Stable Probability Densities using FFTs and Newton-Cote Rules in Scipy

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August 14, 2018

Abstract

Stable distributions are an important type of heavy-tailed distribution. In Scipy 1.2 Jones u. a. (2001–) there are various implementations for calculating Stable probability densities including direct numerical integration and the rectangle rule FFT implementation as described by Mittnik u. a. (1999). FFT implementations are useful when estimating distribution parameters using MLE. An alternative FFT implementation is a Simpson's rule approach as described by Wang und Zhang (2008). Here we explore this approach along with analogues. We find optimal configurations and utilise symbolic algebra package Sympy Meurer u. a. (2017) to verify any derivations. The approach discussed here is applicable to other distributions with different characteristic functions.

1 Introduction

In what follows we use the following definition of stable characteristic function where we set location and scale to 0 and 1 respectively:

$$\phi(t) = e^{-(-i\beta\omega(t,\alpha)\operatorname{sign}(t)+1)|t|^{\alpha}} \tag{1}$$

where

$$\omega(t,\alpha) = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right) & \text{for } \alpha \neq 1 \\ -\frac{2\log(|t|)}{\pi} & \text{otherwise} \end{cases} \tag{2}$$

The density function then is:

$$f(x) = \frac{\int_{-\infty}^{\infty} \phi(t)e^{-itx}\,dt}{2\pi} \tag{3}$$

2 FFT method derivation

The FFT method comprises of the following steps.

1. Approximate the integral in (3) to a finite range:

$$\mathrm{f_L}(x) = \frac{\int_{-L}^{L} \phi(t)e^{-itx}\,dt}{2\pi} \tag{4}$$

   The relative error introduced by this truncation is derived in Wang und Zhang (2008) and is bounded below $2e^{-L}$.

2. The truncated interval is then divided into N subintervals of equal length $h = \frac{2L}{N}$. The segmented integral now taking the form:

$$\mathrm{f_L}\left(x\right) = \sum_{j=0}^{N-1} \frac{\int_{-L+hj}^{-L+h(j+1)} \phi(t)e^{-itx}\,dt}{2\pi} \tag{5}$$

3. Now each integral can be estimated using various methods. In Mittnik u. a. (1999) a rectangle rule with varying h is used. In Wang und Zhang (2008) a Simpsons rule is applied. Here we will explore variations of Simpsons rule with the n-degree approximation of f_L being $f_{L,n}$.

$$\mathrm{f_{L,n}}\left(x\right) = \sum_{j=0}^{N-1} \frac{h\sum_{k=0}^{n-1} C(n,k)\phi\left(t_{k,j}\right)e^{-it_{k,j}x}}{2\pi\sum_{k=0}^{n-1} C(n,k)} \tag{6}$$

   where $C(n,k)$ is a number from the Cotesian sequence and $t_{k,j} = -L + hj + \frac{hk}{n-1}$. See appendix for details.

The sum (6) can be expressed as Discrete Fourier Transform (DFT) which has fast machine implementations as developed by Cooley und Tukey (1965).

The Discrete Fourier Transform is defined as

$$\mathrm{DFT}\left(y_j, l\right) = \sum_{j=0}^{N-1} y_j e^{-\frac{2i\pi jl}{N}} \tag{7}$$

To utilise DFT in (6) we set $y_j = \phi\left(t_{k,j}\right)e^{i\pi j}$ and divide x into N intervals $x_l = \frac{\pi\left(-\frac{N}{2}+l\right)}{L}$, $l = 0, \ldots, N-1$. We also move the inner sum out:

$$\mathrm{f_{L,n}}\left(x_l\right) = \frac{h\sum_{k=0}^{n-1} C(n,k)\sum_{j=0}^{N-1} \phi\left(t_{k,j}\right)e^{-it_{k,j}x_l}}{2\pi\sum_{k=0}^{n-1} C(n,k)} \tag{8}$$

We now calculate a factor G needed to transform our original summand into a DFT summand. Taking summands from (8) and (7):

$$G\phi\left(t_{k,j}\right)e^{i\pi j}e^{-\frac{2i\pi jl}{N}} = \phi\left(t_{k,j}\right)e^{-it_{k,j}x_l} \tag{9}$$

Solving:

$$G = e^{\frac{i\left(-\pi Nj - Nt_{k,j}x_l + 2\pi jl\right)}{N}} \tag{10}$$

and by making substitutions $t_{k,j} = -L + hj + \frac{hk}{n-1}$, $h = \frac{2L}{N}$ and $x_l = \frac{\pi\left(-\frac{N}{2}+l\right)}{L}$ we obtain:

$$log(G) = \frac{i\pi Nk}{Nn - N} - \frac{i\pi N}{2} - \frac{2i\pi kl}{Nn - N} + i\pi l \tag{11}$$

We note that by choosing N divisible by 4 we can ignore term $\frac{i\pi N}{2}$. In fact, N will need to be a power of 2 to fully utilize the divide and conquer algorithm of Cooley und Tukey (1965).

$$log(G) = \frac{i\pi k}{n-1} + i\pi l - \frac{2i\pi kl}{N\left(n-1\right)} \tag{12}$$

Finally subtituting all back into (8)

$$\mathrm{f}_{\mathrm{L,n}}\left(x_l\right) = \frac{h \sum_{k=0}^{n-1} C(n,k) \sum_{j=0}^{N-1} \phi\left(t_{k,j}\right) e^{i\pi j} e^{-\frac{2i\pi jl}{N}} e^{\frac{i\pi k}{n-1}+i\pi l-\frac{2i\pi kl}{N(n-1)}}}{2\pi \sum_{k=0}^{n-1} C(n,k)} \tag{13}$$

and then substituting DFT expression back in:

$$\mathrm{f}_{\mathrm{L,n}}\left(x_l\right) = \frac{h \sum_{k=0}^{n-1} (-1)^l C(n,k)\,\mathrm{DFT}\left((-1)^j \phi\left(t_{k,j}\right), l\right) e^{\frac{i\pi k}{n-1}-\frac{2i\pi kl}{N(n-1)}}}{2\pi \sum_{k=0}^{n-1} C(n,k)} \tag{14}$$

2.1 Example $n = 3$

This is the rule derived by Wang und Zhang (2008).

$$\begin{aligned}
\mathrm{f}_{\mathrm{L,n}}\left(x_l\right) = \frac{h}{12\pi} \Big[& (-1)^l\,\mathrm{DFT}\left((-1)^j \phi(-L+hj), l\right) \\
& +4\,(-1)^l\, i\,\mathrm{DFT}\left((-1)^j \phi\left(-L+hj+\frac{h}{2}\right), l\right) e^{-\frac{i\pi l}{N}} \\
& -(-1)^l\,\mathrm{DFT}\left((-1)^j \phi(-L+hj+h), l\right) e^{-\frac{2i\pi l}{N}} \Big]
\end{aligned}$$

3 Analysis of results

Several configurations were run within a linear space of $\alpha \in (0,2]$ and $\beta \in [-1,1]$ parameters with step 0.1 and for $x \in [-10,10]$ with step 1. This range for x was considered suitable as most of the density lies within $[-4,4]$ according to Mittnik u. a. (1999). The output was compared against that of Nolan (2018) stablec executable with any nonsensical probability values from stablec being discarded. Each configuration was also timed. For the following results we selected 6 configurations chosen either as useful for comparison or because of useful properties.

