Announcements

Seminar announcement

In-class midterm on Thursday. Closed book. No devices. No notes. I'll provide scratch paper. **Be sure to bring a pen or pencil**.

The midterm will take place in 3 different rooms across campus. Your room depends on your last name:

- Last names starting with A-F go to **Stiteler Hall room B26**
- Last names starting with G-L go to **Claire Fagin Hall, room 118**
- Last names starting with M-Z go to **Towne 100 (here)**

The TAs will lead a midterm review session tonight at 8pm in Wu and Chen.

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2.3 Quicksort

- *quicksort*
- *selection*
- *duplicate keys*
- *system sorts*

Selection

Goal. Given an array of N items, find the k^{th} smallest item.

Ex. Min $(k = 0)$, max $(k = N - 1)$, median $(k = N/2)$.

Applications.

- Order statistics.
- Find the "top k."

Use theory as a guide.

- Easy $N \log N$ upper bound. How?
- Easy N upper bound for $k = 1, 2, 3$. How?
- Easy N lower bound. Why?

Which is true?

- $N \log N$ lower bound? ← is selection as hard as sorting?
- N upper bound? ← is there a linear-time algorithm?

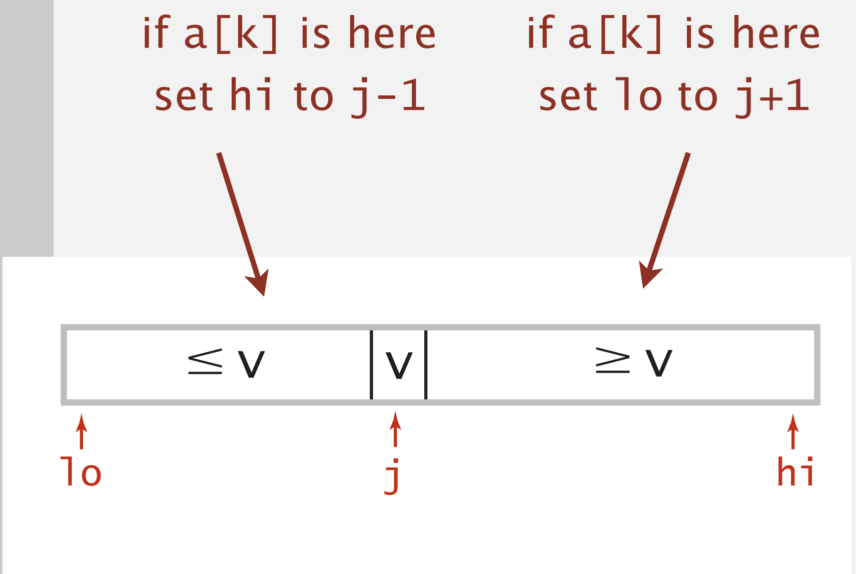
Quick-select

Partition array so that:

- Entry `a[j]` is in place.
- No larger entry to the left of `j`.
- No smaller entry to the right of `j`.

Repeat in one subarray, depending on `j`; finished when `j` equals `k`.

```
public static Comparable select(Comparable[] a, int k)
{
    StdRandom.shuffle(a);
    int lo = 0, hi = a.length - 1;
    while (hi > lo)
    {
       int j = partition(a, lo, hi);
       if      (j < k) lo = j + 1;
       else if (j > k) hi = j - 1;
       else            return a[k];
    }
    return a[k];
}
```



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select element of rank k = 5

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
50	21	28	65	39	59	56	22	95	12	90	53	32	77	33

k = 5

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partition on leftmost entry

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partitioned array

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can safely ignore right subarray

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stop: partitioning item is at index k

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12	21	22	32	28	33	39	50	95	56	90	53	59	77	65

k = 5

Quick-select: mathematical analysis

Proposition. Quick-select takes linear time on average.

Pf sketch.

- Intuitively, each partitioning step splits array approximately in half: $N + N/2 + N/4 + \ldots + 1 \sim 2N$ compares.
- Formal analysis similar to quicksort analysis yields:

$$
\begin{aligned}
C_N &= 2\,N + 2\,k \ln (N/k) + 2\,(N-k) \ln (N/(N-k)) \\
&\leq (2 + 2 \ln 2)\,N
\end{aligned}
$$

- Ex: $(2 + 2 \ln 2)\,N \approx 3.38\,N$ compares to find median ($k = N/2$).

Theoretical context for selection

Proposition. [Blum, Floyd, Pratt, Rivest, Tarjan, 1973] Compare-based selection algorithm whose worst-case running time is linear.

Time Bounds for Selection

by .

Manuel Blum, Robert W. Floyd, Vaughan Pratt,
Ronald L. Rivest, and Robert E. Tarjan

Abstract

The number of comparisons required to select the i-th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm -- PICK. Specifically, no more than 5.4305 n comparisons are ever required. This bound is improved for

Remark. Constants are high $\Rightarrow$ not used in practice.

Use theory as a guide.

- Still worthwhile to seek practical linear-time (worst-case) algorithm.
- Until one is discovered, use quick-select (if you don't need a full sort).

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2.3 Quicksort

- *quicksort*
- *selection*
- *duplicate keys*
- *system sorts*

Duplicate keys

Often, purpose of sort is to bring items with equal keys together.

- Sort population by age.
- Remove duplicates from mailing list.
- Sort job applicants by college attended.

Typical characteristics of such applications.

- Huge array.
- Small number of key values.

```
Chicago 09:25:52
Chicago 09:03:13
Chicago 09:21:05
Chicago 09:19:46
Chicago 09:19:32
Chicago 09:00:00
Chicago 09:35:21
Chicago 09:00:59
Houston 09:01:10
Houston 09:00:13
Phoenix 09:37:44
Phoenix 09:00:03
Phoenix 09:14:25
Seattle 09:10:25
Seattle 09:36:14
Seattle 09:22:43
Seattle 09:10:11
Seattle 09:22:54
```

key

War story (system sort in C)

A beautiful bug report. [Allan Wilks and Rick Becker, 1991]

```
We found that qsort is unbearably slow on "organ-pipe" inputs like "01233210":

main (int argc, char**argv) {
   int n = atoi(argv[1]), i, x[100000];
   for (i = 0; i < n; i++)
      x[i] = i;
   for ( ; i < 2*n; i++)
      x[i] = 2*n-i-1;
   qsort(x, 2*n, sizeof(int), intcmp);
}

Here are the timings on our machine:
$ time a.out 2000
real    5.85s
$ time a.out 4000
real   21.64s
$time a.out 8000
real   85.11s
```

War story (system sort in C)

Bug. A `qsort()` call that should have taken seconds was taking minutes.

At the time, almost all `qsort()` implementations based on those in:

- Version 7 Unix (1979): quadratic time to sort organ-pipe arrays.
- BSD Unix (1983): quadratic time to sort random arrays of 0s and 1s.

Duplicate keys: stop on equal keys

Our partitioning subroutine stops both scans on equal keys.

scan until ≥ P

scan until ≤ P

P	G	E	P	A	Q	B	P	Y	C	O	U	P	Z	S	R

Q. Why not continue scans on equal keys?

scan until > P

scan until < P

P	G	E	P	A	Q	B	P	Y	C	O	U	P	Z	S	R

Partitioning an array with all equal keys

i	j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
		A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
1	15	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
1	15	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
2	14	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
2	14	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
3	13	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
3	13	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
4	12	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
4	12	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
5	11	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
5	11	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
6	10	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
6	10	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
7	9	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
7	9	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
	8	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
	8	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A

Duplicate keys: partitioning strategies

Bad. Don't stop scans on equal keys.

