mathrm{Exp}_0^{\mathrm{hyp}}}_{|T_0P} = \mathrm{Exp}_0^P$ for every polydisk $P \subset M$, $0 \in P$, where Exp_0^P is given by (3). In particular ${\mathrm{Exp}_0^{\mathrm{hyp}}}_{|T_0N} = \mathrm{Exp}_0^N$ for every complex and totally geodesic submanifold $N \subset M$ through 0.*

Consider now the Hermitian symmetric spaces of compact type (HSSCT). Let us consider first the compact duals of Examples 1 and 2.

Example 3. Let $\mathbb{C}P^n$ be the complex projective space endowed with the Fubini–Study metric g^{FS}, namely the metric whose associated Kähler form is given by

$$\omega^{FS} = \frac{i}{2}\partial\bar{\partial}\log\left(|Z_0|^2 + \cdots + |Z_n|^2\right)$$

for a choice of homogeneous coordinates $Z_0, \ldots, Z_n$. Let $p_0 = [1, 0, \ldots, 0]$ and consider the affine chart $U_0 = \{Z_0 \neq 0\}$. Thus we have the following inclusions

$$\mathbb{C}H^n \subset \mathbb{C}^n \cong U_0 \subset \mathbb{C}P^n, \tag{4}$$

where we are identifying U_0 with $\mathbb{C}^n$ via the affine coordinates

$$U_0 \to \mathbb{C}^n : [Z_0, \ldots, Z_N] \mapsto \left(z_1 = \frac{Z_1}{Z_0}, \ldots, z_n = \frac{Z_n}{Z_0}\right).$$

Under this identification we make no distinction between the point p_0 and the origin $0 \in \mathbb{C}^n$. Calabi's diastasis function $D_0^{FS} : U_0 \to \mathbb{R}$ around $p_0 \equiv 0$ is given by

$$D_0^{FS}(z) = \log\left(1 + |z|^2\right).$$

Observe that D_0^{FS} blows up at the points belonging to $\mathbb{C}P^n \setminus U_0$ which is the cut locus of p_0 with respect to the Fubini–Study metric. We denote this set by $\mathrm{Cut}_0(\mathbb{C}P^n)$.

It is not hard to verify that the map

$$\mathrm{Exp}_0^{FS} : T_0\mathbb{C}P^n \to \mathbb{C}P^n \setminus \mathrm{Cut}_0(\mathbb{C}P^n)$$

given by

$$\mathrm{Exp}_0^{FS}(v) = \sqrt{e^{|v|^2} - 1}\,\frac{v}{|v|}, \quad \mathrm{Exp}_0^{FS}(0) = 0,$$

is a diastatic exponential at 0, namely it satisfies $\left(d\,\mathrm{Exp}_0^{FS}\right)_0 = \mathrm{id}_{T_0\mathbb{C}P^n}$ and

$$D_0^{FS}\left(\mathrm{Exp}_0^{FS}(v)\right) = g_0^{FS}(v, v) = |v|^2, \quad \forall v \in T_0\mathbb{C}P^n.$$

Example 4. Let $P^* = \left(\mathbb{C}P^1\right)^\ell$ be a (dual) polydisk. If z_k, for $k = 1, \ldots, \ell$, denotes the affine coordinate in each factor of P^* and $v = (v_1, \ldots, v_\ell) \in T_0M^* \cong \mathbb{C}^\ell$ then it is immediate to see that the diastasis $D_0^{P^*} : P^* \to \mathbb{R}$, the exponential map $\exp_0^{P^*} : T_0P^* \to P^*$ and a diastatic exponential $\mathrm{Exp}_0^{P^*} : T_0P^* \to P^*$ at the origin are given respectively by:

$$D_0^{P^*}(z) = \sum_{k=1}^{\ell} \log\left(1 + |z_k|^2\right),$$

$$\exp_0^{P^*}(v) = \left(\tan(|v_1|)\,\frac{v_1}{|v_1|}, \ldots, \tan(|v_\ell|)\,\frac{v_\ell}{|v_\ell|}\right), \quad \exp_0(0) = 0,$$

