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# SPARSE AND GROUP REGRESSION MODELS IN PORTFOLIO OPTIMIZATION

(ANNEX)

# Contents

# 1 Description of Sections

## 1.1 Background Research

Given the complex nature of this research topic, there was a lot background research required to obtain an intuitive understanding on the models and concepts related. This section aims to provide the reader with the core knowledge required on the financial, mathematical, statistical and algorithmic concepts, definitions and formulations related to this research topic.

Understanding on several financial concepts will be important to provide knowledge on the application of this model. This section provides the mathematical notations and definitions that will be consistent throughout the paper, as well as financial definitions and formulations, such Volatility, Financial Index, Modern Portfolio Theory, Portfolio Optimization, Index Tracking, Value-at-Risk, Conditional-Value-at-Risk, between others.

It will also be important to cover the machine learning and algorithmic models that are present in this research project. These are broken down and analyzed in order to provide an intuitive and clear knowledge. The models used in this paper consist of feature-level regression models (i.e. sum of absolute values, sum of squares, ridge regression (L0+L2-norm) CVaR and Lasso), and sparse-inducing group-level regression models (Group Lasso, Sparse Group Lasso and Multiple Sparse Group Lasso). The algorithmic approaches taken to impose the cardinality constraints in the simple non-sparse models are full-search and greedy algorithms.

## 1.2 Implementation

The aim of this section is to collect results and observe the behaviour these feature and group level regression models. The main objective in this paper is to discover the effects of taking into account classification information in financial data, as opposed to considering the features as individual entities. We strive to obtain results that can show whether using this extra information will allow for better results.

Each individual regression model is applied to Market Index Data for a specific window of time. Experiments are based completely on the Index Tracking portfolio optimization problem. Our results are quantified on the output Tracking Error, which is equivalent to the test error of our model. In few words, the Index Tracking minimization problem consists on finding a subset of stocks from a specific portfolio that behaves as similar as possible to the original portfolio (i.e. has the lowest tracking error).

Each model contains several constants that need to be adjusted to provide reliable and relevant results – some of these constants define level of sparsity to be achieved, as well as other characteristics. Several experiments will be carried out to find efficient values for these variable to provide reliable results.

Although stochastic data will be used initially, the main financial data set used in this paper has been downloaded using Yahoo finance API and consists of the stocks comprising the FTSE100 (Pronounced 'footsie' 100) – which is a share index of the 100 companies listed on the London Stock Exchange with the highest market capitalization.

The implementation will consist of two main sections – the implementation of feature-level regression models, and the implementation of group-level regression models.

### 1.3 Analysis

In this section, this paper aims to provide a thorough and in-depth analysis of the results obtained in the implementation section.

The objective of this section is to provide an insight based on the observations drawn from the implementation that would allow for an objective perspective on how using grouping information affects results obtained. In order to provide such insight, this section contains thorough analysis on the behaviour of both sets of models applied in the implementation – feature and group level regression models. Intuition should be provided to the reader through comparing the benefits and drawbacks observed on each of these implementations.

To conclude this section, a real-life application of the implementations in this paper will be given. The objective of this is to provide a more clear understanding on the usefulness of these models in the real financial world. This sub-section should also provide an idea on whether the use of classification information in finance is good choice.

### 1.4 Expansion

Based in the results and analysis a new idea arose from the rich amount of clustering information available for financial information. It was observed that the Sparse Group Lasso model was limited to only one category. In this section a new concept is proposed to encourage further research in this area of machine learning. The concept is basically a Multiple Sparse Group Model, and as the name implies this concept aims to consider multiple categories during the regression process, allowing for consideration for categorizing characteristics of financial instruments such as type of instrument, sector, industry, volatility, historical return correlation, etc.

### 1.5 Conclusion

We finalize this paper by providing a brief overview of the results obtained, and the final observations made. This section is very important as it would be ideal that this paper can provide a base, or an addition to current research. Conclusions made in this paper are

# 2 Expanded Background of Literature Search

## 2.1 Mathematical Definitions (Redefined)

### 2.1.1 General

We will use $\sigma$ to refer to the volatility of a financial instrument (i.e. the standard deviation of the daily returns of a specific financial instrument).

Norms will be referred as L$\varphi$–norms, where $\varphi \in \{1,2, \ldots\}$ as $\|\boldsymbol{x}\|_{\varphi} = [\sum_{i=1}^{n} |x_i|^{\varphi}]^{\frac{1}{\varphi}}$.

The statistical and mathematical notations used in this paper are all considered within a specific time window t = 1, ...,T and a number of assets n.

Input data is of the form $\mathbf{R}_t \in \mathbb{R}^n = (\mathbf{R}_{t,1}, \mathbf{R}_{t,2}, \ldots, \mathbf{R}_{t,n})^T$ where $\mathbf{R}$ is an $\mathbb{R}^{T\times n}$ matrix where each column is a vector of returns of all the assets of the portfolio at time t.

Our target data is of the form $\boldsymbol{I} = \boldsymbol{R}_{\boldsymbol{t}}^{T}\boldsymbol{\pi}'$, where $\boldsymbol{I}$ is a Market Index (i.e. a share index of the n companies listed on a specific Stock Exchange with the highest market capitalization) and $\boldsymbol{\pi} \in \mathbb{R}^n = (\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_n)^T$ is the proportions of each stock $\boldsymbol{\pi}_i$ – in this case $\boldsymbol{\pi}' = 1/n$.

In all our regression problems, $\boldsymbol{\pi}$ will be the parameter to be learned. It is important to note that throughout all the models of this paper, the constraint $\boldsymbol{\pi} > 0$ will always be present, as it enseures that no shortings are done. Finally, as this variable contains percentage values, another constraint that will be held at all times is $\sum_{=0}^{n} \boldsymbol{\pi} = 1$.

### 2.1.2 Group Notation

When dealing with groups, we will have $m$ groups of size $p_i \in \boldsymbol{p}$, where $\sum_{i=1}^{n} p_i = n$.

Our stocks will be grouped in these m groups of size $p_i$, which means our data $\mathbf{R}$ is in groups $\overline{\boldsymbol{R}} = (\overline{\boldsymbol{R}}_{\boldsymbol{1}}, \overline{\boldsymbol{R}}_{\boldsymbol{2}}, \ldots, \overline{\boldsymbol{R}}_{\boldsymbol{m}})$, where $\overline{\boldsymbol{R}}_{\boldsymbol{i}} \subseteq \mathbf{R}$ and $\overline{\boldsymbol{R}}_{i,t} \in \mathbb{R}^{p_i} = (\overline{\boldsymbol{R}}_{i,t,1}, \ldots, \overline{\boldsymbol{R}}_{i,t,p_i})^T$.

Although our Index Portfolio would be referenced with the same variable $\mathbf{I}$, a new group definition is now introduced as $\boldsymbol{I} = \overline{\boldsymbol{R}}_{\boldsymbol{1}}\overline{\boldsymbol{\pi}}_{\boldsymbol{1}} + \overline{\boldsymbol{R}}_{\boldsymbol{2}}\overline{\boldsymbol{\pi}}_{\boldsymbol{2}} + \cdots + \overline{\boldsymbol{R}}_{\boldsymbol{m}}\overline{\boldsymbol{\pi}}_{\boldsymbol{m}}$, where $\overline{\boldsymbol{\pi}}_{\boldsymbol{i}} \in \mathbb{R}^{p_i}$ are the parameters to learn that belong to the stocks in group $i \in \{1,2, \ldots, m\}$.

For the new group formulation proposed we will need to introduce the concept of superscripting (same as in arrays in code) in order to obtain specific elements in matrices and vectors through indexing. In this paper we will use the pseudo-code notation $\boldsymbol{A}[\,\boldsymbol{x},\boldsymbol{y}\,]$, where **X** is a matrix of any size, **x** and **y** are vectors of the same size containing the positions in the Matrix to be superscripted, and they are of the form $\boldsymbol{x_i},\boldsymbol{y_i} \in \mathbb{R} \quad \wedge (\forall x,y.\ |x| < |\boldsymbol{A}[\,x,:\,]| \quad \wedge\ y < |\boldsymbol{A}[\,x,:\,]|) \wedge (\forall x,y.\ \max(x) < |\boldsymbol{A}[\,x,:\,]| \wedge \max(y) < |\boldsymbol{A}[\,x,:\,]|))$. A column can be used to denote that all columns or rows are selected, so $\boldsymbol{A}[\,:\,,\boldsymbol{y}\,]$ would superscript all the row elements for columns contained in the indexes **y**.

For indexing we will use a variable $\boldsymbol{\Psi_g} \in \boldsymbol{\psi}$ that will denote the indexes of the stocks of each group. For example, if group $\boldsymbol{\bar{\pi}_g}$ contains stocks 3, 5, 6, etc, then $\boldsymbol{\Psi_g}$ = {3, 5, 6, ...}. This allows to introduce the logical biconditional $(\boldsymbol{\bar{\pi}_g} \subseteq \boldsymbol{\pi}) \Leftrightarrow (\boldsymbol{\pi}[\,1,\boldsymbol{\Psi_g}\,] = \boldsymbol{\bar{\pi}_g})$.

The final notation required for the new proposed formula is the sum of all the columns or rows as $sum(\boldsymbol{A},1) = \sum_{i=1}^{r} \boldsymbol{A}[\,:\,,i\,] \wedge sum(\boldsymbol{A},1) = \sum_{i=1}^{c} \boldsymbol{A}[\,i,:\,]$ where r is the number of rows in **A** and c is the number of columns in **A**.

## 2.2 Financial Definitions

### *2.2.1 Basic financial terms*

- **Financial risk** - potential loss or uncertainty from an investment, and can be measured through mathematical and statistical models.
- **Volatility** - A factor that is often used when measuring financial risk previously defined as σ. The volatility of a financial instrument is the standard deviation of the historical returns.
- **Portfolio –** A group of financial assets such as bonds, stocks where an amount of money is invested
- **Market Index** – An aggregate of stocks from a financial market that aim to represent an entire stock market.

### *2.2.2 Portfolio Diversification*

In finance it is a very important concern to obtain diversification when dealing with investments. What is meant by diversification is to distribute risk by investing in several (preferably uncorrelated) financial instruments. This way, if the price of a group of financial instruments goes down, losses would not be as bad, as a diversified portfolio would have investments in numerous disjoint groups.

### *2.2.3 Stochastic Simulation Models*

In this paper it was a very important concern to find an effective way to simulate financial data. The models that were considered were the GARCH, ARIMA, EULER and a Monte Carlo approach. The approach taken was the Monte Carlo approach

#### *2.2.4 GARCH Prediction Model*

This model is derived from the family of AutoRegressive Conditional Hteroskedacity (ARCH) models, and it is used to model time series – hence why it is often applied in finance. In the case of the GARCH model, it is just an ARCH variation which is assumed for the error variance.

