Programming Challenges (GB21802)

Week 5 - Graph Part I: Basics

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(last updated: May 6, 2021)

Version 2020.1

Graphs in Computer Science

Graph Data structures explain the relationships between data, and are used in several real world problems:

- Geography and Maps;
  - Pathing between locations;
  - Cycles and Tours;
- Human Networks;
  - Social Networks;
  - Citation Clusters;
- State Machines;
  - Program Pipelines;
  - Library Requirements;
- Natural Language;
  - Graph Grammars;

Graph Algorithms: Week 4 and 5

Graphs Part I (This Week)

- Graphs Data Structure;
- Depth First Search and Breadth First Search;
- Graph Search Problems (DFS and BFS);
- Minimum Spanning Tree: Kruskal and Prim Algorithms;

Graphs Part II (Next Week)

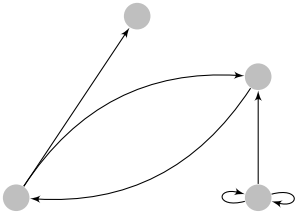
- Single Sourse Shortest Path (Djikstra);
- All Pairs Shortest Path (Floyd-Warshall);
- Network Flow;
- Bipartite Graph Matching;

What is a graph?

A graph $G = \{V, E\}$ is composed of a set of **vertices** V, which are connected to a set of **edges** E. Each edge connects exactly two vertices.

Edge and vertices characteristics:

- An edge can be **directed** or **undirected**;
- An edge or a vertice can have **weights** or **labels**;
- **Self-edges** and **multi-edges**;
- A graph can be **connected** or **disconnected**;



Common questions about graphs

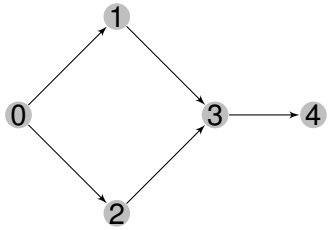
Some tasks are common across graph problems:

- Test if a path exist between vertice V_i and V_j (test if they are **connected**)
- Test the shortest path between vertice V_i and V_j
  - With or without weights
  - Test if there is more than one path
- Add or remove vertices or edges from a graph;
- Test characteristics of a graph;
- etc.;

UVA 11902 – Dominator

A vertice V_i **dominates** V_j if all paths $V_0 \rightarrow V_j$ must include V_i.

- **input**: A directed graph $\{V, E\}$;
- **output**: A table with which vertices dominate each other;



Input

```
1
5
0 1 1 0 0
0 0 0 1 0
0 0 0 1 0
0 0 0 0 1
0 0 0 0 0
```

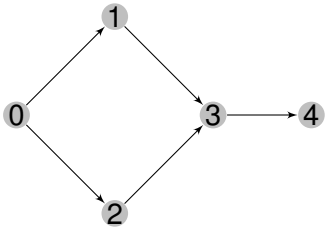
Output

```
Case 1:
+---------+
|Y|Y|Y|Y|Y|
+---------+
|N|Y|N|N|N|
+---------+
|N|N|Y|N|N|
+---------+
|N|N|N|Y|Y|
+---------+
|N|N|N|N|Y|
```

UVA 11902 – Dominator

Quiz: How do we solve this problem?

- How do we store the graph from the input?
- How do we determine "dominator" status?



Input

```
1
5
0 1 1 0 0
0 0 0 1 0
0 0 0 1 0
0 0 0 0 1
0 0 0 0 0
```

Output

```
Case 1:
+---------+
|Y|Y|Y|Y|Y|
+---------+
|N|Y|N|N|N|
+---------+
|N|N|Y|N|N|
+---------+
|N|N|N|Y|Y|
+---------+
|N|N|N|N|Y|
```

Storing the Dominator Graph

Adjacency Matrix: stores the connection between certices

```
int adj[100][100];

for (int i = 0; i < n; i++)
  for (int j = 0; i < n; j++)
    cin >> adj[i][j]; // 0 if no edge, 1 if edge
```

- Pros:
  - Easy to program;
  - Quickly access any edge;
- Cons:
  - Cannot store multigraph;
  - Wastes memory with sparse graphs;
  - Time $O(V)$ to calculate number of neighbors;

Storing the Dominator Graph

Adjacency List: stores edge list for each Vertex

```
typedef pair<int,int> edge; // pair: <neighbor, weight>
typedef vector<edge> neighb;// all neighbors of V_i
vector<neighb> AdjList;     // all V_i

int e;
for (int i = 0; i < n; i++)
  for (int j = 0; j < n; j++)
    cin >> e;
    if (e == 1) { AdjList[i].push_back(pair(j,1)); }
```

- **Pro**:
  - Memory efficient if the graph is sparse;
  - Can store multigraph;
- **Cons**:
  - $O(\log(V))$ to test if two vertices are adjacent;

Storing the Dominator Graph

Edge List

```
pair <int,int> edge; // Edge between i and j
vector<pair <int,edge>> Elist; // All edges;

int e;
for (int i = 0; i < n; i++)
  for (int j = 0; j < n; j++)
    cin >> e;
    if (e == 1)
      Elist.push_back(pair(1, pair(i,j)));
```

- Not very common;
- Has to sort edges to find the neighbors of a Vertice;

Graph Search: BFS and DFS

- Basic algorithms to visit every vertice in a graph.
- Many graph algorithms require BFS or DFS;
- So learn to do these with your eyes closed;

