CENG 384 - Signals and Systems for Computer Engineers
Spring 2024
Homework 2

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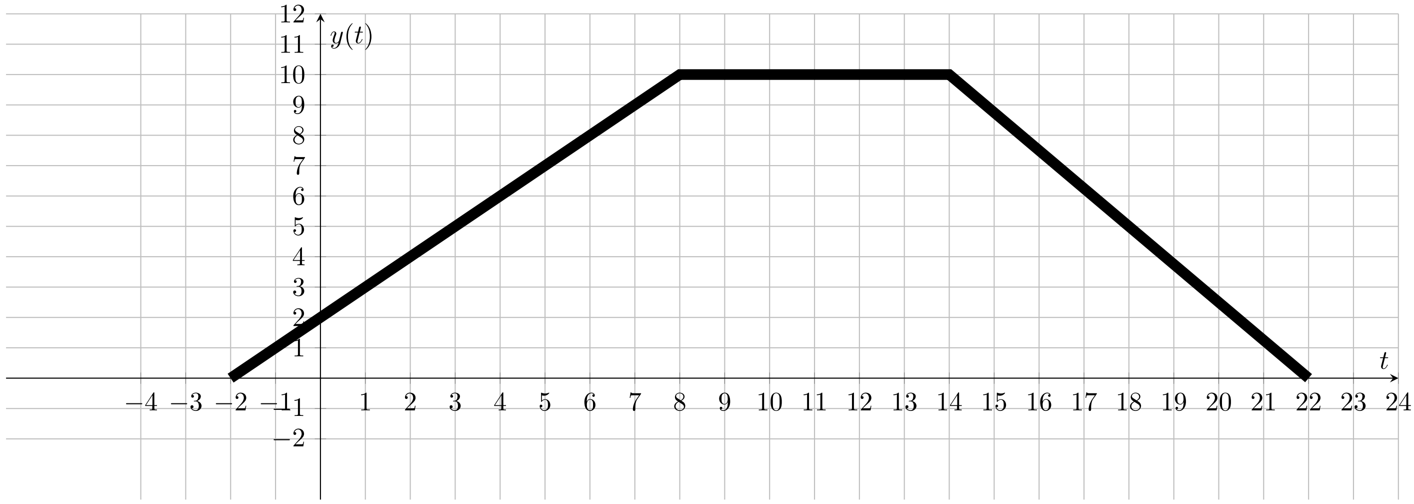
May 10, 2024

1.

$$y(t) = \int_{\infty}^{-\infty} x(\tau) * h(t-\tau)d\tau$$

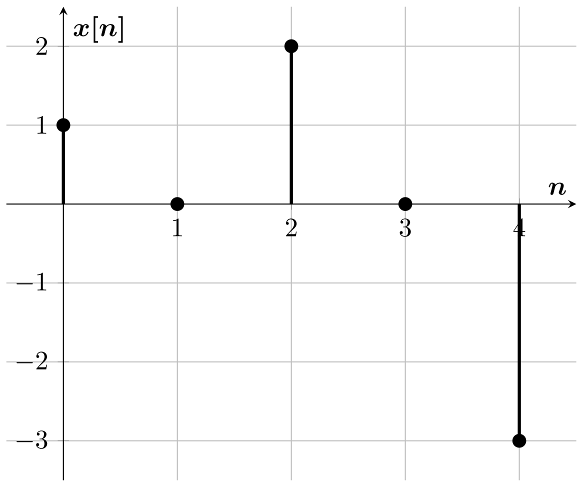
But instead of solving this integral we can do a visualisation that we have seen during lecture. We can slide the x(t) through h(t) starting from ∞ up to $-\infty$. And draw the corresponding graph. And when we do that sliding operation we can see that the convolution integral equals to the signal below:

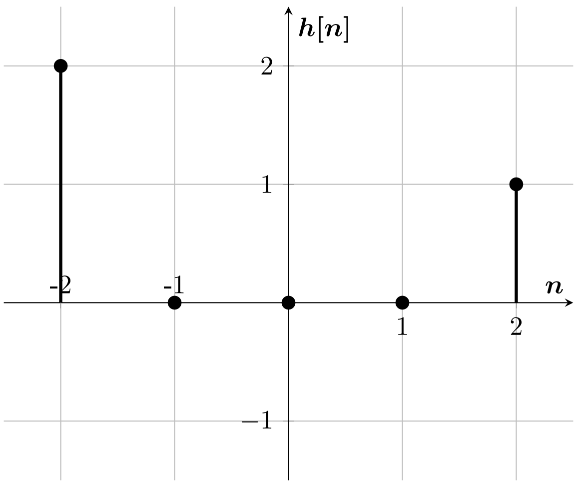


2. We found the answers using convolution formula

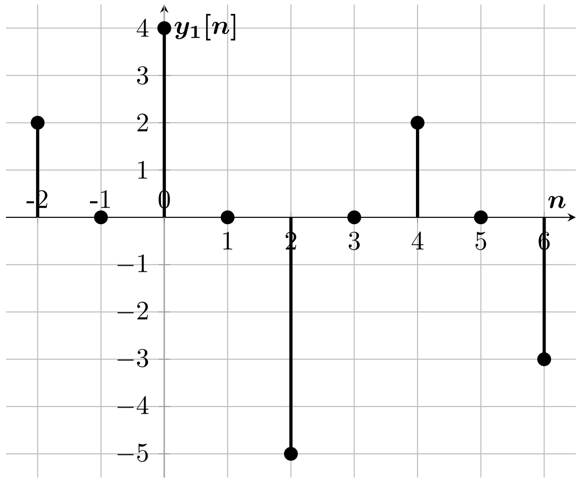
$$y[n] = \sum_{k=-\infty}^{\infty} x[k] * h[n-k]$$

for DT systems , and flip-slide-multiply-sum method

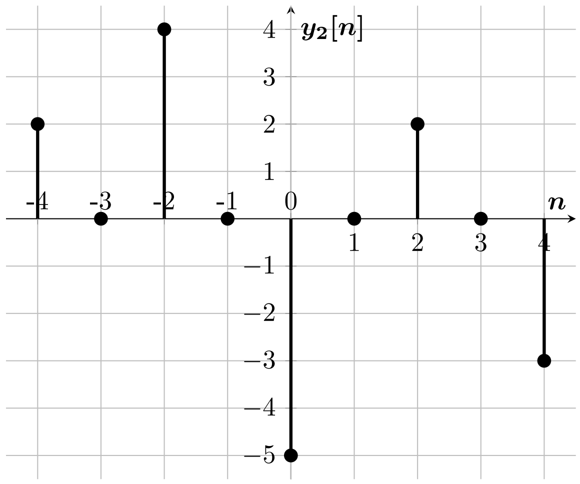




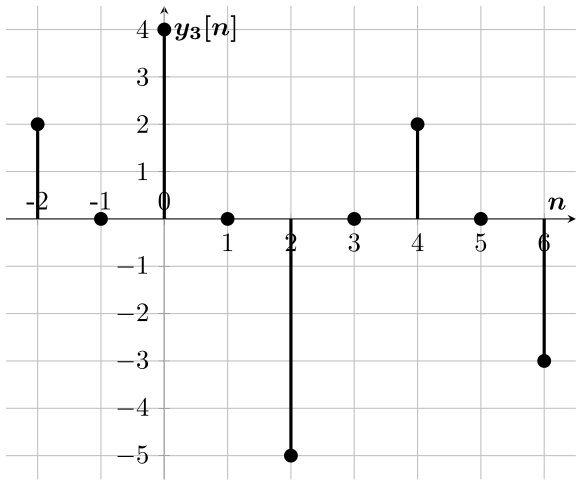
(a)



(b)



(c)



3. (a) To find the impulse response $h[n]$ of the system we can replace $x[n]$ with $\delta[n]$. So we found out that: $h[n] = 0.2 \times \delta[n-1] + \delta[n]$

   (b) $y[n] = 0.2 \times \delta[n-3] + \delta[n-2]$

   (c) A system is BIBO stable if every bounded input produces a bounded output. To check that we can check whether or not $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$.

   Actually it is clear that the above summation equals to $\sum_{k=-\infty}^{\infty} |h[k]| = 1.2 < \infty$. So we have proved that this system is BIBO stable.

   (d) The system is not memoryless. Because the output y[n] doesn't only depend on the input at time n. It also depends on the input time n-1. So the system has memory.

   (e) If a system (which has an impulse response $h[n]$) is invertible, then there exists an inverse system (which has an impulse response $h_1[n]$), which satisfies the equation $h_1[n] * h[n] = \delta[n]$.

