MODULE 3 : TRIGONOMETRY

Introduction

Trigonometry is a branch of mathematics involving the measurement of sides and angles of triangles, and relationships between sides and angles of triangles. The word 'trigonometry' is derived from Greek 'trigonon' (triangle) and 'metron' (measure).

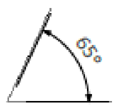
Triginometry is used in astronomy, surveying and navigation, often to find an inaccessible distance, such as that between Earth and the moon for example.

Angles

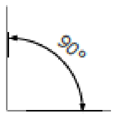
An angle is a measure of the rotation between 2 straight lines.

Angle Definitions:

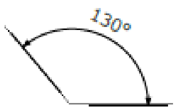
- ***acute*** angles are less than 90°



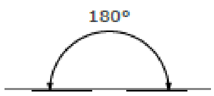
- right angle equal 90°



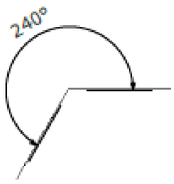
- ***obtuse*** angles are greater than 90° but less than 180°

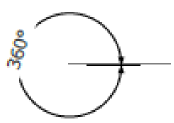


- straight line angles equal 180°



- ***reflex*** angles are greater than 180° but less than 360°

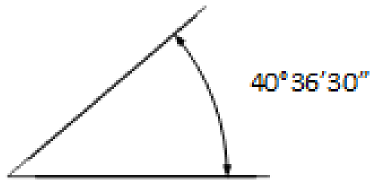




- angles at a point sum 360°

Angles are measured in:

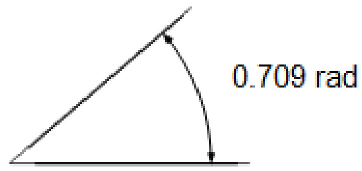
- degrees, minutes, seconds
- radians (arc)



θ radians = arc/radius

360° = 2π radians

Or approximately 1rad = 57.3°



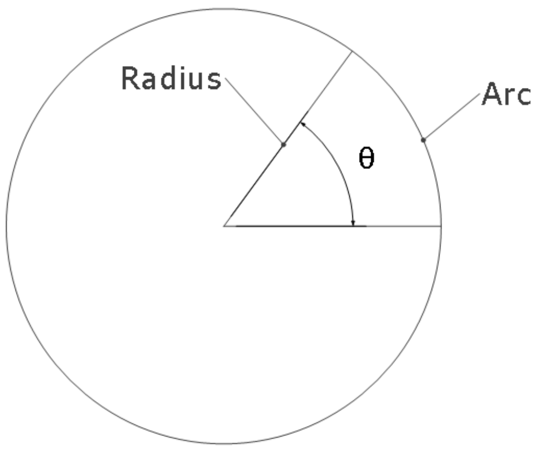
Radian Measure

An angle can be measured in degrees, minutes and seconds or radians

The radian measure of an angle can be obtained by the formula:

$$\theta\ radians = \frac{arc}{radius}$$

The symbol for angles in radians is 'radian' or 'rad' or in some cases θ^c (c for circular measure)

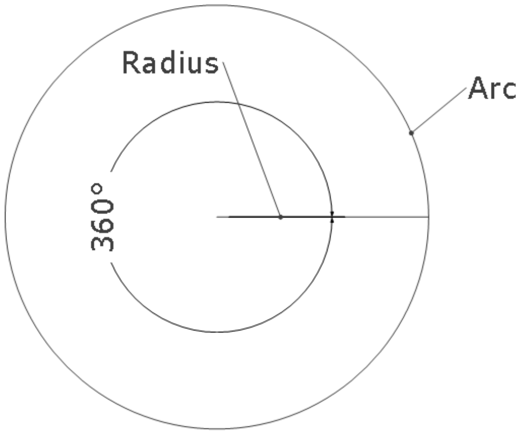


Radian measure is useful to obtain an angle without using a protractor as it can be obtained just by measuring lengths of the arc and the radius. Radians are quite commonly used in engineering, for example to express angular velocity. Radians are sometimes found convenient to use as they can be represented in the form of **π .**

Note: A unit called 'grads' (100 grads = 90°) was introduced to try and obtain metric angular measure but was dropped although it still appears on scientific calculators.

Converting Radians to Degrees and vice versa

Consider a circle and the angle at its centre



The angle at the centre in degrees

$\theta° = 360°$

The angle at the centre could also be measured in radians

$$\theta = \frac{arc}{radius} = \frac{2\pi r}{r} = 2\pi$$

So $360^o = 2\pi$ radians

or $180^o = \pi$ radians or $57.3^o = 1$ radian

So, to convert to radians (from degrees):	To convert to degrees (from radians):
$\theta\ radians = \frac{degrees\ \times 2\pi}{360}$	$\theta\ degrees = \frac{radians\ \times 360}{2\pi}$

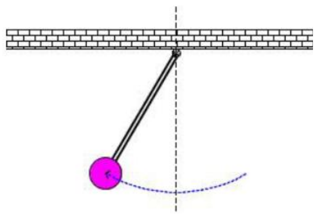
Example 1: Convert 36° to radians.

$$\frac{36 \times 2\pi}{360} = 0.628\ radians$$

Example 2: Convert 1.4 radians to degrees.

$$\frac{1.4 \times 360}{2\pi} = 80.21°$$

Example 3: A 600mm long pendulum swings through an angle of 72 degrees. What is the length of the arc through which the pendumus swings?



$$length\ of\ arc = \frac{72}{360}\ of\ a\ full\ circumference$$

$$\therefore arc\ length = \frac{72}{360} \times 2\pi r = \frac{72}{360} \times 2\pi \times 600 = 754\ mm$$

Example 4: A 600mm long pendulum swings through an arc of 500mm. Through what angle (in degrees) does the pendulum swing?

$$a\ full\ circumference\ of\ radius\ 600 = 2\pi \times 600 = 3770\ mm$$

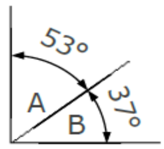
$$\frac{arc}{circumference} \times 360° = \frac{500}{3770} \times 360 = 47.7°$$

Properties of Angles created with Parallel Straight Lines

Complementary and Supplementary angles:

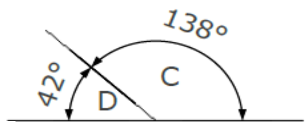
- Complementary angles add up to 90°

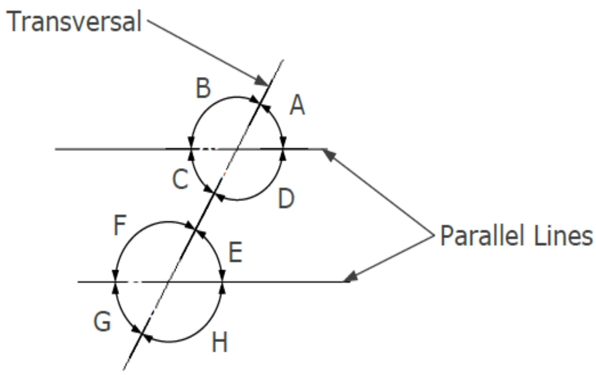
  $\angle A + \angle B = 90°$

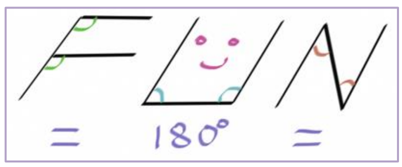


- Supplementary angles add up to 180°

  $\angle C + \angle D = 180°$







Alternate angles are equal.

- Angles C and E are alternate and equal.
- Angles D and F are alternate and equal.

$\angle C = \angle E$ $\quad\quad$ $\angle D = \angle F$

Corresponding angles are equal.

- The following pairs of angles are corresponding and are equal:

$\angle A = \angle E$ $\quad\quad$ $\angle C = \angle G$

$\angle B = \angle F$ $\quad\quad$ $\angle D = \angle H$

Interior angles sum 180 degrees (are supplementary):

- The following pairs of angles are Interior angles and are supplementary:

  $\angle C + \angle F$=180°

  $\angle D + \angle E$=180°

When two straight lines cross each other the **vertically opposite angles are equal**.

