Ridge regularization for spatial auto-regressive models with multicollinearity issues.

Simulations results of comparison of RRSEM with existing methods

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In our paper titled "Ridge Regularization for Spatial Autoregressive Models with Multicollinearity Issues," we introduce a novel approach for conducting Ridge regression in the context of spatial autoregressive models.

In this document, we provide the complete results of the simulations conducted within the framework of the dependent variable determined by the following equation:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + (I_n - \lambda W)^{-1}\boldsymbol{\varepsilon},$$

We consider eight highly correlated covariates generated as described in Section 5. Here, we present two scenarios: the deterministic scenario where the covariates were generated once for all the simulations and the stochastic scenario where the covariates were generated for each simulation.

The SEM model, defined in equation 2 of the paper, is generated for five values of the dependence parameter $\lambda \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

The following tables display the average bias, average variance, and average mean squared error (MSE) of the eight regression coefficient estimates and the dependence parameter estimates computed across 500 simulations for each value of the dependence parameter (λ). These estimates are computed using different estimation algorithms: OLS, ordinary SEM, ordinary Ridge, Spatially Filtered Ridge Regression (SFRR), and our methodology, named Ridge Regression for SEM Models (RRSEM). We present the results in two sections, the first one is dedicated to the results of deterministic covariates, and the second one to the results of stochastic covariates.

Tables 16-18 and 34-36 compile the average results for all the regression coefficients.

Deterministic case

Results $\lambda = 0.1$

Table 1: Coefficient bias for $\lambda = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.105	0.189	-0.008	0.064	-0.002	-0.001	-0.164	-0.008	NA
SEM	0.113	0.206	-0.008	0.069	-0.003	-0.001	-0.179	-0.008	-0.007
RR	-0.568	-0.459	-0.066	-1.019	-0.027	0.093	-0.248	0.203	NA
SFRR	-0.611	-0.462	-0.056	-0.956	-0.061	0.045	-0.266	0.178	-0.007
RRSEM	-0.458	-0.495	-0.045	-0.717	-0.012	0.065	0.015	0.131	-0.007

Table 2: Coefficient variance for $\lambda = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	5.630	19.506	0.015	4.324	0.016	0.034	12.793	0.084	NA
SEM	5.569	19.354	0.015	4.264	0.016	0.033	12.740	0.083	0.002
RR	0.038	0.026	0.010	0.131	0.006	0.006	0.063	0.012	NA
SFRR	0.045	0.027	0.010	0.167	0.006	0.007	0.066	0.013	0.002
RRSEM	0.106	0.066	0.012	0.480	0.007	0.010	0.296	0.031	0.002

Table 3: Coefficient MSE for $\lambda = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	5.641	19.542	0.015	4.328	0.016	0.034	12.819	0.084	NA
SEM	5.582	19.397	0.015	4.268	0.016	0.033	12.772	0.083	0.002
RR	0.361	0.237	0.014	1.169	0.007	0.015	0.125	0.053	NA
SFRR	0.419	0.240	0.013	1.081	0.010	0.009	0.137	0.044	0.002
RRSEM	0.316	0.312	0.014	0.995	0.007	0.014	0.296	0.048	0.002

Results $\lambda = 0.3$

Table 4: Coefficient bias for $\lambda = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.106	0.187	-0.008	0.066	-0.002	-0.001	-0.159	-0.009	NA
SEM	0.115	0.212	-0.008	0.069	-0.003	-0.001	-0.186	-0.007	-0.006
RR	-0.571	-0.462	-0.066	-1.023	-0.027	0.093	-0.247	0.204	NA
SFRR	-0.718	-0.490	-0.038	-0.848	-0.134	-0.055	-0.300	0.125	-0.006
RRSEM	-0.450	-0.494	-0.044	-0.699	-0.011	0.063	0.026	0.127	-0.006

Table 5: Coefficient variance for $\lambda = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	6.296	21.849	0.016	4.814	0.018	0.038	14.295	0.091	NA
SEM	5.716	20.036	0.015	4.310	0.017	0.034	13.290	0.083	0.002
RR	0.035	0.028	0.011	0.119	0.007	0.007	0.069	0.012	NA
SFRR	0.043	0.029	0.011	0.154	0.006	0.007	0.068	0.013	0.002
RRSEM	0.116	0.078	0.012	0.512	0.007	0.010	0.318	0.033	0.002