[$\sim \frac{1}{2} N^2$ compares when all keys equal]

B A A B A B B B C C C   A A A A A A A A A A A

Good. Stop scans on equal keys.

[$\sim N \lg N$ compares when all keys equal]

B A A B A B C C B C B   A A A A A A A A A A A

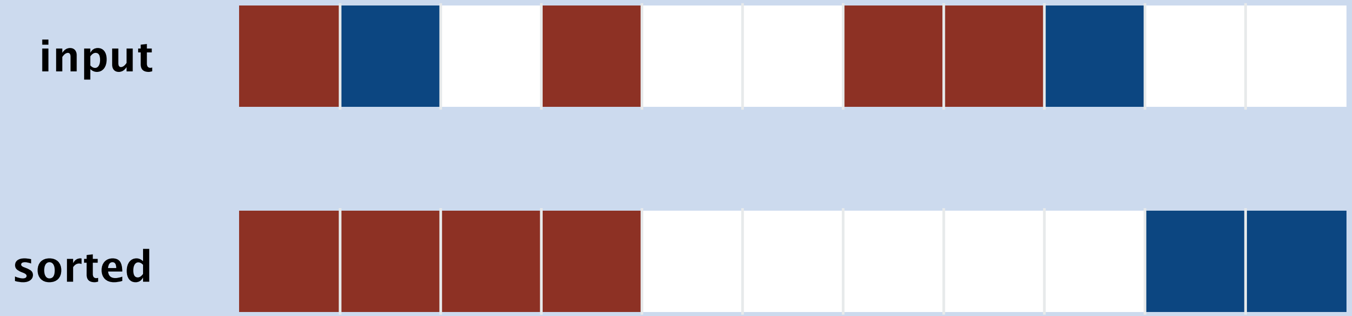
Better. Put all equal keys in place. How?

[$\sim N$ compares when all keys equal]

A A A B B B B B C C C   A A A A A A A A A A A

Dutch National Flag Problem

Problem. [Edsger Dijkstra] Given an array of N buckets, each containing a red, white, or blue pebble, sort them by color.



Operations allowed.

- $swap(i, j)$: swap the pebble in bucket i with the pebble in bucket j.
- $color(i)$: color of pebble in bucket i.

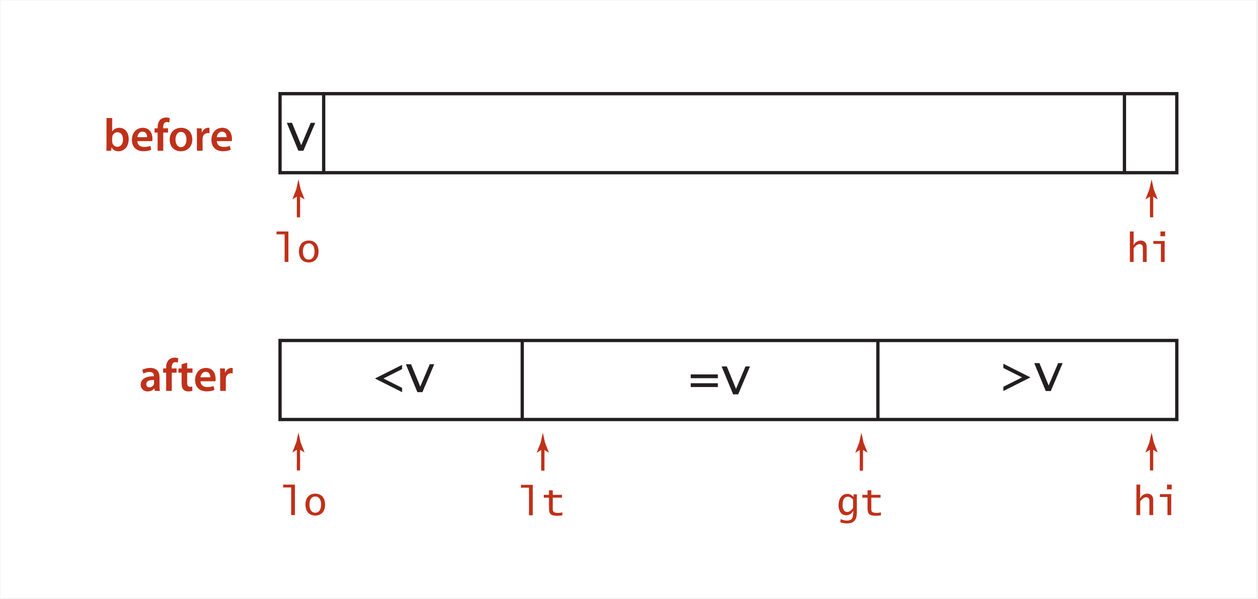
Requirements.

- Exactly N calls to $color()$.
- At most N calls to $swap()$.
- Constant extra space.

3-way partitioning

Goal. Partition array into three parts so that:

- Entries between `lt` and `gt` equal to the partition item.
- No larger entries to left of `lt`.
- No smaller entries to right of `gt`.



Dutch national flag problem. [Edsger Dijkstra]

- Conventional wisdom until mid 1990s: not worth doing.
- Now incorporated into C library `qsort()` and Java 6 system sort.

Dijkstra 3-way partitioning demo

- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i]  < v)`: exchange `a[lt]` with `a[i]`; increment both `lt` and `i`
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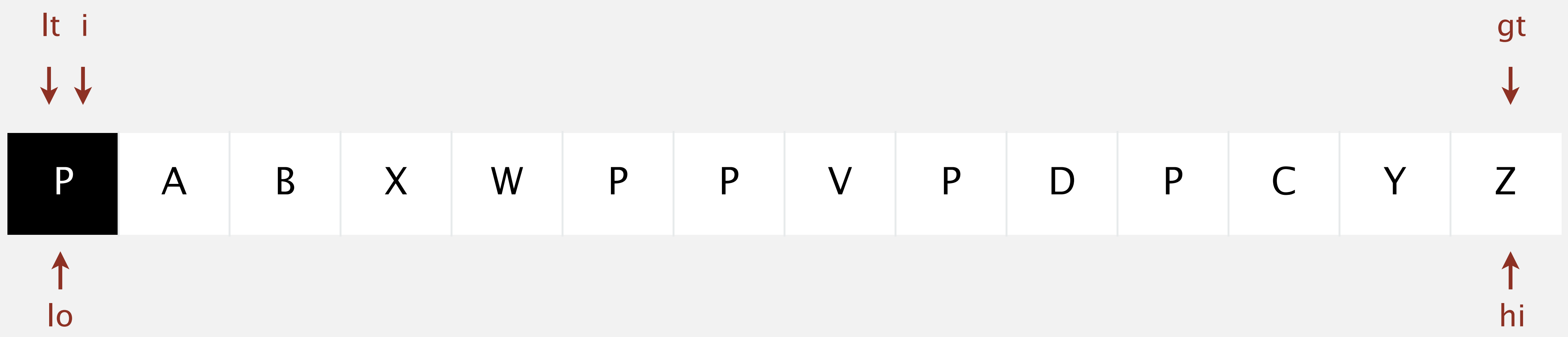
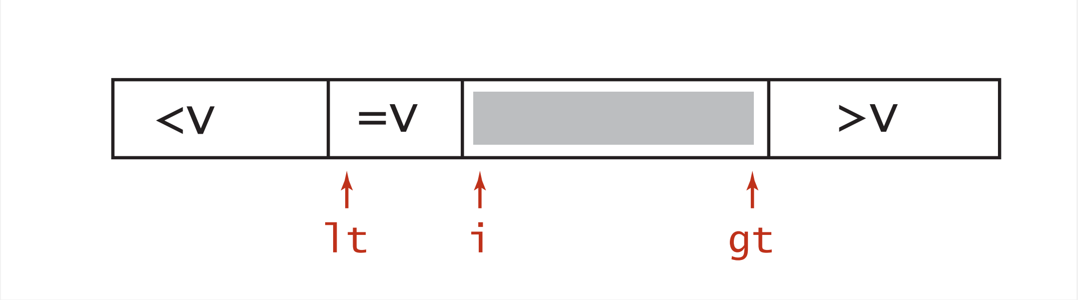


