

Numbers

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0.1 Notes

1. The notebooks are largely self-contained, i.e, if you see a symbol there will be an explanation about it at some point in the notebook.
 - Most often there will be links to the cell where the symbols are explained
 - If the symbols are not explained in this notebook, a reference to the appropriate notebook will be provided
2. **Github does a poor job of rendering this notebook.** The online render of this notebook is missing links, symbols, and notations are badly formatted. It is advised that you clone a local copy (or download the notebook) and open it locally.
3. **See the Collections notebook before this notebook to gain familiarity with set notations**

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1.1 Importing Libraries

```
[1]: import random
import math
import cmath
```

Real numbers

Absolute value

The absolute value or modulus of a real number x is the non-negative value of x without regard to its sign, and it is denoted by: $|x|$

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

For more information on the piecewise conditional braces see the [Functions notebook](#)

```
[2]: abs(2), abs(-2)
```

```
[2]: (2, 2)
```

Plus minus

The plus minus notation is used to encapsulate the positive and negative value of a real number together as a shorthand. It may be denoted for a real number x using:

$$\pm x$$

or

$$\mp x$$

Both notations mean the same. But the second notation may be called **minus plus** notation. For example the solutions to the equation $x^2 - 4 = 0$ is 2 and -2 , which may be written together as ± 2

```
[3]: eq = lambda x: x**2 - 4
eq(2), eq(-2)
```

```
[3]: (0, 0)
```

When the *plus minus* symbol is repeated, it can make the meaning ambiguous, for example:

$$\pm 1 \pm \sqrt{3}$$

may mean:

$$1 + \sqrt{3}, 1 - \sqrt{3}, -1 + \sqrt{3}, -1 - \sqrt{3}$$

But when the *minus plus* symbol is used with the *plus minus* notation, the meaning may be quite clear:

$$\mp 1 \pm \sqrt{3}$$

may mean:

$$-1 + \sqrt{3}, 1 - \sqrt{3}$$

Context can be important in these cases

Defined

The defined symbol can be used to show the definition of a variable. For example: Let's assume we define a variable x that is defined as $1 - \sqrt{2}$, we can denote that as:

$$x := 1 - \sqrt{2}$$

or

$$x \stackrel{\text{def}}{=} 1 - \sqrt{2}$$

or

$$x \triangleq 1 - \sqrt{2}$$

```
[4]: x = 1 - math.sqrt(2)
```

```
x
```

```
[4]: -0.41421356237309515
```

Some authors use the $:=$ symbol for iterative definition. Let i be the current iteration, then the iterative rule can be written as

$$x := x + 1$$

which is traditionally written as

$$x_{i+1} = x_i + 1$$

```
[5]: x = 10
for i in range(10):
    x = x + 1
```

```
x
```

```
[5]: 20
```

Identically equal to

The identically equal to symbol can be used to show that an assertion is equal for **all** values, and is denoted by: \equiv

For example:

$$x \in \mathbb{R}, \sin^2 x + \cos^2 x \equiv 1$$

is true for **all** values of x . Compare this with the assertion $x^2 - 4 = 0$ which is true only for *some* values of x .

(See: Real set)

```
[6]: # sin^2 x + cos^2 x equivalent to 1
for i in range(5):
    x = random.random()
    print(round(math.sin(x)**2, 4) + round(math.cos(x)**2, 4) == 1)
```

True
True
True
True
True

```
[7]: # x^2 - 4 is equal to 0
for x in [2, -2, 3]:
    print(x**2 - 4 == 0)
```

True
True
False

Subset of the Reals

Real set

A set containing real numbers is called a real number set. It is represented by:

$$\mathbb{R}$$

So we can represent variable x can only be a real number with the representation:

$$x \in \mathbb{R}$$

```
[8]: R = (float,int)
x = random.random()

type(x) == R[0] or type(x) == R[1]
```

[8]: True

Integer set

A set containing only integers is called an integer set. It is represented by:

$$\mathbb{Z}$$

So we can represent variable x can only be an integer with the representation:

$$x \in \mathbb{Z}$$

```
[9]: Z = int
     x = random.randint(-10000,10000)

     type(x) == Z
```

[9]: True

Natural number set

The definition of natural number is not standard, but the definition that is preferred is that a natural number is a non-negative integer. It is denoted by:

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

However, some authors and standards prefer not to include 0 such that $\mathbb{N} = \{1, 2, \dots\}$

```
[10]: Z = int
      x = 5

      type(x) == Z and x >= 0
```

[10]: True

Rational number set

Rational numbers are those numbers that can be expressed as a ratio of integers $\frac{a}{b}$ where $b \neq 0$ and is denoted by: \mathbb{Q}

$$\mathbb{Q} = \{x \in \mathbb{R} \mid x = a/b \text{ where } a, b \in \mathbb{Z} \text{ and } b \neq 0\}$$

```
[11]: def b_nq_zero():
      b_val = random.randint(-10000,10000)
      while b_val == 0:
          b_val = random.randint(-10000,10000)
      return b_val
```

```
def a():
    return random.randint(-10000,10000)

Q_example = [a()/b_nq_zero() for i in range(5)]

Q_example
```

```
[11]: [2.2196162046908317,
       2.0015333503705595,
       -1.5165736790285527,
       -0.708507670850767,
       2.808454810495627]
```

