Code Verification for Collisional Plasma Dynamics

Brian A. Freno

Christopher H. Moore

William J. McDoniel

Duncan A. O. McGregor

Sandia National Laboratories

U.S. DEPARTMENT OF ENERGY

National Nuclear Security Administration

Sandia National Laboratories

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Collisional Plasma Dynamics

- Important for many scientific and engineering applications
    - **Hypersonic & reentry air plasmas** affecting heat loads and radiation
    - **Plasma devices** such as lightning arrester connectors and plasma switches
    - **Pulsed power** for simulating Sandia flagship experimental facilities
    - **Semiconductor & thin-film plasmas** for etching and deposition

- Modeled via particle-in-cell (PIC) with collision algorithm (MCC/DSMC)
    - Solve Maxwell's equations to compute electromagnetic fields on grid
    - Solve particle equations of motion due to Lorentz force and collisions
    - Interpolate EM fields onto particles, distribute particle properties onto grid
    - Model particle collisions with direct simulation Monte Carlo (DSMC)

Code Verification

- Code verification assesses correctness of numerical-method implementation

- Most rigorously measures rate at which error decreases with refinement

- Error requires exact solution – usually unavailable

- Manufactured solutions are popular alternative: $\mathbf{r}(\mathbf{u}) = \mathbf{0} \rightarrow \mathbf{r}(\mathbf{u}) = \mathbf{r}(\mathbf{u}^M)$
  - Manufacture an arbitrary solution $\mathbf{u}^M$
  - Insert manufactured solution into equations to get residual term $\mathbf{r}(\mathbf{u}^M)$
  - Add residual term to equations to make manufactured solution a solution

- For *collisionless* plasma dynamics, few instances of code verification exist

- Significant code-verification challenges with *collisional* plasma dynamics:
  - Discretization errors from space and time discretization
  - Statistical sampling error from finite number of computational particles
  - Stochasticity from collision modeling – considering random subset of collisions

Existing Work: Collisionless Plasma Dynamics

- Riva et al., *Physics of Plasmas* (2017), `10.1063/1.4977917`
    - Modify particle weights to achieve manufactured distribution function
        - Particles move independently of manufactured distribution function
        - Particle weights modified according to Vlasov equation
    - 1D, electrons
    - Measure electric field error
    - Multiple approaches with varying expense to measure dist. function error
    - Many runs per discretization

- Tranquilli et al., *Journal of Computational Physics* (2022), `10.1016/j.jcp.2021.110751`
    - Extend the approach of Riva et al. to 2D, electrons and ions
    - Measure charge density, electric field, and electric potential errors
    - Derive expected convergence rates for statistical sampling errors in fields
    - Argue against the need to measure error in distribution function
    - Single run per discretization

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This Work: Collisional Plasma Dynamics

- Apply method of manufactured solutions to equations of motion
  - Avoid potentially negative weights – weights are unmodified
  - Obtain manufactured particle positions and velocities at each time step
    - Inversely query cumulative manufactured distribution function
- Balance collision algorithm velocity change with manufactured source term
  - Average outcomes from multiple collision-algorithm runs at each time step
  - Compute analytical expected change in velocity due to collisions
    - Manufacture cross section and anisotropy
- Apply method of manufactured solutions to Poisson equation
  - Manufacture electric scalar potential
- Compute field errors and particle errors
  - Single run per discretization
- Demonstrate approach for collisional and collisionless plasma dynamics

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  - Particle-in-Cell Method Overview
  - Equations of Motion for Charged Particles
  - Maxwell's Equations

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Particle-in-Cell Method Overview

- Place (weighted) computational particles randomly in phase space (according to distribution function)

- Interpolate particle charge onto spatial mesh

- Solve Maxwell's equations on spatial mesh for electromagnetic fields

- Interpolate electric field onto particles

- For each particle, integrate equations of motion due to
  - Lorentz force from electromagnetic fields
  - Collisions between particles

Equations of Motion for Charged Particles (single species, electrostatic)

Equations of motion for each particle p:

$$\dot{w}_p(t) = 0, \qquad \dot{\mathbf{x}}_p(t) = \mathbf{v}_p(t), \qquad \dot{\mathbf{v}}_p(t) = \frac{\mathbf{F}_p(t)}{m} + \left(\frac{\Delta \mathbf{v}_p(t)}{\Delta t}\right)_{\text{coll}}$$

- w_p, $\mathbf{x}_p$, and $\mathbf{v}_p$ are computational particle weight, position, and velocity
- $\mathbf{F}_p(t) = q\mathbf{E}_p(t)$ is electrostatic Lorentz force, $\mathbf{E}_p(t) = \mathbf{E}(\mathbf{x}_p(t), t)$
- $\mathbf{E}$ is electric field
- m and q are species mass and charge
- $(\Delta \mathbf{v}_p / \Delta t)_{\text{coll}}$ is instantaneous change in $\mathbf{v}_p$ per Δt due to collision algorithm

Increasing N_p, distribution function evolution approaches Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}$$

- $f(\mathbf{x}_p, \mathbf{v}_p, t)$ is particle distribution function, $(\partial f / \partial t)_{\text{coll}}$ is collision term

Maxwell's Equations (electrostatic case – negligible magnetic field)

Gauss's law	$\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$	$\rightarrow$

$$\Delta \phi = -\frac{\rho}{\epsilon_0}$$

Gauss's law for magnetism $\nabla \cdot \mathbf{B} = 0$

$\uparrow$

Faraday's law of induction $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}\,{\color{red}0}$ $\rightarrow$ $\mathbf{E} = -\nabla \phi$