Depth First Search – DFS

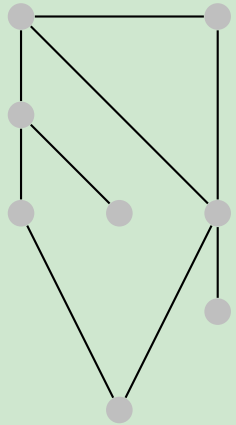
- Visit the first edge in the next vertice;
- No guarantee of order, tends to go away from start;
- Easy to implement with recursion;

Breadth First Search – BFS

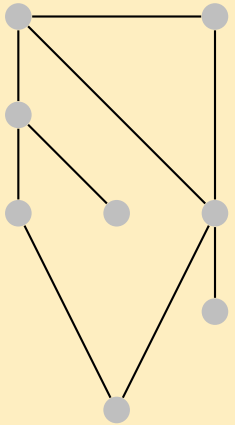
- First visit the vertices closer to starting point;
- Place new edges on a FIFO, and search with a loop;

BFS/DFS: Visualization

DFS

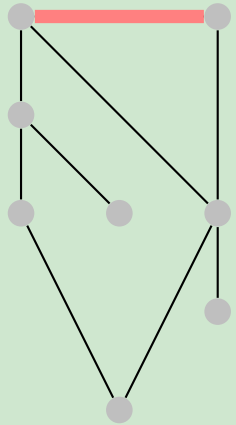


BFS

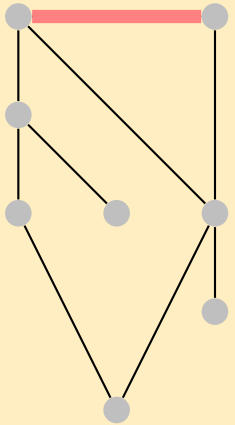


BFS/DFS: Visualization

DFS

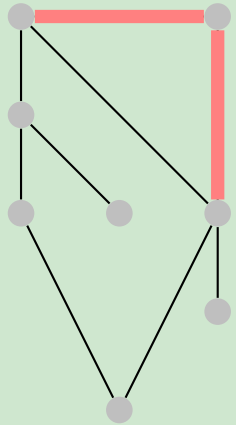


BFS

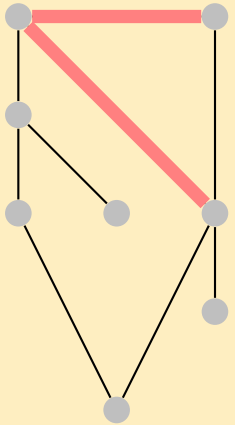


BFS/DFS: Visualization

DFS

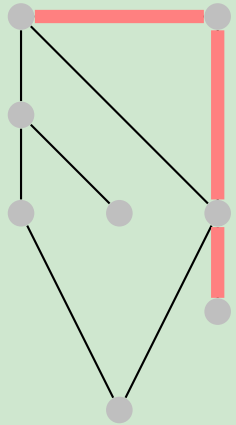


BFS

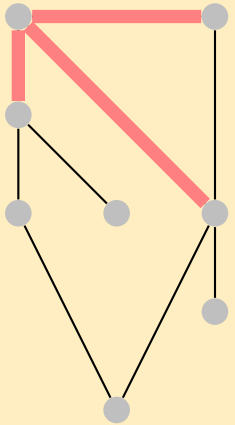


BFS/DFS: Visualization

DFS

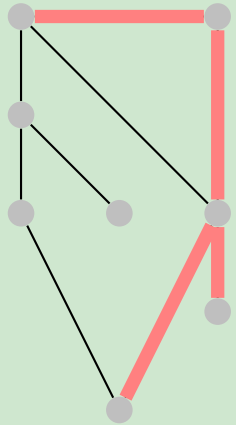


BFS

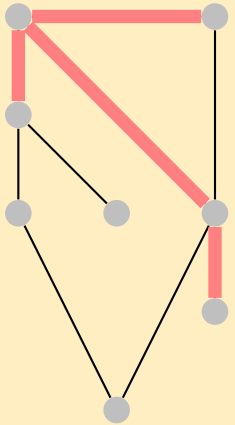


BFS/DFS: Visualization

DFS

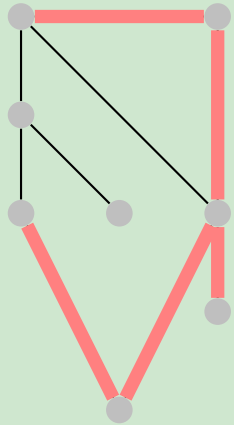


BFS

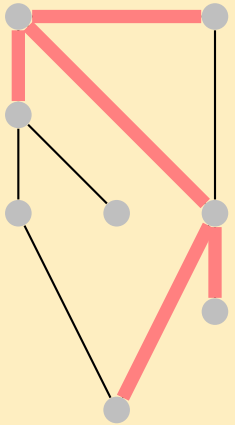


BFS/DFS: Visualization

DFS

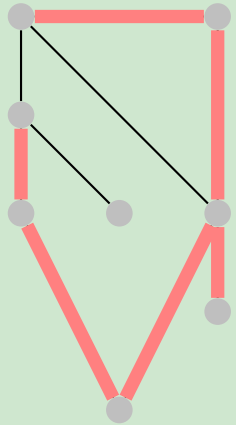


BFS

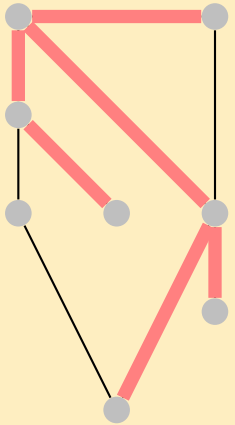


BFS/DFS: Visualization

DFS



BFS



BFS/DFS: Visualization

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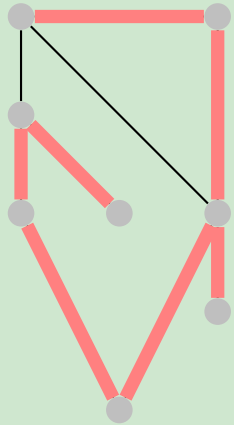
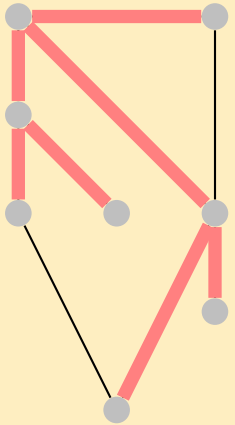


BFS



DFS Implementation

DFS (Using Adjacency List)

```
vector<int> dfs_vis; // visited nodes, init to 0

void dfs(int v) {
   dfs_vis[v] = 1;
   for (int i; i < AdjList[v].size(); i++)
   {
      edge u = AdjList[v][i]; // u = neighb, weight
      // do something...
      if (dfs_vis[u.first] == 0)
         dfs(v.first);
   }
}
dfs(start_vertice);
```

BFS Implementation

BFS (Using adjacency List)

```
vector<int> bfs_vis;   // visited nodes; init to 0
queue<int> q;          // list of vertices to visit;
q.push(start_vertice); // Start BFS

while(!q.empty()) {
  int u = q.front(); q.pop();
  bfs_vis[u] = 1;
  // Do something...
  for (int i = 0; i < AdjList[v].size(); i++) {
    edge e = AdjList[v][i];
    if (bfs_vis[e.first] == 0)
      q.push(e.first);
  }
}
```

BFS and DFS

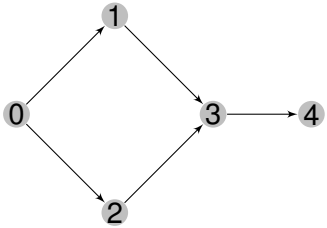