#### *2.2.5 Autoregressive Integrated Moving Average*

#### *2.2.6 Monte Carlo*

The Monte Carlo model is based in a simulation model that is provided by the Matlab core library, and it is based in generating a normally distributed random value that represents the distribution of each stock at each point in time. This model proved to be an accurate and simple approach and this was the reason why this model was chosen.

In matlab, the algorithm is defined as:

**Algorithm 1 - Monte Carlo Implementation**

```
for i=1:numberStocks
    time = 0;
    volatility = volbase + rand()*volvar;
    prices = [];

    while time < timeToExpiry
      time = time + sampleRate;
      drift = (riskFreeRate - dividend - volatility*volatility/2)*sampleRate;
      perturbation = volatility*sqrt( sampleRate )*randn();
      price = price*exp(drift + perturbation);
      times = [times time];
      prices = [prices, price];
    end

    all_prices = [all_prices; prices];
  end
```

#### 2.2.7 Modern Portfolio Theory

In finance, portfolio optimization is an extremely popular and revisited concept since its debut in the original paper by [1] Markowitz(1952), 'Porfolio Theory'. More widely known as Modern Portfolio Theory, it consists of a minimization on the risk for a given level of expected return **r** through choosing the proportions (weights) of the assets **π** that comprise the portfolio. The formulation introduced in this paper is defined as follows:

**Equation 1 – MPT Minimization**

$$\min_{\pi} \quad \boldsymbol{\pi}^T \mathbf{R} \boldsymbol{\pi}$$
$$\text{s.t.} \quad \boldsymbol{r}^{\mathrm{T}} \boldsymbol{\pi} = \gamma \quad \sum_{i=1}^{n} \pi_i = 1$$

Using input data over a range of time **T** for **n** assets, this Markowitz portfolio optimization formula minimizes on the portfolio variance given by $\boldsymbol{\pi}^T \mathbf{R} \boldsymbol{\pi}$, where **π** is the proportion invested on each asset, and $\mathbf{R} \in \mathbb{R}^{\mathrm{nxn}}$ is a covariance matrix of asset is the parameter to be learned. The amount of expected return is given by $\gamma$, and a vector of expected returns for each asset is given by $\boldsymbol{r} \in \mathbb{R}^{\mathrm{n}}$. This model revolutionized portfolio theory since its debut, and has been thoroughly revisited since.

#### 2.2.8 Tracking Error

In this paper, we will base all our calculations in a very commonly used loss function – the Tracking Error. Tracking error is very intuitive, especially when seen graphically – it basically measures how similar a portfolio A behaves to a portfolio B. We will refer to tracking error as $\xi_{\varphi,\mathrm{t}}(\boldsymbol{\pi})$ defined as:

**Equation 2 - Tracking Error**

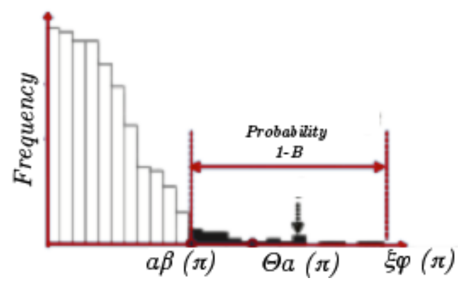
$$\xi_{\varphi,t}(\boldsymbol{\pi}) = \frac{1}{T} \sqrt[1/\varphi]{\left[ \sum_{t=1}^{T} \left| \boldsymbol{I}_t - \boldsymbol{R}_t^{\mathrm{T}} \boldsymbol{\pi} \right|^{\varphi} \right]}$$

This tracking error function will also be used to measure the results of the accuracy of our model compared to other models proposed.

### 2.3 Background in Regression Applications for Finance

In order to obtain an understanding on how regression models can be effectively implemented in financial datasets, we require to introduce some financial concepts before this, including Value-at-Risk and it's variations.

Figure 1




### *2.3.1 Value-at-Risk*

A very important risk management model in finance is the Value-at-Risk model, or VaR [2] Philippe (1996). The most common application of Value-at-Risk is the β%-VaR. Given a percentage β% and a set of historical prices, we can build a frequency table which we assume has normal distribution, and then we can estimate the loss that we are β% sure that will not be exceeded.

Mathematically, given a confidence index α based on β% (i.e. 95% would be 2 standard deviations), a mean μ and a volatility σ, we can compute the model presented in [6] Mina et al. β%-VaR from a portfolio as follows:

**Equation 3 - Β%-VaR**

$$\beta\%VaR = \alpha * (\sigma - \mu)$$

Value-at-Risk is excellent when dealing with raw calculations from a given volatility, however it lacks of convexity characteristics, making it hard to optimize, and hence unattractive for minimization problems.

### *2.3.2 Expected Shortfall*

β-Conditional-Value-at-Risk, also known as Expected Shortfall, is the conditional expectation of losses above β%-VaR – in other words, the probability that a specific loss will exceed the β%-VaR. Unlike VaR, CVaR has many desired characteristics such as convexity, that will allow for the definitions required for using this as a minimization problem.

To define this, we base ourselves in the mathematical notation in [Akiko 2010] and in the logic of [4] Rockafellar R. et al. (2000). We initially base ourselves in a regression problem to obtain a linear function approximator y = (w,x) + b from m = 1, ..., M samples $(x_i, y_i)$. Our variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ are our input and output values respectively and $w \in \mathbb{R}^n$ and $b \in \mathbb{R}$ are the variables to be learned.

### *2.3.3 β-VaR*

Once defined our loss function, in this case, Tracking Error, we assume we an get probability of a specific result of $\xi\varphi(\boldsymbol{\pi})$ by computing $\rho(\xi\varphi(\boldsymbol{\pi}))$. This allows us to define a function $\Theta_\alpha(\boldsymbol{\pi})$ that gives us the cumulative distribution for our probability loss function $\rho(\xi\varphi(\boldsymbol{\pi}))$ given a given confidence α. This is, in other words, the probability of $\xi_\varphi(\boldsymbol{\pi})$ not exceeding a threshold given by α. The function $\theta_\alpha(\boldsymbol{\pi})$ is defined as follows:

**Equation 4**

$$\Theta_\alpha(\boldsymbol{\pi}) = \int_{\rho\left(\xi_{\varphi,t}(\boldsymbol{\pi})\right) \leq \alpha} \rho\left(\xi_{\varphi,t}(\boldsymbol{\pi})\right) \, d\, \xi_{\varphi,t}(\boldsymbol{\pi})$$

Finally, now that we can calculate the cumulative distribution of our loss function $\xi\varphi(\boldsymbol{\pi})$ we can obtain the definition of our β-VaR formula $\alpha_\beta(\boldsymbol{\pi})$ defined as:

**Equation 5**

$$\alpha_\beta(\boldsymbol{\pi}) = \min_{\alpha} \; \Theta_\alpha(\boldsymbol{\pi}) \geq \beta$$

Now that we have obtained β-VaR, or the lowest percentage amount that we will be β% sure that will not be exceeded, we can define the CVaR as the average loss **exceeding** VaR. Now that we have reached this definition, we can bring up the proof in [4] Rockafellar R. (2000), which shows the convex nature of CVaR, and allows for a minimization definition as follows:

**Equation 6**

$$\min_{\boldsymbol{\pi},b,\alpha} \quad \alpha + \frac{1}{(1-\beta)T} \sum_{t=1}^{T} |\xi_{\varphi,t}(\boldsymbol{\pi}) - \alpha|^{+}$$

Where $|t|^{+} = t$ when t > 0 and $|t|^{+} = 0$ when $t \leq 0$.

## 2.4 Feature Regression Models

Now that the necessary formulations were introduced to provide the reader with a core understanding on some of the minimization functions in financial data, as well as an idea on their potential applications we can proceed to discuss the regression models that will be used for single-feature analysis in this paper.

### *2.4.1 CVaR Minimization*

It is proven in Akiko [REFERENCE 1] that this CVaR minimization is equivalent to the Support Vector Regression algorithm, implying optimality, and also includes a proof that allows us to introduce the variable $\mathbf{z_t}$, where $\mathbf{z_t}$ allows us to use the definition of $|\,x\,|^+$ by a simple rearrangement of constraints, hence simplify the implementation of the problem in our algorithm massively. This allows us to get our final definition of our CVaR minimization formula as follows:

**Equation 7**

$$\min_{\pi,\alpha,z} \quad \alpha + \frac{1}{(1-\beta)T}\sum_{t=1}^{T} \boldsymbol{z_t}$$

$$\text{s.t.} \qquad \boldsymbol{z_t} - \xi_{\varphi,t}(\boldsymbol{\pi}) + \alpha \geq 0$$

$$\boldsymbol{z_t} \geq 0, \quad i \in \mathrm{T},$$

It is worth bringing up the fact that although the formulation provided in AKIKO [REFERENCE 1], called the Norm Constrained CVaR (NCCVaR) implements a sparsity inducing variable C2 which induces sparsity in the set of weights. Although the NCCVaR formulation won't be used, a lot of the proofs and concepts present in this paper were of great help in order to be able to use the CVaR minimization formula efficiently. This method will be referred to as *CVaR.*

### *2.4.2 Lasso Regression (Expanded)*

Before proceeding to the introduction of the group regression methods, it is necessary to introduce some core concepts in regards to sparsity model – one crucial one is the Lasso Regression model.

The Lasso regression model is comprised of a sum of squares, plus a scaled sum of the absolute value of the parameters – using our variables for Index, Return portfolio and weights, our model would be defined as follows.

**Equation 8**

$$\min_{\pi} \varrho(\boldsymbol{\pi}) = \sum_{i=1}^{T} \left| \boldsymbol{I}_t - \boldsymbol{R}_t^{\mathrm{T}}\boldsymbol{\pi} \right| + \lambda\,|\boldsymbol{\pi}|$$

$$\text{s.t.}$$

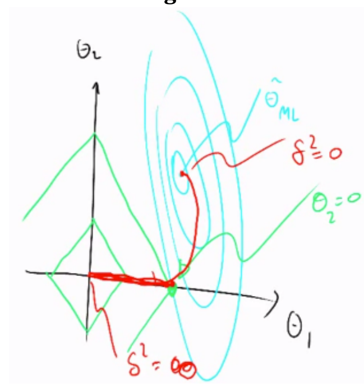
$$\sum_{i=1}^{T} \boldsymbol{\pi_i} = 1, \quad \boldsymbol{\pi_i} > \mathbf{0}$$

At first sight, it may seem very similar to the ridge (L0+L2-norm) model introduced previously, however, what makes this model stand out is the second coefficient –

namely the scaled sum of the absolute value of the parameters, which, different from the past model, this coefficient controls the sparsity of our final $\boldsymbol{\pi}$.