Ampère's circuital law $\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \right)$

- Charge conservation $\dfrac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$

- Charge density $\rho(\mathbf{x}, t) = q \displaystyle\int_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}$

- Electric current density $\mathbf{J}(\mathbf{x}, t) = q \displaystyle\int_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{v}, t) \mathbf{v} d\mathbf{v}$

- ϵ_0 and μ_0 are permittivity and permeability of free space

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- Manufactured Solutions for Collisional Plasma Dynamics
  - Manufactured Particle Distribution Function
  - Manufactured Solutions through the Equations of Motion
  - Manufactured Source Term for Binary Elastic Collisions
  - Manufactured Solutions for the Poisson Equation

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Manufactured Particle Distribution Function

Assume f^M takes the form of 3D analog of previous work:

$$f^M(\mathbf{x}, \mathbf{v}, t) = f_{\mathbf{x}}(\mathbf{x}, t) f_{\mathbf{v}}(\mathbf{v}, t),$$

where

$$f_{\mathbf{v}}(\mathbf{v}, t) = \prod_{i=1}^{3} f_{v_i}(v_i, t), \quad f_{v_i}(v_i, t) = \frac{2}{\sqrt{\pi}} \frac{v_i^2}{\hat{v}_i(t)^3} e^{-v_i^2/\hat{v}_i(t)^2}, \quad \int_{-\infty}^{\infty} f_{v_i}(v_i, t) dv_i = 1,$$

and

$$f_{\mathbf{x}}(\mathbf{x}, t) = N \prod_{i=1}^{3} f_{x_i}(x_i, t), \qquad \int_{0}^{L_{x_i}} f_{x_i}(x_i, t) dx_i = 1, \qquad \int_{V} f_{\mathbf{x}}(\mathbf{x}, t) d\mathbf{x} = N$$

- N is the number of physical particles in the volume $V = \prod_{i=1}^{3} L_{x_i}$
- Separability of $f_{\mathbf{x}}(\mathbf{x}, t)$ imposed for convenience
- $f_{\mathbf{v}}(\mathbf{v}, t)$ is deliberately non-Maxwellian, $\hat{v}_i(t)$ incorporates time variation

Manufactured Solutions through the Equations of Motion

Apply method of manufactured solutions to equations of motion:

$$\dot{\mathbf{x}}_p = \mathbf{v}_p + \dot{\mathbf{x}}_p^M - \mathbf{v}_p^M, \qquad \dot{\mathbf{v}}_p = \frac{q}{m}\mathbf{E}_p + \left(\frac{\Delta \mathbf{v}_p}{\Delta t}\right)_{\text{coll}} + \dot{\mathbf{v}}_p^M - \frac{q}{m}\mathbf{E}_p^M - \left(\frac{\Delta \mathbf{v}_p^M}{\Delta t}\right)_{\text{coll}}$$

- Avoids negative weights
- At $t = 0$, for component i, take uniform random samples $\xi_{x_{i_p}}$, $\xi_{v_{i_p}} \in [0, 1]$
- Inversely query cumulative dist. functions to obtain $x_{i_p}^M(t)$ and $v_{i_p}^M(t)$:

$$F_{x_i}\big(x_{i_p}^M(t), t\big) = \xi_{x_{i_p}}, \qquad F_{v_i}\big(v_{i_p}^M(t), t\big) = \xi_{v_{i_p}}$$

- Differentiate to obtain $\dot{x}_{i_p}^M(t)$ and $\dot{v}_{i_p}^M(t)$
- In general, $\dot{\mathbf{x}}_p^M \neq \mathbf{v}_p^M$
- $(\Delta \mathbf{v}_p^M / \Delta t)_{\text{coll}}$ is manufactured collision term

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Manufactured Source Term for Collisions

Velocity equation

$$\dot{\mathbf{v}}_p = \frac{q}{m}\left(\mathbf{E}_p - \mathbf{E}_p^M\right) + \frac{(\Delta\mathbf{v}_p)_{\text{coll}} - (\Delta\mathbf{v}_p^M)_{\text{coll}}}{\Delta t} + \dot{\mathbf{v}}_p^M$$

Requires velocity changes due to collisions at each time step:

- $(\Delta\mathbf{v}_p)_{\text{coll}}$ due to stochastic collision algorithm

- $(\Delta\mathbf{v}_p^M)_{\text{coll}}$ due to corresponding deterministic manufactured source term

At each time step, make collision algorithm outcome less stochastic:

- Run collision algorithm N_{avg} independent times, average velocity change

- Replace $(\Delta\mathbf{v}_p)_{\text{coll}}$ with $\langle\Delta\mathbf{v}_p\rangle_{\text{coll}} = \dfrac{1}{N_{\text{avg}}}\displaystyle\sum_{k=1}^{N_{\text{avg}}}(\Delta\mathbf{v}_p^k)$

- Derive expected velocity change for each particle: $(\Delta\mathbf{v}_p^M)_{\text{coll}} = \langle\Delta\mathbf{v}_p^M\rangle_{\text{coll}}$

Expected Change in Velocity (binary elastic collisions)

Post-collision velocities are obtained from momentum and energy conservation:

$$\mathbf{v}'_p = \frac{1}{2}(\mathbf{v}_q + \mathbf{v}_p - g\mathbf{n}), \qquad \mathbf{v}'_q = \frac{1}{2}(\mathbf{v}_q + \mathbf{v}_p + g\mathbf{n}), \qquad \mathbf{n} = \left\{ \begin{array}{c} \cos\epsilon\sin\chi \\ \sin\epsilon\sin\chi \\ \cos\chi \end{array} \right\},$$

where $g = |\mathbf{v}_p - \mathbf{v}_q| = |\mathbf{v}'_p - \mathbf{v}'_q|$ is relative speed

