

# CODE-VERIFICATION TECHNIQUES FOR AN ARBITRARY-DEPTH ELECTROMAGNETIC SLOT MODEL

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# Outline

- Introduction
- Governing Equations
- Code-Verification Approaches
- Numerical Examples
- Summary

## Outline

- Introduction
    - Electromagnetic Integral Equations
    - Verification and Validation
    - Error Sources
    - This Work
  - Governing Equations
  - Code-Verification Approaches
  - Numerical Examples
  - Summary

## Electromagnetic Integral Equations

- Are commonly used to model electromagnetic scattering and radiation
  - Relate surface current to incident electric and/or magnetic field
  - Discretize surface of electromagnetic scatterer with elements
  - Evaluate 4D reaction integrals over 2D test and source elements
  - Contain singular integrands when test and source elements are near

# Electromagnetic Aperture and Slot Models

- EM penetration occurs through openings of otherwise closed surfaces
  - Penetration may occur intentionally or unintentionally
  - Slot connects exterior surface of scatterer to interior surface of cavity
  - Model slot as wires carrying magnetic current located at apertures
    - Exterior surface interacts with exterior wire
    - Interior surface interacts with interior wire
    - Exterior and interior wires interact with each other
    - Exterior and interior surfaces do not interact directly

## Verification and Validation

Credibility of computational physics codes requires verification and validation

- **Validation** assesses how well models represent physical phenomena
    - Compare computational results with experimental results
    - Assess suitability of models, model error, and bounds of validity
  - **Verification** assesses accuracy of numerical solutions against expectations
    - *Solution verification* estimates numerical error for particular solution
    - *Code verification* verifies correctness of numerical-method implementation

## Code Verification

- Code verification most rigorously assesses rate at which error decreases
  - Error requires exact solution – usually unavailable
  - Manufactured solutions are popular alternative
    - Manufacture an arbitrary solution
    - Insert manufactured solution into governing equations to get residual term
    - Add residual term to equations to coerce solution to manufactured solution
  - For integral equations, few instances of code verification exist
  - Analytical differentiation is straightforward – analytical integration is not
  - Numerical integration is necessary, generally incurs an approximation error
  - Therefore, manufactured source term may have its own numerical error

# Error Sources in the Electromagnetic Integral Equations

### 3 sources of numerical error:

- **Domain discretization:** Representation of curved surfaces with planar elements
    - Second-order error for curved surfaces, no error for planar surfaces
    - Error reduced with curved elements
  - **Solution discretization:** Representation of solution or operators
    - Common in solution to differential, integral, and integro-differential equations
    - Finite number of basis functions to approximate solution
    - Finite samples queried to approximate underlying equation operators
  - **Numerical integration:** Quadrature
    - Analytical integration is not always possible
    - For well-behaved integrands,
      - Expect integration error at least same order as solution-discretization error
      - Less rigorously, error should decrease with more quadrature points
    - For (nearly) singular integrands, **monotonic convergence is not assured**

# This Work

## Isolate solution-discretization error

- Manufacture solution
- Eliminate numerical-integration error by manufacturing Green's function
- Mitigate contamination from discontinuity due to wire–surface interaction

## Isolate numerical-integration error

- Manufacture solution
- Cancel solution-discretization error using basis functions

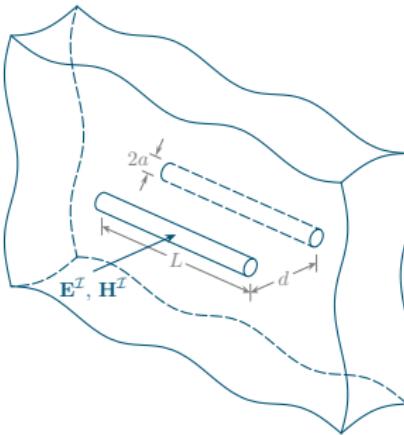
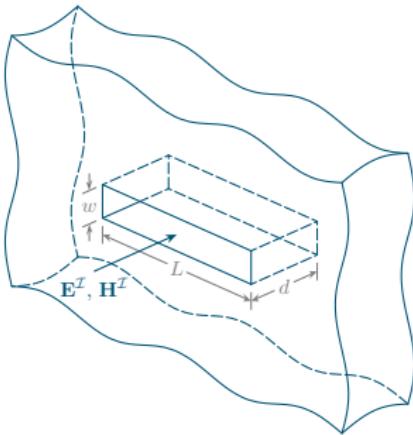
## Avoid domain-discretization error

- Consider only planar surfaces
- Previously provided approaches to account for domain-discretization error

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  - Overview
  - The Electric-Field Integral Equation
  - The Slot Equation
  - Discretization
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# Arbitrary-Depth Slot Model Overview



- Electromagnetic scatterer of arbitrary depth  $d$  encloses a cavity
- Exterior is connected to interior by rectangularly prismatic slot with  $L \gg w$  (left)
- Slot is replaced with two thin wires at apertures that carry magnetic current (right)
- Exterior and interior surfaces interact with wires, not each other
- Wires interact with each other through waveguide model
- EFIE solved on each surface, slot equation solved for wires

# The Electric-Field Integral Equation

In time-harmonic form,  $\mathbf{E}^S$  computed from  $\mathbf{J}$  and  $\mathbf{M}$

Scattered electric field       $\mathbf{E}^S(\mathbf{x}) = -\left(j\omega \mathbf{A}(\mathbf{x}) + \nabla \Phi(\mathbf{x}) + \frac{1}{\epsilon} \nabla \times \mathbf{F}(\mathbf{x})\right)$

Magnetic vector potential       $\mathbf{A}(\mathbf{x}) = \mu \int_{S'} \mathbf{J}(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') dS'$

Electric scalar potential       $\Phi(\mathbf{x}) = \frac{j}{\epsilon\omega} \int_{S'} \nabla' \cdot \mathbf{J}(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') dS'$

Electric vector potential       $\mathbf{F}(\mathbf{x}) = \epsilon \int_{S'} \mathbf{M}(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') dS'$

Green's function       $G(\mathbf{x}, \mathbf{x}') = \frac{e^{-jkR}}{4\pi R}, \quad R = |\mathbf{x} - \mathbf{x}'|$

Singularity when  $R \rightarrow 0$

$\mathbf{J}$  and  $\mathbf{M}$  are electric and magnetic surface current densities