To visualize this we'll take n=2 as an example, which is pictured in the figure 3 on the right, where the x axis holds the value for our $\boldsymbol{\pi}_1$ and the y axis holds the value for $\boldsymbol{\pi}_2$.

**Figure 2**

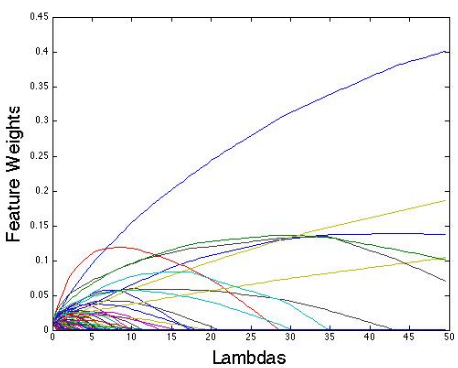



The center of the green diamond represents the point of $\varrho(\boldsymbol{\pi})$ when $\lambda = \infty$, and the center of the blue circle represents the point when $\tau = 0$.

The red path denotes the value $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ take as $\tau$ changes value. It can be observed that when $\tau = 0$, $\theta_2$ will be equal to the maximum likelihood of $\boldsymbol{\pi}_2$, denoted in blue by $\hat{\boldsymbol{\pi}}_{ML}$ which is the result of the first term of our equation. When $\tau = \infty$, the values of $\boldsymbol{\pi}$ will need to be as small as possible in order to minimize our cost function, which will result in our point in the origin.

By differentiating our cost function and equating it to zero, we find that many of our variables become zero – this is shown in Figure 2

**Figure 3**




Going back to figure 3 we can observe this behaviour when our blue circle intersects the corner of the green diamond. It is proven that intersections between these two are very likely to happen in the corners – which implies that one of our parameters has taken the value of zero – in this example, our parameter $\theta_2$ takes the value of zero when the red line intersects the x axis.

Intuitively, we will build an algorithm that initially takes a guess for all our parameters, then it will proceed to calculate the optimal value of each of the parameters using the partial derivative of our cost function, and repeat this until it converges.

**Equation 9**

$$\frac{\partial J(\theta)}{\partial \theta_j} = 2\sum_{i=1}^{m} x^2\theta_j - 2\sum_{i=0}^{m}(y_i - x_{i-j}\theta_j)x_{ij} + \frac{\partial \delta}{\partial \theta_j}|\theta_j|$$

The reason why the last term has not been derived yet is because it consists of an absolute value – for this, subdifferentials will be used. In order to simplify the formulation shown, we will make the following assignments.

**Equation 10**

$$a_j = 2\sum_{i=1}^{m} x^2$$

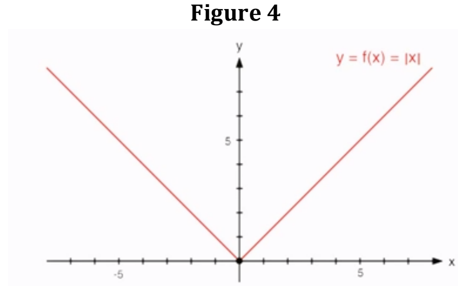
$$c_j = 2\sum_{i=0}^{m} (y_i - x_{i-j}\theta_j)x_{ij}$$

This would give us a much simpler formula to expand on its subdifferential, which consist of the following:

**Equation 11**

$$\frac{\partial\, J(\theta)}{\partial\, \theta_j} = a_j\theta_j + c_j + \frac{\partial \delta}{\partial \theta_j}|\theta_j|$$

**Figure 4**




Basically, with a subdifferential, we have a point which cannot be differentiated normally, as we would require a single gradient to explain a subset of gradients – in Figure 3, our point would be the origin, and we would have to define the differential of our formula as:

**Equation 12**

$$f(x) = |x|$$

$$\partial f(x) = \begin{cases} \{-1\} & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ \{+1\} & \text{if } x > 0 \end{cases}$$

So, our formulas would consist of exactly the same format, but instead we would use our variables a, b and $\theta_j$ as follows:

**Equation 13**

$$= \begin{cases} \{a_j\theta_j - c_j - \delta^2\} & \text{if } \theta_j < 0 \\ [-c_j - \delta^2, -c_j + \delta^2] & \text{if } \theta_j = 0 \\ \{a_j\theta_j - c_j + \delta^2\} & \text{if } \theta_j > 0 \end{cases}$$

Now, by setting our values to zero, our estimate $\hat{\theta}$ can be calculated by setting these differentials to zero for each of the $\theta_j$, which gives us the following estimates:

**Equation 14**

$$\widehat{\theta}_j = \begin{cases} (c_j + \delta^2)/a_j & \text{if } c_j < -\delta^2 \\ 0 & \text{if } c_j \in [-\delta^2, \delta^2] \\ (c_j - \delta^2)/a_j & \text{if } c_j > \delta^2 \end{cases}$$

Knowing this estimate now allows us to implement the simple Lasso Algorithm in a very simple manner:

**Algorithm 2 - Lasso PseudoCode**

1. **Initialize** $\theta$ (Can be either random, or with something like ridge, etc)
2. **Repeat** until converged
    a. **For** j = 1,2, … , d DO
        i. $a_j = 2\sum_{i=1}^{m} x^2$
        ii. $c_j = 2\sum_{i=0}^{m}(y_i - x_{i-j}\theta_j)x_{ij}$
        iii. **if** $(c_j < -\delta^2)$
            1. $\theta_j = (c_j + \delta^2)/a_j$
        iv. **elseif** $(c_j > -\delta^2)$
            1. $\theta_j = (c_j - \delta^2)/a_j$
        v. **else**
            1. $\theta_j = 0$

## 2.5 Model Selection Approaches (Expanded)

#### *2.5.1.1 Forward Search Application (Expanded)*

Due to the exponential complexity of the problem in [0] Mahesan et al. (2013) (i.e. to find an optimal subset of stocks from a given Market Index), it was very limited when it comes to even relatively small portfolios. Given that the average Market Index is composed of at least 100 distinct assets, this makes the implementation of a full search is almost impossible.

For this reason, a greedy approach was taken - namely, this was done through a forward model selection algorithm, where one optimal stock is chosen and added to the subset on each step until the subset reaches the desired cardinality.

Formally, the model is formulated as a subset selection, where each step, a locally optimal asset is chosen and added to the portfolio. The algorithm proposed in [0] Mahesan et al. (2013) is converted into a subset selection problem defined as follows:

**Algorithm 1** Greedy Algorithm

*Initialize:* $S_0 = \emptyset$ and $k = 1$. Set $f^\tau(\widetilde{\boldsymbol{\pi}}^\tau_{S_0})$ to a value large enough.
**while** $k \le C_0$ **do**
  For all $s \in N \setminus S_{k-1}$, compute $\widetilde{\boldsymbol{\pi}}^\tau_{S_{k-1}\cup\{s\}}$
  Select $s^*$ such that $\min_{s\in N\setminus S_{k-1}} f^\tau(\widetilde{\boldsymbol{\pi}}^\tau_{S_{k-1}\cup\{s\}})$ and set $\widetilde{\boldsymbol{\pi}}^\tau_{S_k} \leftarrow \widetilde{\boldsymbol{\pi}}^\tau_{S_{k-1}\cup\{s^*\}}$.
  Set $S_k \leftarrow S_{k-1} \cup \{s^*\}$ and $k \leftarrow k+1$.
**end while**
$\boldsymbol{\pi}^\tau_G \leftarrow \widetilde{\boldsymbol{\pi}}^\tau_{S_{C_0}}$ and $S^\tau_G \leftarrow S_{C_0}$.
**return** $\boldsymbol{\pi}^\tau_G$ and $S^\tau_G$.

To apply this algorithm, we begin with an empty set, then for all the available stocks, we add one and run the L0+L1-norm model on each. We then calculate the tracking error for each, and select the asset with the lowest tracking error as our local optimal. We do this until the cardinality of our set reaches the L-0 constraint $C_0$. This greedy forward search algorithm will be referred to as GFS throughout this paper.

## 2.6 Group Regression Approaches

### *2.6.1 Model Selection Approach (Expanded)*

Before considering complex group regression models, this paper considers an approach based in the approach in [0] Mahesan et al. (2013) that initially inspired this research project – mainly it is being referred to the forward-search approach that was taken to select a subset of stocks from a Market Index with the lowest tracking error.

As it was revisited in the last section, the approach taken in [0] was a greedy, forward-search approach, and this was the choice over a full-search approach due to the NP characteristics of the subset selection problem. Given that stocks are considered individually, and the Market Indexes vary from 100 to 500 stocks, the time complexity of this problem would be exponential on the number of stocks to consider, which would not allow for a full-search approach.

This section proposes to reduce this individual stock subset selection problem reduces to a group subset selection problem, and assuming that our group number does not consists of a large amount of groups (e.g. more than 20), a full-search approach is actually possible, as we could have a full subset selection on our groups.

Similar to the previous algorithm, we can select any regression model and implement a full-search group subset selection as follows:

**Algorithm 1** Greedy Algorithm

*Initialize:* $S_0 = \emptyset$ and $k = 1$. Set $f^{\tau}(\widetilde{\boldsymbol{\pi}}^{\tau}_{S_0})$ to a value large enough.
**while** $k \leq C_0$ **do**
  For all $s \in N \setminus S_{k-1}$, compute $\widetilde{\boldsymbol{\pi}}^{\tau}_{S_{k-1}\cup\{s\}}$
  Select $s^*$ such that $\min_{s\in N\setminus S_{k-1}} f^{\tau}(\widetilde{\boldsymbol{\pi}}^{\tau}_{S_{k-1}\cup\{s\}})$ and set $\widetilde{\boldsymbol{\pi}}^{\tau}_{S_k} \leftarrow \widetilde{\boldsymbol{\pi}}^{\tau}_{S_{k-1}\cup\{s^*\}}$.
  Set $S_k \leftarrow S_{k-1} \cup \{s^*\}$ and $k \leftarrow k+1$.
**end while**
$\boldsymbol{\pi}^{\tau}_G \leftarrow \widetilde{\boldsymbol{\pi}}^{\tau}_{S_{C_0}}$ and $S^{\tau}_G \leftarrow S_{C_0}$.
**return** $\boldsymbol{\pi}^{\tau}_G$ and $S^{\tau}_G$.

