Ridge regularization for spatial auto-regressive models with multicollinearity issues.

Simulations results of comparison of RRSAR with existing methods

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In our paper titled "Ridge Regularization for Spatial Autoregressive Models with Multicollinearity Issues," we introduce a novel approach for conducting Ridge regression in the context of spatial autoregressive models.

In this document, we provide the complete results of the simulations conducted within the framework of the dependent variable determined by the following equation:

$$\mathbf{Y} = (I_n - \rho W)^{-1}\mathbf{X}\boldsymbol{\beta} + (I_n - \rho W)^{-1}\boldsymbol{\varepsilon},$$

We consider eight highly correlated covariates generated as described in Section 5. Here, we present two scenarios: the deterministic scenario where the covariates were generated once for all the simulations and the stochastic scenario where the covariates were generated for each simulation.

The SAR model, defined in equation 2 of the paper, is generated for five values of the dependence parameter $\rho \in (0.1, 0.3, 0.5, 0.7, 0.9)$.

The following tables display the average bias, average variance, and average mean squared error (MSE) of the eight regression coefficient estimates and the dependence parameter estimates computed across 500 simulations for each value of the dependence parameter (ρ). These estimates are computed using different estimation algorithms: OLS, ordinary SAR, ordinary Ridge, Spatially Filtered Ridge Regression (SFRR), and our methodology, named Ridge Regression for SAR Models (RRSAR). We present the results in two sections, the first one for the results of the deterministic covariates, and the second one for the results of the stochastic covariates.

Tables 16-18 and 34-36 compile the average results for all the regression coefficients.

Deterministic

Results $\rho = 0.1$

Table 1: Coefficient bias for $\rho = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	-0.527	-0.162	-0.908	-0.942	-0.938	-0.942	-1.312	-0.955	NA
SAR	-0.581	-0.251	-0.909	-0.949	-0.940	-0.943	-1.280	-0.955	-0.006
RR	-1.004	-0.914	-0.908	-1.016	-0.934	-0.923	-1.022	-0.921	NA
SFRR	-1.007	-0.922	-0.909	-1.015	-0.937	-0.927	-1.021	-0.925	-0.006
RRSAR	-0.974	-0.952	-0.913	-1.024	-0.913	-0.905	-0.942	-0.882	-0.093

Table 2: Coefficient variance for $\rho = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.426	1.046	0.000	0.010	0.000	0.001	0.156	0.002	NA
SAR	0.399	0.980	0.000	0.010	0.000	0.001	0.146	0.002	0.001
RR	0.003	0.006	0.000	0.000	0.000	0.000	0.001	0.000	NA
SFRR	0.002	0.005	0.000	0.000	0.000	0.000	0.001	0.000	0.001
RRSAR	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 3: Coefficient MSE for $\rho = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.703	1.073	0.825	0.897	0.880	0.887	1.876	0.913	NA
SAR	0.736	1.043	0.826	0.910	0.884	0.890	1.784	0.915	0.001
RR	1.012	0.842	0.825	1.033	0.873	0.853	1.046	0.848	NA
SFRR	1.017	0.855	0.826	1.031	0.878	0.859	1.044	0.856	0.001
RRSAR	0.949	0.906	0.833	1.049	0.834	0.820	0.888	0.778	0.009

Results $\rho = 0.3$

Table 4: Coefficient bias for $\rho = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	-0.539	-0.202	-0.931	-0.942	-0.951	-0.955	-1.294	-0.966	NA
SAR	-0.696	-0.456	-0.934	-0.963	-0.957	-0.959	-1.203	-0.968	-0.005
RR	-0.997	-0.922	-0.932	-1.014	-0.947	-0.937	-1.017	-0.933	NA
SFRR	-1.005	-0.943	-0.934	-1.011	-0.954	-0.947	-1.016	-0.946	-0.005
RRSAR	-0.979	-0.962	-0.935	-1.020	-0.934	-0.927	-0.954	-0.908	-0.159