Method	Description
Scipy-Best	Corresponds to current Scipy 1.2 'best' implementation utilizing direct numeric integration of CF or Zolaterev's piecewise approach.
Mitnik-17-3	FFT approach as described by Mitnik using 2^{17} points. Note that Mitnik originally recommended 2^{13} points.
Wang-17-3	FFT approach as described by Wang using 2^{17} points.
F7-17-7	General FFT approach with 7 degree simpsons integration using 2^{17} points and 7th degree interpolation.
F11-17-11	General FFT approach with 11 degree simpsons integration using 2^{17} points and 11th degree interpolation.
F11-17-11-G	General FFT approach with 11 degree simpsons integration using 2^{17} points and 11th degree interpolation. Calculated on GPU.

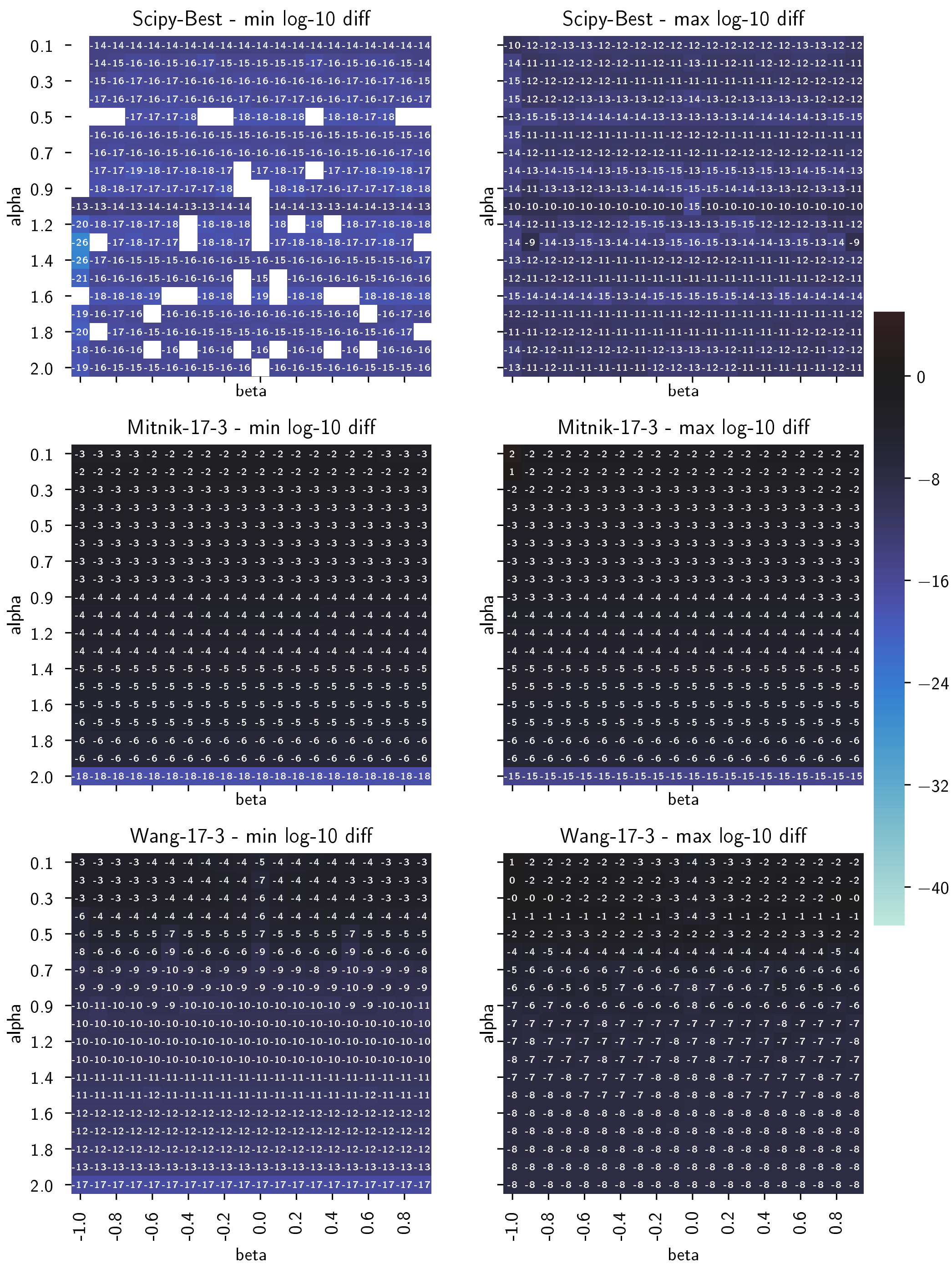

Figure 1: Base 10 logarithm of minimum and maximum difference for Scipy-Best, Mitnik-17-3 and Wang-17-3 against Nolan's stablec for a range of x values.