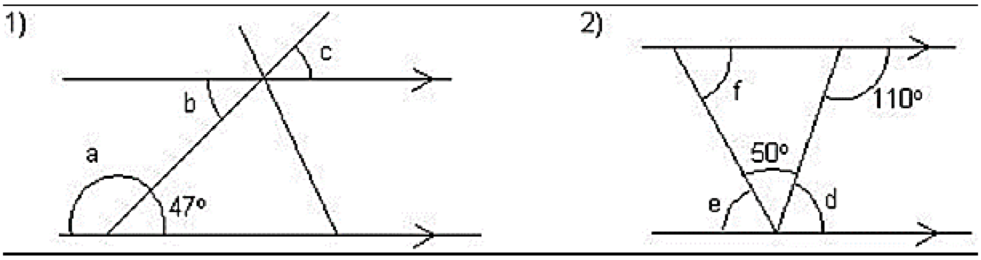
- The following pairs of angles are vertically opposite angles and are equal:

  $\angle A = \angle C$ $\angle E = \angle G$

  $\angle B = \angle D$ $\angle F = \angle H$

Exercise 3.1 Properties of angles

Determine the size of the following angles



3) Convert $40°$ to radians giving your answer to 3 d.p.

4) Convert 2.5 radians to degrees giving your answer to 1 d.p.

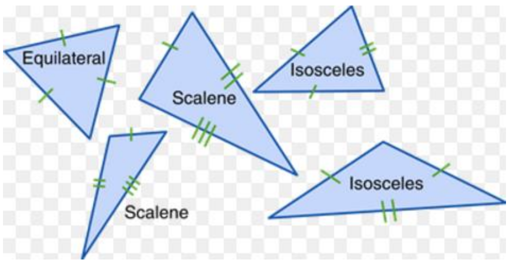
Types of Triangles

A triangle is a figure enclosed by three straight lines.

The sum of the three angles in a triangle is equal to 180 degrees.

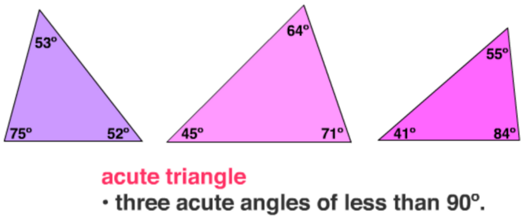
Types of triangles based on side lengths:

- Isosceles - two sides are equal, two angles are equal
- Equilateral - all three sides are equal, all angles are equal
- Scalene - unequal sides and unequal angles.

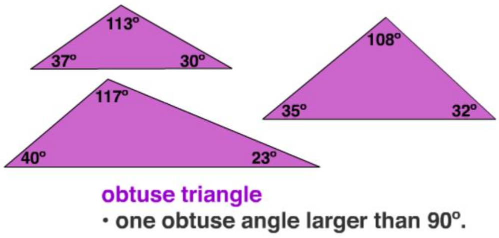


Types of triangles based on largest angle:

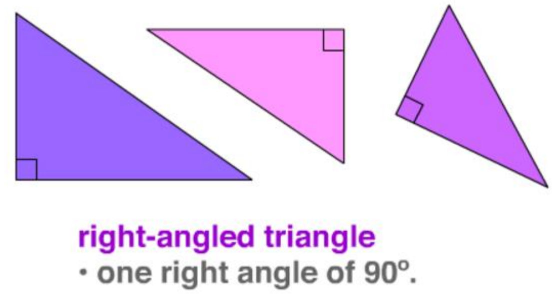
Acute



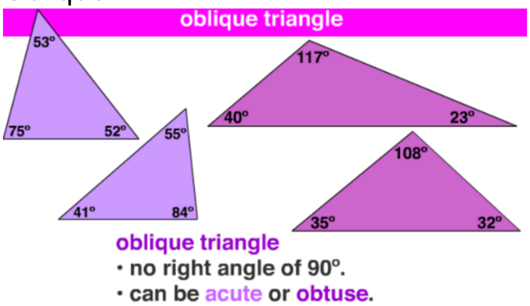
Obtuse



Right-angled

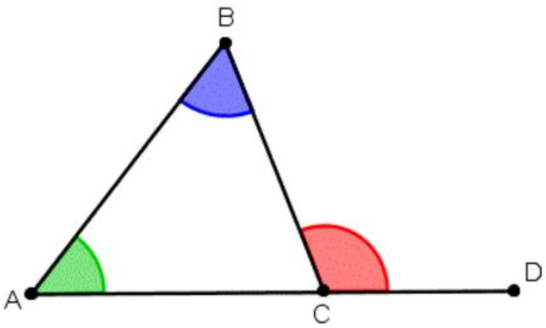


Oblique

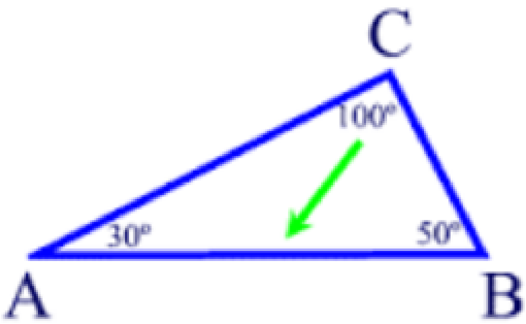


The side opposite the right angle is known as the **Hypotenuse.** This is always the longest side.

Other Properties of Triangles:



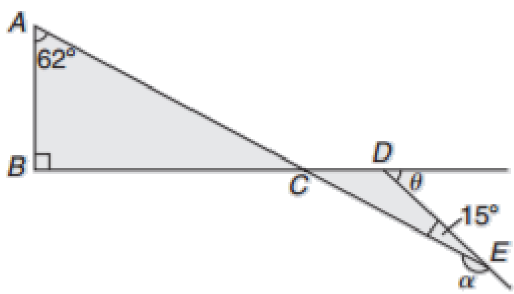
The exterior angle of a triangle (C) equals the sum of the interior opposite angles (A and B).



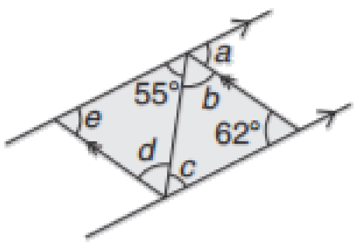
The largest side is always opposite the largest angle.

Exercise 3.2 Angles in Triangles

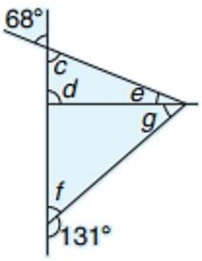
1. Find angles α and θ giving angle reasons for each step of logic.



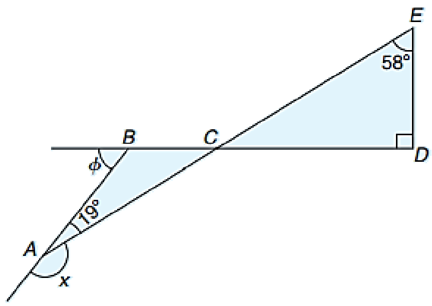
2. Find angles a,b,c,d and e, giving angle reasons for each step of logic.



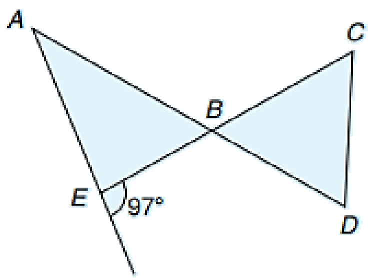
3. Given that angle d is a right angle, find angles c, e, f, and g giving angle reasons for each step of logic.



4. Find angles x, φ giving angle reasons for each step of logic.

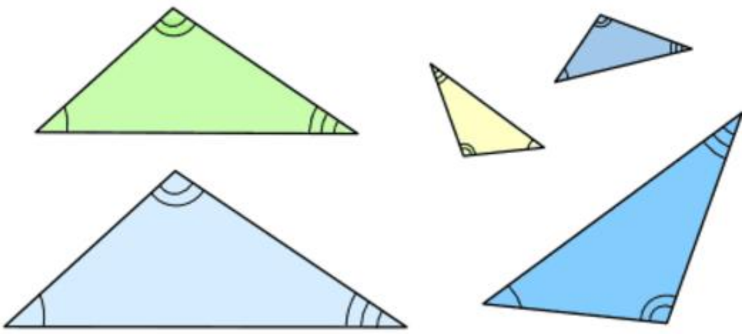


5. BCD is an equilateral triangle. Determine all interior angles of triangle ABE giving angle reasons for each step of logic.

