CENG 384 - Signals and Systems for Computer Engineers
Spring 2024
Homework 3

Sekmen, Aytaç
e2575983@ceng.metu.edu.tr

YILMAZ, Başar
e2644409@ceng.metu.edu.tr

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1. Using synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jw_0 kt} \tag{1}$$

In our case the equation can be expressed as replacing a_k:

$$x(t) = \sum_{k=-\infty}^{\infty} e^{jw_0 kt} - 2 * \sum_{k=-\infty}^{\infty} e^{j2w_0 kt} \tag{2}$$

We know the from the Fourier Series table:

$$\mathcal{F}(\sum_{k=-\infty}^{\infty} e^{jw_0 kt}) = 1 \tag{3}$$

and also we know that (T is the period of the signal and $w_0 = \frac{2\pi}{T}$):

$$\mathcal{F}(\sum_{l=-\infty}^{\infty} \delta(t - lT)) = \frac{1}{T} * \int_T (\sum_{l=-\infty}^{\infty} \delta(t - lT)) * e^{-jw_0 kt} dt = \frac{1}{T} * e^{-jw_0 klt} = \frac{1}{T} \tag{4}$$

Then it can be inferred:

$$\mathcal{F}(T * \sum_{l=-\infty}^{\infty} \delta(t - lT)) = 1 \tag{5}$$

Using equations 3, 5 we can find the first term of the equation 2, we know ($T = \dfrac{2\pi}{w_0}$):

$$\sum_{k=-\infty}^{\infty} e^{jw_0 kt} = T * \sum_{l=-\infty}^{\infty} \delta(t - lT) = \frac{2\pi}{w_0} * \sum_{l=-\infty}^{\infty} \delta(t - lT) = \mathbf{4} * \sum_{\mathbf{l}=-\infty}^{\infty} \delta(\mathbf{t} - \mathbf{4l}) \tag{6}$$

Now same procedure can be applied to the second term of the equation 2:

$$\sum_{k=-\infty}^{\infty} e^{j2w_0 kt} = T * \sum_{l=-\infty}^{\infty} \delta(t - 2lT) = \frac{2\pi}{w_0} * \sum_{l=-\infty}^{\infty} \delta(t - 2lT) = \mathbf{2} * \sum_{\mathbf{l}=-\infty}^{\infty} \delta(\mathbf{t} - \mathbf{2l}) \tag{7}$$

Finally, using equations 6 and 7, we can write the signal x(t) as an impulse train:

$$x(t) = 4 * \sum_{l=-\infty}^{\infty} \delta(t - 4l) - 4 * \sum_{l=-\infty}^{\infty} \delta(t - 2l) = 4 * \sum_{l=-\infty}^{\infty} \delta(t - 4l) - 4 * \sum_{l=-\infty}^{\infty} [\delta(t - 4l) + \delta(t - 4l + 2)] = -\mathbf{4} \sum_{\mathbf{k}=-\infty}^{\infty} \delta(\mathbf{t} - \mathbf{4l} + \mathbf{2}) \tag{8}$$

2.

a)

$$a_k = \frac{1}{4}\int_0^2 2t \cdot e^{-j\frac{\pi}{2}kt}\,dt + \frac{1}{4}\int_2^4 (4-t)e^{-j\frac{\pi}{2}kt}\,dt$$

$$\pi t = x \qquad \pi(t-2) = x \quad 2-t = \frac{-x}{\pi}$$

$$\pi dt = dx \qquad \pi dt = dx \quad dt = dx$$

$$dt = \frac{dx}{\pi}$$

$$= \frac{1}{4\pi}\int_0^{2\pi} \frac{2x}{\pi} \cdot e^{\frac{-kxi}{2}}\,dx + \frac{1}{4\pi}\int_0^{2\pi} (2 - x/\pi) \cdot e^{\frac{-kxi}{2} - \cancel{2k\pi i}}\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} \underbrace{\left(2+\frac{x}{\pi}\right)}_{u} \underbrace{\cos\frac{kx}{2}\,dx}_{dv} - i\,\frac{1}{4\pi}\int_0^{2\pi} \underbrace{\left(2+\frac{x}{\pi}\right)}_{u} \underbrace{\sin\frac{kx}{2}\,dx}_{dv}$$

$$du = \frac{dx}{\pi} \quad v = \frac{2}{k}\cdot\sin\left(\frac{kx}{2}\right) \qquad du = \frac{1}{\pi}dx, \; v = \frac{-2}{k}\cos\left(\frac{kx}{2}\right)$$

