Cosserat Test suite

CSIRO

July 12, 2016

Tests involving Cosserat mechanics are described. The notation is defined in the Theory manual.

Contents

1 Cosserat glide: elasticity **4**

2 Cosserat tension **6**

3 Beam bending 1 **7**

3.1 Stiff Cosserat joints 7

3.2 Slippery Cosserat joints 9

4 Beam bending 2 **11**

1 Cosserat glide: elasticity

Forest[1] describes a "glide" test of Cosserat elasticity in his Appendix A. The test involves a 3D material, but all quantities are assumed to be functions of the $x_2 = y$ direction only. The displacement field, u, and Cosserat rotation, θ^{c} are assumed to obey

$$\begin{aligned} u_i &= (u_x(y),\ 0,\ 0)\ , && (1.1) \\ \theta^{\mathrm{c}}_i &= (0,\ 0,\ \theta^{\mathrm{c}}_z(y))\ . && (1.2) \end{aligned}$$

These mean that the strain tensor, γ, and curvature tensor, κ, are

$$\gamma = \begin{pmatrix} 0 & \theta^{\mathrm{c}}_z + \nabla_y u_x & 0 \\ -\theta^{\mathrm{c}}_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \kappa = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \nabla_y \theta^{\mathrm{c}}_z & 0 \end{pmatrix} \tag{1.3}$$

The solution below has boundary conditions $u_x(0) = 0 = \theta^{\mathrm{c}}_z(y)$.

The stress and couple-stress tensors are assumed to be zero, except for the following components:

$$\begin{aligned} \sigma_{yx} &\neq 0\ , && (1.4) \\ m_{zy} &\neq 0\ . && (1.5) \end{aligned}$$

This is depicted in Figure 1.1. If standard (non-Cosserat, Cauchy) elasticity were being used, the solution is the trivial $u_x = 0$. However, using Cosserat elasticity a nontrivial solution is found. Physically this setup corresponds to a 3D object subjected to a moment m_{zy} that rotates the Cosserat grains.

Isotropic elasticity is assumed, with the additional assumption that the couple-stress equation only involves two moduli:

$$\begin{aligned} \sigma_{ij} &= \lambda\delta_{ij}\mathrm{Tr}\gamma + 2\mu\gamma_{(ij)} + 2\alpha\gamma_{[ij]}\ , \\ m_{ij} &= \beta\delta_{ij}\mathrm{Tr}\kappa + 2\varepsilon\kappa_{(ij)} + 2\varepsilon\kappa_{[ij]}\ . && (1.6) \end{aligned}$$

With these assumptions, the moment and force balance equations reduce to a second-order ODE that has solution

$$\begin{aligned} \theta^{\mathrm{c}}_z &= B\sinh(\omega_e y)\ , && (1.7) \\ u_x &= \frac{2\alpha B}{\omega_e(\mu+\alpha)}(1-\cosh(\omega_e y))\ , && (1.8) \\ m_{zy} &= 2B\varepsilon\omega_e\cosh(\omega_e y)\ , && (1.9) \\ \sigma_{yx} &= -\frac{4\mu\alpha}{\mu+\alpha}B\sinh(\omega_e y) && (1.10) \end{aligned}$$

[1] S Forest "Mechanics of Cosserat media An introduction". Available from http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.154.4476&rep=rep1&type=pdf

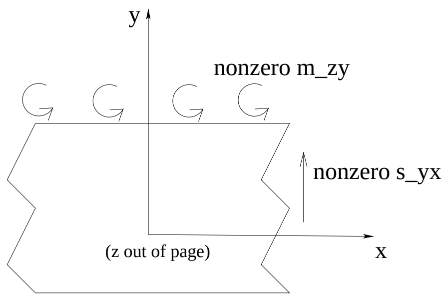

Figure 1.1: The Cosserat glide experiment

with B being an arbitrary constant of integration, and

$$\omega_e = \sqrt{\frac{2\mu\alpha}{\varepsilon(\mu+\alpha)}} \, . \tag{1.11}$$

Forest's notation is slightly different: for α he writes μ_c, and for ε he writes β.

