Cosserat Mechanics Theory Manual

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Contents

1 A 2D primer to Cosserat elasticity

Consider the infinitesimal area element shown in Figure 1.1. The force density (force/unit length in 2D, or force/unit area in 3D), and moment density on a surface with outward normal **n** are

$$\text{force density}_i = \sigma_{ij} n_j \quad \text{and} \quad \text{moment density}_i = m_{ij} n_j \ ,$$

respectively. Force and Moment balance lead to the equations of equilibrium

$$\begin{aligned} F_1 + \partial_1 \sigma_{11} + \partial_2 \sigma_{12} &= 0 \ , \\ F_2 + \partial_1 \sigma_{21} + \partial_2 \sigma_{22} &= 0 \ , \\ M + \partial_1 m_{31} + \partial_2 m_{32} + \sigma_{21} - \sigma_{12} &= 0 \ . \end{aligned}$$

These will be rederived in a more general setting in 3D in Section 2

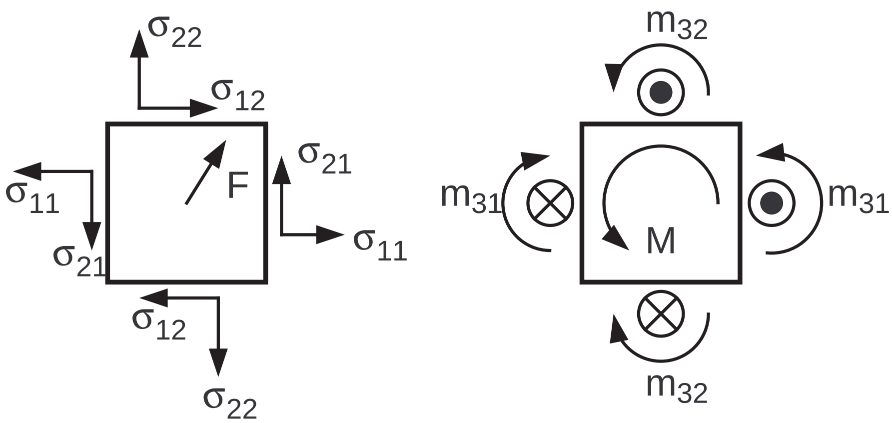

Figure 1.1: The infinitesimal area element in 2D Cosserat elasticity. In addition to the standard situation, shown on the left, moments are applied to the area element. M is a moment/unit area (or moment/unit volume in 3D), and m are moments/unit length (or moments/unit area in 3D). The filled-in-dot or crossed-dot is the standard pictorial method for representing a moment “coming out of or into the paper (respectively)” using the right-hand screw convention.

The micro-moments, M, are considered to produce micro-rotations in the material. One can think of the material being made up of grains which are stuck together with an elastic glue. The positions and orientations of the grains can be measured.

This is described mathematically as follows. The centres of mass of the grains move according to

$$x \to x + u_x \quad \text{and} \quad y \to y + u_y \ ,$$

in the standard way. This also rotates the grains relative to their initial orientation through an angle

$$\theta = \tfrac{1}{2}(\partial_x u_y - \partial_y u_x) \ .$$

For example, in Figure 1.2, the grains have moved according to

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \approx \begin{pmatrix} x-\theta y \\ y+\theta x \end{pmatrix} \ . \tag{1.1}$$

(The approximation holds for small θ.) This has rotated the grains through an angle of θ about the origin, and if standard elasticity were being used, this would be the end of story. However, now suppose that the individual grains can additionally rotate without deforming the surrounding continuum (called 'glue' above), as also depicted in Figure 1.2. Denote by θ^c the angle between the original orientation (before deformation by $\mathbf{u}$), and the final orientation. Then

$$\theta^{\mathrm{rel}} = \theta^c - \theta \ ,$$

is the relative rotation between how the grains are actually oriented, and how they would be oriented if only the deformation $\mathbf{u}$ had been used.

