Ridge regularization for spatial auto-regressive models with multicollinearity issues.

Simulations results of comparison of RRSAR with existing methods

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In our paper titled "Ridge Regularization for Spatial Autoregressive Models with Multicollinearity Issues," we introduce a novel approach for conducting Ridge regression in the context of spatial autoregressive models.

In this document, we provide the complete results of the simulations conducted within the framework of the dependent variable determined by the following equation:

$$\mathbf{Y} = (I_n - \rho W)^{-1}\mathbf{X}\boldsymbol{\beta} + (I_n - \rho W)^{-1}\boldsymbol{\varepsilon},$$

We consider eight highly correlated covariates generated as described in Section 5. Here, we present two scenarios: the deterministic scenario where the covariates were generated once for all the simulations and the stochastic scenario where the covariates were generated for each simulation.

The SAR model, defined in equation 1 of the paper, is generated for five values of the dependence parameter $\rho \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

The following tables display the average bias, average variance, and average mean squared error (MSE) of the eight regression coefficient estimates and the dependence parameter estimates computed across 500 simulations for each value of the dependence parameter (ρ). These estimates are computed using different estimation algorithms: OLS, ordinary SAR, ordinary Ridge, Spatially Filtered Ridge Regression (SFRR), and our methodology, named Ridge Regression for SAR Models (RRSAR). We present the results in two sections, the first one is dedicated to the results of deterministic covariates, and the second one to the results of stochastic covariates.

Tables 16-18 and 34-36 compile the average results for all the regression coefficients.

Deterministic case

Results $\rho = 0.1$

Table 1: Coefficient bias for $\rho = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	-0.012	-0.061	-0.030	-0.209	0.040	0.077	-0.008	0.053	NA
SAR	0.103	0.188	-0.008	0.059	-0.002	0.000	-0.165	-0.007	-0.001
RR	-0.522	-0.452	-0.076	-1.101	0.012	0.151	-0.238	0.239	NA
SFRR	-0.568	-0.461	-0.065	-1.016	-0.026	0.093	-0.243	0.203	-0.001
RRSAR	-0.461	-0.496	-0.046	-0.726	-0.012	0.066	0.011	0.133	-0.001

Table 2: Coefficient variance for $\rho = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	5.630	19.506	0.015	4.324	0.016	0.034	12.793	0.084	NA
SAR	5.476	18.966	0.015	4.223	0.016	0.033	12.448	0.083	0.000
RR	0.031	0.021	0.010	0.101	0.006	0.006	0.061	0.011	NA
SFRR	0.040	0.026	0.010	0.144	0.006	0.006	0.070	0.012	0.000
RRSAR	0.101	0.060	0.012	0.463	0.007	0.009	0.284	0.030	0.000

Table 3: Coefficient MSE for $\rho = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	5.631	19.510	0.016	4.367	0.018	0.040	12.793	0.087	NA
SAR	5.486	19.001	0.015	4.227	0.016	0.033	12.475	0.083	0.000
RR	0.303	0.225	0.015	1.312	0.006	0.029	0.117	0.068	NA
SFRR	0.363	0.239	0.014	1.178	0.007	0.015	0.129	0.053	0.000
RRSAR	0.314	0.306	0.014	0.991	0.007	0.014	0.284	0.048	0.000

Results $\rho = 0.3$

Table 4: Coefficient bias for $\rho = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	-0.505	-1.107	-0.074	-1.044	0.157	0.293	0.741	0.232	NA
SAR	0.102	0.185	-0.008	0.058	-0.002	0.000	-0.163	-0.007	-0.001
RR	-0.358	-0.442	-0.087	-1.223	0.118	0.301	-0.135	0.321	NA
SFRR	-0.566	-0.460	-0.065	-1.013	-0.026	0.093	-0.242	0.203	-0.001
RRSAR	-0.461	-0.496	-0.046	-0.727	-0.011	0.067	0.011	0.133	-0.001

Table 5: Coefficient variance for $\rho = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	6.296	21.849	0.016	4.814	0.018	0.038	14.295	0.091	NA
SAR	5.482	18.991	0.015	4.230	0.016	0.033	12.464	0.083	0.000
RR	0.032	0.021	0.010	0.095	0.007	0.007	0.079	0.014	NA
SFRR	0.041	0.026	0.010	0.146	0.006	0.006	0.070	0.012	0.000
RRSAR	0.101	0.060	0.012	0.463	0.007	0.009	0.285	0.030	0.000

