CENG 384 - Signals and Systems for Computer Engineers
Spring 2024
Homework 4

Sekmen, Aytaç
e2575983@ceng.metu.edu.tr

Yılmaz, Başar
e2644409@ceng.metu.edu.tr

June 15, 2024

---

1. (a) From left to right, if we assume the first integrator gives output $z(t)$ then:

$$\frac{dz(t)}{dt} = x(t) - 6y(t) \tag{1}$$

For the second integrator, the output is $y(t)$ then we can write:

$$\frac{dy(t)}{dt} = 4x(t) - 5y(t) + z(t) \tag{2}$$

Differentiate the equation 2 with respect to t:

$$\frac{d^2y(t)}{dt^2} = 4\frac{dx(t)}{dt} - 5\frac{dy(t)}{dt} + \frac{dz(t)}{dt} \tag{3}$$

Using the equation 1 in equation 3:

$$\frac{d^2y(t)}{dt^2} = 4\frac{dx(t)}{dt} - 5\frac{dy(t)}{dt} + x(t) - 6y(t) \tag{4}$$

Then the final differential equation is:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 4\frac{dx(t)}{dt} + x(t) \tag{5}$$

(b) In order to find the frequency response of the system, we can find the impulse response of the system then take the Fourier Transform of the impulse response. But in our case it is not easy to find the impulse response of the system. So we can use the convolution property of the Fourier Transform. The frequency response can be represented as:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{4j\omega + 1}{(j\omega)^2 + 5j\omega + 6} \tag{1}$$

(c) We need to take the inverse fourier transform of the frequency response to find the impulse response of the system. The frequency response can be written as:

$$H(j\omega) = \frac{4j\omega + 1}{(j\omega)^2 + 5j\omega + 6} = \frac{4j\omega + 1}{(j\omega + 2)(j\omega + 3)} = \frac{A}{j\omega + 2} + \frac{B}{j\omega + 3} \tag{1}$$

We can find the values of A and B by multiplying both sides with respectively -7 and 11. Then we can take the inverse Fourier Transform of the frequency response using look-up table to find the impulse response of the system.

$$h(t) = -7e^{-2t}u(t) + 11e^{-3t}u(t) \tag{2}$$

(d) We can find the output of the system by convolving the input signal with the impulse response of the system. The input signal can be represented as:

$$y(t) = x(t) * h(t) \tag{1}$$

We can do this by using the convolution property of the Fourier Transform. The output signal can be represented as:

$$Y(j\omega) = X(j\omega)H(j\omega) \tag{2}$$

$$X(j\omega) = \frac{1}{4}\frac{1}{j\omega + \frac{1}{4}} \tag{3}$$

$$Y(jw) = \frac{1}{4}\frac{1}{j\omega + \frac{1}{4}}\left(\frac{-7}{j\omega + 2} + \frac{11}{j\omega + 3}\right) = \frac{1}{4}\left(\frac{-7}{(j\omega + 2)(jw + \frac{1}{4})} + \frac{11}{(j\omega + 3)(jw + \frac{1}{4})}\right) \tag{4}$$

Using the similar transformation as the c)

$$Y(jw) = \frac{1}{4}\left(\frac{A}{j\omega + \frac{1}{4}} + \frac{B}{j\omega + 2} + \frac{C}{j\omega + \frac{1}{4}} + \frac{D}{j\omega + 3}\right) \tag{5}$$

We get the values of the A, B, C and D by multiplying both sides with respectively -4, 4, 4 and -4. Then the fina version:

$$Y(jw) = \frac{1}{4}\left(\frac{4}{j\omega + 2} - \frac{4}{j\omega + 3}\right) = \frac{1}{j\omega + 2} - \frac{1}{j\omega + 3} \tag{6}$$

Then taking the inverse Fourier Transform of the $Y(j\omega)$, we can find the output signal $y(t)$.

$$y(t) = e^{-2t}u(t) - e^{-3t}u(t) \tag{7}$$

2. (a) We know that:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 4}{(j\omega)^2 + 5j\omega + 6} \quad (1)$$

$$Y(j\omega)(j\omega)^2 + 5j\omega Y(j\omega) + 6Y(j\omega) = X(j\omega)(j\omega) + 4X(j\omega) \quad (2)$$

Using the inverse Fourier Transform, we can find the differential equation representing the system:

$$y''(t) + 5y'(t) + 6y(t) = x'(t) + 4x(t) \quad (3)$$

(b) In order to find the impulse response we can use the convolution property of the Fourier Transform. The frequency response can be represented as:

$$H(j\omega) = \frac{j\omega + 4}{(j\omega)^2 + 5j\omega + 6} = \frac{j\omega + 4}{(j\omega + 2)(j\omega + 3)} = \frac{A}{j\omega + 3} + \frac{B}{j\omega + 2} \quad (1)$$

where $A = -1$ and $B = 2$. Then the impulse response can be found by taking the inverse Fourier Transform of the frequency response.

