# Programming Challenges (GB21802)

## Week 9 - Computational Geometry

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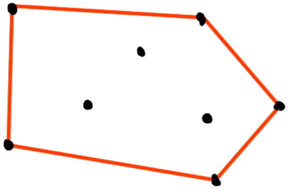
# Computational Geometry

## What is it?

Computational Geometry problems involve answering questions about lines, points and angles. Some example questions:

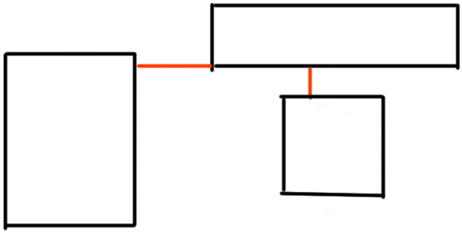
Given $N$ points $(s_1, s_2, s_3, \dots, s_N)$, what is the area of the poligon that covers all the points?



# Computational Geometry
## What is it?

Computational Geometry problems involve answering questions about lines, points and angles. Some example questions:

Given $N$ rectangles, $x_1, y_1, w_1, h_1; \ldots; x_N, y_N, w_N, h_N$, what is the length of lines needed to connect them?

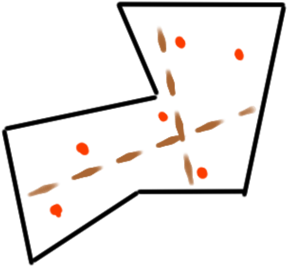


# Computational Geometry

## What is it?

Computational Geometry problems involve answering questions about lines, points and angles. Some example questions:

Given a polygon, and $N$ points, what is the line that divides the polygon in equal areas, so that the same number of points are in each area?



# Computational Geometry

## The good and the bad

- Good: Geometry problems are fun
- Good: You have to draw pretty pictures
- Good: Mostly algorithms from high school
- Good: Code is highly re-usable

- Bad: You have to write a lot of code (in the beginning!)
- Bad: Very easy to get WE...

# Easy Mistakes in Geometry Problems

## Problem 1 – Special Cases

- Multiple points in the same place;
- Collinear points;
- Vertical lines;
- Parallel Lines;
- Intersection at end of segment;
- etc;

## Problem 2 – Precision Errors

- Functions require many multiplications and divisions;
- Easy to propagate floating point errors;

# Easy Mistakes in Geometry Problems

## Dealing Special Cases

- Make sure to add special cases to your library functions;

## Solving Precision Errors

- If possible, convert all values to integers
- Use an EPSILON constant for comparisons:

```
if (float.1 == float.2) then                  // NO
if (fabs(float.1 - float.2) < EPS) then // YES!
```

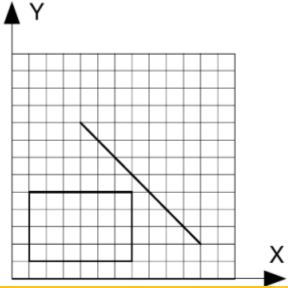
# Class Outline

- Example Problems;
- Basic Geometric Functions;
- Circles;
- Triangles;
- Polygons;

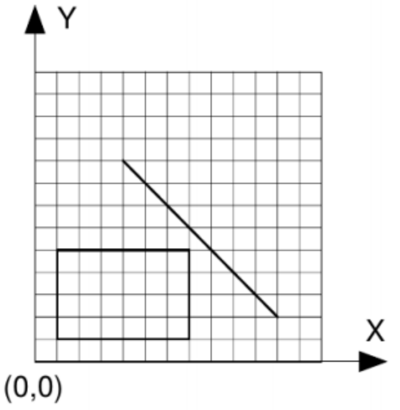
# Problem Example: UVA 191 – Intersection

## Summary

- **Input**
  A rectangle and a line:
  xstart ystart xend yend xleft ytop xright ybottom
- **Output**
  T - if the line intersects the rectangle
  F - if the line does not intersect the rectangle