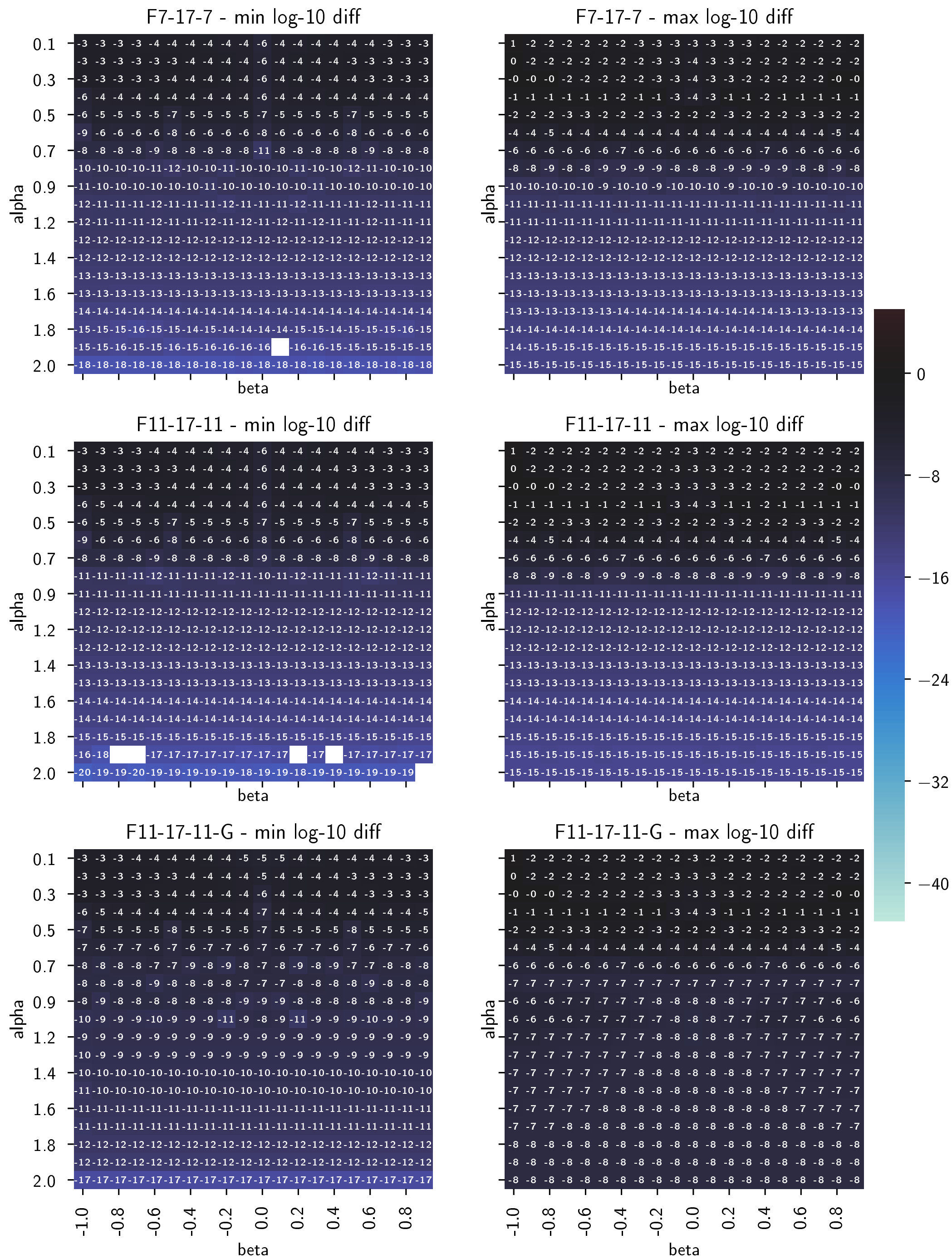

Figure 2: Base 10 logarithm of minimum and maximum difference for F7-17-7, F11-17-11 and F11-17-11-G against Nolan's stablec for a range of x values.

In figure 1 we can see Scipy's "best" method that uses direct numerical integration (DNI) shows consistent differences with Nolan's sample data to more than 10 decimal places or better. For Mitnik's approach we see accuracy is not very good although a surprising jump at $\alpha = 2$ which perhaps is because of the unusual selection of interval when approximating the integral. Wang's approach improves significantly over Mitnik's although it seems the spread between minimum and maximum differences is larger. Based on this information alone we suggest applying Wang's method instead of Mitnik's in Scipy's FFT approach as this would be a significant improvement.

In figure 2 we see how increasing the degree of the interpolating polynomial along the spline interpolation one can again significantly improve accuracy for the non-GPU category. The final pair of charts show the equivalent calculation using a GPU where the min/max spread appears larger most likely because of the GPUs 32-bit float architecture.

All figures demonstrate that the FFT methods lose accuracy usually for $\alpha < 1$ and as $\alpha \to 0$. For financial applications empirical studies show $\alpha \in [1.6, 1.9]$ (Mittnik u. a. (1999)) so FFT approximations will be useful. Also note that all FFT figures skip $\alpha = 1$ as this point is problematic for the FFT approximation. See the tables in appendix for detailed data.

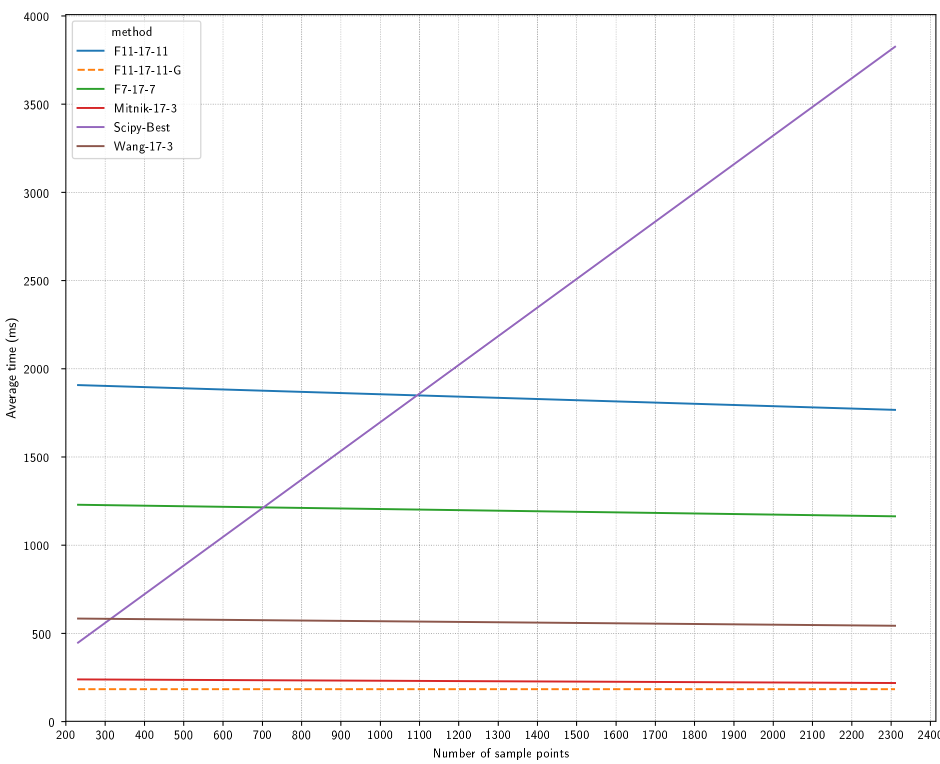

Figure 3: Average time in milliseconds for running several iterations for each method.

In figure 3 we explore average timings of each method. Benchmarks captured process_time() for each method on an Intel Core i7-6700HQ @ 2.6GHz CPU and a Nvidia GeForce GTX 970M GPU. All code ran using Scipy 1.2 beta and Numpy 1.14.5 compiled using OpenBLAS.

The GPU method F11-17-11-G is drawn with a dashed line to highlight that the timings are estimates based on real GPU calculations. Multiple kernels, one for each FFT summand, were sent to the GPU device for processing and data returned. The linear sum being calculated on the CPU. The estimate simply assumes that the linear calculation can be done on the GPU by reducing the device-memory copy time accordingly.

We note that processing time for Scipy's direct numerical approach "best" increases linearly with the number of sample points as expected whereas the FFT methods are near constant especially after calculating the initial pre-interpolation samples points. The location of the line intersections suggests thresholds where one might choose to use FFT over DNI for one's desired accuracy. Visual inspection suggests switching to Wang-17-3 when one requires 300+ points, to F7-17-7 if 700+ points required and F11-17-11 if one needs more than 1100 points.

