Moment maps, scalar curvature and quantization of Kähler manifolds

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Abstract

Building on Donaldson's work on constant scalar curvature metrics, we study the space of regular Kähler metrics $\mathcal{E}_\omega$, i.e. those for which deformation quantization has been defined by Cahen, Gutt and Rawnsley. After giving, in Section 2 and 3 a review of Donaldson's moment map approach, we study the "essential" uniqueness of balanced basis (i.e. of coherent states) in a more general setting (Theorem 2.5). We then study the space $\mathcal{E}_\omega$ in Section 4 and we show in Section 5 how all the tools needed can be defined also in the case of non-compact manifolds.
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1 Introduction

Let (M, ω) be a polarized Kähler manifold with polarization L, namely a compact Kähler manifold endowed with a holomorphic line bundle L whose first Chern class $c_1(L) = [\omega]_{dR}$. One can define two natural subspaces of $\mathcal{C}_\omega$, the set of Kähler forms on M cohomologous to ω. The first one, denoted by $\mathcal{S}_\omega$, is the space of constant scalar curvature metrics cohomologous to ω and the second one is the space $\mathcal{E}_\omega$ consisting of Kähler forms cohomologous to ω such that their Tian's function is constant (see formula (15) for its definition, compare also Sect. 4).

While $\mathcal{S}_\omega$ has an obvious geometric interest, $\mathcal{E}_\omega$ has been proved of great importance in the theory of quantization. Indeed, for a Kähler manifold with

$\mathcal{E}_\omega$ non-empty Cahen, Gutt and Rawnsley [3] have shown how to generalize Berezin's quantization procedure.
In [9] Donaldson, using the concept of balanced basis of $H^0(L)$ studies the interplay between the two spaces above. Under the hypothesis that $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ is discrete he shows that there exists at most one Kähler metric in $\mathcal{S}_\omega$ and in $\mathcal{E}_\omega$ (see Theorem 2.1 and Corollary 2.3 below).

Moreover, he shows (see Theorem 2.2 below) that if $\mathcal{S}_\omega$ is non empty then L^m, the m-th tensor power of L, is stable, for m sufficiently large. Since Kähler-Einstein metrics have constant scalar curvature, Donaldson's result confirms in one direction the well-known conjecture of Yau [24] which asserts that the existence of a Kähler-Einstein metric is equivalent to the stability in the sense of geometric invariant theory. Additional evidence for Yau's conjecture had been provided earlier by Tian (see [21], [22] and [23]), who showed that the existence of constant scalar curvature metrics implies K-stability and CM-stability.
The aim of this paper is two fold. First, we study the space $\mathcal{E}_\omega$ without Donaldson's assumption on $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$. Our main result is Theorem 2.5 below where we generalize Donaldson's result on balanced bases. As a consequence (cfr. Theorem 4.1 below) we have that two cohomologous Kähler forms in $\mathcal{E}_\omega$ belong to the same orbit under the action of the group of biholomorphisms of M which lift to the line bundle L. In Theorem 4.3, we also give a description of the space $\mathcal{E}_\omega$ in the case of compact coadjoint orbits. Second, by using the quantization tools developed by Cahen, Gutt and Rawnsley we show that all the spaces which come in this study can be defined also for non compact manifolds and we explicitly compute the Tian's function in some cases. We end this paper discussing the difficulties we run into trying to study the geometry of the space $\mathcal{E}_\omega$ in this situation.
The paper is organized as follows. In Section 2 we describe Donaldson's results about balanced bases and we state our main Theorem 2.5 which is proved in Section 3. Section 4 is devoted to the study of the space $\mathcal{E}_\omega$. Finally, in Section 5, we treat the non compact case.

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2 Preliminaries and statements of the main results

Consider the complex projective space $\mathbb{C}P^N$ with standard homogeneous coordinates $[z_0, \dots, z_N]$ and the matrix valued function on $\mathbb{C}P^N$ given by:

$$B_{jk} = \frac{z_j \overline{z_k}}{\sum_l |z_l|^2}, \ \ j, k = 0, \dots, N.$$

Let $V \subset \mathbb{C}P^N$ be a projective variety. We define $M(V)$ to be the skew-adjoint $(N+1) \times (N+1)$ matrix with entries

$$M(V)_{jk} = i \int_V B_{jk} d\mu_V, \ \ j, k = 0, \dots N, \tag{1}$$

where $d\mu_V$ is the standard measure on V induced by the Fubini–Study form Ω_{FS} on $\mathbb{C}P^N$, namely $d\mu_V = \frac{\Omega_{FS}^n}{n!}|_V$, where n is the complex dimension of V. Recall that

$$\Omega_{FS} = \frac{i}{2\pi} \partial\bar{\partial} \log \sum_{l=0}^{N} |z_l|^2 \tag{2}$$

for the homogeneous coordinate system $[z_0, \dots, z_N]$ in $\mathbb{C}P^N$.

Donaldson christened a projective variety $V \subset \mathbb{C}P^N$ *balanced* if $M(V)$ is a multiple of the identity matrix. By formula (1) this multiple is a purely imaginary number $i\lambda$ with $\lambda > 0$. Hence we deduce that $V \subset \mathbb{C}P^N$ is balanced iff there exists a positive real number λ such that

$$\int_V B_{jk} d\mu_V = \lambda \delta_{jk}, \ \ j, k = 0, \dots N. \tag{3}$$

The relevance of this condition in the algebro-geometric context is given by Luo's Theorem [17] who proved that a balanced variety is stable in the sense of Hilbert–Mumford (the converse is still an open question).

The previous definition can be extended to the case of polarized manifolds. So, let (M, L) be a polarized manifold with polarization L. This means that M is a compact complex n-dimensional manifold endowed with a holomorphic line bundle L whose first Chern class $c_1(L)$ can be represented by a Kähler form ω on M, namely $c_1(L) = [\omega]_{dR}$. The line bundle L is called a polarization of M. For a positive integer m, let $L^m = L^{\otimes m}$ be the m-th tensor power of L and denote by $H^0(L^m)$ the space of holomorphic sections of L^m. Kodaira embeddding theorem asserts that for m sufficiently large the holomorphic sections of L^m define a projective embedding

$$i_m : M \to \mathbb{P}(H^0(L^m)^*). \tag{4}$$

A choice of a basis $(s_0^m, \ldots, s_{d_m}^m)$ in $H^0(L^m)$ identifies $\mathbb{P}(H^0(L^m)^*)$ with the standard $\mathbb{C}P^{d_m}$, where $d_m = \dim H^0(L^m) - 1$ is given with the help of Riemann–Roch formula. We say that the pair (M, L^m) is balanced if one can choose a basis in $H^0(L^m)$ such that $V = i_m(M)$ is a balanced variety in $\mathbb{C}P^{d_m}$. In this case we call this basis a *balanced basis* of $H^0(L^m)$. In order to see how this definition depends on the basis $(s_0^m, \ldots, s_{d_m-1}^m)$ we write the Kodaira embedding (4) more explicitly. Thus, let $\sigma : U \to L^m$ be a trivializing holomorphic section of L^m on the open set $U \subset M$ and define the map

$$i_\sigma : U \to \mathbb{C}^{d_m+1} \setminus \{0\} : x \mapsto \left(\frac{s_0^m(x)}{\sigma(x)}, \ldots, \frac{s_{d_m}^m(x)}{\sigma(x)}\right). \tag{5}$$

If $\tau : V \to L^m$ is another holomorphic trivialization then there exists a non-vanishing holomorphic function f on $U \cap V$ such that $\sigma(x) = f(x)\tau(x)$. Then, the Kodaira embedding written in the basis $\underline{s}^m = (s_0^m, \ldots, s_{d_m}^m)$ denoted by $i_m(\underline{s}^m)$, is the holomorphic embedding

$$i_m(\underline{s}^m) : M \to \mathbb{P}(H^0(L^m)^*) \cong \mathbb{C}P^{d_m}, \tag{6}$$

whose local expression in the open set U is given by (5).

