

# A Christofides-based approach to the travelling salesman problem in the unit cube

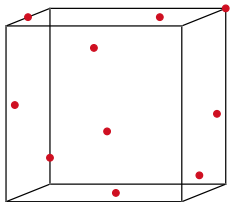
Alexey Gordeev

Umeå University, Sweden

August 28, 2025

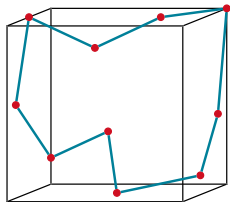
# TSP in the unit cube

find *Hamiltonian cycle* on  $X \subseteq [0, 1]^k$  with  $\min. \sum |e|^m$



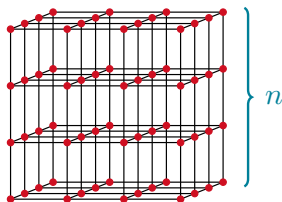
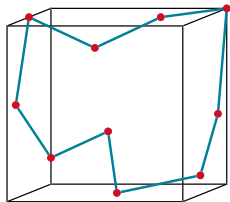
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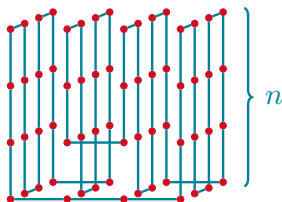
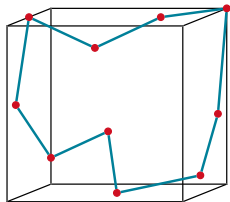
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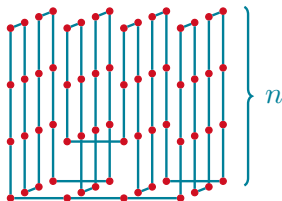
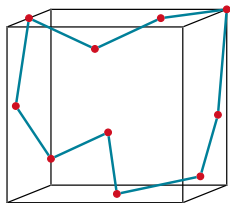
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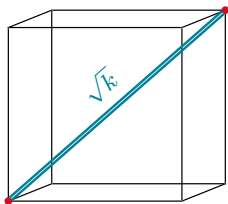
$$\sum |e|^m \approx \frac{n^k}{n^m} \xrightarrow{n \rightarrow \infty} \begin{cases} \infty & \text{if } k > m, \\ 0 & \text{if } k < m, \\ \mathbf{1} & \text{if } k = m. \end{cases}$$

## Bollobás–Meir conjecture

$$\forall \text{ finite } X \subseteq [0, 1]^k \exists \text{ Ham. cycle } H \text{ on } X: (\sum_H |e|^k)^{1/k} \leq s_k^{\text{HC}}$$

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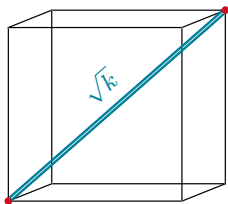
$$\diamond 2^{1/k} \sqrt{k} \leq s_k^{\text{HC}} \leq 9 \cdot \left(\frac{2}{3}\right)^{1/k} \sqrt{k}$$

**Bollobás–Meir 93**



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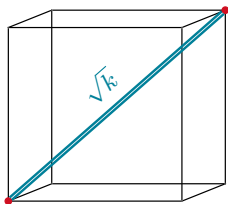
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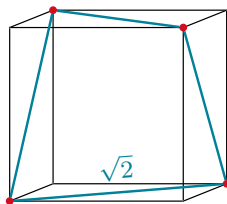
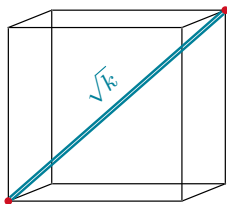
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- $\diamond$  True for  $k = 2$
- $\diamond$  Open for  $k > 2$

**Newman 82**

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**Bollobás–Meir conjecture (upd. Balogh–Clemen–Dumitrescu 24)**

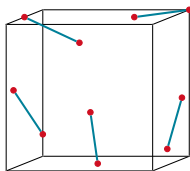
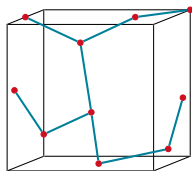
$$s_k^{\text{HC}} = 2^{1/k} \sqrt{k} \text{ for } k \neq 3, \quad s_3^{\text{HC}} = 2^{7/6}$$

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# Results

- $s_k^{ST}$  and  $s_k^{PM}$ : analog.  $s_k^{HC}$  for *spanning trees* and *perfect matchings*



