

# Logic

March 11, 2021

## 0.1 Notes

1. The notebooks are largely self-contained, i.e, if you see a symbol there will be an explanation about it at some point in the notebook.
  - Most often there will be links to the cell where the symbols are explained
  - If the symbols are not explained in this notebook, a reference to the appropriate notebook will be provided
2. **Github does a poor job of rendering this notebook.** The online render of this notebook is missing links, symbols, and notations are badly formatted. It is advised that you clone a local copy (or download the notebook) and open it locally.

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## 1.1 Importing Libraries

```
[1]: import random
import math
```

---

### Boolean Operations

#### And

The Boolean And operation is denoted using  $\wedge$

$$p \wedge q$$

```
[2]: p = bool()
q = bool()

p and q
```

```
[2]: False
```

---

#### Or

The Boolean Or operation is denoted using  $\vee$

$$p \vee q$$

```
[3]: p = bool()
q = bool()

p or q
```

```
[3]: False
```

---

#### Not

The Boolean Not operation is denoted using  $\sim$  or  $\neg$  and sometimes just the text ‘not’ is used:

$$\sim p$$

or

$$\neg p$$

or

$$\text{not } p$$

```
[4]: p = bool()

not p
```

```
[4]: True
```

---

### Exclusive Or

The Exclusive Or operation is denoted using  $\underline{\vee}$  or  $\oplus$  or just XOR

$$p \underline{\vee} q$$

or

$$p \oplus q$$

or

$$p \text{ XOR } q$$

```
[5]: p = bool()
q = bool()

#Shorter code since bool Xor equivalent to bitwise Xor
XOR_short = p ^ q

#Longer code by definition
XOR_long = (not q and p) or (not p and q)

XOR_short, XOR_long
```

```
[5]: (False, False)
```

---

### Nand

The Nand operation is denoted using  $\bar{\wedge}$

$$p \bar{\wedge} q$$

The Nand operator can be expanded to:  $p \bar{\wedge} q = \neg(p \vee q)$

```
[6]: p = bool()
q = bool()

not (p and q)
```

```
[6]: True
```

---

## Proof Symbols

### Implies

In mathematical proofs, the term ‘implies’ means ‘if  $a$  then  $b$ ’ or ‘ $a$  implies  $b$ ’ and this is denoted using the symbol:  $\Rightarrow$

$$a \Rightarrow b$$

Here  $a$  and  $b$  are any mathematical concepts (or *logical predicates*)

Logically,  $a \Rightarrow b$  is equivalent to  $b \vee \neg a$

(See: Or)

```
[7]: boo = [True, False]
print('a implies b truth table\n')
for a in boo:
    for b in boo:
        print('a: ', a, 'b: ', b, ', a implies b :', b or (not a))
```

a implies b truth table

```
a:  True b:  True , a implies b : True
a:  True b:  False , a implies b : False
a:  False b:  True , a implies b : True
a:  False b:  False , a implies b : True
```

To prove a theorem of this form, you must prove that  $b$  is true whenever  $a$  is true. Example: if  $x$  is greater than or equal to 4, then  $2^x \geq x^2$

$$\forall x \in \mathbb{R}, x \geq 4 \Rightarrow 2^x \geq x^2$$

i.e: if  $a = (x \geq 4)$ , then  $b = (2^x \geq x^2)$

Notice from the above truth table that  $a$  may be False when  $b$  is True. However,  $b$  **must** be True when  $a$  is True. Hence, we can prove that  $b$  is True whenever  $a$  is True for  $a \Rightarrow b$  but also, if we show that  $b$  is False when  $a$  is True, we have invalidated the  $a \Rightarrow b$  statement.

