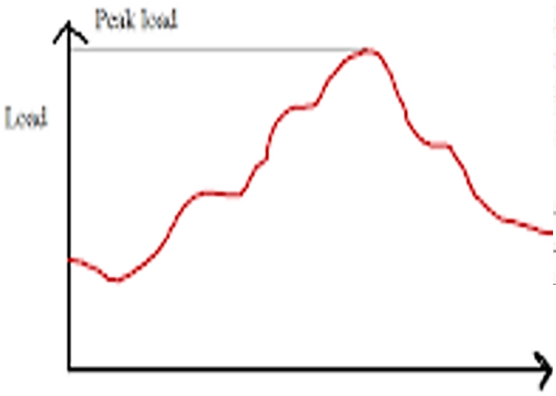
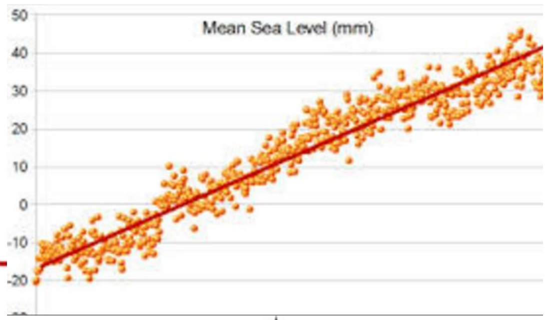
MODULE 5 : GRAPHS

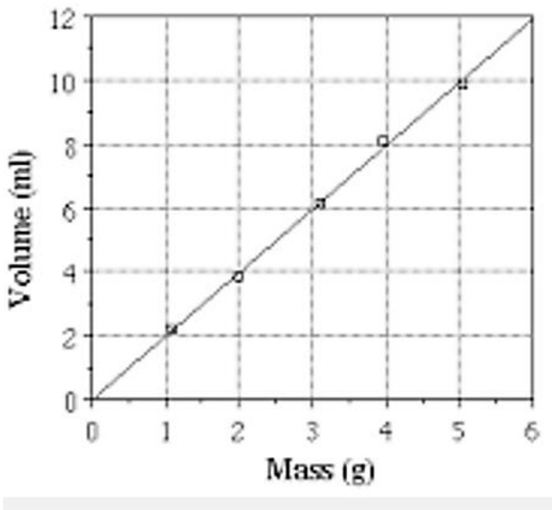
Graphs are a pictorial way of showing how one variable amount changes with another.

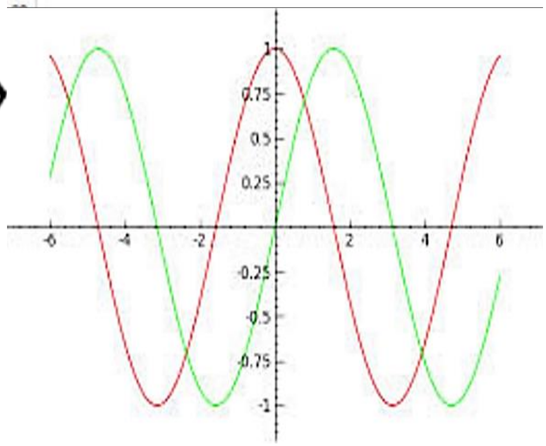
They may consist of straight lines, stepped lines, shapes or curves.

Every curve, no matter how complicated can be represented by an equation, the complexity of which depends on the shape of the curve.









This course covers simple straight lines, (one or two on a graph) plus some simple curves such as Sine , Cosine, x^2 (quadratics), x^3 (cubics) and circles.

Axis (Axes-plural, also called co-ordinate plane):

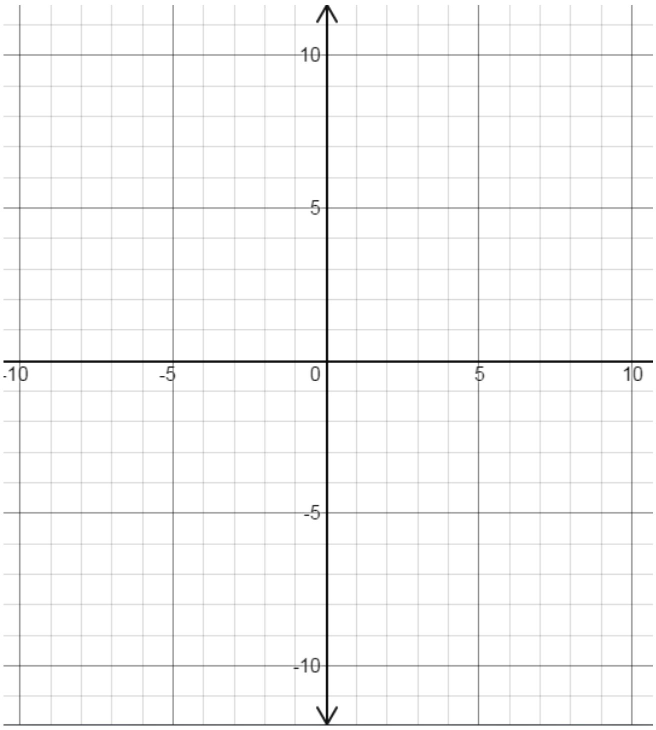
On a simple graph the horizontal axis is called the x–axis and the vertical axis the y–axis. The intersection of the two main axes is called the origin and is the zero point of x and y values. Above and to the right of the origin is positive, below and to the left is negative. Points on a graph are called 'co-ordinates'. The convention to describe a point on a graph is (x,y).

Functions:

An equation to be plotted is also known as a 'function', and denoted f(x). That is, the variable displayed on the vertical axis is a function of x, or changes as x changes.
x is known as the independent variable and y as the dependent variable. .
Depending on the values of x and y, only certain parts of the graph need to be shown. For accuracy the scale should be as large as possible. It is a good idea to calculate approximate maximum values of x and y before drawing the graph so the best **scale** can be determined. If it is required to physically measure angles or lengths from the graph the two scales must be the same. Usually however these values are better calculated from the actual values of x and y so the scales can be different. For example when plotting the value of x squared from 1 to 5, the x scale can have only five divisions but the y scale will need 25. If plotting x cubed the y scale would have to be at least 125 units.

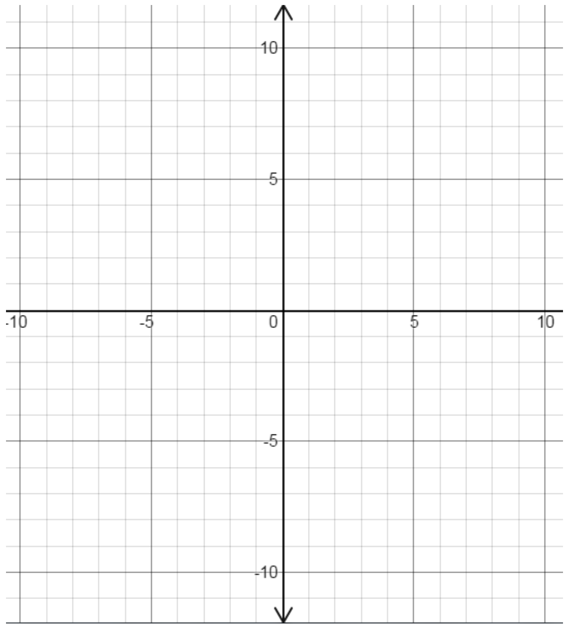
Co-ordinates

Example: Plot and label the following points on this axis (also called a Cartesian Plane),:
A (3, 7)
B (-5, 2)
C (-8, -4)
D (6, -10)



On the **Cartesian Plane** shown, calculate and plot x and y values that comply with the equation *y=2x+1* .

