Announcements

Upcoming review sessions:

- Project Review Session on Sunday December 6th (likely 4-6pm)
- Exam Review Session on Sunday December 13

Term project due Tuesday

Last lecture is Tuesday

Final exam is Tuesday, December 15th at 9am. 3 locations again.
1 page of handwritten notes will be allowed again.

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6.5 REDUCTIONS

- *introduction*
- *designing algorithms*
- *establishing lower bounds*
- *classifying problems*

Overview

Main topics.

- Reduction: relationship between two problems.
- Algorithm design: paradigms for solving problems.

Shifting gears.

- From individual problems to problem-solving models.
- From linear/quadratic to polynomial/exponential scale.
- From implementation details to conceptual frameworks.

Goals.

- Place algorithms and techniques we've studied in a larger context.
- Introduce you to important and essential ideas.
- Inspire you to learn more about algorithms!

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6.5 Reductions

- *introduction*
- *designing algorithms*
- *establishing lower bounds*
- *classifying problems*

Bird's-eye view

Goal. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	*min, max, median, Burrows-Wheeler transform, …*
linearithmic	$N \log N$	*sorting, element distinctness, closest pair, Euclidean MST, …*
quadratic	N^2	?
⋮	⋮	⋮
exponential	c^N	?

Frustrating news. Huge number of problems have defied classification.

Bird's-eye view

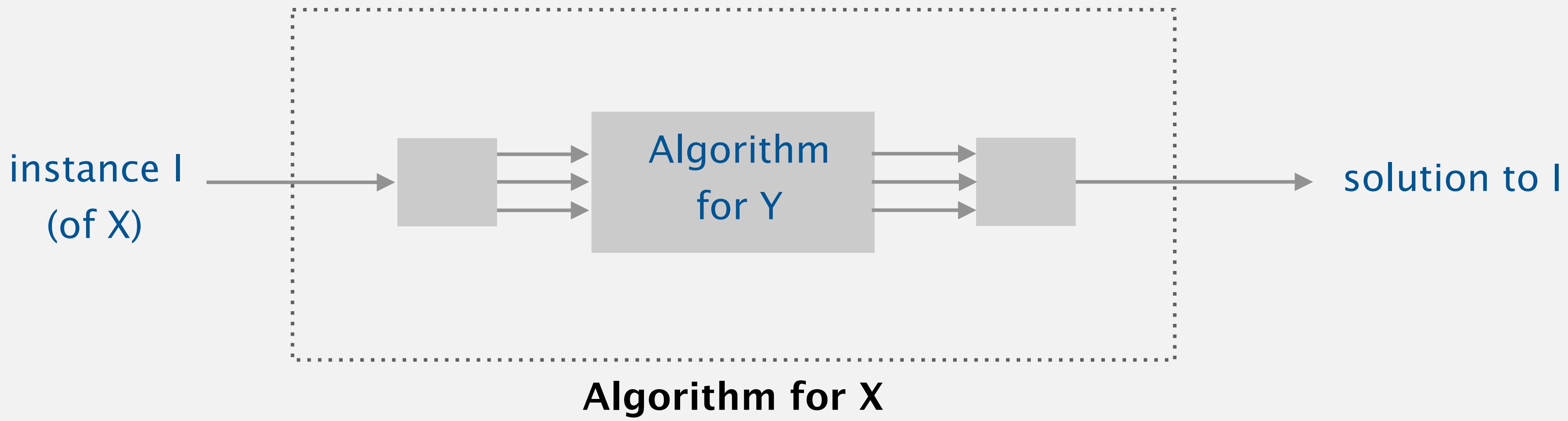
Goal. Classify problems according to computational requirements.

Goal. Suppose we could (could not) solve problem X efficiently.
What else could (could not) we solve efficiently?

" Give me a lever long enough and a fulcrum on which to place it, and I shall move the world. " — *Archimedes*

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



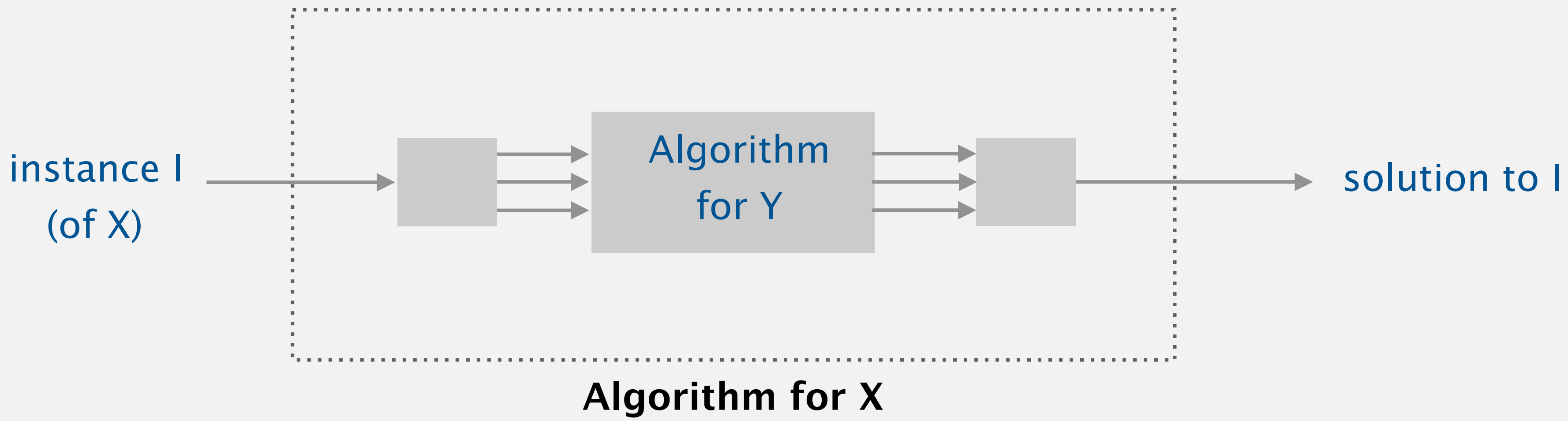
Cost of solving X = total cost of solving Y + cost of reduction.