Consider the Kähler form ω_m on M induced by the Fubini–Study form on $\mathbb{C}P^{d_m}$ namely:

$$\omega_m = i_m(\underline{s}^m)^*(\Omega_{FS}) = \frac{i}{2\pi}\partial\bar{\partial}\log\sum_{l=0}^{d_m}\left|\frac{s_l^m(x)}{\sigma(x)}\right|^2 \tag{7}$$

and the smooth function b_{jk}^m on M with values in the set of $(d_m+1)\times(d_m+1)$ complex matrices defined by:

$$b_{jk}^m(x) = \frac{\frac{s_j^m(x)}{\sigma(x)}\overline{\frac{s_k^m(x)}{\sigma(x)}}}{\sum_{l=0}^{d_m}\left|\frac{s_l^m(x)}{\sigma(x)}\right|^2}. \tag{8}$$

(The expression (8) is given in terms of the trivializing holomorphic section $\sigma : U \to L^m$ but it is straightforward to verify that it is defined on the whole M.) It follows from the definition of balanced variety (cfr. formula (3)) that $s_0^m, \ldots, s_{d_m}^m$ is a balanced basis of $H^0(L^m)$ iff there exists a positive real number λ such that:

$$\int_M b_{jk}^m \frac{\omega_m^n}{n!} = \lambda\delta_{jk}, \quad j, k = 0, \ldots, d_m. \tag{9}$$

Consider now the Hermitian metric $h_m : L^m \times L^m \to \mathbb{C}$ define on a pair of points $(q, q^{'}) \in L^m \times L^m$ by the formula:

$$h_m(q, q^{'}) = \frac{1}{\lambda} \frac{\frac{q}{\sigma(x)} \overline{\frac{q^{'}}{\sigma(x)}}}{\sum_{l=0}^{d_m} |\frac{s_l^m(x)}{\sigma(x)}|^2}. \tag{10}$$

Observe that the Ricci curvature form $\mathrm{Ric}(h_m)$ of h_m is equal to ω_m (see also formula (13) in Sect. 3). Formulae (8), (9) and (10) tell us that a basis $(s_0^m, \dots, s_{d_m}^m)$ is balanced iff

$$\langle s_j^m, s_k^m \rangle_{h_m} = \int_M h_m(s_j^m(x), s_k^m(x)) \frac{\omega_m^n}{n!} = \delta_{jk},$$

or equivalently, iff $(s_0^m, \dots, s_{d_m}^m)$ is an orthonormal basis for $(H^0(L^m), \langle \cdot, \cdot \rangle_{h_m})$, where $\langle \cdot, \cdot \rangle_{h_m}$ is the L^2-product in $H^0(L^m)$ given by:

$$\langle s^m, t^m \rangle_{h_m} = \int_M h_m(s^m(x), t^m(x)) \frac{\omega_m^n}{n!}, \ \forall s^m, t^m \in H^0(L^m). \tag{11}$$

Observe that by replacing h_m by $\frac{h_m}{c}, c > 0$, and by taking $\sqrt{c}\underline{s}^m = (\sqrt{c}s_0^m, \dots, \sqrt{c}s_{d_m}^m)$ we still obtain a balanced basis. The same happens by acting, in the natural way, on $\underline{s}^m$ by an element U of the unitary group $U(d_m + 1)$. We denote this action by $U \cdot \underline{s}^m$.

The first important Donaldson's result in [9] asserts that if the group $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ is discrete these are the only way to make the pair (M, L^m) balanced. Here $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ denotes the group biholomorphisms of M which lift to holomorphic bundles maps $L \to L$ modulo the trivial automorphism group $\mathbb{C}^*$. More precisely, using the same notations as above, the following holds (see Theorem 1 in [9]):

Theorem 2.1 *(Donaldson) Assume that $\frac{Aut(M,L)}{\mathbb{C}^*}$ is discrete. Let $\underline{s}^m$ and $\tilde{\underline{s}}^m$ be two balanced bases of $H^0(L^m)$. Then there exist $U \in U(d_m + 1)$ and $c \in \mathbb{R}^+$ such that*

$$\underline{s}^m = c\, U \cdot \tilde{\underline{s}}^m.$$

Thus, under the hypothesis that $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ is discrete the sequence of Kähler forms ω_m given by (7) is uniquely determined. Indeed if $\tilde{\omega}_m$ is the sequence determined by the balanced basis $\tilde{\underline{s}}^m$, we get:

$$\omega_m = i_m(\underline{s}^m)^*(\Omega_{FS}) = i_m(c\, U \cdot \tilde{\underline{s}}^m)^*(\Omega_{FS}) = i_m(\tilde{\underline{s}}^m)^*(\Omega_{FS}) = \tilde{\omega}_m. \tag{12}$$

Donaldson's main results about this sequence are summarized in the following (see Theorem 2 and 3 in [9]):

Theorem 2.2 *(Donaldson) Assume that* $\frac{Aut(M,L)}{\mathbb{C}^*}$ *is discrete.*

(i) If, for m *sufficiently large, the pair* (M, L^m) *is balanced and the sequence* $\frac{\omega_m}{m}$ C^∞*-converges to some limit* ω_∞ *as* $m \to \infty$*, then* ω_∞ *has constant scalar curvature;*

(ii) if ω_∞ *is a Kähler form in* $c_1(L)$ *with constant scalar curvature. Then, for* m *sufficiently large the pair* (M, L^m) *is balanced and the sequence* $\frac{\omega_m}{m}$ C^∞*-converges to* ω_∞ *as* $m \to \infty$*.*

Consequently, under the assumption of (ii) in the previous theorem and by the above mentioned result of Luo , L^m is stable for m sufficiently large.

From the Kähler geometry point of view Theorem 2.2 implies the following:

Corollary 2.3 *(Donaldson) Assume that* $\frac{Aut(M,L)}{\mathbb{C}^*}$ *is discrete. Then there exists at most one Kähler form* ω *on* M *with* $c_1(L) = [\omega]_{dR}$ *having constant scalar curvature.*

It is worth to mention that Chen [8] proved the uniqueness result of the previous corollary under the hypothesis $c_1(M) < 0$. Hence Corollary 2.3 is an extension of Chen's result , being $\mathrm{Aut}(M)$, and hence $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$, finite if $c_1(M) < 0$ (see e.g. Theorem 2.1 in [12]).

In this paper we study what happens when we drop the hypothesis that $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ is discrete.

The following example shows that the hypothesis on $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ in Theorems 2.1 and 2.2 can not be dropped.

Example 2.4 Consider the complex projective space $M = \mathbb{C}P^n$ equipped with the hyperplane bundle L. We have $c_1(L) = \omega$, being ω the Fubini–Study form on M. In this case every automorphism lifts to a bundle maps $\hat{F} : L \to L$ and we obviously have $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*} = \mathrm{Aut}(M) = \mathrm{PGL}(N+1, \mathbb{C})$. One can easy verify that $s_j = z_j, j = 0, \ldots, N$ is a balanced basis of $H^0(L)$, where we are identifying $H^0(L)$ with the space of homogeneous polynomials in the variable $z_0, \ldots, z_n$. Take any $F \in \mathrm{Aut}(M)$, an easy calculation shows that $\tilde{s}_j = \hat{F} \cdot s_j$ is still a balanced basis of $H^0(L)$. If we take $F \in \mathrm{Aut}(M) \setminus$