(See: For all)

**For more information on the real set and the belongs to notation see the Collections notebook**

```
[8]: print('if a then b')

X    = [-10.00,-2.20,0.00,2.00,3.10,4.00,5.5,6.00,7.8] #Subset of R used as an
    ↪ example

a_implies_b = []
```

```

for x in X:
    condition_a = x >= 4
    condition_b = 2**x >= x**2
    print('x :', x, ', x>=4, a: ', condition_a, ', 2^x>=x^2, b: ', condition_b)

    a_implies_b.append(condition_b or (not condition_a))

print('\nCompared with the truth table above')
print('a implies b: ', all(a_implies_b))
#Atleast for this subset of R

```

```

if a then b
x : -10.0 , x>=4, a: False , 2^x>=x^2, b: False
x : -2.2 , x>=4, a: False , 2^x>=x^2, b: False
x : 0.0 , x>=4, a: False , 2^x>=x^2, b: True
x : 2.0 , x>=4, a: False , 2^x>=x^2, b: True
x : 3.1 , x>=4, a: False , 2^x>=x^2, b: False
x : 4.0 , x>=4, a: True , 2^x>=x^2, b: True
x : 5.5 , x>=4, a: True , 2^x>=x^2, b: True
x : 6.0 , x>=4, a: True , 2^x>=x^2, b: True
x : 7.8 , x>=4, a: True , 2^x>=x^2, b: True

```

Compared with the truth table above  
a implies b: True

---

## Implied by

In mathematical proofs, the term ‘implied by’ means ‘if  $b$  then  $a$ ’ or ‘ $a$  is implied by  $b$ ’ and this is denoted using the symbol:  $\Leftarrow$

$$a \Leftarrow b$$

Here  $a$  and  $b$  are any mathematical concepts (or *logical predicates*)

Logically,  $a \Leftarrow b$  is equivalent to  $a \vee \neg b$

(See: Or)

```

[9]: boo = [True, False]
print('a implied by b truth table\n')
for a in boo:
    for b in boo:
        print('a: ', a, 'b: ', b, ', a implied by b :', a or (not b))

```

a implied by b truth table

```

a: True b: True , a implied by b : True
a: True b: False , a implied by b : True

```

a: False b: True , a implied by b : False  
a: False b: False , a implied by b : True

To prove a theorem of this form, you must prove that  $a$  true whenever  $b$  is true. To explain the concept, lets expand on the previous example, but here let's assume that the opposite condition is true, i.e:  $x \geq 4$  is implied by  $2^x \geq x^2$

$$\forall x \in \mathbb{R}, x \geq 4 \Leftarrow 2^x \geq x^2$$

So based on our truth table above we have:  $a = (x \geq 4), b = (2^x \geq x^2)$ . Interestingly, if this is true we would have proved that  $x \geq 4$  if and only if  $2^x \geq x^2$  (See: if and only if)

(See: For all)

**For more information on the real set and the belongs to notation see the Collections notebook**

Now based on the truth table above if we observe  $a$  is False when  $b$  is True we have essentially disproved the implied by assertion:

```
[10]: print('if b then a')

X    = [0.00,2.00] #Subset of R used as an example

a_implied_by_b = []

for x in X:
    condition_a = x >= 4
    condition_b = 2**x >= x**2
    print('x :', x, ', 2^x>=x^2, b: ', condition_b, ', x>=4, a: ', condition_a,)

    a_implied_by_b.append(condition_a or (not condition_b))

print('\nCompared with the truth table above')
print('a implied by b: ', all(a_implied_by_b))
```

```
if b then a
x : 0.0 , 2^x>=x^2, b: True , x>=4, a: False
x : 2.0 , 2^x>=x^2, b: True , x>=4, a: False
```

Compared with the truth table above  
a implied by b: False

---

## If and only if

In mathematical proofs, the term 'if and only if' means 'if  $a$  then  $b$ ' as well as 'if  $b$  then  $a$ ' and this is denoted using the symbol:  $\Longleftrightarrow$  but it is also sometimes abbreviated as: iff

$$a \Longleftrightarrow b$$

or

$a \text{ iff } b$

The concept of iff is also logically equivalent to  $(a \Rightarrow b) \wedge (b \Rightarrow a)$

(See: And and Implies)

Here  $a$  and  $b$  are any mathematical concepts (or *logical predicates*).

```
[11]: boo = [True, False]
print('a iff b truth table\n')
for a in boo:
    for b in boo:
        print('a: ', a, 'b: ', b, ', a iff b :', (b or (not a)) and (a or (not b)))
```

a iff b truth table

```
a: True b: True , a iff b : True
a: True b: False , a iff b : False
a: False b: True , a iff b : False
a: False b: False , a iff b : True
```

