MODULE 6: CALCULUS

This is a branch of Maths which deals with how things change in very small ways or during very short intervals. We will only look at very basic calculus, specifically "Differential Calculus" to find the slope or "gradient" of a curved graph at any point, for example the graph of x^2 or x^3.

Note the development of calculus is credited to both Sir Isaac Newton and a German mathematician Leibnitz. Both were working independently on similar projects.

Variables and Constants

In the formula for volume of a sphere $V = 4/3\ \pi r^3$ the values of V and r vary with different size of sphere so are variable. 4/3 and π are constants.

The volume V depends upon the radius so is called the "dependant" variable.

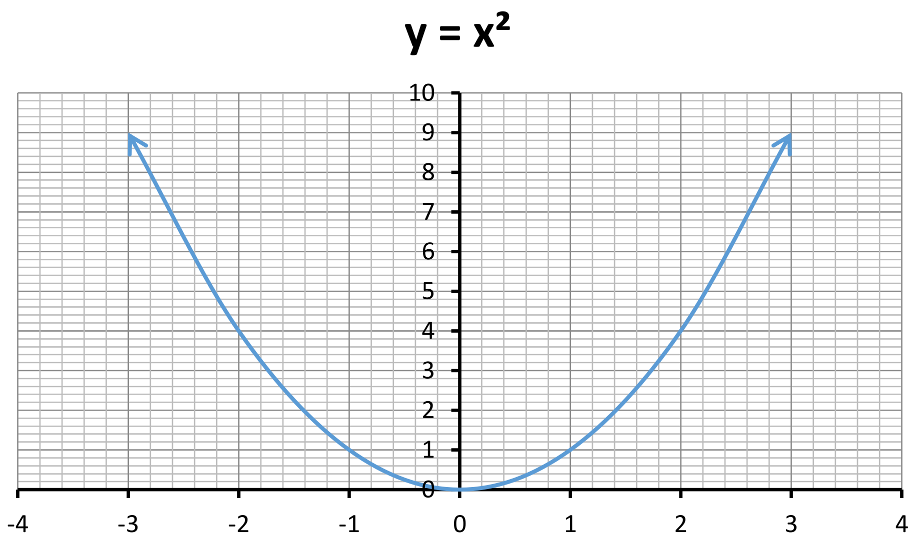
The size of r the radius causes V to change so r is the "independent" variable.

Functions

If one variable changes in a specific way according to the other variable we say that one is a function of the other. e.g. Sines, cosines and tangents are functions of the angles.

This is written as $y = f(x)$ for "y is a function of x"

Plotting a function on a graph x^2 can be plotted between the values of –3 and + 3 as follows.



Obviously the slope of the graph varies at each position, being zero at the bottom of the curve when x = 0 and increasing towards the sides. At any point we could estimate the slope by drawing a small triangle and seeing how much y changes for a change of x. As with straight line graphs, y/x = Tan Slope.

By making the triangle smaller and smaller we the hypotenuse more closely represents the actual slope, but the x and y are smaller so harder to measure.

Calculus reduces the sides x and y mathematically so they are infinitesimally small, thus giving the actual slope at that point.

To represent these very small x and y sides we use a special notation:

$\frac{\delta y}{\delta x}$ or $\frac{dy}{dx}$ pronounced "dee y by dee x"

The symbol δ is the Greek letter delta, often used for "difference". This is used when we are talking initially about very small changes or increments taken to the limits of infinity.

If other letters than x and y are being used in the graph, for example s for distance and t for time, the expression would be $\frac{\delta s}{\delta t}$ ("dee ess by dee tee")

In a straight line graph where $y = mx + c$ the coefficient m represents the gradient.

Differential Coefficient (Differentiation)

Using Calculus, it can be shown that for any curve of the form y = f(x) (y is a function of x); the slope at any point can be found by "differentiating" the equation. The resulting function is called the 'gradient function' $\frac{dy}{dx}$.

For simple powers such as x^2 or x^3 etc. Differentiating is done by multiplying the coefficient of x by the power and subtracting 1 from the power.