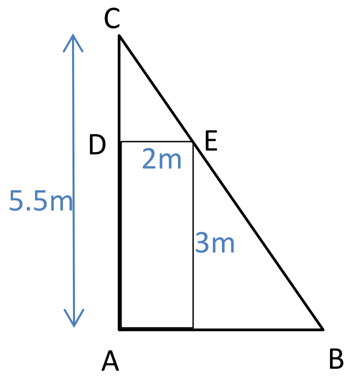


Similar Triangles

Two triangles are said to be similar if the angles of one triangle are equal to the angles of the other (even though the length of their sides may differ).



Worked Example



Using the measurements shown, determine length AB.

Length CD=5.5-3=2.5

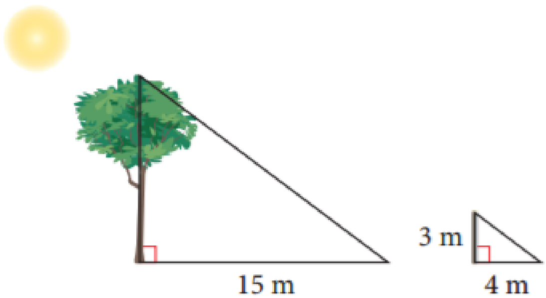
Since triangles ABC and CDE are similar:

$$\frac{AB}{5.5} = \frac{2}{2.5}$$

$$AB = \frac{2 \times 5.5}{2.5} = 4.4\ m$$

Exercise 3.3 Solving Problems using Similar Triangles

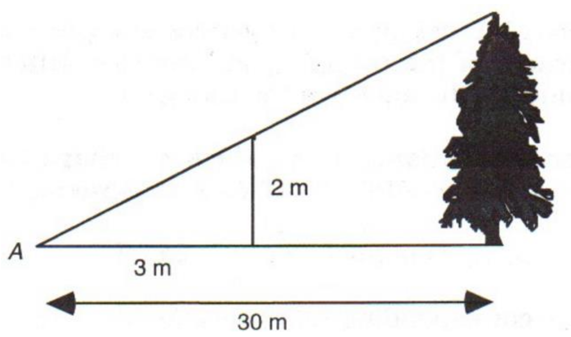
1. A 3m long pole, erected vertically, casts a shadow 4m in length. At the same time, a tree casts a 15m length shadow.



a) Find the height of the tree.

b) How much higher is the tree than the pole.

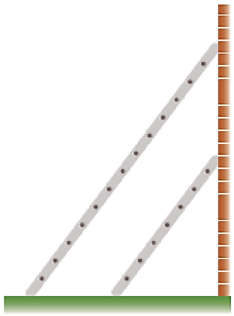
2. A 2m long pole, erected vertically, casts a shadow 3m in length. At the same time, a tree casts a 30m length shadow.



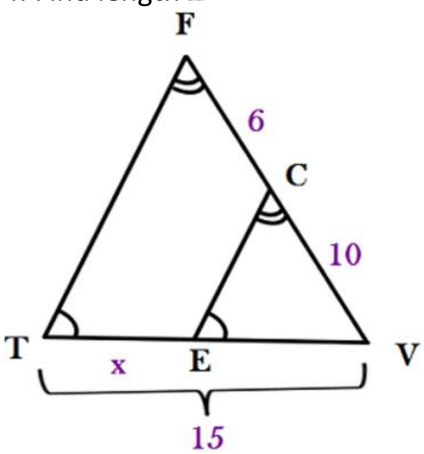
a) Find the height of the tree.

b) How much higher is the tree than the pole.

3. Two ladders are leaning against a wall. They both make the same angle with the ground. The 3m ladder reaches 2.4m vertically up the wall. How much further up the wall does the 5.4m ladder reach?



4. Find length x



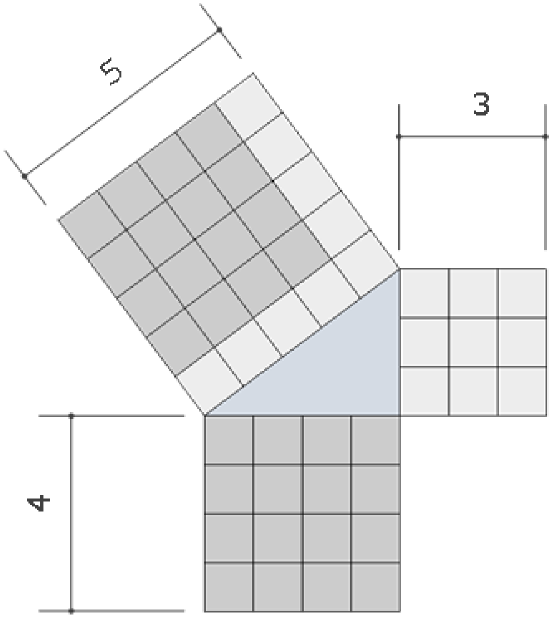
5. Right angled triangles ABC and DEF are similar. The lengths of the sides of ABC are 510 mm, 680 mm and 850 mm. The length of the longest side of DEF is 2040 mm. Find the length of the shortest side of DEF.

Theorem of Pythagoras

This theorem is named after ancient Greek mathematician Pythagoras (c580-500BC), though it probably predates him, as evidence suggests ancient Babylonians were using this theorem over 4000 years ago.

IMPORTANT: This theorem only applies to Right angled triangles

In any right triangle, the area of the square whose side is the hypotenuse (the side opposite the right angle) is equal to the sum of the areas of the squares whose sides are the two legs (the two sides that meet at a right angle).



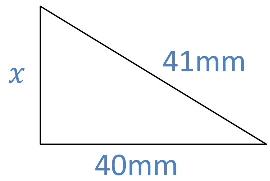
$a^2 + b^2 = c^2$

$3^2 + 4^2 = 5^2$

$9 + 16 = 25$

Example

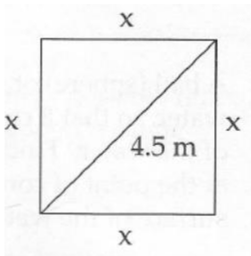
Find the length of the missing side, x of this right-angled triangle.



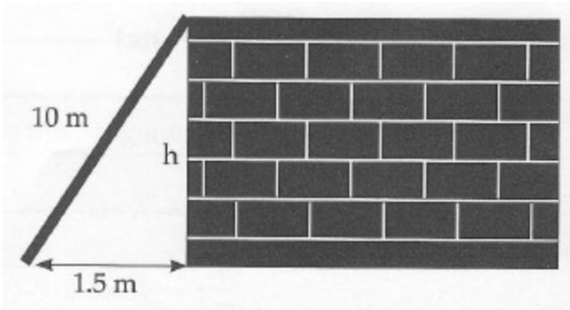
$$x^2 = 41^2 - 40^2$$
$$x^2 = 81$$
$$x = 9\ mm$$

Exercise 3.4 Solving Problems using Pythagoras Theorem

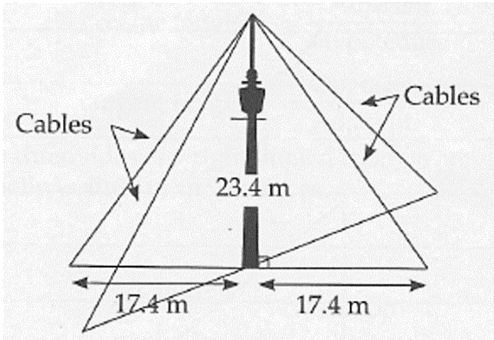
1. A square has a diagonal of 4.5 m. Find the length of its sides.



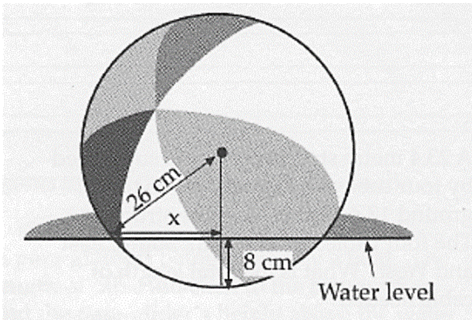
2. A 10 metre long ladder leans against a wall. If the base of the ladder is 1.5 m from the wall, find the height, h that the ladder reaches up the wall.