Table 6: Coefficient MSE for $\rho = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	6.551	23.075	0.022	5.903	0.042	0.123	14.844	0.145	NA
SAR	5.492	19.025	0.015	4.233	0.016	0.033	12.490	0.083	0.000
RR	0.160	0.216	0.018	1.590	0.021	0.098	0.097	0.117	NA
SFRR	0.361	0.238	0.014	1.172	0.007	0.015	0.128	0.053	0.000
RRSAR	0.314	0.307	0.014	0.991	0.007	0.014	0.285	0.048	0.000

Results $\rho = 0.5$

Table 7: Coefficient bias for $\rho = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	-1.508	-3.269	-0.120	-2.473	0.339	0.635	2.389	0.527	NA
SAR	0.101	0.183	-0.008	0.058	-0.002	0.000	-0.161	-0.007	-0.001
RR	-0.103	-0.446	-0.087	-1.425	0.270	0.538	0.025	0.468	NA
SFRR	-0.565	-0.459	-0.065	-1.011	-0.026	0.093	-0.241	0.202	-0.001
RRSAR	-0.461	-0.496	-0.046	-0.726	-0.011	0.067	0.013	0.133	-0.001

Table 8: Coefficient variance for $\rho = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	7.775	27.047	0.019	5.921	0.022	0.047	17.642	0.107	NA
SAR	5.488	19.017	0.015	4.235	0.016	0.033	12.481	0.083	0.000
RR	0.026	0.022	0.014	0.084	0.009	0.009	0.102	0.014	NA
SFRR	0.041	0.026	0.010	0.147	0.006	0.006	0.070	0.013	0.000
RRSAR	0.101	0.060	0.012	0.462	0.007	0.009	0.285	0.030	0.000

Table 9: Coefficient MSE for $\rho = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	10.050	37.730	0.034	12.035	0.137	0.450	23.351	0.385	NA
SAR	5.498	19.050	0.015	4.239	0.016	0.033	12.507	0.083	0.000
RR	0.036	0.220	0.021	2.114	0.082	0.298	0.103	0.233	NA
SFRR	0.361	0.237	0.014	1.170	0.007	0.015	0.128	0.053	0.000
RRSAR	0.313	0.307	0.014	0.988	0.007	0.014	0.285	0.048	0.000

Results $\rho = 0.7$

Table 10: Coefficient bias for $\rho = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	-3.473	-7.719	-0.171	-5.095	0.653	1.248	5.915	1.051	NA
SAR	0.101	0.182	-0.008	0.058	-0.002	0.000	-0.160	-0.007	-0.000
RR	0.395	-0.446	-0.066	-1.836	0.506	0.962	0.245	0.761	NA
SFRR	-0.566	-0.460	-0.065	-1.011	-0.026	0.093	-0.241	0.202	-0.000
RRSAR	-0.461	-0.497	-0.046	-0.725	-0.011	0.067	0.013	0.133	-0.001

Table 11: Coefficient variance for $\rho = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	11.594	40.476	0.028	8.804	0.031	0.072	26.289	0.149	NA
SAR	5.493	19.037	0.015	4.239	0.016	0.033	12.496	0.083	0.000
RR	0.028	0.020	0.021	0.081	0.013	0.014	0.092	0.017	NA

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
SFRR	0.041	0.026	0.010	0.146	0.006	0.006	0.070	0.012	0.000
RRSAR	0.101	0.060	0.012	0.462	0.007	0.009	0.285	0.030	0.000

Table 12: Coefficient MSE for $\rho = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	23.658	100.060	0.057	34.767	0.458	1.629	61.274	1.254	NA
SAR	5.503	19.071	0.015	4.242	0.016	0.033	12.521	0.083	0.000
RR	0.184	0.219	0.025	3.452	0.270	0.939	0.152	0.596	NA
SFRR	0.361	0.238	0.014	1.169	0.007	0.015	0.128	0.053	0.000
RRSAR	0.313	0.307	0.014	0.988	0.007	0.014	0.285	0.048	0.000

