GB21802 - Programming Challenges

Week 4 - Graph Problems (Part I)

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Results for the Previous Week

Here are the results for last week:

Change of Plans!

I have moved the class **Dynamic Programming II** from Week 4 to Week 9.

Since you all are so good at DP, I will make the last class a "challenge class" mixing DP with the contents of other classes.

Week 4 and 5 – Outline

This Week - Graph I

- Graph Basics review: Concepts and Data Structure;
- Depth First Search and Breadth First Search;
- Problems you solve with DFS and BFS;
- Minimum Spanning Tree: Kruskal and Prim Algorithms (Monday);

Next Week - Graph II

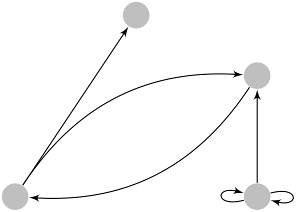
- Single Sourse Shortest Path (Djikstra);
- All Pairs Shortest Path (Floyd Warshall);
- Network Flow and related Problems;
- Bipartite Graph Matching and related Problems;

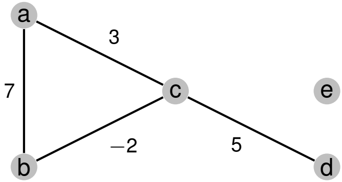
Many variations in graph problems!

Quick Review of Graph Terms (1)

You probably know all of these. If not, ask questions!

- A Graph G is made of a set of vertices V and edges E.
- Edges can be directed (has source and destination vertices);
- Edges can be weighted or not (all weigths = 1);
- Sets of nodes can be connected or disconnected
- Directed Graphs can be Strongly Connected
- Edges can be self-edges, and/or multiple edges

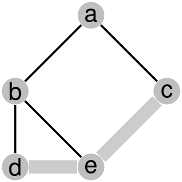


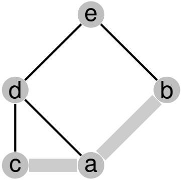


Quick Review of Graph Terms (2)

You probably know all of these. If not, ask questions!

- A path is a set of vertices connected by edges;
- A cycle is a path with first and last vertices identical;
- Labelled graphs and Isomorphic graphs;
- A tree is a acyclical, undirected graph;
- A spanning tree is a subset of edges from E’ that form a tree, connecting all nodes $V \in G$;
- A spamming tree houses very noisy insects in summer;

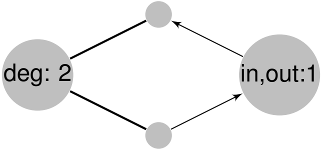


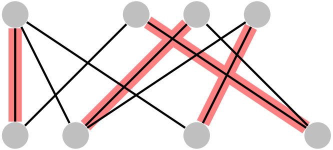


Quick Review of Graph Terms (3)

You probably know all of these. If not, ask questions!

- The degree of a node is the number of edges connected to it;
- Directed graphs have in-degrees and out-degrees;
- A bipartite graph can be divided in two sets of unconnected vertices;
- A Match or Pairing is a set of edges that connects the nodes in the bipartite graph;





Data Structures for Graphs (1)

Adjacency Matrix - Stores connection between Vertices

```
int adj[100][100];
// adj[i][j] is 0 if no edge between i,j
// adj[i][j] is A if edge of weight A links i,j
```

- Pro: Very simple to program, manipulate;
- Con: Cannot store multigraph; Wastes space for sparse graphs; Requires time $O(V)$ to calculate number of neighbors;

Edge List – Stores Edges list for each Vertex

```
typedef pair<int,int> ii;
typedef vector<ii> vii;
vector<vii> AdjList;
```

- Pro: $O(V + E)$ space, efficient if graph is sparse; Can store multigraph;
- Con: A (bit) more code than Adjacency Matrix

Data Structures for Graphs (2)

Edge List

```
vector< pair <int,ii>> Edgelist;
```

Stores a list of all the edges in the graph. Vertices are implicit from the edge list. This is useful for Kruskal's algorithm (which we will see later), but otherwise complicates things.

Implicit Graph

Some graphs do not need to be stored in a special structure if they have very clear rules about when two vertices connect.