Similar to results presented in [0], with this full-search algorithm, we would be able to select a subset of financial instruments from a market index based on prior knowledge on the instruments that allowed us to create the groupings.

## 2.7 Zero-Constrained Models

### *2.7.1 Zero-Constrained Group Lasso Implementation*

Here we present the implementation of the Zero-Constrained Group Lasso implementation which was defined as follows:

$$\min_{\bar{\boldsymbol{\pi}}\in \mathbb{R}^{\boldsymbol{n}}} \quad \left\| \boldsymbol{I} - \boldsymbol{R}^{\mathrm{T}} sum(\widehat{\bar{\boldsymbol{\pi}}}, 2) \right\| + \lambda * sum\left(\sqrt{p^{\mathrm{T}}} * \sqrt{sum(\widehat{\bar{\boldsymbol{\pi}}}.\hat{\ }2,2)}\right)$$

$$\text{s.t.} \qquad \boldsymbol{\pi_i} > \boldsymbol{0} \quad \wedge \quad \widehat{\bar{\boldsymbol{\pi}}}[\, \sim\boldsymbol{\Psi_g} \,, g \,] = 0$$

The implementation was carried out with the help of the CVX library. It receives as input a matrix 'not_group_idx' where each row represents group g of rows containing 1's in the indexes of the stocks that are **not** contained in group $\bar{\boldsymbol{\pi}}_g$, hence not_group_idx is our $\sim\boldsymbol{\Psi_g}$ variable defined previously. It also receives as input the size of the groups as a vector, which represents our variable $\boldsymbol{p}$. Finally, the last argument taken is the lambda coefficient, which determines how much the model will be inducing group sparsity.

With this, our output will be pimat_hat, which represents the variable introduced earlier $\widehat{\bar{\boldsymbol{\pi}}}$ where $\widehat{\bar{\boldsymbol{\pi}}}[\, \sim\boldsymbol{\Psi_g} \,, g \,] = 0$ and $sum(\widehat{\bar{\boldsymbol{\pi}}}, 2) = \boldsymbol{\pi}$.

With this, the GroupLasso function has been implemented in Matlab as follows:

```
Function [pimat_hat] = GroupLasso(not_group_idx, group_sizes, lambda)
group_sizes = sqrt(group_sizes);
cvx_begin
    variable pimat_hat (totalassets,n)

    minimize(sum_square_pos(norm(I_train - R_train'* sum(pimat_hat,2)))
                                    + lambda*norms(pimat_hat,2,1)*sqr_group_sizes)

    subject to
        pimat_hat >= 0
        pimat_hat(not_group_idx) == 0
cvx_end
end
```

### *2.7.2 Zero-Constrained Sparse Group Lasso Implementation*

This definition can be easily expanded into the sparse group formulation by taking the Zero-Constrained Group Lasso definition and expanding it into the Zero-Constrained Group Lasso Implementation, which we define as follows:

$$\min_{\widehat{\widehat{\boldsymbol{\pi}}} \in \mathbb{R}^{\boldsymbol{n}}} \quad \left\|\boldsymbol{I} - \boldsymbol{R}^{\mathrm{T}} sum(\widehat{\widehat{\boldsymbol{\pi}}}, 2)\right\| + \lambda_1 \; sum\left(\sqrt{p^{\mathrm{T}}} * \sqrt{sum(\widehat{\widehat{\boldsymbol{\pi}}}.\,\hat{}\,2{,}2)}\right) + \lambda_2 \left\|sum(\widehat{\widehat{\boldsymbol{\pi}}}, 2)\right\|_{\mathbf{1}}$$

$$\text{s.t.} \qquad \boldsymbol{\pi_i} > \mathbf{0} \quad \wedge \quad \widehat{\widehat{\boldsymbol{\pi}}}\left[\, \sim\boldsymbol{\Psi_g}\, , g \,\right] = 0$$

The only difference in this version is that in this case, the 'lambda' parameter is replaced with two new parameters – namely 'l1_groups', which controls between-group sparsity (Group Lasso characteristic), and 'l2_groups' which controls within group lasso sparsity (Simple Lasso characteristic). The SparseGroupLasso function was implemented in Matlab code as follows:

```
Function [pimat_hat] = SparseGroupLasso(not_group_idx, group_sizes, l1_groups, l2_features)
cvx_begin
    variable pimat_hat (totalassets,n)

    minimize(sum_square_pos(norm(I_train - sum(R_train'* pimat_hat,2)))
                            + l1_groups*norms(pimat_hat,2,1)*sqr_group_sizes
                            + l2_features*sum(abs(pimat_hat (:))) )

    subject to
        pimat_hat >= 0
        pimat_hat(not_group_idx) == 0
cvx_end
end
```

# 3 Alternative Approaches

In this brief subsection, alternatives to the CVX libraries are briefly discussed – Namely the SPAMS and [REFERENCE] libraries.

These two libraries were not implemented in this paper – nevertheless they provided several core concepts that were used, and the research that accompanies these libraries is present in several sections in this paper – including [REFERENCE] and [REFERENCE].

Figure [Figure] on the right shows the results obtained when implementing the Group Lasso Model previously. As it can be observed, the results are notably less accurate than the ones obtained with the CVX library - the reason for this is due to the constraints needed for our implementation – these libraries proved to loose accuracy when implemented with a positivity constraint on the weights.

| Lambda | Zeros | Error |
|---|---|---|
| 0 | 0 | 0.0267 |
| 0.05 | 47 | 2.6464 |
| 0.1 | 47 | 5.2671 |
| 0.15 | 47 | 7.8959 |
| 0.2 | 47 | 10.5329 |
| 0.25 | 72 | 10.9667 |
| 0.3 | 47 | 15.8043 |
| 0.35 | 47 | 18.4401 |
| 0.4 | 47 | 21.0767 |
| 0.45 | 72 | 20.7983 |
| 0.5 | 72 | 23.6749 |
| 0.55 | 72 | 27.4671 |
| 0.6 | 72 | 28.8844 |
| 0.65 | 72 | 30.8535 |
| 0.7 | 47 | 36.8924 |
| 0.75 | 72 | 34.8185 |
| 0.8 | 72 | 38.3813 |
| 0.85 | 47 | 44.8011 |
| 0.9 | 72 | 45.7797 |
| 0.95 | 72 | 44.3252 |
| 1 | 72 | 48.2338 |