invariant



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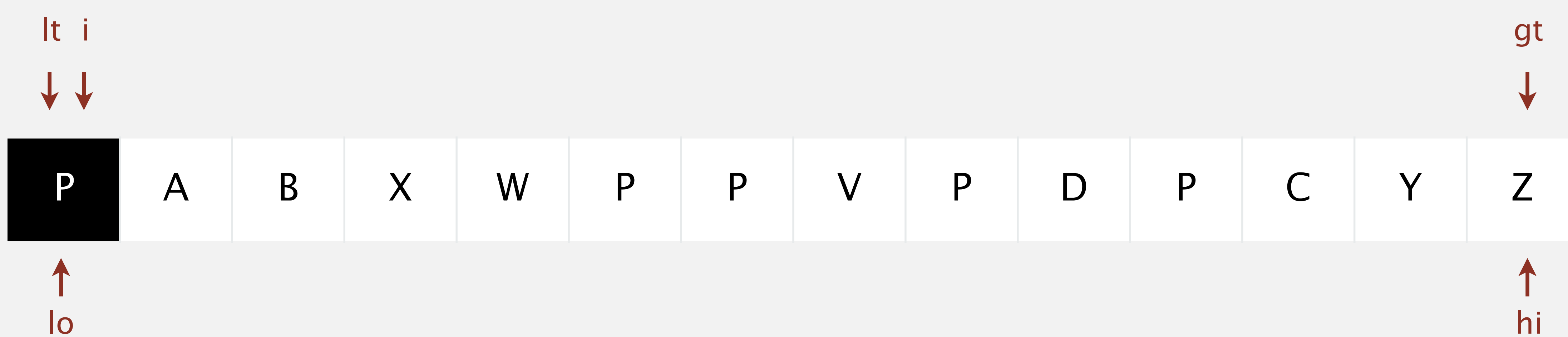

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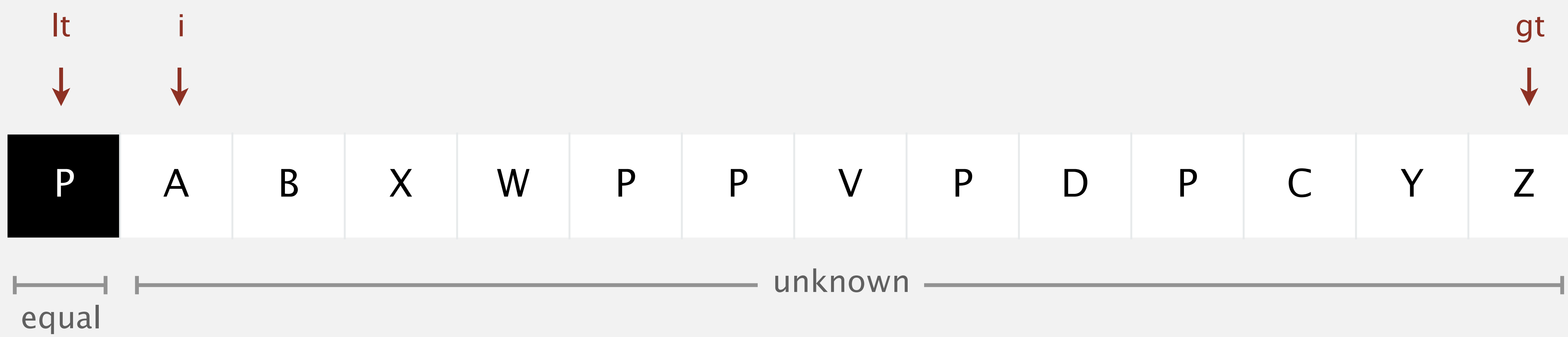

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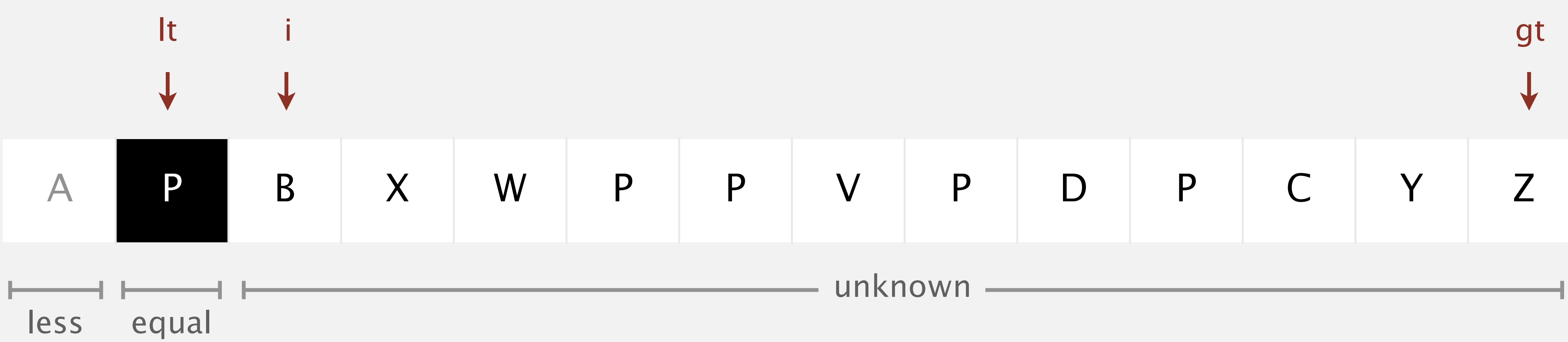