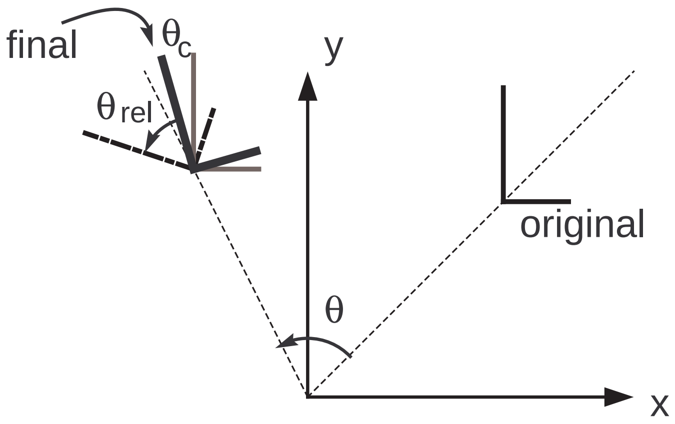

Figure 1.2: An original grain with orientation shown by the "L" attached to it sits on the right. This is deformed by $\mathbf{x} \rightarrow \mathbf{x}+\mathbf{u}$ and the result is shown by a dot-dashed "L". In this case it is a simple rotation through $\theta > 0$ (anticlockwise angles are positive). The grain is then rotated through angle θ^{rel}, which in this case is negative, to the final bold-faced "L" orientation. This final configuration is also obtained by deforming via $\mathbf{u}$ without any rotation of the "L" (resulting in the grey "L"), and then rotating through an angle of θ^c (positive in this case).

Natural measures of infinitesimal strain are

$$\begin{pmatrix} \gamma_{xx} \\ \gamma_{yy} \\ \gamma_{yx} \\ \gamma_{xy} \\ \partial_x \theta^{\mathrm{c}} \\ \partial_y \theta^{\mathrm{c}} \end{pmatrix} \equiv \begin{pmatrix} \partial_x u_x \\ \partial_y u_y \\ \partial_x u_y - \theta^{\mathrm{c}} \\ \partial_y u_x + \theta^{\mathrm{c}} \\ \partial_x \theta^{\mathrm{c}} \\ \partial_y \theta^{\mathrm{c}} \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} - \theta^{\mathrm{rel}} \\ \varepsilon_{xy} + \theta^{\mathrm{rel}} \\ \partial_x \theta^{\mathrm{c}} \\ \partial_y \theta^{\mathrm{c}} \end{pmatrix} \tag{1.2}$$

These are natural for the following reasons. Importantly, these measures transform covariantly under both $x \to -x$ and $y \to -y$. This is because under each of these $\theta^{\mathrm{rel}} \to -\theta^{\mathrm{rel}}$, which may be understood from a number of different perspectives, such as handedness change under these transformations. Hence the quantity γ_{ij} transfoms naturally. Moreover, performing a rigid rotation of the material (see Eq. (1.1)) gives

$$u_x = -\theta y \ , \quad u_y = \theta x \ , \quad \theta^{\mathrm{rel}} = 0 \ , \quad \theta^{\mathrm{c}} = \theta \ ,$$

which leaves γ_{ij} invariant. This is important, as such a rigid transformation should not cost any energy (it is simply a change of reference frame). This is why, for instance $\partial_y u_x + \theta^{\mathrm{c}}$ is more natural than $\partial_y u_x + \theta^{\mathrm{rel}}$.

2 Cosserat elasticity in 3D

The author would like to thank Ioannis Stefanou (ENPC, France) for his help in writing parts of this Chapter.

Consider an infinitesimal 3D volume element with body force per unit volume **F** and moment per unit volume **M**. Denote the stress tensor by σ_{ij} and the couple-stress pseudo tensor by m_{ij}, so that the force density and moment density acting on a surface with outward normal **n** are

$$\text{force density}_i = \sigma_{ij}n_j \quad \text{and} \quad \text{moment density}_i = m_{ij}n_j \ ,$$

respectively. The situation is shown graphically in 2D in Figure 1.1 and the 3D extension is obvious. Equilibrium of the volume V is equivalent to requiring

$$\begin{aligned} 0 &= -\int_V \rho\ddot{x}_i + \int_{\partial V} \sigma_{ij}n_j + \int_V F_i \ , \\ 0 &= -\int_V \rho\varepsilon_{ijk}x_j\ddot{x}_k + \tilde{\rho}\ddot{\theta}_c = \int_{\partial V} \left(\varepsilon_{ijk}x_j\sigma_{kl}n_l + m_{ij}n_j\right) + \int_V \left(\varepsilon_{ijk}x_jF_k + M_i\right) \ . \end{aligned}$$

