CIS 160 - Fall 2012 **(instructor Val Tannen)**

Exams, Mock and Actual, with Solutions

MOCK MIDTERM 1

1. (30 pts) For each statement below, decide whether it is true or false. In each case attach a *very brief* explanation of your answer.

   (a) Let A, B be finite sets with $|A| = 2$ and $|B| = 3$. There are more functions $A \to B$ than functions $B \to A$, true or false?

   Answer: TRUE. Indeed there are $3^2 = 9$ functions $A \to B$ but only $2^3 = 8$ functions $B \to A$.

   (b) Let X, Y be nonempty finite sets such that $|Y| = 1$ and such that there exists an injection $f : X \to Y$. Then $|X| = 1$, true or false?

   Answer: TRUE. Since there exists an injection $X \to Y$ it must be the case that $|X| \leq |Y|$ so $|X| \leq 1$. Because X is nonempty $|X| = 1$.

   (c) There are as many sequences of bits of length 100 that start with a 0 as sequences of bits of length 100 that end with a 1, true or false?

   Answer: TRUE. Indeed in both cases we can have either 0 or 1 in each of 99 positions. There are 2^{99} sequences of bits of length 100 that start with a 0 and there are 2^{99} sequences of bits of length 100 that end with a 1.

   (d) Let S be the set of the first 100 natural numbers: $S = 0, 1, \ldots, 99$. There are as many subsets of S of size 40 that contain the number 40 as subsets of S of size 40 that do not contain the number 40, true or false?

   Answer: FALSE. For the subsets that contain 40 we can choose the other 39 elements out of the remaining 99 so there are $\binom{99}{39}$ such subsets. For the subsets that do not contain the number 40 we can choose all 40 out of the remaining 99 so there are $\binom{99}{40}$. Now

   $$\binom{99}{39} = \frac{99!}{60! \cdot 39!} = \frac{1}{60} \cdot \frac{99!}{59! \cdot 39!}$$

   but

   $$\binom{99}{40} = \frac{99!}{59! \cdot 40!} = \frac{1}{40} \cdot \frac{99!}{59! \cdot 39!}$$

   so they are different numbers.

   (e) Let $R \subseteq A \times B$ be a binary relation with finite nonempty domain and codomain and let $S \subseteq A$. Then $|R(S)| = |S|$, true or false?

   Answer: FALSE. For example we can take $R = \emptyset$ and then $R(S)$ is always empty, even when S is nonempty. A more straightforward example would be $X = \{1\}, Y = \{2, 3\}, R = \{(1, 2), (1, 3)\$, S = \{1\}$ so $R(S) = \{2, 3\}$.

(f) Let X, Y be two nonemptry finite sets. If there exists a k-to-one function ($k \geq 1$) with domain X and codomain Y then $|X|$ is divisible by $|Y|$, true or false?

Answer: TRUE. Indeed the existence of a k-to-one function $X \to Y$ implies that $|X| = k \cdot |Y|$. Therefore $|X|$ is divisible by $|Y|$.

2. (30 points) Let $f : X \to Y$ and A, B be two subsets of X. Prove that

$$A \subseteq B \quad \Longrightarrow f(A) \subseteq f(B)$$

Answer: Assume $A \subseteq B$. We prove that $f(A) \subseteq f(B)$:

Let $y \in f(A)$

Then there exists $x \in A$ s.t. $f(x) = y$

Because $A \subseteq B$ we have $x \in B$

$f(x) \in f(B)$

3. (30 points) Let A, B, C, D be sets. Consider the following predicate (call it $P(A, B, C, D)$)

$$\text{if } A \subseteq C \text{ and } B \subseteq D \text{ then } A - B \subseteq C - D$$

(a) This predicate is an implication. Write down its *contrapositive* without using any *conjunction* (any "and").

Answer: if $A - B \not\subseteq C - D$ then $A \not\subseteq C$ or $B \not\subseteq D$

(b) This predicate does not hold for arbitrary A, B, C, D. Give an example of such sets for which the predicate is false.

Answer: $A = \{1, 2\}, B = \{3\}, C = \{1, 2, 3\}, D = \{2, 3\}$
Indeed $A \subseteq C$ and $B \subseteq D$ but $A - B = \{1, 2\}$ and $C - D = \{1\}$ so $A - B \not\subseteq C - D$.

(c) Now give an example of sets A, B, C, D for which the predicate is true.

Answer: $A = B = \emptyset$ and you can take any sets for C and D.

(d) Below are two statements. *Circle* the one that is true.

Answer: $\forall A, B, C, D \; P(A, B, C, D)$ $\quad\quad$ $\boxed{\exists A, B, C, D \; P(A, B, C, D)}$

4. (20 points) In the remote town of Plictisitor a local ordinance prevents inhabitants from having first names, they can only have last names. These last names must start with an upper case letter followed by one to three lower case letters followed by a number between 1 and 22 (to accomodate families, you see). The lower case letters must be distinct among themselves but they can be the same letter as the upper case at the beginning of the names. Moreover, no two inhabitants can have the same name. The alphabet used in Plictisitor has 31 letters, with lower and upper case for each of them.

   What is the maximum population of Plictisitor? (Just give it as an arithmetical expression since you cannot use a calculator.)

   Answer:

   There are 31 ways to pick the first (upper case) letter

   Then we have one, or else two or else three distinct lower case letters. These three cases are not overlapping so we count the ways in each case and add the three results (by the Sum Rule).

   If it's just one letter then there are 31 ways to pick it. If it's two letters then they are permutations of 2 out of 31 so there are $31 \cdot 30$ ways to pick them. If it's three letters there are $31 \cdot 30 \cdot 29$ ways to pick them. Adding the three cases we get $31 + 31 \cdot 30 + 31 \cdot 30 \cdot 29$.

   Finally, there are 22 ways to pick the number at the end.

   By the Product Rule, the total number or possible names is

   $$31 \cdot (31 + 31 \cdot 30 + 31 \cdot 30 \cdot 29) \cdot 22$$

   Since all names in Plictisitor must be distinct, this is the maximum population.

   (You are not expected to calculate such things in the exam but if you are curious the maximum population of Plictisitor is 1,731,722 so in spite of the idiotic ordinance there is room for growth :)

5. (10 pts) Consider the function $f : \mathbb{N} \to \mathbb{N}$ where $f(x)$ equals the number of sequences of bits with x 0's and x 1's. Prove that f is injective.

   Answer: We prove $\forall x_1, x_2 \in \mathbb{N} \;\; x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$.

   Let $x_1, x_2 \in \mathbb{N}$ arbitrary

   Assume $x_1 \neq x_2$ (and hope to show $f(x_1) \neq f(x_2)$).

   Case 1 $x_1 < x_2$

   Let S_1 be the set of sequences of bits with x_1 0's and x_1 1's and let S_2 be the set of sequences of bits with x_2 0's and x_2 1's. According to the definition of f we have $f(x_1) = |S_1| \; f(x_2) = |S_2|$.

   Consider the function $\sigma : S_1 \to S_2$ that takes a sequence $s \in S_1$ and adds at the end $x_2 - x_1$ 0's followed by $x_2 - x_1$ 1's. Verify that it has indeed the codomain S_2: $\sigma(s)$ has $x_1 + (x_2 - x_1) = x_2$ 0's and similarly x_2 1's. Now, σ is an injection because when $\sigma(s) = \sigma(s')$ we can just erase the last $2(x_2 - x_1)$ positions and we get $s = s'$.

Consider $\sigma(S_1)$. Because σ is an injection, we have a bijection between S_1 and $\sigma(S_1)$ so $|S_1| = |\sigma(S_1)|$. $\sigma(S_1)$ is a subset of S_2 but in fact it is a *proper* subset because there are elements of S_2 that are not in $\sigma(S_1)$: for example sequences in which the last $2(x_2 - x_1)$ positions have $0101\cdots01$ (instead of $00\cdots011\cdots1$ as is the case with the sequences in $\sigma(S_1)$). Therefore $|\sigma(S_1)| < |S_2|$ (strict inequality).

We conclude that $|S_1| < |S_2|$ therefore $f(x_1) \neq f(x_2)$.

Case 2 $x_2 < x_1$ is similar.

Alternative solution (sketch)

Figure out that $f(x) = \frac{(2x)!}{(x!)^2}$.

Prove that f is strictly increasing. A "fast" proof of this would just check that $\forall x\ f(x) < f(x+1)$. A detailed proof would fix an arbitrary x_1 and show by *induction* on x_2 that if $x_2 < x_2$ then $f(x_2) < f(x_2)$.

Then show that strictly increasing implies injective.

MIDTERM 1

1. (25 pts) For each statement below, decide whether it is true or false. In each case attach a *very brief* explanation of your answer.

   (a) The function $f : \mathbb{N} \to \mathbb{N}\ \ f(x) = 2^x$ is a bijection, true or false?

   Answer: FALSE. Because it is not a surjection. Elements which are not powers of 2 are not mapped to by f.

   (b) Let X, Y be nonempty finite sets such that $|X| = 16$ and $|Y| = 3$. Then, there can be no 4-to-one function with domain X and codomain Y, true or false?

   Answer: TRUE. Because if there was such a function then $16 = |X| = 4|Y| = 4 \cdot 3 = 12$.

   (c) Any injective function has an inverse, true or false?

   Answer: FALSE. Only bijections have inverses. An injective function that is not also a surjection cannot have an inverse.

   (d) Let S be the set of the first 100 natural numbers: $S = 0, 1, \ldots, 99$. There are as many subsets of S of size 50 that contain the number 50 as subsets of S of size 50 that do not contain the number 50, true or false?

   Answer: TRUE. The number of subsets that contain 50 is $\binom{99}{49}$. The number of subsets that do not contain 50 is $\binom{99}{50}$. These numbers are equal because $99 = 49 + 50$.

   (e) Suppose that A, B, C are finite nonempty sets with an even number of elements and that A and B are disjoint. Then $|(A \cup B) \times C|$ is divisible by 4, true or false?

   Answer: TRUE. By the sum rule $|A \cup B| = |A| + |B|$ so it's also even. Then, by the product rule $|(A \cup B) \times C| = (|A| + |B|)|C|$, the product of two even numbers, so it's divisible by 4.

2. (22 points) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be two functions such that for all $x \in \mathbb{R}$ we have $(g \circ f)(x) = 2x + 3$. Prove that f is an injection.

   Answer: We need to prove that $\forall x_1, x_2 \in \mathbb{R} \;\; f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

   Let $x_1, x_2 \in \mathbb{R}$ arbitrary.