The velocity change for particle p is $\Delta\mathbf{v}_p = \mathbf{v}'_p - \mathbf{v}_p = \frac{1}{2}(\mathbf{v}_q - \mathbf{v}_p - g\mathbf{n})$

Compute expected velocity change across possible collision partners:

$$\langle\Delta\mathbf{v}_p^M\rangle_{\mathrm{coll}} = \frac{\frac{1}{2}\int_V \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\pi} P_{\mathrm{coll}}(g)(N_p^{\mathrm{cell}}-1)(\mathbf{v}_q - \mathbf{v}_p - g\mathbf{n})f^M(\mathbf{x},\mathbf{v}_q,t)p(\chi,\epsilon)d\chi d\epsilon d\mathbf{v}_q d\mathbf{x}}{\int_V \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\pi} f^M(\mathbf{x},\mathbf{v}_q,t)p(\chi,\epsilon)d\chi d\epsilon d\mathbf{v}_q d\mathbf{x}}$$

$P_{\mathrm{coll}}(g) = \frac{\sigma(g)gw\Delta t}{\Delta V}$ is collision likelihood, $p(\chi,\epsilon)$ is probability density function, $\sigma(g)$ is cross section

$\langle\Delta\mathbf{v}_p^M\rangle_{\mathrm{coll}}$ is deterministic, should be computed analytically – complicated by g

$\rightarrow$ Manufacture anisotropy and cross section

Manufactured Anisotropy (to avoid dependency on g due to $\mathbf{n}p$)

Model probability density function as separable: $p(\chi, \epsilon) = p_\chi(\chi)p_\epsilon(\epsilon)$, where

$$\int_0^{2\pi} \int_0^\pi p(\chi, \epsilon)d\chi d\epsilon = 1, \qquad \int_0^{2\pi} p_\epsilon(\epsilon)d\epsilon = 1, \qquad \int_0^\pi p_\chi(\chi)d\chi = 1$$

Azimuthally symmetric scattering: $p_\epsilon(\epsilon) = \dfrac{1}{2\pi}$

In expression for $\left\langle \Delta \mathbf{v}_p^M \right\rangle_{\mathrm{coll}}$,

$$g \int_0^{2\pi} \int_0^\pi \mathbf{n}p(\chi, \epsilon)d\chi d\epsilon = \frac{g}{2\pi} \int_0^{2\pi} \int_0^\pi \mathbf{n}p_\chi(\chi)d\chi d\epsilon = g \left\{ 0,\ 0,\ \underbrace{\int_0^\pi p_\chi(\chi)\cos\chi d\chi}_{=0} \right\}$$

Avoid dependency on g from anisotropy: $\displaystyle\int_0^\pi p_\chi(\chi)\cos\chi d\chi = 0$

Avoid isotropy: $p_\chi(\chi) \neq \dfrac{\sin\chi}{2}$

For $F_{p_\chi}^{-1}$, use ansatz $p_\chi(\chi) = (C_0 + C_1\cos\chi + C_2\cos^2\chi + C_3\cos^3\chi)\sin\chi$

C_2 and C_3 satisfy constraint, C_0 and C_1 minimize $\displaystyle\int_0^\pi \left(p_\chi(\chi) - \bar{p}_\chi(\chi)\right)^2 d\chi$

Manufactured Cross Section (to evaluate $\langle \Delta \mathbf{v}_p^M \rangle_{\text{coll}}$ analytically)

With $\displaystyle\int_0^\pi p_\chi(\chi)\cos\chi \, d\chi = 0$,

$$\langle \Delta \mathbf{v}_p^M \rangle_{\text{coll}} = \frac{w\Delta t(N_p^{\text{cell}}-1)}{2\Delta V} \int_{-\infty}^{\infty} \sigma(g)g(\mathbf{v}_q - \mathbf{v}_p)f_\mathbf{v}(\mathbf{v}_q,t)d\mathbf{v}_q$$

If $\displaystyle\sigma(g) = \sum_{n=0}^{N_\sigma - 1}\sigma_n g^{2n-1}$,

$$\langle \Delta \mathbf{v}_p^M \rangle_{\text{coll}} = \frac{w\Delta t(N_p^{\text{cell}}-1)}{2\Delta V} \sum_{n=0}^{N_\sigma - 1}\sigma_n \mathbf{f}_n(\mathbf{v}_p,t),$$

where $\displaystyle\mathbf{f}_n(\mathbf{v}_p,t) = \int_{-\infty}^{\infty} g^{2n}(\mathbf{v}_q - \mathbf{v}_p)f_\mathbf{v}(\mathbf{v}_q,t)d\mathbf{v}_q$ can be computed analytically

Numerical and manufactured source terms balanced with manufactured p_χ and σ

Manufactured Solutions for the Poisson Equation

Manufacture the electric scalar potential $\phi^M(\mathbf{x}, t)$ and solve

$$\Delta\phi = -\frac{\rho}{\epsilon_0} + \Delta\phi^M + \frac{\rho^M}{\epsilon_0},$$

where $\Delta\phi^M$ is evaluated analytically

Evaluate manufactured charge density analytically as well:

$$\rho^M(\mathbf{x}, t) = q \int_{-\infty}^{\infty} f^M(\mathbf{x}, \mathbf{v}, t)d\mathbf{v} = q f_{\mathbf{x}}(\mathbf{x}, t)$$