The MOOSE simulation uses 100 elements in the y direction, with $\mu = 2$, $\alpha = 3$ and $\varepsilon = 0.6$. This gives $w_e = 2$. Preset boundary conditions at $y = 0$ and $y = 1$ are used, and the system relaxes to the equilirium solution within 1 iteration. Figure 1.2 reveals that the MOOSE simulation agrees with expectations. The displacements agree well for the 100-element simulation, but the stress components agree less well (this is not really observable to the eye in Figure 1.2) and even a non-zero σ_{xy} appears. However, as the number of elements is increased, the stresses tend to the analytical formulae given above.

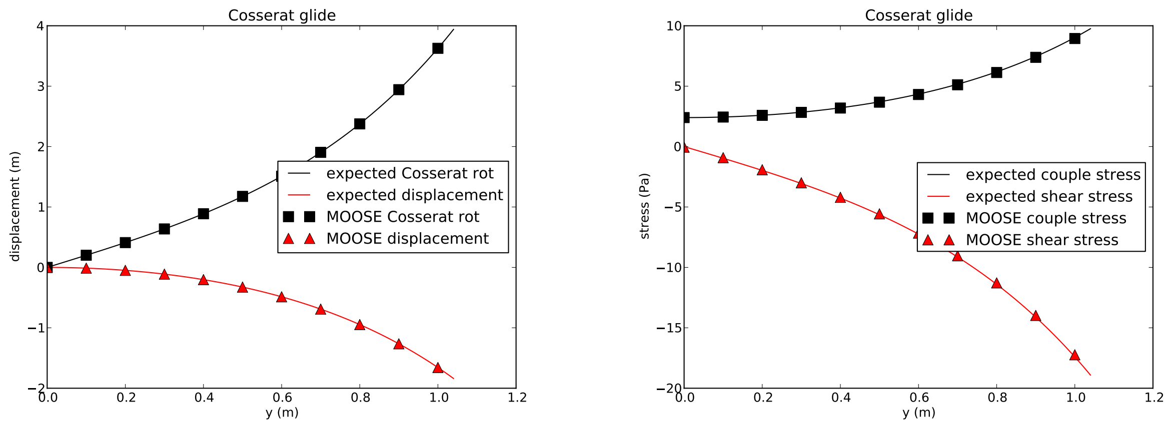

Figure 1.2: Results from the elastic Cosserat glide test. Left: displacements. Right: stresses

2 Cosserat tension

This is a simple test where a 3D sample is subjected to a normal load on its top surface. The sample is allowed to shrink in directions perpendicular to the force, via the Poisson's ratio. Specifically, all components of the stress tensor are zero except for

$$\sigma_{22} \neq 0 \; , \tag{2.1}$$

(which is constant). There are no Cosserat rotations involved:

$$m = 0 = \kappa \; . \tag{2.2}$$

A general isotropic elasticity tensor is assumed so that the constitutive relation reads

$$\sigma_{ij} = \lambda \delta_{ij} \mathrm{Tr}\gamma + 2\mu\gamma_{(ij)} + 2\alpha\gamma_{[ij]} \; . \tag{2.3}$$

The solution is identical to the standard (non-Cosserat) case, which is independent of α, and has strain components

$$\epsilon_{22} = \frac{(\lambda+\mu)}{\mu(3\lambda+2\mu)}\sigma_{22} \quad \text{and} \quad \epsilon_{11} = \epsilon_{33} = -\frac{\lambda}{2(\mu+\lambda)}\epsilon_{22} \; . \tag{2.4}$$

MOOSE generates this solution exactly.

3 Beam bending 1

A cantilever beam is held fixed at one end, and bent by a surface traction applied at its other end. As shown in Figure 3.1, the beam has length L, width $2c$, and a surface traction of τ is applied to its free end.

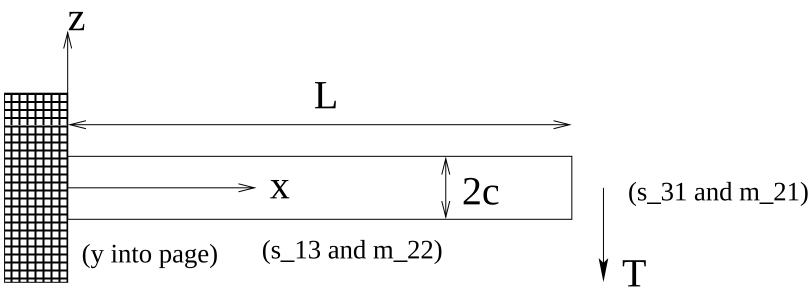

Figure 3.1: A cantilever beam of length L and width $2c$ is held fixed at one end ($x = 0$) and subjected to a surface traction τ at the other end ($x = L$). For reference in the text, some stress and moment-stress components have been shown: σ_{31} acts on the $x = L$ plane in the z direction; m_{21} acts on the $x = L$ plane to rotate around the y axis; σ_{13} acts on the $z = c$ plane in the x direction; m_{22} acts on the $z = c$ plane to rotate around the y axis.