- $h_1[n] * (0.2 \times \delta[n-1] + \delta[n]) = \delta[n]$ And since convolution has disributive property we can do this:
- $0.2(h_1[n] * \delta[n-1]) + h_1[n] * \delta[n]) = \delta[n]$ And since convolution with $\delta[n]$ gives the same signal again, we can do this:
- $0.2(h_1[n] * \delta[n-1]) + h_1[n]) = \delta[n]$. And convolution with $\delta[n-1]$ gives the 1 unit shifted right version of signal, then we can do this:
- $0.2 \times h_1[n-1] + h_1[n] = \delta[n]$. Now we have achieved a equation. Using that equation we should find its values:
  - For n=-1:
    $0.2 \times h_1[-2] + h_1[-1]) = 0$
  - For n=0:
    $0.2 \times h_1[-1] + h_1[0] = 1$
  - For n=1:
    $0.2 \times h_1[0] + h_1[1]) = 0$
  - For n=2:
    $0.2 \times h_1[1] + h_1[2]) = 0$
- From these equations we can assume that $h_1[n] = 0$ for $n < 0$ (This assumption is actually true, because whether we set them 0 or not there will be an inverse system that satisfies the convolution above: $h_1[n] * h[n] = \delta[n]$, and one another reason, we have choosen them being 0 is because the the first system is actually DT causal LTI system, which means we ignore the inputs/outputs for $n < 0$). After assuming that, it becomes clear that $h_1[0] = 1$, $h_1[1] = -0.2$, $h_1[2] = 0.04$ and it goes like that. So we found an impulse response $h_1[n] = \sum_{k=0}^{\infty}(\delta[n-k] * (-0.2)^k)$ which is the impulse response of the inverse system.
- To sum up, we found an impulse response $h_1[n]$ which satisfies the equation $h_1[n] * h[n] = \delta[n]$ and its corresponding inverse system which is $y[n] = \sum_{k=0}^{\infty}(x[n-k] * (-0.2)^k)$

Conclusion: Yes, the above system is invertible and its inverse system is:
$y[n] = \sum_{k=0}^{\infty}(x[n-k] * (-0.2)^k)$.

4. (a) $H(\lambda) = \frac{\sum_{k=0}^{m} b_k \lambda^k}{\sum_{k=0}^{n} a_k \lambda^k}$ Then the differential equation becomes:

$$y''(t) - 2y'(t) + y(t) = 2x'(t)$$

(b) Since the input is zero, we are looking for a homogeneous solution. Since it is an LTI system, we can assume that the solution is of the form $y(t) = Ae^{\lambda t}$. Substituting this into the differential equation, we get:

$$\lambda^2 Ae^{\lambda t} - 2\lambda Ae^{\lambda t} + Ae^{\lambda t} = 0$$

$$A(\lambda^2 - 2\lambda + 1)e^{\lambda t} = 0$$

Since $e^{\lambda t}$ is never zero, we have:

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

$$\lambda = 1 \ (double\ root)$$

So the solution is $y(t) = c_1 e^t + c_2 te^t$ when $x(t) = 0$.

(c) The homogeneous solution is $y(t) = c_1 e^t + c_2 te^t$. The particular solution is $y_p(t) = (At + B) * u(t)$.

$$y_p(t) = (At + B) * u(t)$$

$$y_p'(t) = A * u(t)$$

$$y_p''(t) = 0$$

Substituting these into the differential equation, we get:

$$0 - 2A + At + B = 4$$

$$At - 2A + B = 4$$

$$A = 0,\ B = 4$$

So the particular solution is $y_p(t) = 4u(t)$. The general solution using the superposition principle is $y(t) = c_1 e^t + c_2 te^t + 4u(t)$. Since the system is initially at rest, $y(0) = 0$ and $y'(0) = 0$.

$$y(0) = c_1 + 0 + 4 = 0$$

$$c_1 = -4$$

$$y'(0) = c_1 + c_2 + 4 = 0$$

$$-4 + c_2 = 0$$

$$c_2 = 4$$

So the solution is $\boldsymbol{y(t) = -4e^t + 4te^t + 4u(t)}$.

5. (a) In order to find the impulse response of the system, we need to find the output of the system when the input is an impulse. If we substitute $x(t) = \delta(t)$ into the equation we get:

$$y[n] = \frac{1}{5}y[n-1] + 2\delta[n-2]$$

It can easily be seen that if $n < 2$ then impulse response $h[n] = 0$. Since the system is initially at rest, $y[0] = 0$ and $\delta[n-2]$ is 0 for $n \neq 2$. So the impulse response is $h[n] = 0$ for $n < 2$. For $n = 2$:

$$y[n] = \frac{1}{5}y[1] + 2$$

since $y[1] = 0$:

$$y[n] = 2$$

so $h[2] = 2$. For $n = 3$:

$$y[n] = \frac{1}{5}y[2] + 0$$

since $y[2] = 2$:

$$y[n] = 2 * \frac{1}{5}$$

so $h[3] = \frac{2}{5}$. So we have

$$h[n] = 2 \times (\frac{1}{5})^{n-2} \times u[n-2]$$

(b) In order to find the transfer function of the system first we assume that $x[n]$ is an exponential signal, such that:

$$x[n] = e^{\lambda n}$$

Then we can write the output of the system as:

$$y[n] = H(\lambda) * e^{\lambda n}$$

Using our differential equation, we can write:

$$H(\lambda)e^{\lambda n} = \frac{1}{5}H(\lambda)e^{\lambda(n-1)} + 2\lambda e^{\lambda(n-2)}$$