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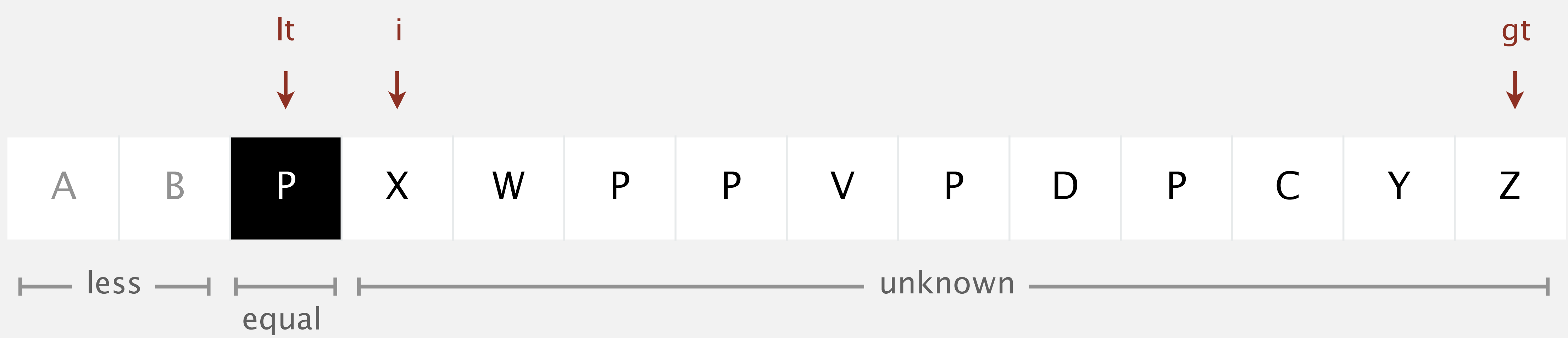

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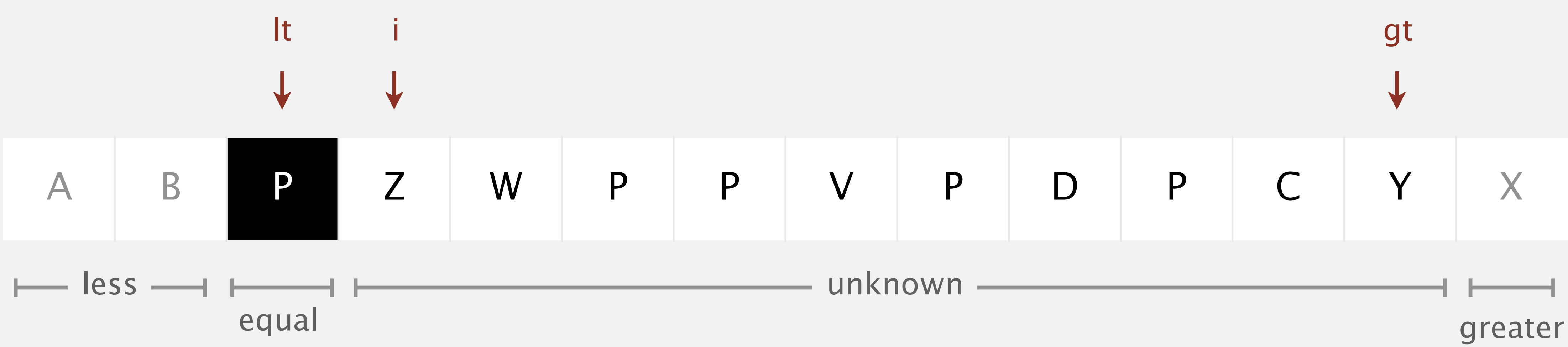

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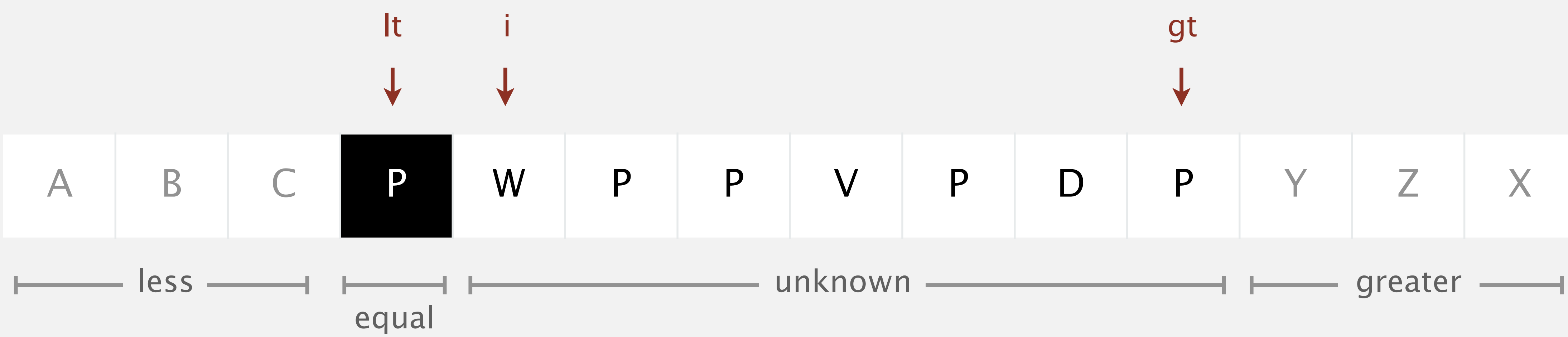

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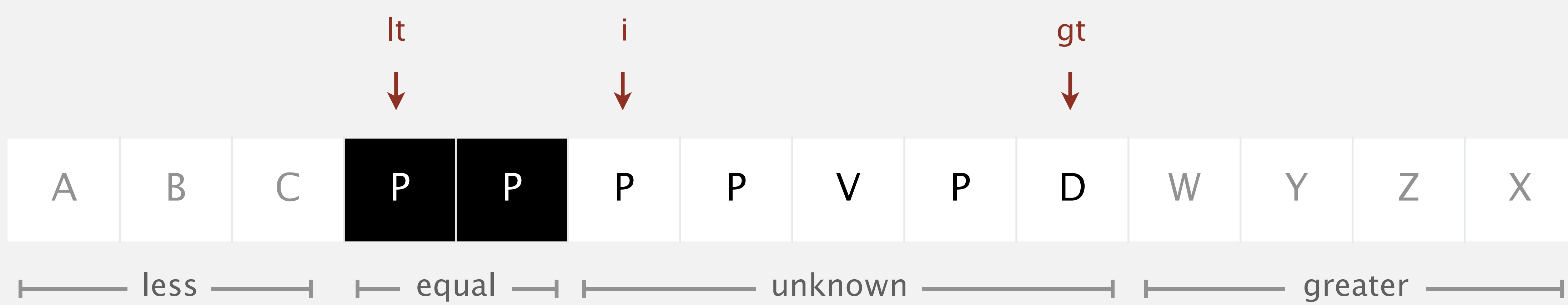

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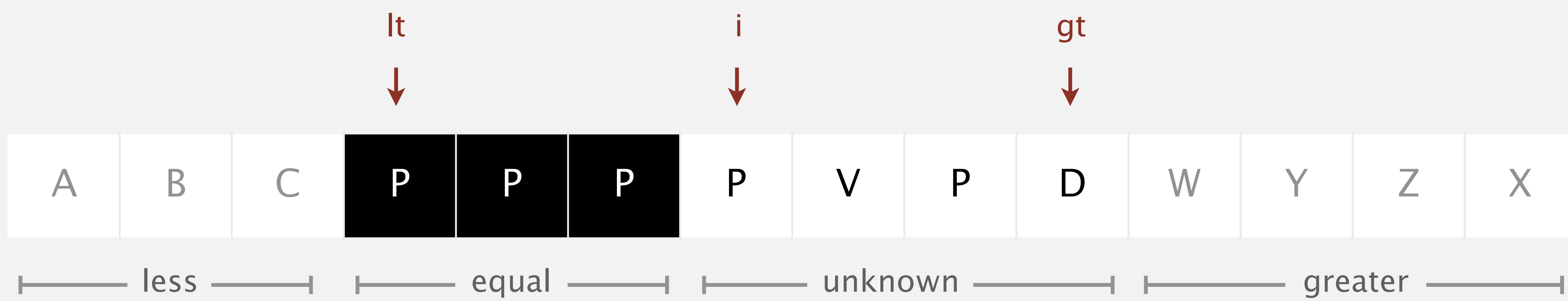