3. A 23.4 m structure is supported by 4 cables. The cables are anchored to the ground North, South, East and West of the structure. The distance along the ground from the point where each cable is anchored to the base of the structure is 17.4 m. Find the length of each cable.



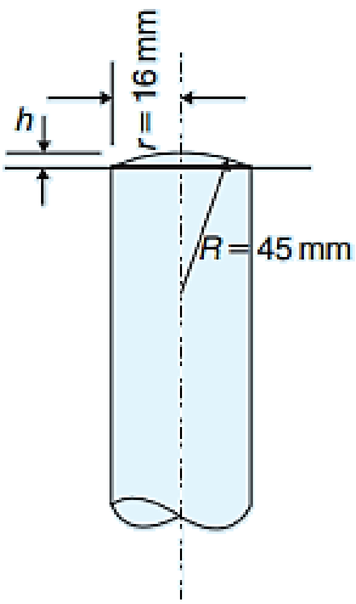
4. A ball (sphere) of radius 26 cm is floating in water, so that 8cm of the ball is below the level of the water. Find the length x



5. Two ships leave port at the same time. One travels due West at 18.4 knots and the other due South at 27.6 knots. Calculate how far apart the ships are after 4 hours.

(Hint: Recall 1 knot = 1 nautical mile per hour)

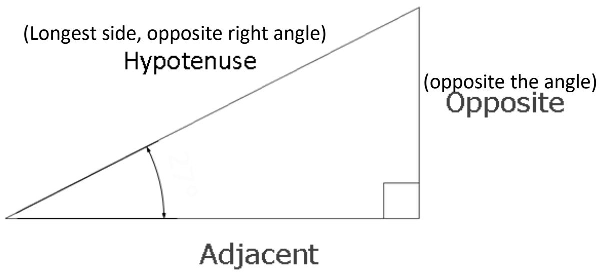
6. The sketch shows a bolt rounded off at one end. Determine the dimension h

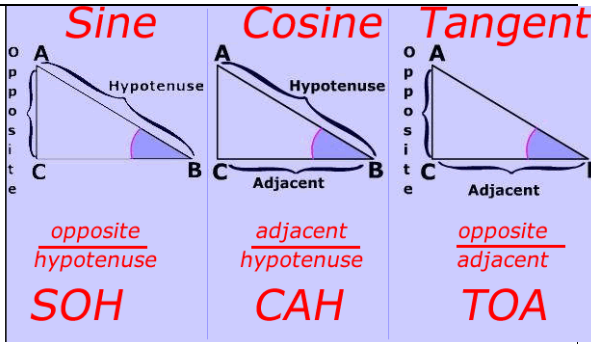


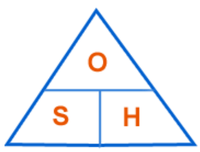
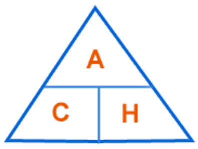
Trigonometric Ratios

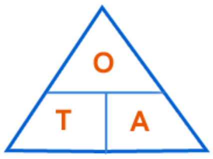
Trigonometric Ratios for Angles between 0° and 90°

The ratio of the lengths of the sides of right-angled triangles are called trigonometric ratios. Every angle has its own value of Sine (sin), Cosine (cos) and Tangent (tan).

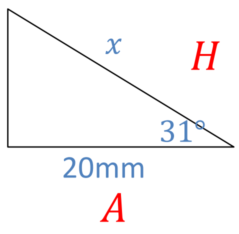


Trig ratio values can be obtained by: • geometric drawing of triangles and measuring sides. • using trigonometric tables • using scientific calculators.	

Trig Function	**Ratio**	**Anogram**	**Triangle Pic** *(cover the term you are trying to find and read the formula required)*
Sine	$\frac{Opp}{Hyp}$	SOH	
Cosine	$\frac{Adj}{Hyp}$	CAH	

Tangent	$\frac{Opp}{Adj}$	TOA	

Example 1 Find length x



<u>Step1:</u> Label the triangle to choose which trig ratio to use

<u>Step 2</u>: Write formula to use

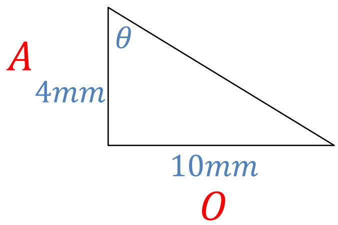
$$H = \frac{A}{cos31}$$

<u>Step 3</u>: Substitute values and solve

$$H = \frac{20}{cos31}$$

$$H = 23.3\ mm$$

Example 2 Find angle θ



<u>Step1:</u> Label the triangle to choose which trig ratio to use

<u>Step 2</u>: Write formula to use

$$tan\theta = \frac{O}{A}$$

<u>Step 3</u>: Substitute values and solve

$$tan\theta = \frac{10}{4}$$

$$\theta = tan^{-1}\left(\frac{10}{4}\right)$$

$$\theta = 68.2°$$

Trigonometry Exercise 3.5

The following two questions refer to right angled triangles. Draw and label each triangle before finding the remaining sides and angles.

1.a) Triangle ABC ∠ACB = 90° AB = 223 mm ∠BAC = 22°

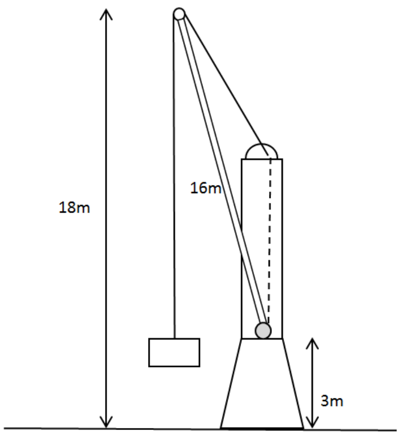
1 b) Triangle PQR ∠PRQ = 90° QR = 1.86 km PQ = 2.52 km

2. A vessel is 45 miles due North of a point of land. What course must she steer, and what distance must she travel to reach a point 85 miles due East of the same point of land?

3. A ladder is placed against a vertical bulkhead. If the ladder is 4 metres in length and is at an angle of 70° to the deck, how far from the deck is the top of the ladder?

4. A ship leaves port (P) on a course of 050°(T) at 15 knots. Calculate how far North of port (P) and how far East of port (P) the ship will be after travelling for 3 hours.

5. A 16 metre long derrick is rigged so that the derrick head is 18 metres above the deck. If the derrick heel is 3 metres above the deck, what angle does the derrick make from the vertical?



6. A ship leaves Port A and steams True West for 72 miles. She then turns to True South and steams for 48 miles to arrive at Port B. How far is Port B from Port A?

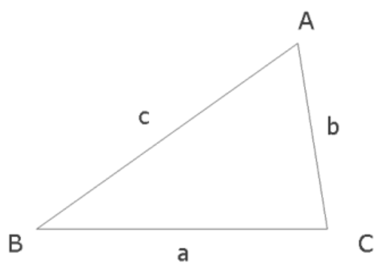
7. When a boat is 110 metres from the base of a beacon, the angle of elevation from the boat to the top of the beacon is found to be 15° 35’20’’ (15.589°). If the boat is then moved to a point 200 metres away from the foot of the beacon, frind the new angle of elevation to the top of the beacon?

