Accelerated Failure Time

Avinash Barnwal

May 2019

1 Error - Normal Distribution

Assume the data follow below model:-

$$
\begin{aligned}
\log y_i &= x_i{}' \beta + z_i \sigma \\
\log \hat{y_i} &= x_i{}' \hat{\beta} \\
\eta &= x_i{}' \hat{\beta}
\end{aligned}
\tag{1}
$$

where y_i is the uncensored response and $\hat{y_i}$ is the predicted value for i-th observation and σ is the standard deviation of the error.

1.1 Uncensored Data

1.1.1 Loss Function

$$
L_i = -\log \text{lik}_i = -log(f_{Y_i}(y_i|\eta))
\tag{2}
$$

where, $f_\eta(\eta)$ is the probability density function(pdf) of η.

$$
\begin{aligned}
z_i &= \frac{\log y_i - \eta}{\sigma} \sim f \\
f(z) &= \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \\
f^{'}(z) &= -zf(z) \\
f^{''}(z) &= -f(z) - zf^{'}(z)
\end{aligned}
\tag{3}
$$

Now using change of variable for probability density function(pdf). We can write the below equations.

$$
\begin{aligned}
f_Y(y|\eta) &= f_Z(z)\frac{\partial z}{\partial y} \\
f_Y(y|\eta) &= \frac{\exp^{-\frac{(\log(y)-\log(\hat{y_i}))^2}{2\sigma^2}})}{y\sigma\sqrt{2\pi}} \\
L_i &= -log(\frac{\exp^{-\frac{(log(y_i)-\log(\hat{y_i}))^2}{2\sigma^2}}}{y_i\sigma\sqrt{2\pi}}) \\
L_i &= -log(\frac{\exp^{-}(\frac{(log(y_i)-\eta)^2}{2\sigma^2}}{y_i\sigma\sqrt{2\pi}})
\end{aligned} \tag{4}
$$

1.1.2 Negative Gradient

In Gradient Boosting and Xgboost, we need to calculate negative gradient of loss function with respect to real-value prediction which is $\hat{\eta}$. Here i am changing between $\hat{\eta_i}$ to $\hat{\eta}$ to make it general.

$$
\begin{aligned}
-\frac{\partial L_i}{\partial \hat{\eta}} &= \frac{\partial log(f_{Y_i}(y_i|\eta))}{\partial \eta} \\
&= \frac{1}{f_{Y_i}(y_i|\eta)} * \frac{\partial f_{Y_i}(y_i|\eta)}{\partial \eta}
\end{aligned} \tag{5}
$$

Using change of variable between z and η. We can write the below in terms of z.

$$
\frac{\partial f_Y(y|\eta)}{\partial \eta} = \frac{\partial}{\partial \eta}[f_Z(z)\frac{\partial z}{\partial y}] \tag{6}
$$

Using product rule of differentiation, we can split $f_Z(z)$ and $\frac{\partial z}{\partial \eta}$

$$
= \frac{\partial f_Z(z)}{\partial \eta}\frac{\partial z}{\partial y} + f_Z(z)\frac{\partial^2 z}{\partial \eta \partial y} \tag{7}
$$

$$
= \frac{\partial f_Z(z)}{\partial z}\frac{\partial z}{\partial \eta}\frac{\partial z}{\partial y} + f_Z(z)\frac{\partial^2 z}{\partial \eta \partial y} \tag{8}
$$

As we know $\frac{\partial f_Z(z)}{\partial z} = -zf_Z(z)$

Now, we will calculate the gradient of z with respect to η as we have $\frac{\partial z^2}{\partial \eta \partial y}$ in the equation 8.

$$
\frac{\partial z}{\partial \eta} = \frac{-1}{\sigma} \tag{9}
$$

$$
\frac{\partial z}{\partial y} = \frac{1}{y\sigma} \tag{10}
$$

We also need double differentiation of z with respect to η as we have $\frac{\partial^2 z}{\partial \eta^2}$ in the equation 8.