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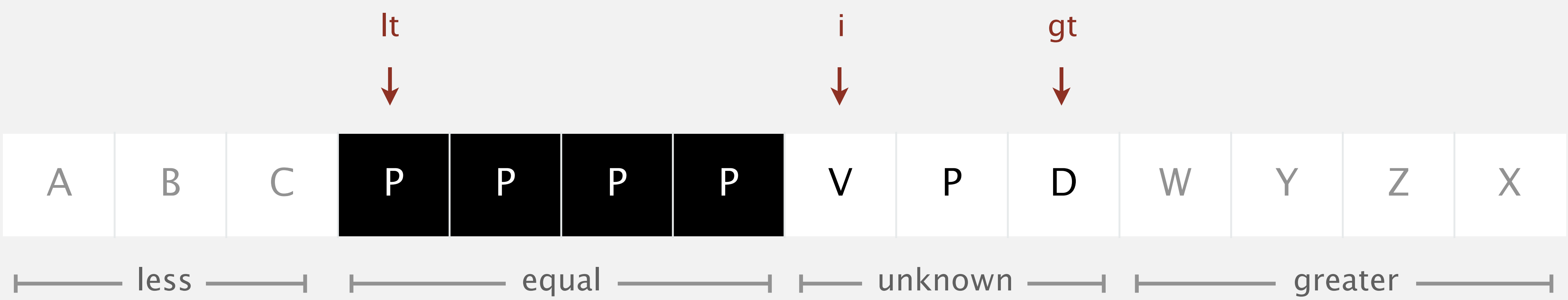

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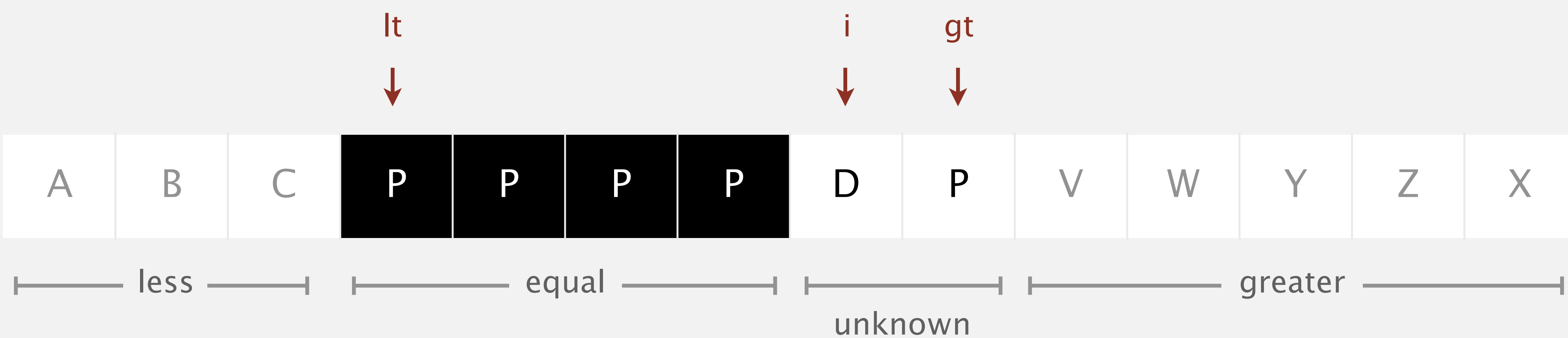

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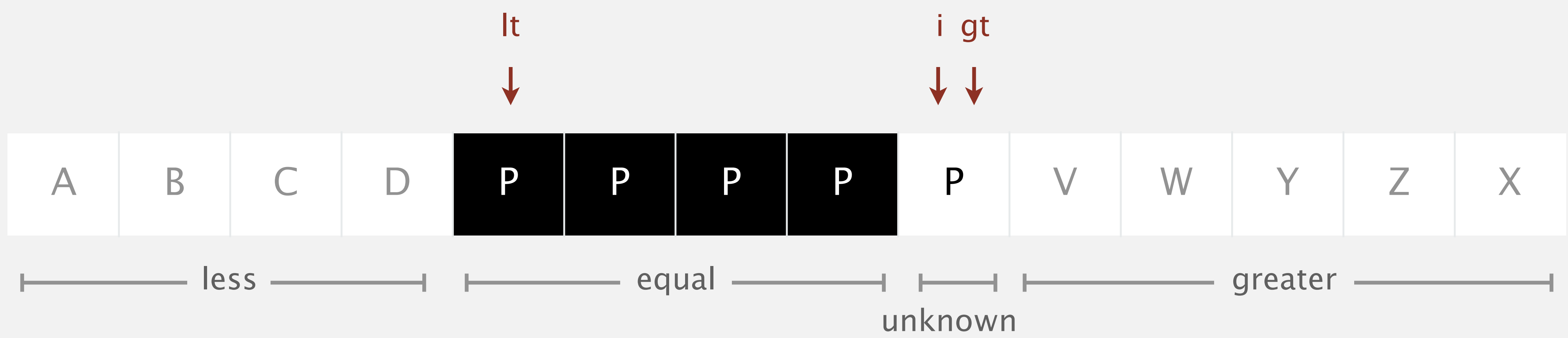

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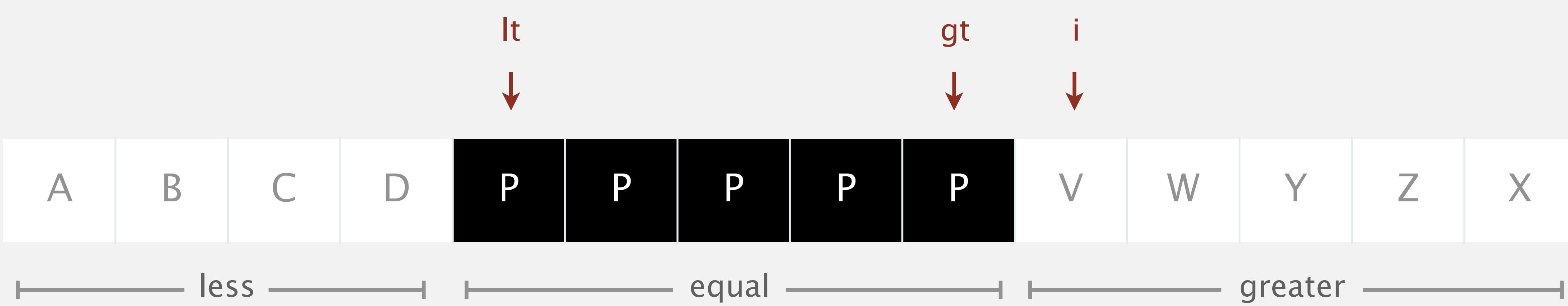

