Mapping of a plane defined in $\mathbb{R}^3$ onto $\mathbb{R}^2$

September 15, 2017

1 General problem

The aim is to obtain a transformation $\vec{r}(x,y,z) \in \mathbb{R}^3 \Rightarrow \vec{w}(u,v) \in \mathbb{R}^2$, for $\vec{r}(x,y,z)$ lying on a given plane, which maintains the distances, i.e. $\|\vec{r}_2 - \vec{r}_1\| = \|\vec{w}_2 - \vec{w}_1\|$, where $\vec{r}_1$ and $\vec{r}_2$ stand for any two points lying in the plane.

We are interested in planes containing the axes origin $(0,0,0)$, i.e. whose points $\vec{r}(x,y,z)$ satisfy the equation:

$$A\,x + B\,y + C\,z = 0 \tag{1}$$ {eq:1}

or equivalently:

$$\vec{R} \cdot \vec{r} = 0 \qquad\qquad \vec{R} = (A,B,C) \tag{2}$$ {eq:2}

For this purpose, we will choose two orthogonal unit vectors, $\vec{w}_u$ and $\vec{w}_v$, which are also orthogonal $\vec{R}$:

$$\vec{R} \cdot \vec{w}_u = \vec{R} \cdot \vec{w}_v = \vec{w}_u \cdot \vec{w}_v = 0 \quad \text{and} \quad \vec{w}_u \cdot \vec{w}_u = \vec{w}_v \cdot \vec{w}_v = 1 \tag{3}$$ {eq:3}

We start by choosing two unnormalized vectors with the orthogonality requirements. For instance:

$$\vec{w}'_u = (-B, A, 0) \tag{4}$$ {eq:4}

and

$$\vec{w}'_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A & B & C \\ -B & A & 0 \end{vmatrix} = -AC\,\vec{i} - BC\,\vec{j} + (A^2+B^2)\,\vec{k} = (-AC, -BC, A^2+B^2) \tag{5}$$ {eq:5}

It is easy to prove that:

$$\vec{w}'_u \cdot \vec{w}'_u = A^2 + B^2 \quad \text{and} \quad \vec{w}_v \cdot \vec{w}_v = (A^2+B^2)\,(A^2+B^2+C^2) \tag{6}$$ {eq:6}

so that the unit vectors seeked are:

$$\vec{w}_u = \frac{1}{(A^2+B^2)^{1/2}}\,(-B, A, 0) \tag{7}$$ {eq:7}

and

$$\vec{w}_v = \frac{1}{[(A^2+B^2)\,(A^2+B^2+C^2)]^{1/2}}\,(-AC, -BC, A^2+B^2) \tag{8}$$ {eq:8}

The mapping is given by: $\vec{r} = u\,\vec{w}_u + v\,\vec{w}_v$, or equivalently (u,v) are the coordinates of $\vec{r}$ in $\mathbb{R}^2$. Thus:

$$
\begin{aligned}
x &= -\frac{B}{(A^2+B^2)^{1/2}}\,u - \frac{AC}{[(A^2+B^2)\,(A^2+B^2+C^2)]^{1/2}}\,v \\
y &= \frac{A}{(A^2+B^2)^{1/2}}\,u - \frac{BC}{[(A^2+B^2)\,(A^2+B^2+C^2)]^{1/2}}\,v \\
z &= 0\,u + \left(\frac{A^2+B^2}{A^2+B^2+C^2}\right)^{1/2} v
\end{aligned}
\tag{9}
$$

{eq:9}

It can be proved (see notebook `symmetry_planes.nb`) that $x^2 + y^2 + z^2 = u^2 + v^2$ which is the condition imposed to the mapping.

2 Particular cases

The general solution proposed in the previous section can be impractical or even unfeasible for some particular cases, in which other solutions are more suitable, namely:

Case 1: $A = 0,\ B = 0,\ C \neq 0$: $\quad \vec{w}_u = (1,0,0), \quad \vec{w}_u = (0,1,0)$

Case 2: $A = 0,\ B \neq 0,\ C = 0$: $\quad \vec{w}_u = (1,0,0), \quad \vec{w}_u = (0,0,1)$

Case 3: $A \neq 0,\ B = 0,\ C = 0$: $\quad \vec{w}_u = (0,1,0), \quad \vec{w}_u = (0,0,1)$

In other cases, the general solution is valid but can be greatly simplified:

Case 4: $A \neq 0,\ B \neq 0,\ C = 0$: $\quad \vec{w}_u = (-\frac{B}{(A^2+B^2)^{1/2}}, \frac{A}{(A^2+B^2)^{1/2}}, 0), \quad \vec{w}_v = (0,0,1)$

Case 5: $A \neq 0,\ B = 0,\ C \neq 0$: $\quad \vec{w}_u = (0,1,0), \quad \vec{w}_v = (-\frac{C}{(A^2+C^2)^{1/2}}, 0, \frac{A}{(A^2+C^2)^{1/2}})$

Case 6: $A = 0,\ B \neq 0,\ C \neq 0$: $\quad \vec{w}_u = (-1,0,0), \quad \vec{w}_v = (0, -\frac{C}{(B^2+C^2)^{1/2}}, \frac{B}{(A^2+C^2)^{1/2}})$