$$= \frac{1}{4\pi}\left(\frac{2}{k}\left(2+\frac{x}{\pi}\right) \cdot \sin\frac{kx}{2}\Big|_{x=0}^{2\pi} - \frac{2}{k\pi}\int_0^{2\pi}\sin\left(\frac{kx}{2}\right)dx\right) - \frac{1}{4\pi}i\left(-\frac{2}{k}\left(2+\frac{x}{\pi}\right)\cos\frac{kx}{2}\Big|_0^{2\pi} + \ldots\right.$$

$$\left.\frac{2}{k\pi}\int_0^{2\pi}\cos\frac{kx}{2}\,dx\right) = \frac{1}{k\pi^2}\cdot\frac{2}{k}\cdot\cos\frac{kx}{2}\Big|_{x=0}^{2\pi} - \frac{1}{4\pi}i\left(\frac{8(-1)^{k+1}+4}{k} + \frac{4}{k^2\pi}\sin\frac{kx}{2}\Big|_0^{2\pi}\right)$$

$$a_k = \frac{4}{k^2\pi^2}\cdot\left((-1)^k\cdot(-i\pi k+1)+1\right) + \frac{8}{k^2\pi^2}\left((-1)^k\cdot(i\pi k+1)+1\right)$$

b)

$$F\left\{\frac{dx(t)}{dt}\right\} \Longrightarrow jw_0 k a_k = j\frac{\pi}{2}\cdot k\left[\frac{4}{(k\pi)^2}\left((-1)^k(-j\pi k+1)+1\right) + \frac{8}{(k\pi)^2}\left((-1)^k(j\pi k+1)+1\right)\right]$$

$$w_0 = \pi/2$$

Figure 1: Solution of Q2

3. (a) $x_1[n]$ can be written as follows using euler's formula:

$$x_1[n] = \cos(\frac{\pi}{2}n) = \frac{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}}{2}$$

Fundamental period of $x_1[n]$ is $N_1 = 4$. So we can find the spectral coefficients of $x_1[n]$ as follows:

$$a_k = \frac{1}{N_1}\sum_{n=0}^{N_1-1} x_1[n]e^{-j\frac{2\pi}{N_1}kn}$$

which is equal to:

$$a_1 = \frac{1}{2},\ a_0 = 0,\ a_{-1} = \frac{1}{2}$$

$x_2[n]$ can be written as follows using euler's formula:

$$x_2[n] = \sin(\frac{\pi}{2}n) = \frac{e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}}{2j}$$

Fundamental period of $x_2[n]$ is $N_2 = 4$. So we can find the specular coeffecients of $x_2[n]$ as follows:

$$b_k = \frac{1}{N_2}\sum_{n=0}^{N_2-1} x_2[n]e^{-j\frac{2\pi}{N_2}kn}$$

which is equal to:

$$b_1 = \frac{1}{2j},\ b_0 = 0,\ b_{-1} = \frac{-1}{2j}$$

$x_3[n]$ can be written as follows using Euler's formula:

$$x_3[n] = x_1[n]x_2[n] = \sin(\frac{\pi}{2}n)\cos(\frac{\pi}{2}n) = \frac{1}{2}\sin(\pi n) = \frac{1}{2}\sin(\pi n) = 0$$

Fundamental period of $x_3[n]$ is $N_3 = 2$. So we can find the specular coeffecients of $x_3[n]$ as follows:

$$c_k = \frac{1}{N_3}\sum_{n=0}^{N_3-1} x_3[n]e^{-j\frac{2\pi}{N_3}kn}$$

which is equal to:

$$c_k = 0$$

(b) Using the multiplication property, we can write the of c_k as follows:

$$c_k = \sum_{l=0}^{N-1} a_l b_{k-l}$$

where $N = 4$.

$$c_1 = a_0b_2 + a_1b_1 + a_2b_0 + a_3b_{-1} = 0$$
$$c_0 = a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0 = 0$$
$$c_2 = a_0b_1 + a_1b_0 + a_2b_{-1} + a_3b_{-2} = 0$$
$$c_3 = a_0b_0 + a_1b_{-1} + a_2b_{-2} + a_3b_{-3} = 0$$
$$c_k = 0$$

4. Since the periods of first and second cosine expressions are different we should analyze the spectral coefficients for them seperately:

Using euler equation, we get these two equations:

$$a_k^1 = cos(\frac{k\pi}{3}) = \frac{1}{2}\left(e^{(\frac{kj\pi}{3})} + e^{(-\frac{kj\pi}{3})}\right) \tag{1}$$

$$a_k^2 = cos(\frac{k\pi}{4}) = \frac{1}{2}\left(e^{(\frac{kj\pi}{4})} + e^{(-\frac{kj\pi}{4})}\right) \tag{2}$$