perhaps many calls to Y
on problems of different sizes
(typically only 1 call)

preprocessing and postprocessing
(typically less than cost of solving Y)

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



Ex 1. [finding the median reduces to sorting]

To find the median of N items:

- Sort N items.
- Return item in the middle.

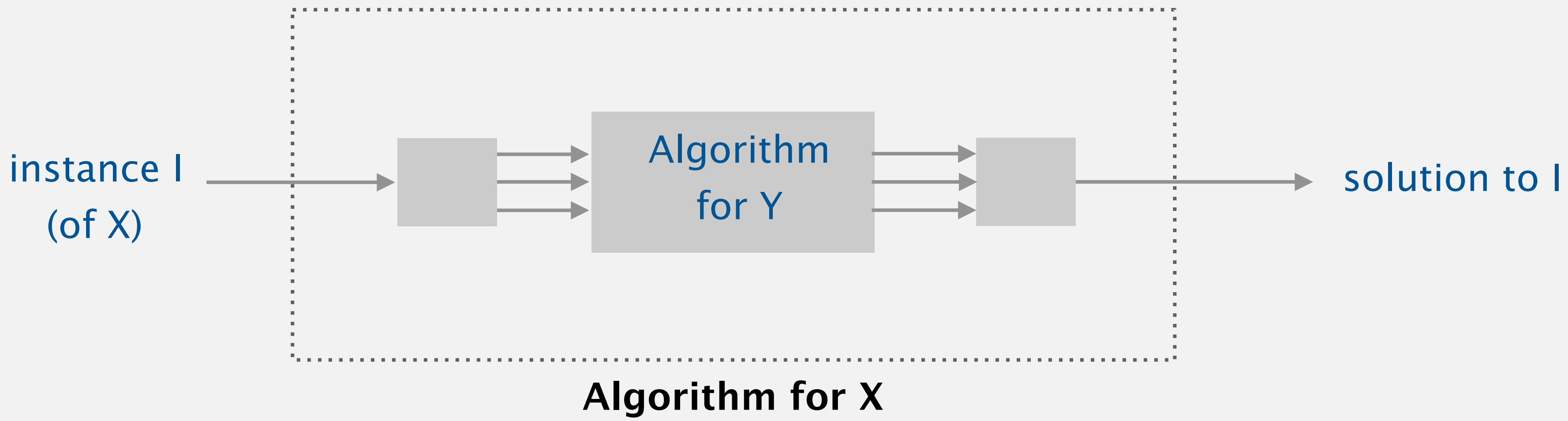
cost of sorting

cost of reduction

Cost of finding the median. $N \log N + 1$.

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



Ex 2. [element distinctness reduces to sorting]

To solve element distinctness on N items:

- Sort N items.
- Check adjacent pairs for equality.

cost of sorting

cost of reduction

Cost of element distinctness. $N \log N + N$.

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6.5 Reductions

- introduction
- **designing algorithms**
- establishing lower bounds
- classifying problems

Reduction: design algorithms

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.

Design algorithm. Given an algorithm for Y, can also solve X.

Example reductions.

- Finding the median reduces to sorting
- Element distinctness reduces to sorting
- Arbitrage reduces to negative cycles.
- Seam carving reduces to shortest paths in a DAG.

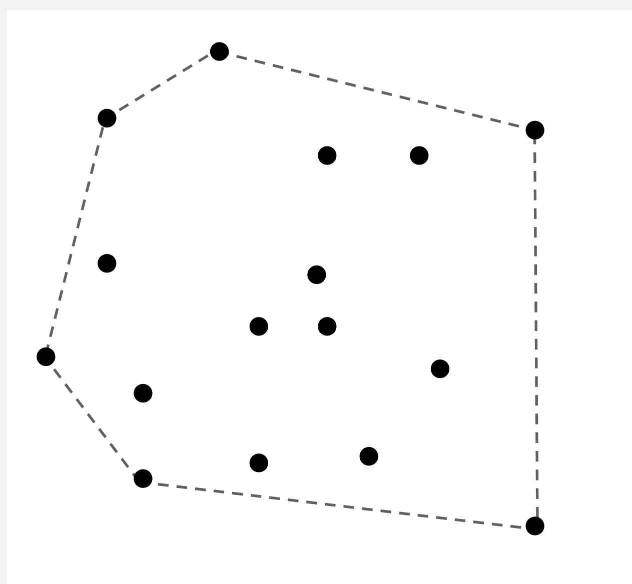
Mentality. Since I know how to solve Y, can I use that algorithm to solve X?

programmer's version: I have code for Y. Can I use it for X?

Convex hull reduces to sorting

Sorting. Given N distinct integers, rearrange them in ascending order.

Convex hull. Given N points in the plane, identify the extreme points of the convex hull (in counterclockwise order).



convex hull

1251432
2861534
3988818
8111033
13546464
89885444
43434213
34435312

sorting

Proposition. Convex hull reduces to sorting.