Examples:

- A square grid;
- Knight's chess moves;
- Two vertices i, j connect if $i + j$ is prime;

Searching in a Graph: BFS and DFS

Almost all graph problems involve visiting each of its vertices in some form. There are two approaches for visiting the nodes in a graph:

Depth First Search – DFS

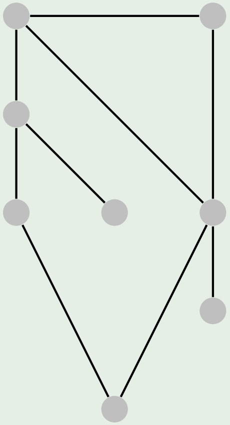
DFS is commonly implemented as a recursive search. For every node visited, immediately visit the first edge in it, backtracking when a loop is reached, or no more edges can be followed.

Breadth First Search – BFS

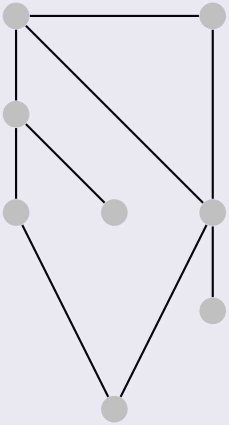
BFS is commonly implemented iterating over a FIFO queue. For every node visited, all new edges are put on the back of the queue. Visit the next edge at the top of the queue.