Independent variable (x)	*Dependent variable (y)*
-3	***-5***
-2	
-1	
0	
1	
2	***5***
3	



Line Graphs - Linear Equations

Gradient (slope, steepness, rate of change):

In general, the equation for a straight line is written in the form

$y = mx + c$; For example, $y = 2x + 6$

Note: it is important to recognize this form of equation. It might sometimes be written in other similar forms such as

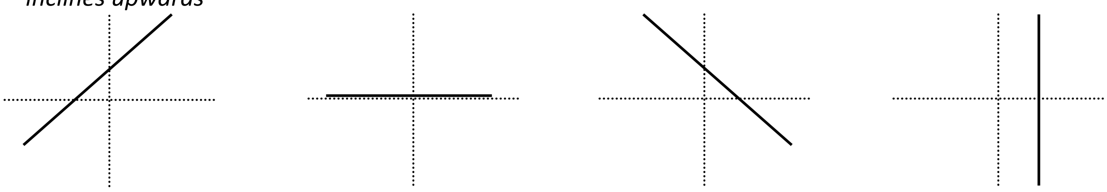
$y = Ax + B$ *or* $y = Q + Px$ *or* $y = mx + b$

The m value in the equation $y = mx + c$ is a number (or fraction/decimal) and represents the slope (or steepness or gradient) of the line graph.

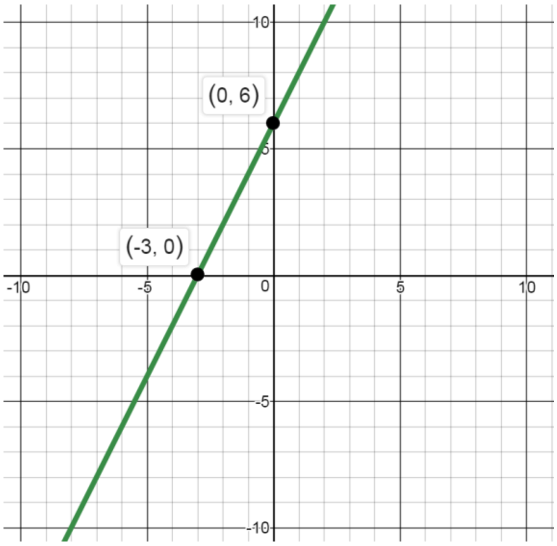
If the slope of the line is upwards from bottom left to top right, the gradient is said to be positive and the coefficient will be positive. If the line slopes down from top left to bottom right it is said to be negative.

(From left to right) inclines upwards

(From left to right) inclines downwards



+ve gradient	gradient=0	–ve gradient	gradient= ∞
y= mx + c	y = c	y= mx + c	x = c



Example: $m = 2$ in the equation

$y = 2x + 6$

Note that for every unit across the graph rises 2 units. This is sometimes referred to as $\frac{rise}{run}$

$\mathbf{y - intercept:}$

The c value in the equation $y = mx + c$ is a number (constant) and represents the y −intercept (or point that the line graph crosses the $y - axis$).

e.g. $c = 6$ in the equation $y = 2x + 6$

The y −intercept can always be found by setting $x = 0$ in any linear graph equation.

e.g. setting x=0 in the equation $y = 2x + 6$ gives $y = 6$

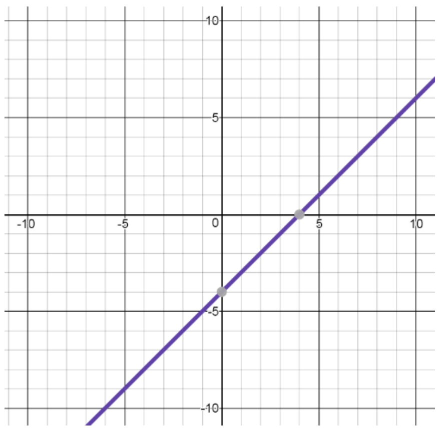
e.g. setting x=0 in the equation $3y = 9 - 2x$ gives $3y = 9$ or $y = 3$

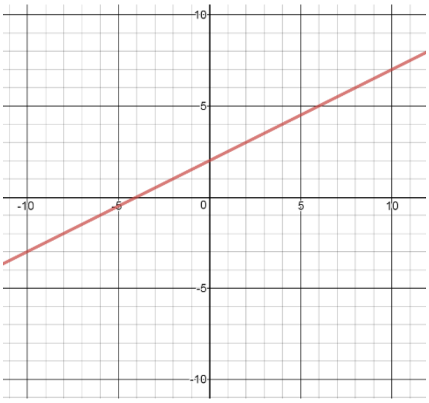
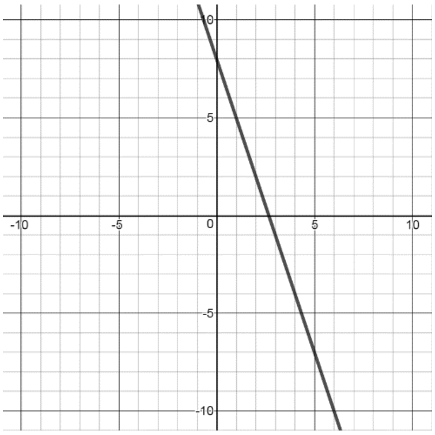
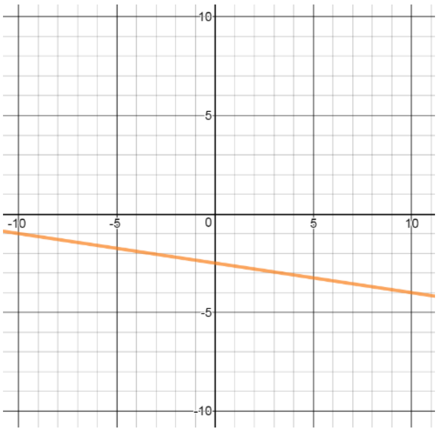
$Example$: $y = x - 4$,

$slope = 1$,

$y - int = -4$,

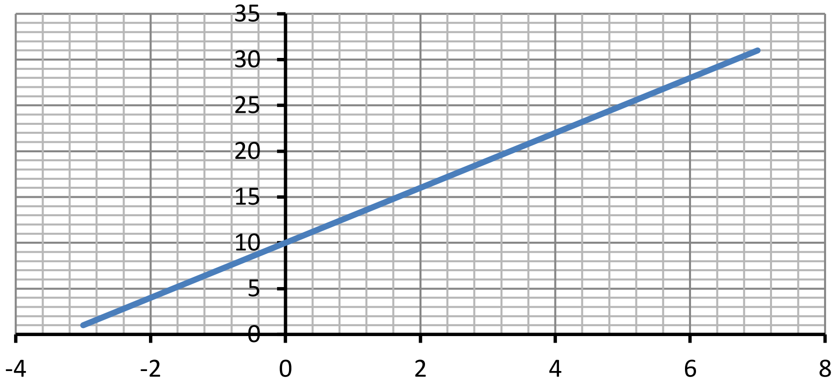
$x - int = 4$



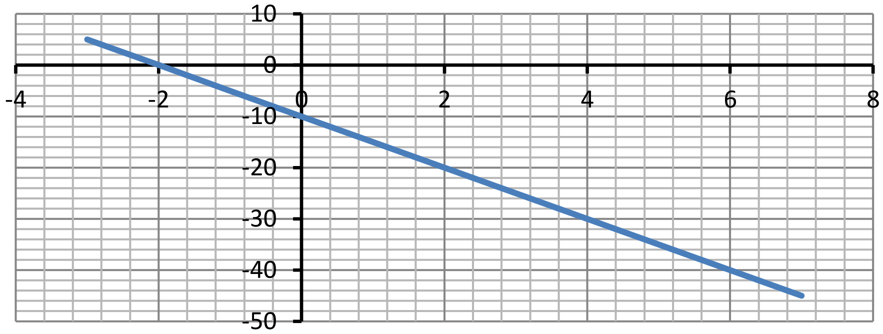
$Example$: $y = 0.5x + 2$, $slope = 0.5$, $y - int = 2$, $x - int = -4$	
$Example$: $y = -3x + 8$, $slope = -3$, $y - int = 8$, $x - int = \frac{8}{3}$	
$Example$: $2y + 0.4x = -5$, $rearrange\ equation\ to$ $y = -0.2x - 2.5$ $slope = -0.2$, $y - int = 2.5$, $x - int = -12.5$	

Note that the scales for each axis may not be the same. When calculating a gradient check you read values off each axis correctly.