$S' = S$  is surface of scatterer

$\mu$  and  $\epsilon$  are permeability and permittivity of surrounding medium

$k = \omega\sqrt{\mu\epsilon}$  is wavenumber

# The Electric-Field Integral Equation (continued)

Compute  $\mathbf{J}$  and  $\mathbf{M}$  from incident electric field  $\mathbf{E}^I$   $(\mathbf{n} \times (\mathbf{E}^S + \mathbf{E}^I)) = Z_s \mathbf{n} \times \mathbf{J})$

Discretize surface with triangles, approximate  $\mathbf{J}$  with RWG basis functions:

$$\mathbf{J}_h(\mathbf{x}) = \sum_{j=1}^{n_b} J_j \boldsymbol{\Lambda}_j(\mathbf{x})$$

Project EFIE onto linear, vector-valued RWG basis functions

Express  $\mathbf{M}$  in terms of filament magnetic current  $\mathbf{I}_m$

Discretize wire with bars, approximate  $\mathbf{I}_m$  with 1D basis functions:

$$\mathbf{I}_h(s) = \sum_{j=1}^{n_b^m} I_j \boldsymbol{\Lambda}_j^m(s)$$

# The Slot Equation

The magnetic current along the slot is modeled using a waveguide model:

$$\mathbf{s} \cdot (\mathbf{J}^\pm \times \mathbf{n}^\pm) + \frac{j\omega\epsilon}{2wL(k^2 - \beta_x^2)} \sum_{p=1}^{\infty} \beta_{y_p} \int_0^L \sin\left(\frac{p\pi s}{L}\right) \sin\left(\frac{p\pi s'}{L}\right) \times$$

$$(\pm [I_m^-(s') - I_m^+(s')] \tan(\beta_{y_p} d/2) + [I_m^+(s') + I_m^-(s')] \cot(\beta_{y_p} d/2)) ds' = 0$$

$$I_m(0) = I_m(L) = 0$$

$\mathbf{s}$  is the direction of the wire ( $\mathbf{I}_m = I_m(s)\mathbf{s}$ )

Superscripts – and + denote exterior and interior

Effective wire radius  $a$  obtained from conformal mapping using  $w$  and  $d$

$\beta_\alpha$  is the propagation constant in the  $\alpha$  direction

Project slot equation onto 1D basis functions

## Discretized Problem

Find  $\mathbf{J}_h \in \mathbb{V}_h$  and  $\mathbf{I}_h \in \mathbb{V}_h^m$ , such that

$$a_{\mathcal{E},\mathcal{E}}(\mathbf{J}_h, \boldsymbol{\Lambda}_i) + a_{\mathcal{E},\mathcal{M}}(\mathbf{I}_h, \boldsymbol{\Lambda}_i) = b_{\mathcal{E}}(\mathbf{E}^{\mathcal{T}}, \boldsymbol{\Lambda}_i) \quad \text{for } i = 1, \dots, n_b \quad (\text{EFIE})$$

$$a_{\mathcal{M},\mathcal{E}}(\mathbf{J}_h, \boldsymbol{\Lambda}_i^m) + a_{\mathcal{M},\mathcal{M}}(\mathbf{I}_h, \boldsymbol{\Lambda}_i^m) = 0 \quad \text{for } i = 1, \dots, n_b^m \quad (\text{Slot})$$

Evaluate EFIE on exterior and interior surfaces:  $n_b^- + n_b^+$  unknowns for  $\mathbf{J}_h$

Evaluate slot equation on exterior and interior wires:  $n_b^{m-} + n_b^{m+}$  unknowns for  $\mathbf{I}_h$

# Matrix–Vector Form

In matrix–vector form, solve for  $\mathcal{J}^h$ :

$$\mathbf{Z}\mathcal{J}^h = \mathbf{V}$$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{A}^- & \mathbf{0} & \mathbf{B}^- & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^+ & \mathbf{0} & \mathbf{B}^+ \\ \mathbf{C}^- & \mathbf{0} & \mathbf{D}_\sim^- & \mathbf{D}_\not\sim^- \\ \mathbf{0} & \mathbf{C}^+ & \mathbf{D}_\not\sim^+ & \mathbf{D}_\sim^+ \end{bmatrix}, \quad \mathcal{J}^h = \begin{Bmatrix} \mathbf{J}^{h-} \\ \mathbf{J}^{h+} \\ \mathbf{I}^{h-} \\ \mathbf{I}^{h+} \end{Bmatrix}, \quad \mathbf{V} = \begin{Bmatrix} \mathbf{V}^{\mathcal{E}-} \\ \mathbf{V}^{\mathcal{E}+} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix},$$

Impedance matrix                      Current vector                      Excitation vector

where

$$A_{i,j} = a_{\mathcal{E},\mathcal{E}}(\Lambda_j, \Lambda_i), \quad B_{i,j} = a_{\mathcal{E},\mathcal{M}}(\Lambda_j^m, \Lambda_i), \quad C_{i,j} = a_{\mathcal{M},\mathcal{E}}(\Lambda_j, \Lambda_i^m), \quad D_{i,j} = a_{\mathcal{M},\mathcal{M}}(\Lambda_j^m, \Lambda_i^m),$$

$$J_j^h = J_j, \quad I_j^h = I_j, \quad V_j^{\mathcal{E}} = b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}}, \Lambda_i)$$

More compactly:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}, \quad \mathcal{J}^h = \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix}, \quad \mathbf{V} = \begin{Bmatrix} \mathbf{V}^{\mathcal{E}} \\ \mathbf{0} \end{Bmatrix}$$

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  - Solution-Discretization Error
  - Numerical-Integration Error
  - Manufactured Green's Function
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# Manufactured Solutions for the EFIE

Continuous:  $r_{\mathcal{E}_i}(\mathbf{J}, \mathbf{I}_m) = a_{\mathcal{E}, \mathcal{E}}(\mathbf{J}, \boldsymbol{\Lambda}_i) + a_{\mathcal{E}, \mathcal{M}}(\mathbf{I}_m, \boldsymbol{\Lambda}_i) - b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}}, \boldsymbol{\Lambda}_i) = 0$