$$\frac{\partial^2 z}{\partial \eta \partial y} = 0 \tag{11}$$

$$= -\frac{f_Z^{'}(z)}{y\sigma^2} \tag{12}$$

Now going back to original equation of calculating negative gradient of loss function with respect to η. In equation 9, we have calculated the second part of negative gradient of loss function with respect to η which is $\frac{\partial f_{Y_i}(y_i|\eta)}{\partial \eta}$. By replacing the 12th equation in the negative gradient of loss function w.r.t. to η, we get the results as follows:-

$$\begin{aligned}
-\frac{\partial L_i}{\partial \eta} &= \frac{1}{f_{Y_i}(y_i|\eta)} * \frac{\partial f_{Y_i}(y_i|\eta)}{\partial \eta} \\
&= \frac{1}{f_{Y_i}(y_i|\eta)} * \frac{-f_{Z_i}^{'}(z_i)}{y\sigma^2} \\
&= \frac{-f_{Z_i}^{'}(z_i)}{\sigma f_{Z_i}(z_i|\eta)}
\end{aligned} \tag{13}$$

1.1.3 Hessian

Hessian is the second derivative of Loss with respect to η. As we have already calculated the negative gradient, further we need to take one more partial derivative with respect to η of negative gradient with negative sign.

$$\begin{aligned}
\frac{\partial^2 L_i}{\partial \eta^2} &= \frac{\partial}{\partial \eta}\frac{\partial L_i}{\partial \eta} \\
&= \frac{\partial}{\partial \eta}\frac{f_{Z_i}^{'}(z_i)}{\sigma f_{Z_i}(z_i|\eta)} \\
&= \frac{\partial}{\partial z_i}\frac{f_{Z_i}^{'}(z_i)}{\sigma f_{Z_i}(z_i|\eta)}\frac{\partial z_i}{\partial \eta} \\
&= \frac{\partial}{\partial z_i}\frac{f_{Z_i}^{'}(z_i)}{\sigma f_{Z_i}(z_i|\eta)}\frac{\partial z_i}{\partial \eta} \\
&= -\frac{f_{Z_i}(z_i) f_{Z_i}^{''}(z_i) - [f_{Z_i}^{'}(z_i)]^2}{\sigma^2 f_{Z_i}^2(z_i)}
\end{aligned} \tag{14}$$

1.2 Left Censored Data

1.2.1 Loss Function

$$L_i = -\log \text{lik}_i = -log(F_{Y_i}(y_i|\eta)) \tag{15}$$

where, F is the cdf of $Y_i|\eta$.

Therefore, equation 15 can be written as below using property of **??**

$$L_i = -log(\phi(z)) \tag{16}$$

1.2.2 Negative Gradient

$$-\frac{\partial L_i}{\partial \eta} = -\frac{\partial - log(\phi(z))}{\partial \eta} \tag{17}$$

$$-\frac{\partial L_i}{\partial \eta} = \frac{\partial log(\phi(z))}{\partial \eta} \tag{18}$$

$$= \frac{\phi^{'}(z_i)}{\phi(z_i)} \frac{\partial z_i}{\partial \eta} \tag{19}$$

Therefore, combining results of chain rule and pdf, below is the final result for negative gradient,

$$-\frac{\partial L_i}{\partial \eta} = \frac{-f(z_i)}{\sigma \phi(z_i)} \tag{20}$$

1.2.3 Hessian

Hessian is the second derivative of Loss with respect to η. As we have already calculated the negative gradient, further we need to take one more partial derivative with respect to η of negative gradient with negative sign.

$$\frac{\partial^2 L_i}{\partial \eta^2} = \frac{\partial}{\partial \eta} \frac{\partial L_i}{\partial \eta} \tag{21}$$

After inputting the values of negative gradient calculated to above equation.

$$\begin{aligned} &= \frac{\partial}{\partial \eta} * \frac{f(z_i)}{\sigma \phi(z_i)} \\ &= \frac{\partial}{\partial z_i} \frac{f(z_i)}{\sigma \phi(z_i)} * \frac{\partial z_i}{\partial \eta} \\ &= \frac{\phi(z_i) f^{'}(z_i) - f^2(z_i)}{\sigma \phi^2(z_i)} * \frac{-1}{\sigma} \\ &= -\frac{\phi(z_i) f^{'}(z_i) - f^2(z_i)}{\sigma^2 \phi^2(z_i)} \end{aligned} \tag{22}$$

1.3 Right Censored Data

1.3.1 Loss Function

$$L_i = -\log \text{lik}_i = -log(1 - F_{Y_i}(y_i|\eta)) \tag{23}$$

where, F is the cdf of $Y_i|\eta$.