The Sine Rule (FOR ANY TRIANGLE)

Trigonometric ratios (SOH CAH TOA) and Pythagoras theorem cannot be used to find missing sides or angles of non-right-angled triangles. Instead, the Sine and Cosine rules may be used.

$$\frac{a}{Sin\ A} = \frac{b}{Sin\ B} = \frac{c}{Sin\ C}$$

where angle A is opposite side a
where angle B is opposite side b

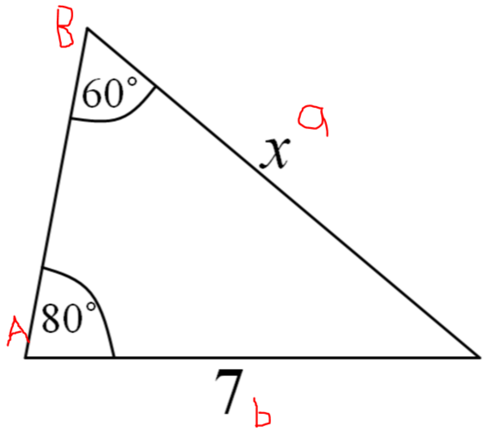


This rule can be used when:

- 1 side and any 2 angles are given
- 2 sides and an angle opposite one of them are given

Example 1

Find missing side x in this triangle



Step1: *Label the side you are trying to find 'a' and label other angles and sides.*

Step 3: *Write formula to use*

$$\frac{a}{Sin\ A} = \frac{b}{Sin\ B}$$

Step4: *Substitute values and solve*

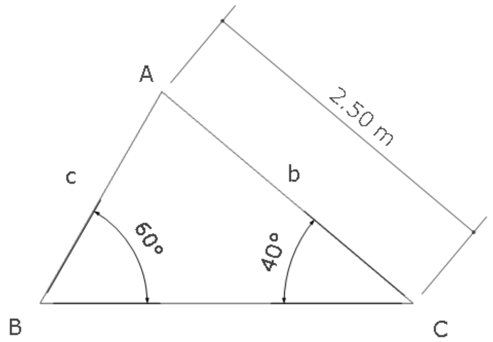
$$\frac{a}{Sin\ 80} = \frac{7}{Sin\ 60}$$

$$a = \frac{7 \times Sin80}{Sin\ 60}$$

$$a = 7.96$$

Example 2

In the triangle given below, find side c



$$\frac{b}{Sin\ B} = \frac{c}{Sin\ C}$$

$$\frac{2.5}{Sin\ 60°} = \frac{c}{Sin\ 40°}$$

$$\frac{2.5}{Sin\ 60°} = \frac{c}{Sin\ 40°}$$

$$c = \frac{2.5\ \times 0.643}{0.866}$$

$$c = \ 1.856\, m$$

Example 3

Find missing angle $m°$ in this triangle

$$\frac{SinA}{a} = \frac{SinB}{b}$$

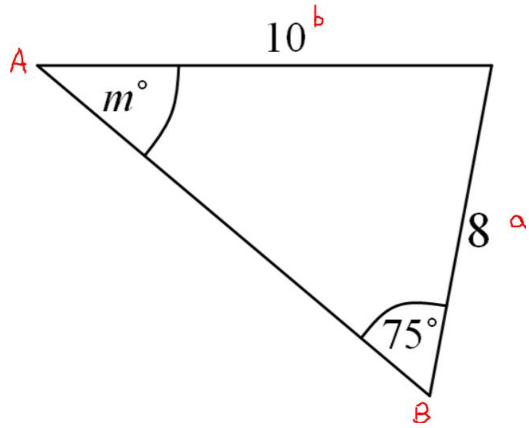
$$\frac{SinA}{8} = \frac{Sin75}{10}$$

$$SinA = \frac{8 \times Sin75}{10}$$

$$SinA = 0.7727$$

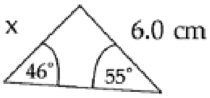
$$A = Sin^{-1}(0.7727)$$

$$A = 50.6°$$

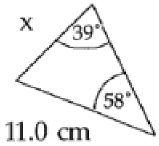


Trigonometry Exercise 3.6 Sine Rule Skills

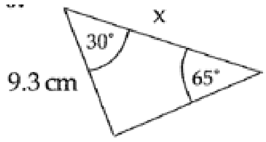
1. Find side x.



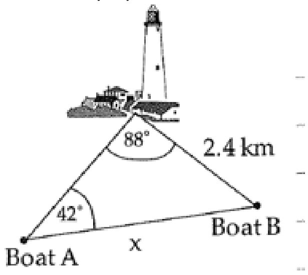
2. Find side x.



3. Find side x.



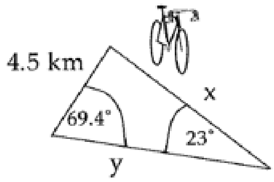
4. Two boats are offshore from a lighthouse as shown below. Calculate the distance, x, between the 2 boats.



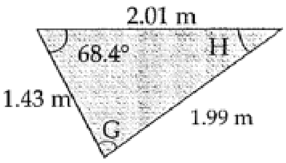
5. A bike race consists of a triangular course as shown.

Calculate lengths x and y

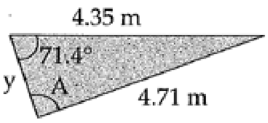
Find the total distance of the race.



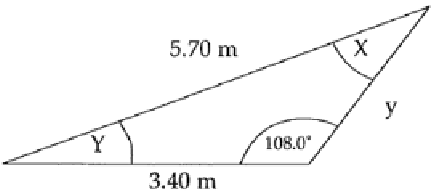
6. Find angles G and H.



7. Find angle A and side y.



8. Find angles X and Y and side y.



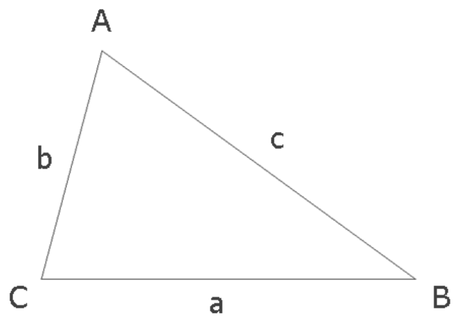
The Cosine Rule (FOR ANY TRIANGLE)

To find the length of unknown side 'a', the cosine rule is

$$a^2 = b^2 + c^2 - 2bc\ \cos A$$

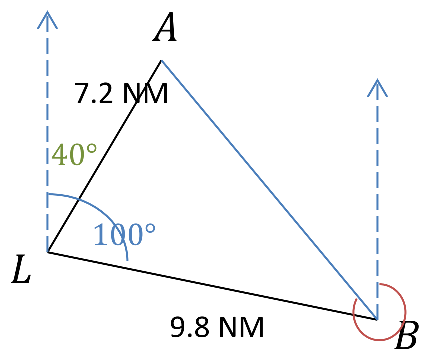
By transposing this equation we can derive the rule to find unknown angle 'A':

$$Cos\ A = \frac{b^2 + c^2 - a^2}{2bc}$$

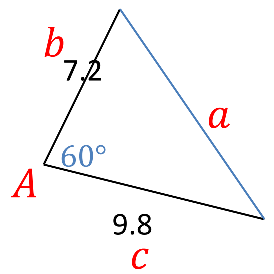


Example 1

From a lighthouse, vessel A is 7.2 NM away on a bearing of 40° and vessel B is 9.8 NM away on a bearing of 100°. Find the distance between A and B.



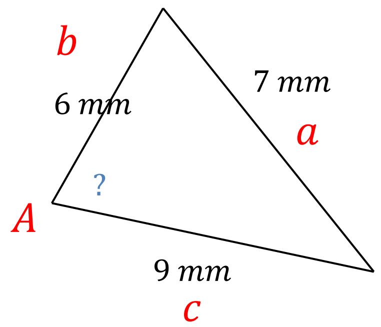
<u>Step1:</u> Draw sketch of the problem.
<u>Step2:</u> Add useful angles and lengths to help determine which trig ratio to use. Note angle $ALB = 100° - 40° = 60°$. *(Perhaps redraw the sketch to ensure correct labelling for trig formula).*
<u>Step 3</u>: Write formula to use
$a^2 = b^2 + c^2 - 2bcCosA$
<u>Step4</u>: Substitute values and solve
$a^2 = 7.2^2 + 9.8^2 - 2 \times 7.2 \times 9.8 \times Cos60$
$a^2 = 51.84 + 96.04 - 70.56$
$a = \sqrt{77.32}$
$a = 8.8\ NM\ (1\ d.p.)$



Example 2

A triangle has sides of lengths $9mm$, $6mm$ and $7mm$. Determine the size of all the internal angle between the $9mm$ and $6mm$ sides.
<u>Step1:</u> Draw sketch of the problem.
<u>Step2:</u> Add useful angles and lengths to help determine which trig ratio to use. Remember to label the angle you are looking for 'A' and opposite side 'a' (to ensure correct substitution into the trig formula).
<u>Step 3</u>: Write formula to use

$$CosA = \frac{b^2 + c^2 - a^2}{2bc}$$

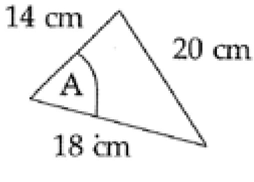


Step4: Substitute values and solve

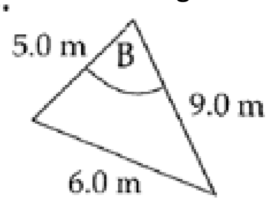
$$CosA = \frac{6^2 + 9^2 - 7^2}{2 \times 6 \times 9}$$
$$CosA = \frac{36 + 81 - 49}{108}$$
$$CosA = 0.6296$$
$$A = Cos^{-1}(0.6296)$$
$$A = 51°$$

Trigonometry Exercise 3.7 Cosine Rule Skills

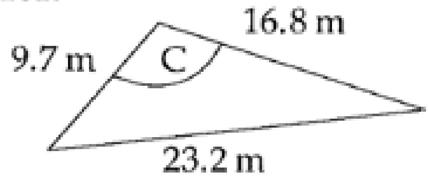
1. Calculate angle A.