$$\mathrm{Exp}_0^{P^*}(v) = \left(\sqrt{e^{|v_1|^2} - 1}\,\frac{v_1}{|v_1|}, \ldots, \sqrt{e^{|v_\ell|^2} - 1}\,\frac{v_\ell}{|v_\ell|}\right), \quad \mathrm{Exp}_0^{P^*}(0) = 0. \tag{5}$$

Given an arbitrary HSSNT M of genus g let denote by M^* its compact dual equipped with the Fubini–Study metric g^{FS}, namely the pull-back of the Fubini–Study metric of $\mathbb{C}P^N$ via the Borel–Weil embedding $M^* \to \mathbb{C}P^N$ (see [5] for details). Let $0 \in M^*$ be a fixed point and denote by $\mathrm{Cut}_0(M^*)$ the cut locus of 0 with respect to the Fubini–Study metric. In the irreducible case the Kähler form ω^{FS} associated to g^{FS} is given (in the *affine* chart $M^* \setminus \mathrm{Cut}_0(M^*)$) by

$$\omega^{FS} = \frac{i}{2g}\partial\bar{\partial}\log K_{M^*},$$

where

$$K_{M^*}(z,\bar{z}) = 1/K_M(z,-\bar{z}). \tag{6}$$

We call K_{M^*} the *reproducing kernel* of M^*. Notice that K_{M^*} is the weighted Bergman kernel for the (finite dimensional) complex Hilbert space consisting of holomorphic functions f on $M^*\setminus\mathrm{Cut}_0(M) \subset M^*$ such that $\int_{M^*\setminus\mathrm{Cut}_0(M)} |f|^2 \left(\omega^{FS}\right)^n < \infty$ (see [8] and also [7] for a nice characterization of symmetric spaces in terms of K_{M^*}). Notice that when $M = \mathbb{C}H^n$ then $g = n+1$, $K_M(z,\bar{z}) = \left(1-|z|^2\right)^{-(n+1)}$, $K_{M^*}(z,\bar{z}) = \left(1+|z|^2\right)^{n+1}$ and the Borel–Weil embedding is the identity of $\mathbb{C}P^n$.

Observe that, as in the previous examples, D_0^{FS} is globally defined in $M^* \setminus \mathrm{Cut}_0(M^*)$ (see [17] for a proof) and it blows up at the points in $\mathrm{Cut}_0(M^*)$. Moreover

$$D_0^{FS}(z) = \frac{1}{g}\log K_{M^*}(z,\bar{z}), \quad z \in M^* \setminus \mathrm{Cut}_0(M^*).$$

Furthermore (see e.g. [18]) $M^* \setminus \mathrm{Cut}_0(M^*)$ is globally biholomorphic to T_0M and if 0 denote the origin of M one has the following inclusions (analogous of (4))

$$M \subset T_0M = T_0M^* \cong M^* \setminus \mathrm{Cut}_0(M^*) \subset M^*. \tag{7}$$

We are now in the position to state our second result which is the dual counterpart of Theorem 1.

Theorem 2. *Let $\left(M^*, g^{FS}\right)$ be an HSSCT. Then there exists a globally defined diastatic exponential $\mathrm{Exp}_0^{FS} : T_0M^* \to M^* \setminus \mathrm{Cut}_0(M^*)$ which is uniquely determined by the fact that for every (dual) polydisk $P^* = \left(\mathbb{C}P^1\right)^s \subset M^*$ its restriction to T_0P^* equals the map $\mathrm{Exp}_0^{P^*}$ given by (5). In particular $\mathrm{Exp}_0^{FS}{}_{|T_0N^*} = \mathrm{Exp}_0^{N^*}$ for every complex and totally geodesic submanifold $N^* \subset M^*$ through 0.*