Example : y = 3x + 10



Example: y = -5x - 10



Exercise 5.1 Drawing Linear Graphs from Equations

Draw graphs of each of the following equations on a 10x10 grid and determine the gradient, y-intercept, x-intercept.

1. $y = 3x + 2$

2. $y = 3x - 5$

3. $y = \frac{1}{3}x - 1$

4. $y = -2x + 1$

5. $y = -\frac{3}{4}x + 2$

6. $y = 4$

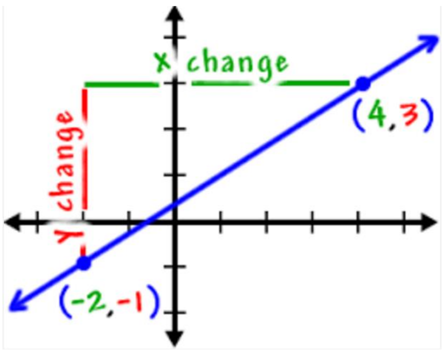
Finding the equation of a straight line graph:

Given two co-ordinates (x_1,y_1) and (x_2,y_2) it is possible to plot the line and find the equation $y = mx + c$

Remember the slope (gradient) of the graph is the number in front of x term (m) and the constant is the y=intercept (where the line crosses the y-axis).

Recall, to find the slope, draw a right-angled triangle using the straight line graph as the hypotenuse. Calculate the length of the vertical and horizontal sides (according to the scale of the graph axes). Take the difference between the (vertical) y values and divide by the difference between the (horizontal) x values. $i.e. \frac{rise}{run}$.

Calculating the slope of the line (m):



If the line is taken as the hypotenuse of a right angled triangle, the value of the vertical side is the difference in two y values and the horizontal side is the value of two x values.

Slope = $\frac{(change\ in\ y)}{(change\ inx)} = \frac{\Delta y}{\Delta x} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$

$= \frac{4}{6} = \frac{2}{3}$

Find the slope. Then substitute it into the general line equation $y = mx + c$ using either of the given (x,y) co-ordinates. This allows y-intercept c to be determined.

Example: Find the equation of the line joining coordinates (–5, –20) and (+8, 15)

Slope m $= \frac{(y_2 - y_1)}{(x_2 - x_1)}$

$= \frac{15 - -20}{8 - -5}$

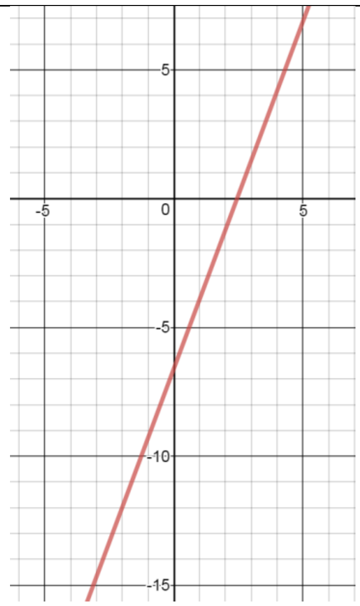
$= \frac{35}{13}$

$= 2.7$

y-intercept

Substitute x and y values, using coordinate x =8 and y =15

$$y = mx + c$$
$$15 = 2.7 \times (8) + c$$
$$15 - 21.6 = c$$
$$c = -6.6$$
$$\therefore y = 2.7x - 6.6$$



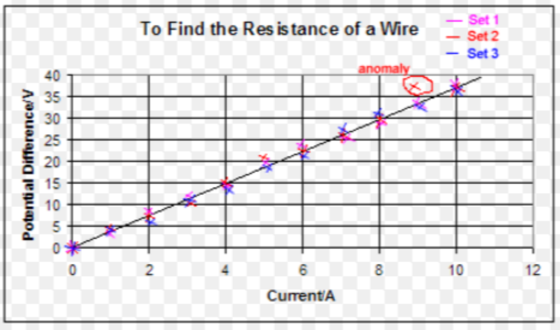
Exercise 5.2 Drawing Linear Graphs from Two Co-ordinates

Find the equation of the line passing through each set of co-ordinates. Plot the graph for each.

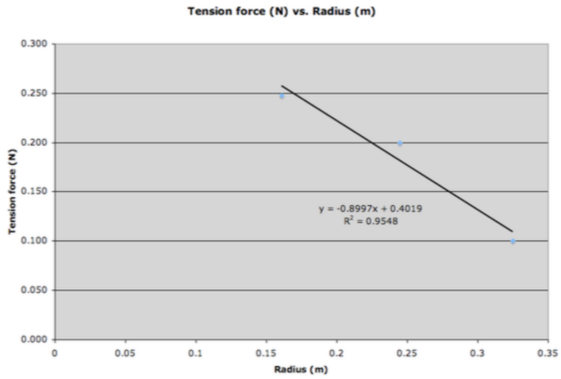
1. (2,0) and (3,1)
2. (6,5) and (4,12)
3. (–3,8) and (3,–5)
4. (–5,–3) and (3,–2)

Determining Lines of Best Fit for Plotted Data

If a plot of the relationship between two quantities slopes from bottom left to top right as in the graph below, the slope is positive and the quantity plotted on the vertial axis is <u>directly</u> proportional to the quantity on the horizontal axis.



If a plot of the relationship between two quantities slopes from top left to bottom right as in the graph below, the slope is negative and the quantity plotted on the vertical axis is <u>indirectly</u> proportional to the quantity on the horizontal axis.



When experimental data is plotted and appears to generally form a straight line, the equation of this line of best fit (trend line) can be determined and this equation models the relationship between the 2 quantities.

Worked Example:

This table shows a set of readings obtained when a mass of W (grams) is hung from a spring causing it to extend to a length L (cm).

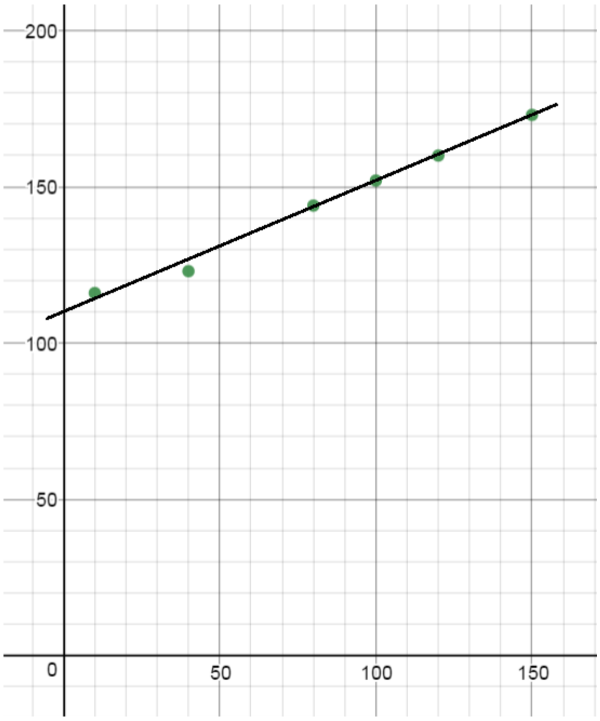
1. Determine the equation of the line of best fit for this data in the form $L = mW + c$.

2. Use you equation of best fit to estimate the extension for a mass of 70 g.

W (g)	10	40	80	100	120	150
L (cm)	116	123	144	152	160	173

1. Notice that the line of best fit cannot go through every data point. This is a trend line determined (visually) which best fits all the data. Some data points may fall above the line and others below.

