GB21802 - Programming Challenges

Week 8 - Computational Geometry

Claus Aranha
caranha@cs.tsukuba.ac.jp

College of Information Science

2015-06-17,20

Last updated June 17, 2016

Last Week Results

Week 6 - Graph II

- Division – 16/31
- What Base is This? – 6/31
- Divisibility of Factors – 11/31
- Triangle Counting – 11/31
- Help my Brother (II) – 4/31
- Marbles – 1/31
- Ocean Deep! Make it Shallow! – 9/31
- Winning Streak – 0/31

- 14 people: 0 problems;
- 6 people: 1-2 problems;
- 6 people: 3-4 problems;
- 4 people: 5-6 problems;
- 1 people: 7-8 problems!

Special Notes

Topic of the Week – Computational Geometry

- Computational Geometry problems are generally considered to be difficult, both in terms of understanding the solution, and programming the solution;
- One trick for these problems is to prepare a large library of basic geometric operations (distances, intersections, angle operations, etc);
  - Focus of this class is the implementation of these operations.
- Special attention is needed to deal with degeneracies;

Degeneracies: Special cases

Two types of degeneracies: Special cases and Precision errors

(some) Special cases:

- Lines parallel to the vertical axis
- Colinear Lines
- Overlapping Segments
- Concave polygons
- Etc...

Good implementations should deal with common special cases.

Degeneracies: Precision errors

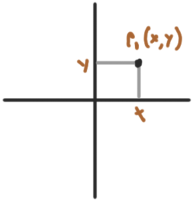
Representation of floating point numbers in computers has a limited precision. So for multiple operations on very small numbers, we may start to see calculation errors.

Some ways to avoid floating point precision errors:

- Whenever possible, convert the float numbers to integers
- Never compare “float x == float y”.
- Instead, use this: “fabs(x - y) < EPS” (float EPS$=$ 0.00000001)

Point Representation

Points are the building blocks of geometric objects. In C/C++, we can represent them using a struct with two members:




```
// When possible, use int coordinates
struct point_i { int x, y;
  point_i() { x = y = 0; }
  point_i(int _x, int _y) : x(_x), y(_y) {}};

// Floating point variation
struct point { double x, y;
  point() { x = y =0.0;}
  point(double _x, double _y) : x(_x), y(_y) {}};
```

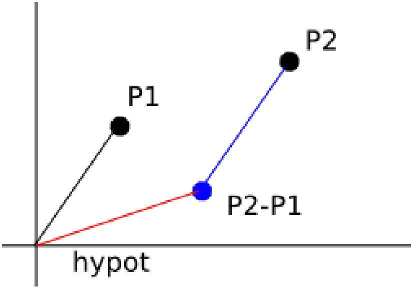
Point Operations

To compare two points, or test for equality, we can overload the *equal* or *less* operator in the point struct.

```
struct point { double x, y;
   point() { x = y = 0.0;
   point(double _x, double _y) : x(_x), y(_y) {}

   // override less than operator -- useful for sorting
   bool operator < (point other) const {
      if (fabs(x - other.x) > EPS)
         return x < other.x;
      return y < other.y; }

   // override equal operator, takes EPS into account
   bool operator == (point other) const {
      return (fabs(x - other.x) < EPS &&
              (fabs(y - other.y) < EPS)); }
   }
```

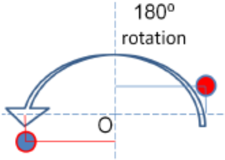
Point: Euclidean Distance

```
#define hypot(dx,dy) sqrt(dx*dx + dy*dy)

double dist(point p1, point p2) {
  return hypot(p1.x - p2.x, p1.y - p2.y);
}
```



Point: Rotation around origin

```
#define PI              3.14159265358979323846  /* pi */
#define DEG_to_RAD(X)  (X*PI)/180.0

// theta is in degrees
point rotate(point p, double theta) {
   double rad = DEG_to_RAD(theta);
   return point(p.x * cos(rad) - p.y * sin(rad),
                p.x * sin(rad) + p.y * cos(rad));}
```



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

Line Representation

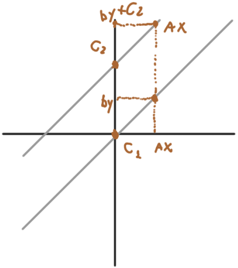
How to represent a line?