# 4 Tracking Error (All Data)

## 4.1 Feature Selection

| | Market | | | | Stochastic | | | | | Market | | | | Stochastic | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Zeros | Abs | Sq. | Ridge | CVaR | Abs | Sq. | Ridge | CVaR | | Abs | Squares | Ridge | CVaR | Abs | Sq. | Ridge | CVaR |
| **0** | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | **50** | 0.055 | 0.048 | 0.032 | 0.046 | 0.236 | 0.189 | 0.169 | 0.229 |
| **1** | 0.007 | 0.002 | 0.006 | 0.002 | 0.013 | 0.036 | 0.029 | 0.073 | **51** | 0.054 | 0.047 | 0.033 | 0.044 | 0.237 | 0.187 | 0.169 | 0.228 |
| **2** | 0.009 | 0.007 | 0.009 | 0.006 | 0.087 | 0.051 | 0.043 | 0.117 | **52** | 0.055 | 0.047 | 0.032 | 0.047 | 0.239 | 0.187 | 0.171 | 0.228 |
| **3** | 0.012 | 0.010 | 0.011 | 0.012 | 0.119 | 0.074 | 0.056 | 0.121 | **53** | 0.054 | 0.047 | 0.033 | 0.046 | 0.241 | 0.187 | 0.177 | 0.227 |
| **4** | 0.014 | 0.011 | 0.012 | 0.013 | 0.120 | 0.085 | 0.073 | 0.125 | **54** | 0.054 | 0.049 | 0.032 | 0.046 | 0.234 | 0.186 | 0.177 | 0.229 |
| **5** | 0.015 | 0.013 | 0.013 | 0.015 | 0.130 | 0.089 | 0.080 | 0.128 | **55** | 0.052 | 0.050 | 0.033 | 0.049 | 0.234 | 0.185 | 0.174 | 0.231 |
| **6** | 0.018 | 0.019 | 0.013 | 0.017 | 0.134 | 0.098 | 0.090 | 0.130 | **56** | 0.058 | 0.050 | 0.034 | 0.052 | 0.234 | 0.184 | 0.176 | 0.233 |
| **7** | 0.021 | 0.024 | 0.013 | 0.019 | 0.145 | 0.101 | 0.094 | 0.142 | **57** | 0.061 | 0.049 | 0.033 | 0.055 | 0.243 | 0.184 | 0.183 | 0.242 |
| **8** | 0.024 | 0.025 | 0.013 | 0.019 | 0.153 | 0.100 | 0.095 | 0.146 | **58** | 0.060 | 0.048 | 0.035 | 0.056 | 0.237 | 0.183 | 0.189 | 0.237 |
| **9** | 0.024 | 0.025 | 0.013 | 0.021 | 0.169 | 0.112 | 0.099 | 0.165 | **59** | 0.060 | 0.048 | 0.034 | 0.052 | 0.238 | 0.183 | 0.198 | 0.248 |
| **10** | 0.025 | 0.027 | 0.014 | 0.022 | 0.176 | 0.121 | 0.109 | 0.174 | **60** | 0.064 | 0.048 | 0.036 | 0.053 | 0.235 | 0.183 | 0.203 | 0.249 |
| **11** | 0.026 | 0.028 | 0.014 | 0.023 | 0.181 | 0.116 | 0.112 | 0.176 | **61** | 0.065 | 0.049 | 0.035 | 0.049 | 0.234 | 0.183 | 0.207 | 0.248 |
| **12** | 0.028 | 0.029 | 0.014 | 0.024 | 0.177 | 0.117 | 0.116 | 0.200 | **62** | 0.064 | 0.049 | 0.038 | 0.051 | 0.230 | 0.187 | 0.213 | 0.247 |
| **13** | 0.026 | 0.029 | 0.015 | 0.026 | 0.175 | 0.119 | 0.118 | 0.213 | **63** | 0.067 | 0.050 | 0.038 | 0.052 | 0.227 | 0.191 | 0.216 | 0.245 |
| **14** | 0.026 | 0.028 | 0.015 | 0.027 | 0.175 | 0.130 | 0.116 | 0.202 | **64** | 0.068 | 0.050 | 0.038 | 0.048 | 0.227 | 0.192 | 0.217 | 0.243 |
| **15** | 0.029 | 0.028 | 0.016 | 0.025 | 0.174 | 0.136 | 0.120 | 0.207 | **65** | 0.067 | 0.050 | 0.038 | 0.047 | 0.227 | 0.197 | 0.221 | 0.243 |
| **16** | 0.031 | 0.029 | 0.017 | 0.025 | 0.198 | 0.135 | 0.125 | 0.188 | **66** | 0.064 | 0.050 | 0.040 | 0.048 | 0.227 | 0.197 | 0.232 | 0.247 |
| **17** | 0.033 | 0.030 | 0.019 | 0.026 | 0.187 | 0.148 | 0.131 | 0.198 | **67** | 0.065 | 0.050 | 0.039 | 0.047 | 0.227 | 0.197 | 0.238 | 0.248 |
| **18** | 0.030 | 0.030 | 0.020 | 0.026 | 0.203 | 0.153 | 0.135 | 0.191 | **68** | 0.068 | 0.048 | 0.041 | 0.045 | 0.227 | 0.197 | 0.246 | 0.255 |
| **19** | 0.034 | 0.033 | 0.019 | 0.026 | 0.209 | 0.155 | 0.138 | 0.194 | **69** | 0.071 | 0.047 | 0.042 | 0.047 | 0.227 | 0.197 | 0.247 | 0.248 |
| **20** | 0.036 | 0.034 | 0.020 | 0.028 | 0.208 | 0.151 | 0.139 | 0.191 | **70** | 0.074 | 0.047 | 0.043 | 0.055 | 0.227 | 0.197 | 0.249 | 0.248 |
| **21** | 0.034 | 0.033 | 0.020 | 0.027 | 0.214 | 0.152 | 0.142 | 0.180 | **71** | 0.074 | 0.048 | 0.045 | 0.055 | 0.227 | 0.197 | 0.255 | 0.252 |
| **22** | 0.034 | 0.033 | 0.020 | 0.032 | 0.213 | 0.164 | 0.144 | 0.179 | **72** | 0.074 | 0.051 | 0.046 | 0.055 | 0.227 | 0.198 | 0.261 | 0.253 |
| **23** | 0.032 | 0.032 | 0.022 | 0.033 | 0.220 | 0.166 | 0.153 | 0.174 | **73** | 0.074 | 0.050 | 0.046 | 0.057 | 0.227 | 0.199 | 0.265 | 0.252 |
| **24** | 0.036 | 0.032 | 0.021 | 0.029 | 0.226 | 0.162 | 0.151 | 0.201 | **74** | 0.078 | 0.052 | 0.048 | 0.060 | 0.227 | 0.204 | 0.278 | 0.252 |
| **25** | 0.037 | 0.034 | 0.023 | 0.031 | 0.231 | 0.158 | 0.152 | 0.201 | **75** | 0.077 | 0.053 | 0.048 | 0.062 | 0.227 | 0.208 | 0.285 | 0.252 |
| **26** | 0.037 | 0.035 | 0.023 | 0.033 | 0.219 | 0.161 | 0.153 | 0.191 | **76** | 0.079 | 0.053 | 0.049 | 0.063 | 0.227 | 0.218 | 0.287 | 0.252 |
| **27** | 0.035 | 0.034 | 0.024 | 0.032 | 0.212 | 0.167 | 0.164 | 0.195 | **77** | 0.077 | 0.055 | 0.051 | 0.068 | 0.228 | 0.220 | 0.291 | 0.252 |
| **28** | 0.033 | 0.034 | 0.024 | 0.032 | 0.208 | 0.175 | 0.166 | 0.211 | **78** | 0.077 | 0.058 | 0.052 | 0.074 | 0.232 | 0.224 | 0.291 | 0.252 |
| **29** | 0.034 | 0.037 | 0.024 | 0.035 | 0.210 | 0.176 | 0.162 | 0.210 | **79** | 0.079 | 0.060 | 0.052 | 0.074 | 0.244 | 0.228 | 0.288 | 0.252 |
| **30** | 0.033 | 0.036 | 0.026 | 0.039 | 0.218 | 0.174 | 0.159 | 0.211 | **80** | 0.078 | 0.064 | 0.056 | 0.077 | 0.245 | 0.232 | 0.287 | 0.252 |
| **31** | 0.036 | 0.037 | 0.027 | 0.040 | 0.226 | 0.174 | 0.161 | 0.206 | **81** | 0.082 | 0.067 | 0.059 | 0.082 | 0.255 | 0.235 | 0.291 | 0.252 |
| **32** | 0.038 | 0.038 | 0.028 | 0.043 | 0.209 | 0.175 | 0.161 | 0.213 | **82** | 0.085 | 0.070 | 0.061 | 0.081 | 0.260 | 0.240 | 0.294 | 0.259 |
| **33** | 0.040 | 0.040 | 0.028 | 0.039 | 0.219 | 0.173 | 0.163 | 0.215 | **83** | 0.091 | 0.074 | 0.061 | 0.079 | 0.265 | 0.244 | 0.291 | 0.263 |
| **34** | 0.038 | 0.038 | 0.029 | 0.041 | 0.219 | 0.172 | 0.168 | 0.212 | **84** | 0.090 | 0.076 | 0.071 | 0.083 | 0.274 | 0.249 | 0.292 | 0.269 |
| **35** | 0.038 | 0.037 | 0.030 | 0.036 | 0.220 | 0.170 | 0.167 | 0.209 | **85** | 0.096 | 0.081 | 0.072 | 0.083 | 0.277 | 0.257 | 0.302 | 0.279 |
| **36** | 0.043 | 0.039 | 0.030 | 0.034 | 0.203 | 0.170 | 0.165 | 0.206 | **86** | 0.099 | 0.084 | 0.078 | 0.087 | 0.288 | 0.264 | 0.306 | 0.288 |
| **37** | 0.044 | 0.041 | 0.031 | 0.040 | 0.205 | 0.175 | 0.168 | 0.198 | **87** | 0.099 | 0.090 | 0.084 | 0.088 | 0.293 | 0.278 | 0.313 | 0.302 |
| **38** | 0.043 | 0.041 | 0.032 | 0.040 | 0.193 | 0.173 | 0.169 | 0.208 | **88** | 0.101 | 0.096 | 0.087 | 0.089 | 0.299 | 0.292 | 0.317 | 0.312 |
| **39** | 0.046 | 0.042 | 0.032 | 0.040 | 0.193 | 0.176 | 0.168 | 0.206 | **89** | 0.106 | 0.102 | 0.095 | 0.096 | 0.317 | 0.307 | 0.325 | 0.319 |
| **40** | 0.045 | 0.042 | 0.032 | 0.041 | 0.185 | 0.190 | 0.167 | 0.212 | **90** | 0.109 | 0.111 | 0.101 | 0.109 | 0.332 | 0.321 | 0.354 | 0.332 |
| **41** | 0.043 | 0.043 | 0.032 | 0.041 | 0.185 | 0.191 | 0.166 | 0.198 | **91** | 0.113 | 0.121 | 0.103 | 0.122 | 0.346 | 0.336 | 0.372 | 0.347 |
| **42** | 0.044 | 0.045 | 0.032 | 0.044 | 0.190 | 0.189 | 0.164 | 0.198 | **92** | 0.124 | 0.131 | 0.112 | 0.137 | 0.370 | 0.356 | 0.400 | 0.359 |
| **43** | 0.046 | 0.046 | 0.032 | 0.042 | 0.198 | 0.187 | 0.167 | 0.206 | **93** | 0.139 | 0.146 | 0.120 | 0.150 | 0.383 | 0.372 | 0.408 | 0.377 |
| **44** | 0.044 | 0.047 | 0.032 | 0.042 | 0.211 | 0.186 | 0.166 | 0.222 | **94** | 0.163 | 0.164 | 0.143 | 0.169 | 0.400 | 0.388 | 0.417 | 0.397 |
| **45** | 0.045 | 0.048 | 0.031 | 0.043 | 0.217 | 0.186 | 0.168 | 0.231 | **95** | 0.189 | 0.188 | 0.182 | 0.188 | 0.416 | 0.414 | 0.444 | 0.420 |
| **46** | 0.046 | 0.047 | 0.032 | 0.044 | 0.237 | 0.186 | 0.170 | 0.238 | **96** | 0.224 | 0.223 | 0.227 | 0.222 | 0.431 | 0.432 | 0.455 | 0.440 |
| **47** | 0.048 | 0.046 | 0.032 | 0.044 | 0.232 | 0.187 | 0.172 | 0.244 | **97** | 0.282 | 0.282 | 0.285 | 0.282 | 0.453 | 0.459 | 0.479 | 0.467 |
| **48** | 0.050 | 0.046 | 0.031 | 0.047 | 0.232 | 0.191 | 0.173 | 0.233 | **98** | 0.415 | 0.416 | 0.415 | 0.415 | 0.482 | 0.483 | 0.499 | 0.484 |

## 4.2 Lasso and Group Lasso

| | Market | | Stochastic | |
|---|---|---|---|---|
| **Lambda** | **Zeros** | **Error** | **Zeros** | **Error** |
| **0** | 0 | 0.000 | 0 | 0.000 |
| **1** | 31 | 0.056 | 22 | 0.071 |
| **1** | 42 | 0.075 | 39 | 0.091 |
| **2** | 54 | 0.088 | 46 | 0.105 |
| **2** | 59 | 0.096 | 51 | 0.120 |
| **3** | 63 | 0.103 | 57 | 0.135 |
| **3** | 65 | 0.108 | 60 | 0.148 |
| **4** | 68 | 0.114 | 64 | 0.159 |
| **4** | 69 | 0.119 | 66 | 0.167 |
| **5** | 71 | 0.124 | 67 | 0.177 |
| **5** | 74 | 0.130 | 69 | 0.188 |
| **6** | 76 | 0.133 | 70 | 0.199 |
| **6** | 77 | 0.137 | 73 | 0.210 |
| **7** | 78 | 0.141 | 73 | 0.221 |
| **7** | 78 | 0.144 | 73 | 0.233 |
| **8** | 79 | 0.148 | 74 | 0.244 |
| **8** | 80 | 0.151 | 74 | 0.256 |
| **9** | 80 | 0.155 | 75 | 0.269 |
| **9** | 80 | 0.160 | 75 | 0.281 |
| **10** | 81 | 0.166 | 75 | 0.293 |
| **10** | 81 | 0.172 | 76 | 0.306 |
| **11** | 83 | 0.176 | 76 | 0.318 |
| **11** | 83 | 0.181 | 77 | 0.331 |
| **12** | 84 | 0.186 | 78 | 0.344 |
| **12** | 84 | 0.191 | 79 | 0.357 |
| **13** | 84 | 0.195 | 80 | 0.369 |
| **13** | 85 | 0.200 | 82 | 0.381 |
| **14** | 86 | 0.203 | 82 | 0.393 |
| **14** | 86 | 0.206 | 83 | 0.405 |
| **15** | 86 | 0.210 | 84 | 0.417 |
| **15** | 86 | 0.213 | 85 | 0.428 |
| **16** | 86 | 0.217 | 88 | 0.437 |
| **16** | 86 | 0.220 | 88 | 0.445 |
| **17** | 87 | 0.224 | 88 | 0.452 |
| **17** | 89 | 0.228 | 88 | 0.460 |
| **18** | 89 | 0.232 | 90 | 0.467 |
| **18** | 89 | 0.235 | 90 | 0.474 |
| **19** | 89 | 0.239 | 90 | 0.481 |
| **19** | 89 | 0.243 | 90 | 0.488 |
| **20** | 89 | 0.247 | 90 | 0.495 |
| **20** | 90 | 0.251 | 90 | 0.502 |
| **21** | 90 | 0.254 | 91 | 0.509 |
| **21** | 91 | 0.255 | 91 | 0.516 |
| **22** | 91 | 0.257 | 91 | 0.522 |
| **22** | 91 | 0.258 | 91 | 0.529 |
| **23** | 91 | 0.260 | 91 | 0.537 |
| **23** | 91 | 0.262 | 91 | 0.544 |
| **24** | 91 | 0.264 | 91 | 0.551 |
| **24** | 91 | 0.266 | 91 | 0.558 |
| **25** | 91 | 0.268 | 91 | 0.565 |