$PU(N+1)$ where $PU(N+1)$ is the projective unitary group we immediately see that its lift $\hat{F}$ does not act unitarily on $H^0(L)$ and so the hypothesis on $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ in Theorem 2.1 is necessary. More generally, for any non-negative integer m the elements of the form

$$s_j^m = \sqrt{\frac{m!}{j_0! \cdots j_n!}} z_0^{j_0} \dots z_n^{j_n}, \ j_0 + \cdots j_n = m,$$

where j is a multi-index is a balanced basis of $H^0(L^m)$ (which we are identifying with the space of homogeneous polynomials of degree m) and the Kodaira map in this basis

$$i_m(\underline{s}^m) : M \to \mathbb{C}P^{N(m)}, \ N(m) = \begin{pmatrix} N+m \\ N \end{pmatrix}$$

satisfies $i_m(\underline{s}^m)^*(\Omega_{FS}) = m\omega$, being Ω_{FS} the Fubini–Study form on $\mathbb{C}P^{N(m)}$. It is worth to mention that the map $i_m(\underline{s}^m)$ is obtained by rescaling the Veronese embedding and has been introduce by Calabi in [7]. As before if we take $F \in \mathrm{Aut}(M) \setminus PU(N+1)$ then $\tilde{s}_j^m = \hat{F} \cdot s_j$ is a balanced basis of $H^0(L^m)$ for any lift $\hat{F} \in \mathrm{Aut}(L^m)$ of F. Moreover, one can easily verify that $i_m(\underline{\tilde{s}}^m)^*(\Omega_{FS}) = mF^{*-1}(\omega) \neq m\omega$. Finally, the sequence of Kähler forms $\omega_{2m} = 2m\omega, \ \omega_{2m+1} = (2m+1)F^{*-1}(\omega), \ m = 1, 2 \ldots$, shows that the hypothesis on $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ in Theorem 2.2 is necessary (cfr. also Conjecture 1 below). In Section 4 we generalize all the considerations of this example to the case homogeneous and simply connected Kähler manifolds.

The main result of this paper is the following Theorem which generalize Theorem 2.1:

Theorem 2.5 *Let $\underline{s}^m$ and $\underline{\tilde{s}}^m$ be two balanced bases of $H^0(L^m)$. Then there exist $U \in U(d_m+1)$, $c \in \mathbb{R}^+$ and $\hat{F} \in Aut(L^m)$ such that*

$$\underline{s}^m = c\, U \cdot \hat{F} \cdot \underline{\tilde{s}}^m,$$

where

$$(\hat{F} \cdot \underline{s}^m)_j = \hat{F} \circ s_j^m \circ F^{-1} \ \ j = 0, \ldots, d_m,$$

and where $F \in Aut(M, L^m)$ is the biholomorphism underlying $\hat{F}$.

It follows from this theorem that, without Donaldson's assumption on $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$, there could exist different sequences ω_m and $\tilde{\omega}_m$ coming from different balanced bases of $H^0(L^m)$. Nevertheless, a careful reading of the proof of (i) in Theorem 2.2 shows that once we fix such a sequence, say ω_m, then $\frac{\omega_m}{m}$ C^∞-converges to a constant scalar curvature Kähler form ω_∞. About the proof of (ii) in Theorem 2.2 we point out that the assumption on $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ is used in different places. Nevertheless, we believe the following holds true:
Conjecture 1: *if ω_∞ is a Kähler form in $c_1(L)$ with constant scalar curvature. Then, for m sufficiently large the pair (M, L^m) is balanced and one can find a sequence $\omega_m = i_m(\underline{s}^m)^*(\Omega_{FS})$ for a balanced basis $\underline{s}^m$ of $H^0(L^m)$ such that $\frac{\omega_m}{m}$ C^∞-converges to ω_∞ as $m \to \infty$.*

3 Moment maps and the proof of Theorem 2.5

One of the main ingredient to prove Theorem 2.5 and Donaldson's theorems above is summarized in the following (see [9] and also [11]):

Proposition 3.1 *Let G be a group acting on a Kähler manifold $(\mathcal{H}, \Omega)$ by preserving the Kähler form Ω, let $Lie(G)$ be its Lie algebra and let $\mu_G : \mathcal{H} \to Lie(G)^*$ be the corresponding moment map. Assume that $G^{\mathbb{C}}$, the complexification of G, also acts on $\mathcal{H}$. Then the following hold:*

(i) let k be an element in the center of $Lie(G)^*$ fixed by the coadjoint action. Then for every $\xi \in \mu_G^{-1}(k)$ we have

$$\mu_G^{-1}(k) \cap (G^{\mathbb{C}} \cdot \xi) = G \cdot \xi,$$

namely the $G^{\mathbb{C}}$-orbit of ξ intersects the level set $\mu_G^{-1}(k)$ in the G-orbit of ξ.

(ii) let $Stab_G(\xi)$ and $Stab_{G^{\mathbb{C}}}(\xi)$ denote the stabilizers of a point $\xi \in \mathcal{H}$ under the action of G and $G^{\mathbb{C}}$ respectively and let $Stab_G^0(\xi)$ and $Stab_{G^{\mathbb{C}}}^0(\xi)$ be the identity components of these groups. Then the inclusion of $Stab_G(\xi)$ in $Stab_{G^{\mathbb{C}}}(\xi)$ induces an isomorphism between the discrete groups $\frac{Stab_G(\xi)}{Stab_G^0(\xi)}$ and $\frac{Stab_{G^{\mathbb{C}}}(\xi)}{Stab_{G^{\mathbb{C}}}^0(\xi)}$.

Throughout this section we fix a natural number m and we will assume that the line bundle L^m is very ample, namely the Kodaira map defined by a basis in $H^0(L^m)$ is an embedding. In order to avoid heavy notations we put $L = L^m$. Fix a basis $\underline{s} = (s_0, \ldots, s_N)$ and let ω_0 be the Kähler form on M given by $i(\underline{s})^*(\Omega)$, where $i(\underline{s}) : M \to \mathbb{C}P^N$ is the Kodaira map with respect to the basis $\underline{s}$, $N = \dim H^0(L) - 1$. Let h_0 be the Hermitian metric on L such that $\omega_0 = \mathrm{Ric}(h_0)$, where $\mathrm{Ric}(h_0)$ is the Ricci curvature form, namely, the 2-form on M defined in terms of the Hermitian metric h_0 by the equation:

$$\mathrm{Ric}(h_0) = -\frac{i}{2\pi}\partial\bar{\partial}\log h_0(\sigma(x), \sigma(x)), \tag{13}$$

for a trivializing holomorphic section $\sigma : U \subset M \to L \setminus \{0\}$ of L. (ω_0 and h_0 play the role of ω_m and h_m given by the formulae (7) and (10) respectively). The Hermitian metric h_0 is defined up to the a positive constant factor. With this notation we then have that the polarized manifold (M, L) is balanced with respect to the the above basis $\underline{s} = (s_0, \ldots, s_N)$ iff $\underline{s}$ is orthonormal with respect to the L^2-product

$$\langle s, t\rangle_{h_0} = \int_M h_0(s(x), t(x))\frac{\omega_0^n}{n!}, \quad \forall s, t \in H^0(L). \tag{14}$$

We start by giving an equivalent criterion expressing that the pair (M, L) is balanced in terms of cohomologous Kähler forms. Let ω be any Kähler form on M such that $c_1(L) = [\omega]_{dR}$, Define a the smooth function on M by the formula

$$\epsilon_\omega(x) = \sum_{j=0}^{N} h(s_j(x), s_j(x)), \quad x \in M, \tag{15}$$

where $s_0, \ldots, s_N$ is an orthonormal basis for $(H^0(L), \langle\cdot,\cdot\rangle_h)$ and h is an Hermitian metric on L whose Ricci curvature equals ω. It is easy to verify, as the notation suggests, that the function ϵ_ω depends only on the Kähler form ω and not on the Hermitian metric h with $\mathrm{Ric}(h) = \omega$ or on the orthonormal basis chosen. The above mentioned criterion is expressed in terms of this function by the following:

Proposition 3.2 *Let (M, L) be a polarized manifold The following assertions are equivalent:*

(i) (M, L) is balanced;

(ii) there exists a Kähler form ω on M with $c_1(L) = [\omega]_{dR}$ such that ϵ_ω equals a constant.