4 Bibliography

References

[Cooley und Tukey 1965] COOLEY, James W. ; TUKEY, John W.: An algorithm for the machine calculation of complex Fourier series. In: *Mathematics of computation* 19 (1965), Nr. 90, S. 297–301

[Johnson 1914] JOHNSON, William W.: *On Cotesian Numbers: Their History, Computation, and Values to N*. 1914

[Jones u. a. 2001–] JONES, Eric ; OLIPHANT, Travis ; PETERSON, Pearu u. a.: *SciPy: Open source scientific tools for Python*. 2001–. – URL http://www.scipy.org/. – [Online; accessed <today>]

[Meurer u. a. 2017] MEURER, Aaron ; SMITH, Christopher P. ; PAPROCKI, Mateusz ; ČERTÍK, Ondřej ; KIRPICHEV, Sergey B. ; ROCKLIN, Matthew ; KUMAR, AMiT ; IVANOV, Sergiu ; MOORE, Jason K. ; SINGH, Sartaj ; RATHNAYAKE, Thilina ; VIG, Sean ; GRANGER, Brian E. ; MULLER, Richard P. ; BONAZZI, Francesco ; GUPTA, Harsh ; VATS, Shivam ; JOHANSSON, Fredrik ; PEDREGOSA, Fabian ; CURRY, Matthew J. ; TERREL, Andy R. ; ROUČKA, Štěpán ; SABOO, Ashutosh ; FERNANDO, Isuru ; KULAL, Sumith ; CIMRMAN, Robert ; SCOPATZ, Anthony: SymPy: symbolic computing in Python. In: *PeerJ Computer Science* 3 (2017), Januar, S. e103. – URL https://doi.org/10.7717/peerj-cs.103. – ISSN 2376-5992

[Mittnik u. a. 1999] MITTNIK, Stefan ; DOGANOGLU, Toker ; CHENYAO, David: Computing the probability density function of the stable Paretian distribution. In: *Mathematical and Computer Modelling* 29 (1999), Nr. 10-12, S. 235–240

[Nolan 2018] NOLAN, J. P.: *Stable Distributions - Models for Heavy Tailed Data*. Boston : Birkhauser, 2018. – In progress, Chapter 1 online at http://fs2.american.edu/jpnolan/www/stable/stable.html

[Wang und Zhang 2008] WANG, Li ; ZHANG, Ji-Hong: Simpson's rule based FFT method to compute densities of stable distribution. (2008)

5 Appendix

5.1 Simpsons method derivation and general approach

To estimate the integral $\int_a^b f(r)dr$, one divides the domain of f into $n-1$ regions and generates a polynomial that interpolates at each of the n points.

$$f_n(r) = \sum_{k=0}^{n-1} f_k \prod_{\substack{l=0}}^{\substack{n-1\\ l\neq k}} \frac{r-r_l}{r_k-r_l}$$

where $f_k = f(r_k)$. Evaluating the area of each rectangle segment.

$$\int_a^b f(r)dr \approx \int_a^b f_n(r)dr = [F_n(r)]_a^b$$

where F_n is the primitive integral of f_n and $r_k = a + k\frac{b-a}{n-1}$

5.2 Example $n = 3$

Otherwise known as Simpsons rule.

$$f_n(r) = \frac{f_0\left(r-r_1\right)\left(r-r_2\right)}{\left(r_0-r_1\right)\left(r_0-r_2\right)} + \frac{f_1\left(r-r_0\right)\left(r-r_2\right)}{\left(-r_0+r_1\right)\left(r_1-r_2\right)} + \frac{f_2\left(r-r_0\right)\left(r-r_1\right)}{\left(-r_0+r_2\right)\left(-r_1+r_2\right)}$$

$$\begin{aligned} F_n(r) = & \frac{r\left(-a^2bf_1 + a^2f_2m + ab^2f_1 - af_2m^2 - b^2f_0m + bf_0m^2\right)}{-a^2b + a^2m + ab^2 - am^2 - b^2m + bm^2} \\ & + \frac{r^2\left(a^2f_1 - a^2f_2 + b^2f_0 - b^2f_1 - f_0m^2 + f_2m^2\right)}{-2a^2b + 2a^2m + 2ab^2 - 2am^2 - 2b^2m + 2bm^2} \\ & + \frac{r^3\left(-af_1 + af_2 - bf_0 + bf_1 + f_0m - f_2m\right)}{-3a^2b + 3a^2m + 3ab^2 - 3am^2 - 3b^2m + 3bm^2} \end{aligned}$$

where $m = \frac{a+b}{2}$ for brevity. Finally,

$$\int_a^b f(r)dr \approx -\frac{(a-b)\left(f_0 + 4f_1 + f_2\right)}{6}$$

with $r_k = \left[a, \quad \frac{a}{2} + \frac{b}{2}, \quad b\right]$.

5.3 General result using Cotesian numbers

Rather than computing the above estimation each time one can refer to the general formula as described in Johnson (1914):

$$\int_a^b f(r)dr \approx \frac{b-a}{\sum_k C(n,k)} \sum_{k=0}^{n-1} C(n,k) f(r_k)$$

where $r_k = a + k\frac{b-a}{n-1}$ and $C(n,k)$ is the k-th in the n-th sequence of the Cotesian numbers.

5.4 Cotesian numbers $C(n,k)$

The following are taken from Johnson (1914).

k n	0	1	2	3	4	5	6	7	8	9	10
1	1										
2	1	1									
3	1	4	1								
4	1	3	3	1							
5	7	32	12	32	7						
6	19	75	50	50	75	19					
7	41	216	27	272	27	216	41				
8	751	3,577	1,323	2,989	2,989	1,323	3,577	751			
9	989	5,888	-928	10,496	-4,540	10,496	-928	5,888	989		
10	2,857	15,741	1,080	19,344	5,778	5,778	19,344	1,080	15,741	2,857	
11	16,067	106,300	-48,525	272,400	-260,550	427,368	-260,550	272,400	-48,525	106,300	16,067