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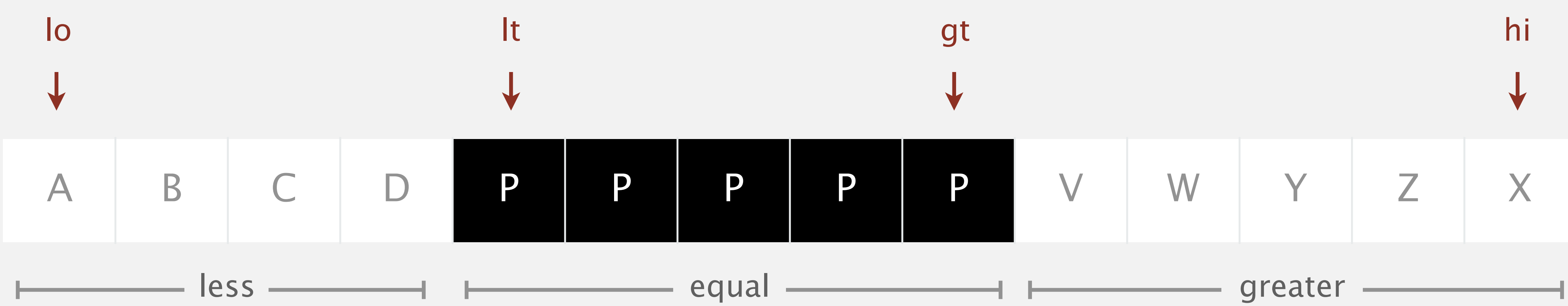
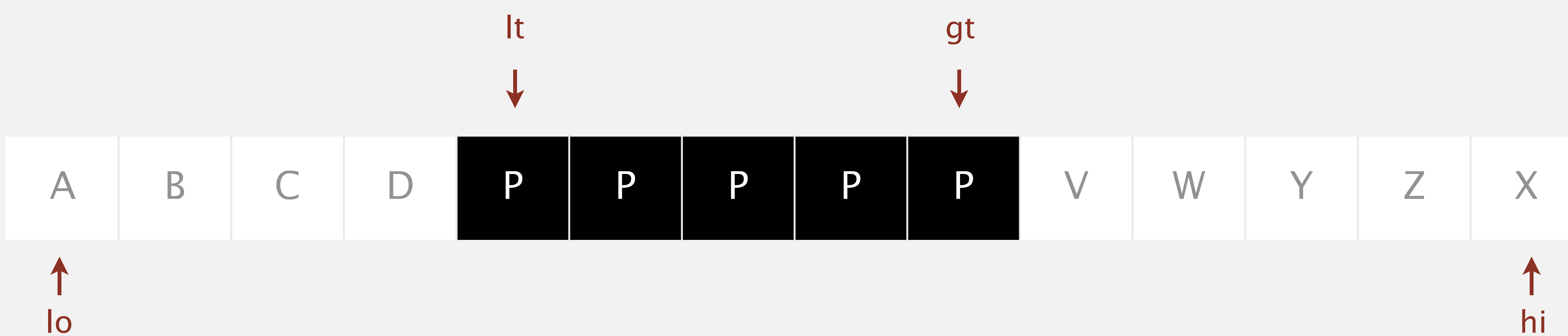
- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i]  < v)`: exchange `a[lt]` with `a[i]`; increment both `lt` and `i`
  - `(a[i]  > v)`: exchange `a[gt]` with `a[i]`; decrement `gt`
  - `(a[i] == v)`: increment `i`

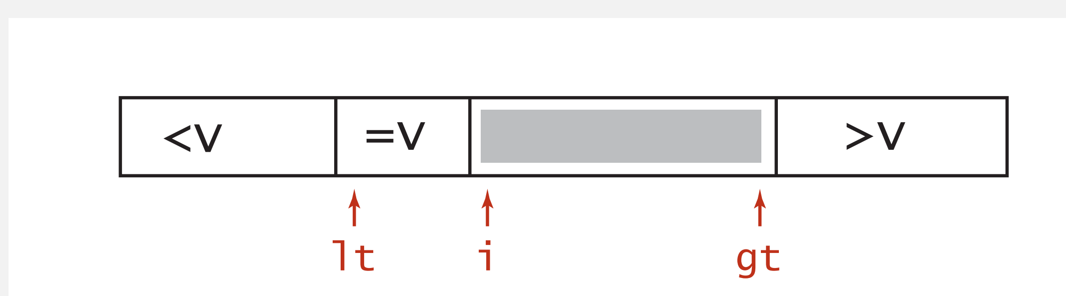


Dijkstra 3-way partitioning demo

- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i]  < v)`: exchange `a[lt]` with `a[i]`; increment both `lt` and `i`
  - `(a[i]  > v)`: exchange `a[gt]` with `a[i]`; decrement `gt`
  - `(a[i] == v)`: increment `i`