Proof: Observe that for any Kähler form ω on M with $[\omega]_{dR} = c_1(L)$ the following formula holds:

$$i(\underline{s})^*(\Omega_{FS}) = \omega + \frac{i}{2\pi}\partial\bar{\partial}\log\epsilon_\omega, \tag{16}$$

where $\underline{s} = (s_0, \ldots, s_N)$ is an orthonormal basis with respect to the L^2-product $\langle\cdot,\cdot\rangle_h$ and where h is the Hermitian metric on L satisfying $\mathrm{Ric}(h) = \omega$. The equivalence between assertions of (i) and (ii) is now immediate. $\square$

Remark 3.3 In the hypothesis that ϵ_ω is constant its value can be easily calculated as $\epsilon_\omega = \frac{N+1}{\mathrm{vol}(M,\omega)}$. Indeed

$$N + 1 = \int_M \sum_{j=0}^{N} h(s_j, s_j)\frac{\omega^n}{n!} = \int_M \epsilon_\omega \frac{\omega^n}{n!} = \epsilon_\omega \,\mathrm{vol}(M, \omega).$$

Observe also that if $\tilde{\omega}$ is another Kähler form in the same cohomology class of ω with $\epsilon_{\tilde{\omega}}$ constant then $\epsilon_{\tilde{\omega}} = \epsilon_\omega = \frac{N+1}{\mathrm{vol}(M,\omega)}$.

Remark 3.4 We can express Donaldson's theorem 2.1 in terms of cohomology classes by saying that *if two cohomologous Kähler forms ω and $\tilde{\omega}$ on M, with $[\omega]_{dR} = [\tilde{\omega}]_{dR} = c_1(L)$ are such that ϵ_ω and $\epsilon_{\tilde{\omega}}$ are constants and $\frac{Aut(M,L)}{\mathbb{C}^*}$ is discrete, then $\omega = \tilde{\omega}$* (cfr. formula (12) above).

The ingenious idea of Donaldson was to relate the balanced condition to the level set of an appropriate moment map and to apply Proposition 3.1. Therefore one needs to find a Kähler manifold $(\mathcal{H}, \Omega)$, a group G acting on $\mathcal{H}$ in such a way that a balanced basis of $H^0(L)$ belongs to the level set of $\mu_G : M \to \mathrm{Lie}(G)^*$, where μ_G is the moment map for this action.

The infinite dimensional manifold $\mathcal{H}$

Until now we have thought of M and of L as complex manifolds equipped with fixed complex structures, say I_0 and J_0 respectively. Observe that I_0 belongs to $\mathcal{I}_{int}$ the set of all ω_0-compatible complex structures on M. This set consist of all the almost complex structure I on M such that, for all vector fields X, Y on M the following hold:

$$N_I(X, Y) = 0, \;\; \omega_0(X, IX) > 0, \;\; \omega_0(IX, IY) = \omega_0(X, Y),$$

where N_I denoted the Nijenhius tensor on M associated to the complex structure I. In order to vary the complex structure J_0 on L, we need a brief digression on holomorphic and Chern connections. Consider two complex structures I on M and J on L. Let $\Gamma(L)$ be the space of smooth sections on L and let s be in $\Gamma(L)$. Given a trivializing (I,J)-holomorphic section $\sigma : U \to L$, i.e. $d\sigma \circ I = J \circ d\sigma$, let f be the smooth complex valued function on U such that $s = f\sigma$. Define the map

$$\bar{\partial}_{I,J} : \Gamma(L) \to \Gamma(\Omega^{0,1}(M) \otimes L), \ : s \mapsto \bar{\partial}_I f \otimes \sigma,$$

where $\bar{\partial}_I : \Gamma(TM) \to \Gamma(T^{0,1}M)$ is the usual operator associated to the complex structure I. Observe that $s \in \Gamma(L)$ is (I,J)-holomorphic iff $\bar{\partial}_{I,J}(s) = 0$. Given a connection ∇ on L, the decomposition of 1-forms into $(1,0)$-part and $(0,1)$-part with respect to I induced a decomposition of $\nabla = \nabla^{0,1} + \nabla^{1,0}$. We say that a connection ∇ on L is (I,J)-*holomorphic* if its $(0,1)$-part (with respect to I) equals $\bar{\partial}_{I,J}$, i.e. $\nabla^{0,1} = \bar{\partial}_{I,J}$.

We refer to [13] p. 85 for the proof of the following:

Proposition 3.5 *Let (M,I) be a complex manifold and let L be a smooth complex line bundle on M equipped with a connection ∇ such that its curvature is purely of type $(1,1)$. Then there exists a unique complex structure J on L such that ∇ is (I,J)-holomorphic.*

We denote the complex structure given by the previous proposition by $J_{I,\nabla}$. One immediately obtains the following:

Corollary 3.6 *Let (M,I) be a complex manifold and let L be be a smooth complex line bundle on M equipped with two connection ∇ and $\tilde{\nabla}$ such that their curvatures are of type $(1,1)$. Suppose that there exists a diffeomorphism $F : M \to M$ which admits a lift $\hat{F} : L \to L$ such that $\hat{F}^*(\tilde{\nabla}) = \nabla$. Then*

$$J_{F\cdot I,\tilde{\nabla}} = d\hat{F} \circ J_{I,\nabla} \circ d\hat{F}^{-1},$$

where $F \cdot I$ is the complex structure on M defined by $F \cdot I = dF \circ I \circ dF^{-1}$. In other words if F is $(I, F\cdot I)$-holomorphic and $\hat{F}$ preserves the connections then $\hat{F}$ is $(J_{I,\nabla}\ , J_{F\cdot I,\tilde{\nabla}})$-holomorphic.

We now fix the Chern connection on L associated to the triple (h_0, I_0, J_0), namely the unique (I_0, J_0)-holomorphic connection ∇^0 on L which is compatible with the Hermitian metric h_0 (see e.g. [10] p. 73). With the notation above we obviously have $J_0 = J_{I_0,\nabla^0}$. Observe also that the curvature of ∇^0 equals $-2\pi i\omega_0$ hence is of type $(1,1)$. To simplify the notation we give the following:

Definition 3.7 *Given a complex structure I on the complex manifold M we say that a section $s \in \Gamma(L)$ is holomorphic with respect to the complex structure I if s is (I, J_{I,∇^0})-holomorphic, being J_{I,∇^0} the unique complex structure on L given by Proposition 3.5.*

We are now in the position to define our manifold $\mathcal{H}$ as the subset of $\Gamma(L)^{N+1} \times \mathcal{I}_{int}$ consisting of pairs $(\underline{s}\ , I)$ where $\underline{s} = (s_0, \ldots, s_N)$ are $\mathbb{C}$-linearly independent elements of $\Gamma(L)$ which are holomorphic with respect to the (ω_0-compatible) complex structure I.

The group G, its action on $\mathcal{H}$ and the moment map μ_G

Let $\mathcal{G}$ be the group of Hermitian bundle maps of L which preserves the Chern connection ∇^0, namely the C^∞ bundle-maps $\hat{F} : L \to L$ such that $\hat{F}^*(h_0) = h_0$ and $\hat{F}^*(\nabla^0) = \nabla^0$. Observe that if $\hat{F}$ belongs to $\mathcal{G}$ then its underlying map $F : M \to M$ is a simplectomorphism, namely $F^*(\omega_0) = \omega_0$. If I is ω_0-compatible then for all vector fields X, Y on M one has:

$$\omega_0((F \cdot I)(X), X) = \omega_0(IdF^{-1}(X), dF^{-1}(X)) > 0,$$

$$\omega_0((F \cdot I)(X), (F \cdot I)(Y)) = \omega_0(IdF^{-1}(X), IdF^{-1}(Y)) = \omega_0(X, Y),$$

which show that $F \cdot I$ is ω_0-compatible. Moreover, let s be a section in $\Gamma(L)$ holomorphic with respect to the complex structure I then, it is easily seen, using Corollary 3.6, that $\hat{F} \cdot s = \hat{F} \circ s \circ F^{-1}$ is holomorphic with respect to $F \cdot I$, for all $\hat{F} \in \mathcal{G}$. Then we have the following:

Proposition 3.8 *The group $\mathcal{G}$ acts on $\mathcal{H}$ by*

$$\hat{F} \cdot (\underline{s}\ , I) = (\hat{F} \cdot \underline{s}\ , F \cdot I) = (\hat{F} \circ \underline{s} \circ F^{-1}, dF \circ I \circ dF^{-1}),$$

where, for $\underline{s} = (s_0, \ldots, s_N)$,

$$\hat{F} \circ \underline{s} \circ F^{-1} = (\hat{F} \circ s_0 \circ F^{-1}, \ldots, \hat{F} \circ s_N \circ F^{-1}).$$

There is another group acting naturally on $\mathcal{H}$. This is the finite dimensional group $SU(N+1)$ acting on $(\underline{s}\ , I) \in \mathcal{H}$ by

$$U \cdot (\underline{s}\ , I) = (U \cdot \underline{s}\ , I),\ U \in SU(N+1),$$

and hence leaving unchanged the complex structure I. The actions of the groups $\mathcal{G}$ and $SU(N+1)$ on $\mathcal{H}$ commutes hence these give rise to an action of the product group $G = \mathcal{G} \times SU(N+1)$ on $\mathcal{H}$. We refer to Donaldson for the proof of the following:

Theorem 3.9 *(Donaldson) The space $\mathcal{H}$ admits a Kähler form Ω invariant by the action of G. The corresponding moment map*

$$\mu_G : \mathcal{H} \to C^\infty(M, \mathbb{C}) \oplus su(N+1),$$

is given by:

$$\mu_G(\underline{s}\ , I) = (\frac{1}{2}\Delta\epsilon + \epsilon, i(\langle s_j, s_k\rangle_{h_0} - \frac{\sum_{l=0}^N \|s_l\|^2_{h_0}}{N+1}\delta_{jk})), \tag{17}$$

where ϵ is the smooth function on M defined by $\epsilon(x) = \sum_{j=0}^N h_0(s_j(x), s_j(x))$, Δ is the Laplacian with respect to the metric g_0 associated to ω_0 (and I_0) and $su(N+1)$ denotes the space of traceless skew-Hermitian matrices.

The complexification $G^{\mathbb{C}}$ of G

In order to apply Proposition 3.1 to our case we need to understand what is $G^{\mathbb{C}}$ for $G = \mathcal{G} \times SU(N+1)$. The complexification of $SU(N+1)$ is $SL(N+1, \mathbb{C})$, namely the set of non-singular matrices with determinant 1. So it remains to give a meaning to $\mathcal{G}^{\mathbb{C}}$. Even if one is not able to define $\mathcal{G}^{\mathbb{C}}$ one can formally identify the $\mathcal{G}^{\mathbb{C}}$-orbit of a point $(\underline{s}\ , I) \in \mathcal{H}$ with the element in the equivalence class of $(\underline{s}\ , I)$ under the following equivalence relation (see Proposition 11 in [9] for details). We declare $(\underline{s}\ , I) \in \mathcal{H}$ equivalent to $(\underline{s}^{'}\ , I^{'}) \in \mathcal{H}$ iff there exists a bundle map $\hat{F} : L \to L$ (not necessarily preserving ∇^0 or h_0) such that $\underline{s} = \hat{F} \cdot \underline{s}^{'}$ and $I = F \cdot I^{'}$ (F as above denotes the underlying map of $\hat{F}$). Roughly speaking one can think of $\mathcal{G}^{\mathbb{C}}$ as the set of C^∞-bundle maps $\hat{F} : L \to L$ such that if $(\underline{s}\ , I)$ belongs to $\mathcal{H}$ then $(\hat{F} \cdot \underline{s}\ , F \cdot I)$ still belongs to $\mathcal{H}$. Proposition 3.1 in our case reads as:

Proposition 3.10 *Let (C, O) be in $C^\infty(M, \mathbb{C}) \oplus su(N+1)$, where C is a constant and O is the $(N+1) \times (N+1)$ zero matrix, and let $(\underline{s}\ , I)$ be in $\mu_G^{-1}((C, O))$. Then the $G^{\mathbb{C}}$-orbit of (s, I) intersects $\mu_G^{-1}((C, O))$ in the G-orbit of $(\underline{s}\ , I)$ or, equivalently, up to the G-action there exists a unique element in $(G^{\mathbb{C}} \cdot (\underline{s}\ , I)) \cap \mu_G^{-1}((C, O))$.*

We are now in the position to prove Theorem 2.5. Observe that with the notation so far and by Proposition 3.2 we are reduced to prove the following: *let $(\underline{s}\ , I_0)$ and $(\underline{\tilde{s}}\ , I_0)$ be two elements in $\mathcal{H}$ and let $\omega_0 = i(\underline{s})^*(\Omega_{FS})$ and $\tilde{\omega} = i(\underline{\tilde{s}})^*(\Omega_{FS})$ be the induced Kähler forms via the Kodaira embedding. Suppose* *that ϵ_{ω_0} and $\epsilon_{\tilde{\omega}}$ are constant functions. Then there exist $U \in U(N+1)$, $c \in \mathbb{R}^+$ and $\hat{F} \in Aut(L)$ such that*

$$\underline{s} = c\, U \cdot \hat{F} \cdot \underline{\tilde{s}}. \tag{18}$$

Let $\mathrm{vol}(M, \omega_0) = \int_M \frac{\omega_0^n}{n!}$. Consider $K = (\frac{N+1}{\mathrm{vol}(M,\omega_0)}, O)$ as an element in $C^\infty(M, \mathbb{C}) \oplus su(N+1)$ and observe that from formula (17) and by the fact that $\epsilon_{\omega_0} = \frac{N+1}{\mathrm{vol}(M,\omega_0)}$ it follows that $\mu_G(\underline{s}\,, I_0) = K$. Let A be in $GL(N+1, \mathbb{C})$ such that $A \cdot \underline{\tilde{s}} = \underline{s}$. Let z be a $N+1$-root of $\det A$ and define $\underline{\tilde{t}} = z^{-1}\underline{\tilde{s}}$. Then there exists $T \in SL(N+1, \mathbb{C})$ such that $T \cdot \underline{\tilde{t}} = \underline{s}$. Let $F_1 : M \to M$ be a diffeomorphism such that $F_1^*(\omega_0) = \tilde{\omega}$. Take a lift $\hat{F}_1 : L \to L$ of F_1 such that $\hat{F}_1^*(h_0) = \lambda \tilde{h}$, $\lambda = |z|^2$, where $\mathrm{Ric}(h_0) = \omega_0$ and $\mathrm{Ric}(\tilde{h}) = \tilde{\omega}$. Now observe that the connection $\tilde{\nabla} = \hat{F}^*(\nabla^0)$ has curvature equals $-2\pi i \tilde{\omega}$ and, therefore, by Corollary 3.6 one easily deduces that $(\hat{F}_1 \cdot \underline{\tilde{t}}\,, F_1 \cdot I_0)$ is an element of $\mathcal{H}$, namely $\hat{F}_1 \cdot \underline{\tilde{t}}$ is holomorphic with respect to $F_1 \cdot I_0$. Moreover $(\underline{s}\,, I_0)$ and $(\hat{F}_1 \cdot \underline{\tilde{t}}\,, F_1 \cdot I_0)$ belong to the same $\mathcal{G}^{\mathbb{C}}$-orbit since $(\hat{F}_1 \cdot \underline{\tilde{t}}\,, F_1 \cdot I_0)$ is obtained by acting on $(\underline{s}\,, I_0)$ with $(id_{\mathcal{G}^{\mathbb{C}}}, T)$ followed by $(\hat{F}_1, id_{SL(N+1,\mathbb{C})})$. We claim that $\mu_G(\hat{F}_1 \cdot \underline{\tilde{t}}\,, F_1 \cdot I_0) = K$. Indeed, since $\epsilon_{\tilde{\omega}} = \sum_{j=0}^N \tilde{h}(\tilde{s}_j, \tilde{s}_j) = \frac{N+1}{\mathrm{vol}(M,\tilde{\omega})} = \frac{N+1}{\mathrm{vol}(M,\omega_0)}$, one gets:

$$\begin{aligned} \sum_{j=0}^N h_0(\hat{F}_1 \cdot \tilde{t}_j, \hat{F}_1 \cdot \tilde{t}_j) &= \sum_{j=0}^N \frac{1}{\lambda}(\hat{F}_1^* h_0)(\tilde{s}_j \circ F_1^{-1}, \tilde{s}_j \circ F_1^{-1}) \\ &= \sum_{j=0}^N \tilde{h}(\tilde{s}_j \circ F_1^{-1}, \tilde{s}_j \circ F_1^{-1}) = \frac{N+1}{\mathrm{vol}(M,\omega_0)}. \end{aligned}$$

and

$$\int_M h_0(\hat{F}_1 \cdot \tilde{t}_j, \hat{F}_1 \cdot \tilde{t}_k) \frac{\omega_0^n}{n!} = \int_M \hat{F}_1^{-1*}(\tilde{h}(\tilde{s}_j, \tilde{s}_k) \frac{\tilde{\omega}^n}{n!}) = \langle \tilde{s}_j, \tilde{s}_k \rangle_{\tilde{h}} = \delta_{jk},$$

which prove our claim. Hence, by Proposition 3.10 there exists $(\hat{F}_2, V) \in G = \mathcal{G} \times SU(N+1)$ such that:

$$(\underline{s}\,, I_0) = (z^{-1}\; V \cdot (\hat{F}_2 \circ \hat{F}_1) \cdot \underline{\tilde{s}}\,, (F_2 \circ F_1) \cdot I_0).$$

By taking $F = F_2 \circ F_1$ we then have that $I_0 = F \cdot I_0$, i.e. F and its lift $\hat{F} = \hat{F}_2 \circ \hat{F}_1$ are holomorphic maps. By writing z^{-1} in polar coordinates $z^{-1} = ce^{i\theta}$ and by putting $U = e^{i\theta}V \in U(N+1)$ we obtain equality (18) as desired.

Remark 3.11 In order to reobtain Donaldson's theorem 2.1 one must show that, if $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ is discrete, then $\hat{F} \in \mathrm{Aut}(L)$ in formula (18) acts on $(H^0(L), \langle\cdot,\cdot\rangle_{h_0})$ as a unitary transformation. This can be shown as follows. First, it is immediate to show that $\mathrm{Stab}_{G^{\mathbb{C}}}(\underline{s}\ , I_0)$, the stabilizer of the point $(\underline{s}\ , I_0)$ for the $G^{\mathbb{C}}$-action, coincides with the set of $(F, \hat{F}) \in \mathrm{Aut}(M, L)$ which acts with determinant 1 on $H^0(L)$. Second, if $\hat{F}$ and $c\hat{F}$, $c \in \mathbb{C}^*$ both act on $H^0(L)$ with determinant 1 then c is forced to be an $N+1$ root of unity. Hence, if we denote by $\mathcal{R}$ the group of the $N+1$-roots of unity, the inclusion $\mathrm{Stab}_{G^{\mathbb{C}}}(\underline{s}\ , I_0) \subset \mathrm{Aut}(M, L)$ induces an inclusion between the quotient spaces

$$\frac{\mathrm{Stab}_{G^{\mathbb{C}}}(\underline{s}\ , I_0)}{\mathcal{R}} \subset \frac{\mathrm{Aut}(M, L)}{\mathbb{C}^*}.$$

If the latter is discrete it follows that $\mathrm{Stab}_{G^{\mathbb{C}}}(\underline{s}\ , I_0)$ is discrete, being $\mathcal{R}$ finite. By (ii) of Proposition 3.1 we then have that $\mathrm{Stab}_{G}(\underline{s}\ , I_0) = \mathrm{Stab}_{G^{\mathbb{C}}}(\underline{s}\ , I_0)$, where $\mathrm{Stab}_{G}(\underline{s}\ , I_0)$ denotes the stabilizer of the point $(\underline{s}\ , I_0)$ for the G-action. Finally, observe that $\hat{F} \in \mathrm{Aut}(M, L)$ in formula (18) satisfies $\hat{F} \cdot \underline{\tilde{t}} = (V^{-1}T) \cdot \underline{\tilde{t}}$ which implies that it acts on $H^0(L)$ with determinant 1 and so $(F, \hat{F})$ belongs to $\mathrm{Stab}_{G^{\mathbb{C}}}(\underline{s}\ , I_0) = \mathrm{Stab}_{G}(\underline{s}\ , I_0)$. Since the group $\mathcal{G}$ preserves h_0, $\hat{F}$ acts on $(H^0(L), \langle\cdot,\cdot\rangle_{h_0})$ as a unitary transformation and this concludes the proof of Donaldson's theorem.

4 On the constancy of ϵ_ω in a fixed cohomology class

In this section we study some geometric properties of the function ϵ_ω. We are mainly interested in the case when ϵ_ω is constant. Therefore, let M be a complex manifold endowed with a holomorphic line bundle L, and let ω be a Kähler form on M with $[\omega]_{dR} = c_1(L)$ and ϵ_ω equals a constant. Let denote by $\mathcal{C}_\omega$ the set of Kähler forms on M cohomologous to ω. We want to study the set $\mathcal{E}_\omega$ consisting of the $\tilde{\omega} \in \mathcal{C}_\omega$ such that $\epsilon_{\tilde{\omega}}$ is constant. We believe that an understanding of the "topology" of this set could give interesting information about (M, ω). We start with the following Theorem which shows that two elements in $\mathcal{E}_\omega$ belongs to the same $\mathrm{Aut}(M, L)$-orbit.

Theorem 4.1 *Let M be a compact complex manifold M endowed with a holomorphic line bundle L. Let $\tilde{\omega}$ be in $\mathcal{E}_\omega$. Then there exists $F \in Aut(M, L)$ such that $F^*(\omega) = \tilde{\omega}$. If $\frac{Aut(M,L)}{\mathbb{C}^*}$ is discrete then $\omega = \tilde{\omega}$.*

Proof: For the first part, it is enough to observe that by formula (18) (which holds with our assumption with $\omega = \omega_0$) one gets:

$$\omega = i(\underline{s})^*(\Omega_{FS}) = i(c\,U \cdot \hat{F} \cdot \underline{\tilde{s}})^*(\Omega_{FS}) = F^{-1*}(i(\underline{s})^*(\Omega_{FS})) = F^{-1*}(\tilde{\omega}).$$

The second part follows from Remark 3.11 above. Indeed under the assumption that $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ is discrete, we can take $\hat{F}$ in $\mathcal{G}$ and so its underlying map F belongs to $\mathrm{Aut}(M,L) \cap \mathrm{Symp}(M,\omega)$. □

Corollary 4.2 *Let ω be a Kähler form on $\mathbb{C}P^N$ with ϵ_ω constant. Then there exists a natural number m and $F \in PGL(N+1,\mathbb{C}) = Aut(\mathbb{C}P^N)$ such that $F^*(\omega) = m\Omega_{FS}$.*

Proof: Since the second Betti number of $\mathbb{C}P^N$ is 1 there exists a natural number m such that ω is cohomologous to $m\Omega_{FS}$ and thus the conclusion by Theorem 4.1 (see also Example 2.4). □

Let us denote by $\mathrm{Aut}(M,L,\mathcal{C}_\omega)$ the set of maps in $\mathrm{Aut}(M,L)$ which preserves $\mathcal{C}_\omega$. Observe that if $\mathrm{Aut}(M,L)$ is connected then $\mathrm{Aut}(M,L,\mathcal{C}_\omega) = \mathrm{Aut}(M,L)$. Further denote by $\mathrm{Aut}(M,L,\mathcal{E}_\omega)$ the subset of $\mathrm{Aut}(M,L,\mathcal{C}_\omega)$ which preserves $\mathcal{E}_\omega$. We have the natural inclusion

$$\mathrm{Symp}(M,\omega) \cap \mathrm{Aut}(M,L) \subset \mathrm{Aut}(M,L,\mathcal{E}_\omega)$$