5.5 Differences to Nolan's cstable executable

The following tables

alpha	method beta	F11-17-11	F11-17-11-G	F7-17-7	Mitnik-17-3	Scipy-Best	Wang-17-3
0.2	-1.0	6.9e-04	6.8e-04	7.2e-04	3.9e-03	0.0e+00	7.1e-04
	-0.8	5.7e-04	5.5e-04	5.9e-04	3.9e-03	9.1e-16	5.9e-04
	-0.6	4.3e-04	4.2e-04	4.5e-04	3.9e-03	7.3e-17	4.5e-04
	-0.4	2.9e-04	2.8e-04	3.1e-04	4.0e-03	2.4e-16	3.0e-04
	-0.2	1.5e-04	1.3e-04	1.5e-04	4.1e-03	7.8e-16	1.5e-04
	0.0	1.5e-06	8.0e-06	2.1e-06	4.3e-03	4.8e-16	3.1e-07
	0.2	1.5e-04	1.3e-04	1.5e-04	4.1e-03	7.8e-16	1.5e-04
	0.4	2.9e-04	2.8e-04	3.1e-04	4.0e-03	2.4e-16	3.0e-04
	0.6	4.3e-04	4.2e-04	4.5e-04	3.9e-03	7.3e-17	4.5e-04
	0.8	5.7e-04	5.5e-04	5.9e-04	3.9e-03	9.0e-16	5.9e-04
0.4	-1.0	1.2e-06	2.1e-06	1.7e-06	2.0e-03	0.0e+00	6.7e-07
	-0.8	6.3e-05	6.1e-05	6.6e-05	2.0e-03	3.8e-17	6.6e-05
	-0.6	1.1e-04	1.0e-04	1.1e-04	2.0e-03	5.6e-17	1.1e-04
	-0.4	1.1e-04	1.0e-04	1.1e-04	2.0e-03	5.0e-17	1.1e-04
	-0.2	6.6e-05	6.4e-05	6.9e-05	2.0e-03	3.8e-17	6.8e-05
	0.0	5.1e-07	4.1e-08	6.0e-07	2.0e-03	1.7e-16	9.7e-07
	0.2	6.6e-05	6.4e-05	6.9e-05	2.0e-03	2.9e-17	6.8e-05
	0.4	1.1e-04	1.0e-04	1.1e-04	2.0e-03	5.0e-17	1.1e-04
	0.6	1.1e-04	1.0e-04	1.1e-04	2.0e-03	8.2e-17	1.1e-04
	0.8	6.3e-05	6.1e-05	6.6e-05	2.0e-03	3.8e-17	6.6e-05
0.6	-1.0	1.0e-09	1.7e-07	9.6e-10	8.8e-04	0.0e+00	2.8e-08
	-0.8	5.6e-07	2.6e-07	5.8e-07	8.8e-04	2.2e-16	5.7e-07
	-0.6	8.9e-07	5.0e-07	9.2e-07	8.8e-04	2.8e-16	8.8e-07
	-0.4	9.0e-07	5.5e-07	9.3e-07	8.9e-04	5.6e-17	8.9e-07

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alpha	method beta	F11-17-11	F11-17-11-G	F7-17-7	Mitnik-17-3	Scipy-Best	Wang-17-3
	-0.2	5.5e-07	1.8e-07	5.7e-07	9.1e-04	3.3e-16	5.4e-07
	0.0	1.5e-08	9.8e-08	2.0e-08	9.2e-04	1.7e-16	1.8e-09
	0.2	5.5e-07	1.8e-07	5.7e-07	9.1e-04	6.7e-16	5.4e-07
	0.4	9.0e-07	5.5e-07	9.3e-07	8.9e-04	9.4e-16	8.9e-07
	0.6	8.9e-07	5.0e-07	9.2e-07	8.8e-04	2.8e-16	8.8e-07
	0.8	5.6e-07	2.6e-07	5.8e-07	8.8e-04	3.3e-16	5.7e-07
0.8	-1.0	1.0e-11	1.4e-08	3.5e-11	3.7e-04	0.0e+00	1.7e-09
	-0.8	2.8e-11	1.6e-08	7.5e-11	3.7e-04	4.6e-18	2.0e-09
	-0.6	5.9e-13	5.1e-10	3.1e-11	3.8e-04	1.1e-18	1.6e-09
	-0.4	1.4e-11	1.6e-08	6.0e-11	3.8e-04	1.7e-18	1.1e-09
	-0.2	2.6e-12	2.8e-08	7.1e-12	3.9e-04	3.5e-18	2.8e-10
	0.0	4.4e-11	3.8e-08	2.6e-10	3.9e-04	4.3e-18	1.0e-09
	0.2	2.6e-12	2.8e-08	7.1e-12	3.9e-04	3.5e-18	2.8e-10
	0.4	1.4e-11	1.6e-08	6.0e-11	3.8e-04	6.1e-18	1.1e-09
	0.6	5.9e-13	5.1e-10	3.1e-11	3.8e-04	1.1e-18	1.6e-09
	0.8	2.8e-11	1.6e-08	7.5e-11	3.7e-04	6.5e-19	2.0e-09
1.0	-1.0	nan	nan	nan	nan	6.5e-14	nan
	-0.8	nan	nan	nan	nan	4.0e-15	nan
	-0.6	nan	nan	nan	nan	3.2e-14	nan
	-0.4	nan	nan	nan	nan	5.1e-14	nan
	-0.2	nan	nan	nan	nan	4.0e-15	nan
	0.0	nan	nan	nan	nan	0.0e+00	nan
	0.2	nan	nan	nan	nan	3.9e-15	nan
	0.4	nan	nan	nan	nan	5.1e-14	nan
	0.6	nan	nan	nan	nan	3.2e-14	nan
	0.8	nan	nan	nan	nan	4.0e-15	nan
1.2	-1.0	9.1e-13	6.6e-10	2.6e-12	5.2e-05	8.0e-27	6.4e-11
	-0.8	9.1e-13	1.1e-09	3.5e-12	5.5e-05	3.5e-18	6.4e-11
	-0.6	9.1e-13	1.5e-09	3.1e-12	5.9e-05	4.3e-18	6.4e-11
	-0.4	9.1e-13	1.5e-09	3.4e-12	6.2e-05	0.0e+00	6.4e-11
	-0.2	9.1e-13	1.5e-09	3.6e-12	6.4e-05	8.7e-19	6.4e-11
	0.0	9.1e-13	1.5e-09	3.6e-12	6.5e-05	0.0e+00	6.4e-11
	0.2	9.1e-13	1.5e-09	3.6e-12	6.4e-05	8.7e-19	6.4e-11
	0.4	9.1e-13	1.5e-09	3.4e-12	6.2e-05	1.3e-18	6.4e-11
	0.6	9.1e-13	1.5e-09	3.1e-12	5.9e-05	4.3e-18	6.4e-11
	0.8	9.1e-13	1.1e-09	3.5e-12	5.5e-05	6.1e-18	6.4e-11
1.4	-1.0	1.5e-13	8.6e-11	6.4e-13	1.9e-05	2.0e-21	1.7e-11
	-0.8	1.5e-13	2.1e-10	6.7e-13	2.0e-05	5.6e-17	1.7e-11
	-0.6	1.5e-13	2.1e-10	6.1e-13	2.2e-05	4.2e-16	1.7e-11
	-0.4	1.5e-13	2.1e-10	6.1e-13	2.3e-05	2.5e-16	1.7e-11
	-0.2	1.5e-13	2.1e-10	6.8e-13	2.4e-05	1.1e-16	1.7e-11
	0.0	1.5e-13	2.1e-10	6.4e-13	2.4e-05	3.3e-16	1.7e-11
	0.2	1.5e-13	2.1e-10	6.8e-13	2.4e-05	1.1e-16	1.7e-11