BFS/DFS: Visualization

DFS

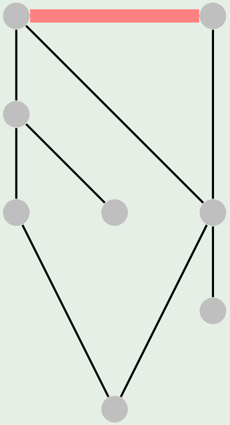


BFS

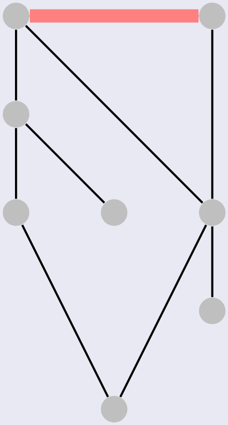


BFS/DFS: Visualization

DFS

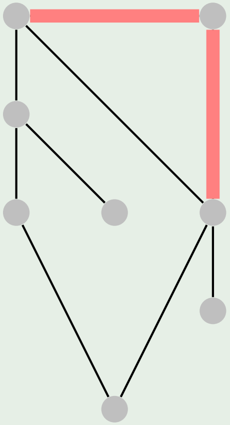


BFS

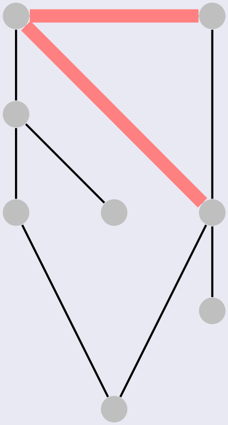


BFS/DFS: Visualization

DFS

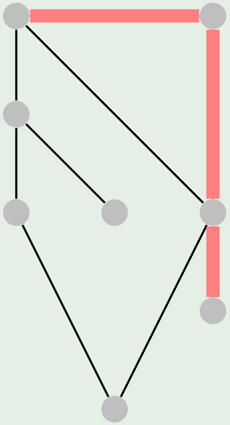


BFS

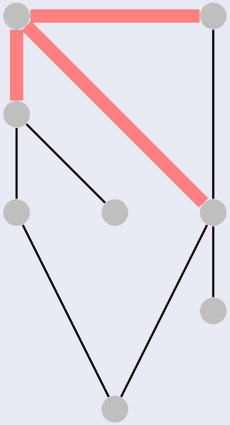


BFS/DFS: Visualization

DFS

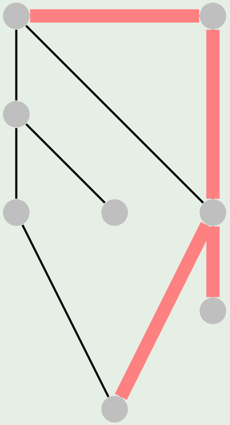


BFS

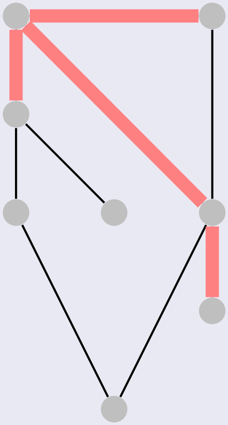


BFS/DFS: Visualization

DFS

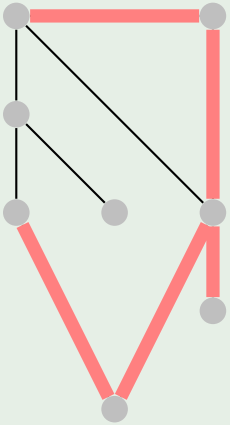


BFS

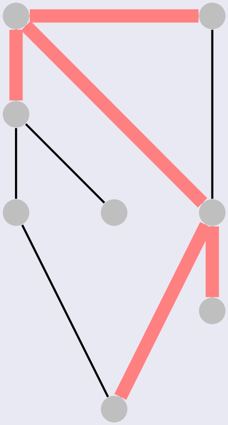


BFS/DFS: Visualization

DFS

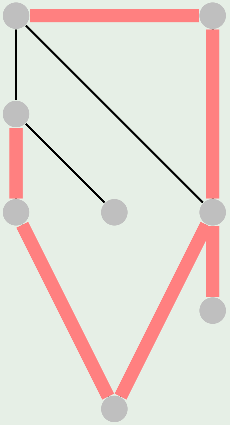


BFS

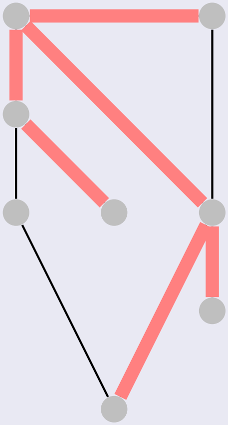


BFS/DFS: Visualization

DFS

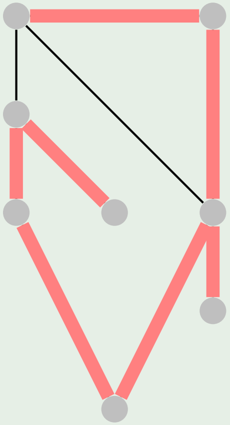


BFS

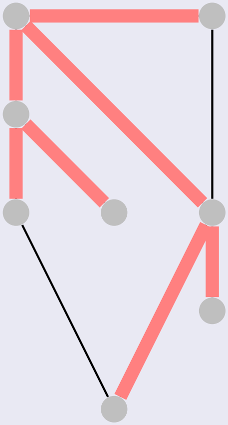


BFS/DFS: Visualization

DFS



BFS



BFS/DFS: Implementation

There are many ways to implement BFS/DFS, here is a suggestion.

DFS

```
vector<int> dfs_vis; // initially all set to UNVISITED
void dfs(int v) {
   dfs_vis = VISITED;
   for (int i; i < (int)Adj_list[v].size(); i++) {
      pair <int,int> u = Adj_list[u][i];
      if (dfs_vis[u.first] == UNVISITED) dfs(v.first)
}}
```

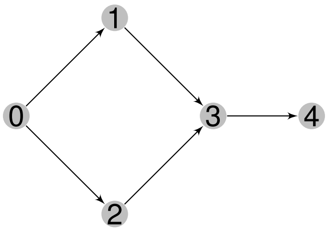
BFS

```
vector<int> d(V,INF); d[s] = 0; queue<int> q; q.push(s);
while(!q.empty()) {
   u = q.front(); q.pop();
   for (int i=0; i < (int)Adj_list[q].size(); i++) {
   pair <int,int> v = Adj_list[u][i]; //same as dfs
   if (d[v.first] == INF) {
      d[v.first] = d[u] + 1; q.push(v.first);
}}}
```

Simple BFS/DFS – UVA 11902: Dominator

Problem Summary

Vertex X dominates vertex Y if every path from a start vertex 0 to Y must go through X. Determine which nodes dominate which other.