   Assume $f(x_1) = f(x_2)$

   $g(f(x_1)) = g(f(x_2))$

   $(g \circ f)(x_1)) = (g \circ f)(x_2))$

   $2x_1 + 3 = 2x_2 + 3$

   $x_1 = x_2$

3. (23 points) Let X, Y be two nonempty sets and $f : X \rightarrow Y$ a function. Let $\text{pow}(X)$ be the set whose elements are all the subsets (proper or not) of X. Define

$$g : Y \rightarrow \text{pow}(X) \qquad g(y) \;=\; \{x \in X | f(x) = y\}$$

   Let $z \in Y$ arbitrary. Prove by contrapositive that if $g(z) = \emptyset$ then $z \notin$ range(f)

   Answer: The contrapositive is

$$z \in \text{range(f)} \quad \Rightarrow \quad g(z) \neq \emptyset$$

   Assume $z \in$ range(f)

   Then there exists $x \in X$ s.t. $f(x) = z$

   $x \in g(z)$

   Therefore $g(z) \neq \emptyset$

4. (21 points) Let E the set of strictly positive even natural numbers (strictly positive means that 0 is not in E) and O the set of odd natural numbers and consider the binary relation $R \subseteq E \times O$ defined by

$$R \;=\; \{(m, n) | m \text{ is a multiple of } n\}$$

   (a) Explain why R is total. (No proof required.)

   Answer: Because for any $m \in E$ we have $(m, 1) \in R$.

   (b) Explain why R is not functional. (No proof required.)

   Answer: Because both $(30, 3) \in R$ and $(30, 5) \in R$.

5. (21 pts) Consider sequences of bits such as m of the bits are 0, where $m \geq 2$, n of the bits are 1, where $n \geq 1$, and are such that they start with a 1 and end with two 0's. How many such sequences are there? (Give the answer and an explanation of how you figured it out. No proofs required.)

   Answer: These sequences have length $m + n$. Three of their positions have fixed elements in them that leaves $m + n - 3$ positions in which we can put $m - 2$ 0's and $n - 1$ 1's in every possible way.

Each way is determined by choosing (say) the positions in which to put the 0's. So the answer is $\binom{m+n-3}{m-2}$.

(Sanity check; when $m = 2$ and $n = 1$ there is exactly one such sequence, namely "100". And indeed $\binom{0}{0} = 1$ (because 0!=1).)

6. (8 pts) Let w be the number in position 20 of row 50 of Pascal's triangle. (Both positions and rows are counted from 1. So, for example, row 2 has two positions, 1 and 2 and in both of them is the number 1.) What is the number in position 22 of row 52? (Give it as an expression in w. No explanation is required. No proofs are required. No partial credit will be given.)

Answer:

$$\frac{(51 \cdot 50)w}{21 \cdot 20}$$

(No more than was is above was required as an answer but to see how to figure it out observe that the number in position p of row r is $\binom{r-1}{p-1}$.)

MOCK MIDTERM 2

1. (30 pts) For each statement below, decide whether it is true or false. In each case attach a *very brief* explanation of your answer.

   (a) When proving properties of Fibonacci numbers it is easier to use strong induction than ordinary induction, true or false?

   Answer: TRUE. That's because the fundamental relationship between Fibonacci numbers is given by $F_{n+1} = F_n + F_{n-1}$ and therefore in the induction step we are likely to need the induction hypothesis not just for n but also for $n - 1$.

   (b) The well-ordering principle says that any nonempty set of real numbers has a least element, true or false?

   Answer: FALSE. Here is a nonempty set of real numbers that does not have a least element: the interval $(0, 1]$! The well-ordering principle is not about real numbers it is about natural numbers.

   (c) In a uniform finite probability space all events have the same probability, true or false?

   Answer: FALSE. Only the individual outcomes has the same probability. Events that consists of different numbers of outcomes will have different probabilities.

   (d) Consider a uniform finite probability space whose sample space has 100 outcomes. In this space, an event of probability 1/4 consists of 40 outcomes, true or false?

   Answer: FALSE. In such a space each outcome has probability 0.01. An event of probability $1/4 = 0.25$ will consist of 25 outcomes.

(e) Consider a finite probability space with 10 outcomes. 9 of the outcomes have probability 0.01. Then, at least half of the events of this space have a probability bigger than 0.9, true or false?

Answer: TRUE. Let ω be the other outcome. It has probability $1 - 9 \cdot 0.01 = 1 - 0.09 = 0.91$. Therefore, if $\omega \in E$ then $\Pr[E] \geq 0.91 > 0.9$. The space has a total of 2^{10} events and 2^9 of them contain ω. But 2^9 is half of 2^{10}.

(f) The 10th Fibonacci number is smaller than 1024, true or false?

Answer: TRUE. We proved in class by strong induction that $\forall n \in \mathbb{N}\ F_n < 2^n$. Hence $F_{10} < 2^{10} = 1024$.

2. (15 points) Let A be the set of all sequences of bits of length 10 and B be the set of all sequences of bits of length 5. Consider an arbitrary function $f : A \to B$ and the statement "there exist at least m sequences in A that are mapped by f to the same sequence in B". What is the maximum value that one can give to m and still be sure that the statement is true no matter how f maps elements?

   Answer: We know that $|A| = 2^{10}$ and $|B| = 2^5$. By the generalized pigeonhole principle the maximum m such that there exist at least m sequences in A that are mapped by f to the same sequence in B is the maximum m such that $|A| > (m-1)|B|$, that is the maximum m such that $m - 1 < 2^{10}/2^5 = 2^5 = 32$ therefore $m < 33$, so the maximum m is 32.

3. (20 points) Let A, B, C be finite sets such that $B \cap C = \emptyset$. Using the principle of inclusion-exclusion prove that

$$|A \cup B \cup C| + |A| = |A \cup B| + |A \cup C|$$

   Answer: By the principle of inclusion-exclusion

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

   Since $A \cap B \cap C \subseteq B \cap C$ and $B \cap C = \emptyset$ it follows that also $A \cap B \cap C = \emptyset$. So in this case the inclusion-exclusion equality above reduces to

$$(*) \qquad |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |C \cap A|$$

   Again by inclusion exclusion, we have

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ |A \cup C| &= |A| + |C| - |A \cap C| \end{aligned}$$

   Adding both sides and using the equality $(*)$ above we get

$$|A \cup B| + |A \cup C| = |A| + |B| - |A \cap B| + |A| + |C| - |C \cap A| = |A| + |A \cup B \cup C|$$

   which proves the desired equality.

4. (20 points) Prove that if $E \cap F \cap G = \emptyset$ then $\Pr[E \cup F|G] = \Pr[E|G] + \Pr[F|G]$.

   A couple of facts about sets that you can assume known and use in your proof:

   $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$

   $(E \cap G) \cap (F \cap G) = E \cap F \cap G$

   Answer: By the definition of conditional probability

$$\Pr[E \cup F|G] = \frac{\Pr[(E \cup F) \cap G]}{\Pr[G]}$$

   Now, using the first fact that we can assume known and then using the Sum Rule for probabilities

$$\Pr[(E \cup F) \cap G] = \Pr[(E \cap G) \cup (F \cap G)] = \Pr[E \cap G] + \Pr[F \cap G)] - \Pr[(E \cap G) \cap (F \cap G)]$$

   Using the second fact that we can assume known and the assumption $E \cap F \cap G = \emptyset$ we have

$$\Pr[(E \cap G) \cap (F \cap G)] = \Pr[E \cap F \cap G] = \Pr[\emptyset] = 0$$

   Therefore

$$\Pr[E \cup F|G] = \frac{\Pr[E \cap G] + \Pr[F \cap G)]}{\Pr[G]} = \frac{\Pr[E \cap G]}{\Pr[G]} + \frac{\Pr[F \cap G)]}{\Pr[G]} = \Pr[E|G] + \Pr[F|G]$$

5. (20 pts)

   Prove by ordinary induction on n that for any $n \in \mathbb{N},\ n \geq 1$ we have

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

   Answer:

   (BASE CASE) $n = 1$.

$$\sum_{i=1}^{1} i^2 = 1 \quad \text{and} \quad \frac{1(1+1)(2 \cdot 1 + 1)}{6} = 1$$

   so the base case holds. (INDUCTION STEP) Let $k \geq 1$ arbitrary and assume the INDUCTION HYPOTHESIS:

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

   We want to prove that this implies

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

The left hand side of this desired equality equals

$$\sum_{i=1}^{k} i^2 + (k+1)^2$$

By induction hypothesis and a bit of algebra this further equals

$$\frac{k(k+1)(2k+1)}{6}+(k+1)^2 = (k+1)\frac{k(2k+1)+6(k+1)}{6} = \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

as desired.

6. (15 pts)

Prove by ordinary induction on n that for any $n \in \mathbb{N},\ n \geq 1$ we have

$$\sum_{i=1}^{n} (-1)^i \cdot i = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Answer:

(BASE CASE) $n = 1$.

$$\sum_{i=1}^{1} (-1)^i \cdot i = -1 \quad \text{and 1 being odd} \qquad -\frac{1+1}{2} = -1$$

so the base case checks out.

(INDUCTION STEP) Let $k \geq 1$ arbitrary and assume the INDUCTION HYPOTHESIS:

$$\sum_{i=1}^{k} (-1)^i \cdot i = \begin{cases} \frac{k}{2} & \text{if } k \text{ is even} \\ -\frac{k+1}{2} & \text{if } k \text{ is odd} \end{cases}$$

We wish to show that this implies

$$\sum_{i=1}^{k+1} (-1)^i \cdot i = \begin{cases} \frac{k+1}{2} & \text{if } k+1 \text{ is even} \\ -\frac{(k+1)+1}{2} & \text{if } k+1 \text{ is odd} \end{cases}$$

We write the left hand side as

$$\sum_{i=1}^{k} (-1)^i \cdot i + (-1)^{k+1}(k+1)$$

so that we have two case

(CASE 1) $k+1$ is even.

Then k is odd and by induction hypothesis

$$\sum_{i=1}^{k}(-1)^i \cdot i + (-1)^{k+1}(k+1) = -\frac{k+1}{2} + (k+1) = \frac{k+1}{2}$$

so this case checks out.

(CASE 2) $k+1$ is odd.

Then k is even and by induction hypothesis

$$\sum_{i=1}^{k}(-1)^i \cdot i + (-1)^{k+1}(k+1) = \frac{k}{2} - (k+1) = \frac{-k-2}{2} = -\frac{k+2}{2}$$

so this case checks out also.

MIDTERM 2

1. (30 pts) For each statement below, decide whether it is true or false. In each case attach a *very brief* explanation of your answer.

   (a) When proving properties of Fibonacci numbers it is easier to use strong induction than ordinary induction, true or false?