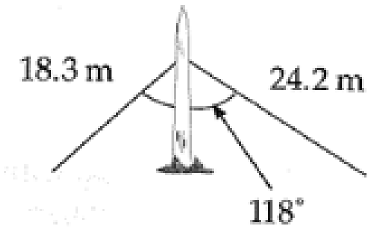
2. Calculate angle B.



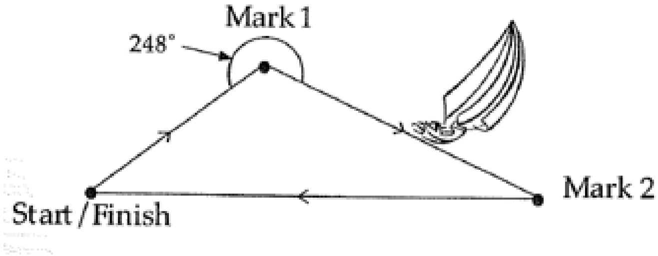
3. Calculate angle C.



4. A radio mast is held by two wires anchored to the ground. One of the wires is 18.3 m long and the other wire is 24.2 m long. If the angle between the two wires is 118°, calculate the distance between the ropes at ground level.

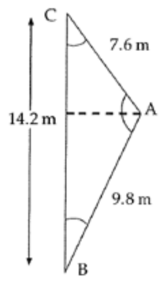


5. This diagram shows a yacht race course. The distance from the start to Mark 1 is 3.2 km. The distance from Mark 1 to Mark 2 is 5.9 km.



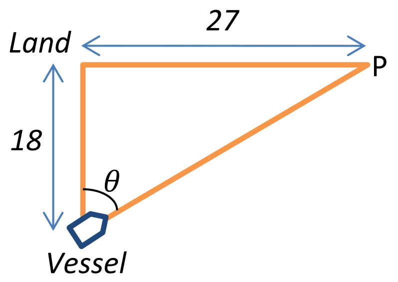
a) Calculate the distance form Mark 2 to the Finish.
b) If a yacht averages 3.8 km/h, how long will it take to complete the course?
c) Calculate the angle through which the yacht must turn at Mark 2 in order to head for the finish.

6. Calculate angles A, B and C



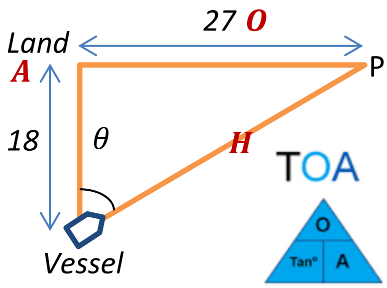
Worked Examples For Solving Trigonometry Problems in Context

Worked Example 1: A vessel is 18 NM due S of a point of land. What course must she steer to reach a point 27 NM due E of the point of land?



The vessel must steer a course as shown in the diagram (from the starting position 18NM due south of the land to a point (labelled 'P') 27 NM east of the land).

The angle θ can be found using trig (sine rule or right angle trig (SOH CAH TOA).



<u>Method 1</u> (right angle triangle trig)

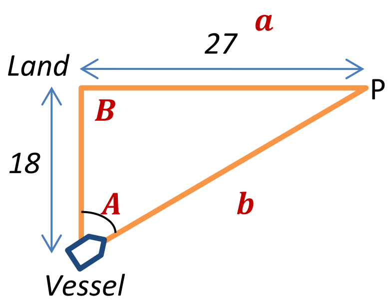
1. Label the Hypotenuse (H), Opposite (O) and Adjacent (A) sides of the triangle.
2. Determine which trig function to use. In this case, use Tan because there are length for sides 'O' and 'A' (TOA)

$$tan\theta = \frac{O}{A}$$

$$tan\theta = \frac{27}{18}$$

$$\theta = tan^{-1}\left(\frac{27}{18}\right)$$

$$\theta = 56.3°$$



<u>Method 2</u> (sine rule for any triangle)

1. Label the angle you are trying to find 'A'
2. Label the side opposite this angle 'a'
3. Label the other known angle 'b' and its opposite side 'B'
4. In this case we need to determine length 'b' using Pythagoras theorem before we can use the sine rule

$$b^2 = 18^2 + 27^2$$

$$b = \sqrt{1053}$$

$$b = 32.45\ NM$$

5. Solve for angle 'A' using the sine rule

$$\frac{SinA}{a} = \frac{SinB}{b}$$

$$\frac{SinA}{27} = \frac{Sin90}{32.45}$$

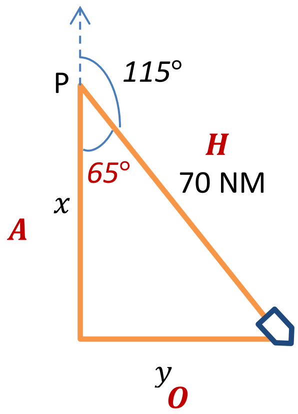
$$SinA = \frac{27 \times Sin90}{32.45}$$

$$SinA = 0.832$$

$$A = sin^{-1}(0.832)$$

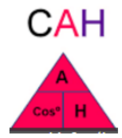
$$A = 56.3°$$

Worked Example 2: A ship leaves port P sailing on a course of 115°T. Calculate how far South of P and how far East of P the ship is after 70 NM?

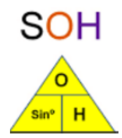


<u>Method 1</u> (right angle triangle trig)

1. Label any internal angles (65°) and ‘A’, ‘H’,’O’ sides.

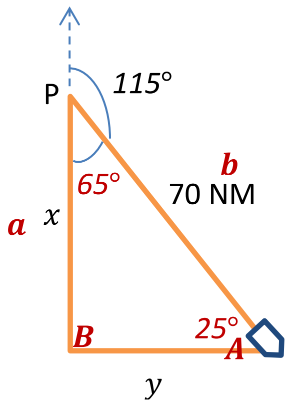


2. Choose trig function to find x
   $A = cos\theta \times H$
   $A = cos65 \times 70$
   $A = 29.6\ NM$



3. Choose trig function to find y
   $O = sin\theta \times H$
   $O = sin65 \times 70$
   $O = 63.4\ NM$

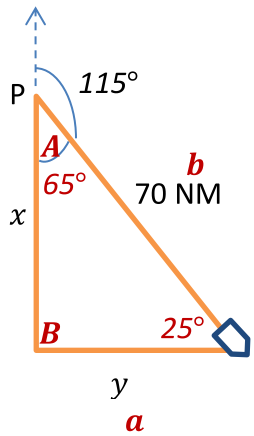
<u>Method 2</u> (sine rule for any triangle)



Finding x

1. Label the x side you are trying to find ‘a’
2. Determine the angle opposite side ‘a’ and label it ‘A’
3. Label the other known angle ‘B’ and its opposite side ‘b’

$$\frac{a}{sinA} = \frac{b}{sinB}$$
$$\frac{a}{sin25} = \frac{70}{sin90}$$
$$a = \frac{70 \times sin25}{sin90}$$
$$a = 29.6\ NM$$

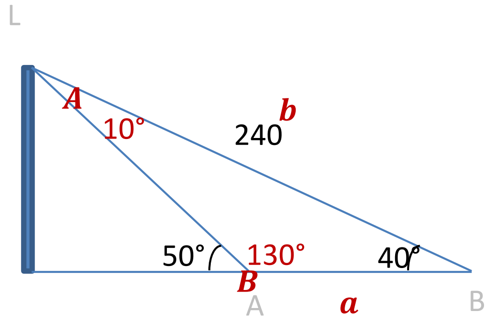


Finding y

1. Re-label the y side you are trying to find ‘a’
2. Determine the angle opposite side ‘a’ and label it ‘A’
3. Label the other known angle ‘B’ and its opposite side ‘b’

$$\frac{a}{sinA} = \frac{b}{sinB}$$
$$\frac{a}{sin65} = \frac{70}{sin90}$$
$$a = \frac{70 \times sin65}{sin90} = 63.4\ NM$$

Worked Example 3: Two swimmers, A and B are swimming in the same direction towards a lighthouse. The distance from B, furthest away, to the top of the lighthouse is 240 metres. The angle of elevation from A is 50°, and from B is 40°. Calculate the distance between A and B to the nearest metre.