The key ingredient for the proof of Theorem 1 and Theorem 2 is the theory of Hermitian positive Jordan triple systems (HPJTS). In [5] this theory has been the main tool to study the link between the symplectic geometry of an Hermitian symmetric space $\left(M, \omega^{\mathrm{hyp}}\right)$ and its dual $\left(M^*, \omega^{FS}\right)$ where ω^{hyp} (resp. ω^{FS}) is the Kähler form associated to g^{hyp} (resp. g^{FS}). The main result proved there, is the following theorem (an alternative proof of this theorem can be found in [6]).

Theorem 3. *Let M be an HSSNT and $B(z,w)$ its associated Bergman operator (see next section). Then the map*

$$\Psi_M : M \to M^* \setminus \mathrm{Cut}_0(M^*), \quad z \mapsto B(z,z)^{-\frac{1}{4}} z, \tag{8}$$

called the symplectic duality map*, is a real analytic diffeomorphism satisfying*

$$\Psi_M^* \omega_0 = \omega^{\mathrm{hyp}}$$

and

$$\Psi_M^* \omega^{FS} = \omega_0,$$

where ω_0 *is the flat Kähler form on* $T_0 M$*. Moreover, for every complex and totally geodesic submanifold* $N \subset M$ *one has* $\Psi_{M|N} = \Psi_N$*.*

Here ω_0 denotes the Kähler form on M obtained by the restriction of the flat Kähler form on $T_0 M = \mathbb{C}^n$.

The following theorem which represents our third result provides a geometric interpretation of the symplectic duality map in terms of diastatic exponentials.

Theorem 4. *Let* M *be a HSSNT and* M^* *be its compact dual. Then the symplectic duality map can be written as*

$$\Psi_M = \mathrm{Exp}_0^{FS} \circ \left(\mathrm{Exp}_0^{\mathrm{hyp}}\right)^{-1} : M \to M^* \setminus \mathrm{Cut}_0\left(M^*\right),$$

where $\mathrm{Exp}_0^{\mathrm{hyp}} : T_0 M \to M$ *and* $\mathrm{Exp}_0^{FS} : T_0 M^* \to M^* \setminus \mathrm{Cut}_0\left(M^*\right)$ *are the diastatic exponentials at* 0 *of* M *and* M^* *respectively.*

Our fourth (and last) result is the following theorem which shows that the "algebraic manipulation" (6) which allows us to pass from K_M to K_{M^*} can be realized via the symplectic duality map.

Theorem 5. *Let* K_M *be the reproducing kernel for an HSSNT and let* K_M^* *be its dual. Then*

$$K_{M^*} \circ \Psi_M = K_M,$$

where $\Psi_M : M \to M^* \setminus \mathrm{Cut}_0\left(M^*\right)$ *is the symplectic duality map.*

The paper contains another section, where, after recalling some standard facts about HSSNT and HPJTS, we prove Theorem 1, Theorem 2, Theorem 4 and Theorem 5.

1. HPJTS and the proofs of the main results

We refer the reader to [15] (see also [14]) for more details of the material on Hermitian positive Jordan triple systems.

1.1. **Definitions and notations.** An Hermitian Jordan triple system is a pair $(\mathcal{M}, \{,,\})$, where $\mathcal{M}$ is a complex vector space and $\{,,\}$ is a map

$$\{,,\} : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$$
$$(u, v, w) \mapsto \{u, v, w\}$$

which is $\mathbb{C}$-bilinear and symmetric in u and w, $\mathbb{C}$-antilinear in v and such that the following Jordan identity holds:

$$\{x, y, \{u, v, w\}\} - \{u, v, \{x, y, w\}\} = \{\{x, y, u\}, v, w\} - \{u, \{v, x, y\}, w\}.$$