Discretized:  $r_{\mathcal{E}_i}(\mathbf{J}_h, \mathbf{I}_h) = a_{\mathcal{E}, \mathcal{E}}(\mathbf{J}_h, \boldsymbol{\Lambda}_i) + a_{\mathcal{E}, \mathcal{M}}(\mathbf{I}_h, \boldsymbol{\Lambda}_i) - b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}}, \boldsymbol{\Lambda}_i) = 0$

Method of manufactured solutions modifies discretized equations:

$$\mathbf{r}_{\mathcal{E}}(\mathbf{J}_h, \mathbf{I}_h) = \mathbf{r}_{\mathcal{E}}(\mathbf{J}_{MS}, \mathbf{I}_{MS})$$

$\mathbf{J}_{MS}$  and  $\mathbf{I}_{MS}$  are manufactured solutions,  $\mathbf{r}_{\mathcal{E}}(\mathbf{J}_{MS}, \mathbf{I}_{MS})$  is computed exactly

New Discretized:  $a_{\mathcal{E}, \mathcal{E}}(\mathbf{J}_h, \boldsymbol{\Lambda}_i) + a_{\mathcal{E}, \mathcal{M}}(\mathbf{I}_h, \boldsymbol{\Lambda}_i) = \underbrace{a_{\mathcal{E}, \mathcal{E}}(\mathbf{J}_{MS}, \boldsymbol{\Lambda}_i) + a_{\mathcal{E}, \mathcal{M}}(\mathbf{I}_{MS}, \boldsymbol{\Lambda}_i)}_{= b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}}, \boldsymbol{\Lambda}_i)}: \text{implement via } \mathbf{E}^{\mathcal{I}}$

$$\begin{aligned} \mathbf{E}^{\mathcal{I}}(\mathbf{x}) &= \frac{j}{\epsilon\omega} \int_{S'} [k^2 \mathbf{J}_{MS}(\mathbf{x}') \mathbf{G}(\mathbf{x}, \mathbf{x}') + \nabla' \cdot \mathbf{J}_{MS}(\mathbf{x}') \nabla \mathbf{G}(\mathbf{x}, \mathbf{x}')] dS' + Z_s \mathbf{J}_{MS}(\mathbf{x}) \\ &\quad - \frac{1}{4} (\mathbf{n}(\mathbf{x}) \times \mathbf{I}_{MS}(\mathbf{x})) \delta_{slot}(\mathbf{x}) + \frac{1}{4\pi} \int_0^L \mathbf{I}_{MS}(s') \times \int_0^{2\pi} \nabla' \mathbf{G}(\mathbf{x}, \mathbf{x}') d\phi' ds' \end{aligned}$$

MMS incorporated through  $\mathbf{E}^{\mathcal{I}}$  – no additional source term required

# Manufactured Solutions for the Slot Equation

Continuous:  $r_{\mathcal{M}_i}(\mathbf{J}_m, \mathbf{I}_m) = a_{\mathcal{M}, \mathcal{E}}(\mathbf{J}_m, \Lambda_i^m) + a_{\mathcal{M}, \mathcal{M}}(\mathbf{I}_m, \Lambda_i^m) = 0$

Discretized:  $r_{\mathcal{M}_i}(\mathbf{J}_h, \mathbf{I}_h) = a_{\mathcal{M}, \mathcal{E}}(\mathbf{J}_h, \Lambda_i^m) + a_{\mathcal{M}, \mathcal{M}}(\mathbf{I}_h, \Lambda_i^m) = 0$

Method of manufactured solutions modifies discretized equations:

$$\mathbf{r}_{\mathcal{M}}(\mathbf{J}_h, \mathbf{I}_h) = \mathbf{r}_{\mathcal{M}}(\mathbf{J}_{MS}, \mathbf{I}_{MS})$$

New Discretized:

$$a_{\mathcal{M}, \mathcal{E}}(\mathbf{J}_h, \Lambda_i^m) + a_{\mathcal{M}, \mathcal{M}}(\mathbf{I}_h, \Lambda_i^m) = \underbrace{a_{\mathcal{M}, \mathcal{E}}(\mathbf{J}_{MS}, \Lambda_i^m) + a_{\mathcal{M}, \mathcal{M}}(\mathbf{I}_{MS}, \Lambda_i^m)}_{= 0: \text{ no source term needed}}$$

Given  $\mathbf{J}_{MS}$ , solve for  $\mathbf{I}_m(s) = I_m(s)\mathbf{s}$  to avoid source term

$$J_s(s) = \sum_{q=1}^{\infty} J_{sq} \sin\left(\frac{q\pi s}{L}\right), \quad J_{sq} = \frac{2}{L} \int_0^L J_s(s) \sin\left(\frac{q\pi s}{L}\right) ds,$$

$$I_m(s) = \sum_{q=1}^{\infty} I_{mq} \sin\left(\frac{q\pi s}{L}\right), \quad I_{mq}^{\pm} = \frac{jw(k^2 - \beta_x^2)}{\beta_{yq}\omega\epsilon} ([J_{sq}^+ + J_{sq}^-] \tan(\beta_{yq}d/2) \mp [J_{sq}^+ - J_{sq}^-] \cot(\beta_{yq}d/2))$$

## Solution-Discretization Error

- Error due to basis-function approximations of solutions:

$$\mathbf{J}_h(\mathbf{x}) = \sum_{j=1}^{n_b} J_j \boldsymbol{\Lambda}_j(\mathbf{x}), \quad \mathbf{I}_h(s) = \sum_{j=1}^{n_b^m} I_j \boldsymbol{\Lambda}_j^m(s)$$

- Measured with discretization errors:  $\mathbf{e}_{\mathbf{J}} = \mathbf{J}^h - \mathbf{J}_n, \quad \mathbf{e}_{\mathbf{I}} = \mathbf{I}^h - \mathbf{I}_s$

$$\|\mathbf{e}_{\mathbf{J}}\| \leq C_{\mathbf{J}} h^{p_{\mathbf{J}}}, \quad \|\mathbf{e}_{\mathbf{I}}\| \leq C_{\mathbf{I}} h^{p_{\mathbf{I}}}$$

$J_{n_j}$ : component of  $\mathbf{J}_{MS}$  flowing from  $T_j^+$  to  $T_j^-$

$I_{s_j}$ : component of  $\mathbf{I}_{MS}$  flowing along  $\mathbf{s}$  at  $s_j$

$C$ : function of solution derivatives

$h$ : measure of mesh size

$p$ : order of accuracy

- Compute  $p$  from  $\|\mathbf{e}\|$  across multiple meshes (expect  $p = 2$  for these bases)
- Avoid numerical-integration error if integrating exactly