Computational Cost

In the full BFS and DFS, you need to check every vertice and every edge in the graph:

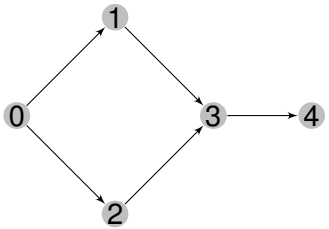
- A BFS/DFS implemented with **Adjacency List**, costs $O(V + E)$.
- A BFS/DFS implemented with **Adjacency Matrix**, costs $O(V^2)$.
  - That's because to visit every edge of a vertice in an Adjacency Matrix, it costs $O(V)$.

Solving the Dominator Problem with DFS

- V_j is dominated by V_j, if all paths from S to V_j pass through V_i;
- In other words, you cannot access V_j from S, if V_i is not available;
- **Algorithm:** Remove V_i, and test if you can access V_j;



Solving the Dominator Problem with DFS



Solution: DFS *N* Times (BFS OK too)

```
// Modified DFS: never visits V_i;
boolean DFS2(S,i);

// init: Processing Root node
DFS2(0,-1);
for (int j = 0; j < N; j++)
  if (VISITED[j])
    DOMINATED[0][j] == 1;

for (int i = 1; i < N; i++) {
  memset(VISITED,0,sizeof(VISITED));
  DFS2(0,i);
  for (int j = 0; j < N; j++)
    if (!VISITED[j] && DOMINATED[0][j])
      DOMINATED[i][j] == 1;
}
```

Common Graph Problems

In this section, we will review several common problems involving graphs in Programming Challenges.

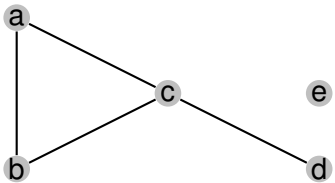
Most of these can be solved with small modifications to DFS or BFS.

- Connected Components;
- Flood Fill;
- Topological Sort;
- Bipartite Checking;
- Articulation Vertices;
- Strongly Connected Components;

Connected Components

Definition

A **connected component** of a (undirected) graph is a subset of vertices C^k where every $V_i \in C^k$ is reachable.



Connected Components

Example Problem

Problem Example: Extra cables

There is a network of N computers. Some of the computers are connected by cables. Computers connected by cables, even if indirectly, are said to be on the **same network**.

What is the minimum number of cables that you need to make sure that all N computers are part of the same network?

Solution: Count the number of Connected Components (C), the answer is $C - 1$.

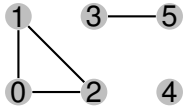
Quiz: How do you implement this?