For $x, y, z \in \mathcal{M}$ considered the following operator

$$T\left(x, y\right) z = \{x, y, z\}$$
$$Q\left(x, z\right) y = \{x, y, z\}$$
$$Q\left(x, x\right) = 2Q\left(x\right)$$
$$B\left(x, y\right) = \mathrm{id}_{\mathcal{M}} - T\left(x, y\right) + Q\left(x\right) Q\left(y\right).$$

The operators $B(x,y)$ and $T(x,y)$ are $\mathbb{C}$-linear, the operator $Q(x)$ is $\mathbb{C}$-antilinear. $B(x,y)$ is called the *Bergman operator*. For $z \in V$, the *odd powers* $z^{(2p+1)}$ of z in the Jordan triple system V are defined by

$$z^{(1)} = z \qquad z^{(2p+1)} = Q(z)\, z^{(2p-1)}.$$

An Hermitian Jordan triple system is called *p*ositive if the Hermitian form

$$(u \mid v) = \operatorname{tr} T(u,v)$$

is positive definite. An element $c \in \mathcal{M}$ is called *tripotent* if $\{c,c,c\} = 2c$. Two tripotents c_1 and c_2 are called *(strongly) orthogonal* if $T(c_1, c_2) = 0$.

1.2. **HSSNT associated to HPJTS.** M. Koecher ([12], [13]) discovered that to every HPJTS $(\mathcal{M}, \{,,\})$ one can associate an Hermitian symmetric space of noncompact type, i.e. a bounded symmetric domain M centered at the origin $0 \in \mathcal{M}$. The domain M is defined as the connected component containing the origin of the set of all $u \in \mathcal{M}$ such that $B(u,u)$ is positive definite with respect to the Hermitian form $(u,v) \mapsto \operatorname{tr} T(u,v)$. *We will always consider such a domain in its (unique up to linear isomorphism) circled realization.* The reproducing kernel K_M of M is given by

$$K_M(z, \bar{z}) = \det B(z,z) \tag{9}$$

and so when M is irreducible

$$\omega^{\mathrm{hyp}} = -\frac{i}{2g} \partial\bar{\partial} \log \det B.$$

The HPJTS $(\mathcal{M}, \{,,\})$ can be recovered by its associated HSSNT M by defining $\mathcal{M} = T_0 M$ (the tangent space to the origin of M) and

$$\{u,v,w\} = -\frac{1}{2}\left(R_0(u,v)\, w + J_0 R_0(u, J_0 v)\, w\right), \tag{10}$$

where R_0 (resp. J_0) is the curvature tensor of the Bergman metric (resp. the complex structure) of M evaluated at the origin. The reader is referred to Proposition III.2.7 in [1] for the proof of (10). For more informations on the correspondence between $HPJTS$ and $HSSNT$ we refer also to p. 85 in Satake's book [16].

1.3. **Totally geodesic submanifolds of HSSNT.** In the proof of our theorems we need the following result.

Proposition 5. *Let M be a HSSNT and let $\mathcal{M}$ be its associated $HPJTS$. Then there exists a one to one correspondence between (complete) complex totally geodesic submanifolds through the origin and sub-$HPJTS$ of $\mathcal{M}$. This correspondence sends $T \subset M$ to $\mathcal{T} \subset \mathcal{M}$, where $\mathcal{T}$ denotes the HPJTS associated to T.*

1.4. **Spectral decomposition and Functional calculus.** Let $\mathcal{M}$ be a HPJTS. Each element $z \in \mathcal{M}$ has a unique *spectral decomposition*

$$z = \lambda_1 c_1 + \cdots + \lambda_s c_s \qquad (0 < \lambda_1 < \cdots < \lambda_s),$$

where $(c_1, \ldots, c_s)$ is a sequence of pairwise orthogonal tripotents and the λ_j are real number called eigenvalues of z. For every $z \in \mathcal{M}$ let $\max\{z\}$ denote the largest eigenvalue of z, then $\max\{\cdot\}$ is a norm on $\mathcal{M}$ called the *spectral norm.* The HSSNT