## Solution-Discretization Error: Discontinuity

- $\delta_{\text{slot}}$  introduces discontinuity due to wire interaction with surface
- Discontinuity impacts  $\mathbf{E}^{\mathcal{I}}$  for MMS
- Discontinuity will contaminate convergence studies:  $\mathcal{O}(h^2) \rightarrow \mathcal{O}(h)$
- Discontinuity denoted by  $\mathbf{B}_1$  in  $\mathbf{Z} = \begin{bmatrix} \mathbf{A} & (\mathbf{B}_1 + \mathbf{B}_2) \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$
- Since  $\mathbf{B}_1 = -\frac{1}{4}\mathbf{C}^T$ , use  $\mathbf{C}$  to cancel contribution from  $\mathbf{B}_1$  and modify  $\mathbf{E}^{\mathcal{I}}$ :

$$\mathbf{Z} = \begin{bmatrix} \mathbf{A} & (\cancel{\mathbf{B}_1} + \mathbf{B}_2) \\ \mathbf{C} & \mathbf{D} \end{bmatrix},$$

$$\mathbf{E}^{\mathcal{I}} = \frac{j}{\epsilon\omega} \int_{S'} [k^2 \mathbf{J}_{\text{MS}} G + \nabla' \cdot \mathbf{J}_{\text{MS}} \nabla G] dS' - \cancel{\frac{1}{4}(\mathbf{n} \times \mathbf{I}_{\text{MS}}) \delta_{\text{slot}}} + \frac{1}{4\pi} \int_0^L \mathbf{I}_{\text{MS}} \times \int_0^{2\pi} \nabla' G d\phi' ds' + Z_s \mathbf{J}_{\text{MS}}$$

- Correctness of  $\mathbf{B}_1$  is assessed by successful removal using  $\mathbf{C}$
- Correctness of  $\mathbf{C}$  is assessed through the mesh-convergence study

# Numerical-Integration Error

- Error due to quadrature integral evaluation  $(\cdot)^q$  on both sides of equations
- Measure numerical-integration error:

$$e_a = \mathcal{J}^H(\mathbf{Z}^q - \mathbf{Z})\mathcal{J}, \quad e_b = \mathcal{J}^H(\mathbf{V}^q - \mathbf{V}),$$

where  $\mathcal{J} = \begin{Bmatrix} \mathbf{J}_n \\ \mathbf{I}_s \end{Bmatrix}$

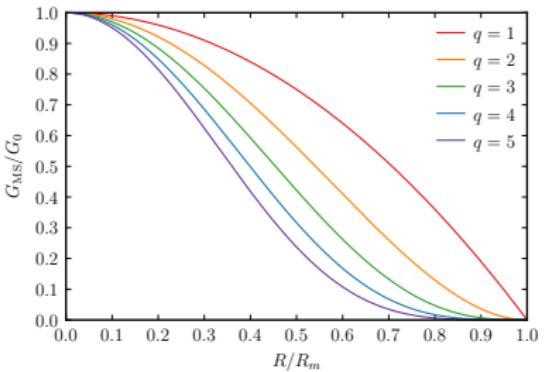
- Solution-discretization error is canceled
- $|e_a| \leq C_a h^{p_a}$  and  $|e_b| \leq C_b h^{p_b}$ 
  - $C$ : function of integrand derivatives
  - $p$ : order of accuracy of quadrature rules
- With multiple meshes, compute  $p$  from  $|e|$

# Manufactured Green's Function

Integrals with  $G$  cannot be computed analytically or, when  $R \rightarrow 0$ , accurately

Inaccurately computing integrals on either side contaminates convergence studies

Manufacture Green's function:  $G_{\text{MS}}(R) = G_0 \left(1 - \frac{R^2}{R_m^2}\right)^q$ ,  $R_m = \max_{\mathbf{x}, \mathbf{x}' \in S} R$  and  $q \in \mathbb{N}$



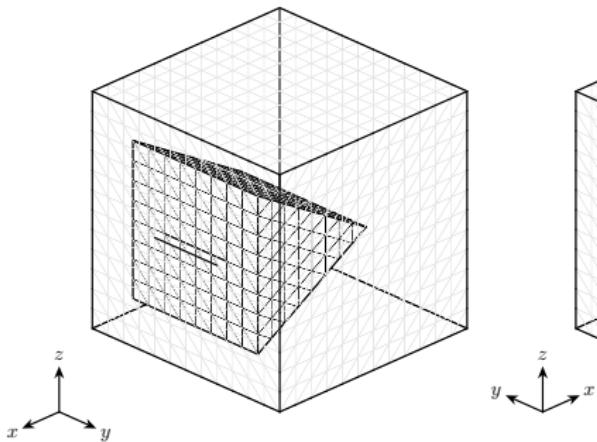
Reasoning:

- 1) Even powers of  $R$  permit integrals to be computed analytically
- 2)  $G_{\text{MS}}$  increases when  $R$  decreases, as with actual  $G$

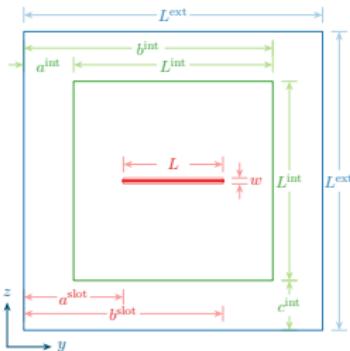
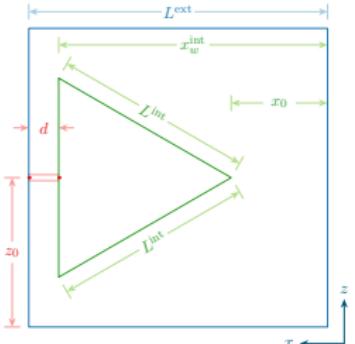
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  - Domain and Parameters
  - Manufactured Surface Current
  - Magnetic Current
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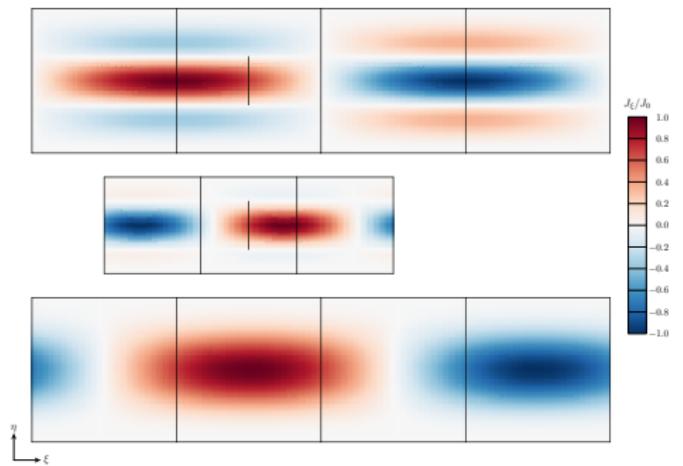
# Cubic Scatterer with a Triangularly Prismatic Cavity