   Answer: TRUE. That's because the fundamental relationship between Fibonacci numbers is given by $F_{n+1} = F_n + F_{n-1}$ and therefore in the induction step we are likely to need the induction hypothesis not just for n but also for $n-1$.

   (b) It follows from the well-ordering principle that any nonempty set of negative integers has a greatest element, true or false?

   Answer: TRUE. Let A be a nonempty set of negative integers and let $\hat{A} = \{-z | z \in A\}$. $\hat{A}$ is a nonempty set of natural numbers and by the well-ordering principle it has a least element, which is then the greatest element of A.

   (c) We write all the permutations of 3 out of the 26 letters of the English alphabet on 2600 pages. Then at least 6 of these permutations must written on the same page, true or false?

   Answer: TRUE. There are $26 \cdot 25 \cdot 24 = 26 \cdot 100 \cdot 6 = 6 \cdot 2600$ permutations of 3 out of the 26 letters. Since $6 \cdot 2600 > 5 \cdot 2600$ it follows from the generalized pigeonhole principle that at least $5 + 1 = 6$ sequences are written on the same page.

   (d) Let E, F be two events in a finite probability space. If $|E| = |F|$ then $\Pr[E] = \Pr[F]$, true or false?

   Answer: FALSE. Since the probability space is not uniform there may be two outcomes ω_1 and ω_2 such that $\Pr[\omega_1] \neq \Pr[\omega_2]$. In this case the events $E = \{\omega_1\}$ and $F = \{\omega_2\}$ both have size 1 but they have different probabilities.

(e) If E, F are two events in a finite probability space such that $\Pr[E \cap F] > 0$ then E and F can be disjoint, true or false?

Answer: FALSE. If they were disjoint then $E \cap F = \emptyset$. Since $\Pr[\emptyset] = 0$ we have a contradiction.

(f) Consider a finite probability space in which for any two events E, F if $|E| = |F|$ then $\Pr[E] = \Pr[F]$. Then this must be a uniform probability space, true or false?

Answer: TRUE. Indeed for any two outcomes ω_1 and ω_2 we have $|\{\omega_1\}| = |\{\omega_2\}|$ therefore $\Pr[\{\omega_1\}] = \Pr[\{\omega_2\}]$ so the two outcomes have the same probability. Since this holds for arbitrary outcomes, the space is uniform.

2. (15 points) Assume that in any given class at Penn at least 85% of the class consists of students born in the USA. What is the minimum enrollement of such a class so that we can be sure that it contains at least *two* students born in the same *state*. (FYI, the USA has 50 states.) Explain how you applied the Pigeonhole Principle to get your answer (you need to identify the pigeons, the pigeonholes and the function that assigns pigeons to the pigeonholes).

Answer: Here the pigeons are the students, the pigeonholes are the US states and the function assigns to each student born in the USA the state she/he was born in. By the pigeonhole principle, to be sure that at least two students are born in the same US state we need to have a minimum of 51 students born in the USA in the class. Therefore the class must have a minimum of

$$51 \cdot \frac{100}{85} = 51 \cdot \frac{20}{17} = 3 \cdot 20 = 60$$

students.

3. (20 points)

Let A, B, C be arbitrary finite sets.

(a) Give the formula that calculates $|A \cup B \cup C|$ according to the rule (principle) of Inclusion-Exclusion.

Answer:

$$(*) \qquad |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

(b) Let $m = |A| + |B| + |C| - |A \cup B \cup C|$ and $n = |A \cap B| + |B \cap C| + |C \cap A|$. Prove that $m \leq n$.

Answer: Rewrite $(*)$ as follows

$$|A \cap B| + |B \cap C| + |A \cap C| = |A| + |B| + |C| - |A \cup B \cup C| + |A \cap B \cap C|$$

that is, $n = m + |A \cap B \cap C|$. Since $|A \cap B \cap C| \geq 0$ it follows that $m \leq n$.

4. (20 points)

(a) Let A, B be two *disjoint* events in a finite probability space. Give the formula that calculates $\Pr[A \cup B]$ according to the Sum Rule (for probabilities).

Answer: $\Pr[A \cup B] = \Pr[A] + \Pr[B]$

(b) Let E, F be two events in a finite probability space such that $\Pr[E \cap F] > 0$. Use the Sum Rule for probabilities to prove that $\Pr[E - F] + \Pr[F - E] < \Pr[E \cup F]$.

A couple of facts about sets that you can assume known. Use any of them as needed in your proofs.

$(E - F) \cup (E \cap F) = E$

$E \cup (F - E) = E \cup F$

$E \cup F = (E - F) \cup (E \cap F) \cup (F - E)$

Answer: Using the Sum Rule, we have, since $E - F$ and $E \cap F$ are disjoint and since $(E - F) \cup (E \cap F) = E$

$$\Pr[E - F] + \Pr[E \cap F] = \Pr[(E - F) \cup (E \cap F)] = \Pr[E]$$

Again using the Sum Rule, the fact that E and $F - E$ are disjoint and the fact that $E \cup (F - E) = E \cup F$, we have

$$\Pr[E] + \Pr[F - E] = \Pr[E \cup (F - E)] = \Pr[E \cup F]$$

Combining the two equalities we get

$$(*) \quad \Pr[E - F] + \Pr[E \cap F] + \Pr[F - E] = \Pr[E \cup F)]$$

Since $\Pr[E \cap F] > 0$ it follows that

$$\Pr[E - F] + \Pr[F - E] < \Pr[E \cup F]$$

Note. An equally valid proof would derive $(*)$ directluy from the (generalized) sum rule using the given fact $E \cup F = (E - F) \cup (E \cap F) \cup (F - E)$ and the fact that $E - F$, $E \cap F$, and $F - E$ are pairwise disjoint.

5. (20 pts)

(a) State the principle of ordinary induction for proving that for any nonzero natural number n we have $P(n)$.

Answer: If $P(1)$ and if $\forall k \in \mathbb{N}, k \geq 1 \;\; P(k) \Rightarrow P(k+1)$ then $\forall n \in \mathbb{N}, n \geq 1 \;\; P(n)$.

(b) Prove by ordinary induction on n that for any $n \in \mathbb{N},\; n \geq 1$ we have

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Answer:

(BASE CASE) $n = 1$. $\quad 1 = 1$ and $\frac{1(1+1)}{2} = 1$. Check.

(INDUCTION STEP) Let $k \in \mathbb{N}, k \geq 1$, arbitrary.

Assume the INDUCTION HYPOTHESIS:

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

and we wish to prove

$$(*) \qquad 1 + 2 + \cdots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$$

By induction hypothesis and a bit of algebra the left hand side of $(*)$ equals

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

which equals the right hand side of $(*)$.

6. (15 pts) Prove by ordinary induction on n that for any $n \in \mathbb{N}$, $n \geq 1$ we have that for any two sets A, B with $|A| = 2$ and $|B| = n$ the number of functions with domain A and codomain B is n^2. (Indeed we have counted functions before and we know that this is the correct count. Here you *must prove it by induction.*)

   Answer:

   (BASE CASE) $n = 1$.

   Let A, B be arbitrary sets with $|A| = 2$ and $|B| = 1$. There is exactly one function from A to B, the one that maps both elements of A to the single element of B. But $1 = 1^2$ so the base case checks out.

   (INDUCTION STEP) Let $k \in \mathbb{N}, k \geq 1$ arbitrary.

   Assume the INDUCTION HYPOTHESIS: for any two sets A, B with $|A| = 2$ and $|B| = k$ the number of functions with domain A and codomain B is k^2. We want to show that this implies: for any two sets A, B with $|A| = 2$ and $|B| = k+1$ the number of functions with domain A and codomain B is $(k+1)^2$.

   Let A, B be arbitrary sets with $|A| = 2$ and $|B| = k+1$. Let's name a_1, a_2 the elements of A and let b be one of the elements of B. We divide the functions $f : A \to B$ into two disjoint sets. W consists of those functions for which $b \notin \text{range}(f)$ and V consists of those functions for which $b \in \text{range}(f)$. Now we have to prove that $|W| + |V| = (k+1)^2$.

   The set W is in bijection with the set of all functions from A to $B - \{b\}$. Since $|B - \{b\}| = k$, by induction hypothesis there are k^2 functions in this set, therefore $|W| = k^2$.

   We further divide the set V of functions $f : A \to B$ for which $b \in \text{range}(f)$ into three disjoint subsets: V_1 which has those f for which $f(a_1) = b$ but $f(a_2) \neq b$, V_2 which has those f for which $f(a_2) = b$ but $f(a_1) \neq b$, V_3 which has those f for which $f(a_1) = f(a_2) = b$.

   For the functions in V_1 there are k choices for $f(a_2)$ so $|V_1| = k$.

   Similarly $|V_2| = k$.

Finally, there is exactly one function in V_3 so $|V_3| = 1$.

It follows that $|V| = |V_1|+|V_2|+|V_3| = k+k+1 = 2k+1$. Therefore $|W|+|V| = k^2+2k+1 = (k+1)^2$ so the induction step checks out.

MOCK MIDTERM 3

1. (30 pts) For each statement below, decide whether it is true or false. In each case attach a *very brief* explanation of your answer.

   (a) Consider events A, B, C in a finite probability space such that $A \perp B$, $B \perp C$, and $A \cap B \perp B \cap C$. Suppose that $\Pr[B] = 1/2$. Assume also that $\Pr[A] \neq 0$, and $\Pr[C] \neq 0$. Then, $\Pr[A \cap B \cap C] = \Pr[A]\Pr[B]\Pr[C]$, true or false?

   Answer: FALSE. $A \perp B \Rightarrow \Pr[A \cap B] = \Pr[A]\Pr[B]$, $B \perp C \Rightarrow \Pr[B \cap C] = \Pr[B]\Pr[C]$, $A \cap B \perp B \cap C \Rightarrow \Pr[(A \cap B) \cap (B \cap C] = \Pr[A \cap B]\Pr[B \cap C]$. Since $(A \cap B) \cap (B \cap C) = A \cap B \cap C$, we have that $\Pr[A \cap B \cap C] = \Pr[A]\Pr[B]\Pr[C]$ is equivalent to $(\Pr[B])^2 = \Pr[B]$ which is false.

   (b) If a recursive definition of data is ambiguous then structural induction is still valid but some recursive definition of function may be invalid, true or false?

   Answer: TRUE. Indeed structural induction is still sound, albeit redundant on those data that have more than one derivation. But function definitions may lead to different values for the same argument.

   (c) A total order can have two different maximal elements, true or false?

   Answer: FALSE. Let a, b be maximal. Suppose $a \neq b$. The order is total so $a \leq b$ or $b \leq a$. In the first case b is not maximal, in the second a is not. In both cases we have contradictions.

   (d) If all antichains of a partial order are empty then the order is total, true or false?