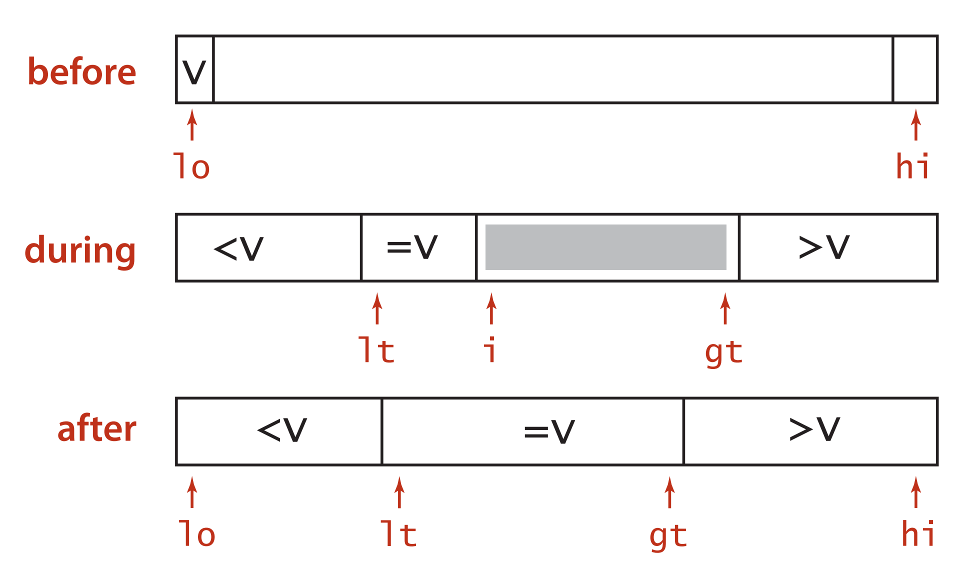
invariant



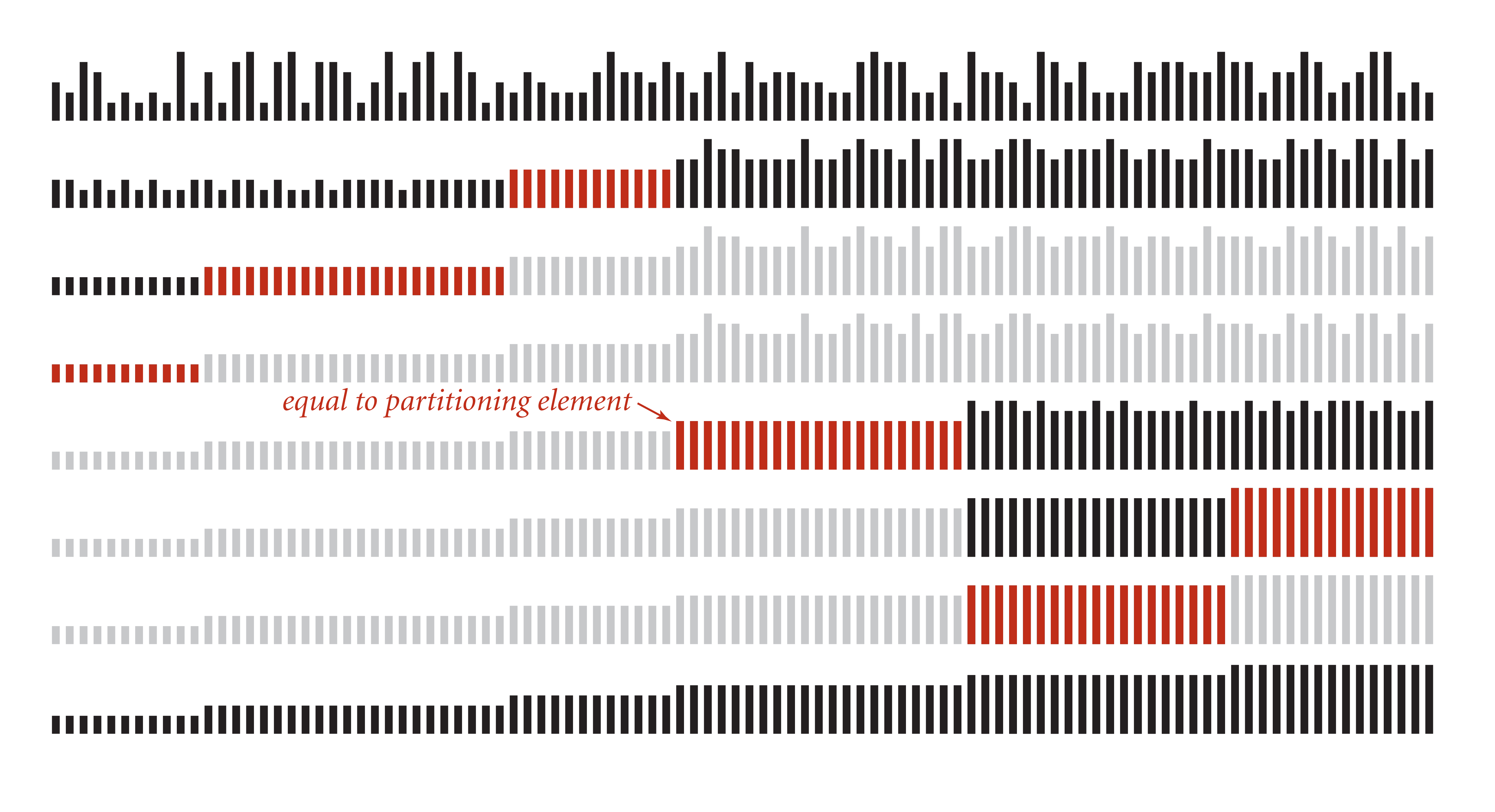
3-way quicksort: Java implementation

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo) return;
   int lt = lo, gt = hi;
   Comparable v = a[lo];
   int i = lo;
   while (i <= gt)
   {
      int cmp = a[i].compareTo(v);
      if      (cmp < 0) exch(a, lt++, i++);
      else if (cmp > 0) exch(a, i, gt--);
      else              i++;
   }

   sort(a, lo, lt - 1);
   sort(a, gt + 1, hi);
}
```



3-way quicksort: visual trace



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2.3 Quicksort

- quicksort
- selection
- duplicate keys
- system sorts

Sorting applications

Sorting algorithms are essential in a broad variety of applications:

- Sort a list of names.
- Organize an MP3 library.
- Display Google PageRank results.
- List RSS feed in reverse chronological order.

obvious applications

- Find the median.
- Identify statistical outliers.
- Binary search in a database.
- Find duplicates in a mailing list.

problems become easy once items are in sorted order

- Data compression.
- Computer graphics.
- Computational biology.
- Load balancing on a parallel computer.

. . .

non-obvious applications

Engineering a system sort (in 1993)

Bentley-McIlroy quicksort.

- Cutoff to insertion sort for small subarrays.
- Partitioning item: median of 3 or Tukey's ninther.
- Partitioning scheme: Bentley-McIlroy 3-way partitioning.

sample 9 items

similar to Dijkstra 3-way partitioning
(but fewer exchanges when not many equal keys)

Engineering a Sort Function

JON L. BENTLEY
M. DOUGLAS McILROY
AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, NJ 07974, U.S.A.

SUMMARY

We recount the history of a new `qsort` function for a C library. Our function is clearer, faster and more robust than existing sorts. It chooses partitioning elements by a new sampling scheme; it partitions by a novel solution to Dijkstra's Dutch National Flag problem; and it swaps efficiently. Its behavior was assessed with timing and debugging testbeds, and with a program to certify performance. The design techniques apply in domains beyond sorting.

Very widely used. C, C++, Java 6, ...

A beautiful mailing list post (Yaroslavskiy, September 2009)

Replacement of quicksort in java.util.Arrays with new dual-pivot quicksort

```
Hello All,

I'd like to share with you new Dual-Pivot Quicksort which is faster than the
known implementations (theoretically and experimental). I'd like to propose
to replace the JDK's Quicksort implementation by new one.

...

The new Dual-Pivot Quicksort uses *two* pivots elements in this manner:

1. Pick an elements P1, P2, called pivots from the array.
2. Assume that P1 <= P2, otherwise swap it.
3. Reorder the array into three parts: those less than the smaller pivot,
   those larger than the larger pivot, and in between are those elements
   between (or equal to) the two pivots.
4. Recursively sort the sub-arrays.

The invariant of the Dual-Pivot Quicksort is:

[ < P1 | P1 <= & <= P2 } > P2 ]

...
```

http://mail.openjdk.java.net/pipermail/core-libs-dev/2009-September/002630.html

A beautiful mailing list post (Yaroslavskiy-Bloch-Bentley, October 2009)

Replacement of quicksort in java.util.Arrays with new dual-pivot quicksort

```
Date: Thu, 29 Oct 2009 11:19:39 +0000
Subject: Replace quicksort in java.util.Arrays with dual-pivot implementation

Changeset: b05abb410c52
Author:    alanb
Date:      2009-10-29 11:18 +0000
URL:       http://hg.openjdk.java.net/jdk7/tl/jdk/rev/b05abb410c52

6880672: Replace quicksort in java.util.Arrays with dual-pivot implementation
Reviewed-by: jjb
Contributed-by: vladimir.yaroslavskiy at sun.com, joshua.bloch at google.com,
jbentley at avaya.com