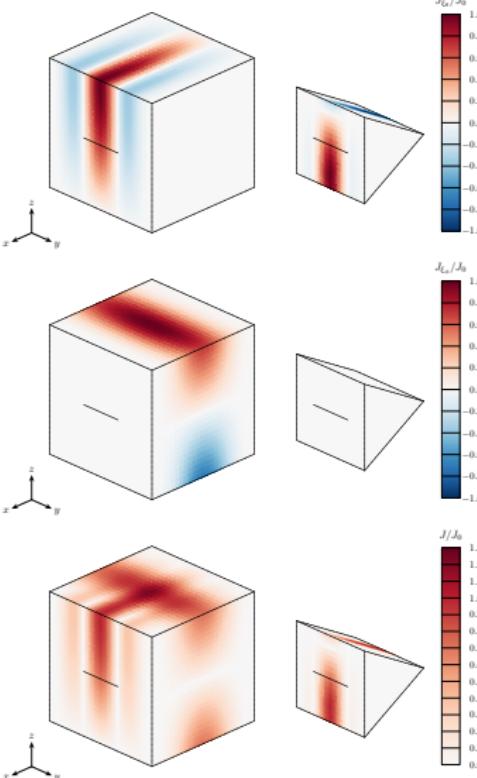
- $L^{\text{ext}} = 1 \text{ m}$ ,  $L^{\text{int}} = 2L^{\text{ext}}/3$ ,  $L = L^{\text{ext}}/3$ ,  $w = L^{\text{ext}}/50$
- $a^{\text{int}} = L^{\text{ext}}/6$ ,  $c^{\text{int}} = L^{\text{ext}}/6$ ,  $z_0 = L^{\text{ext}}/2$
- $\mu = \mu_0$ ,  $\epsilon = \epsilon_0$ ,  $k = 2\pi \text{ m}^{-1}$ ,  $\sigma_{\text{scatterer}} \gg \sigma_{\text{slot}} > \sigma_0 = 0$
- 3 depths:  $d_1 = L^{\text{ext}}/5$ ,  $d_2 = L^{\text{ext}}/10$ ,  $d_3 = L^{\text{ext}}/20$
- 2 Green's functions:  $G_1, G_2$



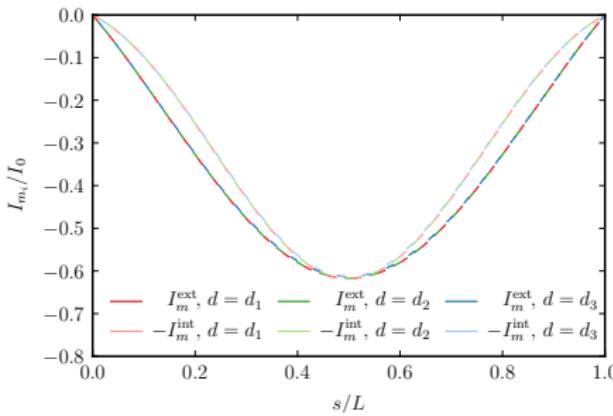
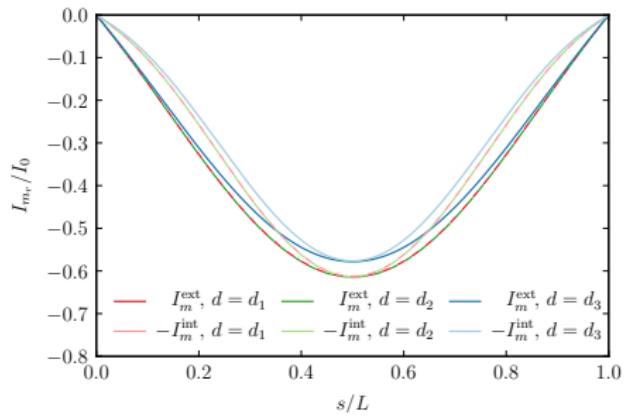
# Manufactured Surface Current



- Manufacture solutions for 2D strips of class  $C^2$
- Wrap strips around lateral surfaces of prisms
- Solutions are product of  $\xi$  and  $\eta$  dependencies
  - $\xi$  dependency: sine function with single period
  - $\eta$  dependency: linear combination of odd-harmonic sine functions
- Current flows along  $\xi$ ; at  $s = \{0, L\}$ ,  $J_\xi = 0$