M associated to $\mathcal{M}$ is the open unit ball in $\mathcal{M}$ centered at the origin (with respect the spectral norm M), i.e.,

$$M = \{z = \sum_{j=1}^{s} \lambda_j c_j \mid \max\{z\} = \max_j\{\lambda_j\} < 1\} \tag{11}$$

Using the spectral decomposition, it is possible to associate to an *odd* function $f : \mathbb{R} \to \mathbb{C}$ a map $F : \mathcal{M} \to \mathcal{M}$ as follows. Let $z \in \mathcal{M}$ and let

$$z = \lambda_1 c_1 + \cdots + \lambda_s c_s, \quad 0 < \lambda_1 < \cdots < \lambda_s$$

be the spectral decomposition of z. Define the map F by

$$F(z) = f(\lambda_1) c_1 + \cdots + f(\lambda_s) c_s. \tag{12}$$

If f is continuous, then F is continuous. If

$$f(t) = \sum_{k=0}^{N} a_k t^{2k+1}$$

is a polynomial, then F is the map defined by

$$F(z) = \sum_{k=0}^{N} a_k z^{(2k+1)} \qquad (z \in \mathcal{M}).$$

If f is analytic, then F is real-analytic. If f is given near 0 by

$$f(t) = \sum_{k=0}^{\infty} a_k t^{2k+1},$$

then F has the Taylor expansion near $0 \in V$:

$$F(z) = \sum_{k=0}^{\infty} a_k z^{(2k+1)}.$$

Example 6. Let $P = \left(\mathbb{C}H^1\right)^\ell \subset \left(\mathbb{C}^\ell, \{,,\}\right)$ be the polydisk embedded in is its associated HPJTS $\left(\mathbb{C}^\ell, \{,,\}\right)$. Define $\tilde{c}_j = \left(0, \ldots, 0, e^{i\theta_j}, 0, \ldots, 0\right)$, $1 \leq j \leq \ell$. The $\tilde{c}_j$ are mutually strongly orthogonal tripotents. Given $z = \left(\rho_1 e^{i\theta_1}, \ldots, \rho_\ell e^{i\theta_\ell}\right) \in \left(\mathbb{C}H^1\right)^\ell$, $z \neq 0$, then up to a permutation of the coordinates, we can assume $0 \leq \rho_1 \leq \rho_2 \leq \cdots \leq \rho_\ell$. Let i_1, $1 \leq i_1 \leq \ell$, the first index such that $\rho_{i_1} \neq 0$ then we can write

$$z = \rho_{i_1}\left(\tilde{c}_{i_1} + \cdots + \tilde{c}_{i_2-1}\right) + \rho_{i_2}\left(\tilde{c}_{i_2} + \cdots + \tilde{c}_{i_3-1}\right) + \cdots + \rho_{i_s}\left(\tilde{c}_{i_s} + \cdots + \tilde{c}_{i_{s+1}-1}\right)$$

with $0 < \rho_{i_1} < \rho_{i_2} < \cdots < \rho_{i_s} = \rho_\ell$ and $i_{s+1} = \ell + 1$. The c_j's, defined by $c_j = \tilde{c}_{i_j} + \cdots + \tilde{c}_{i_{j+1}-1}$, are still mutually strongly orthogonal tripotents and $z = \lambda_1 c_1 + \cdots + \lambda_s c_s$ with $\lambda_j = \rho_{i_j}$, is the spectral decomposition of z. So the diastatic exponential given in (3) can be written as

$$\mathrm{Exp}_0^P(z) = \left(\sqrt{1 - e^{-|z_1|^2}}\frac{z_1}{|z_1|}, \ldots, \sqrt{1 - e^{-|z_\ell|^2}}\frac{z_\ell}{|z_\ell|}\right) = \sum_{j=1}^{s}\left(1 - e^{-\lambda_j^2}\right)^{\frac{1}{2}} c_j$$

and $\mathrm{Exp}_0^P(0) = 0$.