- Two points. Problem: cannot generalize for other points of the line easily;
- $y = mx + c$. Problem: cannot handle vertical lines (m is infinite)
- $ax + by + c = 0$. Better representation for "most" cases.

```
struct line { double a,b,c; };

void pointsToLine(point p1, point p2, line &l) {
   if (fabs(p1.x - p2.x) < EPS {
      l.a = 1.0; l.b = 0.0; l.c = -p1.x; }
   else {
      l.a = -(double) (p1.y-p2.y)/(p1.x-p2.x);
      l.b = 1.0; l.c = -(double) (l.a*p1.x) - p1.y;}
}
```

Line: Parallel and Identical lines

- Two lines are parallel if their coefficients (a, b) are the same;
- Two lines are identical if all coefficients (a, b, c) are the same;
- Remember that we force b to be 0 or 1;




```
bool areParallel(line l1, line l2) {
   return (fabs(l1.a-l2.a) < EPS) &&
          (fabs(l1.b-l2.b) < EPS); }

bool areSame(line l1, line l2) {
   return areParallel(l1,l2) &&
          (fabs(l1.c - l2.c) < EPS); }
```

Line: Intersection

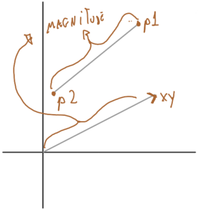
If two lines are not parallel, then they will intersect at a point. This point (x,y) is found by solving the system of two linear equations:

$$a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0$$

```
bool areIntersect(line l1, line l2, point &p) {
   if (areParallel(l1,l2)) return False;

   p.x = (l2.b * l1.c - l1.b * l2.c) /
         (l2.a * l1.b - l1.a * l2.b);
   if (fabs(l1.b) > EPS) // Testing for vertical case
      p.y = -(l1.a * p.x + l1.c);
   else
      p.y = -(l2.a * p.x + l2.c);
   return true; }}
```

Segments and Vectors

- A Line Segment is a line limited by two points and finite length;
- A Vector is a segment with an associated direction;
- Often vectors are represented by a single point (the other assumed to be the origin);

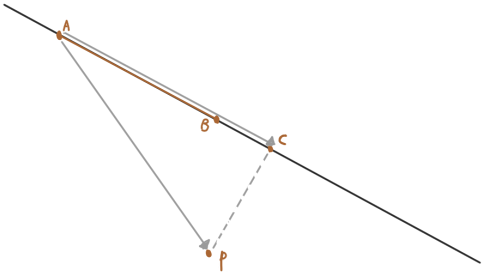



```
struct vec { double x, y;
    vec(double _x, double _y) : x(_x), y(_y) {} };

vec toVec(point a, point b) {
    return vec(b.x - a.x, b.y - a.y); }
vec scale(vec v, double s) {
    return vec(v.x * s, v.y * s); }
point translate(point p, vec v) {
    return point(p.x + v.x , p.y + v.y); }
```

Distance between point and line

Given a point p and a line l, the distance between the point and the line is the distance between p and the c, the closest point in l to p.

We can calculate the position of c by taking the projection of $\bar{a}c$ into l (a, b are points in l).



Distance between point and line

```
double dot(vec a, vec b) {
   return (a.x * b.x + a.y * b.y); }
double norm_sq(vec v) {
   return v.x * v.x + v.y * v.y; }

// Calculates distance of p from line, given
// a,b different points in the line.
double distToLine(point p, point a, point b, point &c) {
  // formula: c = a + u * ab
  vec ap = toVec(a, p), ab = toVec(a, b);
  double u = dot(ap, ab) / norm_sq(ab);
  c = translate(a, scale(ab, u));
  // translate a to c
  return dist(p, c); }
```

Distance between segment and line

If we have a segment *ab* instead of a line, the procedure to calculate the distance is similar, but we need to test if the intersection point falls in the segment.

```
double distToLineSegment(point p, point a,
                         point b, point &c) {
  vec ap = toVec(a, p), ab = toVec(a, b);
  double u = dot(ap, ab) / norm_sq(ab);

  if (u < 0.0) { c = point(a.x, a.y); // closer to a
                 return dist(p, a); }
  if (u > 1.0) { c = point(b.x, b.y); // closer to b
                 return dist(p, b); }

  return distToLine(p, a, b, c); }
```

Angles between segments

angle between two segments ao and ob

```
#import <cmath>

double angle(point a, point o, point b) { // in radians
vec oa = toVector(o, a), ob = toVector(o, b);
return acos(dot(oa, ob)/sqrt(norm_sq(oa)*norm_sq(ob)));}
```

Left/Right test: We can calculate the position of point p in relation to a line l using the cross product.

Take q, r points in l. Magnitude of the cross product pq x pr being positive/zero/negative means that $p \to q \to r$ is a left turn/collinear/right turn.

```
double cross(vec a, vec b) {
  return a.x * b.y - a.y * b.x; }
bool ccw(point p, point q, point r) {
  return cross(toVec(p, q), toVec(p, r)) > 0; }
collinear(point p, point q, point r) {
  return fabs(cross(toVec(p, q), toVec(p, r))) < EPS;
```

Problem Example: UVA – Intersection

Summary

Given two points p_1 and p_2, and a rectangle, test whether the segment p_1p_2 intersects the rectangle.