# Problem Example: UVA 191 – Intersection



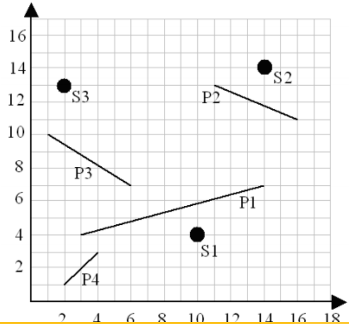

Steps to calculate the solution:

- Test if $p_1$ or $p_2$ are inside the rectangle;
- Test if the segment intersects a side of the rectangle;
- (optional) make a bullet hell game;

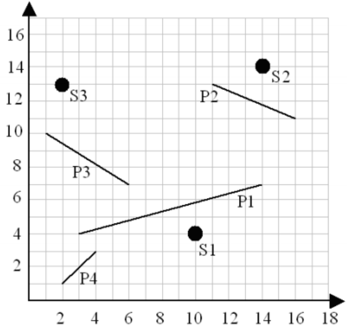
# Problem Example: UVA – Waterfalls

## Summary

- **Input**
  List of line segments in the waterfall
  List of water sources
- **Output**
  X position where each water source falls



# Problem Example: UVA 833 – Waterfalls



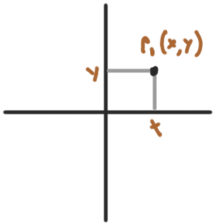

For each water source:

- Identify all segments that it intersects with;
- Select the highest segment;
- Move the source to the bottom of the segment;
- Repeat;

Many opportunities for pruning and pre-computing! (if necessary)

# Implementing Graphical Problems

## Point Representation




### Point Representation

```
struct point_i { int x, y;  // Using int coordinates.
  point_i() { x = y = 0; }
  point_i(int _x, int _y) : x(_x), y(_y) {}};

struct point { double x, y; // Using floats
  point() { x = y = 0.0;}
  point(double _x, double _y) : x(_x), y(_y) {}};
```

# Implementing Graphical Problems

## Comparing and Sorting Points

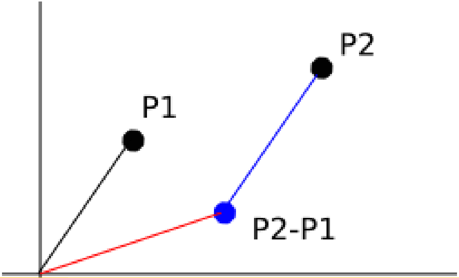
### Point Comparison

```
struct point { double x, y;
   point() { x = y = 0.0;
   point(double _x, double _y) : x(_x), y(_y) {}

   // Sorting by coordinate
   bool operator < (point other) const {
      if (fabs(x - other.x) > EPS)
         return x < other.x;
      return y < other.y; }

   // Equality testing -- Note the use of EPS
   bool operator == (point other) const {
      return (fabs(x - other.x) < EPS &&
              (fabs(y - other.y) < EPS)); }
   }
```

# Basic Library – Points 3

Most common distance measure: Euclidean distance. Sometimes Manhattan distance (Taxicab distance) is also used.

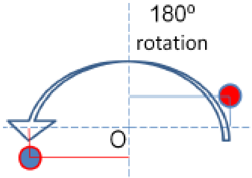
```
#define hypot(dx,dy) sqrt(dx*dx + dy*dy)

double dist(point p1, point p2) {
  return hypot(p1.x - p2.x, p1.y - p2.y); }

double taxicab(point p1, point p2) {
  return fabs(p1.x - p2.x) + fabs(p1.y - p2.y); }
```



# Basic Library – Points 4

## Rotating a point around the origin

```
#define PI            3.14159265358979323846 // Pi const
double PI = 2 * acos(0.0)                    // Better Pi
#define DEG_to_RAD(X) (X*PI)/180.0

// theta is in degrees
point rotate(point p, double theta) {
   double rad = DEG_to_RAD(theta);
   return point(p.x * cos(rad) - p.y * sin(rad),
                p.x * sin(rad) + p.y * cos(rad));}
```




$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

**Quiz:** What do you do if you want to rotate a point around $x_0, y_0$?

# Basic Library – Lines 1

There are many ways to specify a line:

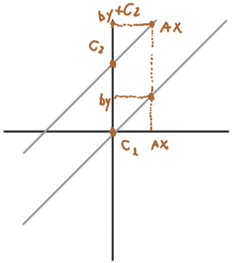
- $ax + by + c = 0$ – useful for most cases.
- $y = mx + c$ – useful for angle manipulation, but special cases
- $x_0, y_0, x_1, y_1$ – two points, not very useful for programming.