Therefore, equation 15 can be written as below using property of **??**

$$L_i = -log(1 - \phi(z_i)) \tag{24}$$

1.3.2 Negative Gradient

$$-\frac{\partial L_i}{\partial \eta} = -\frac{\partial - log(1 - \phi(z_i))}{\partial \eta} \tag{25}$$

$$-\frac{\partial L_i}{\partial \eta} = \frac{\partial log(1 - \phi(z_i))}{\partial \eta} \tag{26}$$

Using similar steps, we have done for left censored.

$$-\frac{\partial L_i}{\partial \eta} = -\frac{\phi'(z_i)}{1 - \phi(z_i)} \frac{\partial z_i}{\partial \eta} \tag{27}$$

which is nothing but negative of the negative gradient of left censored data. Therefore,

$$-\frac{\partial L_i}{\partial \eta} = \frac{f(z_i)}{\sigma(1 - \phi(z_i))} \tag{28}$$

1.3.3 Hessian

$$\frac{\partial^2 L_i}{\partial \eta^2} = -\frac{(1 - \phi(z_i))f'(z_i) + f^2(z_i)}{\sigma^2(1 - \phi(z_i))^2} \tag{29}$$

1.4 Interval Censored Data

1.4.1 Loss Function

This is combination of left censored and right censored data.

$$L_i = -\log \text{lik}_i = -log(F_{Y_i^u}(y_i^u|\eta) - F_{Y_i^l}(y_i^l|\eta)) \tag{30}$$

where, F is the cdf of $Y_i|\eta$, y_i^u is the upper limit of time and y_i^l is the lower limit of the time. Above equation is written in terms of ϕ notations as below.

$$L_i = -log(\phi(z_i^u) - \phi(z_i^l)) \tag{31}$$

1.4.2 Negative Gradient

$$-\frac{\partial L_i}{\partial \eta} = -\frac{\partial - log(\phi(z_i^u) - \phi(z_i^l))}{\partial \eta} \tag{32}$$

$$\begin{aligned} -\frac{\partial L_i}{\partial \eta} &= \frac{\phi'(z_i^u)}{\phi(z_i^u) - \phi(z_i^l)} \frac{\partial z_i^u}{\partial \eta} - \frac{\phi'(z_i^l)}{\phi(z_i^u) - \phi(z_i^l)} \frac{\partial z_i^l}{\partial \eta} \\ &= \frac{f(z_i^u)}{\phi(z_i^u) - \phi(z_i^l)} \frac{-1}{\sigma} - \frac{f(z_i^l)}{\phi(z_i^u) - \phi(z_i^l)} \frac{-1}{\sigma} \\ &= -\frac{f(z_i^u) - f(z_i^l)}{\sigma(\phi(z_i^u) - \phi(z_i^l))} \end{aligned} \tag{33}$$

1.4.3 Hessian

Hessian is the second derivative of Loss with respect to η. As we have already calculated the negative gradient, further we need to take one more partial derivative with respect to η of negative gradient with negative sign.

$$\frac{\partial^2 L_i}{\partial \eta^2} = \frac{\partial}{\partial \eta}\frac{\partial L_i}{\partial \eta} \tag{34}$$

After inputting the values of negative gradient calculated to above equation.

$$= \frac{\partial}{\partial \eta} * \frac{-f(z_i^u) + f(z_i^l)}{\sigma\{\phi(z_i^u) - \phi(z_i^l)\}} \tag{35}$$

Lets consider first part,

$$\begin{aligned} &\frac{\partial}{\partial \eta}\frac{-f(z_i^u)}{\sigma\{\phi(z_i^u) - \phi(z_i^l)\}} \\ &= \frac{\partial}{\partial z_i^u}\frac{-f(z_i^u)}{\sigma\{\phi(z_i^u) - \phi(z_i^l)\}}\frac{\partial z_i^u}{\partial \eta} \\ &= -\frac{\{\phi(z_i^u) - \phi(z_i^l)\}f^{'}(z_i^u) - f^2(z_i^u)}{\sigma\{\phi(z_i^u) - \phi(z_i^l)\}^2}\frac{-1}{\sigma} \\ &= \frac{\{\phi(z_i^u) - \phi(z_i^l)\}f^{'}(z_i^u) - f^2(z_i^u)}{\sigma^2\{\phi(z_i^u) - \phi(z_i^l)\}^2} \end{aligned} \tag{36}$$