These sets are nested as follows:

$$\mathbb{R} \supset \mathbb{Q} \supset \mathbb{Z} \supset \mathbb{N}$$

Higher dimensional spaces

The n -dimensional number coordinate space is the set of all ordered n -tuples of the respective numbers. This is represented with a superscript over the number set notation. For example, the real coordinate space of dimension n is denoted as:

$$\mathbb{R}^n$$

where $\mathbb{R}^n = \{(a_1, a_2, a_3, \dots, a_n) \mid a_i \in \mathbb{R}\}$

And the n -dimensional complex coordinate space (or complex n -space) is denoted as:

$$\mathbb{C}^n$$

where $\mathbb{C}^n = \{(c_1, c_2, c_3, \dots, c_n) \mid c_i \in \mathbb{C}\}$

Additional commonly seen example: Points within a 3-dimensional Euclidean space is considered to form a real coordinate space of dimension three: \mathbb{R}^3 , such that if $\mathbf{x} \in \mathbb{R}^3$, \mathbf{x} may look like $\mathbf{x} = \{(x_1, x_2, x_3)_1, (x_1, x_2, x_3)_2, \dots\}$

For a Vectors perspective of n dimensional real vectors please see the Linear Algebra notebook

```
[12]: # Sampling of points belonging to  $\mathbb{R}^3$  as an illustration

def rand_x():
    return random.uniform(-1000,1000) #ideally we would use a function that
    ↪would produce any random real number

x = []
```

```
for x_i in range(5):
    x.append((rand_x(), rand_x(), rand_x()))

x
```

[12]: [(125.21430358980865, 900.8831623246704, 138.41279753389426),
(368.9157651902808, -136.4450049815133, 386.5798941973917),
(-335.22418750695147, 263.6511427929224, -35.46170633023803),
(-405.9774795043618, 714.5933875155697, -426.0731335007715),
(605.2242667179789, -744.7864483473692, 577.4914734279196)]

Decorations

The number set notations can be decorated with additional symbols which can denote various meanings. Some symbols can mean different things depending on context and author preferences.

Nonnegative and negative elements of the set can be denoted using subscripts or superscripts.

- There are various ways to represent it, commonly used symbols for **non-negative integers** (if the definition of natural numbers is not preferred) are: \mathbb{Z}^* , \mathbb{Z}^+ , $\mathbb{Z}_{\geq 0}$ and similarly **non-negative reals** \mathbb{R}^* , \mathbb{R}^+ , $\mathbb{R}_{\geq 0}$

(See: Natural number set)

In the above notations, the star superscript can mean **various alternate definitions**. For example:

- A star superscript can denote the nonzero elements of the set $\mathbb{R}^* = \{x \in \mathbb{R} \mid x \neq 0\}$
- A star superscript can denote the invertible elements of the set, which is mostly the same as the non-zero elements of the set (since 0 is non invertible) but for integers, $\mathbb{Z}^* = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$. This is also denoted using $\mathbb{Z}^\times = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$
- A star superscript for \mathbb{C}^* can denote $\mathbb{C} \cup \{\infty\}$ although the recommended notation for this is $\hat{\mathbb{C}}$ or $\overline{\mathbb{C}}$. Similarly \mathbb{R}^* can denote $\mathbb{R} \cup \{-\infty, \infty\}$ although $\overline{\mathbb{R}}$ is preferred.

Note: Decorations can be very subject and author specific, so look for context before interpreting.

Cardinality of ints and reals

The cardinality of sets is represented using two pipes, for example: $|A|$ (*See Collections notebook for more info*). But there are special symbols to represent cardinality of integers and reals.

The cardinality of integers is denoted using: \aleph_0

$$\aleph_0 = |\mathbb{Z}|$$

\aleph_0 (read as aleph null) is the smallest transfinite cardinal number and sets with cardinality \aleph_0 are called countable.

The cardinality of real numbers is denoted using: c

$$c = |\mathbb{R}|$$

c is also called the cardinality of the continuum.