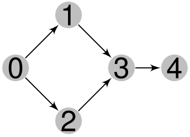
- 0 dominates all nodes;
- 3 dominates 4;
- 1 does not dominate 3;

How do you solve it?

Simple BFS/DFS – UVA 11902: Dominator

Solution

```
DFS(0);
for i in (0:N):
   if i is reached: dominate[0][i] = 1;

for i in (1:N):
   remove i from graph;
   DFS(0)
   for j in (1:N):
       if (j is not reached) and (dominate[0][j] == 1):
           dominate[i][j] = 1
   return i to graph
```

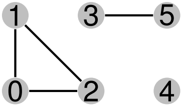


Common Algorithms: Connected Components

With small modifications to BFS/DFS, we can solve many simple problems

Since a single run of DFS/BFS finds all connected nodes, we can use it to find (and count) all the connected components (CC) of an undirected graph.

```
numCC = 0;
dfs_num.assign(V,UNVISITED);
for (int = 0; i < V; i++)
   if (dfs_num[i] == UNVISITED)
      cout << "\nCC " << ++numCC << ":"; dfs(i);
      // modify dfs() to print every node it visits
```




```
CC 1: 0 1 2
CC 2: 3 5
CC 3: 4
```

Flood Fill

A simple twear of the BFS (or DFS) can be used to label/color and count the size of each CC.

“flood fill“ is often used in problems involving implicit 2D grids.

```
####..#
#.###.#
#..@.##
##d.###
#..####
```

```
int dr[] = {1,1,0,-1,-1,-1,0,1}; // trick to explore an
int dc[] = {0,1,1,1,0,-1,-1,-1}; // implicit NESW graph

int floodfill(int y, int x, char c1, char c2) {
  if (y < 0 || y >= R || x < 0 || x >= C) return 0;
  if (grid[y][x] != c1) return 0;
  int ans = 1;
  grid[y][x] = c2;
  for (int d = 0; d < 8; d++)
     ans += floodfill(y+dr[d], x+dc[d], c1, c2);
  return ans;
}
```

Topological Sort (Directed Acyclic Graphs)

A Topological sort is a linear ordering of vertices of a DAG so that vertex u comes before vertex v if edge $u \rightarrow v$ exits in the DAG. Topological Sorts are useful for problems involving the ordering of pre-requisites.

Khan's algorithm for Topological sort (modified edge-BFS)

```
Q = queue(); toposort = list();
for j in edge:
   in_degree[j.destination] += 1
for i in node:
   if in_degree[i] == 0: Q.add(i);
while (Q.size() > 0):
   u = Q.dequeue(); toposort.add(u);
   for i in u.out_edges():
       v = i.destination
       in_degree[v] =- 1
       if in_degree[v] == 0:
          Q.add(v);
```

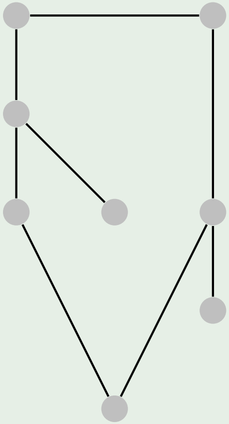
Bipartite Check

To check whether a graph is bipartite, we perform a BFS or DFS on the graph, and set the color of every node to black or white, alternatively. Pay attention to collision conditions.

```
queue<int> q; q.push(s);
vector<int> color(V,INF); color[s] = 0;
bool isBipartite = true;
while (!q.empty() && isBipartite) {
   int u = q.front(); q.pop();
   for (int j=0; j < adj_list[u].size(); j++) {
      pair<int,int> v = adj_list[u][j];
      if (color[v] == INF) {
         color[v.first] = 1 - color[i];
         q.push(v.first);}
      else if (color[v.first] == color[u]) {
         isBipartite = False;
}}}
```