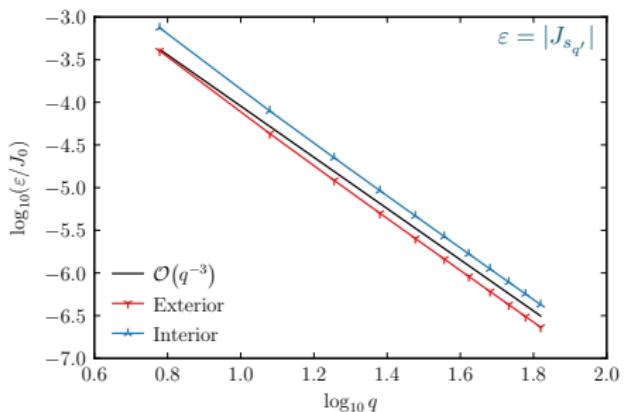
# Magnetic Current



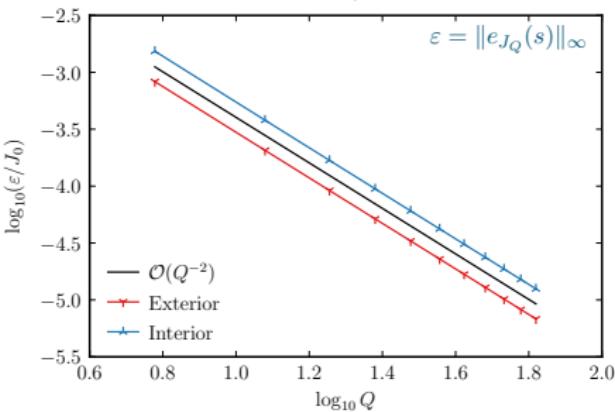
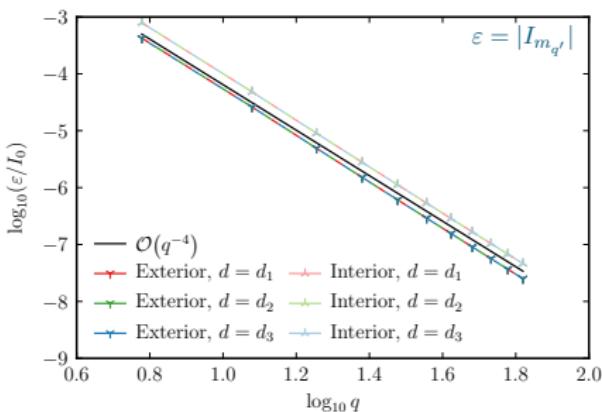
- For a given  $\mathbf{J}_{\text{MS}}$ ,  $\mathbf{I}_m(s) = I_m(s)\mathbf{s}$  is an infinite series that needs to be truncated

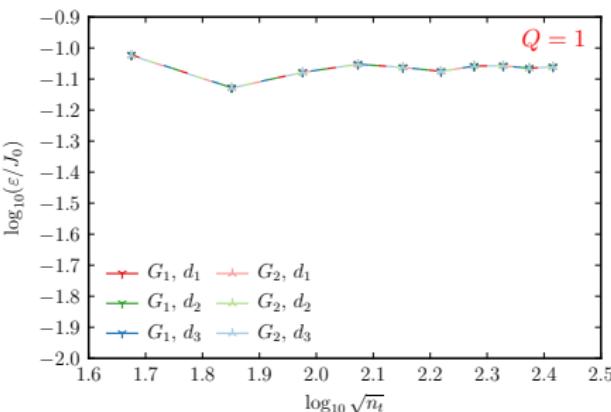
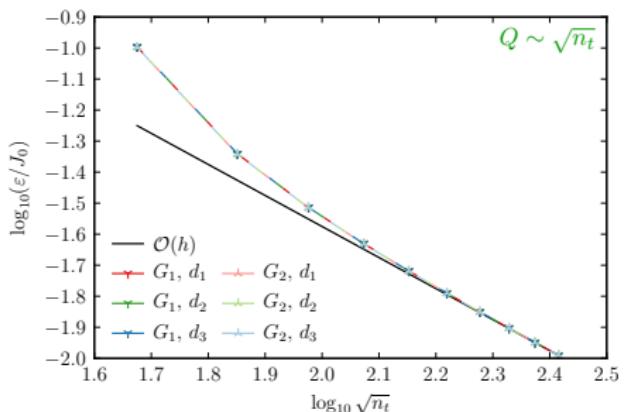
$$I_{m_Q}(s) = \sum_{q=1}^Q I_{m_{q'}} \sin\left(\frac{q'\pi s}{L}\right), \quad q' = 2q - 1$$

# Sine Series Convergence



- $J_s$  sine coefficients are  $\mathcal{O}(q^{-3})$
- $I_m$  sine coefficients are  $\mathcal{O}(q^{-4})$
- $\|e_{J_Q}(s)\|_\infty = \max_{s \in [0, L]} |J_{s_Q}(s) - J_s(s)|$  is  $\mathcal{O}(Q^{-2})$
- $\|e_{I_Q}(s)\|_\infty = \max_{s \in [0, L]} |I_{m_Q}(s) - I_m(s)|$  is  $\mathcal{O}(Q^{-3})$
- Set  $Q \sim \sqrt{n_t}$  to reduce truncation error faster
  - Basis-function error is  $\mathcal{O}(h^2)$
  - $I_m$  truncation error is  $\mathcal{O}(h^3)$

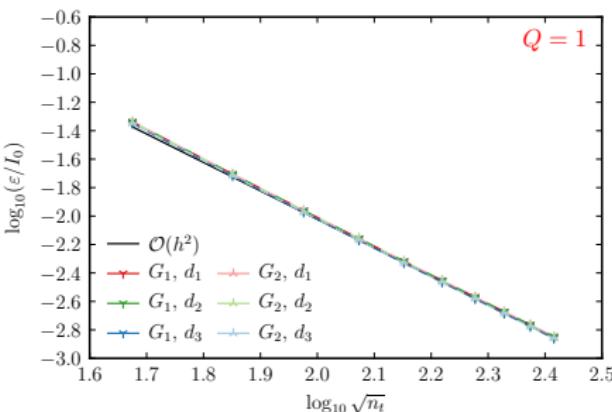
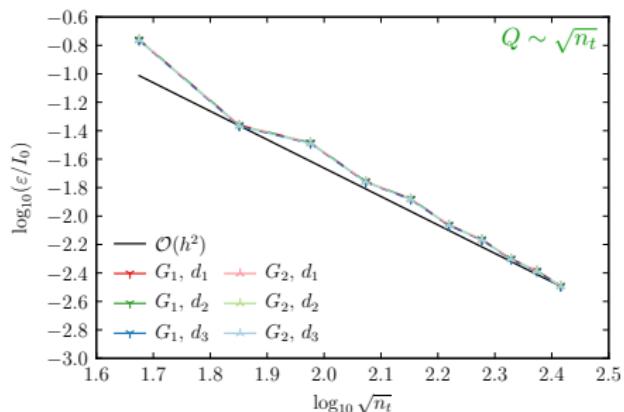


Solution-Discretization Error:  $\varepsilon = \|\mathbf{e}_J\|_\infty$  (with discontinuity)

- Discontinuity present:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\varepsilon \\ \mathbf{0} \end{Bmatrix}$$

- Convergence rates for  $Q = 1$  and  $Q \sim \sqrt{n_t}$  are  $\mathcal{O}(1)$  and  $\mathcal{O}(h)$