Table 6: Coefficient MSE for $\lambda = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	6.307	21.884	0.016	4.819	0.018	0.038	14.321	0.091	NA
SEM	5.729	20.081	0.015	4.315	0.017	0.034	13.325	0.083	0.002
RR	0.361	0.242	0.015	1.166	0.008	0.016	0.130	0.054	NA
SFRR	0.559	0.270	0.012	0.873	0.024	0.010	0.158	0.028	0.002
RRSEM	0.318	0.321	0.014	1.000	0.008	0.014	0.318	0.049	0.002

Results $\lambda = 0.5$

Table 7: Coefficient bias for $\lambda = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.105	0.177	-0.008	0.069	-0.002	-0.001	-0.147	-0.010	NA
SEM	0.115	0.214	-0.007	0.068	-0.003	-0.001	-0.190	-0.006	-0.004
RR	-0.563	-0.460	-0.066	-1.009	-0.026	0.092	-0.239	0.201	NA
SFRR	-0.799	-0.496	-0.015	-0.686	-0.205	-0.160	-0.319	0.062	-0.004
RRSEM	-0.440	-0.490	-0.043	-0.679	-0.011	0.061	0.036	0.122	-0.005

Table 8: Coefficient variance for $\lambda = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	7.775	27.047	0.019	5.921	0.022	0.047	17.642	0.107	NA
SEM	5.758	20.350	0.015	4.280	0.017	0.034	13.591	0.081	0.001
RR	0.050	0.037	0.012	0.167	0.009	0.008	0.095	0.017	NA
SFRR	0.061	0.039	0.012	0.240	0.006	0.008	0.094	0.017	0.001
RRSEM	0.126	0.100	0.012	0.532	0.008	0.010	0.342	0.035	0.001

Table 9: Coefficient MSE for $\lambda = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	7.786	27.078	0.019	5.926	0.022	0.047	17.664	0.107	NA
SEM	5.771	20.396	0.015	4.285	0.017	0.034	13.628	0.081	0.001
RR	0.366	0.248	0.017	1.184	0.009	0.017	0.153	0.057	NA
SFRR	0.698	0.285	0.013	0.711	0.049	0.033	0.196	0.021	0.001
RRSEM	0.319	0.340	0.014	0.993	0.008	0.014	0.344	0.050	0.001

Results $\lambda = 0.7$

Table 10: Coefficient bias for $\lambda = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.091	0.135	-0.009	0.063	-0.001	-0.001	-0.106	-0.012	NA
SEM	0.113	0.213	-0.007	0.066	-0.003	-0.001	-0.191	-0.006	-0.003
RR	-0.569	-0.469	-0.068	-1.025	-0.028	0.094	-0.242	0.204	NA
SFRR	-0.889	-0.501	0.007	-0.550	-0.278	-0.263	-0.358	0.006	-0.003
RRSEM	-0.426	-0.479	-0.041	-0.653	-0.010	0.059	0.042	0.117	-0.004

Table 11: Coefficient variance for $\lambda = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	11.594	40.476	0.028	8.804	0.031	0.072	26.289	0.149	NA
SEM	5.690	20.260	0.014	4.178	0.017	0.033	13.609	0.078	0.001
RR	0.051	0.050	0.017	0.154	0.012	0.013	0.120	0.020	NA
SFRR	0.070	0.035	0.014	0.310	0.006	0.010	0.083	0.018	0.001

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
RRSEM	0.137	0.133	0.012	0.548	0.008	0.011	0.370	0.036	0.001

Table 12: Coefficient MSE for $\lambda = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	11.602	40.494	0.028	8.808	0.031	0.072	26.301	0.149	NA
SEM	5.703	20.305	0.014	4.183	0.017	0.033	13.646	0.078	0.001
RR	0.375	0.270	0.022	1.205	0.013	0.021	0.179	0.061	NA
SFRR	0.860	0.286	0.014	0.613	0.083	0.079	0.211	0.018	0.001
RRSEM	0.318	0.362	0.013	0.974	0.008	0.014	0.371	0.049	0.001