| | Market (Sectors) | | Market (Spectral) | | Stochastic (Spectral) | |
|---|---|---|---|---|---|---|
| **Lambda-1** | **Zeros** | **Error** | **Zeros** | **Error** | **Zeros** | **Error** |
| **0** | 0 | 0.000 | 0 | 0.000 | 0 | 0.000 |
| **1** | 0 | 0.013 | 0 | 0.014 | 22 | 0.071 |
| **1** | 0 | 0.026 | 2 | 0.026 | 39 | 0.091 |
| **2** | 0 | 0.039 | 4 | 0.036 | 46 | 0.105 |
| **2** | 21 | 0.050 | 4 | 0.046 | 51 | 0.120 |
| **3** | 21 | 0.058 | 4 | 0.056 | 57 | 0.135 |
| **3** | 21 | 0.069 | 21 | 0.062 | 60 | 0.148 |
| **4** | 21 | 0.079 | 21 | 0.067 | 64 | 0.159 |
| **4** | 41 | 0.087 | 33 | 0.070 | 66 | 0.167 |
| **5** | 41 | 0.092 | 33 | 0.072 | 67 | 0.177 |
| **5** | 56 | 0.096 | 33 | 0.075 | 69 | 0.188 |
| **6** | 56 | 0.098 | 33 | 0.077 | 70 | 0.199 |
| **6** | 56 | 0.101 | 33 | 0.080 | 73 | 0.210 |
| **7** | 65 | 0.104 | 33 | 0.082 | 73 | 0.221 |
| **7** | 65 | 0.106 | 33 | 0.085 | 73 | 0.233 |
| **8** | 65 | 0.108 | 33 | 0.088 | 74 | 0.244 |
| **8** | 65 | 0.111 | 33 | 0.091 | 74 | 0.256 |
| **9** | 65 | 0.113 | 33 | 0.095 | 75 | 0.269 |
| **9** | 65 | 0.116 | 33 | 0.098 | 75 | 0.281 |
| **10** | 65 | 0.119 | 33 | 0.101 | 75 | 0.293 |
| **10** | 65 | 0.122 | 33 | 0.105 | 76 | 0.306 |
| **11** | 65 | 0.126 | 33 | 0.108 | 76 | 0.318 |
| **11** | 65 | 0.129 | 33 | 0.111 | 77 | 0.331 |
| **12** | 65 | 0.132 | 33 | 0.115 | 78 | 0.344 |
| **12** | 65 | 0.135 | 33 | 0.119 | 79 | 0.357 |
| **13** | 65 | 0.139 | 33 | 0.122 | 80 | 0.369 |
| **13** | 65 | 0.142 | 33 | 0.126 | 82 | 0.381 |
| **14** | 65 | 0.146 | 33 | 0.130 | 82 | 0.393 |
| **14** | 65 | 0.150 | 33 | 0.133 | 83 | 0.405 |
| **15** | 65 | 0.154 | 33 | 0.137 | 84 | 0.417 |
| **15** | 84 | 0.157 | 33 | 0.141 | 85 | 0.428 |
| **16** | 84 | 0.158 | 33 | 0.145 | 88 | 0.437 |
| **16** | 84 | 0.159 | 51 | 0.148 | 88 | 0.445 |
| **17** | 84 | 0.160 | 51 | 0.151 | 88 | 0.452 |
| **17** | 84 | 0.161 | 51 | 0.155 | 88 | 0.460 |
| **18** | 84 | 0.162 | 51 | 0.158 | 90 | 0.467 |
| **18** | 84 | 0.163 | 51 | 0.162 | 90 | 0.474 |
| **19** | 84 | 0.164 | 51 | 0.165 | 90 | 0.481 |
| **19** | 84 | 0.165 | 51 | 0.169 | 90 | 0.488 |
| **20** | 84 | 0.166 | 51 | 0.172 | 90 | 0.495 |
| **20** | 84 | 0.167 | 51 | 0.176 | 90 | 0.502 |
| **21** | 84 | 0.169 | 51 | 0.179 | 91 | 0.509 |
| **21** | 84 | 0.170 | 51 | 0.183 | 91 | 0.516 |
| **22** | 84 | 0.172 | 51 | 0.186 | 91 | 0.522 |
| **22** | 84 | 0.173 | 51 | 0.190 | 91 | 0.529 |
| **23** | 84 | 0.175 | 51 | 0.194 | 91 | 0.537 |
| **23** | 84 | 0.177 | 51 | 0.197 | 91 | 0.544 |
| **24** | 84 | 0.178 | 51 | 0.201 | 91 | 0.551 |
| **24** | 84 | 0.180 | 51 | 0.205 | 91 | 0.558 |
| **25** | 84 | 0.182 | 51 | 0.208 | 91 | 0.565 |

## 4.3 Sparse Group Lasso by Dataset

**5000**

| | | Market (Sectors) | | Market (Spectral) | | Stochastic (Spectral) | |
|---|---|---|---|---|---|---|---|
| | **Lambda-2** | **Lambda-1** | **Error** | **Zeros** | **Error** | **Zeros** | **Error** |
| **0** | **0** | 0 | 0.000 | 0 | 0.000 | 0 | 0.000 |
| **0** | **1** | 9 | 0.035 | 8 | 0.034 | 7 | 0.050 |
| **0** | **1** | 18 | 0.049 | 18 | 0.050 | 18 | 0.069 |
| **0** | **2** | 25 | 0.058 | 23 | 0.058 | 26 | 0.080 |
| **0** | **2** | 28 | 0.066 | 28 | 0.066 | 31 | 0.088 |
| **0** | **3** | 33 | 0.074 | 33 | 0.074 | 38 | 0.096 |
| **0** | **3** | 35 | 0.080 | 36 | 0.080 | 43 | 0.102 |
| **0** | **4** | 39 | 0.084 | 37 | 0.085 | 45 | 0.107 |
| **0** | **4** | 43 | 0.089 | 42 | 0.089 | 46 | 0.113 |
| **0** | **5** | 44 | 0.093 | 46 | 0.094 | 46 | 0.119 |
| **0** | **5** | 47 | 0.097 | 50 | 0.098 | 48 | 0.126 |
| **0** | **6** | 47 | 0.101 | 50 | 0.102 | 50 | 0.132 |
| **0** | **6** | 50 | 0.106 | 50 | 0.106 | 51 | 0.139 |
| **0** | **7** | 50 | 0.110 | 51 | 0.110 | 52 | 0.146 |
| **0** | **7** | 53 | 0.113 | 54 | 0.114 | 55 | 0.152 |
| **0** | **8** | 56 | 0.116 | 56 | 0.118 | 58 | 0.158 |
| **0** | **8** | 61 | 0.119 | 59 | 0.121 | 58 | 0.165 |
| **0** | **9** | 61 | 0.121 | 60 | 0.125 | 58 | 0.172 |
| **0** | **9** | 62 | 0.122 | 61 | 0.128 | 60 | 0.178 |
| **0** | **10** | 62 | 0.124 | 62 | 0.131 | 61 | 0.185 |
| **0** | **10** | 62 | 0.125 | 63 | 0.134 | 61 | 0.191 |
| **0** | **11** | 64 | 0.127 | 64 | 0.136 | 63 | 0.197 |
| **0** | **11** | 64 | 0.128 | 64 | 0.138 | 63 | 0.204 |
| **0** | **12** | 64 | 0.130 | 68 | 0.140 | 63 | 0.210 |
| **0** | **12** | 65 | 0.131 | 69 | 0.142 | 63 | 0.216 |
| **0** | **13** | 65 | 0.133 | 69 | 0.144 | 63 | 0.222 |
| **0** | **13** | 68 | 0.134 | 69 | 0.146 | 63 | 0.228 |
| **0** | **14** | 68 | 0.135 | 69 | 0.149 | 63 | 0.234 |
| **0** | **14** | 70 | 0.137 | 69 | 0.151 | 64 | 0.240 |
| **0** | **15** | 71 | 0.138 | 74 | 0.152 | 64 | 0.246 |
| **0** | **15** | 72 | 0.139 | 74 | 0.154 | 64 | 0.253 |
| **0** | **16** | 72 | 0.141 | 74 | 0.155 | 64 | 0.259 |
| **0** | **16** | 72 | 0.142 | 74 | 0.156 | 64 | 0.265 |
| **0** | **17** | 72 | 0.143 | 74 | 0.157 | 64 | 0.270 |
| **0** | **17** | 72 | 0.145 | 74 | 0.158 | 65 | 0.276 |
| **0** | **18** | 74 | 0.146 | 75 | 0.160 | 66 | 0.282 |
| **0** | **18** | 74 | 0.147 | 76 | 0.161 | 68 | 0.288 |
| **0** | **19** | 74 | 0.148 | 77 | 0.162 | 69 | 0.294 |
| **0** | **19** | 74 | 0.149 | 77 | 0.163 | 69 | 0.299 |
| **0** | **20** | 78 | 0.150 | 77 | 0.165 | 70 | 0.305 |
| **0** | **20** | 79 | 0.151 | 77 | 0.166 | 70 | 0.310 |
| **0** | **21** | 79 | 0.152 | 77 | 0.167 | 70 | 0.315 |
| **0** | **21** | 80 | 0.152 | 77 | 0.169 | 70 | 0.320 |
| **0** | **22** | 80 | 0.153 | 77 | 0.170 | 70 | 0.325 |
| **0** | **22** | 80 | 0.153 | 77 | 0.171 | 70 | 0.331 |
| **0** | **23** | 80 | 0.154 | 78 | 0.173 | 70 | 0.336 |
| **0** | **23** | 81 | 0.155 | 80 | 0.174 | 70 | 0.341 |
| **0** | **24** | 81 | 0.155 | 82 | 0.175 | 70 | 0.347 |
| **0** | **24** | 81 | 0.156 | 82 | 0.176 | 70 | 0.352 |