Connected Components

Algorithm

We can find all connected components by looping through all vertices, and running BFS/DFS on each unvisited vertice;

```
int cables = 0;
for (int = 0; i < N; i++)
   if (dfs_vis[i] == 0) // New component!
   {
      dfs(i);           // Visit vertices
      cables += 1;
   }
cout << "Need "<< cables - 1 <<".\n";
```



Connected Components

UFDS Variant

You could also count Connected Components using the **UFDS** data structure from lecture 2.

It costs $O(E)$ to build the UFDS, and $O(V)$ to count the number of components.

If your problem is dynamic and includes several additions to the graph edges, this might be a good choice, because it is cheaper to recalculate the CCs.

Flood Fill

Problem: The Biggest Island

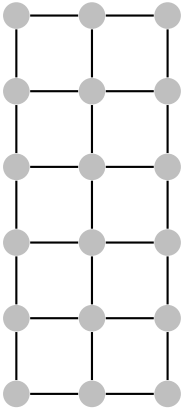
You want to build a large in Minecraft. For this, you need to find the biggest island in the map.

Input: A 2D representation of the map:

```
.....................................
.###.......###.....#.....###.####...
.#####....#####.##.#####.##....#....
.###........###..#...##..#....###...
......###.......###...####...##.....
....####.............######.....###.
....####.......#.......###......###.
.....................................
```

Implicit Graphs

- **Implict Graphs** are data that suggest graph organization. Examples:
  - grids (NSWE connections)
  - maps (distance = weights)
- In some problems, it is not necessary to store the entire graph from the beginning;
- **Grid Floodfill**: Painting images, Walkable tiles in videogames, etc;
- Algorithm is just BFS/DFS with vertex labels;



Flood Fill

Implementation

"Biggest Island" can be solved with BFS/DFS modifying labels;

```
int dr[] = {1,1,0,-1,-1,-1,0,1}; // neighbors for a grid
int dc[] = {0,1,1,1,0,-1,-1,-1}; // with diagonals;

int floodfill(int y, int x) {  // size of one position
  if (y < 0 || y >= R || x < 0 || x >= C) return 0;
  if (grid[y][x] != '#') return 0;
  int size = 1;
  grid[y][x] = '.';    // CHANGE THE MAP (**key**)
  for (int d = 0; d < 8; d++)
     size += floodfill(y+dr[d], x+dc[d]);
  return ans;
}
biggest = 0;
for (int i = 0; i < C; i++)
  for (int j = 0; j < R; j++)
```

Topological Sort

Problem: Preparing a Curriculum

You have a list of courses and requisites. Choose an **ordering** of topics that respect all requisites.

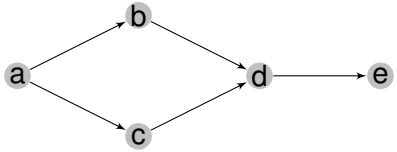
Input: list M topics, and N pairs of topics;
Output: Sorted list of all topics;

```
** Example Input:
5 4 Graphs DP Search Flow Programming
Programming Search
Search DP
Graph Flow
Search Graph

** Example Output:
Course: Programming Search DP Graph Flow
```

Topological Sort Definition

In a **directed graph**, a topological sort is an ordering of vertices where $V_i \prec V_j$ **iff** there is no path $V_j \rightarrow V_i$.



For this graph, a topological sort would be $a \prec b \prec c \prec d \prec e$.

- Toposorts are **not unique**:
  - $a \prec c \prec b \prec d \prec e$ is also a toposort.
- A graph only has a toposort if it has **no cycles**.
- To find the toposort, we use **in-degrees and out-degrees**:
  - a – In-deg: 0; Out-deg: 2;
  - d – In-deg: 2; Out-deg: 1;

Topological Sort Algorithm

Khan's algorithm can be used to find the toposort of a graph. It is composed of a **BFS** where we keep track of the in-degrees of a vertex before adding it to the visit queue.

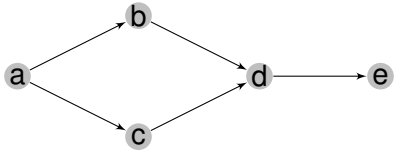
```
queue<int> q; vector<int> toposort;
vector<int> in-deg; // initialize to 0 for all N;

for (int i = 0; i < EdgeList.size(); i++)
  in-deg[EdgeList[i].second]++;
for (int i = 0; i < N; i++)
  if (in-deg[i] == 0) q.push(i); // queue in-deg = 0;

while (!q.empty()) {
  u = q.front(); q.pop(); toposort.push_back(u); // Do something
  for (int i = 0; i < EdgeList[u].size(); i++) {
    d = EdgeList[u][i].first; in-deg[d]--;
    if (in-deg[d] == 0) q.push(d); // queue in-deg = 0;
} }
```

Khan's Algorithm

Simulation



In-deg list:

- a:
- b:
- c:
- d:
- e:

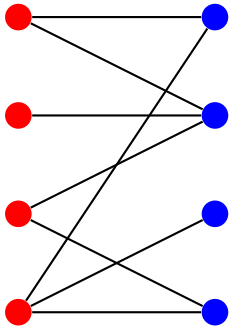
Toposort:

Bipartite Graphs

Definition

Intuitively, a **Bipartite Graph** is one that we can separate between a "left" side and a "right" side.

More generally, a graph (V, E) is bipartite if you can completely partition its vertices in two subsets: V_1 and V_2, so that **there are no edges** connecting two vertices in the same subset.