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Discretization Parameters

- Discretize spatial domain with uniform cells of length Δx_i

- Numerically integrate with time step size Δt

- Represent physical particles with N_p computational particles

- Average the collision algorithm N_{avg} times

Refine these quantities together:

$$\Delta x_i \sim \Delta t \sim h,$$

$$N_p \sim h^{-q}, \qquad N_{\mathrm{avg}} \sim h^{-r}, \qquad N_{\mathrm{cell}} \sim h^{-3}, \qquad N_p^{\mathrm{cell}} \sim h^{-(q-3)}$$

Measure discrete L^1, L^2, and L^∞ norms, refine so error in L^∞ is at most $\mathcal{O}(h^2)$

- Collisional: $q = 7 \rightarrow N_p \sim h^{-7}$, $r = 5$, $N_{\mathrm{avg}} \sim h^{-5}$

- Collisionless: $q = 5 \rightarrow N_p \sim h^{-5}$

Error Sources

Field Quantities

- Trilinear basis functions: $\mathcal{O}(h^2)$ for ϕ, $\mathcal{O}(h^2)$ recovery for $\mathbf{E} = -\nabla\phi$

- Sampling error: $\mathcal{O}(N_p^{-1/2})$ for ϕ, $\mathcal{O}(N_p^{-1/2}h^{-1/2})$ for $\mathbf{E}$ (Tranquilli et al., 2022)

- Particle-position error: $\mathcal{O}(h^{p_x})$ for ϕ and $\mathbf{E}$

Particles (second-order-accurate velocity Verlet time integration)
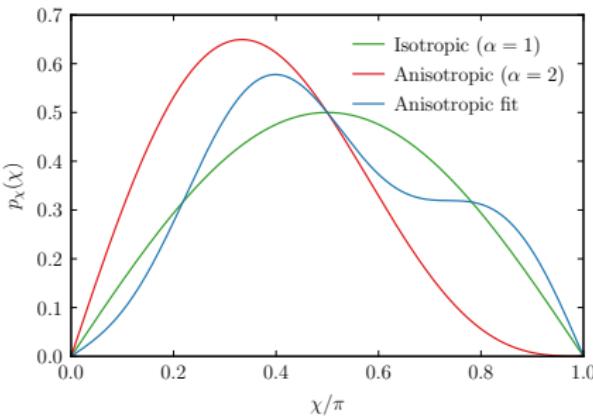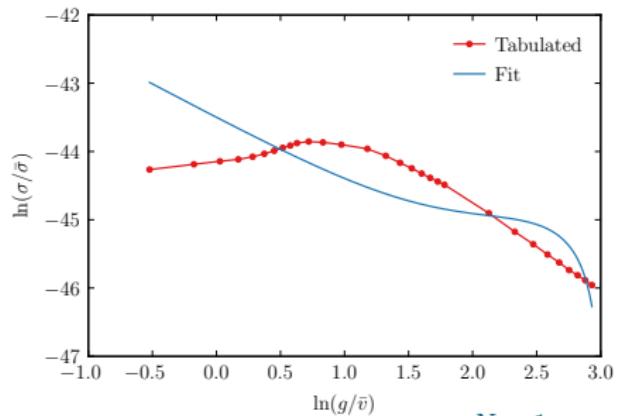
$$\mathbf{v}_p^{n+1/2} = \mathbf{v}_p^n \quad + \frac{1}{2}\left(\frac{q}{m}(\mathbf{E}_p - \mathbf{E}_p^M)^n \quad \Delta t + \langle\Delta\mathbf{v}_p\rangle_{\text{coll}}^n \quad - \langle\Delta\mathbf{v}_p^M\rangle_{\text{coll}}^n \quad + \Delta t(\dot{\mathbf{v}}_p^M)^n \quad\right) + \boldsymbol{\tau}_{\mathbf{v}_p}^n,$$

$$\mathbf{x}_p^{n+1} = \mathbf{x}_p^n \quad + \Delta t\left(\mathbf{v}_p + \dot{\mathbf{x}}_p^M - \mathbf{v}_p^M\right)^{n+1/2} + \boldsymbol{\tau}_{\mathbf{x}_p}^n,$$

$$\mathbf{v}_p^{n+1} = \mathbf{v}_p^{n+1/2} + \frac{1}{2}\left(\frac{q}{m}(\mathbf{E}_p - \mathbf{E}_p^M)^{n+1}\Delta t + \langle\Delta\mathbf{v}_p\rangle_{\text{coll}}^{n+1/2} - \langle\Delta\mathbf{v}_p^M\rangle_{\text{coll}}^{n+1/2} + \Delta t(\dot{\mathbf{v}}_p^M)^{n+1}\right) + \boldsymbol{\tau}_{\mathbf{v}_p}^{n+1/2}$$

- Per-step collision error: $\mathbf{e}_{\text{coll}_p}^n = \langle\Delta\mathbf{v}_p\rangle_{\text{coll}}^n - \langle\Delta\mathbf{v}_p^M\rangle_{\text{coll}}^n$

- Per-step Lorentz-force acceleration error: $\mathbf{e}_{\text{acc}_p}^n = \dfrac{q}{m}(\mathbf{E}_p - \mathbf{E}_p^M)^n\Delta t$