When determining the slope of this line, use two data points which fall exactly on the line for the most accurate result.



Using points (80,144) and (150,173) gives a slope of $\dfrac{(173-144)}{(150-80)} = \dfrac{29}{70}$

$$\implies L = \frac{29}{70}W + c$$

c is determined by substituting a point, e.g. (80,144)

$$144 = \frac{29}{70} \times 80 + c$$

$$144 - 33.1 = c$$

$$c = 110.9$$

$$L = \frac{29}{70}W + 110.9$$

2. Using the equation of best fit, the extension for a mass of 70g is:

$$L = \frac{29}{70} \times 70 + 110.9$$

$$L = 139.9cm$$

Exercise 5.3

1. Plot a graph using the following values and find the line of best fit for this data

x	-2	-1	0	1	2	3
y	10	7	4	1	-2	-5

2. The following data was collected in experiment on a small turbine, where P represents the power developed and m the consumption of steam per hour. Assuming the relationship between P and m can be represented by a straight line, $m = aP + b$, estimate the values of a and b by calculating the line of best fit through this data.

P	20	25	30	35	40	45	50
m	220	265	315	365	410	455	505

3. An experiment with a set of pulley blocks gave the following results:

Effort E (newtons)	9.0	11.0	13.6	17.4	20.8	23.6
Load L (newtons)	15	25	38	57	74	88

Plot a graph of effort (vertically) against load (horizontally) and determine:

a) an equation for the line of best fit.

b) the effort when the load is 30N

c) the load when the effort is 19N.

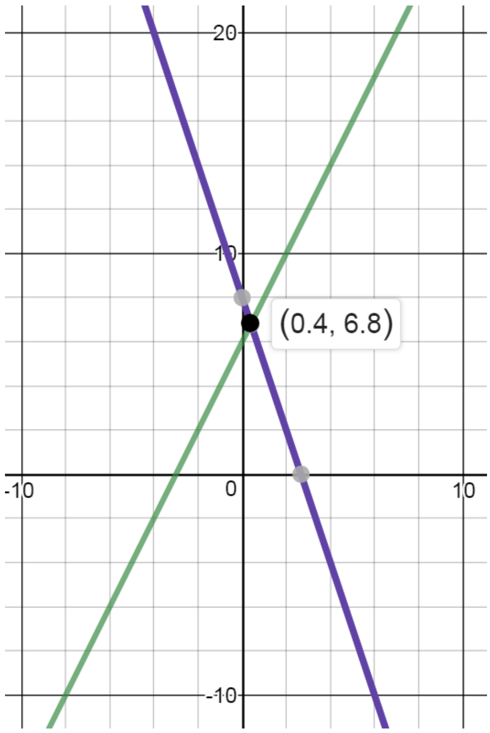
Points of Intersection of Linear Graphs:

If two linear equations are plotted, their intersection is given by the values of x and y which satisfy both equations (simultaneously). This intersection may be determined graphically by reading off the graph (not always possible to do accurately), or the intersection may be determined algebraically by solving both equations simultaneously.

Example:

Find the point of intersection of linear equations $y = -3x + 8$ and $y = 2x + 6$

graphically...



algebraically...

Solving simultaneously:

$y = -3x + 8 \quad and$

$y = 2x + 6$

by substituting for y:

$2x + 6 = -3x + 8$

$2x + 3x = 8 - 6$

$5x = 2$

$x = \frac{2}{5} \; or \; 0.4$

Substituting
$x = 0.4$ into either equation to solve for y

$y = 2 \times 0.4 + 6 = 6.8 \; or$

$y = -3 \times 0.4 + 8 = 6.8$

i.e. point of intersection is $(0.4 \,, 6.8)$

Exercise 5.4 : Finding intersections of lines

By drawing graphs, find the point of intersection of the following pairs of equations (both graphically and algebraically):

1. $y = 3x + 2$ and $y = 3x - 5$
2. $y = \frac{1}{3}x - 1$ and $y = -2x + 1$
3. $y = -\frac{3}{4}x + 2$ and $y = 4$

Curved Graphs

These are formed by non–linear equations, that is equations containing a function such as x^2 or x^3 or sinx, cos x. etc.

When it is necessary to plot a graph, the question may give the limits of x required, e.g. Plot values between x = –5 and x = +8.

While a straight line graph can be plotted using only two points on the line, a curved graph needs many points plotted to get an accurate shape.

It is best to make at table giving values of each term of the equation and hence the value of y to plot for each value of x. Additional points may have to be plotted where the line changes shape quickly.

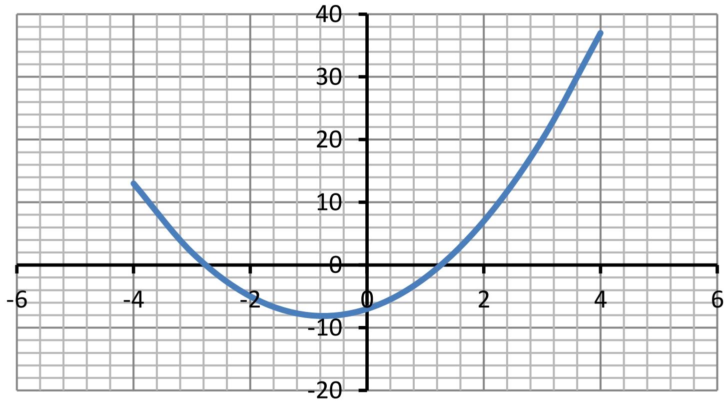
Example. $y = 2x^2 + 3x - 7$ Plot values of y between –4 and + 4

Be careful that a negative number squared always gives a positive answer, check how to do this on your calculator as brackets may have to be used.

i.e. On calculator type $(-4)^2 = +16$ *(not -4^2 which will result in the incorrect answer of -16)*

Value x	$y = 2x^2 + 3x - 7$	(x, y)
–4	$y = 2 \times (-4)^2 + 3 \times (-4) - 7 = 13$	(-4,13)
–3	$y = 2 \times (-3)^2 + 3 \times (-3) - 7 = 2$	(-3,2)
–2	$y = 2 \times (-2)^2 + 3 \times (-2) - 7 = -5$	(-2,-5)
–1	$y = 2 \times (-1)^2 + 3 \times (-1) - 7 = -8$	(-1,-8)
0	$y = 2 \times (0)^2 + 3 \times (0) - 7 = -7$	(0,-7)
1	$y = 2 \times (1)^2 + 3 \times (1) - 7 = -2$	(1,-2)
2	$y = 2 \times (2)^2 + 3 \times (2) - 7 = 7$	(2,7)
3	$y = 2 \times (3)^2 + 3 \times (3) - 7 = 20$	(3,20)
4	$y = 2 \times (4)^2 + 3 \times (4) - 7 = 37$	(4,37)

$y = 2x^2 + 3x - 7$



Practice Example: Plot the graph $y = 2x^2 - 7x + 6$ by completing this table and graphing the resulting co-ordinates.