Bipartite graphs appear in a large number of algorithms. In particular, **flow graphs** (next week) are bipartite graphs.

Most neural networks are bipartite graphs too!
Quiz: How do you test if a graph is bipartite?

Bipartite Check Algorithm

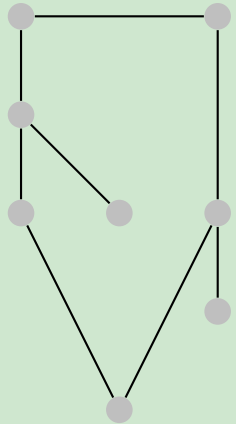
We can check if a graph is bipartite using a BFS/DFS. Every time we visit a vertice, we mark it "0" or "1". If two adjacent vertices are of the same colors, the graph is not bipartite.

```
queue<int> q; q.push(s);
vector<int> color(V, -1); color[s] = 0; // Starting vertex
bool isBipartite = True;

while (!q.empty() && isBipartite) {
   int u = q.front(); q.pop();
   for (int j=0; j < adj_list[u].size(); j++) {
      v = adj_list[u][j].first;
      if (color[v] == -1) {
         color[v] = 1 - color[i];        // Coloring new vertex
         q.push(v.first);}
      else if (color[v.first] == color[u]) {
         isBipartite = False;            // Bipartite collision
}}}
```

Bipartite Check – Visualization

Testing Bipartite property



Bipartite Check – Visualization

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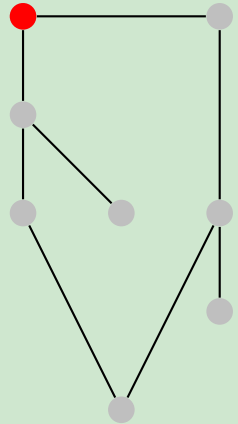

Bipartite Check – Visualization

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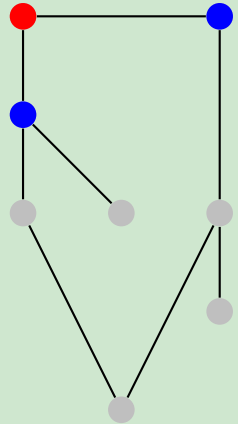

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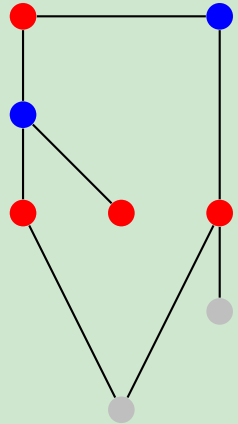

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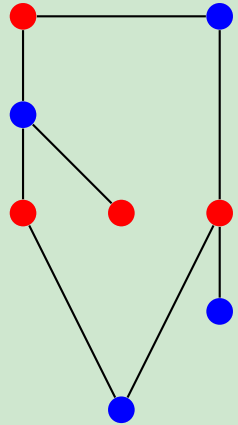

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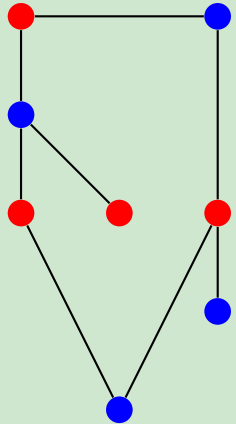
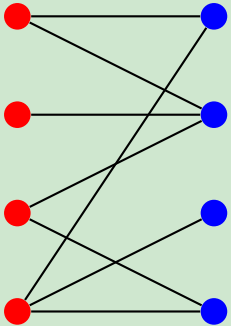


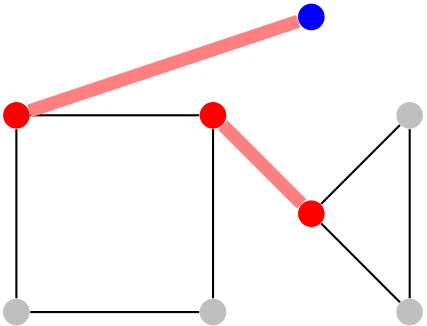
Rearranging the nodes



Articulation Points and Bridges

Definition: In a graph G

- Vertex v_i is an **Articulation Point** if removing v_i makes G disconnected.
- Edge $e_{i,j}$ is a Bridge if removing $e_{i,j}$ makes G disconnected.



Problems and Naive Algorithm

Example Problems

- Find nodes that can be/can not removed from a graph;
- Add extra nodes to "reinforce" a graph;
- Measure the reliability of a network;
- etc;

Complete Search algorithm for Articulation Points $O(V^2 + VE)$

1. Run DFS/BFS, and count the number of CC in the graph;
2. For each vertex v_i, remove v_i and run DFS/BFS again;
3. If the number of CC increases, v is an articulation point;

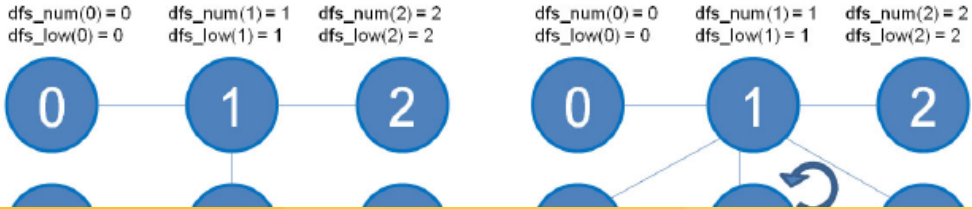
Tarjan's DFS variant for Articulation point (O(V+E))