   Answer: TRUE. If there is just one element, the order is trivially total. If there are more, any two cannot form an antichain so they must be comparable.

   (e) If an undirected graph has n nodes and each node has degree $n-1$ then the graph has exactly $\binom{n}{2}$ edges, true or false?

   Answer: TRUE. Every pair of nodes must be connected by an edge because every node must be adjacent to every other node. Hence there are $\binom{n}{2}$ edges. But there cannot be more than that because any two nodes are already connected by an edge.

   (f) K_n has at least n spanning trees, true or false?

   Answer: TRUE. For each node v, consider the edges that connect v to all the other $n-1$ nodes. These edges form a spanning tree, a different one for each v, thus at least n of them (there are many, many more spanning trees).

2. (15 points) Let $n \geq 1$ be a natural number. Draw a DAG G such that

   - G has exactly $2n+1$ nodes, and
   - among G's nodes there are two, u_0 and u_n, such that there are 2^n paths from u_0 to u_n.

   Answer: The vertices are $u_0, u_1, \ldots, u_n, v_1, \ldots, v_n$ and the edges are $u_i \rightarrow u_{i+1}, i = 0, \ldots, n-1$, $u_i \rightarrow v_{i+1}, i = 0, \ldots, n-1$ and $v_i \rightarrow u_i, i = 1, \ldots, n$,

3. (30 points) Consider an alphabet with four letters, two uppercase and two lowercase: $A = \{D, d, E, e\}$ and a set RR of strings over this alphabet that is recursively defined as follows.

   base case $\lambda \in RR$

   constructor 1 If $s \in RR$ then $Dsd \in RR$.

   constructor 2 If $t \in RR$ then $etE \in RR$.

   (a) Give examples of three different strings of length 6 that belong to RR.

   Answer: $DDDddd$, $eeeEEE$, $DeDdEd$.

   (b) Give the recursive definition of functions $up : RR \rightarrow \mathbb{N}$ and $lo : RR \rightarrow \mathbb{N}$ such that $up(s)$ is the number of upper case letters in s and $lo(s)$ is the number of lower case letters in s.

   Answer:

   base case $up(\lambda) = 0$, $lo(\lambda) = 0$

   constructor 1 $up(Dsd) = 1 + up(s)$, $lo(Dsd) = lo(s) + 1$

   constructor 2 $up(etE) = up(t) + 1$, $lo(etE) = 1 + lo(t)$

   (c) Prove by structural induction that $\forall t \in RR \; up(t) = lo(t)$.

   Answer:

   base case $up(\lambda) = 0 = lo(\lambda)$

   step 1 Suppose $up(s) = lo(s)$. Then $up(Dsd) = 1 + up(s) = lo(s) + 1 = lo(Dsd)$

   step 2 Similar to step 1.

4. (30 points) Let $n \geq 2$ be a natural number and a, b two other things. Consider the undirected graph $G = (V, E)$ where $\{a, 1, 2, \ldots, n, b\}$ and $E = \{\{a, 1\}, \{a, 2\}, \ldots, \{a, n\}\} \cup \{\{1, b\}, \{2, b\}, \ldots, \{n, b\}\}$.

   (a) Count the number of cycles in this graph.

   Answer: $\binom{n}{2}$ (explanation in review session).

   (b) Count the number of paths of length 2 in this graph.

   Answer: $n + 2\binom{n}{2} = n^2$ (explanation in review session).

   (c) What is the maximum length of a path in this graph (this is called the "diameter" of the graph)?

   Answer: 3 (explanation in review session).

(d) Describe what spanning trees look like in this graph. Do this by giving an example from each group of "similar" spanning trees.

Answer:

group 1: $\{\{a,1\},\{a,2\},\ldots,\{a,n\},\{i,b\}\}, i = 1,\ldots,n$
group 2: $\{\{a,i\},\{1,b\},\{2,b\},\ldots,\{n,b\}\}, i = 1,\ldots,n$
group 3: $\{\{a,i\},\{i,b\}$ plus partition the nodes of $\{1,\ldots,n\} - \{i\}$ into two blocks and connect one block to a and the other to b.
(Explanation in review session.)

5. (15 pts) Let A be a set and ρ a strict partial order relation on it. Define another binary relation on A as follows

$$\sigma = \{\ (a,b) \in A \times A \mid a\ \rho\ b \text{ or } a = b\}$$

Prove that σ is transitive, reflexive, and antisymmetric (that is, it is a partial order).

Answer:

(refl) $a\sigma a$ because $a = a$ (warning, ρ is irreflexive!)

(trans) Supose $a\sigma b$ and $b\sigma c$. We have four cases

Case 1: $a\rho b$ and $b\rho c$. Then, because ρ is transitive $a\rho c$ so $a\sigma c$.

Case 2: $a = b$ and $b\rho c$. Then $a\rho c$ so $a\sigma c$.

Case 3: $a\rho b$ and $b = c$. Then $a\rho c$ so $a\sigma c$.

Case 4: $a = b$ and $b = c$. Then $a = c$ so $a\sigma c$.

(antisym) Assume $a\sigma b$ and $b\sigma a$. We just proved that σ is transitive. So $a\sigma a$. Therefore $a\rho a$ or $a = b$. But ρ is irreflexive so only $a = b$ is possible.

MIDTERM 3

1. (30 pts) For each statement below, decide whether it is true or false. In each case attach a *very brief* explanation of your answer.

(a) Consider events A, B, C in a finite probability space such that $A \perp B$, $B \perp C$, and $A\cap B \perp B\cap C$ and such that $\Pr[A] = \Pr[B] = \Pr[C] = 1/2$. Then, $\Pr[A \cap B \cap C] = 1/8$, true or false?

Answer: FALSE.

$$\Pr[A \cap B \cap C] = \Pr[(A \cap B) \cap (B \cap C)] = \Pr[A \cap B]\Pr[B \cap C] = \Pr[A]\Pr[B]\Pr[B]\Pr[C] = \frac{1}{16}$$

(b) There exists an *undirected* graph with 3 vertices, 2 connected components, and 1 edge true or false?

Answer: TRUE. $G = (\{1,2,3\},\{\{1,2\}\}$

(c) Recall the graph K_4, the *undirected* graph with 4 vertices in which any two distinct vertices are connected by an edge. It is possible to remove 4 edges from K_4 such that the remaining graph has exactly 2 connected components, true or false?

Answer: TRUE. There are 6 edges. Remove 4 of them so you leave two that were *not adjacent* in K_4. This leaves two connected components each with one edge. OR, remove all three edges incident to a node v. This leaves a connected component consisting just of v and another which is a triangle. Now further remove one of the edges of the triangle.

(d) Consider the set $X = \{a, b, c\}$, also the set $W = \mathrm{pow}(X) - \{\emptyset, X\}$ and the poset $(W, \subseteq)$. (Note that the elements of W are subsets of X, actually all the subsets except for the empty subset and except for $\{a, b, c\}$.) This poset has exactly as many minimal elements as it has maximal elements, true or false?

Answer: TRUE. W has three maximal elements: $\{a, b\}, \{b, c\}, \{a, c\}$ and three minimal elements: $\{a\}, \{b\}, \{c\}$.

(e) Let A be a set with 100 elements and let ρ be an equivalence relation on A. Then the partition of A that corresponds to ρ has 2^{100} blocks true or false? (Another way to ask the same question is "there are 2^{100} distinct equivalence classes, true or false?")

Answer: FALSE. There can be at most 100 blocks because each block is non-empty and it doesn't overlap with any other block so in each block we can pick an element and it will be a different one from the one picked in another block. So there is some injective function function from the partition to A. Another way to see this is to observe that the function from A to the partition that maps each element to its equivalent class is surjective.

(f) Consider the poset $([0, 1], \leq)$ (that's the interval $[0, 1]$ of real numbers). This poset has infinite chains, true or false?

Answer: TRUE. Here is one: $1/2 < 2/3 < 3/4 < 4/5 < \cdots$. A slicker solution is to observe that the entire interval is an infinite chain :).

2. (30 points) Consider the set M of natural numbers recursively defined as follows.

base case 1 $1 \in M$.

base case 2 $3 \in M$.

constructor If $x \in M$ then $x + 4 \in M$.

(a) Show that $13 \in M$ and also that $19 \in M$

Answer: $13 = (((1) + 4) + 4) + 4 \quad 19 = ((((3) + 4) + 4) + 4) + 4$

(b) Prove by structural induction that $\forall\, x \in M \;\; x$ is *odd*.

Answer:

base case 1 1 is odd.

base case 2 3 is odd.

constructor If x is odd then $x + 4$ is also odd.

(c) Explain (no proof required) why *all* the odd natural numbers must be in M.

Answer: When we divide a natural number by 4 the possible remainders are 0,1,2, and 3.

Let n an odd natural number. Then only 1 and 3 can be remainders when we divide n by 4. So either $n = 4k + 1$ for some $k \in \mathbb{N}$ or $n = 4h + 1$ for some $h \in \mathbb{N}$.

If $n = 4k + 1$ then n can be constructed from base case 1 followed by k applications of the constructor. If $n = 4h+3$ then n can be constructed from base case 2 followed by h applications of the constructor. In both cases, $n \in M$.

3. (20 points) Let $n \geq 2$ be a natural number and a another thing. Consider the *undirected* graph $G = (V, E)$ where $\{a, 1, 2, \ldots, n\}$ and $E = \{\{a, 1\}, \{a, 2\}, \ldots, \{a, n\}\} \cup \{\{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}\}$.

   (a) Draw this graph. Use dot dot dot or use $n = 7$ for the figure.

   Answer:

```
        ------      --------
      /        \   /         \
     /     ---- a ----        \
    /     /    ...    \        \
   /     /             \        \
  1 ---- 2 --   ...  -- n-1 ---- n
```

   (b) Count the number of cycles of length 3. (No explanation or proof is required, just the answer.)

   Answer: $n-1$ (Not required: each such cycle must contain exactly one edge of the form $\{i, i+1\}$ where $i = 1, 2, \ldots, n - 1$.)

   (c) Count the number of paths of length 2 that *start at a*. (No explanation or proof is required, just the answer.)

   Answer: $2n - 2$ (Not required: since we start at a the first edge must be of the form $\{a, i\}$. The second edge then can be either $\{i, i + 1\}$ for $i = 1, \ldots, n - 1$ or $\{i - 1, i\}$ for $i = \{2, \ldots, n\}$. So we have $(n - 1) + (n - 1) = 2n - 2$ possibilites. Check it quickly for $n = 2, 3$.)

   (d) Draw a spanning tree of this graph in which a is a leaf. Use dot dot dot or use $n = 7$ for the figure. (No explanation or proof is required, just the answer.)