Pf. Graham scan algorithm.

cost of sorting

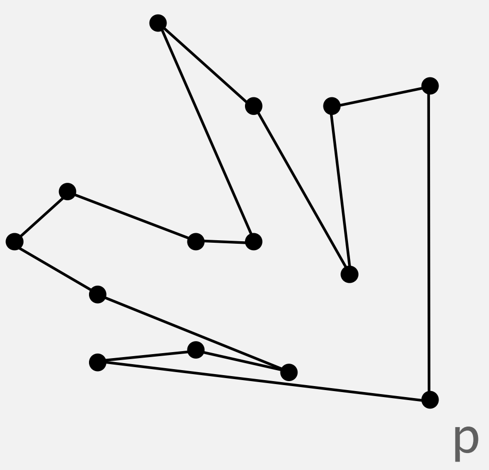
cost of reduction

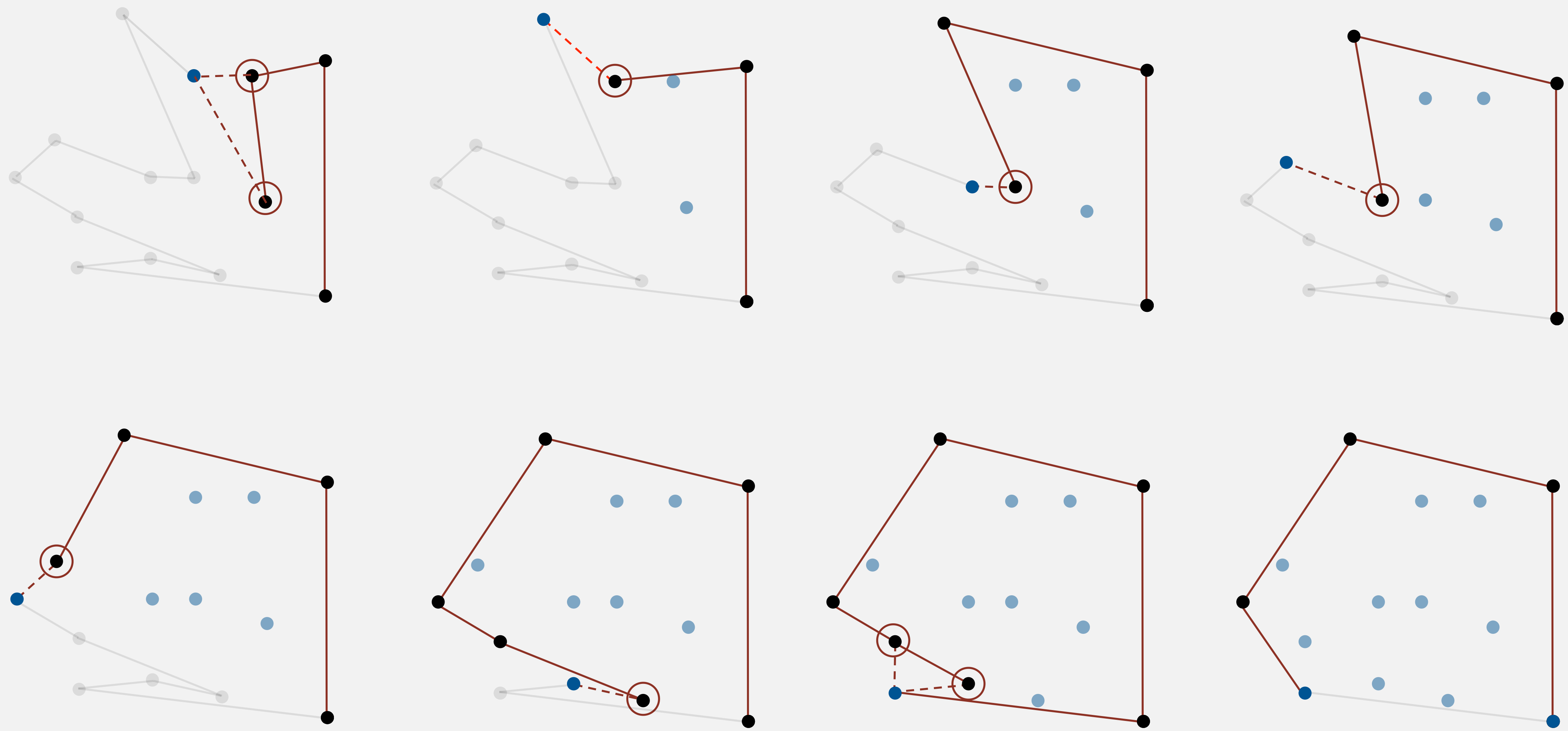
Cost of convex hull. $N \log N + N$.

Graham scan algorithm

Graham scan.