$$h(t) = 2e^{-2t}u(t) - e^{-3t}u(t) \quad (2)$$

(c)

$$Y(j\omega) = X(j\omega)H(j\omega) \quad (1)$$

$X(j\omega)$ can be represented as Fourier Transform of the given $x(t)$:

$$X(j\omega) = \frac{1}{j\omega + 4} - \frac{1}{(j\omega + 4)^2} = \frac{j\omega + 3}{(j\omega + 4)^2} \quad (2)$$

Using the equation 2 in equation 1:

$$Y(j\omega) = \frac{j\omega + 3}{(j\omega + 4)^2}\left(\frac{j\omega + 4}{(j\omega + 2)(j\omega + 3)}\right) = \frac{1}{(j\omega + 4)(j\omega + 2)} \quad (3)$$

Using partial fractions:

$$Y(j\omega) = \frac{A}{j\omega + 4} + \frac{B}{j\omega + 2} \quad (4)$$

where $A = -\frac{1}{2}$ and $B = \frac{1}{2}$.

$$Y(j\omega) = -\frac{1}{2}\frac{1}{j\omega + 4} + \frac{1}{2}\frac{1}{j\omega + 2} \quad (5)$$

(d) In order to find the $y(t)$ using the part c) we can take the inverse Fourier Transform of the $Y(j\omega)$ using look-up table. Then the output signal can be represented as:

$$y(t) = -\frac{1}{2}e^{-4t}u(t) + \frac{1}{2}e^{-2t}u(t) \quad (1)$$

3. (a) $H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw}}$ And by using Discrete Time Fourier transform Table, we can achieve this:

$$X(e^{jw}) = \frac{1}{1 - \frac{2e^{-jw}}{3}} \quad (1)$$

To obtain frequency response for y[n], we can do these:

$$y[n] = \frac{2\left((n+1)(\frac{2}{3})^n - (\frac{2}{3})^n\right)u[n]}{3} \quad (2)$$

$$y[n] = \frac{2(n+1)(\frac{2}{3})^n u[n]}{3} - \frac{2(\frac{2}{3})^n u[n]}{3} \quad (3)$$

And then by using the Fourier Transform Table, we obtain the followings:

$$Y(e^{jw}) = \frac{2}{3}\frac{1}{(1 - \frac{2e^{-jw}}{3})^2} - \frac{2}{3}\frac{1}{1 - \frac{2e^{-jw}}{3}} = (\frac{2}{3})^2\frac{e^{-jw}}{(1 - \frac{2e^{-jw}}{3})^2} \quad (4)$$

And finally frequency response of the whole system is:

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw}} = \frac{(\frac{2}{3})^2 \frac{e^{-jw}}{(1-\frac{2e^{-jw}}{3})^2}}{\frac{1}{1-\frac{2e^{-jw}}{3}}} = (\frac{2}{3})^2 \frac{e^{jw}}{(1-\frac{2e^{-jw}}{3})} \tag{5}$$

(b) Taking inverse fourier transform we obtain the impulse response as such:

$$h[n] = f^{-1}\{H(e^{jw})\} = \frac{4}{9} e^{\left(\frac{2}{3}^{(n-1)}\right)} u[n-1] \tag{6}$$

(c)

$$H(e^{jw}) = (\frac{2}{3})^2 \frac{e^{jw}}{(1-\frac{2e^{-jw}}{3})} = \frac{4e^{jw}}{9-6e^{-jw}} = \frac{Y(e^{jw})}{X(e^{jw}} \tag{7}$$

$$9Ye^{jw} - 6e^{jw}Ye^{jw} = 4e^{jw}Xe^{jw} \tag{8}$$

Finally we got the following difference equation:

$$4x[n-1] = 9y[n] - 6y[n-1] \tag{9}$$

(d)

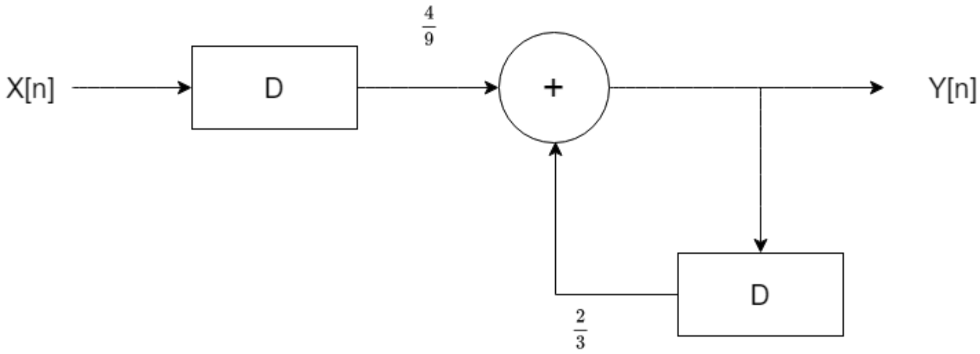

Figure 1: Block Diagram

4. (a) Using the block diagram we get:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n] \tag{1}$$