Strategy

Problem Example: UVA – Intersection

Summary

Given two points p_1 and p_2, and a rectangle, test whether the segment $p_1 p_2$ intersects the rectangle.

Strategy

- Test if points p_1 or p_2 are in the rectangle (easy tests first)
- Test if $p_1 p_2$ intersects with any side of the rectangle.
- "Hard" Way:

Problem Example: UVA – Intersection

Summary

Given two points p_1 and p_2, and a rectangle, test whether the segment $p_1 p_2$ intersects the rectangle.

Strategy

- Test if points p_1 or p_2 are in the rectangle (easy tests first)
- Test if $p_1 p_2$ intersects with any side of the rectangle.
- “Hard” Way:
  - Find the intersection between lines $p_1 p_2$, and top/bottom/left/right
  - Test if the intersection point is in line $p_1 p_2$;
  - Test if the intersection point is in the rectangle;

Problem Example: UVA – Intersection

Summary

Given two points p_1 and p_2, and a rectangle, test whether the segment p_1p_2 intersects the rectangle.

Strategy

- Test if points p_1 or p_2 are in the rectangle (easy tests first)
- Test if p_1p_2 intersects with any side of the rectangle.
- "Hard" Way:
  - Find the intersection between lines p_1p_2, and top/bottom/left/right
  - Test if the intersection point is in line p_1p_2;
  - Test if the intersection point is in the rectangle;
- There is an easier way that takes into account vertical/horizontal sides

Problem Example: UVA – Waterfalls

Summary

Given a list of water sources, and a list of segments, calculate the position that each water source will arrive at the bottom.

Strategy:

- For each water source, calculate all the segments that intersect it (easy because vertical line)
- For each segment, calculate the intersection point - get the highest one.
- New position of the water source is the lowest point of that segment.

Problem Example: UVA – Waterfalls

Summary

Given a list of water sources, and a list of segments, calculate the position that each water source will arrive at the bottom.

Strategy:

- For each water source, calculate all the segments that intersect it (easy because vertical line)
- For each segment, calculate the intersection point - get the highest one.
- New position of the water source is the lowest point of that segment.

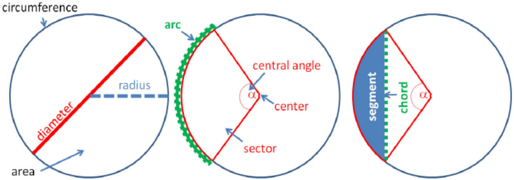
- Problem: No limit of segments or water sources. How do you avoid TLE?

Circles

- A circle is defined by its center (a, b) an its radius r
- The circle contains all points such (x, y) such as $(x - a)^2 + (y - b)^2 \leq r^2$

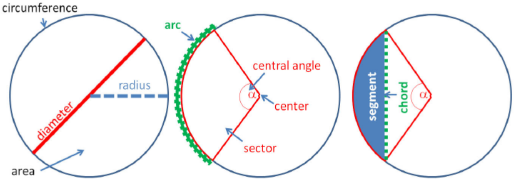
```
int insideCircle(point_i p, point_i c, int r) {
   int dx = p.x-c.x, dy = p.y-c.y;
   int Euc = dx*dx + dy*dy, rSq = r*r;
   return Euc < rSq ? 0 : Euc == rSq ? 1 : 2;
   // 0 - inside, 1 - border, 2- outside
}
```

Circles (2)



- If you are not given π, use $pi = 2{*}\text{acos}(0.0);$
- Diameter: $D = 2r$; Perimeter/Circumference: $C = 2\pi r$; Area: $A = \pi r^2$;
- To calculat the Arc of an angle α (in Degrees), $\frac{\alpha}{360} * C$;

Circles (3)



- A chord of a circle is a segment composed of two points in the circle's border. A circle with radius r and angle α degrees has a chord of length sqrt($2r^2(1-\cos\alpha)$)
- A Sector is the area of the circle that is enclosed by two radius and and arc between them. Area is: $\frac{\alpha}{360}A$
- A Segment is the region enclosed by a chord and an arc.

Problem Example: Area

Summary

Given 4 circles, determine the proportion of points that fall in all four circles.

Triangles!

Polygons!

Problem Discussion

- Sunny Mountains
- Bright Lights
- Rope Crisis in Ropeland
- Bounding Box
- Soya Milk
- SCUD Bursters
- Trash Removal
- The Sultan's Problem

Class Summary

Computational Geometry

- Basic Concepts
- Triangles
- Circles
- Polygons

Final Week: String Problems!