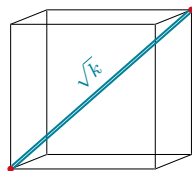
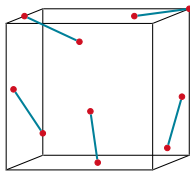
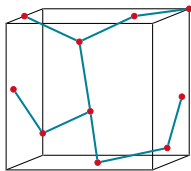
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## Balogh–Clemen–Dumitrescu 24

$$s_k^{\text{ST}} \leq \sqrt{5k} \text{ or } \sqrt{k}(1 + o_k(1))$$

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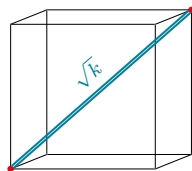
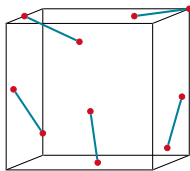
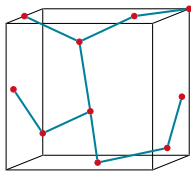
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## G 25+

$$s_k^{\text{PM}} \leq 2^{1/k} \sqrt{2k}, \quad \sqrt{5} \Rightarrow 1.823, \quad 6.709 \Rightarrow 5.059, \quad 2.91 \Rightarrow 2.65$$

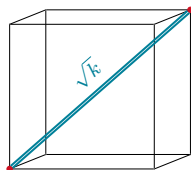
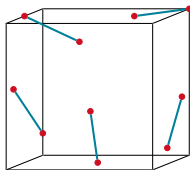
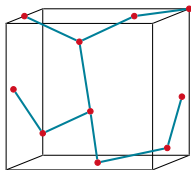
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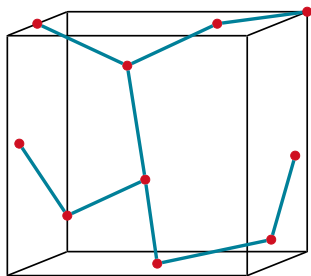
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## G 25++: Bollobás–Meir conjecture holds asymptotically

$$2^{1/k} \sqrt{k} \leq s_k^{\text{HC}} \leq (6(k+1))^{1/k} \sqrt{k} \text{ or } (2 + o_k(1))^{1/k} \sqrt{k}$$

## Tools: ball packing + cycle approximation

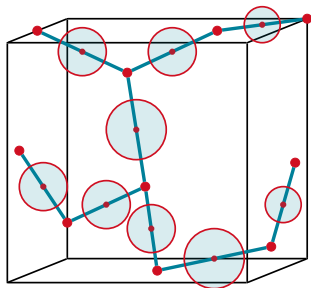
- min. spanning tree  $\mathbf{T}$





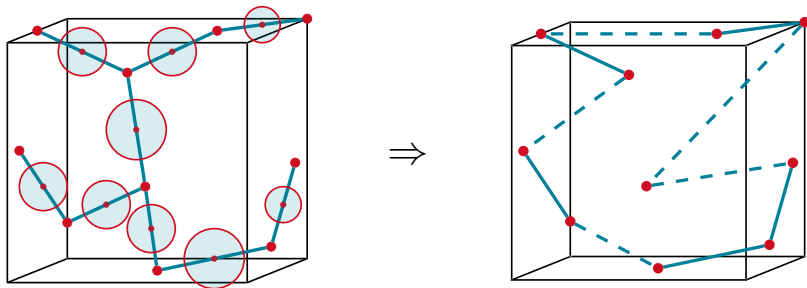
## Tools: ball packing + cycle approximation

- min. spanning tree  $\mathbf{T} \rightarrow \frac{|e|}{4}$ -radius *ball packing*  $\xrightarrow{\text{volume bound}} s_k^{\mathbf{T}} \leq \sqrt{5k}$



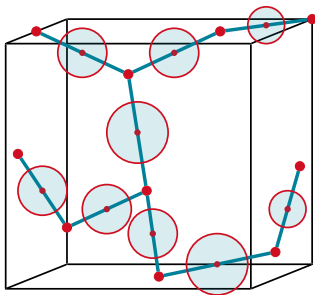
# Tools: ball packing + cycle approximation

- min. spanning tree  $\mathbf{T} \rightarrow \frac{|e|}{4}$ -radius *ball packing*  $\xrightarrow{\text{volume bound}} s_k^{\mathbf{ST}} \leq \sqrt{5k}$
- $\mathbf{T} \Rightarrow$  Hamiltonian cycle  $\mathbf{H}$ :  $s_k^{\mathbf{HC}} \leq (\frac{2}{3})^{1/k} \cdot \mathbf{3} \cdot s_k^{\mathbf{ST}} \leq 6.709 \cdot (\frac{2}{3})^{1/k} \sqrt{k}$