Example, $if\ y = 3x^3\ \ then\ \frac{dy}{dx} = 9x^2$

$if\ y = 3x^2\ \ then\ \frac{dy}{dx} = 6x$

$if\ y = 3x\ \ then\ \frac{dy}{dx} = 3$ *(Any term without a power, such as $3x$ becomes just 3)*

$if\ y = 3\ \ then\ \frac{dy}{dx} = 0$ *(Any term without x, that is the constant, disappears)*

In general, for equation $\boldsymbol{y = ax^n}$ $\boldsymbol{\frac{dy}{dx} = nax^{(n-1)}}$

Notation (and alternative notation):

Original Equation (function)	**First Differential notation**	**Second differentia notation**
y (Leibniz notation)	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
$f(x)$ (Newton notation)	$f'(x)$	$f''(x)$

Example $\quad y \quad = \quad 5x^4 + 2x^3 + 4x^2 + 5x + 7$

$$\frac{dy}{dx} = 20x^3 + 6x^2 + 8x + 5$$

$\frac{dy}{dx}$ is called the 'gradient function' as it represents the gradient of the curve at any point on the curve. By substituting any value of x into the gradient function $\frac{dy}{dx}$ we can find the gradient at that point.

For example in the above, when x = 2 the gradient would be

$$\frac{dy}{dx} = 20 \times (2)^3 + 6 \times (2)^2 + 8(2) + 5 = 205$$

The gradient angle is therefore Tan^{-1}(205)= 89.7°, almost vertical as would be expected with a cube term.

In the previous graph of $y = x^2$, the slope at any point is 2x, so when x= 0.5, the slope is 1 or 45°, when x = 3 the slope is 6 or 80°.

Finding Turning Points:

When an equation of higher powers is graphed it will have one or more changes in direction, known as turning points. The gradient at these turning points is zero. Given this property we can find the coordinates at which turning points occur.

e.g. if the original equation is a cubic (x^3) it will have 2 turning points and its differential will be a quadratic (x^2).

For straight line graphs, the gradient is constant and may be negative or positive, but with curves, the gradient is constantly changing. Inspection of the gradient on each side of a turning point can be used to determine if the turning point is a minimum (in which case, the gradient goes from being negative to positive), or if the turning point is a maximum (in which case, the gradient goes from being positive to negative),

If y increases as x increases, $\frac{dy}{dx}$ is positive. If y decreases as x increases, $\frac{dy}{dx}$ is negative.

Worked Example

1. Plot the graph of y = (x–1) (x–2) (x–3) between x = 0 and x = 4

2. Find the coordinates of the turning points.

Step1: Expand the equation i.e.