- Time-integration truncation errors: $\boldsymbol{\tau}_{\mathbf{v}_p}^n + \boldsymbol{\tau}_{\mathbf{v}_p}^{n+1/2} \sim \mathcal{O}(\Delta t^3)$, $\boldsymbol{\tau}_{\mathbf{x}_p}^n \sim \mathcal{O}(\Delta t^3)$

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Manufactured Cross Section and Anisotropy

- **Cross section** $\sigma(g) = \sum_{n=0}^{N_\sigma - 1} \sigma_n g^{2n-1}, \qquad N_\sigma = 3$

  - Log-scale least squares fitting of data from Itikawa *J. Phys. Chem. Ref.* (2009)
  - Goal is reasonable cross section, not exact fit

- **Anisotropy** $F_{p_\chi}^{-1}, \quad p_\chi(\chi) = (C_0 + C_1 \cos\chi + C_2 \cos^2\chi + C_3 \cos^3\chi)\sin\chi$

  - $\bar{p}_\chi(\chi) = \alpha \cos(\chi/2)^{2\alpha-1} \sin(\chi/2), \qquad 1 \le \alpha \le 2, \qquad$ (variable soft sphere)
  - Isotropic ($\alpha = 1$), anisotropic ($\alpha > 1$)

Manufactured Solutions and Discretizations
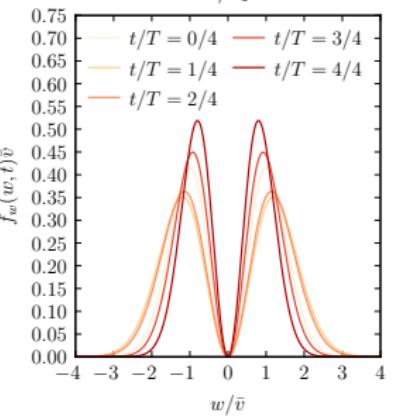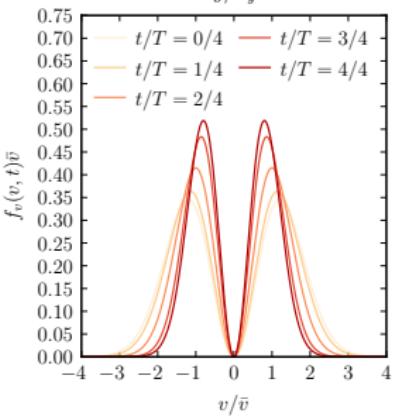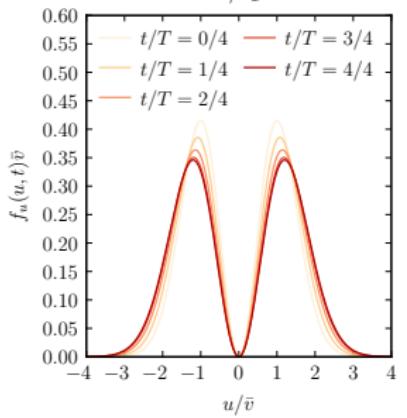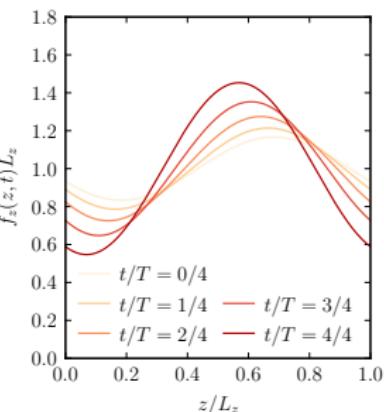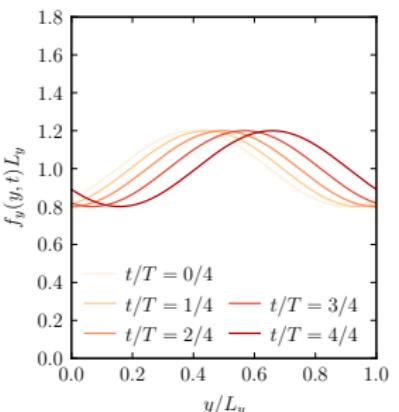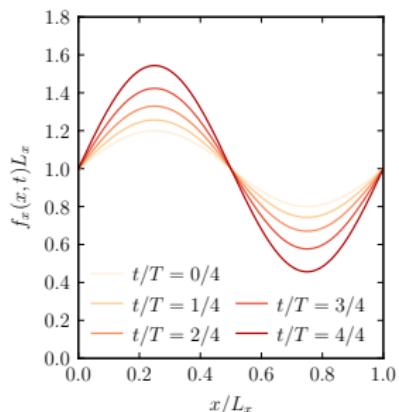
Particle distribution function: $f^M(\mathbf{x}, \mathbf{v}, t) = f_{\mathbf{x}}(\mathbf{x}, t) f_{\mathbf{v}}(\mathbf{v}, t),$
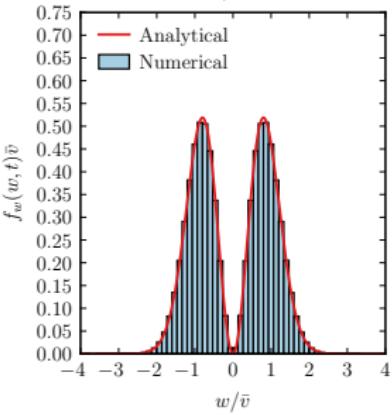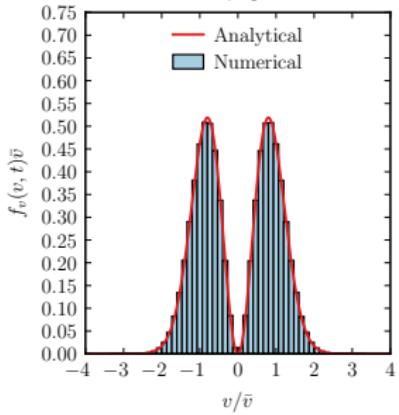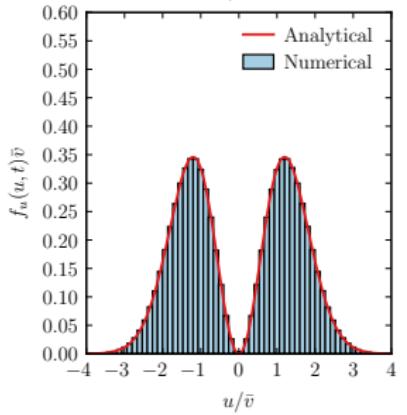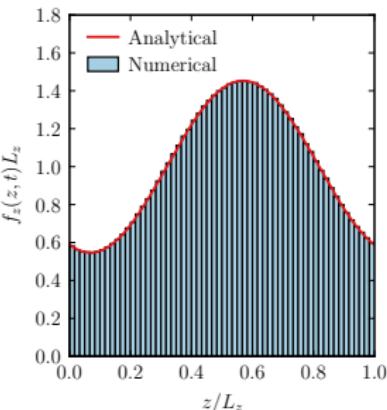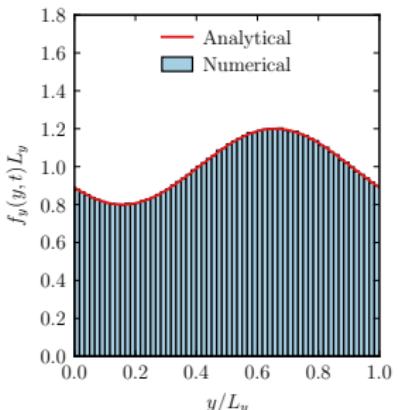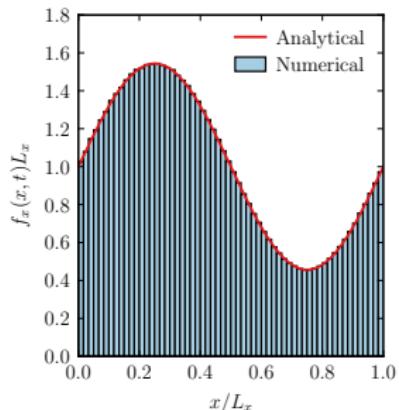
$$f_{\mathbf{x}}(\mathbf{x}, t) = N \prod_{i=1}^{3} f_{x_i}(x_i, t), \qquad f_{\mathbf{v}}(\mathbf{v}, t) = \prod_{i=1}^{3} f_{v_i}(v_i, t),$$