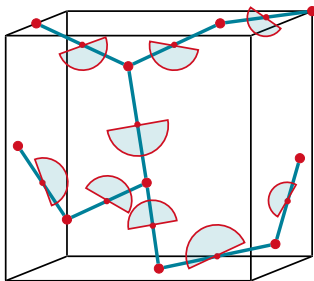
## Half-ball packing argument

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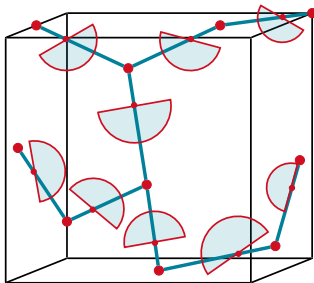
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- ◇ **half-ball** packing:  $s_k^{\mathbf{ST}} \leq 2 \cdot 2^{1/k} \sqrt{k}$
- ◇  $(0.2744 \cdot |e|)$ -radius **half-ball** packing:  $s_k^{\mathbf{ST}} \leq 1.823 \cdot 2^{1/k} \sqrt{k}$
- $s_k^{\mathbf{HC}} \leq 2.65 \sqrt{k} (1 + o_k(1))$



## Christofides approach

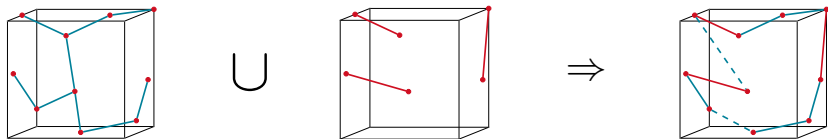
- $T \Rightarrow H$

$\leftrightarrow$

Euclidean TSP 2-approx. algorithm

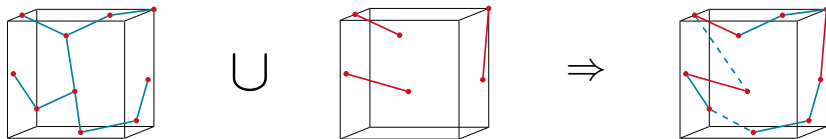
# Christofides approach

- $T \Rightarrow H \Leftrightarrow$  Euclidean TSP 2-approx. algorithm
- $T \cup \text{perfect matching } M \Rightarrow H \Leftrightarrow$  1.5-approx. algorithm **Christofides 76**



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$$\forall a, b, c, d \in \mathbb{R}^k : \left| \frac{a+b}{2} - \frac{c+d}{2} \right|^2 = \frac{|a-c|^2 + |b-d|^2 + |a-d|^2 + |b-c|^2 - |a-b|^2 - |c-d|^2}{4}$$

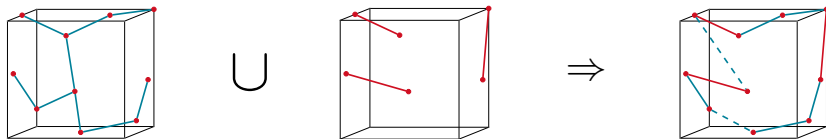
$$\text{Diagram} = \left( \text{Diagram 1} + \text{Diagram 2} - \text{Diagram 3} \right) / 4$$

The diagram shows the vector identity for the Christofides shortcut. It consists of three sub-diagrams: 1) A crossing of two blue lines. 2) Two parallel blue lines. 3) Two parallel red lines. The identity states that the vector from the midpoint of the red segment to the midpoint of the blue segment is equal to the sum of the vectors from the midpoint of the red segment to the endpoints of the blue segment, minus the vector from the midpoint of the blue segment to the endpoints of the red segment, all divided by 4.