When we analyze their period, it is clear that a_k^1 has the period 6 and the a_k^2 has the period 8. To find their input representations we should first represent them in terms of the $period = LCM(6, 8) = 24$ So the equations become like this:

$$a_k^1 = cos(\frac{k\pi}{3}) = \frac{1}{2}\left(e^{(4kj\frac{2\pi}{24})} + e^{(-4kj\frac{2\pi}{24})}\right) \tag{3}$$

$$a_k^2 = cos(\frac{k\pi}{4}) = \frac{1}{2}\left(e^{(3kj\frac{2\pi}{24})} + e^{(-3kj\frac{2\pi}{24})}\right) \tag{4}$$

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jkw_0n} \tag{5}$$

$$w_0 = \frac{2\pi}{T} \tag{6}$$

Then using the equations above we get following equations:

For a_k^1:

$$\frac{1}{24}\sum_{n=<24>} x_1[n]e^{-jk\frac{2\pi}{24}} = \frac{1}{2}\left(e^{(4kj\frac{2\pi}{24})} + e^{(-4kj\frac{2\pi}{24})}\right) \tag{7}$$

From this equation we get that $x_1[4] = x_1[-4] = 12$, $x_1[n] = 0$ for $n \neq \pm 4$, in the duration $-12 \leq n < 12$. And we can show $x_1[n]$ like this:

$$x_1[n] = 12\delta[n-4] + 12\delta[n+4] \tag{8}$$

For a_k^2:

$$\frac{1}{24}\sum_{n=<24>} x_2[n]e^{-jk\frac{2\pi}{24}} = \frac{1}{2}\left(e^{(3kj\frac{2\pi}{24})} + e^{(-3kj\frac{2\pi}{24})}\right) \tag{9}$$

From this equation we get that $x_2[3] = x_2[-3] = 12$, $x_2[n] = 0$ for $n \neq \pm 3$, in the range $-12 \leq n < 12$. And we can show it like this:

$$x_2[n] = 12\delta[n-3] + 12\delta[n+3] \tag{10}$$

So the overall $x[n] = x_1[n] + x_2[n]$ becomes like this:

$$x[n] = 12\delta[n-3] + 12\delta[n+3] + 12\delta[n-4] + 12\delta[n+4] \text{ for the range } -12 \leq n < 12 \tag{11}$$

And the following equation is also true by definition of periodicity:

$$x[n] = x[n+24N] \text{ for } N \in Z \tag{12}$$

5. (a)

To find the fundamental period we should find the smallest integer N such that this equation holds:

$$\sin\left(\frac{6\pi}{13}n + \frac{\pi}{2}\right) = \sin\left(\frac{6\pi}{13}(n+N) + \frac{\pi}{2}\right) \tag{1}$$

For the equation (1) to hold, the following equation should also hold:

$$\frac{6\pi}{13}N = 2k\pi \tag{2}$$

where k is an integer. From equation (2), we can find the smallest N which is the fundamental period and k as follows:

$$N = 13, \quad k = 3 \tag{3}$$

Therefore, the fundamental period of the given signal is 13.

(b) Using the euler equation for sin(wt):

$$sin(wt) = \frac{e^{jwt} - e^{-jwt}}{2j} \tag{1}$$

We get the following equality:

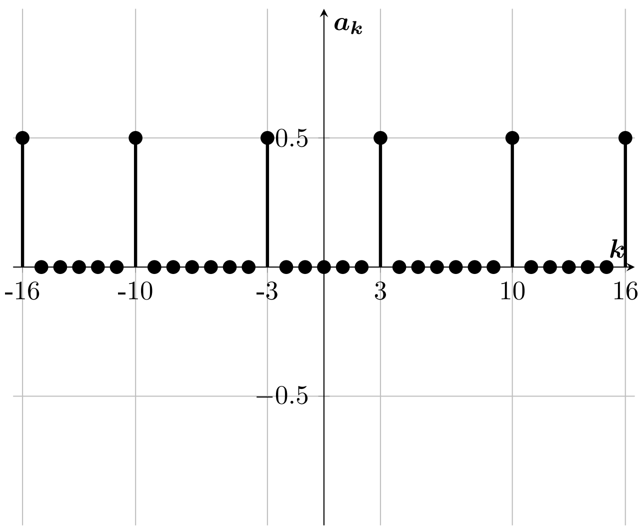
$$e^{\frac{j\pi}{2}} = j, \quad e^{\frac{-j\pi}{2}} = -j, \quad w_0 = \frac{2\pi}{13} \tag{2}$$