<u>Method</u> (using sine rule for any triangle)

1. Label other missing angles in triangle LBA
   Angle LAB=180-50=130°
   Angle ALB=180-130-40=10°
2. Label the distance AB between the swimmers 'a'
3. Determine the angle opposite side 'a' and label it 'A'
4. Label the 240 metre side 'b' and its opposite angle 'B'

$$\frac{a}{sinA} = \frac{b}{sinB}$$

$$\frac{a}{sin10} = \frac{240}{sin130}$$

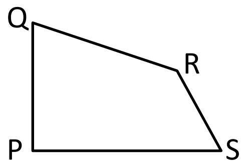
$$a = \frac{240 \times sin10}{sin130}$$

$$a = 54.4\,m$$

Trigonometry Exercise 3.8 General Trigonometry Problems

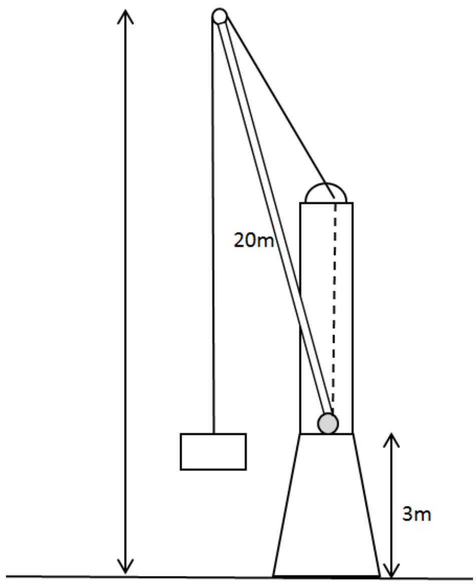
1. A vessel is 18.0 miles due South of a point of land. What course must she steer, and what distance must she travel to reach a point 27 miles due East of the same point of land?
2. A ship leaves port P on a course of 115°(T). Calculate how far south of "P" and how far East the ship will be after travelling for 70 miles.
3. A derrick topped to 32° from the vertical has its heel 4 metres above deck level. If the derrick is 24 metres in length, how high is the derrick head above the deck.
4. A ship leaves port A and steams East (T) for 14.9 miles. She then turns to North (T) and steams a further 10.42 miles thus arriving at port B. Find the direct course and distance from A to B.
5. From a boat the angle of elevation of the top of a lighthouse is 20° 10'. The boat is then rowed directly towards the lighthouse until the elevation is 35° 10'. If the lighthouse is 61.0m high, find how far the boat was rowed.
6. A cube has all its sides measuring 5 cm. Find the distance between the diagonally opposite corners of the box.

7. PQRS is a quadrilateral with PQ = QR = 12, RS = 8 and SP = 20. If angle RSP is 60°, find the angle PQR.



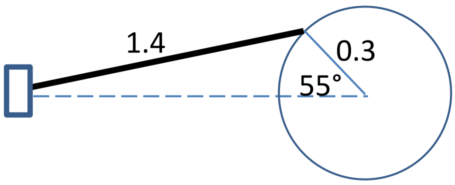
8. A rectangular steel plate has a diagonal line painted on it from one corner to the opposite corner. The diagonal is 23 metres in length, and is at an angle of 32° to the edge of the plate. What is the area of the plate?
9. A ship steams a distance of 55NM in a direction of 140° (T) and then a distance of 75 NM in a direction of 230° (T). Find the resultant displacement of the ship from the point of origin.
10. A ship is steaming along a coastline on a course of 000°(T). A lighthouse is observed bearing 040°(T), distant 7.3 miles. The ship maintains course, and later the same lighthouse is observed to be distant 9.0 miles. What is the bearing of the lighthouse at this second observation?
11. Ship A bears 079°(T) from C and is 107km from C. Ship B bears 101° (T) from C and is I43km from C. How far is A from B?
12. Two ropes attached to the top of a vertical mast 24m high go to cleats on the deck 32m and 18m from the foot of the mast. If the cleats are 36m apart, what is the angle between the ropes when they are taut?
13. A plank of wood 3 metres long leans against a wall with its foot 1 metre from the wall. Another plank of wood, also 3 metres long, rests on the first plank with its other end on the ground a further 1 metre from the wall. What is the angle between this second plank and the ground?

14 A derrick of 20 metres length is topped to 30° from vertical. Its heel is 3 metres above deck level. How high is the derrick head above deck level?



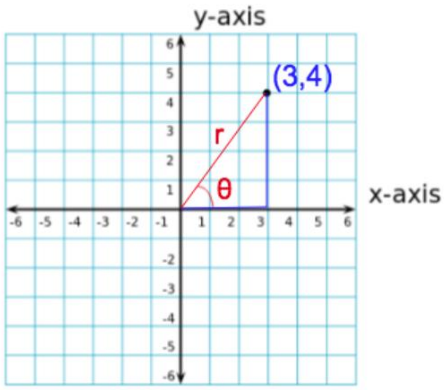
15. A man on top of a 60.50 metre high cliff sees a boat at an angle of 20.25° below his horizon. How far is the boat from the base of the cliff?

16. The crank of an engine is 0.3m long and connecting rod 1.4m long. Given that the crank makes an angle of 55° with the line of stroke of the piston, determine the angle between the connecting rod and the line of stroke.



Polar and Rectangular Coordinates (or Cartesian Coordinates)

Position (x, y) on a Cartesian plane (2 dimensional axes) can also be written in 'Polar' form as the point (r, θ), where r is the range (distance or magnitude from a central point) and θ is the angle measured anti-clockwise from the first quadrant (positive x axis) as shown in the diagram.



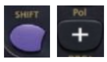
Polar Coordinates (r,θ)
Cartesian Coordinates (x,y)

Therefore, using tirgonometry, the relationship between these two forms of coordinates is
$x = r \,\mathrm{Sin}\,\theta$ and $y = r \,\mathrm{Cos}\theta$
(where θ is measured anticlockwise relative to the positive x-axis).

Converting Polar and Rectangular Coordinates

Example: Convert the position coordinate (3, 4) to polar form.

$r = \sqrt{3^2 + 4^2}$
$r = \sqrt{25}$
$r = 5$
By trig, $4 = 5\sin\theta$
$0.8 = \mathrm{Sin}\theta$
$\mathrm{Sin}^{-1}(0.8) = \theta$
$\theta = 53.13°$

Note: calculators provide functionality for converting between polar and rectangular coordinates using the 'Pol' and 'Rec' keys. . For example, to convert the rectangular coordinate (3,4) into Polar form select  and type in the coodinate $Pol(3,4) =$. The answer in polar form $r = 5, \theta = 53.13$. *Calculators always define θ as measured anticlockwise from the positive x-axis (into quadrant 1); so it follows that negative values of θ measure clockwise from the positive x-axis (into quadrant 4).*

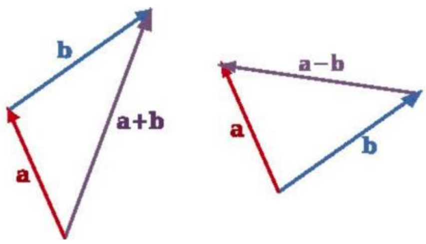
Use of Polar and Rectangular Coordinates

Polar and Rectangular Coordinates are useful when combining vectors to obtain a resultant. This has applications in equation of motion calculations (mechanics) and in navigation to calculate day's work. Any number of vectors can be added together by using the polygon of vectors method. This means adding the tail of one vector to the nose of the previous vector all drawn to scale on a coordinate grid. The resultant vector is drawn from the tail of the first vector to the nose of the last vector. The vectors can be drawn in any order.