3.1 Stiff Cosserat joints

In this section, the beam will be modelled using a layered Cosserat material, but with the Cosserat-joint normal and shear stiffnesses being very large. This means that that Cosserat theory should be identical to standard (non-Cosserat) elasticity.

The beam lies in the x–z plane. Assumed that all displacements are independent of the y direction and $u_y = 0$. The nonzero stress components in this situation are

$$\sigma_{xx} = \frac{3\tau(L-x)z}{2c^3} \quad \text{and} \quad \sigma_{xz} = \sigma_{zx} = \frac{3\tau(z^2-c^2)}{4c^3} \ , \tag{3.1}$$

as well as a nonzero σ_{yy} to ensure the displacements are independent of y. Note that:

- The shear stress σ_{zx} is independent of x, and that $\int_{-c}^{c} \mathrm{d}z\, \sigma_{zx} = -\tau$, as required. (The negative sign indicates the downwards direction.)
- The shear stress σ_{xz} is zero for $z = \pm c$. This is the shear stress on the top and bottom surfaces ($z = \pm c$) of the bar.

- The tension stress, σ_{xx} is zero at $x = L$.
- The equations of equilibrium hold: $\nabla_j \sigma_{ij} = 0$ for all i.

These stresses may be used to obtain the strain components, and finally the displacements.

It is useful for later sections to write the process explicitly. Use an ansatz solution of the form

$$\begin{aligned} u_x &= Az^3 + Bxz(2L - x) \ , && (3.2) \\ u_z &= Cx + Dz^2(L - x) - Fx^2(3L - x) \ . && (3.3) \end{aligned}$$

Here A to F are unknown coefficients, and $u_y = 0 = \theta_c^z = \theta_c^x$. The Cosserat rotation around the y axis, θ_c^y is kept arbitrary for now.

With this ansatz, the nonzero strain components $\gamma_{ij} = \nabla_j u_i + \varepsilon_{ijkl}\theta_c^k$, are

$$\begin{aligned} \gamma_{xx} &= 2Bz(L - x) \ , && (3.4) \\ \gamma_{xz} &= 3Az^2 + Bx(2L - x) - \theta_y^c \ , && (3.5) \\ \gamma_{zx} &= C - Dz^2 - 3Fx(2L - x) + \theta_y^c \ , && (3.6) \\ \gamma_{zz} &= 2Dz(L - x) \ . && (3.7) \end{aligned}$$

In the situation with Cosserat-joint stiffnesses being infinite, $0 = \sigma_{zz} = E_{xxxx}(\gamma_{zz} + \frac{\nu}{1-\nu}\gamma_{xx})$ (the Poisson's ratio is denoted by ν), which implies

$$D = -\frac{B\nu}{1 - \nu} \ . \qquad (3.8)$$

Using this in the equation $\sigma_{xx} = E_{xxxx}(\gamma_{xx} + \frac{\nu}{1-\nu}\gamma_{zz})$ yields

$$B = \frac{3\tau(1 - \nu^2)}{4c^3 E} \ , \qquad (3.9)$$

with E being the Young's modulus. The remaining constants may be identified by inpsecting the equation

$$\sigma_{xz} = G(\gamma_{xz} + \gamma_{xz}) \ , \qquad (3.10)$$

which is true for infinite joint stiffness, where G is the shear modulus $E/2/(1+\nu)$. The result is

$$\begin{aligned} A &= \frac{D}{3} + \frac{\tau}{4c^3 G} \ , && (3.11) \\ C &= -\frac{3\tau}{4cG} \ , && (3.12) \\ F &= B/3 \ . && (3.13) \end{aligned}$$

Evidently, the Cosserat rotation θ_c^y is undetermined. However, if it were anything but a constant, it would induce a nonzero moment stress which is obviously not energetically favourable and violates the boundary conditions. Therefore $\theta_c = 0$ (up to a constant).

Figure 3.2 shows agreement between the above formulae and MOOSE's result.