Similarly, for second part -

$$\begin{aligned} &\frac{\partial}{\partial \eta}\frac{f(z_i^l)}{\sigma\{\phi(z_i^u) - \phi(z_i^l)\}} \\ &= \frac{\partial}{\partial z_i^l}\frac{f(z_i^l)}{\sigma\{\phi(z_i^u) - \phi(z_i^l)\}}\frac{\partial z_i^l}{\partial \eta} \\ &= \frac{\{\phi(z_i^u) - \phi(z_i^l)\}f^{'}(z_i^l) + f^2(z_i^l)}{\sigma\{\phi(z_i^u) - \phi(z_i^l)\}^2}\frac{-1}{\sigma} \\ &= -\frac{\{\phi(z_i^u) - \phi(z_i^l)\}f^{'}(z_i^l) - f^2(z_i^l)}{\sigma^2\{\phi(z_i^u) - \phi(z_i^l)\}^2} \end{aligned} \tag{37}$$

Combining both

$$\frac{\partial^2 L}{\partial \eta^2} = \frac{\{\phi(z_i^u) - \phi(z_i^l)\}\{f^{'}(z_i^u) - f^{'}(z_i^l)\} - \{f^2(z_i^u) - f^2(z_i^l)\}}{\sigma^2\{\phi(z_i^u) - \phi(z_i^l)\}^2} \tag{38}$$

2 Error - Logistic Distribution

2.1 Uncensored Data

2.1.1 Loss Function

$$L_i = -\log \text{lik}_i = -log(f_{Y_i}(y_i|\eta)) \tag{39}$$

where, $f_\eta(\eta)$ is the probability density function(pdf) of η.

$$\begin{aligned} z_i &= \frac{\log y_i - \eta}{\sigma} \sim f \\ f(z) &= \frac{e^z}{(1+e^z)^2} \\ F(z) &= \frac{e^z}{1+e^z} \\ F_{Y_i}(y_i) &= F(z_i) \end{aligned} \tag{40}$$

Now using change of variable for probability density function(pdf). We can write the below equations.

$$\begin{aligned} f_Y(y|\eta) &= f_Z(z)\frac{\partial z}{\partial y} \\ f_Y(y|\eta) &= \frac{\exp(\frac{(\log(y)-\eta))}{\sigma})}{\sigma * y * (1+\exp(\frac{(\log(y)-\eta)}{\sigma}))^2} \\ L_i &= -\log(\frac{\exp(\frac{(\log(y)-\eta)}{\sigma})}{\sigma * y * (1+\exp(\frac{(\log(y)-\eta)}{\sigma}))^2}) \\ &= -\log(\frac{\exp(\frac{(\log(y)-\eta)}{\sigma})}{\sigma * y * (1+\exp(\frac{(\log(y)-\eta)}{\sigma}))^2}) \end{aligned} \tag{41}$$

2.1.2 Negative Gradient

Key results

$$\begin{aligned} w &= e^z \\ f^{'}(z) &= f(z)\frac{1-w}{1+w} \end{aligned} \tag{42}$$

$$f^{''}(z) = f^{'}(z)\frac{1-w}{1+w} - f(z)\frac{1}{1+w} \tag{43}$$

Therefore, $f'(y|\eta)$ is calculated based on chain rule.

$$\frac{\partial f_Y(y|\eta)}{\partial \eta} = \frac{\partial}{\partial \eta}[f_Z(z)\frac{\partial z}{\partial \eta}] \tag{44}$$