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alpha	method beta	F11-17-11	F11-17-11-G	F7-17-7	Mitnik-17-3	Scipy-Best	Wang-17-3
	0.4	1.5e-13	2.1e-10	6.1e-13	2.3e-05	2.5e-16	1.7e-11
	0.6	1.5e-13	2.1e-10	6.1e-13	2.2e-05	4.2e-16	1.7e-11
	0.8	1.5e-13	2.1e-10	6.7e-13	2.0e-05	5.6e-17	1.7e-11
1.6	-1.0	1.6e-14	2.5e-11	8.2e-14	6.1e-06	1.5e-19	2.9e-12
	-0.8	1.6e-14	2.6e-11	6.1e-14	6.5e-06	2.8e-17	2.9e-12
	-0.6	1.6e-14	2.6e-11	5.9e-14	7.0e-06	0.0e+00	2.9e-12
	-0.4	1.6e-14	2.6e-11	7.5e-14	7.5e-06	2.8e-16	2.9e-12
	-0.2	1.6e-14	2.6e-11	7.6e-14	7.8e-06	1.1e-16	2.9e-12
	0.0	1.6e-14	2.6e-11	7.4e-14	7.9e-06	2.2e-16	2.9e-12
	0.2	1.6e-14	2.6e-11	7.7e-14	7.8e-06	1.1e-16	2.9e-12
	0.4	1.6e-14	2.6e-11	7.6e-14	7.5e-06	2.8e-16	2.9e-12
	0.6	1.6e-14	2.6e-11	6.0e-14	7.0e-06	0.0e+00	2.9e-12
	0.8	1.7e-14	2.6e-11	6.2e-14	6.5e-06	2.8e-17	2.9e-12
1.8	-1.0	4.4e-16	2.3e-12	2.0e-15	1.4e-06	6.7e-19	3.4e-13
	-0.8	9.8e-16	2.4e-12	8.0e-16	1.6e-06	2.2e-16	3.4e-13
	-0.6	6.4e-16	2.4e-12	7.2e-16	1.7e-06	0.0e+00	3.4e-13
	-0.4	6.9e-16	2.4e-12	3.5e-15	1.8e-06	0.0e+00	3.4e-13
	-0.2	7.2e-16	2.4e-12	3.4e-15	1.9e-06	1.8e-16	3.4e-13
	0.0	9.6e-16	2.4e-12	4.4e-15	1.9e-06	5.6e-17	3.4e-13
	0.2	8.3e-16	2.4e-12	3.1e-15	1.9e-06	1.8e-16	3.4e-13
	0.4	1.0e-15	2.4e-12	3.3e-15	1.8e-06	0.0e+00	3.4e-13
	0.6	9.6e-16	2.4e-12	1.2e-15	1.7e-06	0.0e+00	3.4e-13
	0.8	6.4e-16	2.4e-12	2.9e-16	1.6e-06	2.2e-16	3.4e-13
2.0	-1.0	1.5e-20	2.6e-17	1.1e-18	1.6e-18	1.0e-21	1.7e-17
	-0.8	1.9e-19	2.7e-17	1.2e-18	1.6e-18	1.0e-21	1.7e-17
	-0.6	1.5e-19	2.6e-17	1.2e-18	1.7e-18	1.0e-21	1.7e-17
	-0.4	6.0e-20	2.6e-17	1.1e-18	1.7e-18	1.0e-21	1.7e-17
	-0.2	1.1e-19	2.6e-17	1.0e-18	1.7e-18	1.0e-21	1.7e-17
	0.0	1.5e-19	2.6e-17	1.2e-18	1.7e-18	1.0e-21	1.7e-17
	0.2	3.2e-19	2.6e-17	1.5e-18	1.7e-18	1.0e-21	1.7e-17
	0.4	2.8e-19	2.6e-17	1.4e-18	1.7e-18	1.0e-21	1.7e-17
	0.6	9.5e-20	2.6e-17	1.3e-18	1.7e-18	1.0e-21	1.7e-17
	0.8	1.7e-19	2.5e-17	1.3e-18	1.6e-18	1.0e-21	1.7e-17

Table 3: Minimum difference from Nolan's stablec output for x.

alpha	method beta	F11-17-11	F11-17-11-G	F7-17-7	Mitnik-17-3	Scipy-Best	Wang-17-3
0.2	-1.0	2.0e+00	2.0e+00	2.0e+00	8.4e+00	1.0e-14	2.0e+00
	-0.8	9.5e-03	9.5e-03	1.0e-02	4.8e-03	3.4e-12	1.2e-02
	-0.6	7.7e-03	7.7e-03	8.1e-03	4.7e-03	4.8e-13	9.1e-03
	-0.4	5.8e-03	5.8e-03	5.6e-03	4.6e-03	7.8e-13	6.2e-03