Table 5: Coefficient variance for $\rho = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.383	0.942	0.000	0.009	0.000	0.001	0.140	0.002	NA
SAR	0.290	0.712	0.000	0.007	0.000	0.000	0.106	0.001	0.000
RR	0.003	0.006	0.000	0.000	0.000	0.000	0.001	0.000	NA
SFRR	0.001	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000
RRSAR	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 6: Coefficient MSE for $\rho = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.674	0.983	0.867	0.897	0.904	0.912	1.816	0.934	NA
SAR	0.773	0.920	0.872	0.934	0.916	0.920	1.554	0.938	0.000
RR	0.997	0.857	0.868	1.028	0.897	0.878	1.035	0.871	NA
SFRR	1.012	0.893	0.872	1.023	0.911	0.897	1.032	0.895	0.000
RRSAR	0.958	0.926	0.875	1.040	0.872	0.860	0.910	0.825	0.026

Results $\rho = 0.5$

Table 7: Coefficient bias for $\rho = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	-0.582	-0.294	-0.954	-0.947	-0.965	-0.969	-1.259	-0.978	NA
SAR	-0.809	-0.659	-0.959	-0.977	-0.973	-0.974	-1.127	-0.980	-0.005
RR	-0.989	-0.935	-0.954	-1.011	-0.962	-0.953	-1.012	-0.949	NA
SFRR	-1.003	-0.964	-0.959	-1.007	-0.971	-0.967	-1.010	-0.966	-0.005
RRSAR	-0.995	-0.985	-0.961	-1.004	-0.966	-0.963	-0.982	-0.958	0.181

Table 8: Coefficient variance for $\rho = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.331	0.813	0.000	0.008	0.000	0.001	0.121	0.002	NA
SAR	0.181	0.446	0.000	0.004	0.000	0.000	0.066	0.001	0.000
RR	0.003	0.007	0.000	0.000	0.000	0.000	0.001	0.000	NA
SFRR	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
RRSAR	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001

Table 9: Coefficient MSE for $\rho = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.669	0.899	0.910	0.905	0.932	0.939	1.705	0.958	NA
SAR	0.836	0.881	0.919	0.958	0.947	0.949	1.337	0.961	0.000
RR	0.982	0.881	0.911	1.021	0.926	0.909	1.025	0.900	NA
SFRR	1.006	0.931	0.919	1.014	0.944	0.935	1.020	0.933	0.000
RRSAR	0.989	0.970	0.923	1.007	0.933	0.928	0.964	0.918	0.034

Results $\rho = 0.7$

Table 10: Coefficient bias for $\rho = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	-0.666	-0.454	-0.975	-0.957	-0.981	-0.983	-1.199	-0.990	NA
SAR	-0.910	-0.839	-0.980	-0.989	-0.987	-0.988	-1.060	-0.990	-0.003
RR	-0.984	-0.953	-0.975	-1.007	-0.979	-0.971	-1.006	-0.967	NA
SFRR	-1.001	-0.982	-0.981	-1.003	-0.987	-0.984	-1.005	-0.984	-0.003
RRSAR	-1.000	-0.995	-0.982	-0.999	-0.985	-0.985	-0.993	-0.984	0.220

Table 11: Coefficient variance for $\rho = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.263	0.646	0.000	0.006	0.000	0.000	0.096	0.001	NA
SAR	0.086	0.210	0.000	0.002	0.000	0.000	0.031	0.000	0.000
RR	0.004	0.008	0.000	0.000	0.000	0.000	0.001	0.000	NA

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
SFRR	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
RRSAR	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 12: Coefficient MSE for $\rho = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.706	0.852	0.951	0.923	0.963	0.966	1.533	0.981	NA
SAR	0.913	0.914	0.961	0.980	0.975	0.976	1.155	0.982	0.000
RR	0.971	0.917	0.951	1.014	0.958	0.942	1.014	0.936	NA
SFRR	1.002	0.965	0.961	1.006	0.973	0.969	1.010	0.968	0.000
RRSAR	1.000	0.990	0.964	0.997	0.970	0.970	0.987	0.968	0.048