The map $F \in \mathrm{Aut}(M,L)$ in Theorem 4.1 belongs to $\mathrm{Aut}(M,L,\mathcal{E}_\omega)$. Moreover F is uniquely determined up to the action (on the left) of the group $\mathrm{Symp}(M,\omega) \cap \mathrm{Aut}(M,L)$. We declare F and G in $\mathrm{Aut}(M,L,\mathcal{E}_\omega)$ equivalent if the belong to the same orbit under the action of this group and we denote by $[F]$ the corresponding equivalence class of F. Hence we can define a bijection:

$$\Phi : \frac{\mathrm{Aut}(M,L,\mathcal{E}_\omega)}{\mathrm{Symp}(M,\omega) \cap \mathrm{Aut}(M,L)} \to \mathcal{E}_\omega : [F] \to F^*(\omega),$$

whose surjectivity follows from the first part Theorem 4.1. Observe also that if $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ is discrete then, from the second part of Theorem 4.1, we deduce that $\mathrm{Symp}(M,\omega) \cap \mathrm{Aut}(M,L) = \mathrm{Aut}(M,L,\mathcal{E}_\omega)$ and hence

$$\Phi(\frac{\mathrm{Aut}(M,L,\mathcal{E}_\omega)}{\mathrm{Symp}(M,\omega) \cap \mathrm{Aut}(M,L)}) = \mathcal{E}_\omega = \{\omega\}.$$

We want to to analyze the case of homogeneous Kähler manifolds (M,ω), namely those manifolds which are acted upon transitively by the group $\mathrm{Symp}(M,\omega) \cap \mathrm{Aut}(M)$. The following theorem can be considered as a generalization of Example 2.4 above.

Theorem 4.3 *Let (M, ω) be a homogeneous and simply-connected Kähler manifold. Then the function ϵ_ω is constant and the map Φ above defines a bijection between $\mathcal{E}_\omega$ and the quotient space $\frac{Aut(M)}{Symp(M,\omega)\cap Aut(M)}$.*

Proof: Let $\mathrm{Aut}(M, L, h)$ be the subgroup of $\mathrm{Aut}(M, L)$ consisting of $(F, \hat{F})$ such that $\hat{F}^*(h) = h$. Being M simply-connected, we have the following equality

$$\mathrm{Aut}(M, L, h) = \mathrm{Symp}(M, \omega) \cap \mathrm{Aut}(M).$$

It is straightforward to verify that the function ϵ_ω is invariant by $\mathrm{Aut}(M, L, h)$ and hence, in the homogeneous case, it reduces to a constant. To prove the second assertions, simply observe that for any $F \in \mathrm{Aut}(M)$ the pair $(M, F^*(\omega))$ is a simply-connected and homogeneous Kähler manifold (the group $F^{-1} \circ (\mathrm{Symp}(M, \omega) \cap \mathrm{Aut}(M)) \circ F$ acts transitively on it) and hence $\epsilon_{F^*(\omega)}$ is constant by the proof of first part. It follows that $\mathrm{Aut}(M) = \mathrm{Aut}(M, L, \mathcal{E}_\omega)$, which concludes the proof of our Theorem. □

Remark 4.4 Note that the condition of simply-connectedness in Theorem 4.3 can not be relaxed. In fact the the n-dimensional complex torus $M = \mathbb{C}^n/\mathbb{Z}^{2n}$ endowed with the flat Kähler form ω is a homogeneous Kähler manifold. On the other hand ω can not be induced by the Fubini–Study metric (see Lemma 2.2 in [19] for a proof) and hence, in particular, ϵ_ω can not constant by formula (16). (See also [14] for the calculation of ϵ_ω in this case).

5 Applications to the theory of quantization and the non-compact case

In the quantum mechanics terminology introduced by Kostant and Souriau a holomorphic Hermitian line bundle (L, h) such that $\mathrm{Ric}(h) = \omega$ is called a *geometric quantization* of the Kähler manifold (M, ω) and L is called the *quantum* line bundle. The function ϵ_ω of the previous section or more generally $\epsilon_{m\omega}$, for a non-negative integer m, plays a fundamental role in the theory of quantization carried out in [2], [3] and [4]. As we have already observed in [1], the function $\epsilon_{m\omega}$ equals the function T_m introduced in [20] which enables Tian to solve a conjecture posed by Yau [24] by proving that the Kähler form ω can be obtained as the limit of Bergmann metrics on M. Tian's Theorem was generalized by Zelditch [25], who, using the theory of

Szegö Kernel on the unit circle bundle L^* over M, proved that there is a asymptotic expansion

$$\epsilon_{m\omega}(x) = m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \cdots, \tag{19}$$

as $m \to \infty$. Later Lu [16], by using Tian's peak section method, gave a detailed description of the smooth coefficients $a_j(x)$. For example he proved that $a_1(x) = \frac{1}{2}\rho$, where ρ is the scalar curvature of the metric g associated to ω. It is worth to mention that these results on $\epsilon_{m\omega}$ together with the moment maps tools, are the key ingredients used by Donaldson in the proof of Theorem 2.2.

The special class of quantizations (L, h) of a Kähler manifolds (M, ω) having ϵ_ω constant are called *regular*. We call a quantization m-regular if $\epsilon_{m\omega}$ is constant for a non negative integer m. The m-regular quantizations enjoy very nice properties which enable Cahen, Gutt and Rawnsley to generalize Berezin's method [2] to the case of compact Kähler manifolds and to obtain a deformation quantization of the Kähler manifold (M, ω).

In view of Proposition 3.2, Donaldson's theorem 2.1 and our Theorem 4.1 one gets the following:

Corollary 5.1 *Let ω and $\tilde{\omega}$ be two cohomologous Kähler forms on a compact complex manifold M both defining regular quantizations (L, h) and $(L, \tilde{h})$ respectively. Then there exists $F \in Aut(M, L)$ such that $F^*(\omega) = \tilde{\omega}$. Moreover if $\frac{Aut(M,L)}{\mathbb{C}^*}$ is discrete then $\omega = \tilde{\omega}$.*

Regarding the existence of regular quantizations we observe that, from Donaldson's theorem 2.2, one immediately gets:

Corollary 5.2 *Given a geometric quantization (L, h) of a compact Kähler manifold (M, ω), such that ω has constant scalar curvature. Suppose $\frac{Aut(M,L)}{\mathbb{C}^*}$ is discrete, then there exists a sequence of Kähler forms ω_m on M all defining regular quantizations such that $\frac{\omega_m}{m}$ C^∞-converges to ω.*

We believe that in the previous corollary the assumption on $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ can be dropped (compare Conjecture 1 above) and this could have interesting consequences for the quantization deformation of constant scalar curvature metrics.

Disregarding the applications to the theory of quantization we believe that the study of the Kähler metrics ω such that $\epsilon_{m\omega}$ is constant for all m deserves further study. Example of this kind of metrics are the homogeneous one on simply-connected manifolds as follows from Theorem 4.3. Observe that such a metrics has necessarily constant scalar curvature, as follows from the above result of Lu (see also [1]). Observe also that the balanced basis $\underline{s}^m$ of $H^0(L^m)$ such that $i_m(\underline{s}^m)^*(\Omega_{FS}) = \omega_m = m\omega$ is very special and the proof of Theorem 2.2 is trivial for this kind of metrics.