These are standard equations, save for the inclusion of $\tilde{\rho}$ which parameterises the inertial effects of rotating, m_{ij} and M_i, and by using the divergence theorem and the arbitrary nature of V, the following equations of equilibrium are obtained:

$$\begin{aligned} 0 &= \nabla_j\sigma_{ij} + F_i \ , \\ 0 &= \nabla_j m_{ij} - \varepsilon_{ijk}\sigma_{jk} + M_i \ . \end{aligned} \tag{2.1}$$

2.1 Principal of virtual work and deformations

Propose that the stress and couple stress are conjugate to displacements u_i and rotations θ_i^c (which are kinematically independent) through the following principal of virtual work

$$\int_{\partial V} (\delta u_i\sigma_{ij}n_j + \delta\theta_i^c m_{ij}n_j) + \int_V (\delta u_i F_i + \delta\theta_i^c M_i) = \int_V \delta\mathcal{E} \ ,$$

where $\mathcal{E}$ is a small change in the potential energy of the medium, and δu_i and $\delta\theta_i^c$ are virtual displacements and rotations of the medium's particles. Using the divergence theorem, the arbitrary nature of V and the equation of force equilibrium results in

$$\delta\mathcal{E} = \sigma_{ij}\left(\nabla_j u_i + \varepsilon_{ijk}\delta\theta_k^c\right) + m_{ij}\nabla_j\delta\theta_i^c = \sigma_{ij}\delta\gamma_{ij} + m_{ij}\delta\kappa_{ij} \ ,$$

where

$$\gamma_{ij} = \nabla_j u_i + \varepsilon_{ijk}\theta_k^c \quad \text{and} \quad \kappa_{ij} = \nabla_j\theta_i^c \ . \tag{2.2}$$

Hence, the energy density is a function of γ_{ij} and κ_{ij}:

$$\mathcal{E} = \mathcal{E}(\gamma_{ij}, \kappa_{ij}) \ ,$$

with variation of the energy resulting in

$$\sigma_{ij} = \frac{\partial \mathcal{E}}{\partial \gamma_{ij}} \quad \text{and} \quad m_{ij} = \frac{\partial \mathcal{E}}{\partial \kappa_{ij}} \ .$$

The deformation measures given in Eq. (1.2) are evident. For instance the antisymmetric part,

$$\gamma_{[12]} = \tfrac{1}{2}(\gamma_{12} - \gamma_{21}) = \tfrac{1}{2}(\partial_2 u_1 - \partial_1 u_2) + \theta^c_3 \ .$$

The form of γ_{ij} given in 2D was motivated through consideration of covariance under certain coordinate transformations, while here its form appears through the 'natural' proposition for the principal of virtual work.

2.2 Hooke's law for an isotropic material

Since γ_{ij} is a proper tensor and κ_{ij} is a pseudo vector, the most general second-order form for the energy density is

$$\mathcal{E} = \tfrac{1}{4}\lambda(\mathrm{Tr}\gamma)^2 + \mu\gamma_{(ij)}\gamma_{(ij)} - \alpha\gamma_{[ij]}\gamma_{[ij]} + \tfrac{1}{4}\beta(\mathrm{Tr}\,\kappa)^2 + \gamma\kappa_{(ij)}\kappa_{(ij)} - \varepsilon\kappa_{[ij]}\kappa_{[ij]} \ .$$

Round brackets indicate symmetric parts while square brackets indicate antisymmetric parts: $\gamma_{(ij)} = \frac{1}{2}(\gamma_{ij} + \gamma_{ji})$ and $\gamma_{[ij]} = \frac{1}{2}(\gamma_{ij} - \gamma_{ji})$. The notation for the moduli is chosen so that λ and μ are the usual Lame constants, and I hope that in the following formulae, the modulus γ can be distinguished from the deformation measure γ_{ij}. The moduli α, β, γ and ε are used in this section *only*.