   Answer: The following works for any i:

```
               a
               |
               |
1 -- ...   -- i -- ... -- n
```

   (Not required: since there are $n + 1$ nodes any spanning trees must have n edges. Moreover since a must be a leaf we can pick only 1 of the n edges incident to a. That means that we must pick all the $n - 1$ bottom edges.)

4. (20 pts) Consider the *directed* graph $G = (V, E)$ where $\{a, 1, 2, 3, 4\}$ and $E = \{(a, 1), (a, 2), (a, 3), (a, 4)\} \cup \{(1, 2), (2, 3), (3, 4)\}$.

(a) Draw this graph. Is this graph a DAG (YES or NO)?

Answer: In the diagram below the direction on the edges goes from top to bottom and from left to right.

```
    ----- a -----
   /     / \     \
  /     /   \     \
 /     /     \     \
1 ---- 2 ---- 3 --- 4
```

YES, this is a DAG.

(b) Write down the adjacency matrix A of this graph.

Answer:

	a	1	2	3	4
a	0	1	1	1	1
1	0	0	1	0	0
2	0	0	0	1	0
3	0	0	0	0	1
4	0	0	0	0	0

(c) Consider $A^2 = A \cdot A$ and $A^3 = A^2 \cdot A$. Consider also $B = A + A^2 + A^3$. *Without actually performing the matrix multiplications* write down the row in B that corresponds to the vertex a. Explain.

Answer: We know from class/textbook that in A^2 the entry in row a and column x is the number of walks of length 2 from a to x.
That means that the row corresponding to a in A^2 is

	a	1	2	3	4
a	0	0	1	1	1
$\dots$					

because there is no walk of length 2 from a to 1, 1 walk of length 2 from a to 2, etc.
Similarly, A^3 contains the number of walks of length 3 so the row corresponding to a in A^3 is

	a	1	2	3	4
a	0	0	0	1	1
$\dots$					

Adding $A + A^2 + A^3$ we get

	a	1	2	3	4
a	0	1	2	3	3
...					

5. (15 points) Let A be a set and ρ a partial order relation on it. Define another binary relation on A as follows

$$\sigma = \{ \ (a, b) \in A \times A \mid a \ \rho \ b \text{ and } a \neq b\}$$

Prove that σ is transitive and irreflexive (that is, it is a strict partial order).

Answer: Let's first prove that σ is irreflexive. Let $a \in A$ arbitrary. Suppose toward a contradiction that $a\sigma a$. Then $a\rho a$ and $a \neq a$, but the second conclusion is false, contradiction.

Now let's prove that σ is transitive. Let $a, b, c \in A$ arbitrary. Assume $a\sigma b$ and $b\sigma c$. Therefore $a\rho b$ and $a \neq b$ and $b\rho c$ and $b \neq c$.

We want to show that $a\sigma c$, that is we want to show that $a\rho c$ and $a \neq c$. The first of these two follows from the transitivity of ρ. We still need to prove the second, that is, $a \neq c$.

Suppose, toward a contradiction that $a = c$. Then we can replace c with a in $b\rho c$ and we obtain $b\rho a$ so we have both $a\rho b$ and $b\rho a$, From the antisymmetry of ρ we obtain $a = b$ which contradicts $a \neq b$.

MOCK FINAL

1. (50 pts) For each statement below, decide whether it is true or false. In each case attach a *very brief* explanation of your answer.

   (a) Let $G = (V, E)$ be a (finite) DAG such that $|V| \geq 3$. Recall that E^+, the transitive closure of E, is a strict partial order on V. Suppose that this strict partial order has a minimum and a maximum. Then, the size of any antichain in this strict partial order is at most $|V| - 2$, true or false?

   Answer: TRUE. If the minimum and the maximum are the same element then $|V| = 1$ which contradict $|V| \geq 3$. Since the minimum m is comparable to any other element it can belong only to one antichain, $\{m\}$ whose size is 1, and $1 \leq |V| - 2$. Same for the maximum. Any other antichain can be either empty and $0 \leq |V| - 2$ or it can contain neither the minimum nor the maximum in which case it can have at most $1 \leq |V| - 2$ elements.

   (b) $G = (V, E)$ be a (finite) connected undirected graph. Then G can have at most $|E| + 1$ vertices, true or false?

   Answer: TRUE. G is connected so it has exactly one connected component. By the theorem in the textbook/class $|E| \geq |V| - 1$ so $|V| \leq |E| + 1$.

(c) Assume that B is a set with 7 elements and that A is a set with 15 elements. Then, for any function $f : A \to B$ there exist at least 3 distinct elements of A that are mapped by f to the same element of B, true or false?

Answer: TRUE. By the generalized piegeonhole principle since $15 > 2 \cdot 7$.

(d) Assume that A, B are finite nonempty sets and $f : A \to B$ is a function such that there exist at least 3 distinct elements of A that are mapped by f to the same element of B. Then $|A| > 2 \cdot |B|$, true or false?

Answer: FALSE. For example $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$ and $f(a_1) = f(a_2) = f(a_3) = b_1$ but $|A| = 3 \leq 4 = 2 \cdot |B|$.

(e) The number of sequences of bits such that three of the bits are 0, three of the bits are 1, and are such that they start with a 1 and end with two 0's is 8, true or false?

Answer: FALSE. Such sequences look like this: $1xyz00$ where two of the bits x, y, z are 1 and the third is 0. This is possible in exactly 3 ways and $3 \neq 8$.

(f) The composition $g \circ f$ of an injection f with a surjection g is always a bijection, true or false?

Answer: FALSE. Counterexample $f : \{1, 2\} \to \{a, b, c\}, f(1) = a, f(2) = b$ and $g : \{a, b, c\} \to \{1, 2\} g(a) = 1, g(b) = 1, g(c) = 2$. Then $g \circ f : \{1, 2\} \to \{1, 2\}, g \circ f(1) = 1, g \circ f(2) = 1$ so $g \circ f$ is not a bijection (in fact it is neither injective nor surjective!)

(g) Let X be a set with m elements and Y be a set with n elements. For any $m \geq 2$ and $n \geq 2$ there are twice as many binary relations with domain X and codomain Y than functions with domain X and codomain Y, true or false?

Answer: FALSE. Since $|X \times Y| = mn$ by the product rule then the number of binary relations with domain X and codomain Y is 2^{mn} since each such relation corresponds to a subset of $X \times Y$. Meanwhile, the number of functions with domain X and codomain Y is $|Y|^{|X|} = n^m$. So, is it possible that $2^{mn} = 2 \cdot n^m$, or equivalently $2^{mn-1} = n^m$ when $m, n \geq 2$? No, and you can stop right here. But let's prove it. Well, what we have implies that n must be even so $n = 2k$ with $2^{mn-1} = 2^m k^m$, that is $2^{m(n-1)-1} = k^m$. Repeat this reasoning $n - 1$ times and we get $2^{m(n-(n-1))-1} = r^m$ for some r. So $2^{m-1} = r^m$ for some r, but this is not possible for $m \geq 2$.

(h) Let X be a finite set with 6 or more elements, and let p be the number of subsets of X of size 2. Then $p \leq 2|X|$, true or false?

Answer: FALSE. Let $n = |n|$. Then $p = \binom{n}{2} = n(n-1)/2$. So is it possible that $n(n-1)/2 \leq 2n$? This is equivalent to $n^2 \leq 5n$ so $n = 0$ or $n \leq 5$. But $n \geq 6$.

(i) Let A be a finite set and let ρ be an equivalence relation on A such that $\forall x \in A\ \exists y \in A\ \ x\rho y$ AND $x \neq y$. Let m be the number of equivalence classes of ρ and assume that $|A|$ is odd. Then $|A| \geq 2m + 1, |A| \geq 2m + 1$, true or false?

Answer: TRUE. The condition $\forall x \in A\ \exists y \in A\ \ x\rho y$ AND $x \neq y$ implies that each equivalence has at least two distinct elements. Since the equivalence classes are disjoint, and by the sum rule, we must have $2m \leq |A|$. Since $|A|$ is odd we must have $2m + 1 \leq |A|$.

(j) Let X be a set with m elements and Y be a set with n elements such that $m > n$. Then, there exist at least n distinct surjective function with domain X and codomain Y, true or false?

Answer: TRUE. If $n = 0$ the statement holds because "there exist at least 0 objects such that blah" is true. Otherwise, let $Y = \{b_1, \ldots, b_n\}$ and $X = \{a_1, a_2, \ldots, a_m\}$ For each $i = 1, \ldots, n$ consider the function f_i that maps $a_1, a_2, \ldots, a_{n-1}$ respectively to $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n$ and $f_i(a_n) = \cdots = f_i(a_m) = b_i$. This function is clealry surjective and we have n such functions $f_1, \ldots, f_n$. Are they distinct? Suppose, toward a contradiction, that for some $i \neq j$ we have $f_i = f_j$. Then, since $b_i = f_i(a_m)$ and $b_j = f_j(a_m)$, we have $b_i = b_j$, contradiction.

2. (15 points) Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $f(x, y) = (x + y, x - y)$ and let $g = f \circ f$. Prove that g is an bijection.

Answer: First we compute $g(x, y) = f(f(x, y)) = f(x+y, x-y) = ((x+y)+(x-y), (x+y)-(x-y)) = (2x, 2y)$.

g is *injective.*

Suppose $g(x_1, y_1) = g(x_2, y_2)$. Then $(2x_1, 2y_1) = (2x_2, 2y_2)$ so $2x_1 = 2x_2$ and $2y_1 = 2y_2$. Therefore $x_1 = x_2$ and $y_1 = y_2$ so $(x_1, y_1) = (x_2, y_2)$.

g is *surjective.*

For any $(z, u) \in \mathbb{R} \times \mathbb{R}$ there exists $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $g(x, y) = (z, u)$, namely, $x = z/2$ and $y = u/2$.

3. (30 points) Let X be a finite nonempty subset of $\mathbb{N}$. Let $|X| = n$ and assume that $n \geq 2$. Let $W = \text{pow}(X) - \{\emptyset\}$. Define a function $q : W \to X$ such that for any $A \in \mathbb{W}$ $q(A)$ is the maximum natural number in A.

(a) Prove that q is surjective.

Answer: For any $a \in A$ we have $\{a\} \in W$ and $q(\{a\}) = a$.

(b) Prove that q injective $\Rightarrow$ $2^n - 1 \leq n$.

Answer: If q is injective then the size of the domain of q is at most the size of its codomain. Since W has all but one of the subsets of X as its elements the size of W is 2^{n-1}. Therefore $2^n - 1 \leq n$.

(c) Prove by ordinary induction on n that for all $n \geq 2$ $n + 1 < 2^n$.