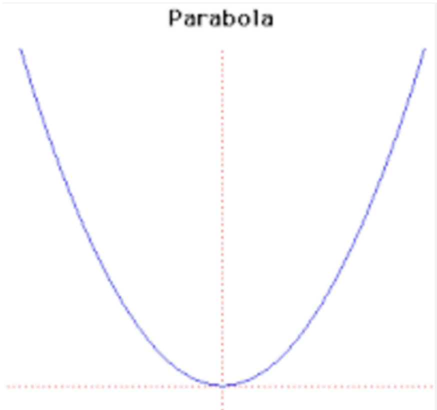
Value x	$y = 2x^2 - 7x + 6$	(x, y)
−1		
0		
1		
2		
3		
4		

Common non-linear Graphs

Parabolas

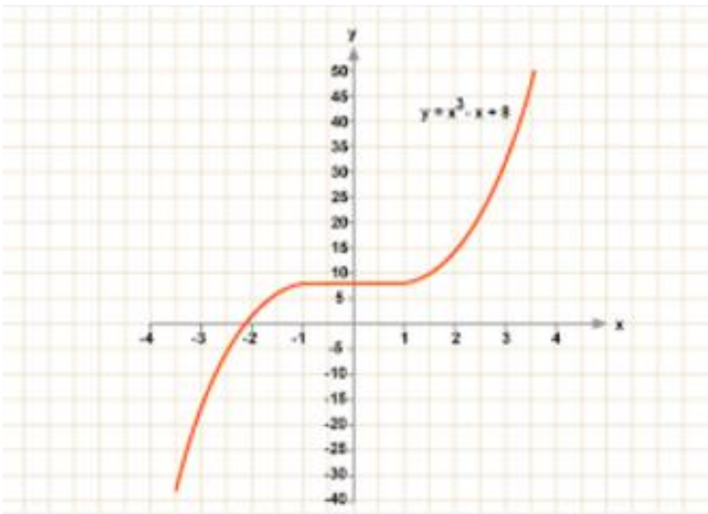
Parabolas are graphs of quadratic functions of the type y = ax^2 + bx + c and have a vertex (turning point) which may be maximum or a minimum.

The solution of an equation is often called the roots of the equation. In the case of a graph, the roots of the graph are the points where the graph cuts the x–axis.



Cubic graphs

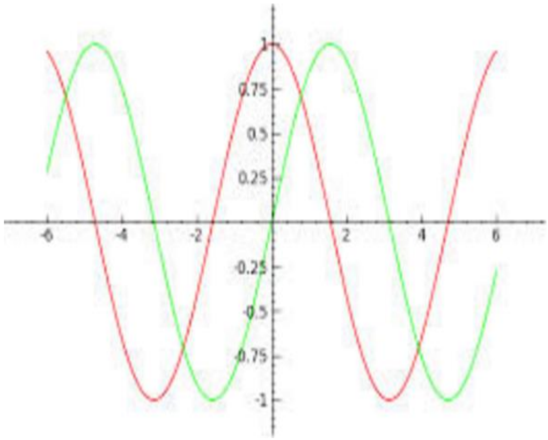
Cubic graphs are graphs of cubic functions of the type y = ax^3 + bx^2 + cx +d and have two vertices (turning points): a maximum *and* a minimum.



Graphs of Trigonometric Functions

The graphs of y = sin x and y = cos x have a characteristic curved shape and are said to be *periodic* as they repeat

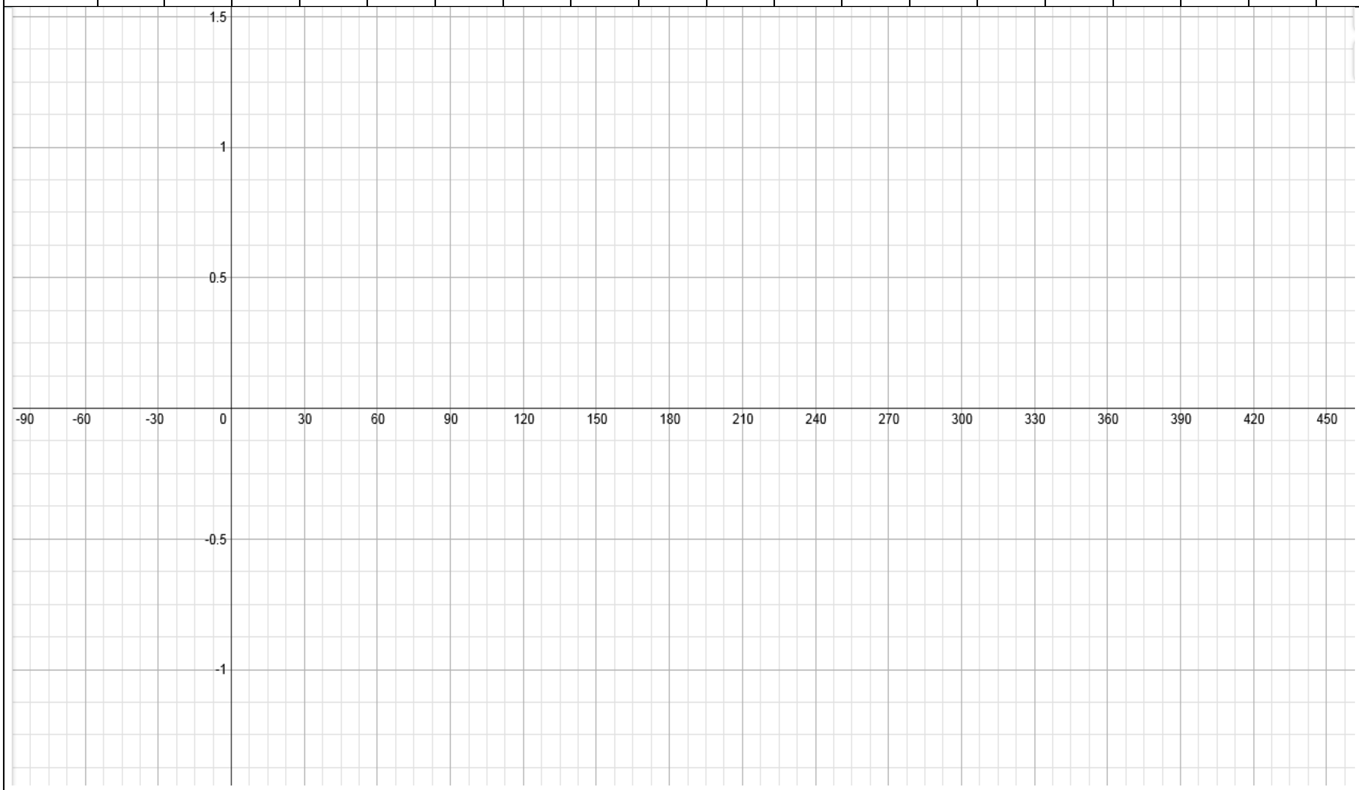
See graphs in the Trigonometry Section of these notes.



Trigonometric Ratios for Angles between 0 and 360°

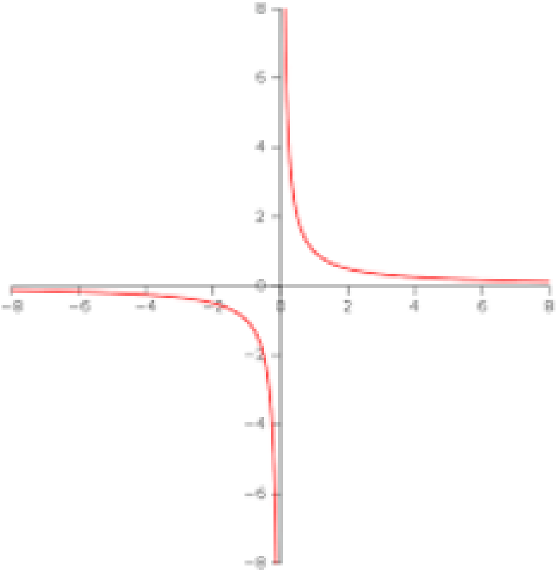
Fill in the table below with trig ratio values using your calculator. Sketch the curves for sin, cos and tan using values in the below. Use different colours for each curve. From the table and from the curves, you will notice that the values repeat themselves.