Results $\rho = 0.9$

Table 13: Coefficient bias for $\rho = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	-0.827	-0.732	-0.993	-0.978	-0.997	-0.996	-1.097	-0.999	NA
SAR	-0.983	-0.969	-0.996	-0.998	-0.998	-0.998	-1.012	-0.998	-0.001
RR	-0.986	-0.982	-0.994	-1.002	-0.996	-0.991	-1.002	-0.989	NA
SFRR	-0.999	-0.995	-0.996	-1.001	-0.997	-0.997	-1.001	-0.997	-0.001
RRSAR	-1.000	-0.999	-0.996	-0.999	-0.997	-0.997	-0.998	-0.997	0.094

Table 14: Coefficient variance for $\rho = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.156	0.383	0.000	0.004	0.000	0.000	0.057	0.001	NA
SAR	0.016	0.040	0.000	0.000	0.000	0.000	0.006	0.000	0.000
RR	0.011	0.021	0.001	0.000	0.001	0.003	0.002	0.003	NA
SFRR	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
RRSAR	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 15: Coefficient MSE for $\rho = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.839	0.919	0.987	0.960	0.994	0.991	1.261	1.000	NA
SAR	0.982	0.979	0.992	0.996	0.995	0.995	1.029	0.996	0.000
RR	0.983	0.985	0.989	1.005	0.993	0.984	1.005	0.982	NA
SFRR	0.999	0.991	0.992	1.001	0.995	0.994	1.003	0.994	0.000
RRSAR	1.001	0.998	0.993	0.998	0.993	0.994	0.997	0.994	0.009

Results average β coefficients

Table 16: Average Coefficient bias

ρ	OLS	SAR	RR	SFRR	RRSAR
0.1	-0.527	-0.581	-1.004	-1.007	-0.974
0.3	-0.539	-0.696	-0.997	-1.005	-0.979
0.5	-0.582	-0.809	-0.989	-1.003	-0.995
0.7	-0.666	-0.910	-0.984	-1.001	-1.000
0.9	-0.827	-0.983	-0.986	-0.999	-1.000

Table 17: Average Coefficient variance

ρ	OLS	SAR	RR	SFRR	RRSAR
0.1	0.426	0.399	0.003	0.002	0
0.3	0.383	0.290	0.003	0.001	0
0.5	0.331	0.181	0.003	0.001	0
0.7	0.263	0.086	0.004	0.000	0
0.9	0.156	0.016	0.011	0.000	0

Table 18: Average Coefficient MSE

ρ	OLS	SAR	RR	SFRR	RRSAR
0.1	0.703	0.736	1.012	1.017	0.949
0.3	0.674	0.773	0.997	1.012	0.958
0.5	0.669	0.836	0.982	1.006	0.989
0.7	0.706	0.913	0.971	1.002	1.000
0.9	0.839	0.982	0.983	0.999	1.001

Stochastic

Results $\rho = 0.1$

Table 19: Coefficient bias for $\rho = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	-0.603	-0.282	-0.906	-0.953	-0.942	-0.938	-1.266	-0.951	NA
SAR	-0.633	-0.336	-0.907	-0.957	-0.944	-0.940	-1.247	-0.954	-0.002
RR	-1.002	-0.911	-0.907	-1.016	-0.939	-0.923	-1.024	-0.923	NA
SFRR	-1.003	-0.919	-0.907	-1.015	-0.942	-0.926	-1.023	-0.928	-0.002
RRSAR	-0.975	-0.952	-0.913	-1.022	-0.914	-0.905	-0.942	-0.880	-0.093

Table 20: Coefficient variance for $\rho = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.447	1.124	0.001	0.011	0.000	0.001	0.166	0.002	NA
SAR	0.388	0.979	0.001	0.009	0.000	0.001	0.145	0.002	0.001
RR	0.005	0.013	0.001	0.000	0.001	0.000	0.002	0.001	NA

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
SFRR	0.003	0.009	0.001	0.000	0.001	0.000	0.001	0.001	0.001
RRSAR	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 21: Coefficient MSE for $\rho = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.811	1.204	0.822	0.920	0.887	0.881	1.768	0.906	NA
SAR	0.789	1.092	0.823	0.925	0.891	0.885	1.700	0.913	0.001
RR	1.008	0.842	0.822	1.032	0.883	0.852	1.050	0.853	NA
SFRR	1.009	0.852	0.823	1.030	0.888	0.858	1.047	0.863	0.001
RRSAR	0.950	0.907	0.834	1.045	0.835	0.819	0.888	0.774	0.009