BASE CASE $n = 2$. The statement becomes $2 + 1 < 2^2$ which is true.

IND. STEP. Let $n \geq 2$ arbitrary. Assume (IND HYP) that $n + 1 < 2^n$.

We want to show that $n + 2 < 2^{n+1}$. Using the IND HYP we have

$$n + 2 = (n + 1) + 1 < 2^n + 1 < 2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$$

(d) Prove that q is not injective.

Answer: By contradiction. If q were injective then by what we proved in part (b) we would have $2^n - 1 \leq n$. Since $n \geq 2$ this would contradict what we proved in part (c).

4. (15 points) Consider sequences of n bits, $n \geq 1$ in which there are exactly k 0's where $k < n$ and every 0 is immediately followed by a 1. How many such sequences are there? (Give the answer and an explanation of how you figured it out. No proofs required.)

   Answer: We have two cases

   CASE 1 $n < 2k$: Here the answer is 0 because we need at least $2k$ bit positions for each of the k 0's to be followed by a 1.

   CASE 2 $n \geq 2k$: Let s be a string of n bits in which there are exactly k 0's and every 0 is immediately followed by a 1. Now s can also be regarded as a string w over al alphabet of two letters: 1's and β's where β is a letter that "stands for" the string 01. For example

$$s = 1010111011 \qquad w = 1\beta\beta 11\beta 1$$

   There will be exactly k occurences of β in w. Therefore, letting x be the number of 1's in w the length of w is $k + x$ β's or 1's and the length of s is $2k + x$ bits. So $2k + x = n$ therefore $x = n - 2k$ and the length of w is $k + x = k + (n - 2k) = n - k$.

   The number of strings of two letters (β's or 1's) of length $n - k$ in which exactly k letters are β is $\binom{n-k}{k}$. And this is the answer.

5. (20 pts) Let $X = \{1, 2, \ldots, 2n\}$. Assume $n \geq 2$. How many nonempty subsets of X contain at most 2 odd numbers? (Give the answer and an explanation of how you figured it out. No proofs required.)

   Answer: We have three disjoint cases that we will then, by the sum rule, add up.

   CASE 1 containg zero odd numbers. There are n even numbers in X and therefore 2^n subsets of X consisting of just even numbers (includes the empty subset).

   CASE 2 containg exactly one odd number. First pick an odd number p. This can be done in n ways. For each p pick a subset (maybe empty) S of the even numbers which can be done. This produces $\{p\} \cup S$ and it can be done in $n \cdot 2^n$ ways.

   CASE 3 containg exactly two distinct odd numbers (this case doesn't exist when $n = 1$ hence the condition $n \geq 2$) First pick two distinct odd numbers p_1, p_2. This can be done in $\binom{n}{2}$ ways. For each p_1, p_2 pick a subset (maybe empty) S of the even numbers which can be done. This produces $\{p_1, p_2\} \cup S$ and it can be done in $\binom{n}{2} \cdot 2^n$ ways.

   The answer to the question is

$$2^n \; + \; n \cdot 2^n \; + \; \binom{n}{2} 2^n = (1 + n + \frac{n^2 - n}{2}) 2^n = (n^2 + n + 2) 2^{n-1}$$

6. (30 pts) Let X, Y, Z be three finite nonempty sets such that $X \cap Y = \emptyset$, $Z \cap Y = \emptyset$, $X \cap Z = \emptyset$ and denote $|X| = m$, $|Y| = n$, $|Z| = p$. Assume that $m < n < p$. Let also $f : X \to Y$ and $g : Y \to Z$ be two functions. Consider the undirected graph $G = (V, E)$ where $V = X \cup Y \cup Z$ and

$$E = \{ \; \{x, f(x)\} \mid x \in X\} \cup \{ \; \{y, g(y)\} \mid y \in Y\}$$

(a) What is $|V|$ and what is $|E|$ (in terms of m, n, p)?. (Give the answer and an explanation of how you figured it out. No proofs required.)

Answer: Clearly $|V| = m + n + p$.

Because f, g are functions there is exactly one set of the form $\{x, f(x)\}$ for each $x \in X$ and exactly one set of the form $\{y, g(y)\}$ for each $y \in Y$. Therefore $|E| = m + n$.

(b) What is the maximum number of nodes of degree 0 that G can have (in terms of m, n, p)?. (Give the answer and an explanation of how you figured it out. No proofs required.)

Answer: Because every node in X and Y is the endpoint of an edge only nodes from Z can have degree 0. They correspond to elements *outside* of the range of g (i.e., the direct image $g(Y)$). To maximize their number we must *minimize* the size of $g(Y)$. This happens when g maps all the elements of Y to one element of Z. Therefore the answer is $p - 1$.

(c) What is the minimum number of nodes of degree 0 that G can have (in terms of m, n, p)?. (Give the answer and an explanation of how you figured it out. No proofs required.)

Answer: Reason like in part (b) then *maximize* the size of $g(Y)$. This happens when g is injective and the size of $g(Y)$ is n. The answer is $p - n$. (Note that we use $n \le p$ otherwise we cannot even define g to be an injection.)

(d) What is the maximum length that a path in G can have? (Give the answer and an explanation of how you figured it out. No proofs required.)

Answer: X is nonempty so either $m = 1$ or $m \ge 2$.

CASE 1 $m = 1$: Let $X = \{a\}$. Since $n > m$ we have $n \ge 2$ so let $b_1, b_2 \in Y$ s.t. $b_1 \ne b_2$. Define $f(a) = b_1$ and $g(b_1) = g(b_2)$. Then we have a path of length 3 in G: $a - - - f(a) = b_1 - - - g(b_1) = g(b_2) - - - b_2$.

Can there be paths of length 4? Since X has just one element there is exactly one edge between X-nodes and Y-nodes. So in any path of length 4 there must be 3 edges between Y-nodes and Z-nodes. At least two of these edges would have to share a Y node, which cannot happen.

So when $m = 1$ the maximum path length is 3.

CASE 2 $m \ge 2$: Let $a_1, a_2 \in X$ s.t. $a_1 \ne a_2$. Since also $n \ge 2$ let $b_1, b_2 \in Y$ s.t. $b_1 \ne b_2$. Define $f(a_1) = b_1$ and $f(a_2) = b_2$ and $g(b_1) = g(b_2)$. Then we have a path of length 4 in G: $a_1 - - - f(a_1) = b_1 - - - g(b_1) = g(b_2) - - - b_2 = f(a_2) - - - a_2$.

Can there be paths of length 5? By the pigeonhole principle In any path with 5 edges there must be 3 edges between X-nodes and Y-nodes or three 3 edges between Y-nodes and Z-nodes. In either case, at least 2 of the 3 edges would have to share an X-node (in the first case) or a Y-node (in the second case). Neither can happen.

So when $m \ge 2$ the maximum path length is 4.

(e) Prove that G is acyclic (that is, it has no cycles of length ≥ 1).

Answer: Consider a cycle of length non-zero in this graph.

Can it have an X-node? The cycle cannot have two distinct edges incident to it because no such edges are in the graph. But neither can just one edge be incident to it (we ruled out self-loops in undirected graphs). So the cycle cannot have X-nodes. Similarly, it cannot have Y-nodes. So it must have only Z-nodes. But there are no edges between Z-nodes.

Therefore no such cycles exist.

7. (35 pts) Consider the 3-letter alphabet $A = \{\ N\ ,\ ]\ ,\ [\ \}$ and the set ROBT of strings over this alphabet defined recursively as follows:

base case $\lambda \in$ ROBT.

constructor If $t_1 \in$ ROBT and $t_2 \in$ ROBT then $[\,t_1\,]\,N\,[\,t_2\,] \in$ ROBT.

Consider also the string $ww = [\,[\,]\,N\,[\,[\,]\,N\,[\,]\,]\,]\,N\,[\,]$.

(a) Show that $ww \in$ ROBT.

Answer: Notice that $ww = [uu]\,N\,[\,]$ where $uu = [\,]\,N\,[vv]$ where $vv = [\,]\,N\,[\,]$. So we can obtain ww by using the base case 4 times and applying the constructor 3 times.

(b) Consider the following recursive definition of a function height : ROBT $\to \mathbb{Z}$:

base case height$(\lambda) = e$.

constructor height$([\,t_1\,]\,N\,[\,t_2\,]) = 1 + \max(\text{height}(t_1), \text{height}(t_2))$.

What should $e \in \mathbb{Z}$ be so that height$(ww) = 2$? For the rest of the problem, use this value for e.

Answer: By the definition using uu and vv defined in part (a) we have

height$(vv) = 1 + \max(e, e) = 1 + e$.

height$(uu) = 1 + \max(e, \text{height}(vv)) = 1 + \max(e, 1 + e) = 1 + 1 + e = 2 + e$

height$(ww) = 1 + \max(\text{height}(uu), e) = 1 + \max(2 + e, e) = 1 + 2 + e = 3 + e$

Now, from height$(ww) = 2$ we obtain $e = -1$. Below we use $e = -1$.

(c) Let's call the substring $[\,]\,N\,[\,]$ a *leaf* and for every string s let's denote by noLeaves(s) the number of occurrences of this substring in s. Prove by structural induction that for all $t \in$ ROBT if height$(t) \geq 0$ then noLeaves$(t) \leq 2^{\text{height}(t)}$.

Answer:

base case height$(\lambda) = -1 \geq 0 \Rightarrow$ noLeaves$(\lambda) \leq \ldots$ The premise of the implication is false so the implication is true.

constructor Let $t_1, t_2 \in ROBT$ arbitrary. Assume (IND HYP)

$$\text{height}(t_1) \geq 0 \Rightarrow \text{noLeaves}(t_1) \leq 2^{\text{height}(t_1)} \quad \text{and} \quad \text{height}(t_2) \geq 0 \Rightarrow \text{noLeaves}(t_2) \leq 2^{\text{height}(t_2)}$$

and we want to prove the statement for the string $t = [\,t_1\,]\,N\,[\,t_2\,]$. Unfortunately, it is possible that t_1 or t_2 or both are λ and for them we cannot use the IND HYP because its premise is false (no free lunch!). So we reason by cases

CASE 1 $t_1 = \lambda, t_2 = \lambda$

Then t is a leaf itself so noLeaves$(t) = 1$. Since height$(t) = 1 + \max(-1, -1) = 0$ the statement holds.

CASE 2 $t_1 = \lambda, t_2 \neq \lambda$

Then, $\text{height}(t) = 1 + \text{height}(t_2)$ and $\text{noLeaves}(t) = \text{noLeaves}(t_2)$. By the IND HYP, $\text{noLeaves}(t_2) \leq 2^{\text{height}(t_2)}$. So

$$\text{noLeaves}(t) = \text{noLeaves}(t_2) \leq 2^{\text{height}(t_2)} < 2^{1+\text{height}(t_2)} = 2^{\text{height}(t)}$$

CASE 3 $t_1 \neq \lambda, t_2 = \lambda$

Like CASE 2.