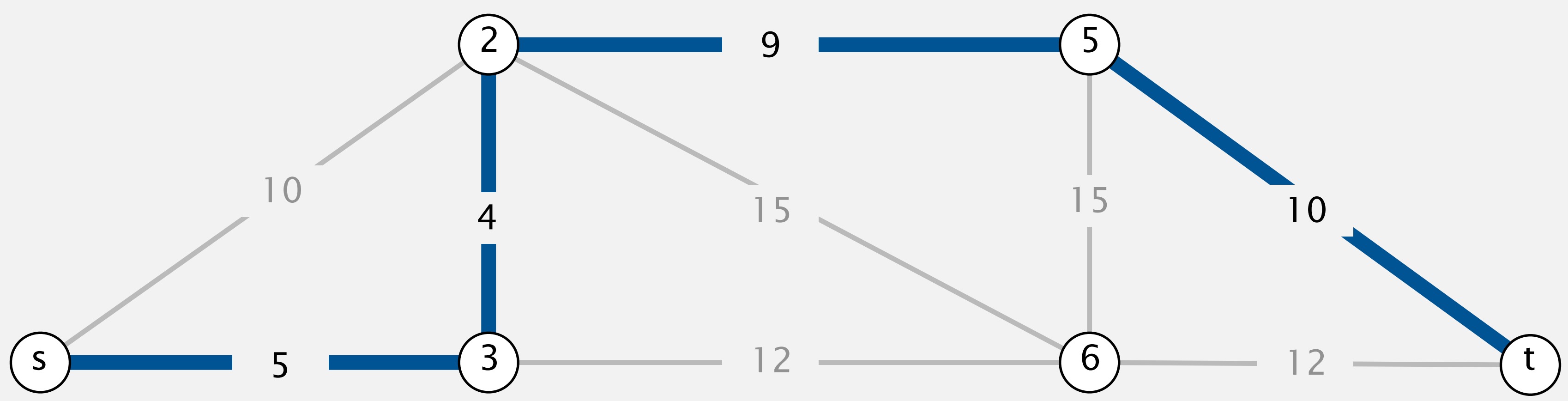
- Choose point p with smallest (or largest) y-coordinate.
- Sort points by polar angle with p to get simple polygon.
- Consider points in order, and discard those that would create a clockwise turn.



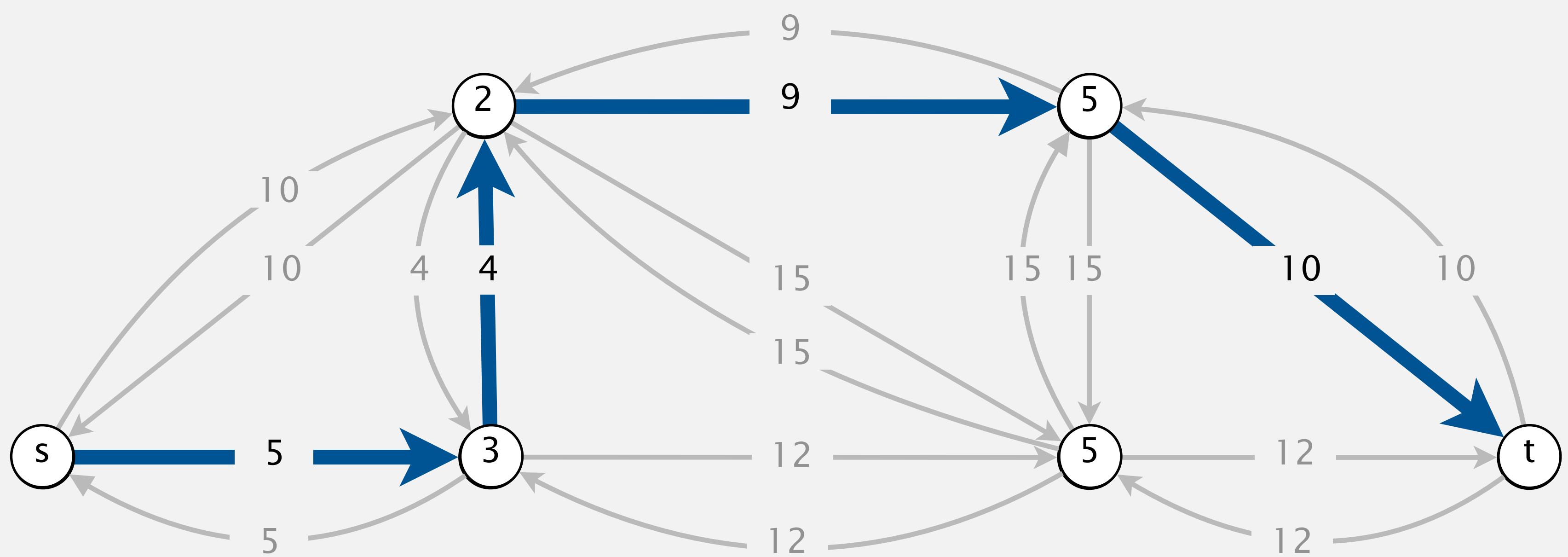


Shortest paths on edge-weighted graphs and digraphs

Proposition. Undirected shortest paths (with nonnegative weights) reduces to directed shortest path.



Pf. Replace each undirected edge by two directed edges.

