

Ridge regularization for spatial auto-regressive models with multicollinearity issues.

Simulations results of comparison of RRSEM with existing methods

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In our paper titled “Ridge Regularization for Spatial Autoregressive Models with Multicollinearity Issues,” we introduce a novel approach for conducting Ridge regression in the context of spatial autoregressive models.

In this document, we provide the complete results of the simulations conducted within the framework of the dependent variable determined by the following equation:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + (I_n - \lambda W)^{-1}\boldsymbol{\epsilon},$$

We consider eight highly correlated covariates generated as described in Section 5. Here, we present two scenarios: the deterministic scenario where the covariates were generated once for all the simulations and the stochastic scenario where the covariates were generated for each simulation.

The SEM model, defined in equation 2 of the paper, is generated for five values of the dependence parameter $\lambda \in (0.1, 0.3, 0.5, 0.7, 0.9)$.

The following tables display the average bias, average variance, and average mean squared error (MSE) of the eight regression coefficient estimates and the dependence parameter estimates computed across 500 simulations for each value of the dependence parameter (λ). These estimates are computed using different estimation algorithms: OLS, ordinary SEM, ordinary Ridge, Spatially Filtered Ridge Regression (SFRR), and our methodology, named Ridge Regression for SEM Models (RRSEM). We present the results in two sections, the first one for the results of the deterministic covariates, and the second one for the results of the stochastic covariates.

Tables 16-18 and 34-36 compile the average results for all the regression coefficients.

Deterministic

Results $\lambda = 0.1$

Table 1: Coefficient bias for $\lambda = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	-0.530	-0.160	-0.897	-0.943	-0.933	-0.936	-1.314	-0.950	NA
SEM	-0.530	-0.159	-0.897	-0.943	-0.933	-0.936	-1.314	-0.950	-0.007
RR	-1.008	-0.912	-0.897	-1.017	-0.929	-0.918	-1.024	-0.916	NA
SFRR	-1.003	-0.914	-0.895	-1.016	-0.937	-0.920	-1.025	-0.925	-0.007
RRSEM	-0.974	-0.950	-0.901	-1.024	-0.906	-0.897	-0.940	-0.874	0.009

Table 2: Coefficient variance for $\lambda = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.480	1.181	0.000	0.012	0.000	0.001	0.176	0.002	NA
SEM	0.483	1.186	0.000	0.012	0.000	0.001	0.176	0.002	0.002
RR	0.003	0.007	0.000	0.000	0.000	0.000	0.001	0.000	NA
SFRR	0.004	0.011	0.001	0.000	0.001	0.001	0.002	0.001	0.002
RRSEM	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002

Table 3: Coefficient MSE for $\lambda = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.761	1.206	0.805	0.900	0.870	0.877	1.903	0.905	NA
SEM	0.763	1.212	0.805	0.900	0.870	0.877	1.904	0.905	0.002
RR	1.019	0.838	0.805	1.035	0.863	0.842	1.051	0.839	NA
SFRR	1.011	0.847	0.801	1.032	0.878	0.848	1.053	0.856	0.002
RRSEM	0.948	0.903	0.812	1.048	0.821	0.805	0.883	0.765	0.002

Results $\lambda = 0.3$ Table 4: Coefficient bias for $\lambda = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	-0.533	-0.164	-0.898	-0.943	-0.933	-0.936	-1.312	-0.950	NA
SEM	-0.530	-0.160	-0.898	-0.943	-0.933	-0.936	-1.314	-0.950	-0.008
RR	-1.008	-0.912	-0.898	-1.017	-0.929	-0.918	-1.024	-0.917	NA
SFRR	-1.004	-0.928	-0.895	-1.014	-0.942	-0.928	-1.024	-0.937	-0.008
RRSEM	-0.974	-0.950	-0.900	-1.023	-0.906	-0.897	-0.940	-0.875	0.001

Table 5: Coefficient variance for $\lambda = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.593	1.458	0.000	0.014	0.000	0.001	0.217	0.003	NA
SEM	0.565	1.388	0.000	0.014	0.000	0.001	0.207	0.003	0.002
RR	0.004	0.009	0.000	0.000	0.000	0.000	0.001	0.000	NA
SFRR	0.005	0.012	0.001	0.000	0.001	0.001	0.002	0.001	0.002
RRSEM	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002