$$y = x^3 - 6x^2 + 11x - 6$$

Step2: differentiate the equation to find the gradient function $\frac{dy}{dx}$

$$\frac{dy}{dx} = 3x^2 - 12\,\mathrm{x} + 11$$

Step3: since the gradient is zero at all turning points, set

$$\frac{dy}{dx} = 0$$

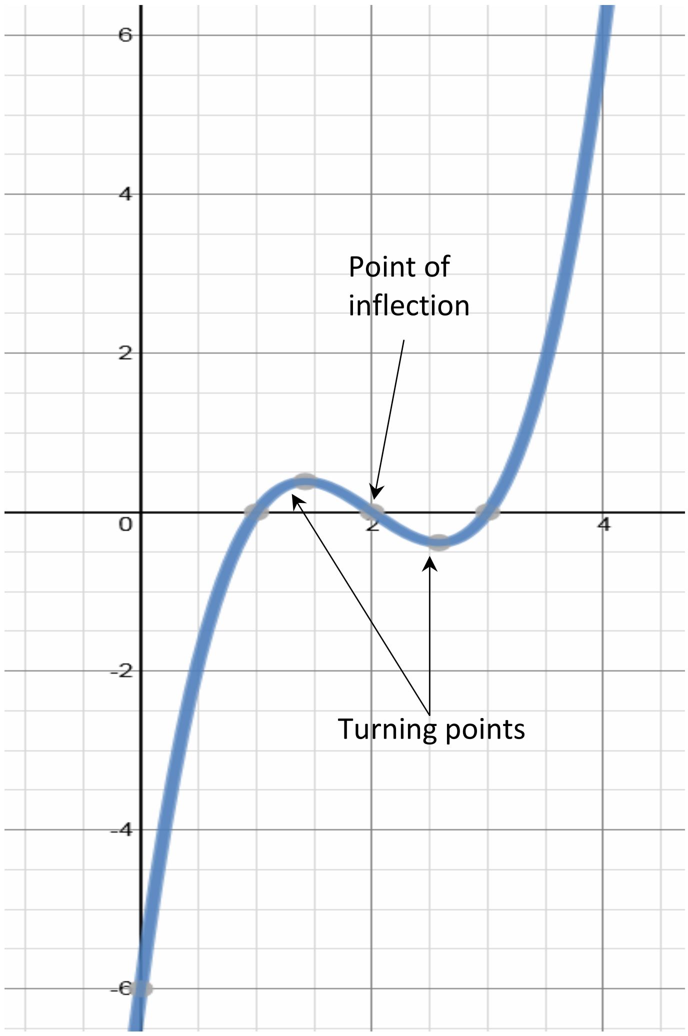
i.e. solve

$$0 = 3x^2 - 12\,\mathrm{x} + 11$$

Using the quadratic formula with a=3, b=(-12), c=11 gives the solutions

$x = 1.423$ and $x = 2.577$

(check this matches the graph)



Step4: Substitute these two x values back into the original equation (curve) to find the associated y coordinates for the turning points.

i.e. for $x = 1.423$, $y = 1.423^3 - 6(1.423)^2 + 11(1.423) - 6 = 0.385$

and for $x = 2.577$, $y = 2.577^3 - 6(2.577)^2 + 11(2.577) - 6 = -0.385$

So, coordinates of the turning points are

$(\mathrm{x} = 1.423, \mathrm{y} = 0.385)$ $(\mathrm{x} = 2.577, \mathrm{y} = -0.385)$

Second Differential.

By differentiating the equation a second time it is possible to determine the nature of the turning points. i.e. whether a turning point is a maximum or a minimum

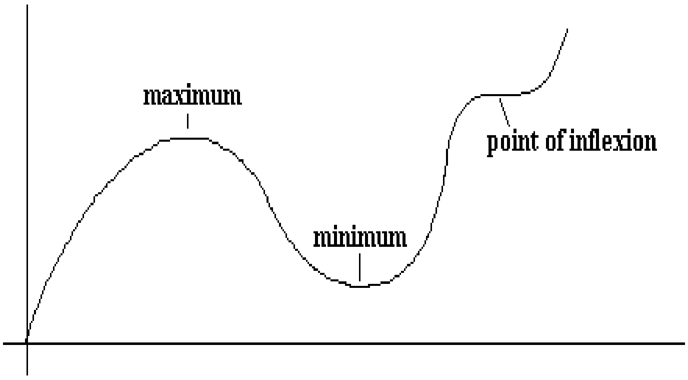
If the 2nd differential is positive ($\frac{d^2y}{dx^2} > 0$) at turning point x, the point is a minimum.

if the 2nd differential is negative ($\frac{d^2y}{dx^2} < 0$) the point is a maximum.

in the previous worked example $\frac{d^2y}{dx^2} = 6x - 12$, when x = 1.423, 6x – 12 is –3.48 (negative) so the turning point at x=1.423 is a maximum (as visible on the graph).

Point of Inflection

A point of inflection is defined to be a point of a curve at which a change in the direction of curvature occurs (from convex to concave). By definition, the second derivative of a curve is equal to zero at a point of inflection. We can use this relationship to find the coordinates of a point of inflection (if it exists).



In the previous worked example, we can find the point of inflection by setting $\frac{d^2y}{dx^2} = 0$ and solving for x

$i.e.\ \frac{d^2y}{dx^2} = 6x - 12,$

solving $0 = 6x - 12, x = 2$ so a point of inflection occurs at $x = 2$

i.e. at the coordinate (2,0)

Calculus Exercise 6.1

Plot the following curves and determine their maximum and minimum points for x and y both graphically and by calculation.

1. $y = x^2 - 2x$ for values of x =–1 to x= 3.

2. $y = x^3 - 6x^2 + 12$ for values of x =–1 to x= 6.

3. $y = 4x^3 + 9x^2 - 12x + 13$ for values of x = –4 to x= 3.

Integration Overview

Recall, by differentiating an equation, the gradient function is found, and this function is used to determine the slope at any point on the curve. Differentiation involves multiplying the coefficients of x by the power, then subtracting 1 from the power.

i.e. $\frac{dy}{dx} = nax^{(n-1)}$

For example: $y = 5x^3 + 3x^2 + 7x - 12$

differentiate $\frac{dy}{dx} = 15x^2 + 6x + 7$

Integration (or anti-differentiation) is the reverse of differentiation.