Angle	**−90**	**−60**	**−30**	**0**	**30**	**60**	**90**	**120**	**150**	**180**	**210**	**240**	**270**	**300**	**330**	**360**	**390**	**420**	**450**
Sin																			
Cos																			
Tan																			

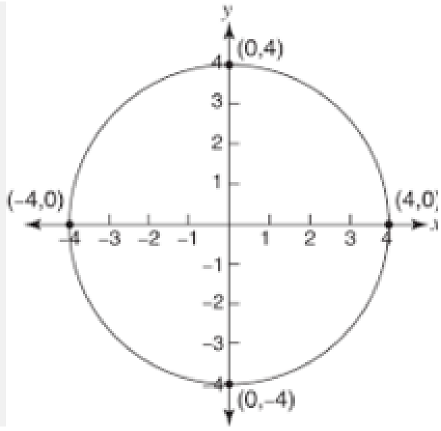


Hyperbolae (for reference only)
Hyperbolae are graphs of functions of the type $y = \dfrac{ax+b}{cx+d}$ where a,b,c and d are constants.

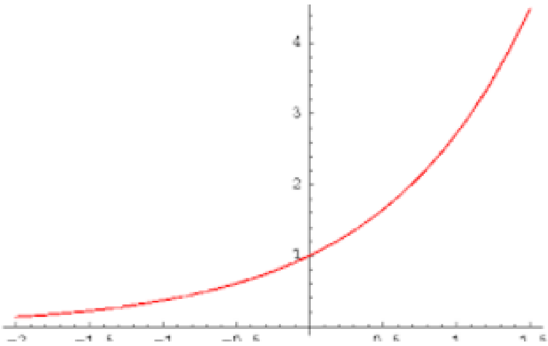
This graph has two separate curves with the ends of each curve approaching horizontal and vertical asymptotes. An ***asymptote*** is a line that a graph gets closer and closer to but never touches.



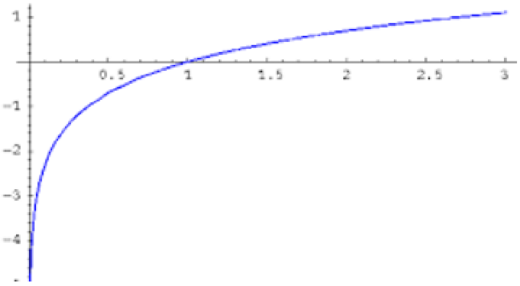
Circles (for reference only)
Circles have an equation in the form $x^2 + y^2 = R^2$ where R = radius. A circle with centre at (a, b) take the form $(x–a)^2 + (y–b)^2 = R^2$



Exponential curves (*for reference only*)
Exponential curves are graphs of the equation $y = a^x$ where a>0
Examples of quantities related by laws of growth and decay



Logarithmic curves (*for reference only*)
Logarithmic Curves are the inverse of exponential curves and have the equationy = $\log_a x$. The two are mirror images of each other for the same value of a.

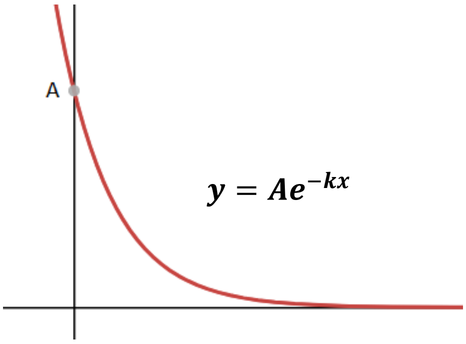


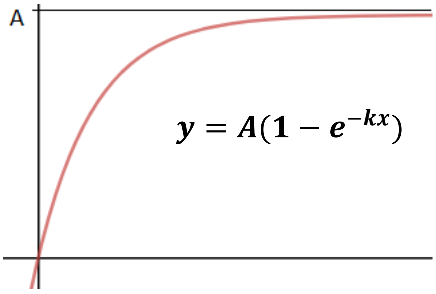
Curves of laws of growth and decay.

$y = Ae^{-kx}$ and $y = A(1 - e^{-kx})$

These laws occur frequently in engineering and science. Examples include:

i. Linear expansion $l = l_0 e^{\alpha\theta}$
ii. Change in electrical resistance with temperature $R_\theta = R_0 e^{\alpha\theta}$
iii. Tension in belts $T_1 = T_0 e^{\mu\theta}$
iv. Newtons law of cooling $\theta = \theta_0 e^{-kt}$
v. Biological growth $y = y_0 e^{kx}$
vi. Atmospheric pressure $p = p_0 e^{-h/c}$
vii. Radioactive decay $N = N_0 e^{-\lambda t}$





Exercise 5.5: Drawing curved graphs using DESMOS graphing calculator

Use www.desmos.com to draw and explore graphs of each of the following equations and name the type of graphs.

1. $y = x^2$
2. $y = (x + 2)^2 + 3$
3. $y = (x-2)(2x-3)$
4. $y = sin\ x$ (from 0° to 360°)
5. $y = cos\ x$ (from 0° to 360°)
6. $y = tan\ x$ (from –90° to + 90°)
7. $y = 2^x$ (not assessed)
8. $y = log_2\ x$ (not assessed)

Exercise 5.6: Plotting intersections of curves and line graphs

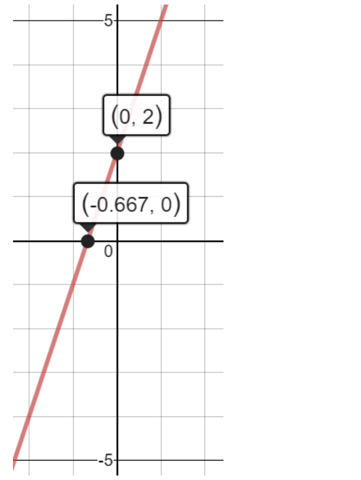
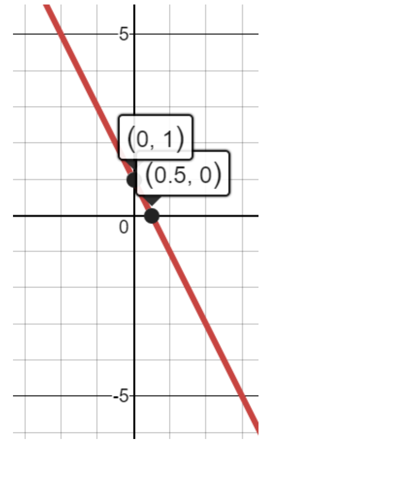
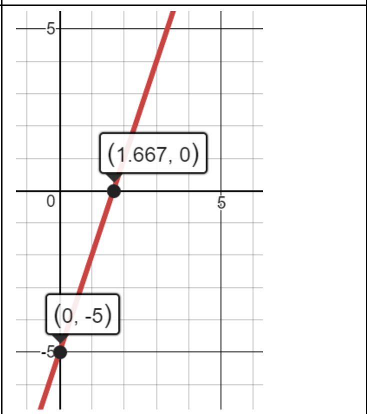
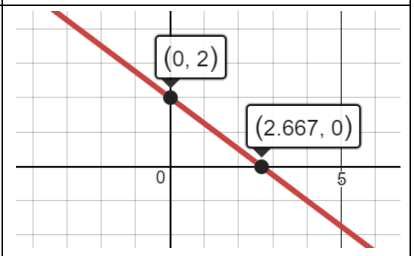
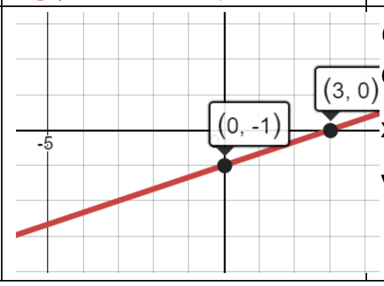
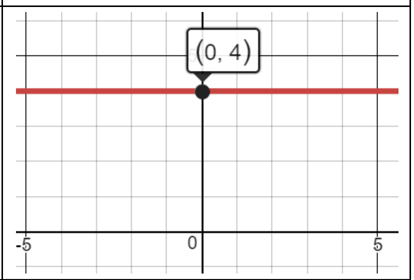
For each of the following equations:

Plot the graphs between x=-5 and x=5. Visually estimate the coordinates of the points of intersection; and solve the equations simultaneously to find their exact points of intersection