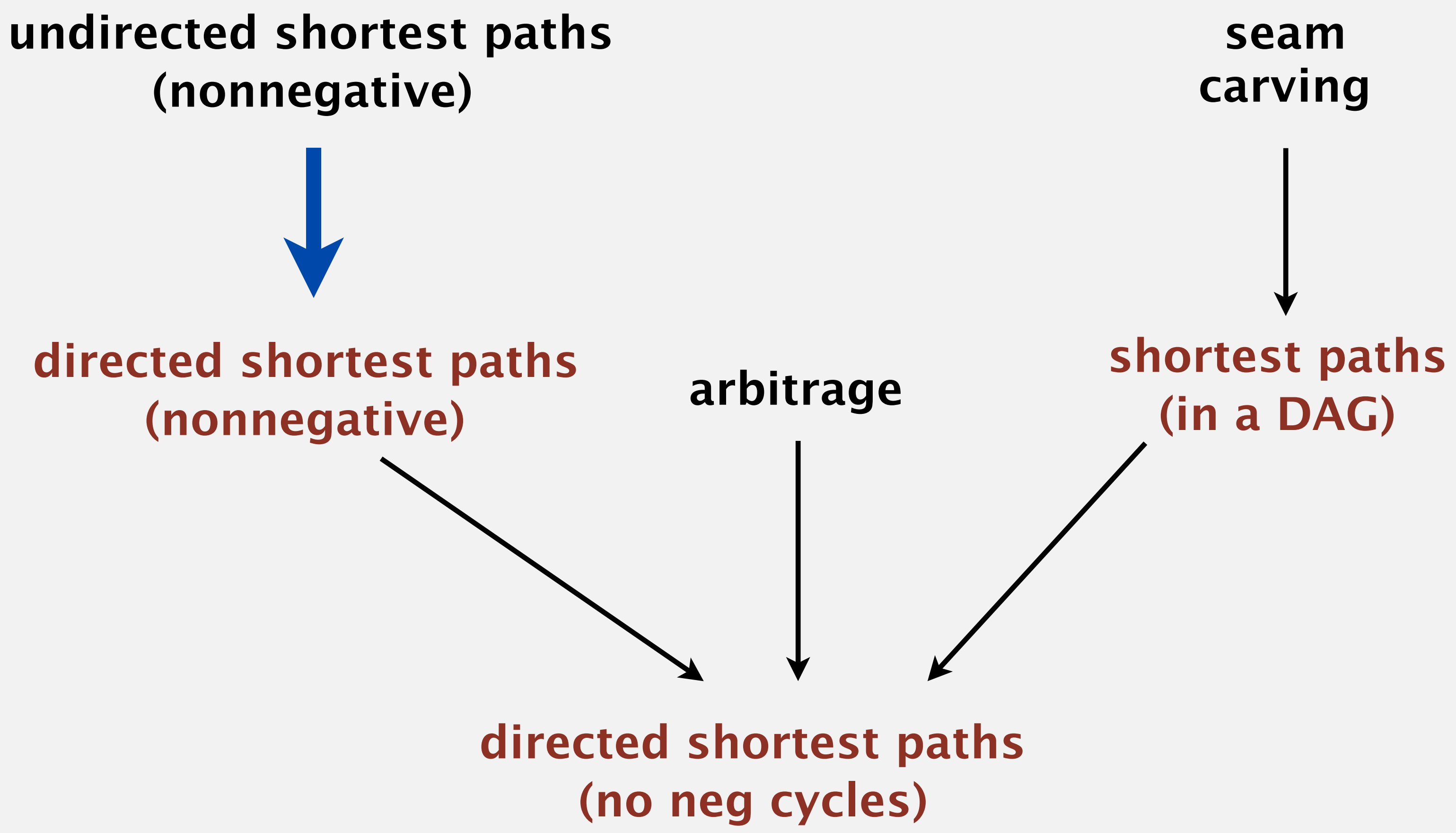


Cost of solving undirected shortest paths. $E \log V + (E + V)$.

cost of Dijkstra

cost of reduction

Some reductions in combinatorial optimization



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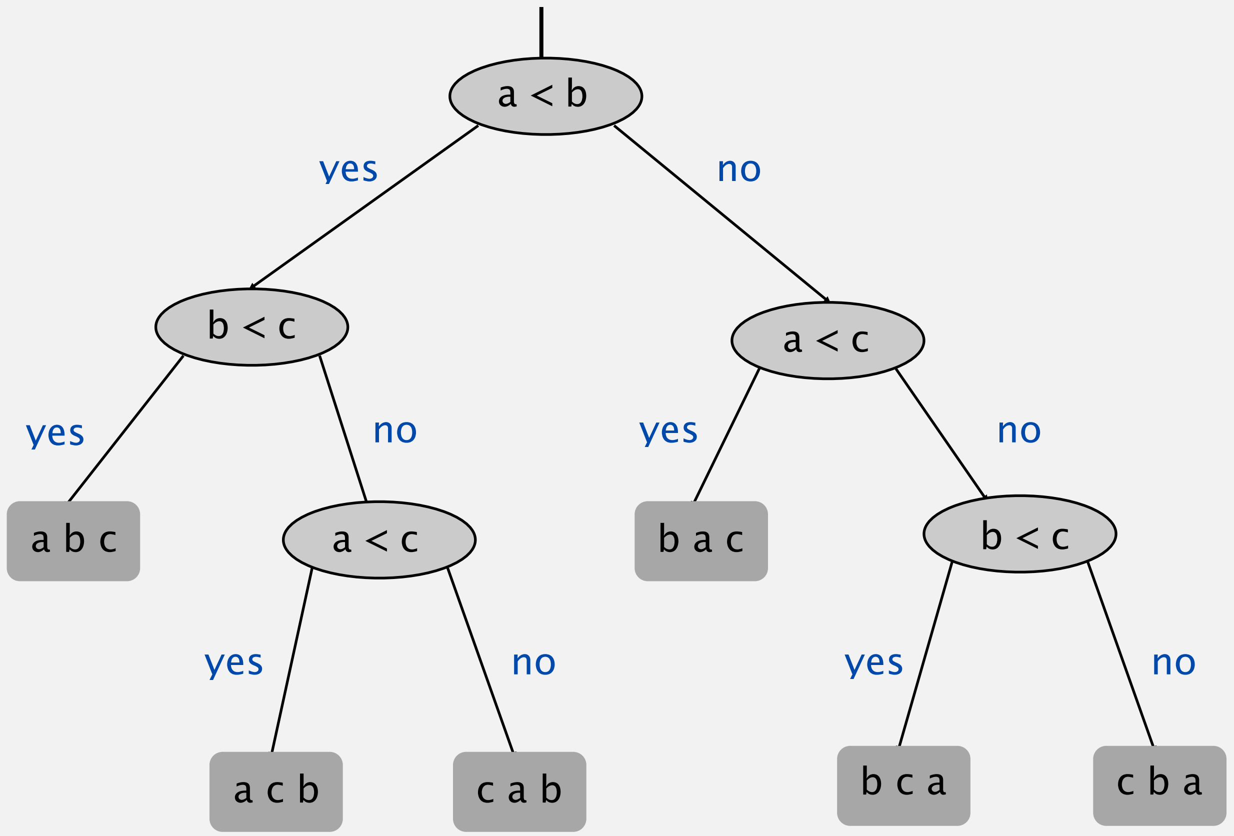
6.5 Reductions

