Programming Challenges

Week 4 - Combinatorics

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What is Combinatorics?

Combinatorics is the mathematics of counting.

Sounds easy, right?

- It begins with easy concepts: How to add or multiply groups;
- But in the end, it involves insight and advanced mathematics;
- It is very useful to simplify complex problems, and to estimate the size of sequences or combinations;

Main definitions in combinatorics

Sequence

$$
\begin{aligned}
S(1) &= 1 = 1 \\
S(2) &= 1 + 2 = 3 \\
S(3) &= 1 + 2 + 3 = 6
\end{aligned}
$$

Main definitions in combinatorics

Recurrence

The recursive form of a sequence:

$$S(n) = S(n-1) + n; S(1) = 1;$$

Closed Form

The non-recursive form of a sequence:

$$S(n) = \frac{n(n+1)}{2}$$

The key in combinatoric problems is usually finding out the recurrence, or the closed form of a sequence.

Combinatorics: What is it good for?

- Of course, some problems are simply to find the recurrence or the closed form of a sequence, or to find a specific value (eg. what is F(300)?)
- But understanding recurrences is important to get an intuition about the size of things. You can look at a computational problem and understand how big it can get.

An intuition in combinatorics is essential to obtain an intuition in computational complexity.

Warmup: Basic Combinatoric techniques

These basic rules are used to derive many of the most advanced combinatoric constructs.

Product Rule

You have 4 shirts and 5 pants. How many different ways can you get dressed?

- Product Rule: We want to combine one element from set A, and one element from set B. There are $|A| \times |B|$ different possibilities.

Warmup: Basic Combinatoric techniques

These basic rules are used to derive many of the most advanced combinatoric constructs.

Sum Rule

The university restaurant has 3 types of curry, 5 types of noodles and 4 types of pasta. How many days does it take to eat one of each type of food?

- Sum Rule: We want to choose one element from either set A or set B. Assuming the sets are independent, there are $|A| + |B|$ possibilities.

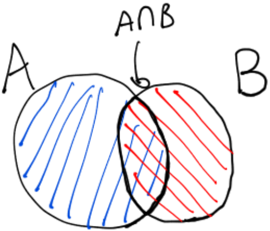
Warmup: Basic Combinatoric Techniques

Intersection and Double Counting

15 students like chocolate. 13 students like vanilla. 5 of these students like both. How many students are there?

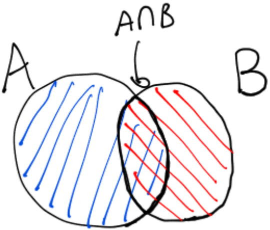
To combine sets that have repeated elements, we have to exclude elements that have been counted twice, and include elements that have been removed twice.

Intersection of sets (2)



How do we count the elements of two intersecting sets?

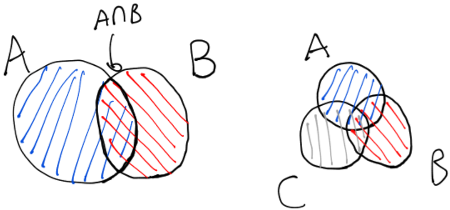
Intersection of sets (2)



How do we count the elements of two intersecting sets?

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Intersection of sets (3)



How do we count the elements of three intersecting sets?

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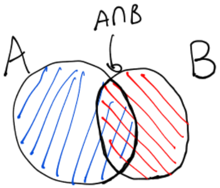
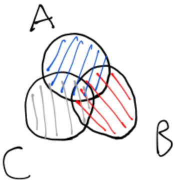
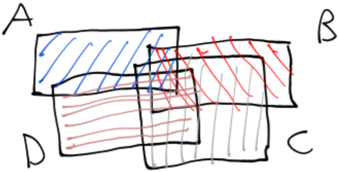




How do we count the elements of three intersecting sets?

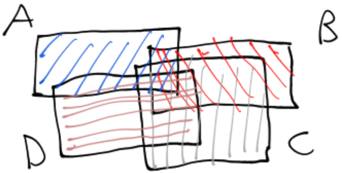
$$|A \cup B \cup C| = |A| + |B| - |A \cap B| + |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Intersection of sets (4)



How do we count the elements of four intersecting sets?

Intersection of sets (4)



How do we count the elements of four intersecting sets?

$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap$

This grows exponentially!

What is this useful for?

Calculating set sizes is essential when estimating how fast a combination can grow!

- Combination sets;
- Problem cases – how many possibilities for the input/output?
- Algorithm complexity – how many combinations of loops?
- Mathematical Proofs – proof by induction!
- Etc. . .

Permutations

Permutations

Arrangement of n items, where every item appears exactly once:
123,132,213,231,312,321

Where do we see them?

- Travelling salesman – output is the permutation of cities in the order that they should be visited.
- Can you give me another example?

Permutations

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How many permutations exist for n?

- $n!$
- $3! = 6$
- $10! = 3628800$
- $20! = 2.432 + e28$

Subsets

Subsets

A selection of n items, where each item can exists or not:
{1},{2},{3},{1,2},{1,3},{2,3},{1,2,3},{}

Where do we use them?

- The backpack problem – Select the items that fit in the backpack.
- Can you give me another example?

Subsets

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{1},{2},{3},{1,2},{1,3},{2,3},{1,2,3},{}

How many subsets are there for n?

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A selection of n items, where each item can exists or not: {1},{2},{3},{1,2},{1,3},{2,3},{1,2,3},{}

How many subsets are there for n?

- 2^n
- $2^3 = 8$
- $2^{10} = 1024$
- $2^{20} = 1048576$

Strings

Strings

A sequence of elements draw from a set with repetition.
string of length 3 from {123}: 222, 321, 121, ...

Where do we see them?

- Game moves (e.g. Prisoner's Dilemma)
- Other examples?