Results $\rho = 0.3$

Table 22: Coefficient bias for $\rho = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	-0.646	-0.363	-0.931	-0.958	-0.953	-0.950	-1.233	-0.958	NA
SAR	-0.733	-0.515	-0.932	-0.969	-0.959	-0.957	-1.180	-0.967	-0.002
RR	-0.999	-0.920	-0.931	-1.013	-0.951	-0.937	-1.019	-0.933	NA
SFRR	-1.002	-0.940	-0.932	-1.011	-0.958	-0.946	-1.017	-0.948	-0.002
RRSAR	-0.981	-0.965	-0.940	-1.017	-0.938	-0.932	-0.957	-0.912	-0.167

Table 23: Coefficient variance for $\rho = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.554	1.382	0.001	0.013	0.001	0.001	0.204	0.003	NA
SAR	0.289	0.730	0.000	0.007	0.000	0.001	0.108	0.002	0.000
RR	0.007	0.018	0.001	0.000	0.001	0.000	0.003	0.001	NA
SFRR	0.002	0.006	0.000	0.000	0.000	0.000	0.001	0.001	0.000
RRSAR	0.000	0.001	0.004	0.000	0.004	0.005	0.002	0.009	0.003

Table 24: Coefficient MSE for $\rho = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.971	1.514	0.866	0.931	0.908	0.904	1.724	0.920	NA
SAR	0.825	0.995	0.869	0.945	0.920	0.916	1.500	0.936	0.000
RR	1.005	0.864	0.867	1.027	0.905	0.878	1.041	0.871	NA
SFRR	1.006	0.890	0.869	1.022	0.918	0.896	1.034	0.899	0.000
RRSAR	0.963	0.932	0.888	1.035	0.885	0.874	0.918	0.841	0.031

Results $\rho = 0.5$

Table 25: Coefficient bias for $\rho = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	-0.706	-0.476	-0.954	-0.965	-0.965	-0.964	-1.189	-0.967	NA
SAR	-0.831	-0.694	-0.957	-0.980	-0.974	-0.973	-1.114	-0.979	-0.002
RR	-0.995	-0.932	-0.954	-1.010	-0.964	-0.953	-1.014	-0.947	NA
SFRR	-1.001	-0.962	-0.957	-1.007	-0.973	-0.966	-1.010	-0.967	-0.002
RRSAR	-0.993	-0.984	-0.960	-1.005	-0.966	-0.962	-0.980	-0.956	0.140

Table 26: Coefficient variance for $\rho = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.829	2.053	0.001	0.020	0.001	0.001	0.303	0.004	NA
SAR	0.187	0.473	0.000	0.004	0.000	0.000	0.070	0.001	0.000
RR	0.013	0.030	0.001	0.000	0.001	0.001	0.004	0.001	NA
SFRR	0.001	0.003	0.000	0.000	0.000	0.000	0.000	0.001	0.000
RRSAR	0.000	0.001	0.000	0.000	0.001	0.001	0.001	0.003	0.016

Table 27: Coefficient MSE for $\rho = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	1.327	2.280	0.911	0.951	0.932	0.930	1.718	0.940	NA
SAR	0.878	0.956	0.917	0.965	0.950	0.947	1.310	0.960	0.000
RR	1.003	0.899	0.911	1.021	0.929	0.908	1.033	0.897	NA
SFRR	1.004	0.929	0.917	1.014	0.948	0.934	1.021	0.936	0.000
RRSAR	0.986	0.968	0.922	1.011	0.933	0.926	0.962	0.917	0.035

Results $\rho = 0.7$

Table 28: Coefficient bias for $\rho = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	-0.786	-0.625	-0.976	-0.974	-0.979	-0.978	-1.134	-0.979	NA
SAR	-0.919	-0.853	-0.980	-0.991	-0.988	-0.987	-1.054	-0.990	-0.002
RR	-0.992	-0.951	-0.976	-1.006	-0.978	-0.971	-1.008	-0.965	NA
SFRR	-1.000	-0.981	-0.980	-1.003	-0.987	-0.984	-1.005	-0.984	-0.002
RRSAR	-0.999	-0.995	-0.981	-1.000	-0.986	-0.984	-0.993	-0.984	0.212