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alpha	method beta	F11-17-11	F11-17-11-G	F7-17-7	Mitnik-17-3	Scipy-Best	Wang-17-3
	-0.2	3.7e-03	3.8e-03	3.0e-03	4.5e-03	2.2e-12	3.3e-03
	0.0	1.6e-03	1.7e-03	2.9e-04	4.3e-03	2.1e-13	2.1e-04
	0.2	3.7e-03	3.8e-03	3.0e-03	4.5e-03	2.2e-12	3.3e-03
	0.4	5.8e-03	5.8e-03	5.6e-03	4.6e-03	7.8e-13	6.2e-03
	0.6	7.7e-03	7.7e-03	8.1e-03	4.7e-03	4.8e-13	9.1e-03
	0.8	9.5e-03	9.5e-03	1.0e-02	4.8e-03	3.4e-12	1.2e-02
0.4	-1.0	6.4e-02	6.4e-02	6.4e-02	2.3e-03	1.2e-15	6.4e-02
	-0.8	1.0e-01	1.0e-01	1.0e-01	2.2e-03	3.1e-12	1.0e-01
	-0.6	8.4e-02	8.4e-02	8.4e-02	2.2e-03	2.3e-13	8.4e-02
	-0.4	7.6e-03	7.6e-03	7.6e-03	2.1e-03	6.9e-14	7.6e-03
	-0.2	8.5e-02	8.5e-02	8.5e-02	2.1e-03	9.7e-13	8.5e-02
	0.0	1.8e-04	1.8e-04	6.8e-05	2.1e-03	7.7e-15	8.7e-05
	0.2	8.5e-02	8.5e-02	8.5e-02	2.1e-03	9.7e-13	8.5e-02
	0.4	7.6e-03	7.6e-03	7.6e-03	2.1e-03	6.9e-14	7.6e-03
	0.6	8.4e-02	8.4e-02	8.4e-02	2.2e-03	2.3e-13	8.4e-02
	0.8	1.0e-01	1.0e-01	1.0e-01	2.2e-03	3.1e-12	1.0e-01
0.6	-1.0	8.1e-05	8.1e-05	8.1e-05	1.1e-03	1.3e-15	8.1e-05
	-0.8	9.8e-06	1.0e-05	1.2e-05	1.1e-03	3.5e-12	1.5e-05
	-0.6	8.2e-05	8.2e-05	8.2e-05	1.0e-03	2.9e-12	8.2e-05
	-0.4	1.7e-04	1.7e-04	1.7e-04	9.9e-04	4.1e-12	1.7e-04
	-0.2	2.4e-04	2.4e-04	2.4e-04	9.6e-04	6.0e-12	2.4e-04
	0.0	2.6e-04	2.6e-04	2.6e-04	9.3e-04	1.9e-12	2.6e-04
	0.2	2.4e-04	2.4e-04	2.4e-04	9.6e-04	6.0e-12	2.4e-04
	0.4	1.7e-04	1.7e-04	1.7e-04	9.9e-04	4.1e-12	1.7e-04
	0.6	8.2e-05	8.2e-05	8.2e-05	1.0e-03	2.9e-12	8.2e-05
	0.8	9.8e-06	1.0e-05	1.2e-05	1.1e-03	3.5e-12	1.5e-05
0.8	-1.0	9.7e-09	8.7e-08	9.9e-09	5.2e-04	4.6e-15	2.7e-06
	-0.8	1.9e-09	6.4e-08	9.4e-10	4.9e-04	5.0e-15	2.3e-06
	-0.6	3.2e-09	6.3e-08	4.9e-09	4.6e-04	1.3e-14	8.7e-07
	-0.4	1.9e-09	6.2e-08	1.7e-09	4.4e-04	1.6e-15	3.1e-07
	-0.2	3.3e-09	6.0e-08	3.1e-09	4.2e-04	1.0e-15	4.3e-07
	0.0	4.1e-09	5.9e-08	3.8e-09	4.0e-04	2.5e-13	2.4e-08
	0.2	3.3e-09	6.0e-08	3.1e-09	4.2e-04	1.0e-15	4.3e-07
	0.4	1.9e-09	6.2e-08	1.7e-09	4.4e-04	1.6e-15	3.1e-07
	0.6	3.2e-09	6.3e-08	4.9e-09	4.6e-04	1.3e-14	8.7e-07
	0.8	1.9e-09	6.4e-08	9.4e-10	4.9e-04	4.0e-15	2.3e-06
1.0	-1.0	nan	nan	nan	nan	1.5e-10	nan
	-0.8	nan	nan	nan	nan	1.7e-10	nan
	-0.6	nan	nan	nan	nan	2.0e-10	nan
	-0.4	nan	nan	nan	nan	1.2e-10	nan
	-0.2	nan	nan	nan	nan	1.2e-10	nan
	0.0	nan	nan	nan	nan	3.3e-16	nan
	0.2	nan	nan	nan	nan	1.2e-10	nan

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alpha	method beta	F11-17-11	F11-17-11-G	F7-17-7	Mitnik-17-3	Scipy-Best	Wang-17-3
	0.4	nan	nan	nan	nan	1.2e-10	nan
	0.6	nan	nan	nan	nan	2.0e-10	nan
	0.8	nan	nan	nan	nan	1.7e-10	nan
1.2	-1.0	9.1e-13	2.3e-07	3.6e-12	7.9e-05	5.5e-15	1.4e-07
	-0.8	9.1e-13	1.5e-07	3.6e-12	7.7e-05	1.7e-14	5.5e-08
	-0.6	9.1e-13	9.4e-08	3.6e-12	7.5e-05	1.9e-15	6.9e-08
	-0.4	9.1e-13	5.7e-08	3.8e-12	7.3e-05	2.7e-14	4.2e-08
	-0.2	9.1e-13	3.0e-08	4.4e-12	7.0e-05	4.5e-14	1.1e-07
	0.0	9.1e-13	1.7e-08	3.6e-12	6.8e-05	1.9e-16	5.7e-08
	0.2	9.1e-13	3.0e-08	4.4e-12	7.0e-05	4.5e-14	1.1e-07
	0.4	9.1e-13	5.7e-08	3.8e-12	7.3e-05	2.6e-14	4.2e-08
	0.6	9.1e-13	9.4e-08	3.6e-12	7.5e-05	1.9e-15	6.9e-08
	0.8	9.1e-13	1.5e-07	3.6e-12	7.7e-05	1.7e-14	5.5e-08
1.4	-1.0	1.5e-13	6.5e-08	7.3e-13	3.1e-05	1.3e-12	6.4e-08
	-0.8	1.5e-13	5.6e-08	8.0e-13	3.0e-05	2.3e-12	4.7e-08
	-0.6	1.5e-13	4.4e-08	7.2e-13	2.9e-05	4.3e-12	3.5e-08
	-0.4	1.5e-13	3.3e-08	7.7e-13	2.8e-05	3.2e-12	5.6e-08
	-0.2	1.5e-13	2.5e-08	7.3e-13	2.6e-05	3.8e-12	2.9e-08
	0.0	1.5e-13	2.0e-08	7.1e-13	2.5e-05	9.9e-12	2.2e-08
	0.2	1.5e-13	2.5e-08	7.3e-13	2.6e-05	3.8e-12	2.9e-08
	0.4	1.5e-13	3.3e-08	7.7e-13	2.8e-05	3.2e-12	5.6e-08
	0.6	1.5e-13	4.4e-08	7.2e-13	2.9e-05	4.3e-12	3.5e-08
	0.8	1.5e-13	5.6e-08	8.0e-13	3.0e-05	2.3e-12	4.7e-08
1.6	-1.0	1.7e-14	4.3e-08	9.0e-14	1.1e-05	2.5e-12	2.2e-08
	-0.8	1.7e-14	3.8e-08	9.9e-14	1.0e-05	3.4e-12	2.6e-08
	-0.6	1.7e-14	3.2e-08	1.0e-13	9.7e-06	4.9e-12	3.0e-08
	-0.4	1.7e-14	2.8e-08	9.6e-14	9.3e-06	7.1e-12	2.9e-08
	-0.2	1.7e-14	2.5e-08	9.3e-14	8.8e-06	4.5e-12	2.2e-08
	0.0	1.7e-14	2.2e-08	9.5e-14	8.4e-06	2.7e-12	1.4e-08
	0.2	1.7e-14	2.5e-08	9.3e-14	8.8e-06	4.5e-12	2.2e-08
	0.4	1.7e-14	2.8e-08	9.7e-14	9.3e-06	7.1e-12	2.9e-08
	0.6	1.7e-14	3.2e-08	1.0e-13	9.7e-06	4.9e-12	3.0e-08
	0.8	1.7e-14	3.8e-08	1.0e-13	1.0e-05	3.4e-12	2.6e-08
1.8	-1.0	1.4e-15	3.2e-08	1.1e-14	2.7e-06	3.1e-14	1.9e-08
	-0.8	1.5e-15	3.0e-08	1.1e-14	2.6e-06	4.9e-13	2.0e-08
	-0.6	1.6e-15	2.8e-08	9.3e-15	2.4e-06	1.1e-12	1.9e-08
	-0.4	1.8e-15	2.7e-08	9.2e-15	2.3e-06	7.1e-13	1.8e-08
	-0.2	1.6e-15	2.5e-08	8.6e-15	2.2e-06	6.7e-13	1.7e-08
	0.0	1.8e-15	2.4e-08	7.7e-15	2.1e-06	9.2e-14	1.5e-08
	0.2	1.4e-15	2.5e-08	8.1e-15	2.2e-06	6.7e-13	1.7e-08
	0.4	1.7e-15	2.7e-08	8.7e-15	2.3e-06	7.1e-13	1.8e-08
	0.6	1.4e-15	2.8e-08	9.2e-15	2.4e-06	1.1e-12	1.9e-08
	0.8	1.7e-15	3.0e-08	1.1e-14	2.6e-06	4.9e-13	2.0e-08