- introduction
- designing algorithms
- **establishing lower bounds**
- classifying problems

Bird's-eye view

Goal. Prove that a problem requires a certain number of steps.

Ex. In decision tree model, any compare-based sorting algorithm requires $\Omega(N \log N)$ compares in the worst case.



argument must apply to all conceivable algorithms

Bad news. Very difficult to establish lower bounds from scratch.

Good news. Spread $\Omega(N \log N)$ lower bound to Y by reducing sorting to Y.

assuming cost of reduction is not too high

Linear-time reductions

Def. Problem X linear-time reduces to problem Y if X can be solved with:

- Linear number of standard computational steps.
- Constant number of calls to Y.

Establish lower bound:

- If X takes $\Omega(N \log N)$ steps, then so does Y.
- If X takes $\Omega(N^2)$ steps, then so does Y.

Mentality.

- If I could easily solve Y, then I could easily solve X.
- I can’t easily solve X.
- Therefore, I can't easily solve Y.

Lower bound for convex hull

Proposition. In quadratic decision tree model, any algorithm for sorting N integers requires $\Omega(N \log N)$ steps.

allows linear or quadratic tests:

$x_i < x_j$ or $(x_j - x_i)(x_k - x_i) - (x_j)(x_j - x_i) < 0$

Proposition. Sorting linear-time reduces to convex hull.

Pf. [see next slide]

lower-bound mentality:
I can't sort in linear time,
so I can't solve convex hull
in linear time either

1251432
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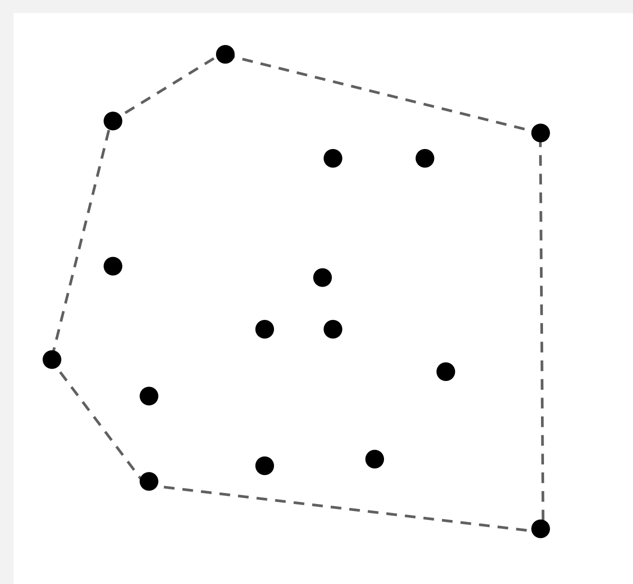
sorting