Varying the energy gives

$$\begin{aligned} \sigma_{ij} &= \lambda\delta_{ij}\mathrm{Tr}\gamma + 2\mu\gamma_{(ij)} + 2\alpha\gamma_{[ij]} \ , \\ m_{ij} &= \beta\delta_{ij}\mathrm{Tr}\,\kappa + 2\gamma\kappa_{(ij)} + 2\varepsilon\kappa_{[ij]} \ . \end{aligned} \tag{2.3}$$

The equations for Cosserat elasticity for an isotropic medium are Eqs. (2.1), (2.2) and (2.3). At each point on the boundary of the material the surface loads and couples, or the displacements and rotations must be specified.

The static equations in terms of u and θ^c are

$$\begin{aligned} 0 &= (\lambda + 2\mu)\nabla(\nabla\cdot u) - (\mu + \alpha)\nabla\times(\nabla\times u) + 2\alpha\nabla\times\theta^c + F \ , \\ 0 &= (\beta + 2\gamma)\nabla(\nabla\cdot\theta^c) - (\gamma + \varepsilon)\nabla\times(\nabla\times\theta^c) + 2\alpha\nabla\times u - 4\alpha\theta^c + M \ . \end{aligned}$$

3 3D layered Cosserat elasticity

Layered Cosserat elasticity is a generalisation of standard (non-Cosserat) transversely-isotropic elasticity. The theory considers the situation where the medium is comprised of a stack of flat layers perpendicular to the z direction. The layers have thickness h and are separated from each other by an interface material. The interface material has thickness h_i, Young's modulus E_i and shear modulus G_i. The theory describes the limit

$$h_\mathrm{i} \to 0 \ , \quad E_\mathrm{i} \to 0 \ , \quad E_\mathrm{i}/h_\mathrm{i} \to hk_n \quad \text{and} \quad G_\mathrm{i}/h_\mathrm{i} \to hk_s \ . \tag{3.1}$$

Here k_n is called the normal stiffness, and k_s is the shear stiffness. As will become more obvious below, the fact that $E_\mathrm{i}/h_\mathrm{i}$ depends on h is natural, since the thicker the layers, the less important the macroscopic effect on the thin interface. The Cosserat rotation around z, $\theta_z^\mathrm{c} = 0$ by definition.

In this situation, the equations are Eqs. (2.1), (2.2), the a-priori definition $\theta_z^\mathrm{c} = 0$, and the consitutive relationships

$$\sigma_{ij} = E_{ijkl}\gamma_{ij} \ , \tag{3.2}$$
$$m_{ij} = B_{ijkl}\kappa_{ij} \ . \tag{3.3}$$

E is called the elasticity tensor, and B is the bending rigidity tensor, and are written below in terms of the layer Young's modulus E, Poisson ratio ν, and the parameters h, k_n and k_s.

E and B may be derived by taking the Equation (3.1) limits of standard equations. For instance, a layered material consisting of homogeneous, isotropic layers in volume ratios α (for the "main", "thick" layers), and α_i (for the interface layers)

$$E_{xxxx} = \alpha E_{xxxx}^\mathrm{main} + \alpha_\mathrm{i} E_{xxxx}^\mathrm{i} - \alpha\alpha_\mathrm{i} \frac{\left(\frac{\nu}{1-\nu}E_{xxxx}^\mathrm{main} - \frac{\nu_\mathrm{i}}{1-\nu_\mathrm{i}}E_{xxxx}^\mathrm{i}\right)^2}{E_{xxxx}^\mathrm{main}E_{xxxx}^\mathrm{i}} E_{zzzz} \ , \tag{3.4}$$

$$E_{xxzz} = \left(\alpha\frac{\nu}{1-\nu} + \alpha^\mathrm{i}\frac{\nu^\mathrm{i}}{1-\nu^\mathrm{i}}\right)E_{zzzz} \ , \tag{3.5}$$

$$E_{zzzz} = \left(\frac{\alpha}{E_{xxxx}^\mathrm{main}} + \frac{\alpha^\mathrm{i}}{E_{xxxx}^\mathrm{i}}\right)^{-1} \ . \tag{3.6}$$