When we simplify the equation:

$$H(\lambda) = \frac{1}{5}H(\lambda)e^{-\lambda} + 2\lambda e^{-2\lambda}$$

$$H(\lambda)[1 - \frac{1}{5}e^{-\lambda}] = 2\lambda e^{-2\lambda}$$

$$H(\lambda) = \frac{2\lambda e^{-2\lambda}}{1 - \frac{1}{5}e^{-\lambda}}$$

$$H(\lambda) = \frac{2\lambda}{e^{2\lambda} - \frac{e^{\lambda}}{5}}$$

(c)

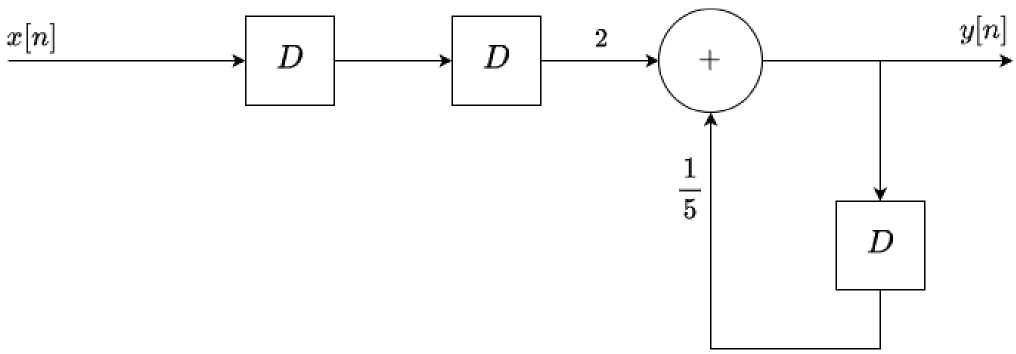


Figure 1: Block diagram of the system for part c

6. (a)

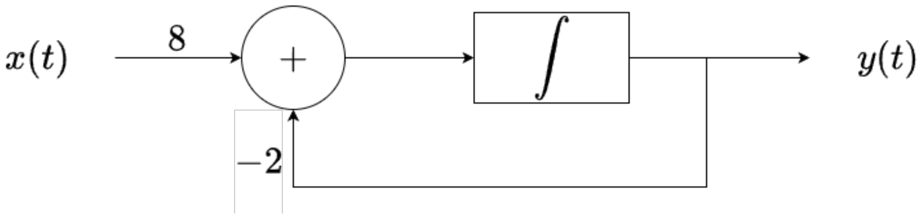


Figure 2: Integrator and Adder Representation

(b)

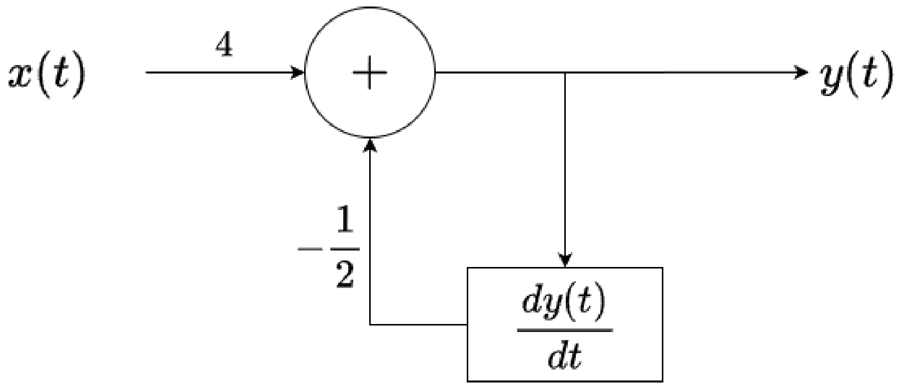


Figure 3: Differentiator and Adder Representation

7.

```
import matplotlib.pyplot as plt

def calculate_output():
    x = [0] * 6  # x[n] = delta[n - 1]
    x[1] = 1     # delta[n - 1] is 1 when n = 1

    y = [0] * 6  # Since the system is causal LTI system y[n]=0 for n<=0, thus We ignored the
                                                          elements for n<0

    # Apply difference equation y[n] = 0.25*y[n - 1] + x[n]
    for n in range(1, 6):
        y[n] = 0.25 * y[n - 1] + x[n]

    return y

# Calculate output sequence y[n]
output_sequence = calculate_output()

# Hocam We're not sure what did you mean by first 5 samples.
# We assumed that you want the samples of the output sequence such that n<=5.
plt.stem(range(6), output_sequence)
plt.xlabel('n')
plt.ylabel('y[n]')
plt.title('Output of Causal LTI System')
plt.grid(True)
plt.show()
```