Potential: $\phi^M(\mathbf{x}, t) = \bar{\phi} e^{t/(2T)} \sin\left(2\pi\left[\dfrac{x}{L_x} - \dfrac{1}{7}\right]\right) \sin\left(2\pi\left[\dfrac{y}{L_y} - \dfrac{1}{5}\right]\right) \sin\left(2\pi\left[\dfrac{z}{L_z} - \dfrac{1}{3}\right]\right)$
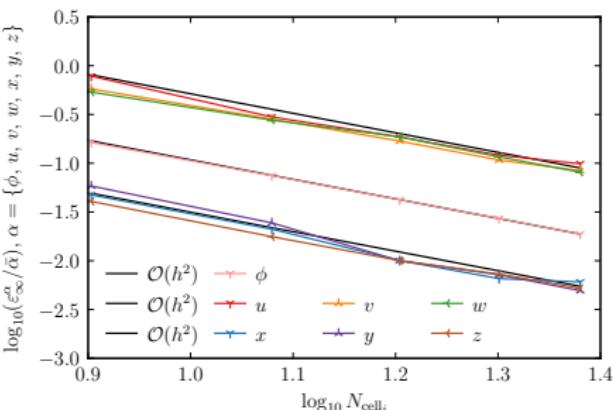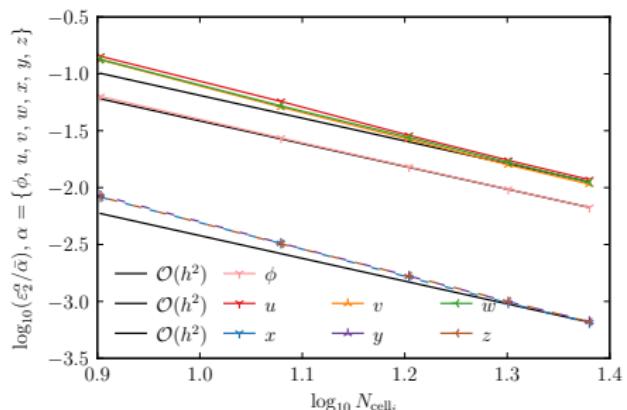
$\bar{v} = 10^6$ m/s, $L_{x_i} = 3/2$ m, $T = L_{x_i}/(10\bar{v}),$ $N = 10^{20}$ particles