Find Articulation Points/Bridges in a single DFS pass: $O(V + E)$

Main idea: Track loops to detect articulations:

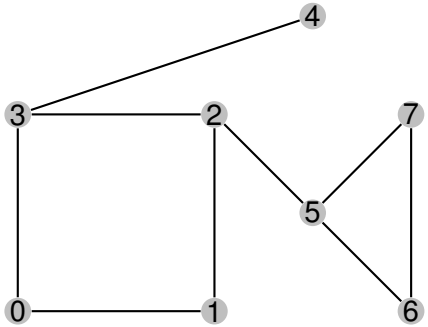
- **dfs_num[i]**: iteration number (DFS order)
- **dfs_low[i]**: lowest dfs_num reachable from i;
  - (not counting direct parent);

For any u, v, if low[v] >= num[u], u is an articulation node (except root) For any u, v, if low[v] > num[u], $e_{u,v}$ is a bridge;



Tarjan's Algorithm for Articulation Point

Simulation



	DFS_NUM	DFS_LOW
0		
1		
2		
3		
4		
5		
6		

Tarjan's Algorithm for Articulation Point

Implementation

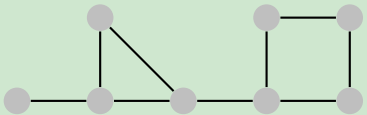
```
void articulation(u){
   dfs_num[u] = dfs_low[u] = IterationCounter++;  // update num[u], init low[u]
   for (int i = 0; i < AdjList[u].size(); i++){
      v = AdjList[u][i];
      if (dfs_num[v.first] == UNVISITED) {        // DFS tree edge
         dfs_parent[v.first] = u;                 // store parent
         if (u == 0) rootTreeEdge++;              // special case for root node

         articulation(v.first);                   // visit next

         // This code happens AFTER we finish the DFS from u;
         if (dfs_low[v.first] >= dfs_num[u])
            articulation_vertex[u] = true;        // do something
         if (dfs_low[v.first > dfs_num[u]])
            bridge[u][v.first] = true;            // do something
         dfs_low[u] = min(dfs_low[u],dfs_low[v.first])
      }
      else if (v.first != dfs_parent[u])          // found a cycle edge
         dfs_low[u] = min(dfs_low[u],dfs_num[v.first]);
  } // for j;
} // articulation(u);

// Starting the algorithm from the main program:
articulation(0);
if (rootTreeEdge > 1)
  articulation_vertex[0] = true;                  // special case for root
```

Strongly Connected Components

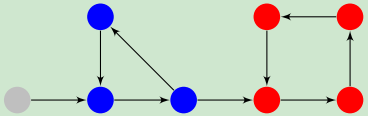
Definition

Given a **directed** graph $G(V, E)$, a **Strongly Connected Component (SCC)** is a subset of vertices V_1 where for every pair of vertices $v_i, v_i \in V_1$, there is both a path $v_i \rightarrow v_j$ and a path $v_j \rightarrow v_i$.

One Connected Component (undirected)



Three Strongly Connected Components (directed)



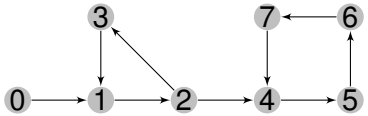
Algorithm for Finding SCCs

We can modify Tarjan's algorithm (for articulation points and bridges) to find Strongly Connected Components:

- Every time we visit a new vertex u, we put u in a stack S;
- Only update dfs_low for vertices with the "visited" flag = 1;
- After visiting all edges of u, check if "dfs_num[u] == dfs_min[u]";
- If the condition is true, u is the root of a new SCC.
- Pop all vertices in S until (and including) u;
- Add all popped vertices to the SCC.

Algorithm for Finding SCCs

Simulation



SCC Stack:

```
0  1  2  3  4  5  6  7
```

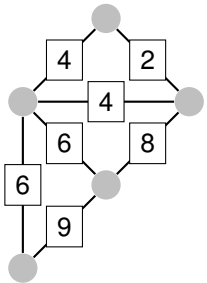
`dfs_low`

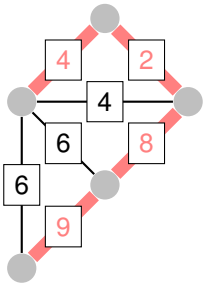
`dfs_sum`

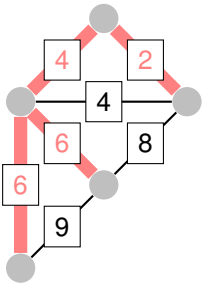
Minimum Spanning Trees (MST) – Definition

A **Spanning Tree** is a subset E' from graph G so that all vertices are connected without cycles.

A Minimum Spanning Tree is a spanning tree where the sum of edge's weights is minimal.