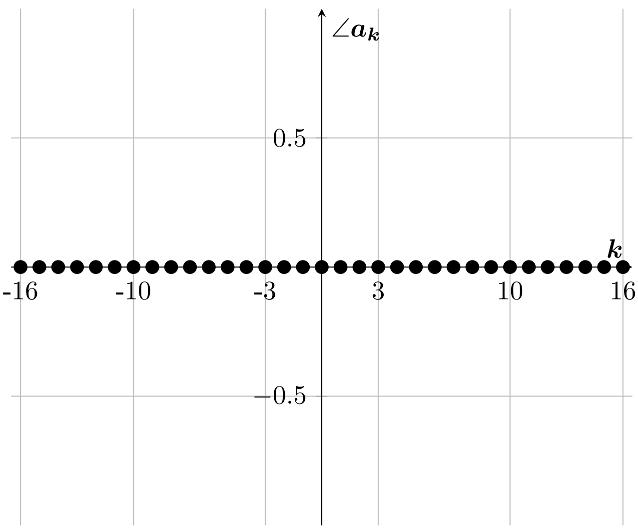
$$sin(\frac{6\pi}{13}n+\frac{\pi}{2}) = \frac{e^{j\left(\frac{6\pi}{13}n+\frac{\pi}{2}\right)} - e^{-j\left(\frac{6n\pi}{13}+\frac{\pi}{2}\right)}}{2j} = \frac{e^{\frac{j\pi}{2}}\left(e^{\frac{6jn\pi}{13}}\right)}{2j} - \frac{e^{\frac{-j\pi}{2}}\left(e^{-j\frac{6\pi}{13}n}\right)}{2j} = \frac{e^{\frac{jn6\pi}{13}} + e^{\frac{-jn6\pi}{13}}}{2} \tag{3}$$

Using the synthesis equation (equation number 4) and the equations 2 and 3, we get spectral coefficients at equation 5:

$$\sum_{k=<N>} a_k e^{jkw_0 n} \tag{4}$$

$$a_{-3} = \frac{1}{2}, \quad a_3 = \frac{1}{2} \tag{5}$$





6. (a) After taking inverse Fourier Transform of the given frequency response we found that:

$$h(t) = \frac{1}{4} \times e^{-\frac{3}{4}t}u(t) \tag{1}$$

(b) Since

$$y(t) = (e^{-5t} - e^{-10t})u(t) \tag{2}$$

We can derive this:

$$Y(jw) = \frac{1}{jw+5} - \frac{1}{jw+10} = \frac{5}{(jw+5) \times (jw+10)} \tag{3}$$

since $H(jw) = \dfrac{1}{4jw+3}$ we have

$$X(jw) = \frac{Y(jw)}{H(jw)} = \frac{(20jw+15)}{(jw+5) \times (jw+10)} \tag{4}$$

To solve this equation we can use partial fractions method:

$$X(jw) = \frac{A}{jw+5} + \frac{B}{jw+10} = \frac{Ajw + 10A + Bjw + 5B}{(jw+5) \times (jw+10)} = \frac{(20jw+15)}{(jw+5) \times (jw+10)} \tag{5}$$

We obtain these two equations:

$$A + B = 20 \tag{6}$$

$$10A + 5B = 15 \tag{7}$$

Solving these two equations together we get $A = -17$ and $B = 37$
So finally we got

$$X(jw) = -17e^{-5t}u(t) + 37e^{-10t}u(t) \tag{8}$$

7.

```python
import numpy as np
import matplotlib.pyplot as plt

# Define the composite continuous-time signal x(t)
def x(t):
    return 0.5 * np.exp(1j * np.pi * t / 3) + 0.5 * np.exp(-1j * np.pi * t)

T = 6 # fundamental period

# Fourier series coef computation using riemann sum like approximation
def fourier_coef(k, num_points=10000):
    t = np.linspace(0, T, num_points, endpoint=False)
    dt = T / num_points
    integral = x(t) * np.exp(-1j * k * np.pi * t / T)
    return (1 / T) * np.sum(integral) * dt


k_values = np.arange(-20, 21)  # I assumed coefecients are -20 <= k <= 20
coef = []
for k in k_values:
coef.append(fourier_coef(k))

plt.figure(figsize=(12, 8))

plt.subplot(1, 2, 1)
plt.stem(k_values, np.abs(coef))  # gets the magnitude
plt.title("Magnitude of Fourier coef")
plt.xlabel("k")
plt.ylabel("Magnitude")

plt.subplot(1, 2, 2)
plt.stem(k_values, np.angle(coef, deg=True))  # gets the phase in degrees
plt.title("Phase of Fourier coef")
plt.xlabel("k")
plt.ylabel("Phase (degrees)")

plt.show()


print(f"Fundamental period: {T} units")

print(
f"Simplified Fourier series representation: x(t) = sum[({coef[0]:}) + sum[({coef[1]:})*cos(pit/{T:}) + ({coef[1]:})*sin(pit/{T:}) + sum[({coef[2]:})*cos(2pit/{T:}) + ({coef[2]:})*sin(2pit/{T:}) + ... + sum[({coef[20]:})*cos(20pit/{T:}) + ({coef[20]:})*sin(20pit/{T:})]]]]"
)
# in the latex code format does not support sigma notation so sum keyword is used instead
```