## Point to Line

```
struct line { double a,b,c; };

void pointsToLine(point p1, point p2, line &l) {
   if (fabs(p1.x - p2.x) < EPS {
      l.a = 1.0; l.b = 0.0; l.c = -p1.x; }
   else {
      l.a = -(double) (p1.y - p2.y)/(p1.x - p2.x);
      l.b = 1.0; l.c = -(double) (l.a*p1.x) - p1.y;}
}
```

# Basic Library – Line 2

- Two lines are parallel if their coefficients ($a$, $b$) are the same;
- Two lines are identical if all coefficients ($a$, $b$, $c$) are the same;
- Remember that we force $b$ to be 0 or 1;




## Parallel and identical lines

```
bool areParallel(line l1, line l2) {
   return (fabs(l1.a-l2.a) < EPS) &&
          (fabs(l1.b-l2.b) < EPS); }

bool areSame(line l1, line l2) {
   return areParallel(l1,l2) &&
          (fabs(l1.c - l2.c) < EPS); }
```

# Basic Library – Line 3

The **intersection** point $x_I, y_I$ is where two lines meet. We can find this point by solving the following system of linear equations:

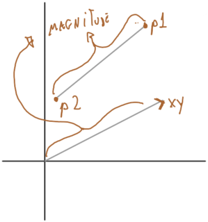
$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

## Computing the intersection

```
bool areIntersect(line l1, line l2, point &p) {
   if (areParallel(l1,l2)) return False;

   p.x = (l2.b * l1.c - l1.b * l2.c) /
         (l2.a * l1.b - l1.a * l2.b);
   if (fabs(l1.b) > EPS) // Testing for vertical case
      p.y = -(l1.a * p.x + l1.c);
   else
      p.y = -(l2.a * p.x + l2.c);
   return true; }}
```

# Basic Library – Vectors

- A Vector indicates direction and length;
- Represented as a $x, y$ point in relation to the origin;
- Operations: Scale, Translation, Addition, Product;

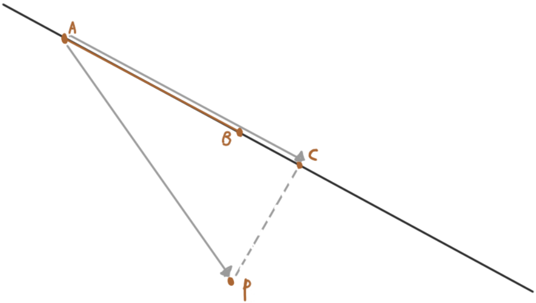



```
struct vec { double x, y;
    vec(double _x, double _y) : x(_x), y(_y) {} };

vec toVec(point a, point b) {
    return vec(b.x - a.x, b.y - a.y); }
vec scale(vec v, double s) {
    return vec(v.x * s, v.y * s); }
point translate(point p, vec v) {
```

# Distance between point and line

Given a point $p$ and a line $l$, the distance between the point and the line is the distance between $p$ and the $c$, the closest point in $l$ to $p$.

We can calculate the position of $c$ by taking the projection of $\bar{a}c$ into $l$ $(a, b$ are points in $l)$.