Table 29: Coefficient variance for $\rho = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	1.467	3.618	0.000	0.035	0.001	0.002	0.534	0.008	NA
SAR	0.093	0.237	0.000	0.002	0.000	0.000	0.035	0.000	0.000
RR	0.022	0.052	0.000	0.000	0.001	0.001	0.008	0.001	NA
SFRR	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
RRSAR	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001

Table 30: Coefficient MSE for $\rho = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	2.084	4.008	0.953	0.984	0.959	0.960	1.819	0.967	NA
SAR	0.938	0.966	0.960	0.983	0.976	0.974	1.147	0.981	0.000
RR	1.007	0.955	0.953	1.013	0.957	0.943	1.024	0.932	NA
SFRR	1.000	0.964	0.960	1.006	0.975	0.968	1.011	0.969	0.000
RRSAR	0.997	0.990	0.963	1.000	0.972	0.969	0.987	0.969	0.046

Results $\rho = 0.9$

Table 31: Coefficient bias for $\rho = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	-0.898	-0.829	-0.994	-0.988	-0.993	-0.993	-1.060	-0.994	NA
SAR	-0.984	-0.970	-0.996	-0.998	-0.998	-0.997	-1.011	-0.998	-0.001
RR	-0.993	-0.979	-0.994	-1.002	-0.992	-0.990	-1.002	-0.987	NA
SFRR	-0.999	-0.995	-0.996	-1.001	-0.997	-0.997	-1.001	-0.997	-0.001
RRSAR	-1.000	-0.999	-0.996	-1.000	-0.997	-0.997	-0.998	-0.997	0.093

Table 32: Coefficient variance for $\rho = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	3.401	8.368	0.001	0.082	0.003	0.006	1.234	0.019	NA
SAR	0.021	0.054	0.000	0.000	0.000	0.000	0.008	0.000	0.000
RR	0.044	0.099	0.001	0.001	0.003	0.001	0.014	0.001	NA
SFRR	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
RRSAR	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 33: Coefficient MSE for $\rho = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	4.207	9.056	0.989	1.057	0.988	0.993	2.357	1.006	NA
SAR	0.988	0.996	0.992	0.997	0.995	0.995	1.030	0.996	0.000
RR	1.030	1.057	0.989	1.006	0.987	0.981	1.019	0.975	NA
SFRR	0.999	0.991	0.992	1.001	0.995	0.994	1.003	0.994	0.000
RRSAR	1.000	0.998	0.993	0.999	0.994	0.994	0.997	0.994	0.009

Results average β coefficients

Table 34: Average Coefficient bias

ρ	OLS	SAR	RR	SFRR	RRSAR
0.1	-0.603	-0.633	-1.002	-1.003	-0.975
0.3	-0.646	-0.733	-0.999	-1.002	-0.981
0.5	-0.706	-0.831	-0.995	-1.001	-0.993

ρ	OLS	SAR	RR	SFRR	RRSAR
0.7	-0.786	-0.919	-0.992	-1.000	-0.999
0.9	-0.898	-0.984	-0.993	-0.999	-1.000

Table 35: Average Coefficient variance

ρ	OLS	SAR	RR	SFRR	RRSAR
0.1	0.447	0.388	0.005	0.003	0.000
0.3	0.554	0.289	0.007	0.002	0.000
0.5	0.829	0.187	0.013	0.001	0.000
0.7	1.467	0.093	0.022	0.000	0.000
0.9	3.401	0.021	0.044	0.000	0.000

Table 36: Average Coefficient MSE

ρ	OLS	SAR	RR	SFRR	RRSAR
0.1	0.811	0.789	1.008	1.009	0.950
0.3	0.971	0.825	1.005	1.006	0.963
0.5	1.327	0.878	1.003	1.004	0.986
0.7	2.084	0.938	1.007	1.000	0.997
0.9	4.207	0.988	1.030	0.999	1.000