convex hull

linear or
quadratic tests

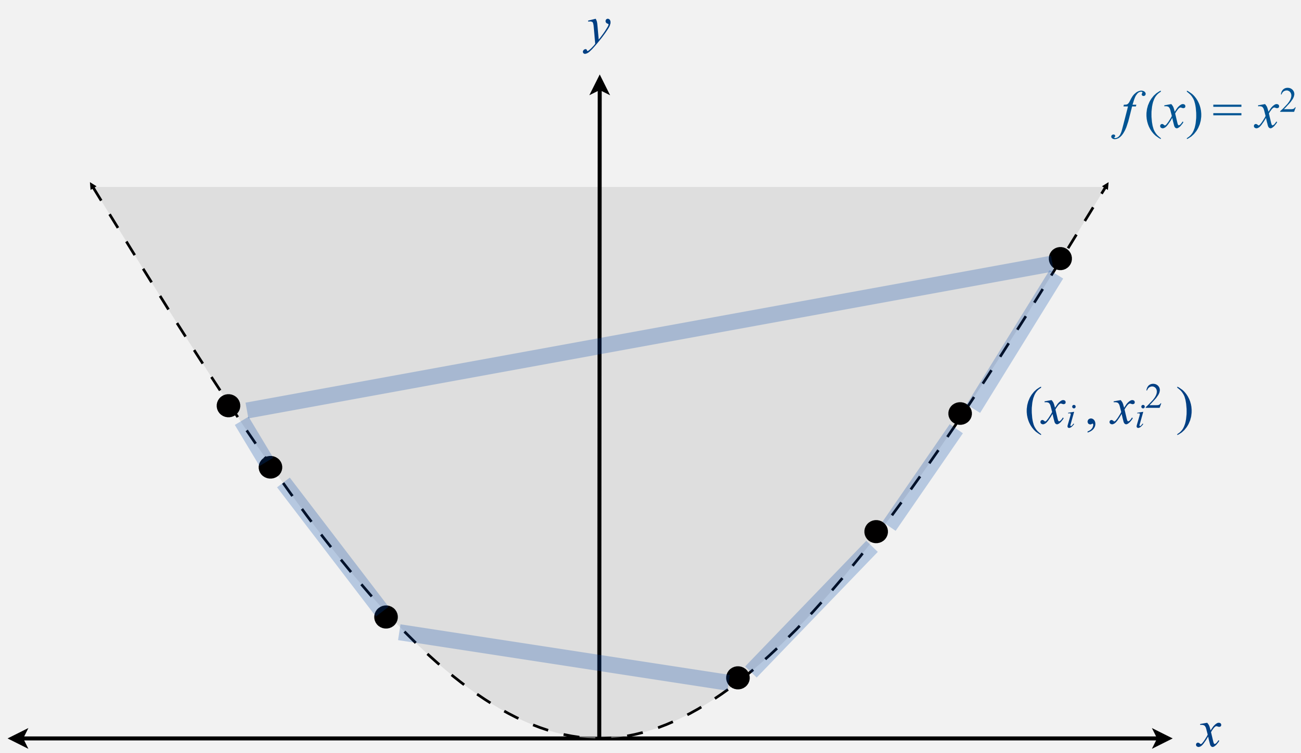
Implication. Any counterclockwise-based convex hull algorithm requires $\Omega(N \log N)$ ops.

Sorting linear-time reduces to convex hull

Proposition. Sorting linear-time reduces to convex hull.

- Sorting instance: $x_1, x_2, \ldots, x_N$.
- Convex hull instance: $(x_1, x_1^2), (x_2, x_2^2), \ldots, (x_N, x_N^2)$.

lower-bound mentality:
I can't sort in linear time,
so I can't solve convex hull
in linear time either



Pf.

- Region $\{ x : x^2 \geq x \}$ is convex $\Rightarrow$ all N points are on hull.
- Starting at point with most negative x, counterclockwise order of hull points yields integers in ascending order.

Establishing lower bounds: summary

Establishing lower bounds through reduction is an important tool in guiding algorithm design efforts.

Q. How to convince yourself no linear-time convex hull algorithm exists?

A1. [hard way] Long futile search for a linear-time algorithm.

A2. [easy way] Linear-time reduction from sorting.

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6.5 Reductions

- *introduction*
- *designing algorithms*
- *establishing lower bounds*
- ***classifying problems***

Classifying problems: summary

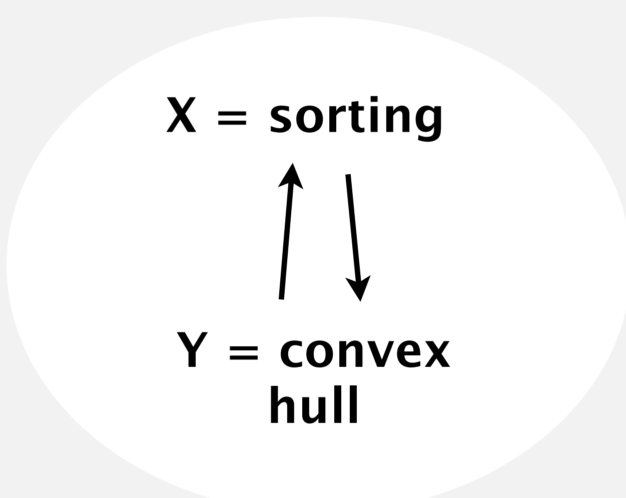
Goal. Problem with algorithm that matches lower bound.

Ex. Sorting and element distinctness have complexity $N \log N$.

Goal'. Prove that two problems X and Y have the same complexity.

- First, show that problem X linear-time reduces to Y.
- Second, show that Y linear-time reduces to X.
- Conclude that X and Y have the same complexity.

even if we don't know what it is!



In one case, reducing convex hull to sorting, gave us a lower bound

In the other case, reducing convex hull to sorting gave us a useful algorithm

Together they are useful because it helps to classify the problems.

Integer arithmetic reductions

Integer multiplication. Given two N-bit integers, compute their product.

Brute force. N^2 bit operations.

								1	1	0	1	0	1	0	1
							×	0	1	1	1	1	1	0	1
								1	1	0	1	0	1	0	1
							0	0	0	0	0	0	0	0	
						1	1	0	1	0	1	0	1		
					1	1	0	1	0	1	0	1			
				1	1	0	1	0	1	0	1				
			1	1	0	1	0	1	0	1					
		1	1	0	1	0	1	0	1						
	0	0	0	0	0	0	0	0							
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1

Integer arithmetic reductions

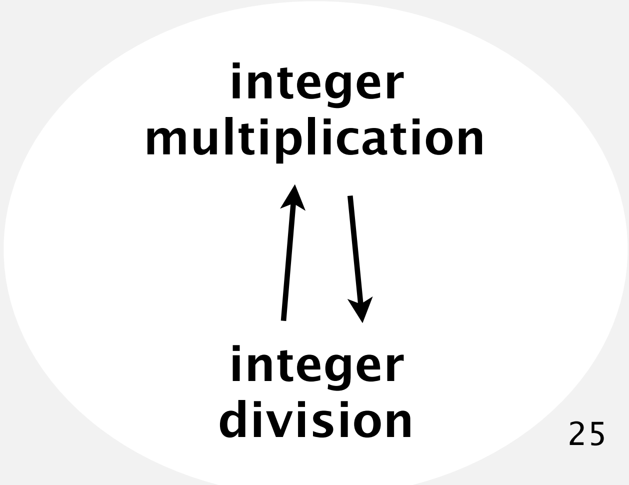
Integer multiplication. Given two N-bit integers, compute their product.