**10000**

| | | Market (Sectors) | | Market (Spectral) | | Stochastic (Spectral) | |
|---|---|---|---|---|---|---|---|
| **Lambda-1** | **Lambda-2** | **Zeros** | **Error** | **Zeros** | **Error** | **Zeros** | **Error** |
| **0** | **0** | 0 | 0.000 | 0 | 0.000 | 0 | 0.000 |
| **0** | **1** | 13 | 0.039 | 12 | 0.039 | 9 | 0.053 |
| **0** | **1** | 26 | 0.055 | 27 | 0.054 | 22 | 0.072 |
| **0** | **2** | 31 | 0.064 | 31 | 0.063 | 30 | 0.083 |
| **0** | **2** | 36 | 0.073 | 37 | 0.073 | 34 | 0.092 |
| **0** | **3** | 42 | 0.082 | 44 | 0.082 | 45 | 0.100 |
| **0** | **3** | 47 | 0.087 | 47 | 0.087 | 47 | 0.106 |
| **0** | **4** | 49 | 0.091 | 52 | 0.091 | 48 | 0.113 |
| **0** | **4** | 54 | 0.095 | 54 | 0.095 | 50 | 0.120 |
| **0** | **5** | 55 | 0.100 | 56 | 0.100 | 51 | 0.128 |
| **0** | **5** | 60 | 0.104 | 60 | 0.104 | 53 | 0.135 |
| **0** | **6** | 60 | 0.107 | 60 | 0.107 | 56 | 0.142 |
| **0** | **6** | 60 | 0.110 | 60 | 0.110 | 59 | 0.149 |
| **0** | **7** | 61 | 0.113 | 63 | 0.114 | 59 | 0.155 |
| **0** | **7** | 62 | 0.116 | 63 | 0.117 | 62 | 0.162 |
| **0** | **8** | 63 | 0.119 | 67 | 0.119 | 63 | 0.169 |
| **0** | **8** | 67 | 0.122 | 67 | 0.122 | 63 | 0.175 |
| **0** | **9** | 67 | 0.124 | 67 | 0.125 | 63 | 0.182 |
| **0** | **9** | 68 | 0.126 | 67 | 0.127 | 64 | 0.189 |
| **0** | **10** | 69 | 0.129 | 69 | 0.130 | 64 | 0.196 |
| **0** | **10** | 69 | 0.131 | 69 | 0.133 | 64 | 0.203 |
| **0** | **11** | 69 | 0.133 | 70 | 0.136 | 66 | 0.209 |
| **0** | **11** | 72 | 0.135 | 71 | 0.138 | 67 | 0.216 |
| **0** | **12** | 72 | 0.137 | 72 | 0.141 | 67 | 0.222 |
| **0** | **12** | 73 | 0.138 | 73 | 0.143 | 68 | 0.229 |
| **0** | **13** | 73 | 0.140 | 74 | 0.145 | 68 | 0.235 |
| **0** | **13** | 74 | 0.141 | 74 | 0.146 | 70 | 0.242 |
| **0** | **14** | 75 | 0.142 | 74 | 0.148 | 70 | 0.248 |
| **0** | **14** | 76 | 0.142 | 74 | 0.150 | 71 | 0.255 |
| **0** | **15** | 76 | 0.143 | 74 | 0.151 | 71 | 0.261 |
| **0** | **15** | 77 | 0.144 | 75 | 0.154 | 71 | 0.267 |
| **0** | **16** | 77 | 0.145 | 75 | 0.156 | 72 | 0.273 |
| **0** | **16** | 78 | 0.145 | 76 | 0.158 | 72 | 0.279 |
| **0** | **17** | 78 | 0.146 | 77 | 0.160 | 72 | 0.285 |
| **0** | **17** | 78 | 0.147 | 77 | 0.163 | 72 | 0.291 |
| **0** | **18** | 78 | 0.148 | 77 | 0.165 | 72 | 0.297 |
| **0** | **18** | 78 | 0.149 | 77 | 0.168 | 73 | 0.303 |
| **0** | **19** | 79 | 0.150 | 77 | 0.170 | 74 | 0.309 |
| **0** | **19** | 79 | 0.152 | 80 | 0.172 | 75 | 0.315 |
| **0** | **20** | 79 | 0.153 | 80 | 0.174 | 75 | 0.321 |
| **0** | **20** | 79 | 0.155 | 80 | 0.176 | 75 | 0.327 |
| **0** | **21** | 79 | 0.157 | 80 | 0.177 | 75 | 0.333 |
| **0** | **21** | 79 | 0.158 | 80 | 0.179 | 75 | 0.339 |
| **0** | **22** | 79 | 0.160 | 81 | 0.182 | 75 | 0.345 |
| **0** | **22** | 79 | 0.163 | 81 | 0.184 | 75 | 0.351 |
| **0** | **23** | 82 | 0.165 | 81 | 0.186 | 76 | 0.358 |
| **0** | **23** | 82 | 0.166 | 81 | 0.188 | 76 | 0.364 |
| **0** | **24** | 82 | 0.168 | 81 | 0.190 | 76 | 0.371 |
| **0** | **24** | 82 | 0.171 | 81 | 0.192 | 76 | 0.377 |

## 4.4 Sparse Group Lasso - by Dataset (Cont...)

| Market (Sectors) | 500 | | 1000 | | 5000 | | 10000 |
|---|---|---|---|---|---|---|---|
| Zeros | Error | Zeros | Error | Zeros | Error | Zeros | Error |
| 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 0 | 0.015 | 0 | 0.020 | 9 | 0.035 | 13 | 0.039 |
| 0 | 0.023 | 0 | 0.029 | 18 | 0.049 | 26 | 0.055 |
| 0 | 0.029 | 1 | 0.037 | 25 | 0.058 | 31 | 0.064 |
| 0 | 0.036 | 1 | 0.044 | 28 | 0.066 | 36 | 0.073 |
| 0 | 0.043 | 1 | 0.050 | 33 | 0.074 | 42 | 0.082 |
| 0 | 0.049 | 3 | 0.056 | 35 | 0.080 | 47 | 0.087 |
| 0 | 0.056 | 3 | 0.062 | 39 | 0.084 | 49 | 0.091 |
| 21 | 0.061 | 5 | 0.068 | 43 | 0.089 | 54 | 0.095 |
| 22 | 0.066 | 22 | 0.073 | 44 | 0.093 | 55 | 0.100 |
| 22 | 0.071 | 23 | 0.077 | 47 | 0.097 | 60 | 0.104 |
| 22 | 0.075 | 28 | 0.082 | 47 | 0.101 | 60 | 0.107 |
| 22 | 0.080 | 28 | 0.086 | 50 | 0.106 | 60 | 0.110 |
| 28 | 0.085 | 28 | 0.090 | 50 | 0.110 | 61 | 0.113 |
| 42 | 0.087 | 31 | 0.094 | 53 | 0.113 | 62 | 0.116 |
| 42 | 0.089 | 48 | 0.097 | 56 | 0.116 | 63 | 0.119 |
| 42 | 0.091 | 48 | 0.099 | 61 | 0.119 | 67 | 0.122 |
| 42 | 0.093 | 48 | 0.100 | 61 | 0.121 | 67 | 0.124 |
| 42 | 0.095 | 48 | 0.102 | 62 | 0.122 | 68 | 0.126 |
| 42 | 0.097 | 48 | 0.103 | 62 | 0.124 | 69 | 0.129 |
| 42 | 0.099 | 48 | 0.105 | 62 | 0.125 | 69 | 0.131 |
| 42 | 0.102 | 48 | 0.107 | 64 | 0.127 | 69 | 0.133 |
| 44 | 0.104 | 48 | 0.108 | 64 | 0.128 | 72 | 0.135 |
| 56 | 0.106 | 48 | 0.109 | 64 | 0.130 | 72 | 0.137 |
| 56 | 0.107 | 48 | 0.111 | 65 | 0.131 | 73 | 0.138 |
| 56 | 0.108 | 48 | 0.112 | 65 | 0.133 | 73 | 0.140 |
| 56 | 0.109 | 48 | 0.114 | 68 | 0.134 | 74 | 0.141 |
| 56 | 0.110 | 49 | 0.115 | 68 | 0.135 | 75 | 0.142 |
| 56 | 0.111 | 49 | 0.117 | 70 | 0.137 | 76 | 0.142 |
| 56 | 0.113 | 49 | 0.118 | 71 | 0.138 | 76 | 0.143 |
| 56 | 0.114 | 49 | 0.120 | 72 | 0.139 | 77 | 0.144 |
| 56 | 0.115 | 49 | 0.121 | 72 | 0.141 | 77 | 0.145 |
| 56 | 0.117 | 49 | 0.122 | 72 | 0.142 | 78 | 0.145 |
| 56 | 0.118 | 49 | 0.124 | 72 | 0.143 | 78 | 0.146 |
| 56 | 0.119 | 50 | 0.125 | 72 | 0.145 | 78 | 0.147 |
| 56 | 0.120 | 50 | 0.127 | 74 | 0.146 | 78 | 0.148 |
| 56 | 0.121 | 50 | 0.128 | 74 | 0.147 | 78 | 0.149 |
| 56 | 0.123 | 50 | 0.129 | 74 | 0.148 | 79 | 0.150 |
| 56 | 0.124 | 60 | 0.131 | 74 | 0.149 | 79 | 0.152 |
| 56 | 0.125 | 62 | 0.131 | 78 | 0.150 | 79 | 0.153 |
| 65 | 0.126 | 62 | 0.132 | 79 | 0.151 | 79 | 0.155 |
| 65 | 0.127 | 62 | 0.133 | 79 | 0.152 | 79 | 0.157 |
| 65 | 0.127 | 62 | 0.133 | 80 | 0.152 | 79 | 0.158 |
| 65 | 0.128 | 62 | 0.134 | 80 | 0.153 | 79 | 0.160 |
| 65 | 0.128 | 62 | 0.135 | 80 | 0.153 | 79 | 0.163 |
| 65 | 0.129 | 62 | 0.135 | 80 | 0.154 | 82 | 0.165 |
| 65 | 0.129 | 62 | 0.136 | 81 | 0.155 | 82 | 0.166 |
| 65 | 0.129 | 62 | 0.136 | 81 | 0.155 | 82 | 0.168 |