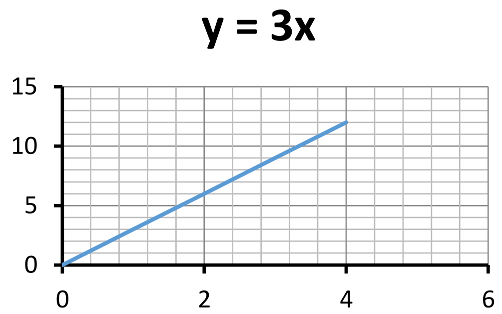
Integration of a curve (function) may be used to find the area below the curve, either from the origin or between two given values of x.

Theory of Integration

Consider the straight line $y = 3x$, whose equation of course follows the rule

$y = mx + c$

This line starts at the origin and rises by $3y$ for every $1x$ (i.e. has a gradient of 3)



The area of the triangle is equal to

$\frac{1}{2}x\,y$

Because we know $y = 3x$ we can substitute this in the above equation so,

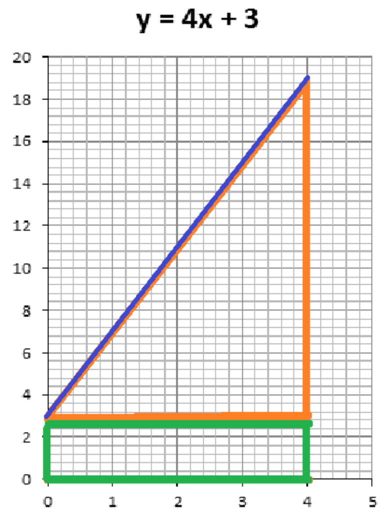
$Area = \frac{1}{2}x \times 3x$

$Area = 1.5 \times x^2$

(Notice that if we differentiate this we get back to the original equation of $y = 3x$)

Example : For equation $y = 4x + 3$,

Area can be calculated by adding the triangle and the rectangle below it:



For the triangle, $y = 4x$

$$Area = \frac{1}{2}\text{base} \times \text{height}$$

$$Area = \frac{1}{2} \times x \times y$$

$$Area = \frac{1}{2} \times x \times (4x)$$

$$Area = 2x^2$$

For the rectangle, $y = 3$,

$$Area = \text{base} \times \text{height}$$

$$Area = x \times y = 3x$$

So general equation for $total\ area = 2x^2 + 3x$

For example, if x = 4 the total area between x = 0 and x = 4 would be

$$2 \times 4^2 + 3 \times 4 = 32 + 12 = 44 \text{ units}^2$$

Integrating Curves:

The principle shown above also works for curved areas.

To integrate each term of an equation, increase its power by one and divide the coefficient by this new power.

Mathematically, the symbol for integration is ∫ dx which stands for "integrating with respect to x".

Comparing Methods of Differentiation and Integration for general equation $y = ax^n$	
In differentiation, $\frac{dy}{dx} = nax^{(n-1)}$ we subtract one from the power we multiply the coefficient by the power any constants (numbers) disappear	In integration, $\int ax^n\,dx = \frac{a}{(n+1)}x^{(n+1)}$ we add one to the power we must divide by the new value of the power we must allow for a constant (number) to reappear, but as we cannot determine its exact value, it is just labelled 'c', the constant of integration

Example 1: Integrate $2x^3$

$$\int 2x^3\,dx$$
$$\frac{2}{(3+1)}x^{(3+1)} + c$$
$$\frac{2}{4}x^4 + c$$
$$0.5x^4 + c$$

A single number such as 5 could be written as $5x^0$ and will integrate to 5x.

Example 2: Integrate $5x^4 + 2x^3 - 4x^2 + 3x + 7$

$$\int 5x^4 + 2x^3 - 4x^2 + 3x + 7\,dx$$

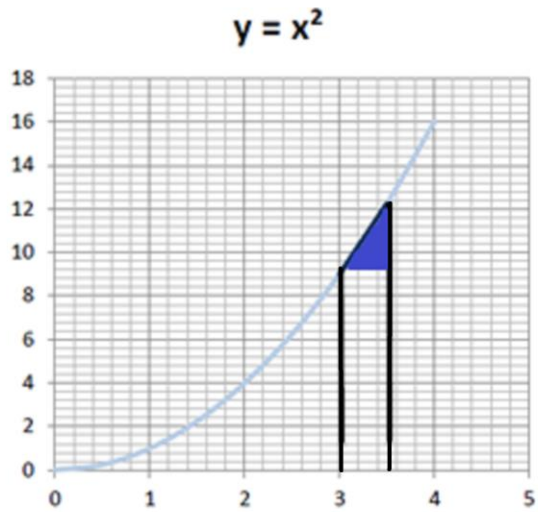
$$\frac{5x^5}{5} + \frac{2x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2} + 7x + c$$