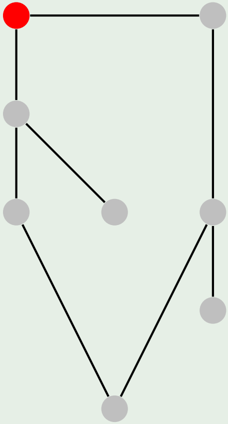
Bipartite Check – Visualization

Testing Bipartite property



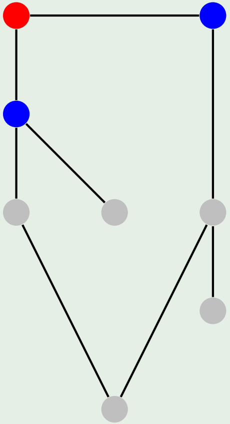
Bipartite Check – Visualization

Testing Bipartite property



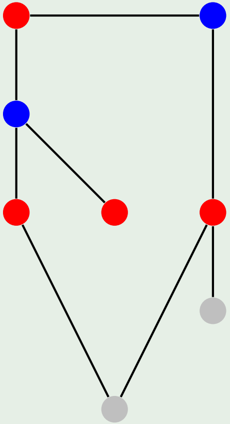
Bipartite Check – Visualization

Testing Bipartite property



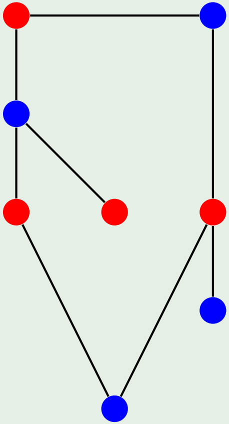
Bipartite Check – Visualization

Testing Bipartite property



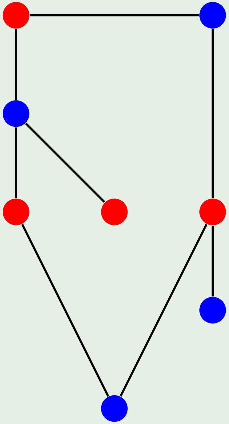
Bipartite Check – Visualization

Testing Bipartite property

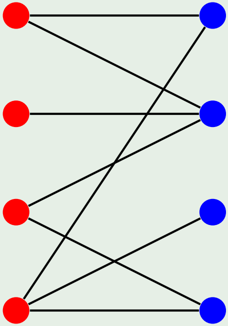


Bipartite Check – Visualization

Testing Bipartite property



Rearranging the nodes

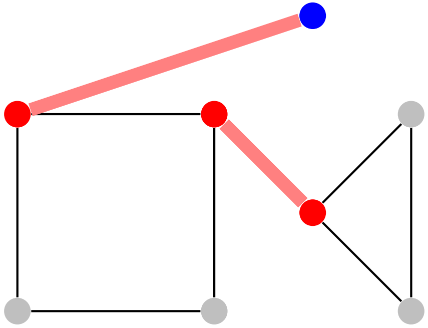


Articulation Points and Bridges

Problem Description

In an undirected graph G:

- A verted V is an Articulation Point if removing V would make G disconnected.
- An edge E is a Bridge if removing E would make G disconnected.



Articulation Points and Bridges: Algorithm

Complete Search algorithm for Articulation Points

1. Run DFS/BFS, and count the number of CC in the graph;
2. For each vertex v, remove v and run DFS/BFS again;
3. If the number of CC increases, v is a connection point;

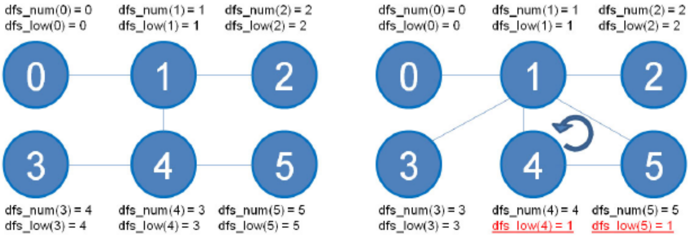
Since DFS/BFS is $O(V + E)$, this algorithm runs in $O(V^2 + EV)$.

... but we can do better!

Tarjan's DFS variant for Articulation point (O(V+E))

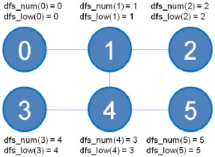
Tarjan Variant: $O(V + E)$

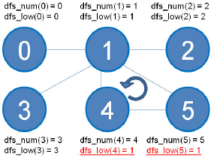
Main idea: Add extra data to the DFS to detect articulations.