Strings

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Strings

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A sequence of elements draw from a set with repetition.
string of length 3 from {123}: 222, 321, 121, ...

How fast do they grow?

- m^n
- $26^5 = 11881376$
- $10^{10} = 10000000000$

Recurrence Relations

Definition

A Recurrence Relation is a function defined in terms of itself.
Recursion – Recurrence – Related words.

Can you see the recursion?

- Tree – A tree has *n* children, which can be leaf nodes or other trees
- List –
- Divide-and-conquer –
- Other examples?

Recurrence Relations

Definition

A Recurrence Relation is a function defined in terms of itself.
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Can you see the recursion?

- Tree – A tree has *n* children, which can be leaf nodes or other trees
- List – an item links to null, or to another List
- Divide-and-conquer – Divide the data, and apply the algorithm to each part
- Other examples?

Recurrence Relations

Why are recurrences important

If we can find the recurrence in a sequence or in a set, we have found a simple algorithm to build that sequence or set.

Recurrences are specially important for Dynamic Programming (which we will talk about in a future class).

Recurrence Relations

Components of a Recurrence Relation

- Starting Condition;
- Recurrence Step;

Example: The Fibonacci Numbers

- Starting Condition: $F(0) = 0; F(1) = 1;$
- Recurrence step: $F(n) = F(n-1) + F(n-2)$

Curiosity

The Fibonacci numbers were created to model the multiplication of rabbits.

Recurrence Relations

Many functions can be easily expressed as recurrences:

- Degrees from polynomials;

$$a_n = a_{n-1} + 1; a_1 = 1;$$

- Exponential ($f(k,n) = k^n$);

$$f(k,n) = k * f(k,n-1); f(k,0) = 1;$$

- Factorial ($f(n) = n!$);

$$f(n) = n * f(n-1); f(0) = 1;$$

Closed Form of Recurrence Relations (1)

Definition

The closed form of a recurrence relation is a formula that describes the result without using the recurrence.

$$F(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

- Closed forms are useful to quickly calculate recurrences when n is very big;
- They can remove precision errors from repeated multiplications;
- They can add approximation errors for small n.
- Calculating closed forms is an art! (Or research topic)

Closed Form of Recurrence Relations (2)

Close Form for the Fibonacci Numbers

$$F(n) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

We can learn things from closed forms

The second term of the closed form is always between 0 and 1, so we can calculate only the first term to estimate the value of a Fibonacci number;

Binomials

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} = \frac{n!}{(n-k)!k!}$$

Very important combinatory sequence

Basic Definition: Number of ways you can make k choices from n elements.

What can you count with binomials

- Probabilities: What are the probabilities of your 5 numbers being chosen from the 60 in the lotto? $\frac{1}{\binom{60}{5}}$
- Paths Across a Grid: How many ways are there to go from the top right of a *nxm* grid to the bottom left, with the smallest amount of steps? $\binom{n+m}{n}$
- Coefficients of $(a+b)^n$: What is the coefficient of $a^k b^{n-k}$? Answer: $\binom{n}{k}$. Can anyone explain why?

Computing the Binomial

Closed form of the binomial

$$\binom{n}{k} = \frac{n!}{(n-k)!(k!)}$$

- What is the problem if you try to calculate that formula in a program?

Computing the Binomial

Closed form of the binomial

$$\binom{n}{k} = \frac{n!}{(n-k)!(k!)}$$

- What is the problem if you try to calculate that formula in a program?
- Repeated multiplication can lead to overflow errors!

Computing the Binomial

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Have you ever played with pascal's triangle?

Computing the Binomial

What can we take from pascal's triangle?

1. Every number is the sum of its "parents";
2. Every line n adds up to 2^n
3. The numbers of line n are the coefficients of $(a+b)^n$

...wait!

We have seen 3 before elsewhere...

Computing the Binomial

Another way to see Pascal's Triangle

$$
\begin{array}{cccc}
 & & & \binom{0}{0} \\
 & & \binom{1}{0} & \binom{1}{1} \\
 & \binom{2}{0} & \binom{2}{1} & \binom{2}{2} \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3}
\end{array}
$$

We know that each element of Pascal's triangle is the sum of its parents. This indicates that the same thing happens for binomials.

Computing the Binomial

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Let's think about the new element n

- What happens if n is part of the sequence k?
- What happens if n is NOT part of the sequence k?
- What do we do with these two sets?

Recursion and Induction

Induction and Recursion are closely related to recurrence relations. If we can find out a correct recurrence relation for a combinatory construct, we can also derive a recursive algorith, its induction, and maybe even its closed form!

Calculating the Recursion

- Try to figure out what the base cases are;
- Plot a some of the smaller values;
- Observe these values for patterns;

Catalan Numbers

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k} = \frac{1}{n+1}\binom{2n}{n}$$

1,1,2,5,14,42,132,. . .

How many valid combinations there are for n pairs of parenthesis?
For $n = 3$, we have 5: ((())), ()(()), (())(), (()()) and ()()().

Can you derive the recurrence above using this information?

Integer Partition

$f(5,5)$ = (5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)

$f(n,k)$ Number of ways that we can sum n, using integers equal or smaller than k. The recurrence is $f(n,k) = f(n-k,k) + f(n,k-1)$, and the base cases are: $f(1,1) = 1$ and $f(n,k) = 0; k > n$. Can you derive this recurrence?

This Week's Problems

- How Many Fibs
- Complete Tree Labeling
- Counting
- Steps

Alert: About Week 5

Alert for Week 5!

There will be no class on 5/18,5/22,5/25. Week 5 only class will be on 5/29.

Because of this, you will have TWO WEEKS to solve week 5's problems (5/16 to 5/31). The lecture notes for week 5 will be online this Friday.