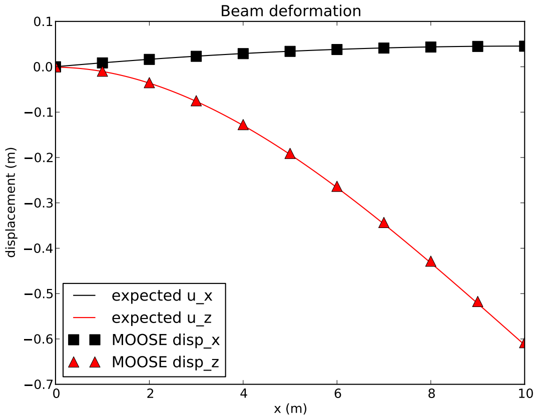

Figure 3.2: Displacements of the bar at $z = 0.5$. Here $L = 10$, $c = 0.5$, $E = 1.2$, $\nu = 0.3$ and $\tau = 0.0002$.

3.2 Slippery Cosserat joints

In this section, the beam is modelled using a layered Cosserat material with large Cosserat-joint normal stiffness but very small Cosserat-joint shear stiffness. This means that the beam deformation should be identical to a number of independent layers of thickness h. Using the Cosserat theory means that we don't need to actually model each layer as a separate finite element! Equation (3.3) yields that the deflection at $z = 0$ is

$$u_z(L,0) = \frac{3\sigma_{zx}}{2G}x + \frac{2\sigma_{xz}(1-\nu^2)}{Eh^2}x^2(3L-x) \ . \tag{3.14}$$

In this equation a constant shear stress σ_{zx} has been used in place of $-\tau/(2c)$, and since the layers are acting independently (like a ream of paper) $2c$ has been replaced by the layer thickness h. Let us derive this result using the layered Cosserat theory.

We seek a solution that depends on x only. The deformation $u_y = 0$ as in the previous section. The only nonzero Cosserat rotation is assumed to be $\theta^{\mathrm{c}}_y \neq 0$.

To the $x = L$ end of the beam apply a uniform shear stress σ_{zx} (independent of y and z). Suppose that all the other stress components are zero, except perhaps for σ_{yy} which ensures the displacements are independent of y. Assume that σ_{zx} is uniform throughout the entire bar. The equations of equilibrium $\nabla_j\sigma_{ij} = 0$ are clearly satisfied.

Suppose that no external moment is applied to the surface of the bar, except perhaps at the clamped end at $x = 0$. Then the equation of equilibrium $\nabla_j m_{ij} = \varepsilon_{ijk}\sigma_{jk}$ implies that $\nabla_j m_{yj} = \sigma_{zx}$, or

$$m_{yx} = \sigma_{zx}(x-L) \ , \tag{3.15}$$

which is zero at $x = L$ as desired. Because θ^{c}_x is assumed to be zero, m_{xy} must also be nonzero:

$$m_{xy} = -\nu m_{yx} \ . \tag{3.16}$$

All other components of the moment stress may be set to zero.

The constitutive equation $m_{ij} = B_{ijkl}\nabla_l\theta^c_k$ implies that

$$\theta^c_y = \tfrac{1}{2}\sigma_{xz}x(x-2L)/B\ , \qquad (3.17)$$

where $B = B_{yxyx} = Eh^2/12/(1-\nu^2)$ (recall that $\tilde{G} = 0$). The other constitutive equations involving m_{ij} are also satisfied (all other moment-stresses and curvatures are zero).

The constitutive equations involving σ_{ij} are either trivially satisfied because the strains are zero or because $\tilde{G} = 0$ (in particular $\sigma_{xz} = 0$). The only nontrivial constitutive equation remaining is $\sigma_{zx} = \frac{1}{2}G\gamma_{zx}$ (valid for $\tilde{G} = 0$). Along with the definition $\gamma_{zx} = \nabla_x u_z + \theta^c_y$, this implies

$$u_z = \frac{2\sigma_{zx}}{G}x - \frac{\sigma_{zx}}{6B}x^2(x-3L) = \frac{2\sigma_{xz}}{G}x + \frac{2\sigma_{xz}(1-\nu^2)}{Eh^2}x^2(3L-x)\ . \qquad (3.18)$$

The second term agrees perfectly with the expected result of Equation (3.14), however, the first term is not identical. To make the match precise, the shear stiffness in the elasticity tensor would have to be $E_{zxzx} = \frac{2}{3}G + \frac{1}{3}\tilde{G}$ rather than $E_{xzxz} = \frac{1}{2}(G+\tilde{G})$, but this is not traditional. Nevertheless, the second term is the leading term for $h/L \to 0$.