Usage Cases for Minimum Spanning Trees

- Problems with MST often ask for a minimal cost to connect all elements in a graph (e.g. minimal infrastructure cost).
- **Variations:** Maximum Spanning Tree, Spanning Forest, Force some edges in advance;

Main algorithms for MST

Two greedy algorithms that add edges to MST:

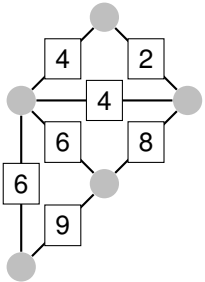
- **Kruskal Algorithm**: based on edge list;
- **Prim's Algorithm**: based on vertex list;

Kruskal's Algorithm

Outline

Kruskal's algorithms sorts all edges by their weight, and try to add each edge to the MST, checking whether adding that edge would create a cycle.

1. Sort all edges;
2. If smallest edge does not create a cycle, add to MST;
3. If smallest edge creates a cycle, remove it from list;
4. Go to 2;



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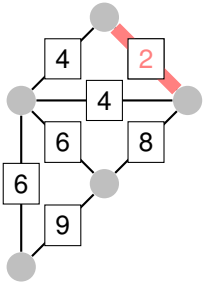

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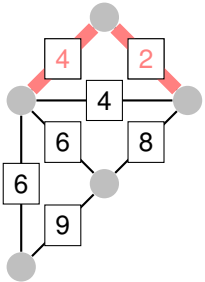

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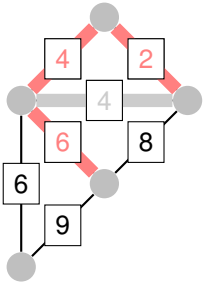

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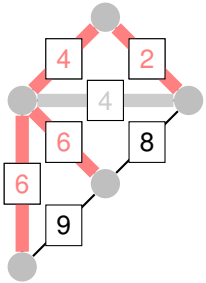

Kruskal's Algorithm – Implementation

```
vector<pair<int, pair<int,int>> Edgelist;
sort(Edgelist.begin(),Edgelist.end());
int mst_cost = 0;
UnionFind UF(V);
  // note 1: Pair object has built-in comparison;
  // note 2: Need to implement UnionSet class;

for (int i = 0; i < Edgelist.size(); i++) {
   pair <int, pair <int,int>> front = Edgelist[i];
   if (!UF.isSameSet(front.second.first,
                     front.second.second)) {
      mst_cost += front.first;
      UF.unionSet(front.second.first,front.second.second)
   }}

cout << "MST Cost: " << mst_cost << "\n"
```

Prim's Algorithm

Outline

Prim's algorith adds nodes to the MST one at a time, and keeps the edges connected to those nodes in a priority queue. It then tests each edge in the priority queue to add more nodes to the MST, avoiding cycles.

1. Add node 0 to MST;
2. Add all edges from new node to Priority Queue;
3. Visit smallest edge in Queue;
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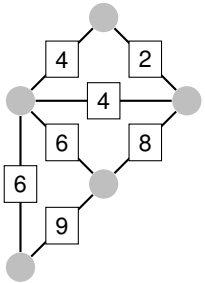

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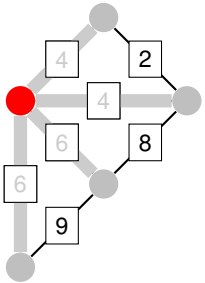

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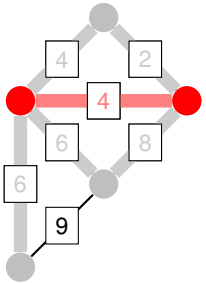

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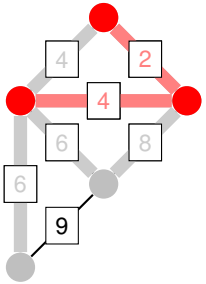

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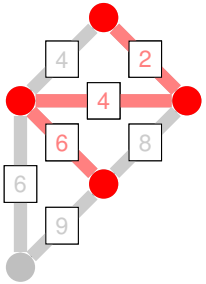

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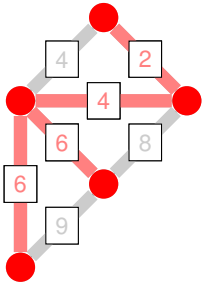


Prim's Algorithm – Implementation

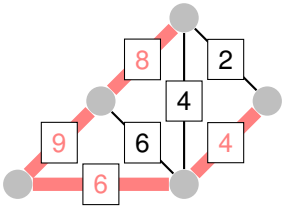
```
vector <int> taken;
priority_queue <pair <int,int>> pq;

void process (int v) {
   taken[v] = 1;
   for (int j = 0; j < (int)AdjList[v].size(); j++) {
      pair <int,int> ve = AdjList[v][j];
      if (!taken[ve.first])
         pq.push(pair <int,int> (-ve.second,-ve.second)
}}
taken.assign(V,0);
process(0);
mst_cost = 0;

while (!pq.empty()) {
  vector <int,int> pq.top(); pq.pop();
  u = -front.secont, w = -front.first;
  if (!taken[u]) mst_cost += w, process(u);
```

MST variant 1 – Maximum Spanning tree

The Maximum Spanning Tree variant requires the spanning tree to have maximum possible weight.