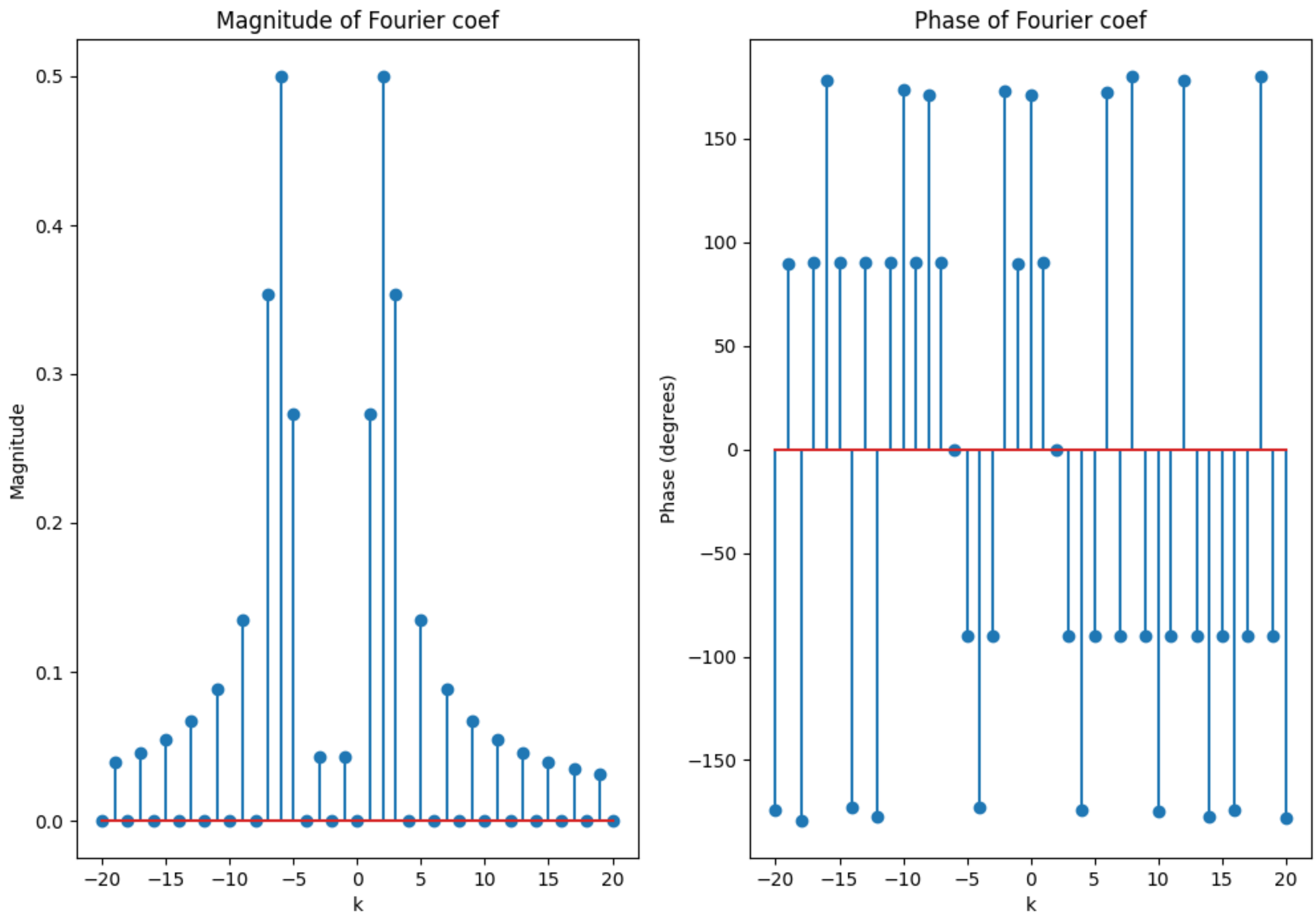


Figure 2: Magnitude and Phase Figures