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alpha	method / beta	F11-17-11	F11-17-11-G	F7-17-7	Mitnik-17-3	Scipy-Best	Wang-17-3
2.0	-1.0	6.4e-16	2.6e-08	1.5e-15	9.0e-16	2.5e-15	1.3e-08
	-0.8	6.4e-16	2.6e-08	1.5e-15	9.0e-16	2.4e-15	1.3e-08
	-0.6	6.4e-16	2.6e-08	1.5e-15	9.0e-16	2.4e-15	1.3e-08
	-0.4	6.5e-16	2.6e-08	1.5e-15	9.0e-16	2.4e-15	1.3e-08
	-0.2	6.1e-16	2.6e-08	1.5e-15	9.0e-16	2.4e-15	1.3e-08
	0.0	6.0e-16	2.6e-08	1.5e-15	9.0e-16	2.4e-15	1.3e-08
	0.2	6.2e-16	2.6e-08	1.5e-15	9.0e-16	2.4e-15	1.3e-08
	0.4	6.2e-16	2.6e-08	1.5e-15	9.0e-16	2.4e-15	1.3e-08
	0.6	6.4e-16	2.6e-08	1.4e-15	9.0e-16	2.4e-15	1.3e-08
	0.8	6.0e-16	2.6e-08	1.4e-15	8.9e-16	2.4e-15	1.3e-08

Table 4: Maximum difference from Nolan's stablec output for x.

5.6 Calculation times

The following displays raw timing data for each method. The GPU timing are based on actual timings but adjusted to take factor in the reduction in device to

	method	F11-17-11		F11-17-11-G		F7-17-7		Mitnik-17-3		Scipy-Best		Wang-17-3	
	count	231	2310	231	2310	231	2310	231	2310	231	2310	231	2310
alpha	beta												
1.2	-1.0	1828	1781	180	180	1203	1187	234	218	281	2484	609	562
	-0.8	1875	1796	185	185	1218	1187	218	218	437	3781	468	546
	-0.6	1875	1781	183	183	1218	1250	234	234	406	3453	656	562
	-0.4	2031	1765	185	185	1296	1218	234	234	406	3421	609	546
	-0.2	1828	1781	185	185	1250	1203	234	218	375	3578	562	562
	0.0	1703	1640	166	166	1125	1078	218	203	468	3609	515	500
	0.2	1875	1765	183	183	1187	1203	250	218	437	3437	546	531
	0.4	1953	1734	177	177	1140	1171	218	203	406	3156	531	562
	0.6	1890	1781	183	183	1234	1203	250	218	437	3484	734	546
	0.8	1968	1765	183	183	1187	1203	265	203	437	3625	609	546
1.4	-1.0	1796	1781	183	183	1187	1187	218	234	234	2078	562	546
	-0.8	1890	1781	185	185	1234	1203	312	218	453	3468	656	546
	-0.6	2062	1781	183	183	1156	1187	250	234	406	3562	562	546
	-0.4	1890	1781	185	185	1218	1234	234	218	437	3781	578	546
	-0.2	1890	1781	185	185	1234	1218	171	250	390	3625	656	531
	0.0	1937	1625	166	166	1078	1125	203	218	406	3515	531	515
	0.2	1843	1734	185	185	1218	1187	296	218	421	3609	546	546
	0.4	1812	1750	180	180	1203	1171	234	203	437	3703	593	531
	0.6	1828	1796	185	185	1203	1171	234	218	421	3359	562	562
	0.8	2203	1734	183	183	1203	1187	218	218	453	3625	578	546
1.6	-1.0	2031	1796	194	194	1203	1140	234	218	312	2859	562	546
	-0.8	1796	1765	183	183	1234	1156	250	218	531	4875	531	546

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	method	F11-17-11		F11-17-11-G		F7-17-7		Mitnik-17-3		Scipy-Best		Wang-17-3	
	count	231	2310	231	2310	231	2310	231	2310	231	2310	231	2310
alpha	beta												
	-0.6	2000	1750	183	183	1156	1125	218	218	515	4421	546	562
	-0.4	1937	1750	180	180	1203	1109	281	218	515	4453	656	531
	-0.2	1796	1750	180	180	1328	1125	234	234	531	4625	578	546
	0.0	1750	1625	166	166	1187	1046	250	218	453	4156	593	500
	0.2	1906	1781	185	185	1171	1156	171	218	515	4375	578	546
	0.4	1890	1750	180	180	1203	1171	265	250	500	4593	531	531
	0.6	1890	1765	185	185	1203	1156	234	203	640	4609	640	515
	0.8	1953	1765	185	185	1187	1203	296	218	562	4859	593	546
1.8	-1.0	2062	1796	191	191	1281	1156	234	218	343	2796	578	546
	-0.8	1875	1812	183	183	1250	1156	265	203	671	5546	531	562
	-0.6	2031	1765	188	188	1171	1156	218	218	625	5203	578	546
	-0.4	1937	1828	188	188	1281	1171	281	218	562	5281	687	562
	-0.2	1859	1812	194	194	1203	1156	234	234	640	5046	562	546
	0.0	1703	1656	169	169	1125	1046	234	203	625	5031	546	515
	0.2	1875	1750	188	188	1312	1125	234	234	515	5140	687	546
	0.4	1953	1796	185	185	1328	1156	250	218	656	4937	734	546
	0.6	1937	1812	183	183	1203	1109	265	234	640	5406	578	546
	0.8	1875	1796	191	191	1328	1156	234	234	781	5500	562	531
2.0	-1.0	1828	1703	175	175	1234	1125	218	171	187	1484	531	531
	-0.8	2078	1750	177	177	1156	1140	218	218	140	1250	531	531
	-0.6	1890	1734	177	177	1281	1125	218	203	140	1312	546	515
	-0.4	1828	1734	175	175	1203	1093	218	203	171	1265	562	546
	-0.2	1906	1734	180	180	1250	1109	218	203	156	1265	546	531
	0.0	1703	1546	158	158	1109	1015	234	171	140	1203	531	484
	0.2	1953	1734	177	177	1171	1109	234	203	171	1281	531	515
	0.4	1828	1718	172	172	1156	1125	250	187	140	1281	578	531
	0.6	1906	1640	177	177	1312	1125	234	203	156	1265	515	515
	0.8	1859	1578	177	177	1156	1109	187	203	203	1250	531	515

Table 5: Average time in milliseconds to calculate a range of x values.