Brute force. N^2 bit operations.

problem	arithmetic	order of growth
integer multiplication	$a \times b$	$M(N)$
integer division	$a \,/\, b,\ a \bmod b$	$M(N)$
integer square	a^2	$M(N)$
integer square root	$\lfloor \sqrt{a} \rfloor$	$M(N)$

integer arithmetic problems with the same complexity as integer multiplication

Q. Is brute-force algorithm optimal?



History of complexity of integer multiplication

year	algorithm	order of growth
?	**brute force**	N^2
1962	**Karatsuba**	$N^{1.585}$
1963	**Toom-3, Toom-4**	$N^{1.465}$, $N^{1.404}$
1966	**Toom-Cook**	$N^{1+\varepsilon}$
1971	**Schönhage–Strassen**	$N \log N \log \log N$
2007	**Fürer**	$N \log N \, 2^{\log^* N}$
?	**?**	N

number of bit operations to multiply two N-bit integers

used in Maple, Mathematica, gcc, cryptography, ...

Remark. GNU Multiple Precision Library uses one of five different algorithm depending on size of operands.

GMP
«Arithmetic without limitations»

Numerical linear algebra reductions

Matrix multiplication. Given two N-by-N matrices, compute their product.

Brute force. N^3 flops.

$$
\begin{array}{c} 0.1 \\ \text{row } i \\ 0.1 \\ 0 \end{array}
\begin{array}{c} \\ \left[\begin{array}{cccc} 0.1 & 0.2 & 0.8 & 0.1 \\ 0.5 & 0.3 & 0.9 & 0.6 \\ 0.1 & 0 & 0.7 & 0.4 \\ 0 & 0.3 & 0.3 & 0.1 \end{array}\right] \end{array}
\times
\begin{array}{cccc} & & \text{column } j & \\ 0.4 & 0.3 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0 & 0.6 \\ 0 & 0 & 0.4 & 0.5 \\ 0.8 & 0.4 & 0.1 & 0.9 \end{array}
=
\begin{array}{c|cccc} & & & j & \\ \hline & 0.16 & 0.11 & 0.34 & 0.62 \\ i & 0.74 & 0.45 & 0.47 & 1.22 \\ & 0.36 & 0.19 & 0.33 & 0.72 \\ & 0.14 & 0.1 & 0.13 & 0.42 \end{array}
$$

$$0.5 \cdot 0.1 + 0.3 \cdot 0.0 + 0.9 \cdot 0.4 + 0.6 \cdot 0.1 = 0.47$$

Numerical linear algebra reductions

Matrix multiplication. Given two N-by-N matrices, compute their product.

Brute force. N^3 flops.

problem	linear algebra	order of growth
matrix multiplication	$A \times B$	$MM(N)$
matrix inversion	A^{-1}	$MM(N)$
determinant	$\| A \|$	$MM(N)$
system of linear equations	$Ax = b$	$MM(N)$
LU decomposition	$A = L\,U$	$MM(N)$
least squares	$\min \|\|Ax - b\|\|_2$	$MM(N)$

numerical linear algebra problems with the same complexity as matrix multiplication

Q. Is brute-force algorithm optimal?

History of complexity of matrix multiplication

year	algorithm	order of growth
?	**brute force**	N^3
1969	**Strassen**	$N^{2.808}$
1978	**Pan**	$N^{2.796}$
1979	**Bini**	$N^{2.780}$
1981	**Schönhage**	$N^{2.522}$
1982	**Romani**	$N^{2.517}$
1982	**Coppersmith-Winograd**	$N^{2.496}$
1986	**Strassen**	$N^{2.479}$
1989	**Coppersmith-Winograd**	$N^{2.376}$
2010	**Strother**	$N^{2.3737}$
2012	**Williams**	$N^{2.372873}$
2014	**de Gall**	$N^{2.372864}$
?	**?**	$N^{2+\varepsilon}$

number of floating-point operations to multiply two N-by-N matrices

Birds-eye view: revised

Goal. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	*min, max, median, Burrows-Wheeler transform, …*
linearithmic	$N \log N$	*sorting, element distinctness, …*
M(N)	?	*integer multiplication, division, square root, …*
MM(N)	?	*matrix multiplication, $Ax = b$, least square, determinant, …*
⋮	⋮	⋮
NP-complete	*probably not* N^b	3-SAT, IND-SET, ILP, …

Good news. Can put many problems into equivalence classes.

Complexity zoo

Complexity class. Set of problems sharing some computational property.

https://complexityzoo.uwaterloo.ca

Bad news. Lots of complexity classes (496 animals in zoo).

Summary

Reductions are important in theory to:

- Design algorithms.
- Establish lower bounds.
- Classify problems according to their computational requirements.

Reductions are important in practice to:

- Design algorithms.
- Design reusable software modules.
  - stacks, queues, priority queues, symbol tables, sets, graphs
  - sorting, regular expressions, suffix arrays
  - MST, shortest paths, maxflow, linear programming
- Determine difficulty of your problem and choose the right tool.