$q = e,$ $m = 3 \times 10^8 m_e,$ $\bar{\phi} = 10^{10}$ V,   periodic BCs

Disc.	$T/\Delta t$	N_{cell_i}	N_{cell}	Collisional			Collisionless	
				N_{avg}	N_p	N_p/N_{cell}	N_p	N_p/N_{cell}
1	8	8	512	32	10,240	20.00	10,240	20.00
2	12	12	1,728	243	174,960	101.25	77,760	45.00
3	16	16	4,096	1,024	1,310,720	320.00	327,680	80.00
4	20	20	8,000	3,125	6,250,000	781.25	1,000,000	125.00
5	24	24	13,824	7,776	22,394,880	1620.00	2,488,320	180.00

Particle Distribution Function $f_M(\mathbf{x}, \mathbf{v}, t) = f_{\mathbf{x}}(\mathbf{x}, t) f_{\mathbf{v}}(\mathbf{v}, t)$
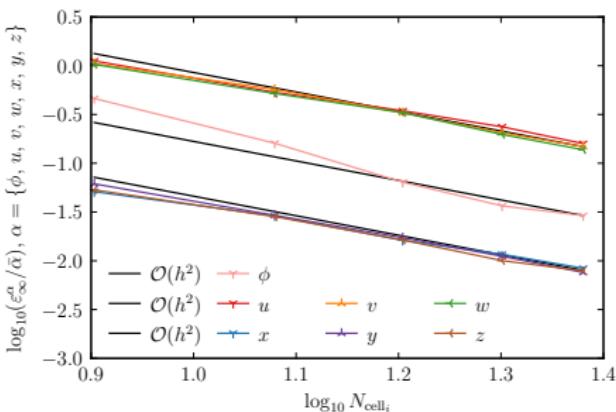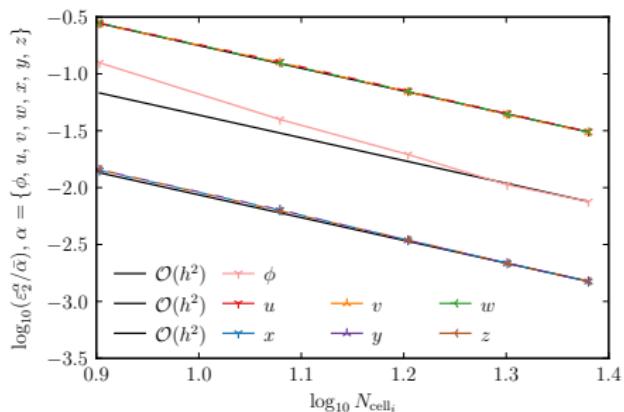
Particle Distribution Function $f_M(\mathbf{x}, \mathbf{v}, t) = f_{\mathbf{x}}(\mathbf{x}, t) f_{\mathbf{v}}(\mathbf{v}, t)$, $t = T$

## Error Convergence at $t = T$: Collisional, Uncoupled
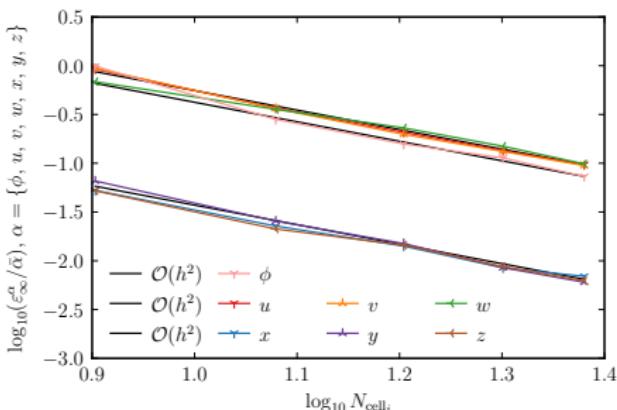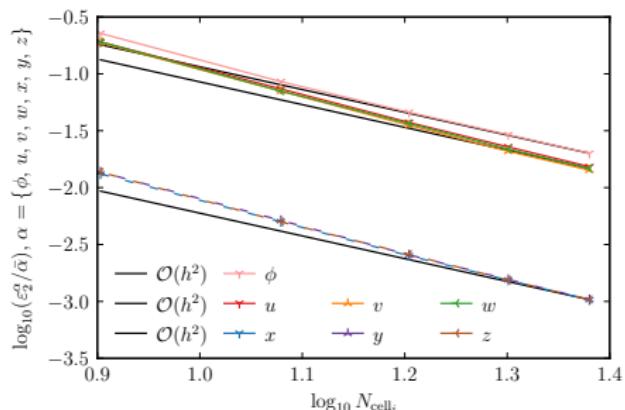
- Particles and fields **uncoupled** (single run)
  - Field does not affect particles ($q/m = 0$)
  - Particles do not affect field ($q = 0$)
- Particle error due to collisions, time integration
  - Velocity and position: $\mathcal{O}(h^2)$ in L^1, L^2, L^∞
- Field error due to basis functions
  - Potential: $\mathcal{O}(h^2)$ in L^1, L^2, L^∞