Results $\lambda = 0.9$

Table 13: Coefficient bias for $\lambda = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	-0.012	-0.115	-0.012	-0.001	0.004	0.003	0.123	-0.012	NA
SEM	0.109	0.207	-0.007	0.062	-0.003	0.000	-0.187	-0.005	-0.003
RR	-0.566	-0.471	-0.072	-1.044	-0.031	0.095	-0.257	0.208	NA
SFRR	-0.979	-0.518	0.030	-0.401	-0.350	-0.367	-0.374	-0.054	-0.003
RRSEM	-0.415	-0.471	-0.039	-0.633	-0.010	0.057	0.046	0.113	-0.003

Table 14: Coefficient variance for $\lambda = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	32.714	114.170	0.077	25.272	0.081	0.225	73.283	0.385	NA
SEM	5.519	19.774	0.014	4.014	0.016	0.032	13.340	0.074	0.000
RR	0.095	0.109	0.043	0.220	0.028	0.038	0.215	0.038	NA
SFRR	0.085	0.041	0.016	0.392	0.006	0.012	0.106	0.021	0.000
RRSEM	0.145	0.161	0.011	0.555	0.008	0.010	0.389	0.036	0.000

Table 15: Coefficient MSE for $\lambda = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	32.714	114.184	0.077	25.272	0.081	0.225	73.298	0.385	NA
SEM	5.531	19.817	0.014	4.018	0.016	0.032	13.375	0.074	0.000
RR	0.416	0.331	0.049	1.310	0.029	0.047	0.281	0.082	NA
SFRR	1.043	0.309	0.017	0.553	0.129	0.147	0.246	0.024	0.000
RRSEM	0.318	0.382	0.013	0.956	0.008	0.014	0.392	0.049	0.000

Average of the β coefficients

Table 16: Average Coefficient bias

λ	OLS	SEM	RR	SFRR	RRSEM
0.1	0.105	0.113	-0.568	-0.611	-0.458
0.3	0.106	0.115	-0.571	-0.718	-0.450
0.5	0.105	0.115	-0.563	-0.799	-0.440
0.7	0.091	0.113	-0.569	-0.889	-0.426
0.9	-0.012	0.109	-0.566	-0.979	-0.415

Table 17: Average Coefficient variance

λ	OLS	SEM	RR	SFRR	RRSEM
0.1	5.630	5.569	0.038	0.045	0.106
0.3	6.296	5.716	0.035	0.043	0.116
0.5	7.775	5.758	0.050	0.061	0.126
0.7	11.594	5.690	0.051	0.070	0.137
0.9	32.714	5.519	0.095	0.085	0.145

Table 18: Average Coefficient MSE

λ	OLS	SEM	RR	SFRR	RRSEM
0.1	5.641	5.582	0.361	0.419	0.316
0.3	6.307	5.729	0.361	0.559	0.318
0.5	7.786	5.771	0.366	0.698	0.319
0.7	11.602	5.703	0.375	0.860	0.318
0.9	32.714	5.531	0.416	1.043	0.318

Stochastic case

Results $\lambda = 0.1$

Table 19: Coefficient bias for $\lambda = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.164	0.251	0.006	0.143	0.000	-0.009	-0.151	-0.023	NA
SEM	0.161	0.248	0.006	0.138	-0.001	-0.009	-0.153	-0.021	-0.009
RR	-0.533	-0.455	-0.045	-0.816	-0.024	0.070	-0.158	0.173	NA
SFRR	-0.576	-0.471	-0.045	-0.760	-0.048	0.034	-0.158	0.144	-0.009
RRSEM	-0.422	-0.491	-0.024	-0.540	-0.008	0.046	0.099	0.103	-0.009

Table 20: Coefficient variance for $\lambda = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	6.277	22.842	0.013	3.504	0.016	0.021	14.551	0.064	NA

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
SEM	6.201	22.558	0.013	3.468	0.016	0.021	14.345	0.063	0.002
RR	0.068	0.034	0.011	0.241	0.005	0.006	0.121	0.017	NA
SFRR	0.070	0.038	0.013	0.247	0.005	0.007	0.133	0.017	0.002
RRSEM	0.131	0.113	0.012	0.458	0.006	0.008	0.342	0.032	0.002