| Market (Spectral) | 500 | | 1000 | | 5000 | | 10000 |
|---|---|---|---|---|---|---|---|
| Zeros | Error | Zeros | Error | Zeros | Error | Zeros | Error |
| 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 8 | 0.034 | 0 | 0.020 | 8 | 0.034 | 12 | 0.039 |
| 18 | 0.050 | 0 | 0.030 | 18 | 0.050 | 27 | 0.054 |
| 23 | 0.058 | 3 | 0.039 | 23 | 0.058 | 31 | 0.063 |
| 28 | 0.066 | 3 | 0.046 | 28 | 0.066 | 37 | 0.073 |
| 33 | 0.074 | 3 | 0.052 | 33 | 0.074 | 44 | 0.082 |
| 36 | 0.080 | 8 | 0.057 | 36 | 0.080 | 47 | 0.087 |
| 37 | 0.085 | 8 | 0.062 | 37 | 0.085 | 52 | 0.091 |
| 42 | 0.089 | 10 | 0.067 | 42 | 0.089 | 54 | 0.095 |
| 46 | 0.094 | 10 | 0.072 | 46 | 0.094 | 56 | 0.100 |
| 50 | 0.098 | 11 | 0.076 | 50 | 0.098 | 60 | 0.104 |
| 50 | 0.102 | 12 | 0.081 | 50 | 0.102 | 60 | 0.107 |
| 50 | 0.106 | 12 | 0.086 | 50 | 0.106 | 60 | 0.110 |
| 51 | 0.110 | 12 | 0.091 | 51 | 0.110 | 63 | 0.114 |
| 54 | 0.114 | 13 | 0.096 | 54 | 0.114 | 63 | 0.117 |
| 56 | 0.118 | 13 | 0.101 | 56 | 0.118 | 67 | 0.119 |
| 59 | 0.121 | 19 | 0.105 | 59 | 0.121 | 67 | 0.122 |
| 60 | 0.125 | 30 | 0.109 | 60 | 0.125 | 67 | 0.125 |
| 61 | 0.128 | 38 | 0.112 | 61 | 0.128 | 67 | 0.127 |
| 62 | 0.131 | 38 | 0.114 | 62 | 0.131 | 69 | 0.130 |
| 63 | 0.134 | 38 | 0.117 | 63 | 0.134 | 69 | 0.133 |
| 64 | 0.136 | 38 | 0.120 | 64 | 0.136 | 70 | 0.136 |
| 64 | 0.138 | 39 | 0.122 | 64 | 0.138 | 71 | 0.138 |
| 68 | 0.140 | 39 | 0.125 | 68 | 0.140 | 72 | 0.141 |
| 69 | 0.142 | 40 | 0.127 | 69 | 0.142 | 73 | 0.143 |
| 69 | 0.144 | 40 | 0.130 | 69 | 0.144 | 74 | 0.145 |
| 69 | 0.146 | 40 | 0.132 | 69 | 0.146 | 74 | 0.146 |
| 69 | 0.149 | 40 | 0.135 | 69 | 0.149 | 74 | 0.148 |
| 69 | 0.151 | 40 | 0.137 | 69 | 0.151 | 74 | 0.150 |
| 74 | 0.152 | 40 | 0.139 | 74 | 0.152 | 74 | 0.151 |
| 74 | 0.154 | 41 | 0.142 | 74 | 0.154 | 75 | 0.154 |
| 74 | 0.155 | 41 | 0.144 | 74 | 0.155 | 75 | 0.156 |
| 74 | 0.156 | 41 | 0.146 | 74 | 0.156 | 76 | 0.158 |
| 74 | 0.157 | 41 | 0.149 | 74 | 0.157 | 77 | 0.160 |
| 74 | 0.158 | 42 | 0.151 | 74 | 0.158 | 77 | 0.163 |
| 75 | 0.160 | 43 | 0.153 | 75 | 0.160 | 77 | 0.165 |
| 76 | 0.161 | 43 | 0.155 | 76 | 0.161 | 77 | 0.168 |
| 77 | 0.162 | 43 | 0.158 | 77 | 0.162 | 77 | 0.170 |
| 77 | 0.163 | 43 | 0.160 | 77 | 0.163 | 80 | 0.172 |
| 77 | 0.165 | 43 | 0.163 | 77 | 0.165 | 80 | 0.174 |
| 77 | 0.166 | 43 | 0.165 | 77 | 0.166 | 80 | 0.176 |
| 77 | 0.167 | 44 | 0.168 | 77 | 0.167 | 80 | 0.177 |
| 77 | 0.169 | 57 | 0.170 | 77 | 0.169 | 80 | 0.179 |
| 77 | 0.170 | 57 | 0.172 | 77 | 0.170 | 81 | 0.182 |
| 77 | 0.171 | 57 | 0.174 | 77 | 0.171 | 81 | 0.184 |
| 78 | 0.173 | 58 | 0.175 | 78 | 0.173 | 81 | 0.186 |
| 80 | 0.174 | 58 | 0.177 | 80 | 0.174 | 81 | 0.188 |
| 82 | 0.175 | 58 | 0.178 | 82 | 0.175 | 81 | 0.190 |

## 4.5 Sparse Group Lasso - by Dataset (Cont...)

| Stochastic (Spectral) | 500 | | 1000 | | 5000 | | 10000 |
|---|---|---|---|---|---|---|---|
| Zeros | Error | Zeros | Error | Zeros | Error | Zeros | Error |
| 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 0 | 0.027 | 0 | 0.036 | 7 | 0.050 | 9 | 0.053 |
| 0 | 0.039 | 3 | 0.052 | 18 | 0.069 | 22 | 0.072 |
| 2 | 0.047 | 7 | 0.062 | 26 | 0.080 | 30 | 0.083 |
| 2 | 0.054 | 7 | 0.070 | 31 | 0.088 | 34 | 0.092 |
| 2 | 0.061 | 16 | 0.077 | 38 | 0.096 | 45 | 0.100 |
| 10 | 0.065 | 17 | 0.081 | 43 | 0.102 | 47 | 0.106 |
| 11 | 0.069 | 18 | 0.086 | 45 | 0.107 | 48 | 0.113 |
| 11 | 0.072 | 18 | 0.090 | 46 | 0.113 | 50 | 0.120 |
| 11 | 0.075 | 18 | 0.095 | 46 | 0.119 | 51 | 0.128 |
| 11 | 0.078 | 19 | 0.099 | 48 | 0.126 | 53 | 0.135 |
| 11 | 0.081 | 19 | 0.103 | 50 | 0.132 | 56 | 0.142 |
| 11 | 0.084 | 19 | 0.108 | 51 | 0.139 | 59 | 0.149 |
| 11 | 0.087 | 19 | 0.112 | 52 | 0.146 | 59 | 0.155 |
| 11 | 0.090 | 19 | 0.116 | 55 | 0.152 | 62 | 0.162 |
| 11 | 0.093 | 22 | 0.121 | 58 | 0.158 | 63 | 0.169 |
| 11 | 0.096 | 24 | 0.125 | 58 | 0.165 | 63 | 0.175 |
| 11 | 0.099 | 24 | 0.129 | 58 | 0.172 | 63 | 0.182 |
| 11 | 0.102 | 24 | 0.132 | 60 | 0.178 | 64 | 0.189 |
| 11 | 0.105 | 25 | 0.136 | 61 | 0.185 | 64 | 0.196 |
| 12 | 0.108 | 25 | 0.140 | 61 | 0.191 | 64 | 0.203 |
| 12 | 0.111 | 25 | 0.144 | 63 | 0.197 | 66 | 0.209 |
| 12 | 0.115 | 25 | 0.148 | 63 | 0.204 | 67 | 0.216 |
| 12 | 0.118 | 25 | 0.152 | 63 | 0.210 | 67 | 0.222 |
| 12 | 0.121 | 25 | 0.156 | 63 | 0.216 | 68 | 0.229 |
| 12 | 0.124 | 26 | 0.159 | 63 | 0.222 | 68 | 0.235 |
| 13 | 0.128 | 27 | 0.163 | 63 | 0.228 | 70 | 0.242 |
| 13 | 0.131 | 27 | 0.167 | 63 | 0.234 | 70 | 0.248 |
| 13 | 0.134 | 27 | 0.171 | 64 | 0.240 | 71 | 0.255 |
| 13 | 0.137 | 27 | 0.175 | 64 | 0.246 | 71 | 0.261 |
| 13 | 0.141 | 27 | 0.178 | 64 | 0.253 | 71 | 0.267 |
| 13 | 0.144 | 28 | 0.182 | 64 | 0.259 | 72 | 0.273 |
| 13 | 0.147 | 28 | 0.186 | 64 | 0.265 | 72 | 0.279 |
| 13 | 0.151 | 28 | 0.189 | 64 | 0.270 | 72 | 0.285 |
| 13 | 0.154 | 28 | 0.193 | 65 | 0.276 | 72 | 0.291 |
| 13 | 0.157 | 29 | 0.197 | 66 | 0.282 | 72 | 0.297 |
| 40 | 0.161 | 29 | 0.201 | 68 | 0.288 | 73 | 0.303 |
| 40 | 0.163 | 29 | 0.204 | 69 | 0.294 | 74 | 0.309 |
| 40 | 0.165 | 31 | 0.208 | 69 | 0.299 | 75 | 0.315 |
| 42 | 0.167 | 51 | 0.212 | 70 | 0.305 | 75 | 0.321 |
| 42 | 0.169 | 51 | 0.214 | 70 | 0.310 | 75 | 0.327 |
| 42 | 0.171 | 51 | 0.216 | 70 | 0.315 | 75 | 0.333 |
| 42 | 0.173 | 51 | 0.218 | 70 | 0.320 | 75 | 0.339 |
| 42 | 0.175 | 51 | 0.221 | 70 | 0.325 | 75 | 0.345 |
| 42 | 0.177 | 51 | 0.223 | 70 | 0.331 | 75 | 0.351 |
| 42 | 0.179 | 51 | 0.225 | 70 | 0.336 | 76 | 0.358 |
| 42 | 0.181 | 51 | 0.228 | 70 | 0.341 | 76 | 0.364 |
| 42 | 0.183 | 51 | 0.230 | 70 | 0.347 | 76 | 0.371 |
| 42 | 0.184 | 52 | 0.233 | 70 | 0.352 | 76 | 0.377 |
| 42 | 0.186 | 52 | 0.235 | 71 | 0.358 | 76 | 0.384 |