# Christofides approach

- $T \Rightarrow H \Leftrightarrow$  Euclidean TSP 2-approx. algorithm
- $T \cup \text{perfect matching } M \Rightarrow H \Leftrightarrow 1.5\text{-approx. algorithm}$  **Christofides 76**



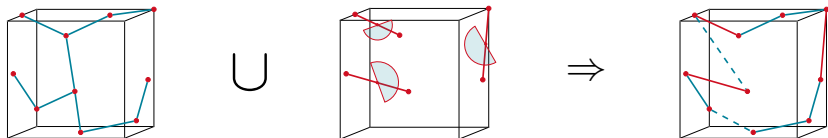
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$$\begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} = \left( \begin{array}{cc} \bullet & \bullet \\ \diagdown & \diagup \\ \bullet & \bullet \end{array} + \begin{array}{cc} \bullet & \bullet \\ \text{---} & \text{---} \\ \bullet & \bullet \end{array} - \begin{array}{cc} \bullet & \bullet \\ \text{---} & \text{---} \\ \bullet & \bullet \end{array} \right) / 4$$

min. perfect matching  $M$ :  $\begin{array}{cc} \bullet & \bullet \\ \diagdown & \diagup \\ \bullet & \bullet \end{array} \geq \begin{array}{cc} \bullet & \bullet \\ \text{---} & \text{---} \\ \bullet & \bullet \end{array}, \begin{array}{cc} \bullet & \bullet \\ \text{---} & \text{---} \\ \bullet & \bullet \end{array} \geq \begin{array}{cc} \bullet & \bullet \\ \text{---} & \text{---} \\ \bullet & \bullet \end{array} \Rightarrow \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \geq \left( \begin{array}{cc} \bullet & \bullet \\ \text{---} & \text{---} \\ \bullet & \bullet \end{array} \right) / 4$

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- $T \cup \text{perfect matching } M \Rightarrow H \iff 1.5\text{-approx. algorithm}$  **Christofides 76**



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◇  $\frac{|e|}{2\sqrt{2}}$ -radius packing:  $s_k^{\text{PM}} \leq 2^{1/k} \sqrt{2k}, \quad s_k^{\text{HC}} \leq 5.059 \cdot (1.28)^{1/k} \sqrt{k}$

## Ball packing for Hamiltonian cycles

$$\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array} = \left( \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array} + \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array} - \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array} \right) / 4$$

## Ball packing for Hamiltonian cycles

$$\mathbf{H} = \left( \begin{array}{c} \bullet \quad \bullet \\ \text{---} \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \text{---} \\ \bullet \quad \bullet \end{array} - \begin{array}{c} \bullet \quad \bullet \\ \text{---} \\ \bullet \quad \bullet \end{array} \right) / 4$$

min. Ham.  
cycle

**H:**



$\geq$



but



$\not\geq$



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$\not\geq$



$\geq$



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$\Rightarrow$

$$\begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \geq \left( \begin{array}{c} \bullet \quad \bullet \\ \hline \bullet \quad \bullet \end{array} \begin{array}{c} \bullet \quad \bullet \\ \hline \bullet \quad \bullet \end{array} \right) / 4$$

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$$s_k^{\text{HC}} \leq 6^{1/k} \sqrt{2k}$$

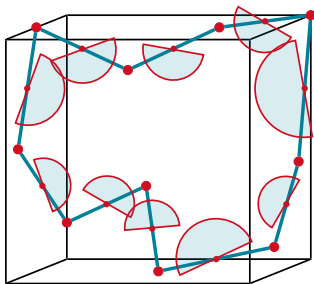
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## Bollobás–Meir conjecture

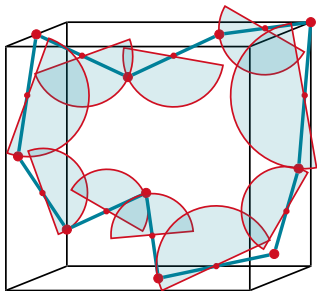
$$s_k^{\text{HC}} = 2^{1/k} \sqrt{k} \text{ for } k \neq 3, \quad s_3^{\text{HC}} = 2^{7/6}$$

◇  $\frac{|e|}{2\sqrt{2}}$ -radius **3-fold** packing:

$$s_k^{\text{HC}} \leq 6^{1/k} \sqrt{2k}$$

◇ *spherical codes*  $\rightarrow$   $\frac{|e|}{2}$ -rad. **3(k+1)-fold**:

$$s_k^{\text{HC}} \leq (6(k+1))^{1/k} \sqrt{k}$$



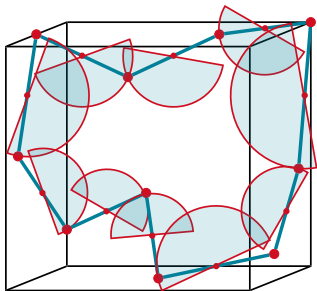


# Bollobás–Meir conjecture holds asymptotically

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- ◇ small/large edges separately:  $2^{1/k} \sqrt{k} \leq s_k^{\text{HC}} \leq (2 + o_k(1))^{1/k} \sqrt{k}$



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