Results $\rho = 0.9$

Table 13: Coefficient bias for $\rho = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	-7.271	-18.810	-0.255	-10.679	1.394	2.811	15.372	2.120	NA
SAR	0.101	0.183	-0.008	0.059	-0.002	0.000	-0.160	-0.007	-0.000
RR	2.281	-0.620	0.009	-2.779	1.016	2.118	1.006	1.466	NA
SFRR	-0.568	-0.460	-0.065	-1.016	-0.027	0.093	-0.244	0.203	-0.000
RRSAR	-0.460	-0.497	-0.046	-0.725	-0.011	0.067	0.013	0.133	-0.001

Table 14: Coefficient variance for $\rho = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	32.714	114.170	0.077	25.272	0.081	0.225	73.283	0.385	NA
SAR	5.487	19.021	0.015	4.232	0.016	0.033	12.488	0.083	0.000
RR	0.087	0.047	0.053	0.185	0.032	0.042	0.206	0.044	NA
SFRR	0.041	0.026	0.010	0.145	0.006	0.006	0.069	0.013	0.000
RRSAR	0.101	0.060	0.012	0.463	0.007	0.009	0.286	0.030	0.000

Table 15: Coefficient MSE for $\rho = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	85.582	467.991	0.142	139.320	2.025	8.124	309.572	4.878	NA
SAR	5.497	19.055	0.015	4.235	0.016	0.033	12.513	0.083	0.000
RR	5.288	0.431	0.053	7.909	1.063	4.530	1.218	2.193	NA
SFRR	0.364	0.238	0.014	1.177	0.007	0.015	0.129	0.054	0.000
RRSAR	0.313	0.307	0.014	0.989	0.007	0.014	0.286	0.048	0.000

Average of the β coefficients

Table 16: Average Coefficient bias

ρ	OLS	SAR	RR	SFRR	RRSAR
0.1	-0.012	0.103	-0.522	-0.568	-0.461
0.3	-0.505	0.102	-0.358	-0.566	-0.461
0.5	-1.508	0.101	-0.103	-0.565	-0.461
0.7	-3.473	0.101	0.395	-0.566	-0.461
0.9	-7.271	0.101	2.281	-0.568	-0.460

Table 17: Average Coefficient variance

ρ	OLS	SAR	RR	SFRR	RRSAR
0.1	5.630	5.476	0.031	0.040	0.101
0.3	6.296	5.482	0.032	0.041	0.101
0.5	7.775	5.488	0.026	0.041	0.101
0.7	11.594	5.493	0.028	0.041	0.101
0.9	32.714	5.487	0.087	0.041	0.101

Table 18: Average Coefficient MSE

ρ	OLS	SAR	RR	SFRR	RRSAR
0.1	5.631	5.486	0.303	0.363	0.314
0.3	6.551	5.492	0.160	0.361	0.314
0.5	10.050	5.498	0.036	0.361	0.313
0.7	23.658	5.503	0.184	0.361	0.313
0.9	85.582	5.497	5.288	0.364	0.313

Stochastic case

Results $\rho = 0.1$

Table 19: Coefficient bias for $\rho = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.044	-0.134	-0.002	-0.003	0.052	0.049	0.230	-0.003	NA
SAR	-0.236	-0.533	-0.005	-0.136	0.025	0.015	0.480	-0.011	-0.000
RR	-0.521	-0.467	-0.051	-0.941	0.012	0.126	-0.156	0.213	NA
SFRR	-0.551	-0.476	-0.051	-0.837	-0.017	0.078	-0.138	0.169	-0.000
RRSAR	-0.445	-0.538	-0.030	-0.556	0.002	0.052	0.149	0.096	-0.000

Table 20: Coefficient variance for $\rho = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	6.739	23.916	0.017	3.559	0.017	0.022	15.466	0.061	NA

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
SAR	5.788	20.657	0.017	3.068	0.015	0.020	13.545	0.054	0.000
RR	0.053	0.028	0.013	0.173	0.005	0.006	0.078	0.014	NA
SFRR	0.053	0.030	0.014	0.187	0.004	0.006	0.079	0.012	0.000
RRSAR	0.109	0.086	0.015	0.432	0.006	0.009	0.333	0.030	0.000