Table 6: Coefficient MSE for $\lambda = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.877	1.485	0.806	0.904	0.871	0.878	1.939	0.906	NA
SEM	0.846	1.414	0.806	0.902	0.871	0.878	1.933	0.906	0.002
RR	1.019	0.840	0.807	1.035	0.864	0.843	1.051	0.840	NA
SFRR	1.012	0.873	0.802	1.027	0.889	0.862	1.051	0.879	0.002
RRSEM	0.949	0.903	0.810	1.046	0.821	0.804	0.884	0.766	0.002

Results $\lambda = 0.5$

Table 7: Coefficient bias for $\lambda = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	-0.541	-0.178	-0.900	-0.944	-0.934	-0.937	-1.307	-0.951	NA
SEM	-0.536	-0.171	-0.900	-0.943	-0.934	-0.938	-1.310	-0.952	-0.008
RR	-1.007	-0.913	-0.900	-1.017	-0.931	-0.920	-1.024	-0.918	NA
SFRR	-1.004	-0.942	-0.897	-1.011	-0.948	-0.937	-1.023	-0.950	-0.008
RRSEM	-0.975	-0.952	-0.900	-1.022	-0.906	-0.897	-0.941	-0.877	-0.003

Table 8: Coefficient variance for $\lambda = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.806	1.980	0.000	0.020	0.001	0.001	0.294	0.004	NA
SEM	0.647	1.590	0.000	0.016	0.000	0.001	0.237	0.003	0.001
RR	0.006	0.013	0.000	0.000	0.000	0.000	0.002	0.000	NA
SFRR	0.006	0.015	0.001	0.000	0.001	0.001	0.002	0.001	0.001
RRSEM	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001

Table 9: Coefficient MSE for $\lambda = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	1.098	2.012	0.809	0.911	0.874	0.880	2.003	0.909	NA
SEM	0.934	1.619	0.809	0.905	0.874	0.880	1.953	0.909	0.002
RR	1.020	0.846	0.810	1.034	0.867	0.846	1.051	0.843	NA
SFRR	1.015	0.903	0.805	1.023	0.900	0.878	1.049	0.903	0.002
RRSEM	0.950	0.906	0.811	1.044	0.822	0.805	0.886	0.770	0.001

Results $\lambda = 0.7$

Table 10: Coefficient bias for $\lambda = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	-0.561	-0.215	-0.904	-0.946	-0.937	-0.940	-1.294	-0.954	NA
SEM	-0.554	-0.204	-0.904	-0.945	-0.937	-0.940	-1.298	-0.954	-0.006
RR	-1.006	-0.915	-0.904	-1.016	-0.934	-0.923	-1.024	-0.922	NA
SFRR	-1.005	-0.958	-0.902	-1.009	-0.955	-0.946	-1.021	-0.963	-0.006
RRSEM	-0.975	-0.956	-0.905	-1.021	-0.910	-0.903	-0.946	-0.886	-0.003

Table 11: Coefficient variance for $\lambda = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	1.299	3.191	0.000	0.032	0.001	0.002	0.474	0.007	NA
SEM	0.707	1.736	0.000	0.017	0.001	0.001	0.259	0.004	0.001
RR	0.011	0.022	0.000	0.000	0.001	0.000	0.003	0.000	NA
SFRR	0.008	0.019	0.001	0.000	0.001	0.001	0.003	0.001	0.001

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
RRSEM	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001

Table 12: Coefficient MSE for $\lambda = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	1.614	3.237	0.818	0.927	0.880	0.886	2.147	0.916	NA
SEM	1.013	1.778	0.817	0.911	0.879	0.885	1.942	0.914	0.001
RR	1.022	0.859	0.818	1.032	0.873	0.853	1.052	0.851	NA
SFRR	1.018	0.937	0.813	1.018	0.914	0.896	1.045	0.928	0.001
RRSEM	0.950	0.914	0.818	1.043	0.829	0.815	0.895	0.786	0.001