Intervals

Intervals indicate membership of elements within a range and are denoted as open or closed using paranthesis or brackets respectively. A paranthesis indicates endpoints that are not included, a bracket indicates an endpoint that is included. For example $[-1, 1]$ indicates all real numbers beginning from -1 inclusive and ending at 1 inclusive. Other examples:

$$[1, 2] = \{x \in \mathbb{R} \mid 1 \leq x \leq 2\}$$

$$[1, 2) = \{x \in \mathbb{R} \mid 1 \leq x < 2\}$$

$$(1, 2] = \{x \in \mathbb{R} \mid 1 < x \leq 2\}$$

$$(1, 2) = \{x \in \mathbb{R} \mid 1 < x < 2\}$$

```
[13]: #[1,2]

for x in [0.5, 1, 1.5, 2, 2.5]:
    print('\nx: ',x)
    print('[1,2]: ', 1<=x<=2)
    print('[1,2): ', 1<=x<2)
    print('(1,2]: ', 1<x<=2)
    print('(1,2): ', 1<x<2)
```

```
x: 0.5
[1,2]: False
[1,2): False
(1,2]: False
(1,2): False
```

```
x: 1
[1,2]: True
[1,2): True
(1,2]: False
(1,2): False
```

```
x: 1.5
[1,2]: True
[1,2): True
(1,2]: True
(1,2): True
```



```

x: 2
[1,2]: True
[1,2): False
(1,2]: True
(1,2): False

```

```

x: 2.5
[1,2]: False
[1,2): False
(1,2]: False
(1,2): False

```

The symbols $-\infty$ and ∞ may be used for the left and right ends to show unbounded ranges:

$$\begin{aligned}
[2, \infty) &= \{x \in \mathbb{R} \mid x \geq 2\} \\
(-\infty, -2) &= \{x \in \mathbb{R} \mid x < -2\} \\
(-\infty, \infty) &= \mathbb{R}
\end{aligned}$$

Some authors use reversed brackets to denote exclusive endpoints, it is the same as using a parenthesis:

$$\begin{aligned}
[a, b[&\text{ is equivalent to } [a, b) \\
]a, b] &\text{ is equivalent to } (a, b]
\end{aligned}$$

Famous numbers

Some constants have their own symbols and used generally enough to be mentioned here.

- Pi is the ratio a circle's circumference to it's diameter and denoted using: π
- Euler's number or the base of natural logarithms is denoted using: e
- The imaginary unit with it's value equal to $\sqrt{-1}$ is denoted by: i
- Physical constants may include speed of light (denoted by c) etc

More info: https://en.wikipedia.org/wiki/Mathematical_constant

Complex numbers

Complex set

A set containing only complex numbers is called a complex number set. It is represented by:

$$\mathbb{C}$$

So we can represent variable z as belonging to a complex number set with the representation:

$$z \in \mathbb{C}$$

```
[14]: C = complex
      z_1 = complex(random.random(), random.random())

      z_2 = 3 + 6j #Can also be defined this way

      type(z_1) == C, type(z_2) == C
```

```
[14]: (True, True)
```

The complex numbers are created using an object i which has the property $i^2 = -1$ sometimes called the imaginary unit. The result is a collection of numbers of the form:

$$\{a + ib \mid a, b \in \mathbb{R}, i^2 = -1\}$$

Some authors prefer writing i after b for ex, $a + bi$, and in electrical engineering literature authors prefer to use j instead of i since i represents electrical current: $a + bj$

Here, a is called the real component and b is called the imaginary component, and it is generally denoted using: \Re for real, and \Im for imaginary.

$$z = a + bi, \Re z = a, \Im z = b$$

This can also be represented using simply the abbreviations as:

$$z = a + bi, \operatorname{Re} z = a, \operatorname{Im} z = b$$

```
[15]: z = complex(3,4)

      z.real, z.imag
```

```
[15]: (3.0, 4.0)
```

```
[16]: z = 3 + 6j

      z.real, z.imag
```

```
[16]: (3.0, 6.0)
```

Polar representation

The complex number $z = a + bi$ can simply be represented in a complex plane as points with coordinates (a, b) . These can also be represented in polar coordinates (r, θ) where simple rectangular to polar transformation rules apply:

$$a = r \cos \theta, b = r \sin \theta$$

Extended to complex numbers, the polar representation of $z = a + ib$ can be denoted as: $re^{i\theta}$ or $r \operatorname{cis}\theta$

$$re^{i\theta} = r[\cos \theta + i \sin \theta] = a + ib$$

or

$$r \operatorname{cis}\theta = r[\cos \theta + i \sin \theta] = a + ib$$

Here, r is the magnitude (See: Magnitude/Absolute value), and θ is the argument (or also called phase angle) and is denoted by $\arg(z)$

$$r = |z| = |a + ib|$$

$$\theta = \arg(z) = \arg(a + ib)$$

```
[17]: z = complex(3,4)
      r = abs(z)
      theta = cmath.phase(z)

      r, theta
```

```
[17]: (5.0, 0.9272952180016122)
```

Magnitude/Absolute value

The magnitude or absolute value of a complex number z is denoted as: $|z|$

$$z = a + bi, |z| = |a + bi| = \sqrt{a^2 + b^2}$$

```
[18]: z = complex(3,4)

      abs(z)
```

```
[18]: 5.0
```

Conjugate

The (complex) conjugate of a complex number z is denoted as: \bar{z}

$$z = a + bi, \bar{z} = \overline{a + bi} = a - bi$$

```
[19]: z = complex(3,4)
      z.conjugate()
```

```
[19]: (3-4j)
```