## Error Convergence at $t = T$: Collisional, One-Way Coupled
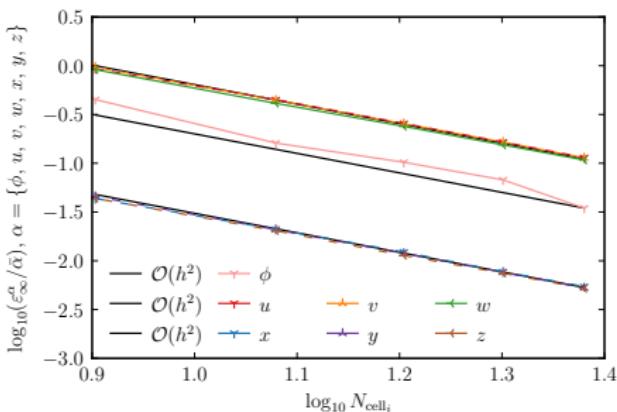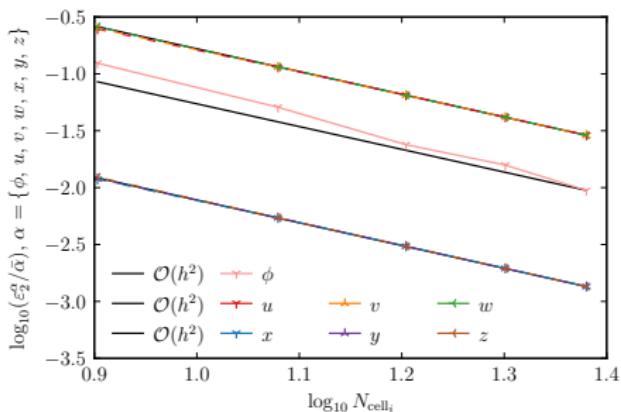
- Particles and fields **one-way coupled** (2 separate runs)
  - Field affects particles but is not affected by particles ($q/m \neq 0$, $q = 0$)
  - Particles affect field but are not affected by field ($q \neq 0$, $q/m = 0$)
- Particle error due to collisions, time integration, field basis function error
  - Velocity and position: $\mathcal{O}(h^2)$ in L^1, L^2, L^∞
- Field error due to basis functions, finite sampling, collisions, integration
  - Potential: $\mathcal{O}(h^2)$ in L^1, L^2, L^∞

Error Convergence at $t = T$: Collisional, Fully Coupled
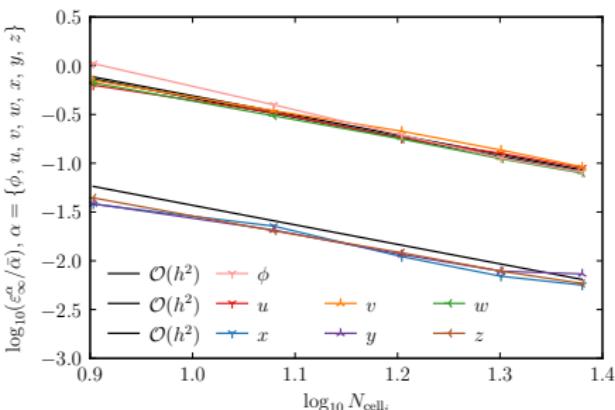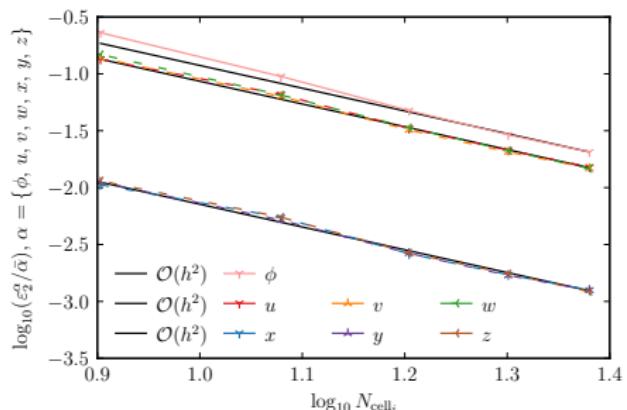
- Particles and fields **fully coupled** (single run)
  - Field affects particles ($q/m \neq 0$)
  - Particles affect field ($q \neq 0$)
- Particle error due to collisions, time integration, all field errors
  - Velocity and position: $\mathcal{O}(h^2)$ in L^1, L^2, L^∞
- Field error due to basis functions, finite sampling, all particle errors
  - Potential: $\mathcal{O}(h^2)$ in L^1, L^2, L^∞

Error Convergence at $t = T$: Collisionless, One-Way Coupled

- Particles and fields **one-way coupled** (2 separate runs)
  - Field affects particles but is not affected by particles ($q/m \neq 0$, $q = 0$)
  - Particles affect field but are not affected by field ($q \neq 0$, $q/m = 0$)
- Particle error due to time integration, field basis function error
  - Velocity and position: $\mathcal{O}(h^2)$ in L^1, L^2, L^∞
- Field error due to basis functions, finite sampling, particle time integration
  - Potential: $\mathcal{O}(h^2)$ in L^1, L^2, L^∞

Error Convergence at $t = T$: Collisionless, Fully Coupled

- Particles and fields **fully coupled** (single run)
  - Field affects particles ($q/m \neq 0$)
  - Particles affect field ($q \neq 0$)
- Particle error due to time integration, all field errors
  - Velocity and position: $\mathcal{O}(h^2)$ in L^1, L^2, L^∞
- Field error due to basis functions, finite sampling, all particle errors
  - Potential: $\mathcal{O}(h^2)$ in L^1, L^2, L^∞

Outline

- Introduction

- Equations

- Manufactured Solutions for Collisional Plasma Dynamics

- Error Analysis

- Numerical Examples

- Summary
  – Closing Remarks

Closing Remarks

- Presented code-verification approach for collisional plasma dynamics

- Added manufactured source terms to equations of motion (weights unmodified)

- Manufactured distribution function, potential, cross section, and anisotropy

- Computed manufactured source terms analytically, averaged collisions

- Ran single simulation per discretization

- Achieved expected convergence rates for collisional and collisionless cases

Questions?   bafreno@sandia.gov   brianfreno.github.io

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