CASE 4 $t_1 \neq \lambda, t_2 \neq \lambda$

Then, $\text{height}(t) = 1 + \max(\text{height}(t_1), \text{height}(t_2))$ and $\text{noLeaves}(t) = \text{noLeaves}(t_1) + \text{noLeaves}(t_2)$ (because any leaf is either completely in t_1 or completely in t_2. Therefore we wish to show that

$$\text{noLeaves}(t_1) + \text{noLeaves}(t_2) \leq 2^{1+\max(\text{height}(t_1),\text{height}(t_2))}$$

By IND HYP $\text{noLeaves}(t_1) \leq 2^{\text{height}(t_2)}$ and $\text{noLeaves}(t_2) \leq 2^{\text{height}(t_2)}$. Adding each side up we get

$$\text{noLeaves}(t_1) + \text{noLeaves}(t_1) \leq 2^{\text{height}(t_1)} + 2^{\text{height}(t_2)}$$

So the desired result now follows from the following general inequality

$$2^{h_1} + 2^{h_2} \leq 2^{\max(h_1,h_2)} + 2^{\max(h_1,h_2)} = 2 \cdot 2^{\max(h_1,h_2)} = 2^{1+\max(h_1,h_2)}$$

(d) Give a recursive definition of a function complete : $\mathbb{N} \to$ ROBT such that for all $n \in \mathbb{N}$ we have $\text{height}(\text{complete}(n)) = n$ and $\text{noLeaves}(\text{complete}(n)) = 2^n$. (That is, give a recursive definition and then prove by ordinary induction on n that indeed your definition satisfies $\text{height}(\text{complete}(n)) = n$ and $\text{noLeaves}(\text{complete}(n)) = 2^n$).

Answer: This is a recursive definition on $\mathbb{N}$! So the base case is 0 and the constructor is "add 1".

base case $\text{complete}(0) = [\,]\, N\, [\,]$ (a leaf).

constructor $\text{complete}(n+1) = [\text{complete}(n)]\, N\, [\text{complete}(n)]$

Proof by ordinary induction that $\forall n\, P(n)$ where $P(n) = \text{height}(\text{complete}(n)) = n$ AND noLeaves(comple
2^n AND $\text{complete}(n) \neq \lambda$. Notice that we "strengthened the induction hypothesys" with the non-empty string part.

BASE CASE $n = 0$. The height of a leaf is 0 (see part (b) in which vv is a leaf) and the number of leaves is 1 so both check out. A leaf is not empty.

IND STEP Let n be a natural number, arbitrary. Assume (IND HYP) that $\text{height}(\text{complete}(n)) = n$ AND $\text{noLeaves}(\text{complete}(n)) = 2^n$ AND $\text{complete}(n) \neq \lambda$.

Now

$$\text{height}(\text{complete}(n{+}1)) = \text{height}(\,[\text{complete}(n)]\, N\, [\text{complete}(n)]\,) = 1{+}\max(\text{height}(\text{complete}(n)), \text{height}$$

Using the IND HYP we get $\text{height}(\text{complete}(n+1)) = n + 1$.

Also

$$\text{noLeaves}(\text{complete}(n{+}1)) = \text{noLeaves}(\,[\text{complete}(n)]\,N\,[\text{complete}(n)]\,) = 2{\cdot}\text{noLeaves}(\text{complete}(n))$$

where in the last step we used from the IND HYP the fact that complete(n) is not empty. Now $2 \cdot 2^n = 2^{n+1}$ so the second part checks out. For the third part, clearly complete($n+1$) is not empty.

8. (30 pts) Sophie is playing the following game:

- First she chooses with equal probability one of the numbers $0, 1, \ldots, 4$, call it a.
- Then she chooses with equal probability one of the *remaining* numbers, call it b.
- Then she adds $a + b = c$ and c is the outcome of her game.

(a) What possible outcomes can Sophie's game have?

Answer: Any natural number between 1 and 7 can be expressed as the sum of two *distinct* numbers between 0 and 4:

$$1 = 0 + 1, 2 = 0 + 2, 3 = 0 + 3, 4 = 0 + 4, 5 = 1 + 4, 6 = 2 + 4, 7 = 3 + 4$$

but 8 (or bigger numbers) cannot. So the outcomes are 1,2,3,4,5,6,7.

(b) Using the four step method (draw the diagram please!) determine the probability of each of the outcomes of the game.

Answer: You draw the diagram. I am just writing the probabilities :)

$$\Pr[1] = 2/20, \Pr[2] = 2/20\Pr[3] = 4/20, \Pr[4] = 4/20, \Pr[5] = 4/20, \Pr[6] = 2/20, \Pr[7] = 2/20$$

(c) What is the probability that the outcome is 2?

Answer: 2/20

(d) What is the probability of the event $\{2, 3\}$?

Answer: $2/20 + 4/20 = 6/20$

(e) What is the conditional probability $\Pr[\{2\}|\{2, 3\}]$?

Answer: $\Pr[\{2\}|\{2, 3\}] = \Pr[\{2\} \cap \{2, 3\}]/\Pr[\{2, 3\}] = \Pr[\{2\}]/\Pr[\{2, 3\}] == (2/20)/(6/20) = 1/3$

(f) Find two events E, F such that neither is empty, neither equals the whole sample space and $E \perp F$.

Answer: $E = \{1, 2\}$ and $F = \{2, 3, 4\}$.
$\Pr[\{1, 2\}] = 4/20$
$\Pr[\{2, 3, 4\}] = 10/20$
$\Pr[\{1, 2\}] \cdot \Pr[\{2, 3, 4\}] = (4/20) \cdot (10/20) = 2/20 = \Pr[\{1, 2\} \cap \{2, 3, 4\}]$

9. (15 pts) Draw an example of a DAG, $G = (V, E)$ such that the strict partial order E^+ has

- a minimum element;
- exactly 2 maximal elements;
- at least one chain of 4 distinct elements; and
- at least one antichain of 3 distinct elements.

Explain (proof not required) why such a DAG cannot be a tree (when we erase the direction on the edges).

Answer: I am just defining the graph, you draw it.

$G = (V, E)$ where $V = \{a, b, c, d, e, f, g\}$ and

$E = \{(a, b), (a, c), (a, d), (b, e), (e, f), (c, g), (d, g)\}$

Minimum is a. Maximal elements are b and c. The chain is $\{a, b, e, f\}$. The antichain can be $\{b, c, d\}$.

Why can't this DAG be a tree (when we erase the direction on the edges)?

Because it has an antichain of size 3 but only 2 maximal elements.

Indeed, every vertex must be comparable to one of the two maximal ones otherwise we get more maximal elements. So each of the 3 elements of the antichain must be comparable to one of the maximal ones. Pigeonholing we see that two of them must be comparable to the *same* maximal element. These two, with the maximal one and with minimum one form a cycle. RIP tree.

FINAL

1. (45 pts) For each statement below, decide whether it is true or false. In each case attach a *very brief* explanation of your answer.

   (a) Let $f : \mathbb{N} \to \mathbb{N}$ and let the subset $A \subseteq \mathbb{N}$ have 100 elements. Then the direct image $f(A)$ must also have 100 elements, true or false?

   Answer: FALSE. It doesn't *have to* because, for example, if we define for any $x \in \mathbb{N}$, $f(x) = 13$ then $f(A) = \{13\}$ so it has just one element.

   (b) The function $f : \mathbb{N} \to \mathbb{N}$ $f(x) = 2^x - 1$ is a bijection, true or false?

   Answer: FALSE. The function f is not a bijection because it is not a surjection: for example there is no x such that $f(x) = 2$.

   (c) Let A, B be events in a finite probability space such that $\Pr[A] = 1/4$ and $\Pr[A \cup B] = 1/2$. Then, $\Pr[B] < 1/4$, true or false?

   Answer: FALSE. We know that $\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$. So $1/2 \leq 1/4 + \Pr[B]$ hence $\Pr[B] \geq 1/4$.

(d) Let A, B, C be events in a finite probability space such that $A \perp B$, $B \perp C$, and $A \cap B \perp B \cap C$ and such that $\Pr[A] = \Pr[B] = \Pr[C] = 1/2$. Then, $\Pr[A \cap B \cap C] = 1/8$, true or false?

Answer: FALSE.

$$\Pr[A \cap B \cap C] = \Pr[(A \cap B) \cap (B \cap C)] = \Pr[A \cap B]\Pr[B \cap C] = \Pr[A]\Pr[B]\Pr[B]\Pr[C] = \frac{1}{16}$$

(e) Consider a finite probability space in which for *any* two events E, F if $|E| = |F|$ then $\Pr[E] = \Pr[F]$. Then this must be a uniform probability space, true or false?

Answer: TRUE. Indeed for any two outcomes ω_1 and ω_2 we have $|\{\omega_1\}| = |\{\omega_2\}|$ therefore $\Pr[\{\omega_1\}] = \Pr[\{\omega_2\}]$ so the two outcomes have the same probability. Since this holds for arbitrary outcomes, the space is uniform.

(f) Consider a poset with exactly two minimal elements. Then, one of these two elements is minimum, true or false?

Answer: FALSE. Let a, b such that $a \neq b$ be the two minimal elements. If a is minimum then $a \leq b$ so b is not minimal. Similar for b, contradiction in both cases.

(g) There exists an *undirected* graph with 3 vertices, 2 connected components, and 1 edge, true or false?

Answer: TRUE. $G = (\{1, 2, 3\}, \{\{1, 2\}\}$

(h) Consider an *undirected* graph with 3 or more vertices and with *exactly* 3 connected components. In order to make this into a connected graph we must add at least 3 edges, true or false?

Answer: FALSE. Two will suffice: add one edge between a node in component 1 and a node in component 2 and one edge between a node in component 2 and a node in component 3. Now nodes in component 1 are connected to nodes in component 3, but by paths that go through component 2.

(i) For any directed acyclic graph (DAG) G, let G^u be the undirected graph obtained from G by keeping the same vertices but erasing the direction of the edges. Then, no matter what DAG G is, if G^u is connected it must be the case that G^u is also a tree, true or false?

Answer: FALSE. Indeed, consider the DAG: $G = (\{1, 2, 3\}, \{(1, 2), (2, 3), (1, 3)\})$. G^u is connected but it is not a tree because $1 - \{1, 2\} - 2 - \{2, 3\} - 3\{3, 1\} - 1$ is a closed walk of length 3 with distinct vertices, i.e., a cycle.