Table 21: Coefficient MSE for $\lambda = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	6.304	22.905	0.013	3.525	0.016	0.022	14.574	0.064	NA
SEM	6.226	22.619	0.013	3.487	0.016	0.021	14.368	0.063	0.002
RR	0.352	0.241	0.013	0.907	0.005	0.011	0.146	0.047	NA
SFRR	0.401	0.260	0.015	0.825	0.007	0.008	0.158	0.038	0.002
RRSEM	0.309	0.354	0.012	0.749	0.006	0.010	0.352	0.043	0.002

Results $\lambda = 0.3$

Table 22: Coefficient bias for $\lambda = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.166	0.251	0.006	0.148	0.000	-0.009	-0.147	-0.024	NA
SEM	0.165	0.259	0.006	0.138	-0.001	-0.009	-0.165	-0.021	-0.008
RR	-0.533	-0.454	-0.046	-0.818	-0.023	0.071	-0.159	0.173	NA
SFRR	-0.703	-0.508	-0.050	-0.710	-0.104	-0.040	-0.216	0.096	-0.008
RRSEM	-0.413	-0.482	-0.024	-0.528	-0.008	0.044	0.097	0.101	-0.008

Table 23: Coefficient variance for $\lambda = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	6.968	25.379	0.015	3.877	0.018	0.024	16.171	0.069	NA
SEM	6.227	22.663	0.013	3.475	0.016	0.021	14.397	0.063	0.002
RR	0.074	0.035	0.012	0.262	0.005	0.007	0.132	0.019	NA
SFRR	0.079	0.049	0.022	0.274	0.007	0.009	0.165	0.021	0.002
RRSEM	0.143	0.142	0.012	0.475	0.006	0.008	0.365	0.033	0.002

Table 24: Coefficient MSE for $\lambda = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	6.996	25.443	0.015	3.899	0.018	0.024	16.193	0.070	NA
SEM	6.254	22.730	0.013	3.495	0.016	0.021	14.424	0.063	0.002
RR	0.358	0.241	0.014	0.930	0.006	0.012	0.158	0.048	NA
SFRR	0.573	0.307	0.025	0.778	0.018	0.011	0.211	0.030	0.002
RRSEM	0.313	0.374	0.012	0.754	0.006	0.010	0.375	0.044	0.002

Results $\lambda = 0.5$

Table 25: Coefficient bias for $\lambda = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.176	0.263	0.007	0.161	0.001	-0.010	-0.149	-0.027	NA
SEM	0.168	0.267	0.006	0.137	-0.002	-0.009	-0.175	-0.020	-0.006
RR	-0.547	-0.454	-0.049	-0.848	-0.024	0.074	-0.184	0.180	NA
SFRR	-0.810	-0.538	-0.051	-0.617	-0.160	-0.118	-0.247	0.039	-0.006
RRSEM	-0.406	-0.475	-0.023	-0.519	-0.008	0.043	0.098	0.099	-0.006

Table 26: Coefficient variance for $\lambda = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	8.484	30.903	0.018	4.716	0.022	0.030	19.667	0.083	NA
SEM	6.126	22.297	0.012	3.412	0.017	0.021	14.151	0.061	0.001
RR	0.075	0.038	0.015	0.259	0.007	0.008	0.132	0.020	NA
SFRR	0.110	0.089	0.040	0.365	0.011	0.014	0.259	0.029	0.001
RRSEM	0.155	0.177	0.012	0.483	0.006	0.008	0.380	0.034	0.001

Table 27: Coefficient MSE for $\lambda = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	8.514	30.973	0.018	4.742	0.022	0.030	19.689	0.084	NA
SEM	6.154	22.368	0.013	3.431	0.017	0.021	14.181	0.061	0.001
RR	0.374	0.243	0.017	0.978	0.008	0.014	0.165	0.052	NA
SFRR	0.766	0.378	0.043	0.745	0.036	0.028	0.320	0.031	0.001
RRSEM	0.320	0.403	0.012	0.752	0.006	0.010	0.390	0.043	0.001

Results $\lambda = 0.7$

Table 28: Coefficient bias for $\lambda = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.199	0.294	0.009	0.189	0.003	-0.013	-0.158	-0.033	NA
SEM	0.168	0.272	0.006	0.134	-0.002	-0.009	-0.183	-0.019	-0.004
RR	-0.559	-0.448	-0.050	-0.876	-0.024	0.076	-0.213	0.187	NA
SFRR	-0.933	-0.574	-0.053	-0.555	-0.217	-0.194	-0.296	-0.011	-0.004
RRSEM	-0.402	-0.470	-0.023	-0.514	-0.008	0.043	0.094	0.098	-0.004