Using product rule of differentiation, we can split $f_Z(z)$ and $\frac{\partial z}{\partial \eta}$

$$= \frac{\partial f_Z(z)}{\partial \eta}\frac{\partial z}{\partial \eta} + f_Z(z)\frac{\partial^2 z}{\partial \eta^2} \tag{45}$$

$$= \frac{\partial f_Z(z)}{\partial z}\frac{\partial z^2}{\partial \eta^2} + f_Z(z)\frac{\partial^2 z}{\partial \eta^2} \tag{46}$$

$$\frac{\partial z}{\partial \eta} = \frac{-1}{\sigma} \tag{47}$$

We also need double differentiation of z with respect to η as we have $\frac{\partial^2 z}{\partial \eta^2}$ in the equation 47.

$$\frac{\partial^2 z}{\partial \eta^2} = 0 \tag{48}$$

$$\frac{\partial z^2}{\partial \eta^2} = \frac{1}{\sigma^2} \tag{49}$$

Therefore,

$$-f^{'}(y|\eta) = -\frac{f^{'}_Z(z)}{\sigma^2} \tag{50}$$

$$\begin{aligned} -\frac{\partial L_i}{\partial \eta} &= \frac{1}{f_{Y_i}(y_i|\eta)} * \frac{\partial f_{Y_i}(y_i|\eta)}{\partial \eta} \\ &= \frac{1}{f_{Y_i}(y_i|\eta)} * \frac{-f^{'}_{Z_i}(z_i)}{y\sigma^2} \\ &= \frac{-f^{'}_{Z_i}(z_i)}{\sigma f_{Z_i}(z_i|\eta)} \end{aligned} \tag{51}$$

2.1.3 Hessian

$$\begin{aligned} \frac{\partial^2 L_i}{\partial \eta^2} &= \frac{\partial}{\partial \eta}\frac{\partial L_i}{\partial \eta} \\ &= \frac{\partial}{\partial \eta}\frac{f^{'}_{Z_i}(z_i)}{\sigma f_{Z_i}(z_i|\eta)} \\ &= \frac{\partial}{\partial z_i}\frac{f^{'}_{Z_i}(z_i)}{\sigma f_{Z_i}(z_i|\eta)}\frac{\partial z_i}{\partial \eta} \\ &= \frac{\partial}{\partial z_i}\frac{f^{'}_{Z_i}(z_i)}{\sigma f_{Z_i}(z_i|\eta)}\frac{\partial z_i}{\partial \eta} \\ &= -\frac{f_{Z_i}(z_i)f^{''}_{Z_i}(z_i) - [f^{'}_{Z_i}(z_i)]^2}{\sigma^2 f^2_{Z_i}(z_i)} \end{aligned} \tag{52}$$

2.2 Left Censored Data

2.2.1 Loss Function

$$L_i = -\log \text{lik}_i = -log(F_{Y_i}(y_i|\eta)) \tag{53}$$

where, F is the cdf of $Y_i|\eta$.

As we know,

$$log(Y_i) \sim Logistic(\eta, \sigma) \tag{54}$$

$$F_{Y_i}(y_i|\eta) = \frac{\exp^{\frac{\log(y_i)-\eta}{\sigma}}}{1+\exp^{\frac{\log(y_i)-\eta}{\sigma}}} \tag{55}$$

2.2.2 Negative Gradient

$$-\frac{\partial L_i}{\partial \eta} = -\frac{\partial - log(F_{Y_i}(y_i|\eta))}{\partial \eta} \tag{56}$$

$$-\frac{\partial L_i}{\partial \eta} = \frac{\partial log(F_{Y_i}(y_i|\eta))}{\partial \eta} \tag{57}$$

$$= \frac{F^{'}_{Y_i}(y_i|\eta)}{F_{Y_i}(y_i|\eta)} \tag{58}$$

$$-\frac{\partial L_i}{\partial \eta} = \frac{-f(z_i)}{\sigma F(z_i)} \tag{59}$$

2.2.3 Hessian

$$\begin{aligned} \frac{\partial^2 L_i}{\partial \eta^2} &= \frac{\partial}{\partial \eta} * \frac{f(z_i)}{\sigma F(z_i)} \\ &= \frac{\partial}{\partial z_i}\frac{f(z_i)}{\sigma F(z_i)} * \frac{\partial z_i}{\partial \eta} \\ &= \frac{F(z_i)f^{'}(z_i) - f^2(z_i)}{\sigma F^2(z_i)} * \frac{-1}{\sigma} \\ &= -\frac{F(z_i)f^{'}(z_i) - f^2(z_i)}{\sigma^2 F^2(z_i)} \end{aligned} \tag{60}$$

2.3 Right Censored Data

2.3.1 Loss Function

$$L_i = -\log \text{lik}_i = -log(1 - F_{Y_i}(y_i|\eta)) \tag{61}$$

where, F is the cdf of $Y_i|\eta$.