The non-compact case

We want now to show that the tools developed in the theory of quantization can be used to extend all the previous definitions to the non-compact case. In what follows, we refer to [18] (see also [5], [6] and [14]) for details and further results. Suppose that (L, h) is a geometric quantization of a (not necessarily compact) Kähler manifold. Consider the space $H^0_h(L) \subset H^0(L)$ consisting of global holomorphic sections s of L which are bounded with respect to

$$\langle s, s\rangle_h = \|s\|^2_h = \int_M h(s(x), s(x))\frac{\omega^n}{n!}.$$

One can show that $H^0_h(L)$ is a separable complex Hilbert space. Let $x \in M$ and $q \in L\setminus\{0\}$ a fixed point of the fiber over x. If one evaluates $s \in H^0_h(L)$ at x, one gets a multiple $\delta_q(s)$ of q, i.e. $s(x) = \delta_q(s)q$. The map $\delta_q : H^0_h(L) \to \mathbb{C}$ is a continuous linear functional [3] hence from Riesz's theorem, there exists a unique $e_q \in H^0_h(L)$ such that $\delta_q(s) = \langle s, e_q\rangle_h, \forall s \in H^0_h(L)$, i.e.

$$s(x) = \langle s, e_q\rangle_h q. \tag{20}$$

It follows, by (20), that

$$e_{cq} = \overline{c}^{-1}e_q, \ \forall c \in \mathbb{C}^*.$$

The holomorphic section $e_q \in H^0_h(L)$ is called the *coherent state* relative to the point q. Thus, one can define a smooth function on M

$$\epsilon(x) = h(q, q)\|e_q\|^2_h, \tag{21}$$

where $q \in L \setminus \{0\}$ is any point on the fiber of x. If $s_j, \ j = 0, \dots N \ N \leq \infty$ is a orthonormal basis for $(H^0_h(L), \langle\cdot,\cdot\rangle_h)$ then one can easily verify that $\epsilon = \sum_{j=0}^N h(s_j, s_j)$ and therefore, in the compact case the function ϵ equals the function ϵ_ω. Suppose the following condition holds:

Condition A: *for every $x \in M$ there exists $s \in H^0_h(L)$ such that $s(x) \neq 0$.*

Then, for every fixed basis $\underline{s}$ of $H^0_h(L)$, one can define, as in formula (6) above, the Kodaira's map:

$$i(\underline{s}) : M \to \mathbb{C}P^N, N \le \infty.$$

One call this map the *coherent states map.* We say that a basis $\underline{s} = (s_0, \dots, s_N)$ is a balanced basis of $H^0_h(L)$ iff $\underline{s}$ is an orthonormal basis of $(H^0_h(L), \langle\cdot,\cdot\rangle_h)$ where $\mathrm{Ric}(h) = \omega$ and $\omega = i(\underline{s})^*(\Omega_{FS})$, where Ω_{FS} is the Fubini–Study form on $\mathbb{C}P^N$ (which can be defined also if $N = +\infty$). As for the compact case we can prove formula (16) above, namely:

$$i(\underline{s})^*(\Omega_{FS}) = \omega + \frac{i}{2\pi}\partial\bar{\partial}\log\epsilon_\omega,$$

Therefore if ϵ_ω is constant then the basis $\underline{s}$ is balanced. (The converse is not generally true since, in the non-compact case, a function f satisfying $\bar{\partial}\partial f = 0$ is not necessarily constant.)

In the following two examples we show that condition A above is satisfied and ϵ_ω is constant.

Example 5.3 Let $\omega = \sum_{j=0}^n dz_j \wedge d\bar{z}_j$ be the standard Kähler form on $\mathbb{C}^n$. The trivial bundle $L = \mathbb{C}^n \times \mathbb{C}$ on $\mathbb{C}^n$ equipped with the Hermitian metric:

$$h_z(w_1, w_2) = e^{-\pi\|z\|^2} w_1\bar{w}_2, \ \ \forall w_1, w_2 \in \mathbb{C},$$

where $\|z\|^2 = |z_1|^2 + \cdots + |z_n|^2$ defines a geometric quantization of $(\mathbb{C}^n, \omega)$. One can easily verify that $t_j = z_1^{j_1} \dots z_n^{j_n}$ is an orthogonal basis for $(H^0_h(L), \langle\cdot,\cdot\rangle_h)$ and

$$\|t_j\|^2_h = \frac{1}{n!}\prod_{l=1}^n \frac{j_l!}{\pi^{j_l}}.$$

Therefore

$$\epsilon_\omega = n! e^{-\pi\|z\|^2} \prod_{l=1}^n \sum_{j_l=0}^{+\infty} \frac{\pi^{j_l}}{l!}|z_l|^{2j_l} = n!.$$

Therefore $s_j = n!\prod_{l=1}^n \frac{\pi^{j_l}}{l!} t_j$ is a balanced basis of $H^0_h(L)$ and the coherent states map is given in this case by:

$$i(\underline{s}) : \mathbb{C}^n \to \mathbb{C}P^\infty : \ (z_1, \dots, z_n) \mapsto [\dots, \sqrt{\prod_{l=1}^n \frac{\pi^{j_l}}{l!}} z_1^{j_1} \dots z_n^{j_n}, \dots]. \qquad (22)$$

Example 5.4 Let $\mathcal{D} = \{z \in \mathbb{C} | |z|^2 < 1\}$ be the unit disk in $\mathbb{C}$ equipped with the hyperbolic form $\omega = \frac{i}{\pi} \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2}$. The trivial bundle $\mathcal{D} \times \mathbb{C} \to \mathcal{D}$ endowed with the Hermitian metric

$$h_z(w_1, w_2) = \frac{1 - |z|^2}{2} w_1 \bar{w}_2, \ \ \forall w_1, w_2 \in \mathbb{C},$$

satisfies $\mathrm{Ric}(h) = \omega$ and so the pair (L, h) is a geometric quantization of $(\mathcal{D}, \omega)$. It is easily seen that the holomorphic function $s_j = \sqrt{j+1} z^j$, $j = 0, 1, \ldots$ defines an orthonormal basis for the Hilbert space $(H^0_h(L), \langle \cdot, \cdot \rangle_h)$. Furthermore $\underline{s} = (\ldots, s_j, \ldots)$ is a balanced basis for $H^0_h(L)$ since

$$\epsilon_\omega(z) = \frac{1 - |z|^2}{2} \sum_{j=0}^{+\infty} (j+1)|z|^{2j} = \frac{1 - |z|^2}{2} \frac{1}{1 - |z|^2} = \frac{1}{2}$$

is constant. The coherent states map is given by:

$$i(\underline{s}) : \mathcal{D} \to \mathbb{C}P^\infty : \ z \mapsto [\ldots, \sqrt{j+1} z^j, \ldots], \ \ j = 0, 1 \ldots. \tag{23}$$

Remark 5.5 Example (5.4) can be generalized to the case of bounded symmetric domains of non-compact type endowed with the Bergmann metric (see [15] and [12]).

Remark 5.6 The maps (22) and (23) were considered also by Calabi [7] in the context of holomorphic and isometric immersions in the infinite dimensional complex projective space.

Remark 5.7 Observe that the constancy of ϵ_ω in the previous examples can also be obtained from Theorem 4.3 above whose proof extends without any change to the non-compact case.

We wonder if the Kähler metrics with ϵ_ω constant, as in the previous examples, enjoy nice properties similar to those in the compact case. For example if ω and $\tilde{\omega}$ are two cohomologous Kähler forms on a complex manifold M such that ϵ_ω and $\epsilon_{\tilde{\omega}}$ are both constant how are the Kähler forms related? One could compare this question with the analogous one posed by Yau (see Section 6 in [24]) for Kähler-Einstein metrics on non-compact manifolds.

One of the main difficulties in answering this kind of questions is that the Hermitian metrics h and $\tilde{h}$ on L such that $\mathrm{Ric}(h) = \omega$ and $\mathrm{Ric}(\tilde{h}) = \tilde{\omega}$ define two Hilbert spaces $H^0_h(L)$ and $H^0_{\tilde{h}}(L)$ which can be different. Thus, two balanced bases $\underline{s}$ and $\underline{\tilde{s}}$ could "live" in different Hilbert spaces and we do not know how to connect them as we did, for example, in Theorem 4.1.

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