Table 29: Coefficient variance for $\lambda = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	12.301	44.751	0.028	6.863	0.033	0.045	28.356	0.118	NA
SEM	5.909	21.495	0.012	3.286	0.016	0.020	13.627	0.058	0.001
RR	0.086	0.051	0.021	0.290	0.011	0.012	0.150	0.023	NA
SFRR	0.141	0.161	0.067	0.438	0.016	0.021	0.384	0.038	0.001
RRSEM	0.166	0.204	0.011	0.485	0.006	0.008	0.385	0.034	0.001

Table 30: Coefficient MSE for $\lambda = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	12.341	44.837	0.028	6.899	0.033	0.045	28.381	0.119	NA
SEM	5.938	21.569	0.012	3.304	0.016	0.020	13.660	0.059	0.001
RR	0.399	0.251	0.024	1.057	0.011	0.018	0.195	0.058	NA
SFRR	1.013	0.490	0.070	0.746	0.063	0.058	0.472	0.038	0.001
RRSEM	0.328	0.425	0.012	0.749	0.006	0.010	0.394	0.043	0.001

Results $\lambda = 0.9$

Table 31: Coefficient bias for $\lambda = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	0.280	0.402	0.017	0.289	0.012	-0.024	-0.195	-0.051	NA
SEM	0.167	0.272	0.005	0.129	-0.003	-0.009	-0.188	-0.017	-0.003
RR	-0.551	-0.414	-0.045	-0.859	-0.017	0.072	-0.233	0.183	NA
SFRR	-1.051	-0.617	-0.054	-0.478	-0.272	-0.271	-0.326	-0.066	-0.003
RRSEM	-0.400	-0.470	-0.022	-0.510	-0.008	0.042	0.098	0.097	-0.003

Table 32: Coefficient variance for $\lambda = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	32.018	115.785	0.078	18.071	0.089	0.124	72.524	0.307	NA
SEM	5.598	20.337	0.011	3.109	0.016	0.019	12.875	0.055	0.000
RR	0.194	0.129	0.056	0.542	0.031	0.033	0.325	0.054	NA
SFRR	0.209	0.288	0.103	0.615	0.024	0.031	0.624	0.054	0.000
RRSEM	0.170	0.220	0.011	0.474	0.006	0.008	0.377	0.033	0.000

Table 33: Coefficient MSE for $\lambda = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	λ
OLS	32.096	115.946	0.079	18.154	0.089	0.124	72.562	0.309	NA
SEM	5.625	20.411	0.011	3.126	0.016	0.019	12.910	0.055	0.000
RR	0.497	0.301	0.058	1.279	0.032	0.038	0.379	0.088	NA
SFRR	1.314	0.669	0.106	0.844	0.098	0.104	0.730	0.058	0.000
RRSEM	0.330	0.441	0.011	0.734	0.006	0.010	0.386	0.042	0.000

Average of the β coefficients

Table 34: Average Coefficient bias

λ	OLS	SEM	RR	SFRR	RRSEM
0.1	0.164	0.161	-0.533	-0.576	-0.422
0.3	0.166	0.165	-0.533	-0.703	-0.413
0.5	0.176	0.168	-0.547	-0.810	-0.406
0.7	0.199	0.168	-0.559	-0.933	-0.402
0.9	0.280	0.167	-0.551	-1.051	-0.400

Table 35: Average Coefficient variance

λ	OLS	SEM	RR	SFRR	RRSEM
0.1	6.277	6.201	0.068	0.070	0.131
0.3	6.968	6.227	0.074	0.079	0.143
0.5	8.484	6.126	0.075	0.110	0.155
0.7	12.301	5.909	0.086	0.141	0.166
0.9	32.018	5.598	0.194	0.209	0.170

Table 36: Average Coefficient MSE

λ	OLS	SEM	RR	SFRR	RRSEM
0.1	6.304	6.226	0.352	0.401	0.309
0.3	6.996	6.254	0.358	0.573	0.313
0.5	8.514	6.154	0.374	0.766	0.320
0.7	12.341	5.938	0.399	1.013	0.328
0.9	32.096	5.625	0.497	1.314	0.330