# Distance between point and line

```
double dot(vec a, vec b) {
   return (a.x * b.x + a.y * b.y); }
double norm_sq(vec v) {
   return v.x * v.x + v.y * v.y; }

// Calculates distance of p from line, given
// a,b different points in the line.
double distToLine(point p, point a, point b, point &c) {
  // formula: c = a + u * ab
  vec ap = toVec(a, p), ab = toVec(a, b);
  double u = dot(ap, ab) / norm_sq(ab);
  c = translate(a, scale(ab, u));
  // translate a to c
  return dist(p, c); }
```

# Distance between point and segment

If we have a segment *ab* instead of a line, the procedure to calculate the distance is similar, but we need to test if the intersection point falls in the segment.

```
double distToLineSegment(point p, point a,
                         point b, point &c) {
  vec ap = toVec(a, p), ab = toVec(a, b);
  double u = dot(ap, ab) / norm_sq(ab);

  if (u < 0.0) { c = point(a.x, a.y); // closer to a
                 return dist(p, a); }
  if (u > 1.0) { c = point(b.x, b.y); // closer to b
                 return dist(p, b); }

  return distToLine(p, a, b, c); }
```

# Angles between segments

**angle between two segments ao and ob**

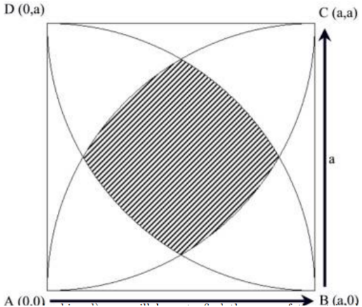
```
#import <cmath>

double angle(point a, point o, point b) { // in radians
vec oa = toVector(o, a), ob = toVector(o, b);
return acos(dot(oa, ob)/sqrt(norm_sq(oa)*norm_sq(ob)));}
```

Left/Right test: We can calculate the position of point *p* in relation to a line *l* using the cross product.

Take *q*, *r* points in *l*. Magnitude of the cross product *pq* x *pr* being positive/zero/negative means that $p \rightarrow q \rightarrow r$ is a left turn/collinear/right turn.

```
double cross(vec a, vec b) {
  return a.x * b.y - a.y * b.x; }
bool ccw(point p, point q, point r) {
  return cross(toVec(p, q), toVec(p, r)) > 0; }
collinear(point p, point q, point r) {
  return fabs(cross(toVec(p, q), toVec(p, r))) < EPS;
```

# Problem Example: UVA 10589 Area



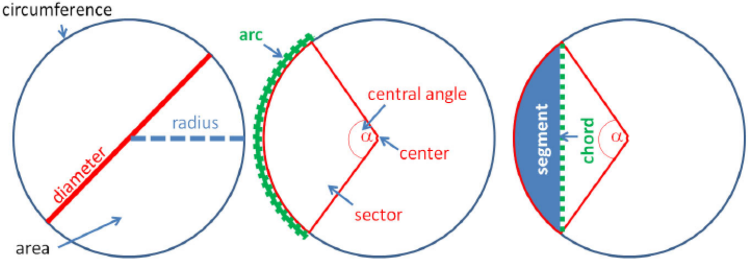

- What is the area of the shaded part of the rectangle?
- You are given the radius of 4 circles, each centered in the corners of the rectangle.

# Circles

- A circle is defined by its center $(a, b)$ an its radius $r$
- The circle contains all points such $(x, y)$ such as $(x-a)^2 + (y-b)^2 \leq r^2$

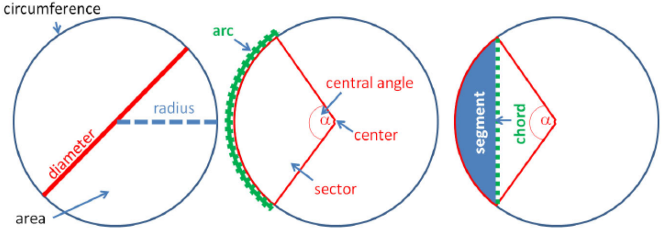
```
int insideCircle(point_i p, point_i c, int r) {
   int dx = p.x-c.x, dy = p.y-c.y;
   int Euc = dx*dx + dy*dy, rSq = r*r;
   return Euc < rSq ? 0 : Euc == rSq ? 1 : 2;
   // 0 - inside, 1 - border, 2- outside
}
```