Results $\lambda = 0.9$

Table 13: Coefficient bias for $\lambda = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	-0.649	-0.369	-0.922	-0.957	-0.949	-0.952	-1.236	-0.963	NA
SEM	-0.636	-0.352	-0.922	-0.955	-0.949	-0.952	-1.242	-0.963	-0.004
RR	-1.002	-0.924	-0.922	-1.012	-0.946	-0.938	-1.022	-0.938	NA
SFRR	-1.005	-0.977	-0.920	-1.006	-0.968	-0.962	-1.016	-0.979	-0.004
RRSEM	-0.977	-0.969	-0.922	-1.017	-0.926	-0.921	-0.960	-0.911	-0.003

Table 14: Coefficient variance for $\lambda = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	3.262	7.990	0.000	0.079	0.004	0.007	1.181	0.018	NA
SEM	0.638	1.572	0.000	0.015	0.001	0.001	0.234	0.003	0.000
RR	0.030	0.060	0.000	0.001	0.003	0.001	0.009	0.001	NA
SFRR	0.009	0.021	0.001	0.000	0.001	0.001	0.003	0.001	0.000
RRSEM	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.001	0.000

Table 15: Coefficient MSE for $\lambda = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	3.683	8.126	0.851	0.995	0.904	0.912	2.708	0.944	NA
SEM	1.043	1.696	0.851	0.928	0.902	0.907	1.777	0.930	0.000
RR	1.033	0.914	0.851	1.025	0.898	0.882	1.053	0.880	NA
SFRR	1.019	0.976	0.847	1.012	0.938	0.926	1.036	0.959	0.000
RRSEM	0.956	0.939	0.851	1.033	0.858	0.848	0.922	0.831	0.000

Results average β coefficients

Table 16: Average Coefficient bias

λ	OLS	SEM	RR	SFRR	RRSEM
0.1	-0.530	-0.530	-1.008	-1.003	-0.974
0.3	-0.533	-0.530	-1.008	-1.004	-0.974
0.5	-0.541	-0.536	-1.007	-1.004	-0.975
0.7	-0.561	-0.554	-1.006	-1.005	-0.975
0.9	-0.649	-0.636	-1.002	-1.005	-0.977

Table 17: Average Coefficient variance

λ	OLS	SEM	RR	SFRR	RRSEM
0.1	0.480	0.483	0.003	0.004	0.000
0.3	0.593	0.565	0.004	0.005	0.000
0.5	0.806	0.647	0.006	0.006	0.000
0.7	1.299	0.707	0.011	0.008	0.000
0.9	3.262	0.638	0.030	0.009	0.000

Table 18: Average Coefficient MSE

λ	OLS	SEM	RR	SFRR	RRSEM
0.1	0.761	0.763	1.019	1.011	0.948
0.3	0.877	0.846	1.019	1.012	0.949
0.5	1.098	0.934	1.020	1.015	0.950
0.7	1.614	1.013	1.022	1.018	0.950
0.9	3.683	1.043	1.033	1.019	0.956

Stochastic

Results $\lambda = 0.1$

Table 19: Coefficient bias for $\lambda = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	-0.590	-0.256	-0.895	-0.952	-0.937	-0.933	-1.277	-0.949	NA
SEM	-0.586	-0.250	-0.895	-0.951	-0.937	-0.933	-1.279	-0.949	-0.009
RR	-1.003	-0.907	-0.895	-1.017	-0.935	-0.917	-1.026	-0.919	NA
SFRR	-1.003	-0.914	-0.895	-1.016	-0.937	-0.920	-1.025	-0.925	-0.009
RRSEM	-0.974	-0.951	-0.901	-1.021	-0.907	-0.896	-0.940	-0.872	0.008

Table 20: Coefficient variance for $\lambda = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.469	1.180	0.001	0.011	0.001	0.001	0.174	0.002	NA
SEM	0.470	1.184	0.001	0.011	0.001	0.001	0.175	0.002	0.002
RR	0.005	0.014	0.001	0.000	0.001	0.000	0.002	0.001	NA
SFRR	0.004	0.011	0.001	0.000	0.001	0.001	0.002	0.001	0.002

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
RRSEM	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002