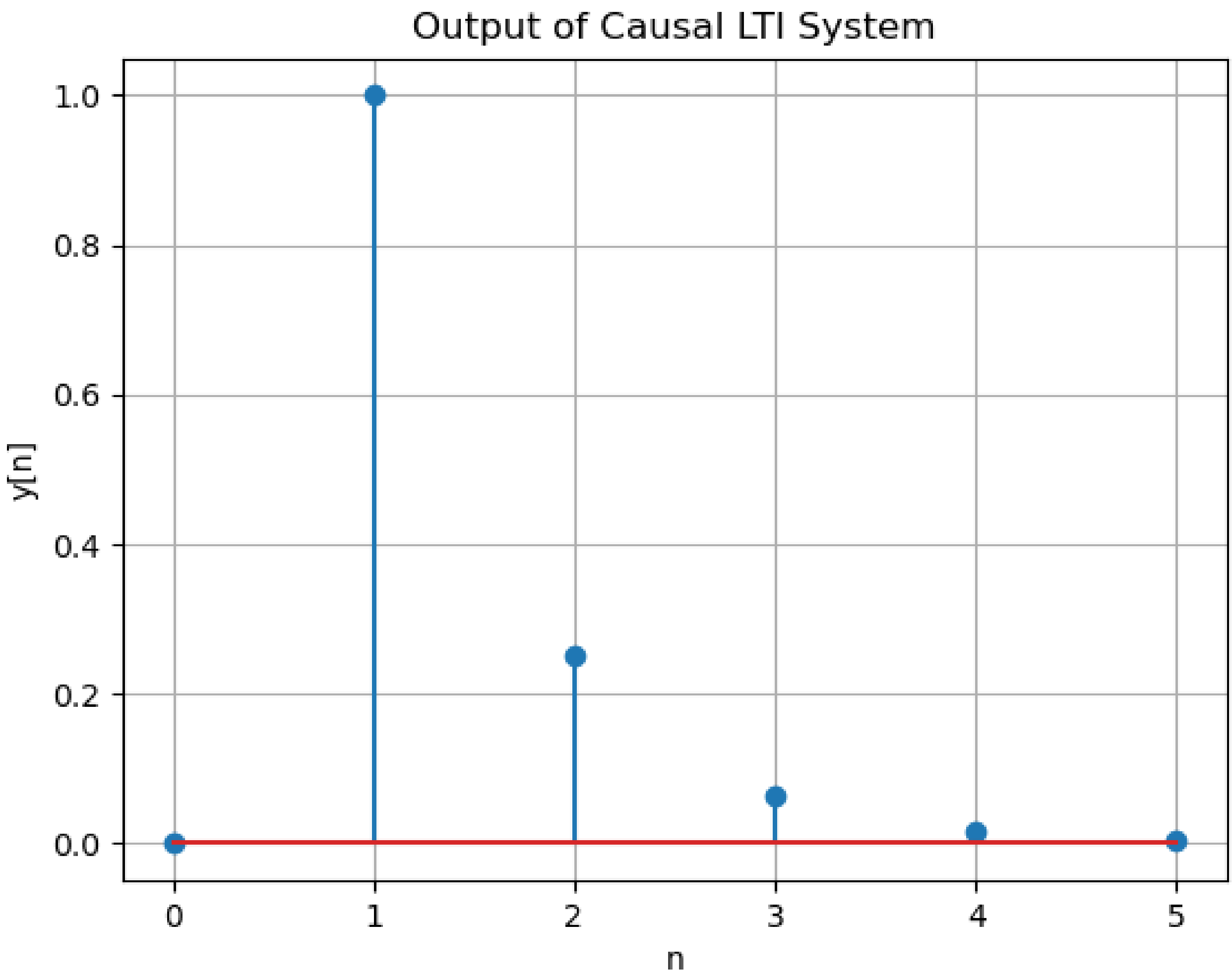


Figure 4: The Plot