Here

$$E_{xxxx}^\mathrm{main} = E\frac{1-\nu}{(1+\nu)(1-2\nu)} \ , \tag{3.7}$$

which is a standard equation from non-Cosserat elasticity written in terms of the "main" layers' Young's modulus E and Poisson's ratio ν. Substituting $\alpha = 1$, $\alpha_\mathrm{i} = h_\mathrm{i}/(h+h_\mathrm{i})$, and taking the limits of Equation (3.1), yields

$$
\begin{aligned}
E_{0000} = E_{1111} &= \frac{E}{1-\nu^2 - \frac{\nu^2(1+\nu)^2}{1-\nu^2+E/(hk_n)}} \ , \\
E_{0011} = E_{1100} &= \frac{\nu}{1-\nu} E_{0000} \ , \\
E_{2222} &= \left(\frac{(1+\nu)(1-2\nu)}{(1-\nu)E} + \frac{1}{hk_n} \right)^{-1} \ , \\
E_{0022} = E_{1122} = E_{2200} = E_{2211} &= \frac{\nu}{1-\nu} E_{2222} \ .
\end{aligned} \tag{3.8}
$$

The shear terms may be motivated in a similar way, and are defined in terms of the shear modulus of the "main" layers, G, and the modified shear, $\tilde{G}$:

$$
G = \frac{E}{2(1+\nu)} \quad \text{and} \quad \tilde{G} = \left(\frac{1}{G} + \frac{1}{hk_s} \right)^{-1} \tag{3.9}
$$

They are[1]

$$
\begin{aligned}
E_{0101} = E_{0110} = E_{1001} = E_{1010} &= G \ , \\
E_{0202} = E_{0220} = E_{2002} = E_{1212} = E_{1221} = E_{2112} &= \tilde{G} \ , \\
E_{2020} = E_{2121} &= \tfrac{1}{2}(G + \tilde{G}) \ .
\end{aligned} \tag{3.10}
$$

These are the only nonzero components of E.

The last equality of Equation (3.10) implies that E does not obey the usual symmetries, ie $E_{ijkl} \neq E_{jikl}$, as should be expected in the Cosserat situation. Equations (3.10) are easily motivated. Firstly, any shear in the (x,y) plane should behave independently of the layers, which motivates the first equation. Now consider the case $k_s = 0$: that is, the Cosserat layers may slide freely over one another ($\tilde{G} = 0$). Shear strains γ_{02} or γ_{12} involves layers sliding over one another. For instance, $u_0 = x_2$ gives $\gamma_{02} = 1$. This should cost zero energy, which is encoded in $E_{ij02} = \tilde{G} = E_{ij01}$. However, a nonzero shear strain γ_{20} or γ_{21} involves shearing the "main" layers, resulting in the last equality of Equations (3.10).

Finally, the only nonzero components of B are defined in terms of the bending rigidity of a layer

$$
D = \frac{Eh^3}{12(1-\nu^2)} \ , \tag{3.11}
$$

and are

$$
\begin{aligned}
B_{0101} = B_{1010} &= \frac{D}{h} \left(\frac{G - \tilde{G}}{G + \tilde{G}} \right) \ , \\
B_{0110} = B_{1001} &= -\nu B_{0101} \ .
\end{aligned} \tag{3.12}
$$

The negative sign in the last equation is not a typo [2].

[1] Some references are the following. DP Adhikary, HB Muhlhaus, AV Dyskin "A numerical study of flexural buckling of foliated rock slopes" International Journal for Numerical and Analytical Methods in Geomechanics 25 (2001) 871–884. DP Adhikary, HB Muhlhaus, AV Dyskin "Modelling the large deformations in stratified media — the Cosserat continuum approach" Mechanics of Cohesive-Frictional Materials 4 (1999) 195–213.

[2] See, for instance the following. R Lakes "Experimental methods for study of Cosserat elasticity solids and other

generalised elastic continua", Equation (4) and discussion following Equation (18). S Forest "Mechanics of Cosserat media: An introduction" Equation (A50) and discussion of (B65)