Table 21: Coefficient MSE for $\lambda = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.816	1.245	0.801	0.918	0.878	0.871	1.804	0.902	NA
SEM	0.813	1.247	0.801	0.916	0.878	0.871	1.811	0.903	0.002
RR	1.010	0.836	0.801	1.034	0.874	0.841	1.055	0.846	NA
SFRR	1.011	0.847	0.801	1.032	0.878	0.848	1.053	0.856	0.002
RRSEM	0.949	0.904	0.813	1.043	0.823	0.804	0.883	0.761	0.002

Results $\lambda = 0.3$

Table 22: Coefficient bias for $\lambda = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	-0.595	-0.265	-0.895	-0.953	-0.937	-0.933	-1.273	-0.949	NA
SEM	-0.581	-0.244	-0.896	-0.951	-0.937	-0.934	-1.281	-0.950	-0.010
RR	-1.002	-0.906	-0.896	-1.016	-0.935	-0.918	-1.026	-0.920	NA
SFRR	-1.004	-0.928	-0.895	-1.014	-0.942	-0.928	-1.024	-0.937	-0.010
RRSEM	-0.974	-0.951	-0.900	-1.021	-0.907	-0.896	-0.940	-0.873	0.001

Table 23: Coefficient variance for $\lambda = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.575	1.442	0.001	0.014	0.001	0.001	0.213	0.003	NA
SEM	0.551	1.386	0.001	0.013	0.001	0.001	0.205	0.003	0.002
RR	0.007	0.018	0.001	0.000	0.001	0.001	0.003	0.001	NA
SFRR	0.005	0.012	0.001	0.000	0.001	0.001	0.002	0.001	0.002
RRSEM	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002

Table 24: Coefficient MSE for $\lambda = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.928	1.512	0.802	0.922	0.879	0.872	1.835	0.903	NA
SEM	0.889	1.445	0.802	0.917	0.879	0.873	1.847	0.905	0.002
RR	1.010	0.839	0.803	1.033	0.875	0.843	1.056	0.848	NA
SFRR	1.012	0.873	0.802	1.027	0.889	0.862	1.051	0.879	0.002
RRSEM	0.949	0.905	0.811	1.043	0.823	0.803	0.884	0.763	0.002

Results $\lambda = 0.5$

Table 25: Coefficient bias for $\lambda = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	-0.605	-0.283	-0.897	-0.954	-0.938	-0.934	-1.267	-0.949	NA
SEM	-0.580	-0.244	-0.897	-0.950	-0.938	-0.935	-1.281	-0.951	-0.009
RR	-1.001	-0.907	-0.897	-1.016	-0.936	-0.919	-1.027	-0.922	NA
SFRR	-1.004	-0.942	-0.897	-1.011	-0.948	-0.937	-1.023	-0.950	-0.009
RRSEM	-0.974	-0.953	-0.901	-1.021	-0.908	-0.897	-0.941	-0.877	-0.003

Table 26: Coefficient variance for $\lambda = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.770	1.926	0.001	0.019	0.001	0.001	0.285	0.004	NA
SEM	0.636	1.599	0.000	0.015	0.001	0.001	0.237	0.003	0.002
RR	0.010	0.025	0.001	0.000	0.001	0.001	0.004	0.001	NA
SFRR	0.006	0.015	0.001	0.000	0.001	0.001	0.002	0.001	0.002
RRSEM	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.001	0.002

Table 27: Coefficient MSE for $\lambda = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	1.137	2.006	0.806	0.929	0.881	0.874	1.889	0.906	NA
SEM	0.973	1.658	0.806	0.918	0.881	0.876	1.879	0.908	0.002
RR	1.012	0.847	0.806	1.032	0.877	0.845	1.058	0.851	NA
SFRR	1.015	0.903	0.805	1.023	0.900	0.878	1.049	0.903	0.002
RRSEM	0.949	0.907	0.812	1.042	0.824	0.805	0.886	0.769	0.002

Results $\lambda = 0.7$ Table 28: Coefficient bias for $\lambda = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	-0.629	-0.325	-0.902	-0.957	-0.941	-0.937	-1.251	-0.951	NA
SEM	-0.590	-0.264	-0.902	-0.951	-0.941	-0.938	-1.274	-0.954	-0.006
RR	-0.999	-0.909	-0.902	-1.015	-0.939	-0.923	-1.026	-0.925	NA
SFRR	-1.005	-0.958	-0.902	-1.009	-0.955	-0.946	-1.021	-0.963	-0.006
RRSEM	-0.974	-0.957	-0.905	-1.020	-0.911	-0.902	-0.946	-0.887	-0.002