Figure 3.3 shows agreement between the Equations (3.17) and (3.18) and MOOSE's result. Figure 3.3 also shows the vertical displacement of the end-point of the bar (at $x = L$) as a function of h. In the Cosserat case there is no need to model multiple layers with multiple finite elements: these results were obtained using 1 element in the z direction.

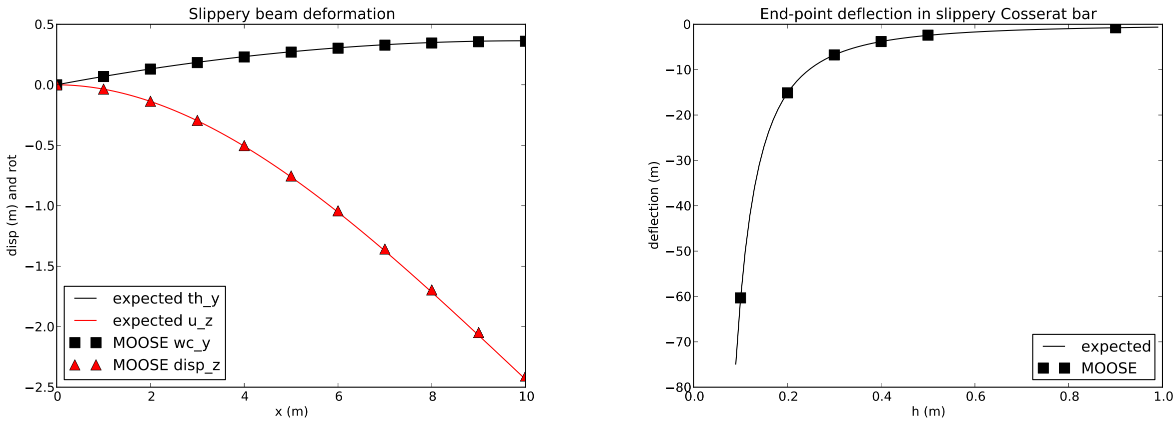

Figure 3.3: Left: displacement and Cosserat rotation of the bar in the "slippery" Cosserat-joint case using $h = 0.5$. Right: vertical displacement of the bar's endpoint (at $x = L$) as a function of Cosserat layer thickness h. In both $L = 10$ $E = 1.2$, $\nu = 0.3$, $c = 1$ and $\sigma_{xz} = -0.0002$.

4 Beam bending 2

A pure bend is applied to a the beam shown in Figure 3.1. In this section, the beam will be modelled using a layered Cosserat material, with small Cosserat joint shear stiffness (so that layers can slip over one another) and with infinite normal stiffness. Some of the material in this section can also be found in Appendix B of Forest.[1]

Suppose that the beam is bent with the following deformations

$$
\begin{aligned}
u_x &= Axz \, , & (4.1)\\
u_y &= Dzy \, , & (4.2)\\
u_z &= -\tfrac{1}{2}Ax^2 + \tfrac{1}{2}D(z^2 - y^2) & (4.3)
\end{aligned}
$$

To first order, in the x–z plane, this is pure bending through an arc of a circle:

$$
(x,z) \to \left((z+R)\frac{x}{\sqrt{x^2+R^2}}, (z+R)\frac{R}{\sqrt{x^2+R^2}} \right) \approx \left(x + xz/R, z - \tfrac{1}{2}x^2/R \right) \, , \tag{4.4}
$$

with $A = 1/R$. The arbitrary constant D and the deformation in the y direction are introduced so that plane stress conditions ($\sigma_{yy} = 0$) can be used.