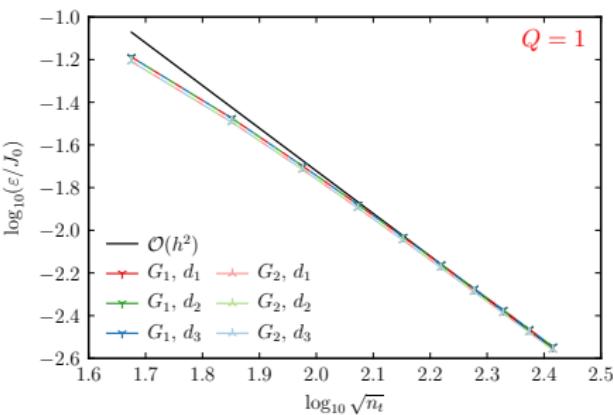
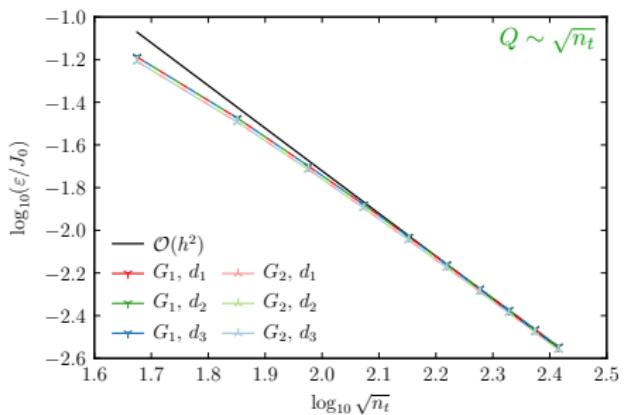
Solution-Discretization Error:  $\varepsilon = \|\mathbf{e}_I\|_\infty$  (with discontinuity)

- Discontinuity present:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\varepsilon \\ \mathbf{0} \end{Bmatrix}$$

- Convergence rates for  $Q = 1$  and  $Q \sim \sqrt{n_t}$  are  $\mathcal{O}(h^2)$

# Solution-Discretization Error: $\varepsilon = \|\mathbf{e}_J\|_\infty$ (without discontinuity)

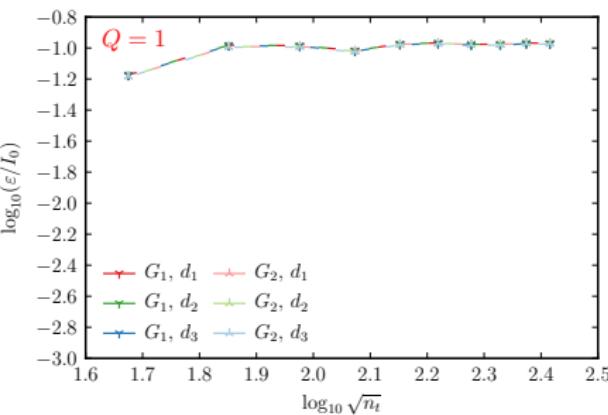
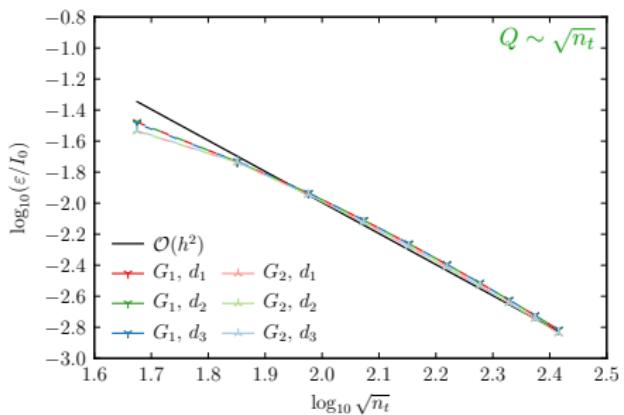


- Discontinuity removed from  $\mathbf{Z}$  using  $\mathbf{C}$ , corresponding MMS source term omitted in  $\mathbf{V}^\varepsilon$ :

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\varepsilon \\ \mathbf{0} \end{Bmatrix} \rightarrow \begin{bmatrix} \mathbf{A} & (\mathbf{B}_1 + \mathbf{B}_2) \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\varepsilon \\ \mathbf{0} \end{Bmatrix}$$

- Convergence rates for  $Q = 1$  and  $Q \sim \sqrt{n_t}$  are  $\mathcal{O}(h^2)$
- Correct implementation of  $\mathbf{B}_1$  suggested by its removal using  $\mathbf{C}$
- Correct implementation of  $\mathbf{C}$  suggested by expected convergence rates

# Solution-Discretization Error: $\varepsilon = \|\mathbf{e}_I\|_\infty$ (without discontinuity)



- Discontinuity removed from  $\mathbf{Z}$  using  $\mathbf{C}$ , corresponding MMS source term omitted in  $\mathbf{V}^\varepsilon$ :

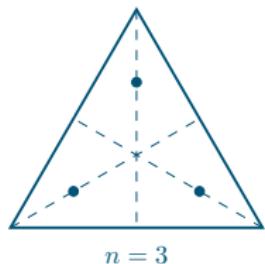
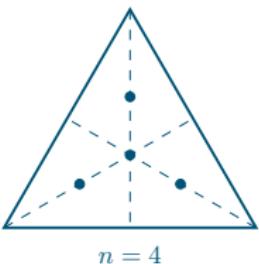
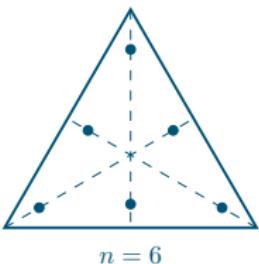
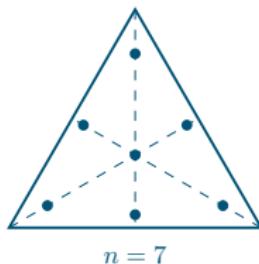
$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\varepsilon \\ \mathbf{0} \end{Bmatrix} \rightarrow \begin{bmatrix} \mathbf{A} & (\mathbf{B}_1 + \mathbf{B}_2) \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\varepsilon \\ \mathbf{0} \end{Bmatrix}$$