Example 1:
Calculate resultant magnitude (r) of $\vec{a} + \vec{b}$

Vector $\vec{a}$: $r = 5\ \theta = 110°$

$\vec{b}$: $r = 6\ \theta = 30°$



Step1: write the vectors in rectangular form using calculator Rec(5,110) and Rec(6,30)

	x	y
Rec(5,110)	-1.7101	4.6985
Rec(6,30)	5.1962	3.0000
sum	3.4861	7.6985

Step2: sum the x and y components.
Step3: convert resultant answer back to Polar coordinates to determine r

$Pol(3.4861, 7.6985) = r = 8.451, \theta = 65.6°$

(Check if you are able to do this calculation in one step using your calculator).

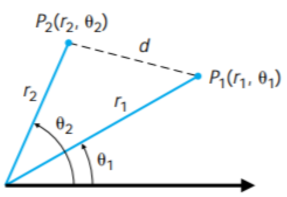
Example 2: Calculate magnitude (r) of $\vec{a} + \overrightarrow{-b}$

Step1: write the vectors in rectangular form using calculator Rec(5,110) and Rec(6,30)

	x	y
Rec(5,110)	-1.7101	4.6985
-Rec(6,30)	-5.1962	-3.0000
sum	-6.9063	1.6985

$Pol(-6.9063, 1.6985) = r = 7.112, \theta = 166.2°$

Example: Find the distance between the two points $P_1\ (4, 30°)\ and\ P_2\ (3, 60°)$



Step1: Recognise this question could be solved using vectors $\vec{r_1} + \overrightarrow{-r_2}$

	x	y
Rec(4,30)	3.4641	2.0000
-Rec(3,60)	-1.5	-2.5981
sum	1.9641	-0.5981

$Pol(1.9641, -0.5981) = d = 2.05, \theta = -16.9°$

Note that $\theta = -16.9°$ is measured clockwise from $positive\ x - axis\ because\ of\ negative\ sign.$

c.f. using the cosine rule to solve this problem:

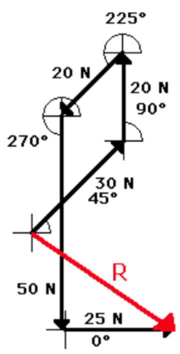
$d^2 = 4^2 + 3^2 - 2 \times 4 \times 3 \times Cos30$

$d^2 = 4.2154 = 2.05$

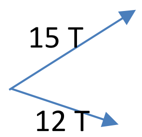
Trigonometry Exercise 3.9

1. Determine the resultant force and direction of a 45N force directed east and a 60N force directed south.

2. Determine the resultant magnitude and direction, R, of the 5 vectors shown in this diagram. (Hint: remember angles in quadrants 3 and 4 (from 180 to 360 degrees) are input as negative θ values measured clockwise from the positive x-axis. e.g. the 20 N force at angle 225° would be input as $\theta = 225° - 360° = -135°$).



3. Two tugs are made fast to the bow of a ship. One tug pulls in a direction 340° with a bollard pull of 12 T while the other pulls towards 040° at 15 T. Find the resultant force on the ship.



4. A force of 200 N due South and another force of 300 N due East each act on an object simultaneously.
    a. Determine the resultant net force.
    b. A third force now acts on the object so that the net force is 0. Determine its magnitude and direction.

5. A boat heads due east across a 100 m wide stream with a velocity of 20.0 m/s. The stream is flowing from north to south at a rate of 5.00 m/s.
    a. What is the resultant velocity of the boat?
    b. How long does it take the boat to reach the other side?
    c. How far downstream is the boat when it reaches the other side?
    d. In which direction should the boat head in order to end up directly across the stream?

ANSWERS TO EXERCISES

Exercise 3.1

1a) 133° 1b) 47° 1c) 47°

2d) 70° 2e) 60° 2f) 60°

3) $0.698\ radians$

4) 143.2°

Exercise 3.2

1.

$\angle ACB = 28^\circ \quad \angle DCE = 28^\circ \quad \angle CDE = 137^\circ$

$\alpha = 165^\circ$

$\theta = 43^\circ$

2.

$a = 62^\circ \quad alt.\angle$

$b = 63^\circ \quad \angle\, straight\ line$

$c = 55^\circ \quad alt.\angle \quad (or\ sum\ \angle\, triangle)$

$d = 63^\circ$

$e = 62^\circ \quad corresp.\ \angle$

3.

$c = 68^\circ,\ d = 90,\ e = 22^\circ,\ f = 49^\circ,\ g = 41^\circ$

4.

$x = 161^\circ \quad \angle\, straight\ line$

$\phi = 51^\circ$

5. $\angle CBD = 60^\circ\ \angle ABE = 60^\circ\ \angle AEB = 83^\circ\ \angle EAB = 37^\circ$

Exercise 3.3

1. a) 11.25 m b) 8.25 m

2. a) 20 m b) 18 m.

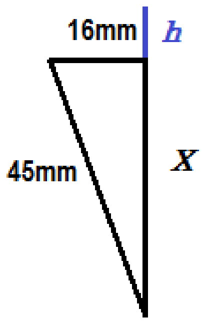
3. 1.92 m

4. 5.625

5. 1224 mm

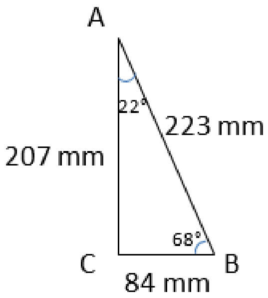
Exercise 3.4

1. $x = 3.2\,m\ \ (1\ d.p.)$
2. $h = 9.9\,m$
3. each cable is $29.16\ m$
4. $x = 18.8cm$
5. 132.68 nautical miles
6. $h = 2.94\,mm$

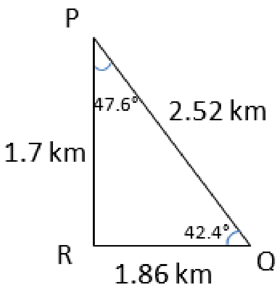


Exercise 3.5

1a)



1b)



2.	117° 53' 50.1" (or 117.9°) 96.176 miles	3.	3.76 metres
4.	28.93 miles North, 34.47 miles East	5.	20.4°
6.	86.53 miles	7.	8.72°

Exercise 3.6

1. 6.8 cm (1 dp.)
2. 14.8 cm (1 dp.)
3. 10.2 cm (1 dp.)

4. 3.6 km (1 dp.)

5. x=10.8 km; y= 11.5 km; Total distance= 26.8 km

6. $G = 69.7° \ and \ H = 41.9°$ OR $H = 41.7° \ and \ G = 69.9°$

7. $A = 61.1°$; $y = 3.66 \ m$

8. $X = 34.6°$ $Y = 37.4°$ $y = 3.64 \ m$

Exercise 3.7

1. $76.2°$
2. $38.9°$
3. $119.8°$
4. 36.6 m
5. a) 7.7 km; b) 4 hours and 25 min (4.417hrs) ; c) $157°$
6. $A = 109°$ $B = 30°$ $C = 41°$

Exercise 3.8

1. $056° \ (T)$, $32.45 \ miles$
2. $29.6° S \ 63.4° \ E$
3. 24.353 m
4. $055° \ (T)$, $18.18 \ miles$
5. 79.5 m (allow rounding error)
6. 8.66 cm
7. $93.2°$
8. $237.73 \ m^2$
9. $194° T$, $93 \ miles$
10. $148.6°$
11. 59.366 km
12. $59.89°$
13. $52.21°$
14. 20.32 m
15. Boat.is 164 m from base of cliff
16. 10.1°

Exercise 3.9

1. 75N $\theta = -53.1° \ (S \ of \ E) or \ 306.9°$
2. R=39.4 N, $\theta = 324°$

3. 23.4 tonnes towards 013.7°
4. a) 361 N @ 56.3° E of S b) 361 N @ 56.3° W of N (opposite direction to resultant)
5. a) 21 m/s @ 14.0° S of E b) 4.9 s c) 24 m d) 14.5° N of E