The Cosserat layers are assumed to rotate along with the deformation, that is

$$
\theta_c^i = \tfrac{1}{2}\varepsilon_{ijk}\partial_j u_k \, , \tag{4.5}
$$

meaning that

$$
\begin{aligned}
\theta_c^x &= -Dy \, , & (4.6)\\
\theta_c^y &= Ax \, , & (4.7)\\
\theta_c^z &= 0 \, . & (4.8)
\end{aligned}
$$

The resulting nonzero strain and curvature components are

$$
\begin{aligned}
\gamma_{xx} &= Ax \, , & (4.9)\\
\gamma_{yy} &= Dz \, , & (4.10)\\
\gamma_{zz} &= Dz \, , & (4.11)\\
\kappa_{xy} &= -D \, , & (4.12)\\
\kappa_{yx} &= A & (4.13)
\end{aligned}
$$

[1] S Forest "Mechanics of Cosserat media An introduction". Available from http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.154.4476&rep=rep1&type=pdf

Finally, choosing

$$D = -\nu A \tag{4.14}$$

(with ν being the Posson's ratio) gives, for layered Cosserat:

$$\sigma_{xx} = AEz\ , \tag{4.15}$$

$$m_{yx} = \tfrac{1}{12}AEh^2\frac{G}{hk_s + G}\ . \tag{4.16}$$

The other stress and moment-stress components are zero. Here $G = \frac{1}{2}E/(1+\nu)$ is the shear modulus, h is the Cosserat layer thickness and k_s is the Cosserat joint shear stiffness.

Equation (4.16) illustrates a key feature in Cosserat mechanics. The nonuniform stress of Equation (4.15) is the standard (non-Cosserat) way of producing pure circular bending in a bar. Imagine that it is applied to a single Cosserat layer of thickness h, as illustrated in Figure 4.1. This generates a moment in the layer:

$$m_{yx} = \frac{1}{hW}\int_{-W/2}^{W/2}\mathrm{d}y\int_{-h/2}^{h/2}\mathrm{d}z AEz = \tfrac{1}{12}AEh^2\ , \tag{4.17}$$

(for a depth W in the y direction). The same result holds for a layer that is not centred at $z = 0$: the nonuniform stress σ_{xx} is decomposed into a uniform stretch and a varying part that induces the micromechanical moment-stress. Equation (4.17) is exactly Equation (4.16) in the limit of zero Cosserat joint shear stiffness. A nonzero shear stiffness reduces the micromechanical moment-stress that is generated in the Cosserat layers. Hence this example has revealed the meaning of the moment-stresses in layered Cosserat mechanics: they are the moments experienced by the layers the orginate from bending of the Cosserat layers.

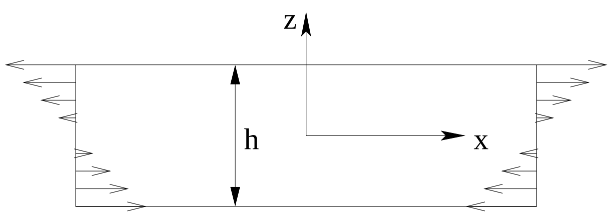

Figure 4.1: Stress $\sigma_{xx} = AEz$ is applied to a single Cosserat layer of thickness h

.

The test suite contains two tests based on the above theory. The tests use $L = 10$, $c = 0.5$, $E = 1.2$, $\nu = 0.3$, $A = 0.0111$ and $h = 2$.

1. The deformations and Cosserat rotations are applied, and the stresses are measured to ensure that they satisfy Equations (4.15) and (4.16). This test uses $k_s = \infty$. This means $m_{yx} = 0.00444$, and MOOSE produces this exactly. MOOSE also produces the result for σ_{xx}, but the accuracy obviously depends on the resolution in the z direction.

2. The $x = 0$ end is held clamped, while σ_{xx} and m_{yz} are applied to the right-hand end ($x = L$). This test uses $k_s = 0.1$: good agreement is obtained, as shown in Figure 4.2

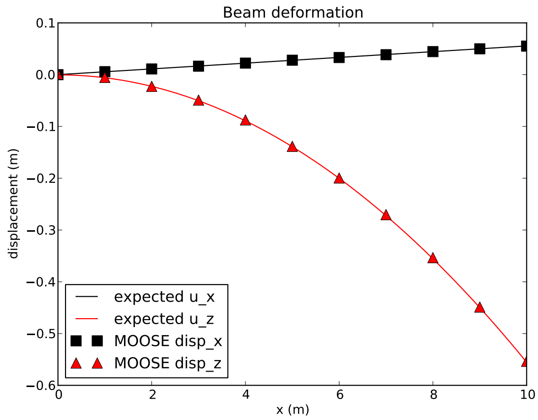

Figure 4.2: Deformations of the beam subjected to nonuniform σ_{xx} and micromechanical moment-stress.