2. (25pts) Give an example of finite sets X, Y and functions $f : X \to Y$ and $g : Y \to X$ such that

- $g \circ f$ is a bijection,
- $g \circ f$ is different from the identity function,
- f is not a surjection,
- g is not an injection, and
- X has exactly 2 elements.

You must justify why your $g \circ f$ is different from the identity function, why your f is not a surjection, and why your g is not an injection. No other proof is required.

Answer: Take $X = \{a, b\}$ and $Y = \{1, 2, 3\}$ and define $f(a) = 2, f(b) = 1$ and $g(1) = a, g(2) = b, g(3) = b$.

Therefore $(g \circ f)(a) = b$ and $(g \circ f)(b) = a$. The problem doesn't require a *proof* that $g \circ f$ is a bijection, we just claim it. Now

(i) $g \circ f$ is different from the identity function because, for example, for the argument $a \in X$ we have $(g \circ f)(a) = g(f(a)) = g(2) = b \neq a = \mathsf{id}_X(a)$.

(ii) f is not a surjection because for $3 \in Y$ there is no $x \in X$ such that $f(x) = 3$.

(iii) g is not an injection because for $2, 3 \in Y$ we have $2 \neq 3$ but $f(2) = b = f(3)$.

3. (15pts) Let A, B, C be arbitrary finite sets. Prove that $|A| + |B| + |C| \geq |A \cap B| + |B \cap C| + |C \cap A|$.

   Answer: The principle of inclusion-exclusion gives

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

   So

$$|A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| = |A \cup B \cup C| - |A \cap B \cap C|$$

   But $|A \cap B \cap C| \subseteq |A \cup B \cup C|$ so $|A \cup B \cup C| \geq |A \cap B \cap C|$. Hence the desired result.

4. (20pts) Let X and A be sets such that $\emptyset \neq A \subseteq X$ and let $f : X \to X$ be a function (warning: the domain and the codomain are the *same* set). Let $f(A)$ be the direct image of A through f. Denote $f(A)$ by B. B is another subset of X so now we can talk about the direct image $f(B)$, in fact we can write $f(f(A))$ instead of $f(B)$. Prove that $f(A) \subseteq A \Rightarrow f(f(A)) \subseteq f(A)$.

   Answer: Assume $f(A) \subseteq A$ and we wish to prove that $f(f(A)) \subseteq f(A)$.

   Let $z \in f(f(A))$. Then there exists $y \in f(A)$ s.t. $z = f(y)$. But $f(A) \subseteq A$ so $y \in A$. Therefore $z = f(y) \in f(A)$.

5. (20pts) Weird Al (WAl) is playing with his coins. The game uses two identical fair *coins* and one *urn*. The outcome of the game is one of H (heads) or T (tails) and is determined as follows:

   - WAl places both coins in an urn.
   - WAl reaches inside the urn and (a) with probability $2/3$ WAl grabs one of the coins and tosses it, OR (b) with probability $1/3$ WAl grabs both coins, then tosses them separately in some order (doesn't matter which order).
   - If WAl has tossed just one coin then whatever that coin shows is the outcome of the game. If WAl has tossed both coins then applying the weird $\otimes$ operation to what the two coins show is the outcome of the game where $T \otimes T = T$, $T \otimes H = H$, $H \otimes T = H$, and $H \otimes H = T$.

(a) Using the four step method (draw the diagram please!) determine the probability of each of the two outcomes of the game.

Answer: The diagram is TBD. But the result is that each of the two outcomes has probability $1/2$.

(b) What simpler game could Weird Al play that would give him exactly the same odds?

Answer: Just toss one coin :).

6. (15pts) Fix $n, k \in \mathbb{N}$ such that $0 < k < n$ and let X be a finite set with n elements. Consider

$$W \;=\; \{\ (A, B) \mid A, B \in \text{pow}(X) \text{ and } A \cap B = \emptyset \text{ and } |A| = k\}$$

Give $|W|$ in terms of n and k. (Give the answer and an explanation of how you figured it out. No proofs required.)

Answer: There are $\binom{n}{k}$ ways to pick the k elements of A. Once A is determined, B can be any of the 2^{n-k} subsets of the remaining $n - k$ elements.

So the answer is $\binom{n}{k}\, 2^{n-k}$.

7. (30pts) Consider the set $M \subseteq \mathbb{N}$ recursively defined as follows.

base case $100 \in M$.

constructor If $x \in M$ then

- if $x \geq 3$ then $x - 3 \in M$
- if $x < 3$ then the constructor does not apply.

(a) Show that $94 \in M$ and also that $1 \in M$

Answer: $94 = (((100) - 3) - 3)$ $\quad 1 = (\cdots((100) - 3) - \cdots - 3)$ where we apply the constructor 33 times.

(b) Prove by structural induction that $\forall\, x \in M$ the remainder of the division of x by 3 is 1.

Answer:

base case $100 = 3 \cdot 33 + 1$

constructor Let x be an arbitrary natural number. Assume (IND HYP) that $x = 3q + 1$. If $x < 3$ the the constructor does not apply and we are done. If $x \geq 3$ we compute $x - 3 = 3(q - 1) + 1$. The remainder is 1 but is it still a natural number? Yes because $x \geq 3$ implies $q \geq 1$.

(c) Explain (no proof required) why *all* the natural numbers less than 100 and whose remainder in the division by 3 is 1 must be in M.

Answer: Any such number has the form $x = 3q + 1$ where $q = 0, \ldots, 33$ and therefore it can be obtained with one use of the base case followed by $33 - q$ uses of the constructor.

8. (20 pts) Consider strings over the 2-letter alphabet $\{a, b\}$ and the set W of these strings such that

- each string starts with an a and end with two b's;
- there are exactly m total number of a's in the string (including the first one) where $m \geq 1$;
- there are exactly n total number of b's in the string (including the last two) where $n \geq 2$.

How many such strings are there (in other words, what is $|W|$)? (Give the answer and an explanation of how you figured it out. No proofs required.)

Answer: These strings have length $m+n$. Three of their positions have fixed elements in them. That leaves $m+n-3$ positions in which we can put $m-1$ a's and $n-2$ b's in every possible way. Each way is determined by choosing (say) the positions in which to put the a's. So the answer is $\binom{m+n-3}{m-1}$.

9. (20pts) Consider the DAG $G=(V,E)$ where $V=\{0,1,2,3,4\}$ and $E=\{(0,1),(0,2),(0,3),(0,4)\} \cup \{(1,2),(2,3),(3,4)\}$. Recall that E^+ is a strict partial order. (In the following questions give the answer and an explanation of how you figured it out. No proofs required.)

   (a) How many maximal elements does E^+ have?

   Answer: E^+ has a maximum, namely 4. Any maximum is also the only maximal element. So the answer is 1.

   (b) How many antichains does E^+ have?

   Answer: We know that the empty subset is an antichain and so are all singleton sets. So that's 1+5=6 antichains.

   E^+ is total. Any two elements are comparable. So E^+ has no antichains of size 2 or more. The answer is 6+0=6.

   (c) How many chains does E^+ have?

   Answer: We already observed that E^+ is total. This means that any subset is a chain (including the empty subset and the singleton subsets which are chains regardless of the totality). There are $2^5=32$ such subsets so the answer is 32.

10. (20pts) Let $n \geq 3$ be a natural number. Consider the *undirected* graph $G=(V,E)$ where $V=\{0,1,2,\ldots,n\}$ and $E=\{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\} \cup \{\{0,1\},\{0,2\},\ldots,\{0,n\}\}$.

    (a) Draw this graph. Use dot dot dot or use $n=5$ for the figure.

    Answer: I am skipping this but think of a "wheel" with with n "spokes" and with 0 in the center.

    (b) Count the number of cycles of length 3 or more. (Give the answer and an explanation of how you figured it out.

    Answer: There is just one cycle that doesn't go through 0 (its length is n).

    Every cycle that goes through 0 must use two distinct spoke edges. There are $\binom{n}{2}$ such pairs. Once such a pair is chosen we have two ways to complete the cycle (clockwise around the wheel rim and counterclockwise around the wheel rim).

So the answer is $1 + \binom{n}{2} \cdot 2 = n^2 - n + 1$

An alternative proof counts the cycles by length. It works but it's easier to mess up :). It's a good exercise to see if you understand the following. There are

n cycles of length $n + 1$,

$n + 1$ cycles of length n,

n cycles of length $n - 1$,

$\ldots$

n cycles of length 3

So the total is $n + (n + 1) + n(n - 3) = n^2 - n + 1$ again.

(c) Count the number of paths of *exactly* length 2.

Answer: Let's first count the paths that start on the wheel rim and go through 0, i.e., they have the form $u - 0 - v$. There are n ways to choose u and for each of these there are $n - 1$ ways to choose $v \neq u$. So we have $n(n - 1)$ such paths.

Next we count the paths that begin at 0. Their first edge can be chosen in n ways and then for the next edge we have two choices: clockwise around the wheel rim or counterclockwise around the rim. This gives $n \cdot 2$ paths that begin at 0. The paths that end at 0 are the reverses of these so we another $n \cdot 2$ paths.

Finally let's count the paths that do not go through 0. There are n ways to choose the start vertex and for each of these the path can go clockwise around the wheel rim or counterclockwise around the rim. So we have $n \cdot 2$ such paths.

So the answer is $n(n - 1) + n \cdot 2 + n \cdot 2 + n \cdot 2 = n^2 + 5n$.

11. (10pts) Let X be a *finite* nonempty set and $f : X \to X$. Let $x \in X$ arbitrary and consider the sequence

$$x, \quad f(x), \quad f(f(x)), \quad \ldots \quad , f(\cdots f(x) \cdots), \quad \ldots$$

Prove that for any $k \geq 2$ there must exist k distinct positions in this sequence in which the same element of X occurs. (This "same" element does not have to be x itself.)

Answer: Let W be the set of functions with domain X and codomain X. Define $p : \mathbb{N} - \{0\} \to W$ as follows: $p(1) = f$ and $p(n + 1) = f \circ p(n)$.

Now define $q : \{1, 2, \ldots, |X|, \ldots, 2|X|, \ldots, k|X|, k|X| + 1\} \to X$ by $q(i) = (p(i))(x)$. Then the result follows from the generalized pigeonhole principle.

Alternative proof which shows more, namely that there exists some element of X, not necessarily x itself, that occurs *infinitely* many times in this sequence. This proof is simpler but it relies on facts about infinite cardinality, a subject that the course did not cover.

Suppose, toward a contradiction, that each element of X that occurs in the sequence appears there only finitely many times. Since X is finite, this would mean that there are only finitely many positions in the sequence. But that's not true: the sequence is infinite (we can keep applying f forever :).