# Circles (2)




- If you are not given $\pi$, use $pi = 2 * \text{acos}(0.0)$;
- Diameter: $D = 2r$; Perimeter/Circumference: $C = 2\pi r$; Area: $A = \pi r^2$;
- To calculat the Arc of an angle $\alpha$ (in Degrees), $\frac{\alpha}{360} * C$;

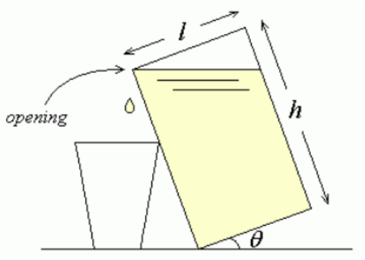
# Circles (3)




- A chord of a circle is a segment composed of two points in the circle's border. A circle with radius $r$ and angle $\alpha$ degrees has a chord of length $\mathrm{sqrt}(2r^2(1 - \cos\alpha))$
- A Sector is the area of the circle that is enclosed by two radius and and arc between them. Area is: $\frac{\alpha}{360}A$
- A Segment is the region enclosed by a chord and an arc.

# Example: UVA 11909 - Soya milk

- **Input**:
  The dimensions of a Milk box, and its inclination:
  $l, w, h, \theta$
- **Output**:
  The amount of milk left in the box.



# Example: UVA 10577 - Bounding Box

Given three vertices of a regular polygon, calculate the minimal square necessary to cover the polygon.

Hint: You don't actually need to calculate any polygons

# Triangle Basics

Any 2 dimensional polygon can be expressed as a combination of triangles. So triangles are important constructs in computational geometry.

## Common Characteristics

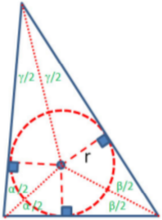
- Triangle Inequality: Sides $a, b, c$ obey $a + b > c$
- Triangle Area: Be $b$ one side of the triangle and $h$ its height, $A = 0.5bh$
- Perimeter: $p = a + b + c$
- Semiperimeter: $s = 0.5p$

## Heron's Formula

We can calculate the area of a triangle based on its sides:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

# Incircle Triangle




## Radius of the Incircle: $r = \text{area}(\Delta)/s$

```
def radiusInCircle(p1,p2,p3):
   ab, bc, cd = dist(p1,p2),dist(p2,p3),
                       dist(p3,p1)
   A = area(ab,bc,ca) % Heron's formula
   P = ab+bc+ca
   return A/(0.5*P)
```

## Finding the center point of the Incircle

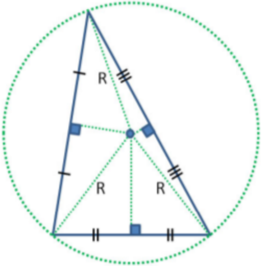
- Check that the three points are not colinear;
- Find the bisection *AP* of the *AB*-*AC* angle;
  - Calculate the point *P* in *BC* that bisects *A*
  - The proportion of *BP* is $(AB/AC)/(1 + AB/AC)$
- Find the bisection $BP'$ of the *BA*-*BC* angle;
- Fint the intersection of $AP$-$BP'$

# Incircle Triangle

## Calculating the Center (Code)

```
int inCircle(point p1, point p2, point p3,
             point &ctr, double &r) {
  r = rInCircle(p1, p2, p3);
  if (fabs(r) < EPS) return 0; // colinear points;
  line l1, l2; // compute these two angle bisectors
  double ratio = dist(p1, p2) / dist(p1, p3);
  point p = translate(p2, scale(toVec(p2, p3),
                    ratio / (1 + ratio)));
  pointsToLine(p1, p, l1);
  ratio = dist(p2, p1) / dist(p2, p3);
  p = translate(p1, scale(toVec(p1, p3),
              ratio / (1 + ratio)));
  pointsToLine(p2, p, l2);
  areIntersect(l1, l2, ctr);
  return 1; }
```

# Excircle Triangle




## Radius of the excircle

A triangle with sides $a, b, c$ and area $A$ has an excircle with radius: $R = abc/4A$.