Table 21: Coefficient MSE for $\rho = 0.1$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	6.741	23.934	0.017	3.559	0.020	0.025	15.518	0.061	NA
SAR	5.844	20.941	0.017	3.086	0.016	0.020	13.775	0.054	0.000
RR	0.324	0.246	0.015	1.058	0.005	0.022	0.103	0.059	NA
SFRR	0.356	0.256	0.017	0.888	0.005	0.012	0.098	0.040	0.000
RRSAR	0.307	0.375	0.016	0.741	0.006	0.012	0.356	0.039	0.000

Results $\rho = 0.3$

Table 22: Coefficient bias for $\rho = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	0.746	0.834	0.015	0.320	0.129	0.146	-0.365	0.032	NA
SAR	-0.233	-0.527	-0.005	-0.133	0.025	0.015	0.476	-0.012	-0.000
RR	-0.360	-0.401	-0.029	-1.066	0.097	0.247	-0.134	0.310	NA
SFRR	-0.550	-0.476	-0.051	-0.837	-0.017	0.078	-0.138	0.169	-0.000
RRSAR	-0.444	-0.538	-0.030	-0.556	0.002	0.052	0.149	0.096	-0.001

Table 23: Coefficient variance for $\rho = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	11.702	41.131	0.030	6.171	0.028	0.037	26.017	0.101	NA
SAR	5.778	20.619	0.017	3.064	0.015	0.020	13.521	0.054	0.000
RR	0.078	0.033	0.024	0.214	0.007	0.009	0.057	0.019	NA
SFRR	0.053	0.030	0.014	0.187	0.004	0.006	0.079	0.012	0.000
RRSAR	0.109	0.086	0.015	0.432	0.006	0.009	0.333	0.030	0.000

Table 24: Coefficient MSE for $\rho = 0.3$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	12.258	41.827	0.030	6.273	0.045	0.059	26.150	0.102	NA
SAR	5.832	20.897	0.017	3.081	0.016	0.020	13.747	0.054	0.000
RR	0.207	0.194	0.024	1.349	0.017	0.070	0.075	0.115	NA
SFRR	0.355	0.256	0.017	0.887	0.005	0.012	0.098	0.040	0.000
RRSAR	0.307	0.375	0.016	0.741	0.006	0.012	0.356	0.039	0.000

Results $\rho = 0.5$

Table 25: Coefficient bias for $\rho = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	1.803	2.227	0.055	0.786	0.254	0.309	-1.201	0.098	NA
SAR	-0.229	-0.520	-0.005	-0.130	0.025	0.015	0.471	-0.012	-0.000
RR	-0.112	-0.374	0.016	-1.261	0.237	0.447	-0.022	0.466	NA
SFRR	-0.547	-0.469	-0.051	-0.833	-0.017	0.077	-0.142	0.169	-0.000
RRSAR	-0.443	-0.537	-0.030	-0.554	0.002	0.052	0.151	0.096	-0.001

Table 26: Coefficient variance for $\rho = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	26.719	93.405	0.080	14.045	0.066	0.088	58.621	0.224	NA
SAR	5.769	20.588	0.017	3.060	0.015	0.020	13.501	0.054	0.000
RR	0.083	0.052	0.064	0.203	0.017	0.019	0.121	0.028	NA
SFRR	0.052	0.032	0.014	0.194	0.005	0.007	0.088	0.013	0.000
RRSAR	0.109	0.086	0.015	0.433	0.005	0.009	0.334	0.030	0.000

Table 27: Coefficient MSE for $\rho = 0.5$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	29.971	98.363	0.083	14.663	0.130	0.183	60.064	0.234	NA
SAR	5.822	20.859	0.017	3.077	0.016	0.020	13.723	0.054	0.000
RR	0.095	0.192	0.064	1.794	0.073	0.219	0.122	0.245	NA
SFRR	0.351	0.252	0.017	0.889	0.005	0.013	0.108	0.041	0.000
RRSAR	0.306	0.375	0.016	0.739	0.006	0.012	0.357	0.039	0.000

Results $\rho = 0.7$

Table 28: Coefficient bias for $\rho = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	3.714	4.565	0.125	1.553	0.486	0.624	-2.593	0.227	NA
SAR	-0.224	-0.512	-0.005	-0.126	0.025	0.014	0.465	-0.012	-0.000
RR	0.441	-0.324	0.096	-1.608	0.494	0.822	0.173	0.745	NA
SFRR	-0.548	-0.468	-0.051	-0.837	-0.017	0.077	-0.147	0.169	-0.000
RRSAR	-0.443	-0.537	-0.030	-0.554	0.002	0.052	0.151	0.096	-0.001