! make/java/java/FILES_java.gmk
! src/share/classes/java/util/Arrays.java
+ src/share/classes/java/util/DualPivotQuicksort.java
```

http://mail.openjdk.java.net/pipermail/compiler-dev/2009-October.txt

System sort in Java 7

`Arrays.sort()`.

- Has one method for objects that are `Comparable`.
- Has an overloaded method for each primitive type.
- Has an overloaded method for use with a `Comparator`.
- Has overloaded methods for sorting subarrays.

Algorithms.

- Dual-pivot quicksort for primitive types.
- Timsort for reference types.

Q. Why use different algorithms for primitive and reference types?

Bottom line. Use the system sort!

Sorting summary

	inplace?	stable?	best	average	worst	remarks
selection	✔		$\frac{1}{2} N^2$	$\frac{1}{2} N^2$	$\frac{1}{2} N^2$	N exchanges
insertion	✔	✔	N	$\frac{1}{4} N^2$	$\frac{1}{2} N^2$	use for small N or partially ordered
shell	✔		$N \log_3 N$	?	$c N^{3/2}$	tight code; subquadratic
merge		✔	$\frac{1}{2} N \lg N$	$N \lg N$	$N \lg N$	$N \log N$ guarantee; stable
timsort		✔	N	$N \lg N$	$N \lg N$	improves mergesort when preexisting order
quick	✔		$N \lg N$	$2 N \ln N$	$\frac{1}{2} N^2$	$N \log N$ probabilistic guarantee; fastest in practice
3-way quick	✔		N	$2 N \ln N$	$\frac{1}{2} N^2$	improves quicksort when duplicate keys
?	✔	✔	N	$N \lg N$	$N \lg N$	holy sorting grail

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1.4 Analysis of Algorithms

- *introduction*
- *observations*
- *mathematical models*
- *order-of-growth classifications*
- ***theory of algorithms***
- *memory*

Types of analyses

Best case. Lower bound on cost.

- Determined by "easiest" input.
- Provides a goal for all inputs.

Worst case. Upper bound on cost.

- Determined by "most difficult" input.
- Provides a guarantee for all inputs.

Average case. Expected cost for random input.

- Need a model for "random" input.
- Provides a way to predict performance.

Ex 1. Array accesses for brute-force 3-SUM.

Best: $\sim \frac{1}{2} N^3$

Average: $\sim \frac{1}{2} N^3$

Worst: $\sim \frac{1}{2} N^3$

Ex 2. Compares for binary search.

Best: ~ 1

Average: $\sim \lg N$

Worst: $\sim \lg N$

Types of analyses

Best case. Lower bound on cost.

Worst case. Upper bound on cost.

Average case. "Expected" cost.

Actual data might not match input model?

- Need to understand input to effectively process it.
- Approach 1: design for the worst case.
- Approach 2: randomize, depend on probabilistic guarantee.

Theory of algorithms

Goals.

- Establish "difficulty" of a problem.
- Develop "optimal" algorithms.

Approach.

- Suppress details in analysis: analyze "to within a constant factor."
- Eliminate variability in input model: focus on the worst case.

Upper bound. Performance guarantee of algorithm for any input.

Lower bound. Proof that no algorithm can do better.

Optimal algorithm. Lower bound = upper bound (to within a constant factor).

Commonly-used notations in the theory of algorithms

notation	provides	example	shorthand for	used to
Big Theta	asymptotic order of growth	$\Theta(N^2)$	$\frac{1}{2} N^2$ $10 N^2$ $5 N^2 + 22 N \log N + 3N$ $\vdots$	classify algorithms
Big O	$\Theta(N^2)$ and smaller	$O(N^2)$	$10 N^2$ $100 N$ $22 N \log N + 3 N$ $\vdots$	develop upper bounds
Big Omega	$\Theta(N^2)$ and larger	$\Omega(N^2)$	$\frac{1}{2} N^2$ N^5 $N^3 + 22 N \log N + 3 N$ $\vdots$	develop lower bounds

Theory of algorithms: example 1

Goals.

- Establish "difficulty" of a problem and develop "optimal" algorithms.
- Ex. 1-SUM = "*Is there a* 0 *in the array?* "

Upper bound. A specific algorithm.

- Ex. Brute-force algorithm for 1-SUM: Look at every array entry.
- Running time of the optimal algorithm for 1-SUM is $O(N)$.

Lower bound. Proof that no algorithm can do better.

- Ex. Have to examine all N entries (any unexamined one might be 0).
- Running time of the optimal algorithm for 1-SUM is $\Omega(N)$.

Optimal algorithm.

- Lower bound equals upper bound (to within a constant factor).
- Ex. Brute-force algorithm for 1-SUM is optimal: its running time is $\Theta(N)$.

Theory of algorithms: example 2

Goals.

- Establish "difficulty" of a problem and develop "optimal" algorithms.
- Ex. 3-SUM.

Upper bound. A specific algorithm.

- Ex. Brute-force algorithm for 3-SUM.
- Running time of the optimal algorithm for 3-SUM is $O(N^3)$.

Theory of algorithms: example 2

Goals.

- Establish "difficulty" of a problem and develop "optimal" algorithms.
- Ex. 3-SUM.

Upper bound. A specific algorithm.

- Ex. Improved algorithm for 3-SUM.
- Running time of the optimal algorithm for 3-SUM is $O(N^2 \log N)$.

Lower bound. Proof that no algorithm can do better.

- Ex. Have to examine all N entries to solve 3-SUM.
- Running time of the optimal algorithm for solving 3-SUM is $\Omega(N)$.

Open problems.

- Optimal algorithm for 3-SUM?
- Subquadratic algorithm for 3-SUM?
- Quadratic lower bound for 3-SUM?

Algorithm design approach

Start.

- Develop an algorithm.
- Prove a lower bound.

Gap?

- Lower the upper bound (discover a new algorithm).
- Raise the lower bound (more difficult).

Golden Age of Algorithm Design.

- 1970s–.
- Steadily decreasing upper bounds for many important problems.
- Many known optimal algorithms.

Caveats.

- Overly pessimistic to focus on worst case?
- Need better than "to within a constant factor" to predict performance.

Commonly-used notations in the theory of algorithms

notation	provides	example	shorthand for	used to
Tilde	leading term	$\sim 10\ N^2$	$10\ N^2$ $10\ N^2 + 22\ N \log N$ $10\ N^2 + 2\ N + 37$	provide approximate model
Big Theta	asymptotic order of growth	$\Theta(N^2)$	$\frac{1}{2}\ N^2$ $10\ N^2$ $5\ N^2 + 22\ N \log N + 3N$	classify algorithms
Big Oh	$\Theta(N^2)$ and smaller	$O(N^2)$	$10\ N^2$ $100\ N$ $22\ N \log N + 3\ N$	develop upper bounds
Big Omega	$\Theta(N^2)$ and larger	$\Omega(N^2)$	$\frac{1}{2}\ N^2$ N^5 $N^3 + 22\ N \log N + 3\ N$	develop lower bounds

Common mistake. Interpreting big-Oh as an approximate model.

This course. Focus on approximate models: use Tilde-notation

Sorting summary

	inplace?	stable?	best	average	worst	remarks
selection	✔		$\frac{1}{2} N^2$	$\frac{1}{2} N^2$	$\frac{1}{2} N^2$	N exchanges
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shell	✔		$N \log_3 N$	?	$c N^{3/2}$	tight code; subquadratic
merge		✔	$\frac{1}{2} N \lg N$	$N \lg N$	$N \lg N$	$N \log N$ guarantee; stable
timsort		✔	N	$N \lg N$	$N \lg N$	improves mergesort when preexisting order
quick	✔		$N \lg N$	$2 N \ln N$	$\frac{1}{2} N^2$	$N \log N$ probabilistic guarantee; fastest in practice
3-way quick	✔		N	$2 N \ln N$	$\frac{1}{2} N^2$	improves quicksort when duplicate keys
?	✔	✔	N	$N \lg N$	$N \lg N$	holy sorting grail