The center of the excircle is the intersection of the *perpendicular bisectors*.

## Trigonometry

- Law of Cosines:
  $c^2 = a^2 + b^2 - 2ab\cos(\gamma)$
  $\gamma = \text{acos}((a^2 + b^2 - c^2/2ab)$
- Law of Sines: ($R$ is the radius of the excircle):
  $a/\sin(\alpha) = b/\sin(\beta) = c/\sin(\gamma) = R$

# Polygons

## Definition

A polygon is a plane figure bounded by a finite sequence of line segments.

## Polygon Representation

- In general we want to sort the points in CW or CCW order
- Adding the first point at the end of the array helps avoid special cases;

```
// 6 points, entered in counter clockwise order;
vector<point> P;
P.push_back(point(1, 1)); // P0
P.push_back(point(3, 3)); // P1
P.push_back(point(9, 1)); // P2
P.push_back(point(12, 4)); // P3
P.push_back(point(9, 7)); // P4
P.push_back(point(1, 7)); // P5
P.push_back(P[0]); // important: loop back
```

# Polygon Algorithms

## Perimeter of a Poligon – sum of distances

```
double perimeter(const vector<point> &P) {
  double result = 0.0;
  for (int i = 0; i < (int)P.size()-1; i++)
     // remember: P[0] = P[P.size()-1]
     result += dist(P[i], P[i+1]);
  return result; }
```

## Area of a Poligon – half the determinant of the XY matrix

```
double area(const vector<point> &P) {
  double result = 0.0, x1, y1, x2, y2;
  for (int i = 0; i < (int)P.size()-1; i++) {
    x1 = P[i].x; x2 = P[i+1].x;
    y1 = P[i].y; y2 = P[i+1].y;
    result += (x1 * y2 - x2 * y1); }
  return fabs(result) / 2.0; }
```

# Polygon – Concave and Convex check

## Convex Polygons

Has NO line segment with ends inside itself that intersects its edges.

Another definition is that all inside angles “turn” the same way.

## Testing for a convex polygon

```
bool isConvex(const vector<point> &P) {
  int sz = (int)P.size();
  if (sz <= 3) return false; // Not a polygon
  bool isLeft = ccw(P[0], P[1], P[2]); //described earlier
  for (int i = 1; i < sz-1; i++)
    if (ccw(P[i],P[i+1],P[(i+2)==sz? 1 : i+2])!=isLeft)
      return false; // works for both left and right
      // different sign -> this polygon is concave
  return true; }
```

# Polygon – Testing Inside or outside

**There are many ways to test if a point $P$ is in a polygon.**

- Winding Algorithm: Sum the angles of all angles $APB$ ($A$, $B$) are points in the polygon. If the sum is $2\pi$. Point is in polygon.
- Ray Casting Algorithm: Draw an segment from $P$ to infinity, and count the number of polygon edges crossed. Odds: Inside. Even: Outside.

**Winding Algorithm Code**

```
bool inPolygon(point pt, const vector<point> &P) {
  if ((int)P.size() == 0) return false;
  double sum = 0;
  for (int i = 0; i < (int)P.size()-1; i++) {
    if (ccw(pt, P[i], P[i+1]))
      sum += angle(P[i], pt, P[i+1]); //left turn/ccw
      else sum -= angle(P[i], pt, P[i+1]); } //right turn/cw
  return fabs(fabs(sum) - 2*PI) < EPS; }
```