$$x^5 + \frac{1x^4}{2} - \frac{4x^3}{3} + \frac{3x^2}{2} + 7x + c$$

$$x^5 + 0.5x^4 - 1.33x^3 + 1.5x^2 + 7x + c$$

Example 2 : (comparing areas calculated numerically and also by integration)

In the diagram the curve is of y=x^2
The strip of area under the curve between $x = 3$ and $x = 3.5$ can be found **numerically** by adding the area of the blue triangle and the area of the rectangular strip below it.
i.e. the area of the blue triangle can be calculated as:
$\frac{1}{2}bh = \frac{1}{2} \times (3.5 - 3) \times (12.25 - 9) = 0.81$
This assumes that the curve is a straight line along the hypotenuse of the triangle.
The area of the rectangular strip below the triangle can be calculated as:
$base \times height = (3.5 - 3) \times 9 = 4.5$
The total area calculated numerically is therefore
$0.81 + 4.5 = 5.3$ units^2



When the area under a curve is found by **integral calculus**, multiple thin strips of width δx are summed, so that the accuracy is greater than when using a numerical method. The area between $x = 3$ and $x = 3.5$ is found by taking the difference between the area under the graph up to $x = 3.5$ and the area under the graph up to $x = 3$.
i.e. by integration,

$$\int_3^{3.5} x^2 \, dx$$

$$= \left[\frac{x^3}{3}\right]_3^{3.5}$$

$$= \left(\frac{3.5^3}{3}\right) - \left(\frac{3^3}{3}\right)$$

$= 14.29 - 9 = 5.29$ units^2

More integration worked examples

Example 1: Find the area below a curve y = x^2 between the values of x = 2 and x = 5

$\int_2^5 x^2 dx$ (The integral of x^2 between the limits of 5 and 2)

This is called a 'definite integral' and the area is calculated and written in the format below

$$\left[\frac{x^3}{3}\right]_2^5$$

Substituting 5 into the equation for x we get the area between 0 and 5, repeat the substitution with x = 3 and subtract to get the area between 2 and 5. i.e.

$$\left[\frac{x^3}{3}\right]_2^5 = \left(\frac{5^3}{3}+c\right)-\left(\frac{2^3}{3}+c\right) = \frac{125}{3}+c-\frac{8}{3}-c = \frac{117}{3} = 39\ sq.units$$

Notice the subtractions causes c to cancel out.

Example 2: Find the area between x = 3 and x = 5 under the curve y = $3x^3 - 2x^2 + 4x + 12$ when x and y units are metres.

$$\int_3^5 3x^3 - 2x^2 + 4x + 12\, dx$$

$$= \left[\frac{3x^4}{4} - \frac{2x^3}{3} + \frac{4x^2}{2} + 12x\right]_3^5$$

$$= \left(\frac{3\times 5^4}{4} - \frac{2\times 5^3}{3} + \frac{4\times 5^2}{2} + 12\times 5\right) - \left(\frac{3\times 3^4}{4} - \frac{2\times 3^3}{3} + \frac{4\times 3^2}{2} + 12\times 3\right)$$

$$= (468.75 - 83.33 + 50 + 60) - (60.75 - 18 + 18 + 36)$$

$$= 495.42 - 96.75 = 398.67\ m^2$$

(Note: In calculus there are special rules (for example the chain rule) for differentiating products which cannot be expanded and also rules for fractions (quotient rule) and trig functions. eg. x^2 Sin x The answer is <u>not</u> the product of the individual differentials. For this basic syllabus we will only deal with expandable products (polynomials))

Calculus Exercise 6.2

1. Differentiate $y = 4x^4 + 2x^3 - 3x^2 + 2x - 5$

2 Differentiate $y = (3x^3 - 4x^2 + 2)(2x^2 - 3x + 3)$

3. Differentiate $y = (x + 2)^3$

4. Differentiate $y = (x^2 - 3)^2$

5. For the curve $y = x^3 - 6x^2 + 11x - 6$ find

   i. the values of x and y when the slope of the graph is zero; and

   ii. the point of inflexion.