We are now in the position to prove our main results. In all the following proofs we can assume, without loss of generality, that M is irreducible. Indeed, in the reducible case the Bergman operator is the product of the Bergman operator of each factor and therefore the same holds true for the diastatic exponential and for the symplectic duality map.

1.5. **Proof of Theorem 1.** Consider the odd smooth function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(t)=\left(1-e^{-t^2}\right)^{\frac{1}{2}}\frac{t}{|t|},\ f(0)=0$$

and the map $F:T_0M\to M\subset T_0M$ associated to f by (12), namely

$$F(z)=\sum_{j=1}^{s}\left(1-e^{-\lambda_j^2}\right)^{\frac{1}{2}}c_j, \tag{13}$$

where $z=\lambda_1c_1+\cdots+\lambda_sc_s$ is the spectral decomposition of $z\in T_0M$. Note that, by (11), $F(T_0M)\subset M$ and (13) is indeed the spectral decomposition of $F(z)$. We will show that $\mathrm{Exp}_0^{\mathrm{hyp}}:=F$ is a diastatic exponential at the origin for M satisfying the conditions of Theorem 1. First,

$$\left(d\,\mathrm{Exp}_0^{\mathrm{hyp}}\right)_0(v)=\lim_{r\to0^+}\frac{d}{dr}\sum_{j=1}^{s}\left(1-e^{-(r\mu_j)^2}\right)^{\frac{1}{2}}d_j=\sum_{j=1}^{s}\mu_jd_j=v,$$

where $v=\mu_1d_1+\cdots+\mu_sd_s$ is the spectral decomposition of $v\in T_0M$. Hence $\mathrm{Exp}_0^{\mathrm{hyp}}$ is a diastatic exponential if one shows that $D_0^{\mathrm{hyp}}\left(\mathrm{Exp}_0^{\mathrm{hyp}}(z)\right)=g_0^{\mathrm{hyp}}(z,z)$. In order to prove this equality observe that (see [15] for a proof)

$$B(z,z)\,c_j=\left(1-\lambda_j^2\right)^2c_j,\quad j=1,\ldots,s, \tag{14}$$

$$\det B(z,z)=\prod_{j=1}^{s}\left(1-\lambda_j^2\right)^g,$$

$$g_0^{\mathrm{hyp}}(z,z)=\frac{1}{g}\,\mathrm{tr}\,T(z,z)=\sum_{j=1}^{s}\lambda_j^2.$$

Thus (9) yields,

$$D_0^{\mathrm{hyp}}(z)=-\frac{1}{g}\log\det B(z,z)=-\log\prod_{j=1}^{s}\left(1-\lambda_j^2\right) \tag{15}$$

and so

$$D_0^{\mathrm{hyp}}\left(\mathrm{Exp}_0^{\mathrm{hyp}}(z)\right)=-\log\prod_{j=1}^{s}\left[1-\left(1-e^{-\lambda_j^2}\right)\right]=\sum_{j=1}^{s}\lambda_j^2=g_0^{\mathrm{hyp}}(z,z),$$

namely the desired equality. Moreover, the map $G:M\subset T_0M\to T_0M$ induced by the odd smooth function

$$g(t)=\left(-\log(1-t^2)\right)^{\frac{1}{2}}\frac{t}{|t|},\ g(0)=0,$$

namely

$$G(z)=\sum_{j=1}^{s}\left(-\log(1-\lambda^2)\right)^{\frac{1}{2}}c_j,$$

is the inverse of $\mathrm{Exp}_0^{\mathrm{hyp}}$ and so $\mathrm{Exp}_0^{\mathrm{hyp}}:T_0M\to M$ is a diffeomorphism.