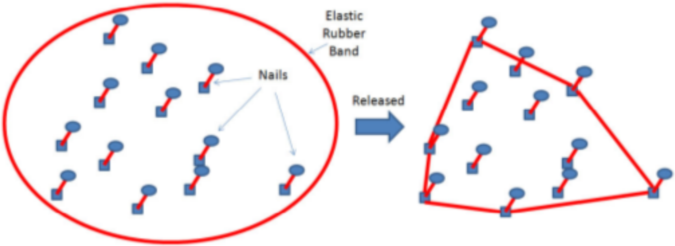
# Polygon – Cutting

To cut $P$ along a line $AB$, we separate the points in $P$ to the left and right of the line.

```
point lineIntersectSeg(point p, point q, point A, point B) {
  double a=B.y-A.y; double b=A.x-B.x; double c=B.x*A.y-A.x*B.y;
  double u=fabs(a*p.x+b*p.y+c); double v=fabs(a*q.x+b*q.y+c);
  return point((p.x*v + q.x*u)/(u+v),
               (p.y*v + q.y*u)/(u+v)); }

vector<point> cutPolygon(point a, point b, const vector<point> &Q){
  vector<point> P;
  for (int i = 0; i < (int)Q.size(); i++) {
    double left1 = cross(toVec(a, b), toVec(a, Q[i])), left2 = 0;
    if (i != (int)Q.size()-1)
      left2 = cross(toVec(a, b), toVec(a, Q[i+1]));
    if (left1 > -EPS)
      P.push_back(Q[i]); //Q[i] is on the left of ab
    if (left1*left2 < -EPS) //edge (Q[i], Q[i+1]) crosses line ab
      P.push_back(lineIntersectSeg(Q[i], Q[i+1], a, b)); }
  if (!P.empty() && !(P.back() == P.front()))
    P.push_back(P.front()); // make P's first point = P's last point
  return P; }
```

# Polygon – Convex Hull

Given a set of points $S$, the convex hull is the polygon $P$ composed of a subset of $S$ so that every point of $S$ is either part of $P$, or inside it.




The main algorithm for calculating the convex hull is *Graham's Scan*.

It's idea is to test each point angle order, to see if the point belongs to the hull.

# Polygon – Graham's Scan

## Helping Functions

```
point pivot(0, 0);

bool angleCmp(point a, point b) { // angle-sorting
  if (collinear(pivot, a, b)) // special case
    return dist(pivot, a) < dist(pivot, b);
  // check which one is closer to X axis
  double d1x = a.x - pivot.x, d1y = a.y - pivot.y;
  double d2x = b.x - pivot.x, d2y = b.y - pivot.y;
  return (atan2(d1y, d1x) - atan2(d2y, d2x)) < 0; }
```

# Polygon – Graham's Scan

## Convex Hull – Initializing the algorithm

```
vector<point> CH(vector<point> P) {
  int i, j, n = (int)P.size();
  // Special Case: Polygon with 3 points
  if (n <= 3) {
    if (!(P[0]==P[n-1])) P.push_back(P[0]);
    return P; }

  // Find Initial Point: Low Y or Right X
  int P0 = 0;
  for (i = 1; i < n; i++)
    if (P[i].y < P[P0].y ||
        (P[i].y == P[P0].y && P[i].x > P[P0].x))
      P0 = i;
  point temp = P[0]; P[0] = P[P0]; P[P0] = temp;
```

# Polygon – Graham's Scan

## Convex Hull – More initialization

```
// second, sort points by angle with pivot P0
pivot = P[0];
sort(++P.begin(), P.end(), angleCmp);

// S holds the Convex Hull
// We initialize it with first three points
vector<point> S;
S.push_back(P[n-1]);
S.push_back(P[0]);
S.push_back(P[1]);

// We start on the third point
i = 2;
```

# Polygon – Graham's Scan

## Convex Hull – Main Loop

```
while (i < n) {
  j = (int)S.size()-1;

  // If the next point is left of CH, keep it.
  // Else, pop the last CH point and try again.

  if (ccw(S[j-1], S[j], P[i]))
    S.push_back(P[i++]);
  else
    S.pop_back();
  }
return S; }
```

# Class Summary

# This Week's Problems

- Sunny Mountains – Line and Points
- Waterfall – Line and Points
- Elevator – Circles and Rectangles
- Colorful Flowers – Circles and Triangles
- Bounding Box – Circles, Triangles and Polygons
- Soya Milk – Rectangle and Triangle
- Trash Removal – Polygon Manipulation
- Board Wrapping – Convex Hull

# About these Slides

# Image Credits I