6. Use www.desmos.com to draw the curve of $y = 3x - x^2$ for values of x between -1 and x=4. Calculate the gradient of the curve when x = 0, 1, 2 and 3. What is the value of x and y when the gradient is zero.

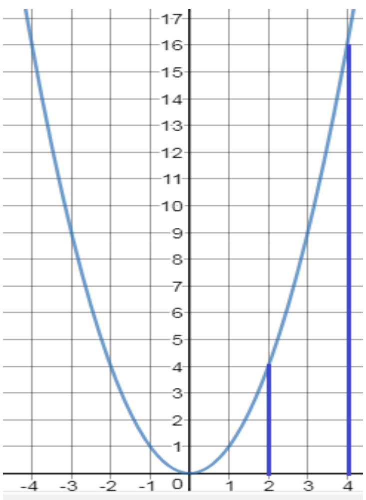
7. Find the max and min co-ordinates (x, y) for the curve $y = 2 - 9x + 6x^2 - x^3$

8. Find the value of x and y at the turning point of curve $y = -x^2 + 3x + 1$

9. An open tank with a square base is to be made of sheet steel with a capacity of $8m^3$. Calculate the minimum area of steel that can be used. *Hint: Write down the formulae for volume and surface area of the tank. Substitute for breadth and depth into units of Length. Area will be minimum when its equation differential is zero. Differentiate it and solve for L.*

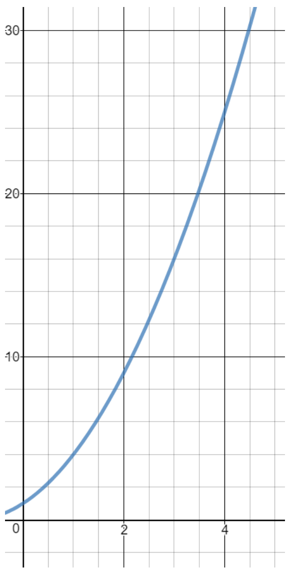
10. If $y = x^2$ metres find the area between the x axis and the ordinates x = 2 and x = 4

$$\int_2^4 x^2\,dx$$

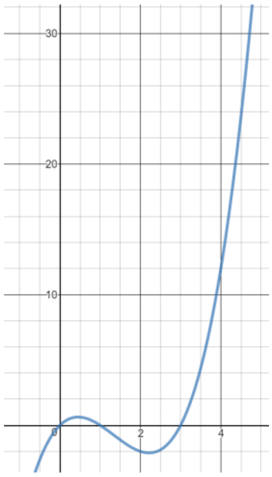


11. Find the area between x = 2 and x = 5 of curve $y = x^2 + 2x + 1$ metres

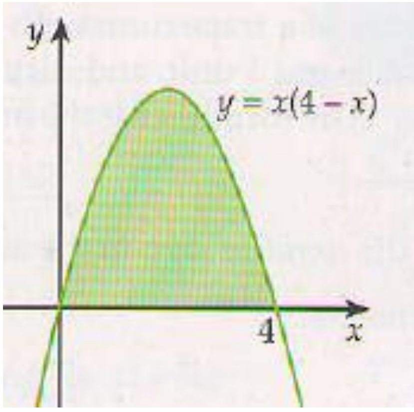
$$\int_2^5 x^2 + 2x + 1\,dx$$



12. Find $\int_1^3 x^3 - 4x^2 + 3x\,dx$



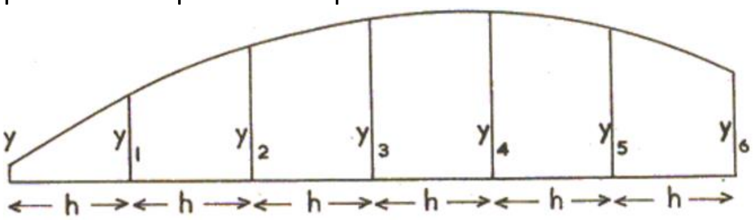
13. Find the shaded area shown below



Simpsons Rule (numerical method for approximating area)

In ship stability we use a numerical method form of integration known as Simpsons Rule to find the area of a ships water plane or the area below the GZ curve.