Table 29: Coefficient variance for $\lambda = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	1.224	3.050	0.001	0.030	0.001	0.002	0.452	0.007	NA
SEM	0.703	1.766	0.000	0.017	0.001	0.001	0.261	0.004	0.001
RR	0.016	0.039	0.001	0.000	0.001	0.001	0.006	0.001	NA
SFRR	0.008	0.019	0.001	0.000	0.001	0.001	0.003	0.001	0.001
RRSEM	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.001	0.001

Table 30: Coefficient MSE for $\lambda = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	1.620	3.156	0.814	0.946	0.886	0.880	2.017	0.912	NA
SEM	1.051	1.836	0.814	0.922	0.887	0.882	1.885	0.914	0.001
RR	1.015	0.865	0.814	1.031	0.883	0.852	1.059	0.858	NA
SFRR	1.018	0.937	0.813	1.018	0.914	0.896	1.045	0.928	0.001
RRSEM	0.949	0.916	0.820	1.041	0.831	0.815	0.896	0.787	0.001

Results $\lambda = 0.9$ Table 31: Coefficient bias for $\lambda = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	-0.715	-0.478	-0.921	-0.968	-0.952	-0.948	-1.194	-0.960	NA
SEM	-0.661	-0.394	-0.921	-0.959	-0.953	-0.951	-1.226	-0.964	-0.003
RR	-0.998	-0.925	-0.921	-1.012	-0.950	-0.938	-1.022	-0.940	NA
SFRR	-1.005	-0.977	-0.920	-1.006	-0.968	-0.962	-1.016	-0.979	-0.003
RRSEM	-0.977	-0.970	-0.923	-1.016	-0.926	-0.921	-0.961	-0.913	-0.002

Table 32: Coefficient variance for $\lambda = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	3.138	7.807	0.001	0.077	0.004	0.006	1.158	0.017	NA
SEM	0.636	1.601	0.000	0.015	0.001	0.001	0.237	0.003	0.000
RR	0.041	0.096	0.001	0.001	0.004	0.001	0.015	0.002	NA
SFRR	0.009	0.021	0.001	0.000	0.001	0.001	0.003	0.001	0.000
RRSEM	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.001	0.000

Table 33: Coefficient MSE for $\lambda = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	3.649	8.035	0.848	1.013	0.910	0.905	2.583	0.938	NA
SEM	1.073	1.756	0.848	0.936	0.908	0.905	1.740	0.932	0.000
RR	1.038	0.951	0.849	1.025	0.907	0.880	1.060	0.886	NA
SFRR	1.019	0.976	0.847	1.012	0.938	0.926	1.036	0.959	0.000
RRSEM	0.954	0.942	0.852	1.032	0.858	0.849	0.924	0.835	0.000

Results average β coefficients

Table 34: Average Coefficient bias

λ	OLS	SEM	RR	SFRR	RRSEM
0.1	-0.590	-0.586	-1.003	-1.003	-0.974
0.3	-0.595	-0.581	-1.002	-1.004	-0.974
0.5	-0.605	-0.580	-1.001	-1.004	-0.974
0.7	-0.629	-0.590	-0.999	-1.005	-0.974

λ	OLS	SEM	RR	SFRR	RRSEM
0.9	-0.715	-0.661	-0.998	-1.005	-0.977

Table 35: Average Coefficient variance

λ	OLS	SEM	RR	SFRR	RRSEM
0.1	0.469	0.470	0.005	0.004	0.000
0.3	0.575	0.551	0.007	0.005	0.000
0.5	0.770	0.636	0.010	0.006	0.000
0.7	1.224	0.703	0.016	0.008	0.000
0.9	3.138	0.636	0.041	0.009	0.000

Table 36: Average Coefficient MSE

λ	OLS	SEM	RR	SFRR	RRSEM
0.1	0.816	0.813	1.010	1.011	0.949
0.3	0.928	0.889	1.010	1.012	0.949
0.5	1.137	0.973	1.012	1.015	0.949
0.7	1.620	1.051	1.015	1.018	0.949
0.9	3.649	1.073	1.038	1.019	0.954