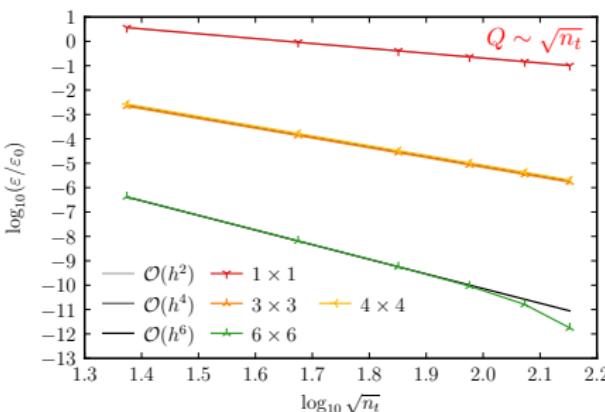
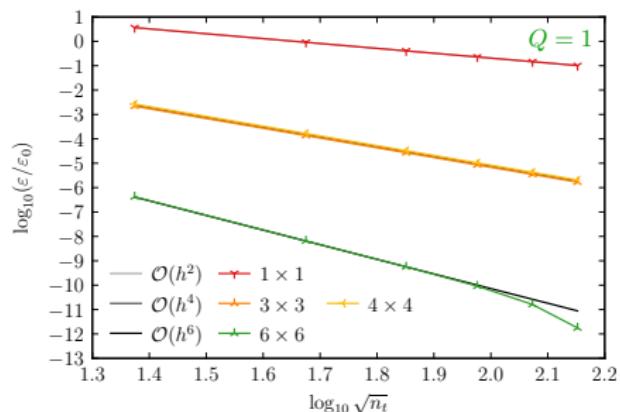
- Convergence rates for  $Q = 1$  and  $Q \sim \sqrt{n_t}$  are  $\mathcal{O}(h^2)$  and  $\mathcal{O}(1)$
- Correct implementation of  $\mathbf{B}_1$  suggested by its removal using  $\mathbf{C}$
- Correct implementation of  $\mathbf{C}$  suggested by expected convergence rates

# Numerical Integration

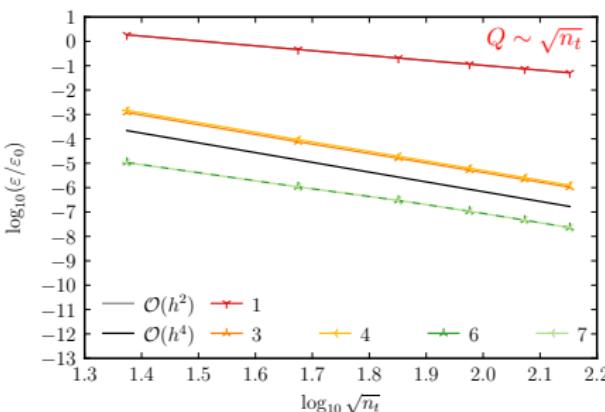
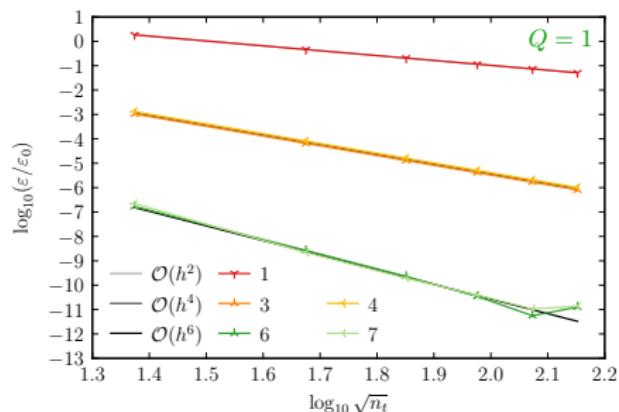
- Surface integrals evaluated using 2D triangle quadrature rules
- Wire integrals evaluated using 1D bar quadrature rules

Maximum integrand degree	Number of 2D points	Number of 1D points	Convergence rate
1	1	1	$\mathcal{O}(h^2)$
2	3	—	$\mathcal{O}(h^4)$
3	4	2	$\mathcal{O}(h^4)$
4	6	—	$\mathcal{O}(h^6)$
5	7	3	$\mathcal{O}(h^6)$

 $n = 3$  $n = 4$  $n = 6$  $n = 7$

Numerical-Integration Error:  $\varepsilon = |e_a|$  ( $G = G_2$ ,  $d = d_1$ )

- 2D points: [number for test integral]  $\times$  [number for source integral]
- 1D points: number of 1D points with same convergence rate as 2D points
- Convergence rates are as expected for  $Q = 1$  and  $Q \sim \sqrt{n_t}$

Numerical-Integration Error:  $\varepsilon = |e_b|$  ( $G = G_2$ ,  $d = d_1$ )

- 2D points: number for test integral
- 1D points: number of 1D points with same convergence rate as 2D points
- Convergence rates are as expected for  $Q = 1$
- Convergence rates are limited to  $\mathcal{O}(h^4)$  for  $Q \sim \sqrt{n_t}$  since  $\left| \int_0^L (I_{m_Q}(s) - I_m(s)) ds \right|$  is  $\mathcal{O}(Q^{-4})$

# Outline

- Introduction
- Governing Equations
- Code-Verification Approaches
- Numerical Examples
- Summary
  - Closing Remarks

## Closing Remarks

3 error sources in electromagnetic integral equations:

- **Domain-discretization error** – avoided
  - Considered planar surfaces
- **Solution-discretization error** – isolated
  - Manufactured  $\mathbf{J}$ , chose  $\mathbf{I}_m$  to avoid source term
  - Manufactured Green's function (to integrate exactly)
  - Removed discontinuity to measure convergence rates without contamination
  - Demonstrated implications of sine series truncation error on convergence
- **Numerical-integration error** – isolated
  - Removed solution-discretization error
  - Demonstrated implications of sine series truncation error on convergence

Achieved expected orders of accuracy

Questions?

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## Additional Information

- B. Freno, N. Matula, W. Johnson  
Manufactured solutions for the method-of-moments implementation of the EFIE  
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- B. Freno, N. Matula, J. Owen, W. Johnson  
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Code verification for practically singular equations  
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- B. Freno, N. Matula, R. Pfeiffer, V. Dang  
Code-verification techniques for an arbitrary-depth electromagnetic slot model  
*Engineering Analysis with Boundary Elements* (2025) [arXiv:2503.04004](https://arxiv.org/abs/2503.04004)