To prove a theorem of this form, you must prove that  $a$  and  $b$  are equivalent. Not only is  $b$  true whenever  $a$  is true, but  $a$  is true whenever  $b$  is true. Example: The integer  $n$  is odd if and only if  $n^2$  is odd.

$$\forall n \in \mathbb{Z}, n \text{ is odd} \iff n^2 \text{ is odd}$$

i.e.  $(a = n \text{ is odd}) \text{ iff } (b = n^2 \text{ is odd})$

(See: For all)

**For more information on the integer set and the belongs to notation see the Collections notebook**

```
[12]: def is_odd(x):
        """returns whether x is Odd"""
        return not (x%2 == 0)

a_iff_b = []

print('if a then b')
N = [random.randint(-1000,1000) for i in range(5)] #Subset of R used as an
        example

for n in N:
    condition_a = is_odd(n)
    condition_b = is_odd(n**2)
    print('n: ', n, ', odd(n), a:', condition_a,
          '\nn2: ', n**2, ', odd(n2), b:', condition_b, '\n')
```

```

    a_iff_b.append((condition_b or (not condition_a)) and (condition_a or (not
    ↪condition_b)))

print('if b then a')
N_new = [random.randint(-1000,1000)**2 for i in range(5)] #Subset of R used as
    ↪an example

for n_squared in N_new:
    n = int(math.sqrt(n_squared))
    condition_b = is_odd(n_squared)
    condition_a = all([is_odd(n), is_odd(-n)])

    print('n2: ',n_squared,', odd(n2), a:', condition_b,
          '\nn: ', n, ', odd(n), b: ', condition_a, '\n')

    a_iff_b.append((condition_b or (not condition_a)) and (condition_a or (not
    ↪condition_b)))

print('\nCompared with the truth table above')
print('a iff b: ',all(a_iff_b))

```

```

if a then b
n:  505 , odd(n), a: True
n2: 255025 , odd(n2), b: True

n: -449 , odd(n), a: True
n2: 201601 , odd(n2), b: True

n:  117 , odd(n), a: True
n2: 13689 , odd(n2), b: True

n:  -33 , odd(n), a: True
n2: 1089 , odd(n2), b: True

n: -690 , odd(n), a: False
n2: 476100 , odd(n2), b: False

if b then a
n2: 14884 , odd(n2), a: False
n:  122 , odd(n), b: False

n2: 67081 , odd(n2), a: True
n:  259 , odd(n), b: True

n2: 82944 , odd(n2), a: False
n:  288 , odd(n), b: False

```



n2: 35721 , odd(n2), a: True  
n: 189 , odd(n), b: True

n2: 123904 , odd(n2), a: False  
n: 352 , odd(n), b: False

Compared with the truth table above  
a iff b: True

---

### Therefore

The term *therefore* is denoted by:  $\therefore$

$$r^2 + \lambda^2 c^2 = 0$$
$$\therefore r = \pm \lambda ci$$

---

### Because

The term *because* is denoted by:  $\because$

$$\because x + 1 = 10 \therefore x = 9$$

---

### Contradiction

Contradiction in a proof is denoted by:  $\Rightarrow \Leftarrow$

Used to show that the supposition was False

---

### End of Proof

The end of a proof is show using the following notations or text:

Just a square box:



a filled square box:



or the text:

QED

---

### Quantifiers

For all

Also called a universal quantifier. The ‘for all’ symbol is used simply to denote that a concept or relation (or *logical predicates*) is applied to every member of the domain. Denoted by  $\forall$

For example: squares of all real numbers are positive or zero can be expressed through:

$$\forall x \in \mathbb{R}, x^2 \geq 0$$

Which can be read as, for all  $x$  belonging to the set of real numbers (essentially any real number), the square of  $x$  is always greater or equal to zero.