(b) Using the shifting and scaling properties of the Fourier Transform, we can find the frequency response of the system. The frequency response can be represented as:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}} \tag{1}$$

(c) In order to find the impulse response of the system, we can take the inverse Fourier Transform of the frequency response. The frequency response can be represented as:

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}} = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})} = \frac{A}{1 - \frac{1}{2}e^{-j\omega}} + \frac{B}{1 - \frac{1}{4}e^{-j\omega}} \tag{1}$$

Where $A = 4$ and $B = -2$. Then the impulse response can be found by taking the inverse Fourier Transform of the frequency response using the look-up table, we can use the transformation $\mathcal{F}\left(a^n u[n]\right) = \dfrac{1}{1 - ae^{j}w}$.

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n] \tag{2}$$

(d) In order to solve the problem using the frequency response, we can use the convolution property of the Fourier Transform:

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \tag{1}$$

We found that the frequency response of the system is:

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}} = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}} \tag{2}$$

The Fourier Transform of the input signal ($x[n] = \left(\frac{1}{4}\right)^n u[n]$) can be represented as:

$$\mathcal{F}(x[n]) = X(e^{j\omega}) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}} \tag{3}$$

We can write what we found on the equations 2 and 3:

$$Y(e^{j\omega}) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}}\left(\frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}\right) = \frac{4}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})} - \frac{2}{(1 - \frac{1}{4}e^{-j\omega})^2} \tag{4}$$

For the first term we apply the partial fraction decomposition:

$$Y(e^{j\omega}) = \frac{A}{1 - \frac{1}{2}e^{-j\omega}} + \frac{B}{1 - \frac{1}{4}e^{-j\omega}} - \frac{2}{(1 - \frac{1}{4}e^{-j\omega})^2} \tag{5}$$

Where $A = 8$ and $B = -4$. Then the final version of the output signal can be found by taking the inverse Fourier Transform of the $Y(e^{j\omega})$.

$$y[n] = 8\left(\frac{1}{2}\right)^n u[n] - 4\left(\frac{1}{4}\right)^n u[n] - 2(n+1)\left(\frac{1}{4}\right)^n u[n] \tag{6}$$

5.
$$h_1[n] = (\frac{1}{3})^n u[n] \tag{1}$$

$$x[n] * h_1[n] + x[n] * h_2[n] = y[n] = x[n] * h[n], \text{ such that } h[n] = h_1[n] + h_2[n] \tag{2}$$

$$H(e^{jw}) = \frac{5e^{-jw} - 12}{(e^{-jw} - 4)(e^{-jw} - 3)} \tag{3}$$

Using partial fraction method, we obtain the following:

$$H(e^{jw}) = \frac{B}{(e^{-jw} - 4)} + \frac{A}{(e^{-jw} - 3)} \tag{4}$$

From the above equation we get the following:

$$4A + 3B = 12, \quad A + B = 5 \tag{5}$$

And $A = -3$, $B = 8$ values hold the equations below. Then we get the equation below:

$$H(e^{jw}) = \frac{8}{(e^{-jw} - 4)} + \frac{-3}{(e^{-jw} - 3)} = \frac{-2}{(1 - \frac{e^{-jw}}{4})} + \frac{1}{(1 - \frac{e^{-jw}}{3})} \tag{6}$$

When we take the inverse Fourier Transform of the equation above, we get the following equation:

$$(\frac{1}{3})^n u[n] - 2(\frac{1}{4})^n u[n] \tag{7}$$

We already know what $h_1[n]$ is. Using that, we get $h_2[n]$ as such:

$$h_2[n] = -2(\frac{1}{4})^n u[n] \tag{8}$$

6.
```python
import numpy as np
import matplotlib.pyplot as plt

min_value = -15
max_value = 15
spaces = 1000

def mySignal(n):
    return np.power(0.5, np.abs(n))

# Range of computation
n_min = -10
n_max = 10
n_values = np.arange(n_min, n_max + 1)

xAxisValues = np.linspace(min_value, max_value, spaces)

# DTFT computation, we round the values to 5 decimal places
# because there appears to be some very small phases in scale of 1e-16
yAxisValues = (np.sum(mySignal(n_values) * np.exp(-1j * np.outer(xAxisValues, n_values)), axis=1)).
                                                       round(5)


plt.figure(figsize=(12, 8))

# Magnitude plot
plt.subplot(2, 1, 1)
plt.plot(xAxisValues, np.abs(yAxisValues))
plt.xlabel('w')
plt.ylabel('|X(e^jw)|')
plt.title('Magnitude of DTFT of x[n] = (0.5)^|n|')
plt.grid(True)

# Phase plot
plt.subplot(2, 1, 2)
plt.plot(xAxisValues, np.angle(yAxisValues))
plt.xlabel('w')
plt.ylabel('phase(X(e^jw))')
plt.title('Phase of DTFT of x[n] = (0.5)^|n|')
```

```
plt.grid(True)

plt.tight_layout()
plt.show()
```

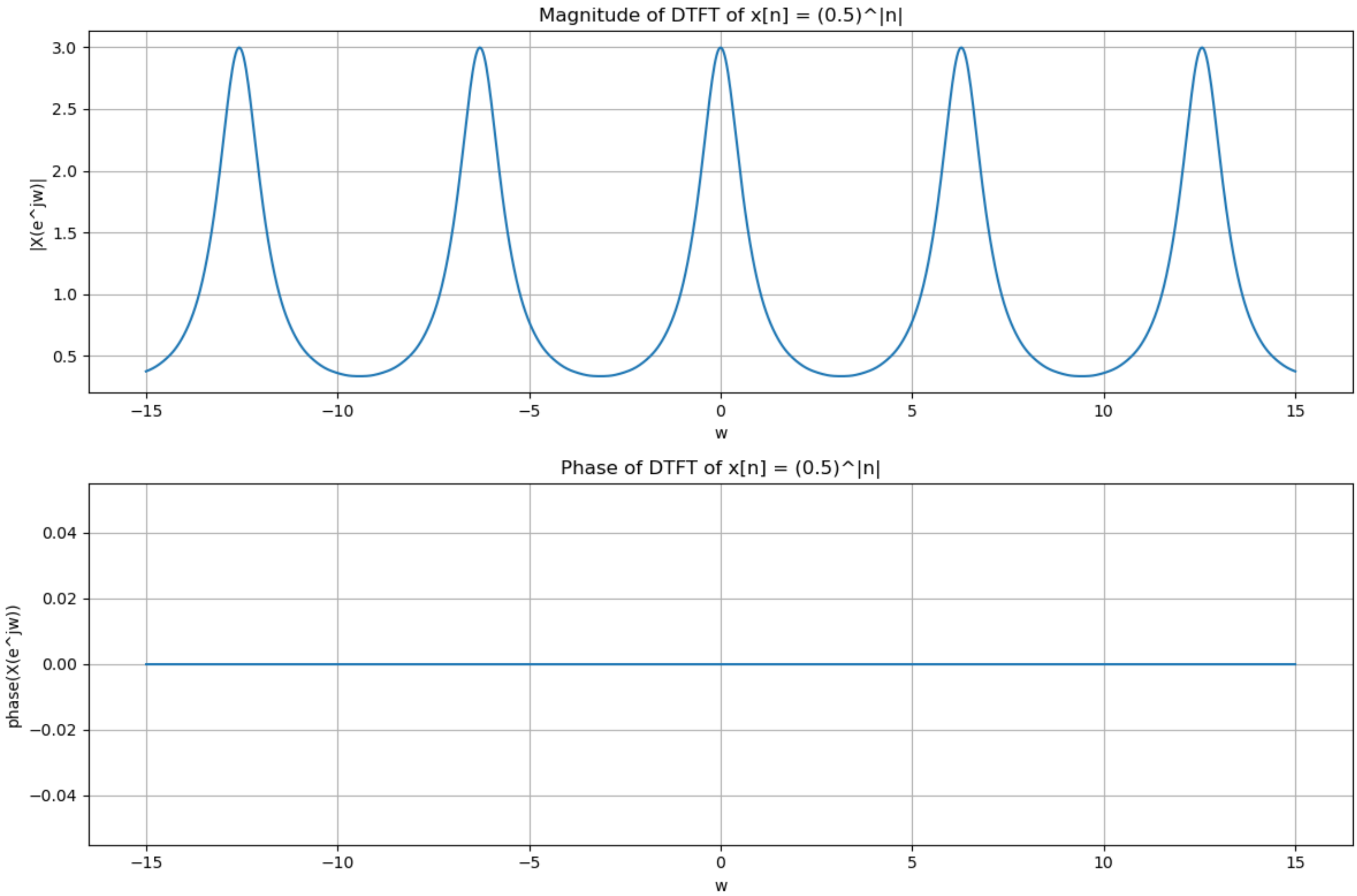

Figure 2: Plot