As we know,

$$log(Y_i) \sim Logistic(\eta, \sigma) \tag{62}$$

$$F_{Y_i}(y_i|\eta) = \frac{\exp^{\frac{\log(y_i)-\eta}{\sigma}}}{1+\exp^{\frac{\log(y_i)-\eta}{\sigma}}} \tag{63}$$

2.3.2 Negative Gradient

$$-\frac{\partial L_i}{\partial \eta} = -\frac{\partial - log(1 - F_{Y_i}(y_i|\eta))}{\partial \eta} \tag{64}$$

$$-\frac{\partial L_i}{\partial \eta} = \frac{\partial log(1 - F_{Y_i}(y_i|\eta))}{\partial \eta} \tag{65}$$

$$= -\frac{F'_{Y_i}(y_i|\eta)}{1 - F_{Y_i}(y_i|\eta)} \tag{66}$$

$$= \frac{f(z_i)}{\sigma(1 - F(z_i))} \tag{67}$$

2.3.3 Hessian

$$\frac{\partial^2 L_i}{\partial \eta^2} = -\frac{(1 - F(z_i))f'(z_i) + f^2(z_i)}{\sigma^2(1 - F(z_i))^2} \tag{68}$$

2.4 Interval Censored Data

2.4.1 Loss Function

This is combination of left censored and right censored data.

$$L_i = -\log \text{lik}_i = -log(F_{Y_i^u}(y_i^u|\eta) - F_{Y_i^l}(y_i^l|\eta)) \tag{69}$$

where, F is the cdf of $Y_i|\eta$, y_i^u is the upper limit of time and y_i^l is the lower limit of the time. Above equation is written in terms of ϕ notations as below.

2.4.2 Negative Gradient

$$-\frac{\partial L_i}{\partial \eta} = -\frac{\partial - log(F_{Y_i^u}(y_i^u|\eta) - F_{Y_i^l}(y_i^l|\eta))}{\partial \eta} \tag{70}$$

$$-\frac{\partial L_i}{\partial \eta} = \frac{F'_{Y_i^u}(y_i^u|\eta) - F'_{Y_i^l}(y_i^l|\eta)}{F_{Y_i^u}(y_i^u|\eta) - F_{Y_i^l}(y_i^l|\eta)} = \frac{F'(z_i^u) - F'(z_i^l)}{F(z_i^u) - F(z_i^l)} \tag{71}$$

2.4.3 Hessian

Hessian is the second derivative of Loss with respect to η. As we have already calculated the negative gradient, further we need to take one more partial derivative with respect to η of negative gradient with negative sign.

$$\frac{\partial^2 L_i}{\partial \eta^2} = \frac{\partial}{\partial \eta}\frac{\partial L_i}{\partial \eta} \tag{72}$$

After inputting the values of negative gradient calculated to above equation.

$$= -\frac{\partial}{\partial\eta} * \frac{F'_{Y_i^u}(y_i^u|\eta) - F'_{Y_i^l}(y_i^l|\eta)}{F_{Y_i^u}(y_i^u|\eta) - F_{Y_i^l}(y_i^l|\eta)} \tag{73}$$

Using division rule using the properties of $f'(z) = -zf(z)$ and **??**. Below is the new equation, we have split the above equation numerator into 2 parts.

$$= -\frac{\{F_{Y_i^u}(y_i^u|\eta) - F_{Y_i^l}(y_i^l|\eta)\}F''_{Y_i^u}(y_i^u|\eta) - \{F'_{Y_i^u}(y_i^u|\eta) - F'_{Y_i^l}(y_i^l|\eta)\}F'_{Y_i^u}(y_i^u|\eta)}{\{F_{Y_i^u}(y_i^u|\eta) - F_{Y_i^l}(y_i^l|\eta)\}^2} \tag{74}$$

Similarly we can do it for second part.