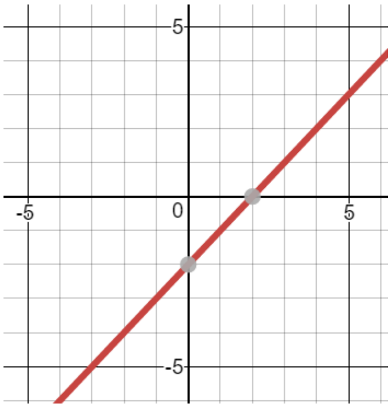
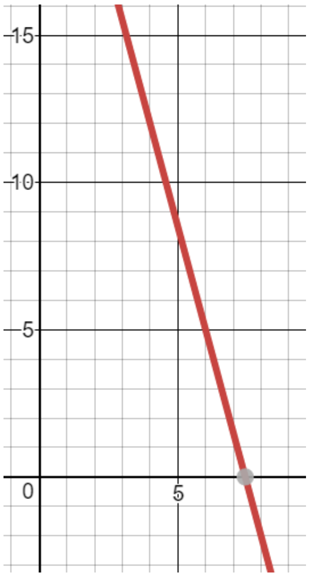
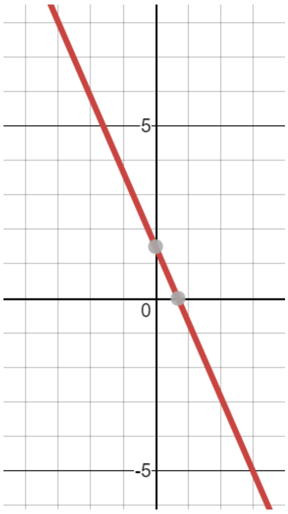
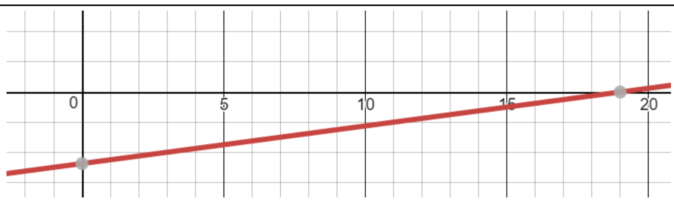
1. $y = -0.5x^2 + 7\ and\ y = 0.5x + 1$
2. $y = (x - 4)(x + 4)\ and\ y = -x - 2$
3. $y = \frac{-x}{2}(x - 3)\ and\ y = -5$

ANSWERS TO EXERCISES

Answers Exercise 5.1

<table>
<tr>
<td>1 .
Gradient=3
x-intercept=$\frac{-2}{3}$
y-intercept=2</td>
<td></td>
<td>4.
Gradient=-2
x-intercept=$\frac{1}{2}$
y-intercept=1</td>
<td></td>
</tr>
<tr>
<td>2.
Gradient=3
x-intercept=$\frac{5}{3}$
y-intercept=-5</td>
<td></td>
<td>5.
Gradient=$-\frac{3}{4}$
x-intercept=$\frac{8}{3}$
y-intercept=2</td>
<td></td>
</tr>
<tr>
<td>3.
Gradient=$\frac{1}{3}$
x-intercept=3
y-intercept=-1</td>
<td></td>
<td>6.
Gradient=0
x-intercept=$\frac{1}{2}$
y-intercept=1</td>
<td></td>
</tr>
</table>

Answers Exercise 5.2

1. $y = x - 2$	
2. $y = -\frac{7}{2}x + 26$ -	
3. $y = -\frac{13}{6}x + \frac{3}{2}$	
4. $y = \frac{1}{8}x - \frac{19}{8}$	