It is very easy to implement the Maximum MST:

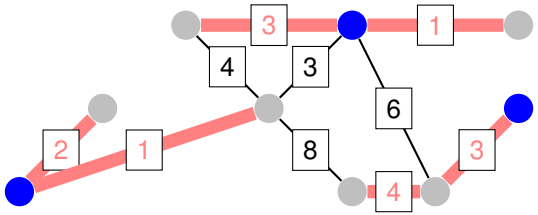
- **Kruskal**: Reverse the sort of the edge list;
- **Prim**: Invert the weight of the priority queue;



MST variant 2 – Minimum Spanning Subgraph, Forest

In this variant, a subset of edges or vertices are pre-selected.

- In the case of pre-selected vertices, add them to the “taken” list in Kruskal’s algorithm before starting;
- In the case of edges, add the end vertices to the “taken” list;



MST Variant 3 – Second Best MST

Problem Definition

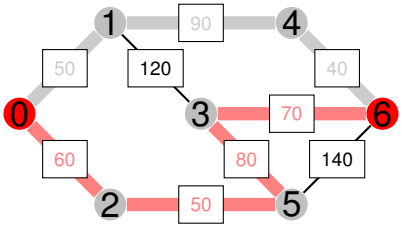
Suppose that you are required to calculate an alternative solution to an MST problem. In this case, you need to find the second cheapest spanning tree.

Simple Algorithm:

- Calculate the MST (using Kruskal or Prim);
- For every edge e_i in the MST:
  - Remove e_i from E;
  - Calculate a new MST;
- Choose the best among the new MSTs as the second-best MST.

QUIZ: How to generalize this algorithm for the n-th best spanning tree?

MST Variant 4 – Minmax path cost



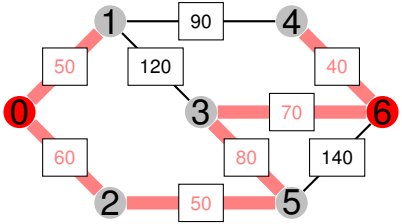
Problem Definition

Regular Cost for a path is the sum of weights of all edges in the path.

Minmax Cost for a path is the maximum weight among all its edges.

Find the path $v_i \rightarrow v_j$ with the smallest **minmax cost**

Finding the Minmax path with MST



Algorithm

- Generate the MST for the graph G.
- Find the path $v_i \rightarrow v_j$ inside the MST.

That's it!

Class Summary

Graph Basics

- Graph Problems come in a large variety of types;
- But Many Algorithms are variations on BFS and DFS;
  - Connected Components and Flood Fill;
  - Topological Sort;
  - Articulation Points and Bridges;
  - Minimum Spanning Trees;
- There are several special cases for graph problems:

Some common special cases

- Graphs with 0 or 1 Vertices; Graphs with 0 nodes;
- Unconnected Graphs;
- Self loops;
- Double edges;

Class Summary

Theme for Next Week

Graph Path Search and Weighted Graphs:

- Shortest Paths (Single Source and All Pairs);
- Network Flow;
- Graph Matching;

Graph Code Library

Graph problems share a lot of common code. I recommend that you prepare a code library as you learn new algorithms.

- Visited node flags and adjacency lists;
- Parent and children lists;
- Different algorithms;
- etc;

This Week's Problems

- Dominator – Discussed in class;
- Forwarding Emails
- Ordering
- Place the Guards
- Doves and Bombs
- Come and Go
- ACM Contest and Blackout
- Ancient Messages

Problem Hints

Forwarding E-mails

Problem Summary

- In a list of N people, every person i forwards e-mails to a single person j.
- If you want to form the longest e-mail chain, where do you start?
- **Output:** The person k that starts the longest chain;

- The time limit is not very large, you should find an $O(n)$ solution;
- How do you deal with loops?

Problem Hints

Ordering and Palace Guards

Ordering

- **Outline:** List all possible orderings of a given Direct Acyclic Graph;
- **Hint:** In the lecture, we saw an algorithm for a single ordering – for this problem, you need to generalize it for all orderings;

Palace Guards

You need to find a guard assignment for the city that obey the rules:

- All roads need **one guard**.
- No road can have **two guards**.

Note that for some inputs a solution is impossible.

Problem Hints

Doves and Bombs / Come and Go

Doves and Bombs

- **Outline:** You must choose one station (vertex) to remove, so that the number of unconnected subgraphs is maximum;
- **Hint:** Do you need to calculate the CCs every time you test a vertex, or is there a better way?

Come and Go

- **Outline:** For a city road map, you must check if that city is strongly connected (you can go from every node i to every node j) or not;
- **Hint:** Can you use the algorithm of strongly connected components for this?

Problem Hints

ACM Contest and Blackout

Outline

Given a graph of a powerplant and several schools, you have to find the **two** cheapest graphs that connect all the nodes.

Output: Cost of cheapest graph, and cost of 2nd cheapest graph.

Hint: The cheapest graph could be found using the MST algorithm. But how do we find the **second** cheapest graph?

Problem Hints

Ancient Message

Outline

- **Input:** A pixel image containing one or more "egyptian letters";
- **Output:** Identify which letters appear in the image;

Hints:

- Note that the input is in hexadecimal format;
- First: Think about how to count the number of characters in an image. This will help you think a way to identify the characters.
- Pay attention to the "Rules" in the problem description.

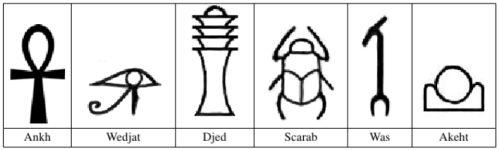


Figure C.2: AKW

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