Table 29: Coefficient variance for $\rho = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	83.771	290.686	0.284	43.817	0.216	0.293	182.404	0.688	NA
SAR	5.765	20.576	0.017	3.059	0.015	0.020	13.494	0.054	0.000
RR	0.167	0.092	0.225	0.242	0.058	0.069	0.223	0.055	NA

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
SFRR	0.052	0.032	0.014	0.197	0.004	0.007	0.088	0.013	0.000
RRSAR	0.109	0.086	0.015	0.432	0.005	0.009	0.333	0.030	0.000

Table 30: Coefficient MSE for $\rho = 0.7$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	97.565	311.530	0.299	46.228	0.452	0.682	189.127	0.740	NA
SAR	5.815	20.838	0.017	3.074	0.016	0.020	13.710	0.054	0.000
RR	0.361	0.197	0.234	2.827	0.303	0.744	0.253	0.610	NA
SFRR	0.353	0.251	0.017	0.898	0.005	0.013	0.109	0.042	0.000
RRSAR	0.306	0.375	0.016	0.738	0.005	0.012	0.356	0.039	0.000

Results $\rho = 0.9$

Table 31: Coefficient bias for $\rho = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	9.092	10.070	0.232	3.174	1.157	1.591	-5.822	0.603	NA
SAR	-0.220	-0.507	-0.004	-0.122	0.025	0.014	0.463	-0.013	-0.000
RR	2.567	-0.369	0.283	-2.515	1.235	1.891	0.928	1.448	NA
SFRR	-0.543	-0.465	-0.050	-0.827	-0.017	0.076	-0.142	0.167	-0.000
RRSAR	-0.443	-0.537	-0.029	-0.553	0.002	0.052	0.151	0.096	-0.000

Table 32: Coefficient variance for $\rho = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	656.366	2223.457	2.220	344.319	1.654	2.393	1384.100	5.332	NA
SAR	5.776	20.635	0.017	3.061	0.015	0.020	13.532	0.054	0.000
RR	1.519	0.450	1.708	1.441	0.463	0.601	1.212	0.404	NA
SFRR	0.054	0.032	0.014	0.203	0.004	0.007	0.088	0.013	0.000
RRSAR	0.109	0.087	0.015	0.431	0.005	0.009	0.333	0.030	0.000

Table 33: Coefficient MSE for $\rho = 0.9$

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	ρ
OLS	739.036	2324.853	2.274	354.390	2.992	4.925	1417.998	5.695	NA
SAR	5.824	20.892	0.017	3.076	0.016	0.020	13.746	0.054	0.000
RR	8.107	0.587	1.788	7.768	1.987	4.177	2.074	2.499	NA
SFRR	0.349	0.249	0.017	0.887	0.005	0.013	0.109	0.041	0.000
RRSAR	0.305	0.375	0.016	0.737	0.005	0.012	0.356	0.039	0.000

Average of the β coefficients

Table 34: Average Coefficient bias

ρ	OLS	SAR	RR	SFRR	RRSAR
0.1	0.044	-0.236	-0.521	-0.551	-0.445
0.3	0.746	-0.233	-0.360	-0.550	-0.444
0.5	1.803	-0.229	-0.112	-0.547	-0.443
0.7	3.714	-0.224	0.441	-0.548	-0.443
0.9	9.092	-0.220	2.567	-0.543	-0.443

Table 35: Average Coefficient variance

ρ	OLS	SAR	RR	SFRR	RRSAR
0.1	6.739	5.788	0.053	0.053	0.109
0.3	11.702	5.778	0.078	0.053	0.109
0.5	26.719	5.769	0.083	0.052	0.109
0.7	83.771	5.765	0.167	0.052	0.109
0.9	656.366	5.776	1.519	0.054	0.109

Table 36: Average Coefficient MSE

ρ	OLS	SAR	RR	SFRR	RRSAR
0.1	6.741	5.844	0.324	0.356	0.307
0.3	12.258	5.832	0.207	0.355	0.307
0.5	29.971	5.822	0.095	0.351	0.306
0.7	97.565	5.815	0.361	0.353	0.306
0.9	739.036	5.824	8.107	0.349	0.305