**For more information on the real set and the belongs to notation see the Collections notebook**

```
[13]: trials = 5

for i in range(trials):
    x = random.uniform(-100000, 100000)**2
    print(x >= 0)
```

True

True

True

True

True

The for all  $\forall$  notations can be extended to denote complex statements. For example the commutative property of addition can be denoted using:

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = y + x$$

---

## There exists

Also called an existential quantifier. This symbol can be interpreted as ‘there exists’, ‘there is at least one’, or ‘for some’ and applied to a mathematical concept (or *logical predicates*); it is denoted by:  $\exists$

For example: there exists at least one real number  $x$  whose square equals 2

$$\exists x \in \mathbb{R}, x^2 = 2$$

This can be read as there exists at least one  $x$  belonging to the real set of numbers such that  $x^2 = 2$ . With this symbol, an assertion is being made about an object’s existence which fulfills a criteria, which is true in this case since both  $x = +\sqrt{2}$  and  $x = -\sqrt{2}$  satisfy this condition.

Sometimes for readability, some authors will use the abbreviation for **such that** (*s.t.*) :

$$\exists x \in \mathbb{R} \text{ s.t. } x^2 = 2$$

**For more information on the real set and the belongs to notation see the Collections notebook**

```
[14]: x_square = 2
      x_1 = math.sqrt(2)
      x_2 = -math.sqrt(2)

      type(x_1) == float, type(x_2) == float

      #There may be more x's but we've shown enough to prove this statement to be true
```

```
[14]: (True, True)
```

---

### There exists uniquely

When the existential quantifier symbol is followed by an exclamation point, it means there exists a **unique** object that fulfills a given criteria:  $\exists!$

For example: there exists a unique real number  $x$  whose square equals 0

$$\exists!x \in \mathbb{R}, x^2 = 0$$

which is true in this case since only  $x = 0$  satisfies this condition.

**For more information on the real set and the belongs to notation see the Collections notebook**

```
[15]: #range(-5,5) is shown as an example, in reality you would have to look at each
      ↪ element
      #in the infinite real set to prove uniqueness
      for x in range(-5,5):
          print('x: ', (x/10), ', x2 == 0', (x/10)**2 == 0)
```

```
x: -0.5 , x2 == 0 False
x: -0.4 , x2 == 0 False
x: -0.3 , x2 == 0 False
x: -0.2 , x2 == 0 False
x: -0.1 , x2 == 0 False
x: 0.0 , x2 == 0 True
x: 0.1 , x2 == 0 False
x: 0.2 , x2 == 0 False
x: 0.3 , x2 == 0 False
x: 0.4 , x2 == 0 False
```

---

### Combining quantifiers

The for all  $\forall$  and exists  $\exists$  notations can be combined to denote complex statements.

For example: For all  $x$  in the real number set, there exists at least one real number  $y$  such that  $x + y = 0$

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 0$$

This statement means that if we were to pick **any** real number  $x$ , we can find at least one number  $y$  so that we get  $x + y = 0$ . We know this to be a True statement since we can always find a number  $y = -x$

For more information on the real set and the belongs to notation see the Collections notebook

```
[16]: R_sample = [random.random() for i in range(10)] #checking for a small subset of
↳the real number
#set as an example. In reality we would have to loop over the entire infinite
↳Real number set
#to check for all x

condition = []

for x in R_sample:
    y = -x #this y will always have the property 'x + y = 0'; for any x in R
    condition.append(x + y == 0)

all(condition) #Check if our condition was tested as True for ALL x in R
```

[16]: True

**Note:** The statement order is very important since it evolves logically and combines to form a logical assertion. The above example was well ordered but consider the following example:

$$\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y = 0$$

In this case we make an assertion that there exists a real number  $y$  which will have the property  $x + y = 0$  for any real number  $x$ . Such a real number  $y$  does not exist and so this assertion is **False**.

```
[17]: R_sample = [random.random() for i in range(10)] #checking for a small subset of
↳the real number
#set as an example. In reality we would have to loop over the entire infinite
↳Real number set
#to check for all y

all_y_conditions = []

for y in R_sample: #Searching for the 'magic' y
    condition = all([x + y == 0 for x in R_sample]) #The magic y has the
↳property 'x + y = 0' for any x
    all_y_conditions.append(condition)

any(all_y_conditions) #Check if we found at least one 'magic' y ?
```

[17]: False