These rules simply assume that the curve of the hull or graphed curve follow a simple parabolic shape whose equation is known.



Therefore the accuracy of the answer depends on how closely the actual curve follows the equation used but is acceptable for most practical applications.

$$Area \approx \frac{1}{3}h[y_0 + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]$$

$$or\ Area \approx \frac{1}{3}h \times (sum\ of\ products)$$

The products for each area can be calculated within a table as follows

y_n	Simpson's Multiplier	Product of Area
y_0	1	$1 \times y_0$
y_1	4	$4 \times y_1$
y_2	2	$2 \times y_2$
y_3	4	$4 \times y_3$
y_4	2	$2 \times y_4$
y_5	4	$4 \times y_5$

y_6	1	$1 \times y_6$

Example: A ship's water-plane is 120 m long. The half-ordinates (measured centre line to the ship's side), measured at equal intervals from aft, are: 0.1 m, 4.5 m , 7.4 m , 7.6 m , 7.5 m, 3.6 m and 0 m. Find the water-plane area.

There are 7 ordinate measurements so there must be 6 intervals. Therefore, each interval $h = \frac{120}{6} = 20\ m$. Use a table (or spreadsheet) to calculate the sum of the Area products:

$half\ ordinate$	Simpson's Multiplier	Product of Area
0.1	1	0.1
4.5	4	18
7.4	2	14.8
7.6	4	30.4
7.5	2	15
3.6	4	14.4
0	1	0
Total		92.7

$$Area \approx \frac{1}{3} h \times (sum\ of\ products)$$

$$Area \approx \frac{1}{3} \times 20 \times 92.7 \approx 618\ m^2$$

The water plane area is double the area calculated above. i.e. $1236\ m^2$

Activity: Create a spreadsheet model to calculate other water plane areas.

Exercise 6.3 : Using Simpson's Rule to Estimate Area

1. The ordinates measured from the centre line to the hull across a ship at the load water line are: 0.2 m, 4.4 m, 6.7 m, 7.2 m, 7.7 m , 8.1 m, 8.0 m, 7.8 m, 5.8 m, 3.6 m and 0 m, and the water plane length is 130m. Find the water-plane area.
2. An object, starting from rest has the following velocities at the times stated. Use Simpson's rule to estimate the distance travelled by the object in 4 seconds.

t (seconds)	0	1	2	3	4
v (m/s)	0	12	24	32	37

ANSWERS

Exercise 6.1

1. (x=1, minimum) 2. Turning points: (0, 12) max and (4, -20) min

3. Turning points: (-2, 41) max and (0.5, 9.75) min

Exercise 6.2

1. $16x^3 + 6x^2 – 6x + 2$ 2. $30x^4 – 68x^3 + 63x^2 – 16x – 6$

3. $3x^2 +12x + 12$ 4. $4x^3 – 12x$

5. turning points (gradient=0) at (1.42, 0.385), (2.58, -0.385), Pt inflexion: (2, 0)

6. gradients are 3, 1, –1, –3; co-ordinate when gradient =(1.5, 2.25)

7. (3, 2), (1, –2) 8. 1.5, 3.25

9. L = B = 2.52m d = 1.26, and minimum area=19.05 m^2

10 18.67 m^2 11. 63. m^2

12. 2.67 sq. units 13. $10\frac{2}{3}$ sq. units

Exercise 6.3

$h = \frac{130}{10} = 13$

$half\ ordinate$	Simpson's Multiplier	Product of Area
0.2	1	0.2
4.4	4	17.6
6.7	2	13.4
7.2	4	28.8
7.7	2	15.4
8.1	4	32.4
8.0	2	16.0
7.8	4	31.2
5.8	2	11.6
3.6	4	14.4
0	1	0
Total		**181**

$$Area \approx \frac{1}{3}h \times (sum\ of\ products)$$

$$Area \approx \frac{1}{3} \times 13 \times 181 \approx 784.3\ m^2$$

Q2

$h = \frac{4}{4} = 1$

$velocity$ (m/s)	Simpson's Multiplier	Products
0	1	0
12	4	48
24	2	48
32	4	128
37	1	37
Total		**261**
$Distance\ travelled\ by\ object = \frac{1}{3} h \times (sum\ of\ products)$ $= \frac{1}{3} \times 1 \times 261 = 87\ m$		