Answers Exercise 5.3

1. $y = -3x + 4$

2. (*Answers will vary depending on line of best fit)* $m = 9.5P + 29.6$

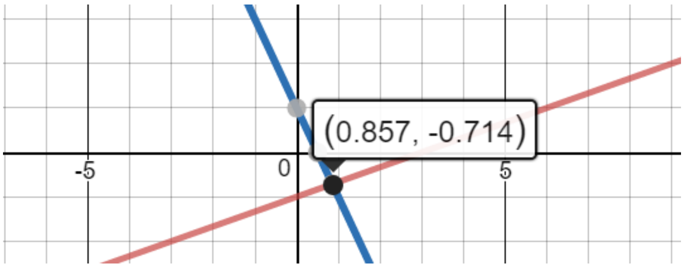
3. a) E=0.2L + 6

   b) 12 N

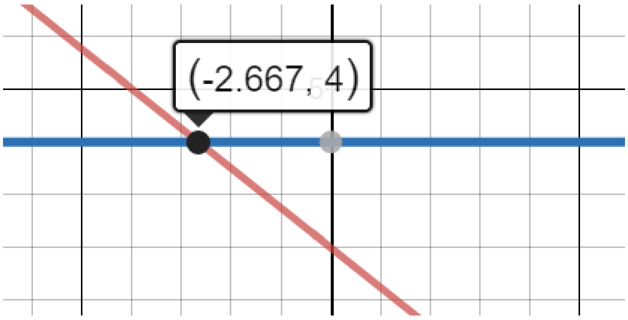
   c) 65 N

Answers Exercise 5.4

1. no intersection since same gradients
2. *(0.86, −0.71)*

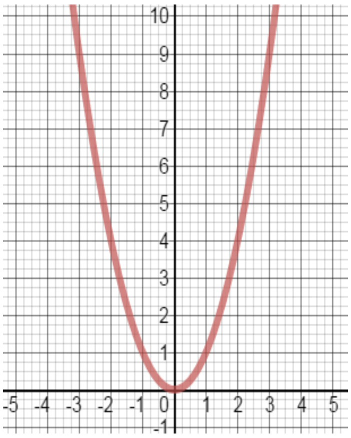


3. *(−2.67, 4)*

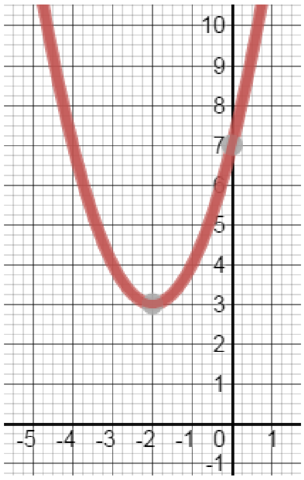


Answers Exercise 5.5

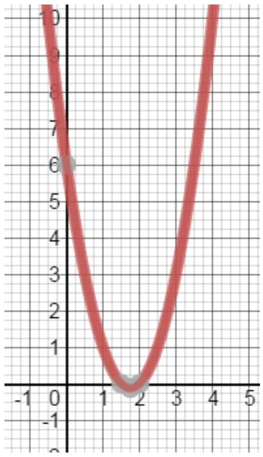
1. $y = x^2$



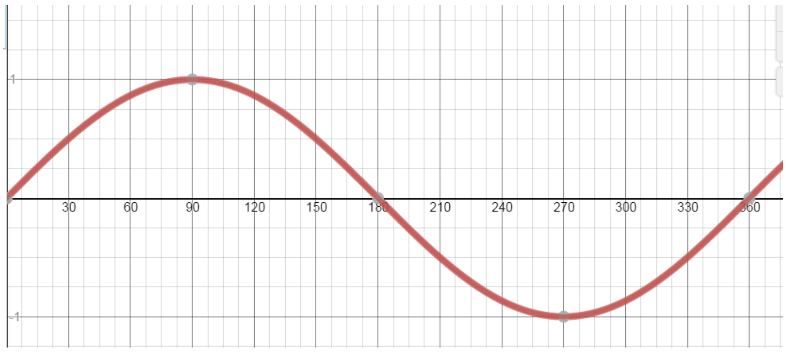
2. $y = (x + 2)^2 + 3$



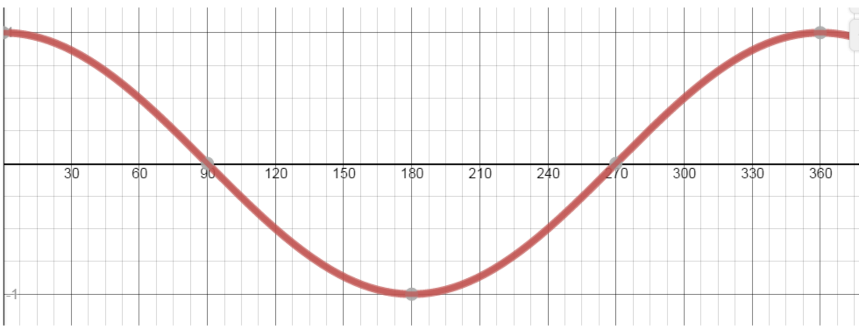
3. $y = (x–2)\ (2x–3)$



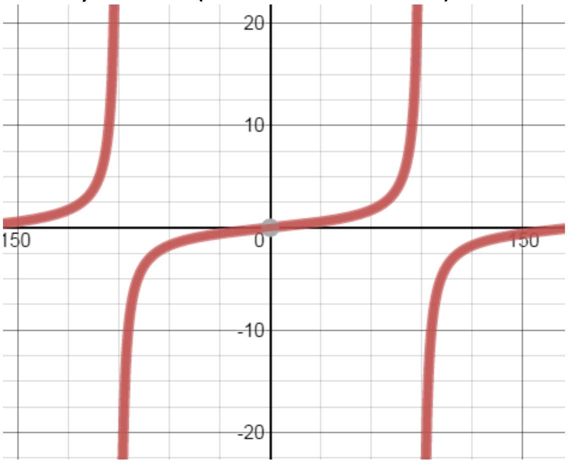
4. y = sin x (from 0° to 360°)



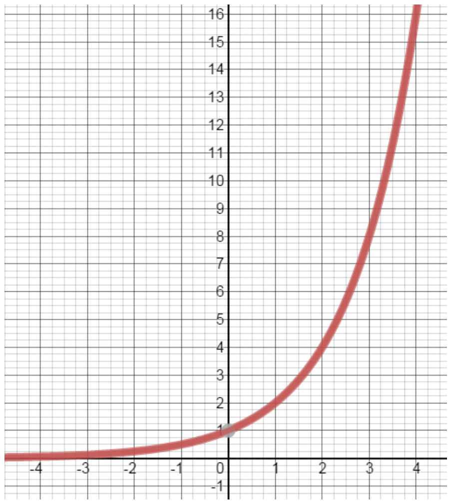
5. y = cos x (from 0° to 360°)



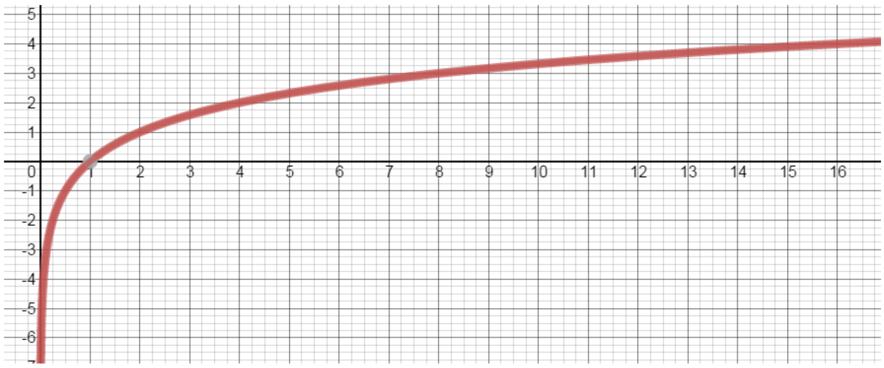
6. $y = \tan x$ (from –90° to + 90°)



7. $y = 2^x$



8. $y = \log_2 x$



Answers Exercise 5.6

1. $-0.5x^2 + 7 = 0.5x + 1$

$0 = 0.5x^2 + 0.5x - 6 \quad a = 0.5, b = 0.5, c = -6$

$$x = \frac{-0.5 \pm \sqrt{0.5^2 - 4 \times 0.5 \times (-6)}}{2 \times 0.5}$$

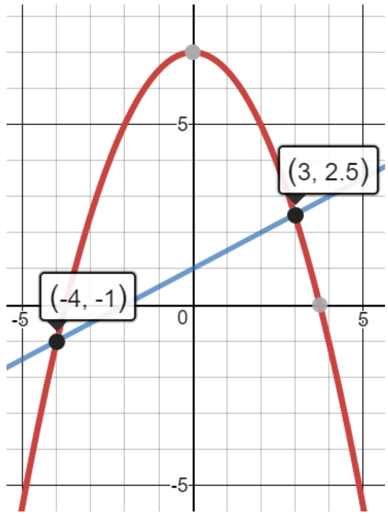
$$x = \frac{-0.5 \pm \sqrt{0.25 + 12}}{1}$$

$$x = \frac{-0.5 \pm \sqrt{12.5}}{1}$$

$$x = \frac{-0.5 \pm 3.5}{1}$$

$x = 3 \text{ or } x = -4$

$when\ x = 3, y = 2.5 \text{ and } when\ x = -4, y = -1$



2.

$(x-4)(x+4) = -x - 2$

$x^2 - 16 = -x - 2$

$x^2 + x - 14 = 0 \quad a = 1, b = 1, c = -14$

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times -14}}{2 \times 1}$$

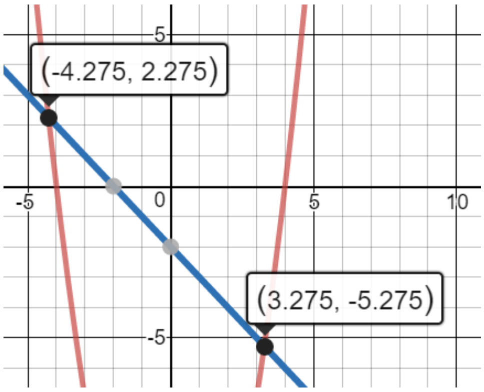
$$x = \frac{-1 \pm \sqrt{1 + 56}}{2 \times 1}$$

$$x = \frac{-1 \pm \sqrt{57}}{2}$$

$$x = \frac{-1 \pm 7.55}{2}$$

$x = 3.275\ or\ x = -4.275$

$when\ x = 3.275, y = -5.275\ and\ when\ x = -4.275, y =$ 2.275



3.

$\frac{-x}{2}(x-3) = -5$

$\frac{-x^2}{2} + \frac{3x}{2} + 5 = 0 \quad a = (-0.5), b = 1.5, c = 5$

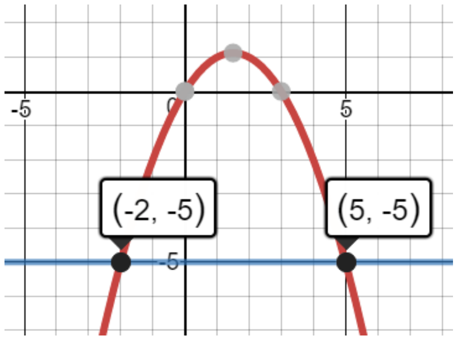
$$x = \frac{-1.5 \pm \sqrt{1.5^2 - 4 \times (-0.5) \times 5}}{2 \times (-0.5)}$$

$$x = \frac{-1.5 \pm \sqrt{2.25 + 10}}{-1}$$

$$x = \frac{-1.5 \pm \sqrt{12.25}}{-1}$$

$$x = \frac{-1.5 \pm 3.5}{-1}$$

$x = -2 \text{ or } x = 5$



$\therefore intersections\ at\ (-2, -5)\ and\ (5, -5)$