In order to prove the second part of the theorem let $P\subset M$ be a polydisk through the origin. Thus equality ${\mathrm{Exp}_0^{\mathrm{hyp}}}_{|T_0P}=\mathrm{Exp}_0^P$ follows by Proposition 5, Example 6 and formula (13). Moreover $\mathrm{Exp}_0^{\mathrm{hyp}}$ is determined by its restriction to polydisks since it is well-known that $\forall z\in T_0M$ there exists a polydisk $P\subset M$ such that $0\in P$ and $z\in T_0P$ (see, e.g. [11] and also [10]).

1.6. **Proof of Theorem 2.** Let $z=\lambda_1c_1+\cdots+\lambda_sc_s$ be a spectral decomposition of $z\in M^*\setminus\mathrm{Cut}_0\left(M^*\right)\cong T_0M$. In analogy with the compact case one has

$$B\left(z,-z\right)c_j=\left(1+\lambda_j^2\right)^2c_j$$

$$\det B\left(z,-z\right)=\prod_{j=1}^{s}\left(1+\lambda_j^2\right)^g.$$

$$g_0^{FS}\left(z,z\right)=\lambda_j^2.$$

Thus, by (6), Calabi's diastasis function at the origin for g^{FS} is given by:

$$\begin{aligned}D_0^{FS}\left(z\right)&=-\frac{1}{g}\log K_{M^*}\left(z,\bar z\right)=\frac{1}{g}\log[K_M\left(z,-\bar z\right)]=\frac{1}{g}\log[\det B\left(z,-z\right)]\\&=\frac{1}{g}\log\prod_{j=1}^{s}\left(1+\lambda_j^2\right)\end{aligned}\tag{16}$$

Define $\mathrm{Exp}_0^{FS}:T_0M^*\cong T_0M\to M^*\setminus\mathrm{Cut}_0\left(M^*\right)\cong T_0M$ as the map associated to the real function $f^*\left(t\right)=\left(e^{t^2}-1\right)^{\frac{1}{2}}\frac{t}{|t|}$ by (12), namely

$$\mathrm{Exp}_0^{FS}\left(z\right)=\sum_{j=1}^{s}\left(e^{\lambda_j^2}-1\right)^{\frac{1}{2}}c_j.\tag{17}$$

Thus, following the same line of the proof of Theorem 1, one can show that Exp_0^{FS} is the diastatic exponential at 0 uniquely determined by its restriction to polydisks.

1.7. **Proof of Theorem 4.** By (8) and (14)

$$\Psi_M\left(z\right)=B\left(z,\bar z\right)^{-\frac{1}{4}}\left(z\right)=\frac{\lambda_j}{\left(1-\lambda_j^2\right)^{\frac{1}{2}}}c_j\tag{18}$$

By the very definition of the diastatic exponential $\mathrm{Exp}_0^{\mathrm{hyp}}$ for the hyperbolic metric its inverse $\left(\mathrm{Exp}_0^{\mathrm{hyp}}\right)^{-1}:M\to T_0M$ read as:

$$\left(\mathrm{Exp}_0^{\mathrm{hyp}}\right)^{-1}\left(z\right)=\sum_{j=1}^{s}\left(-\log\left(1-\lambda_j^2\right)\right)^{\frac{1}{2}}c_j,$$

Then, by (17) and (18),

$$\mathrm{Exp}_0^{FS} \circ \left(\mathrm{Exp}_0^{\mathrm{hyp}}\right)^{-1}(z) = \Psi_M(z)$$

and this concludes the proof of Theorem 4.

1.8. **Proof of Theorem 5.** Since $D_0^{\mathrm{hyp}} = \frac{1}{g}\log K_M$ and $D_0^{FS} = \frac{1}{g}\log K_{M^*}$, equation $K_{M^*} \circ \Psi_M = K_M$ is equivalent to $D_0^{FS} \circ \Psi_M = D_0^{\mathrm{hyp}}$ which is a straightforward consequence of (15), (16) and (18).

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