- dfs_num[]: Recieves the number of the iteration when this node was reached for the first time;
- dfs_low[]: Recieves the lowest dfs_num[] which can be reached if we start the DFS from here;
- For any neighbors *u*, *v*, if dfs_low[*v*] >= dfs_num[*u*], then *u* is an articulation node.

Tarjan's DFS variant for Articulation point (2)





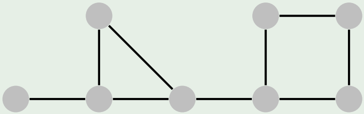

```
void dfs_a(u){
   dfs_num[u] = dfs_low[u] = IterationCounter++;   // dfs_num[u] is a simple counter
   for (int i = 0; i < AdjList[u].size(); i++){
      v = AdjList[u][i];
      if (dfs_num[v] == UNVISITED) {
         dfs_parent[v] = u;                         // store parent
         if (u == 0) rootChildren++:                // special case for root node

         dfs_a(v);
         if (dfs_low[a] >= dfs_num[u])
            articulation_vertex[u] = true;
         dfs_low[u] = min(dfs_low[u],dfs_low[v])

      else if (v != dfs_parent[u])                  // found a cycle edge
         dfs_low[u] = min(dfs_low[u],dfs_num[v])
}}
```

Strongly Connected Components (Directed Graph)

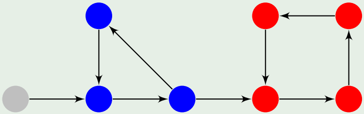
Problem Description

On a directed graph G, a Strongly Connected Component (SCC) is a subset G' where for every pair of nodes $a, b \in G'$, there is both a path $a \to b$ and a path $b \to a$.

One CC



Three SCC



Strongly Connected Components – Algorithm

We can use a simple modification of the algorithm for bridges and articulation points:

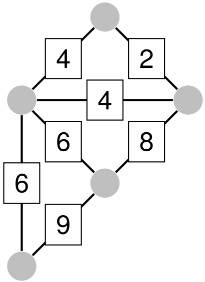
- Every time we visit a new node, put that node in a stack S;
- When we finish visiting a node i, test if dfs_num[i] == dfs_min[i].
- If the above condition is true, i is the root of the SCC. Pop all vertices in the stack as part of the SCC.

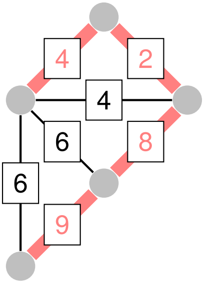
Minimum Spanning Trees (MST)

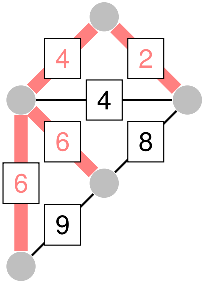
Definition

A Spanning Tree is a subset E' from graph G so that all vertices are connected without cycles.

A Minimum Spanning Tree is a spanning tree where the sum of edge's weights is minimal.







MST – Use cases and Algorithms

Problems using MST

Problems using MST usually involve calculating the minimum costs of infrastructure such as roads or networks.

Some variations may require you to find the maximum spanning tree, or define some edges that must be taken in advance.

Algorithms for MST

The two main algorithms for calculating the MST are the Kruskal's algorithms and the Prim's algorithms.

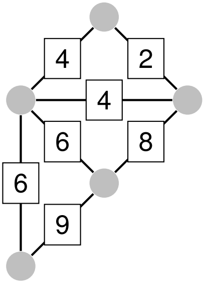
Both are greedy algorithms that add edges to the MST in weight order.

Kruskal's Algorithm

Outline

Kruskal's algorithms sorts all edges by their weight, and try to add each edge to the MST, checking whether adding that edge would create a cycle.

1. Sort all edges;
2. If smallest edge does not create a cycle, add to MST;
3. If smallest edge creates a cycle, remove it from list;
4. Go to 2;



Kruskal's Algorithm

Outline

Kruskal's algorithms sorts all edges by their weight, and try to add each edge to the MST, checking whether adding that edge would create a cycle.

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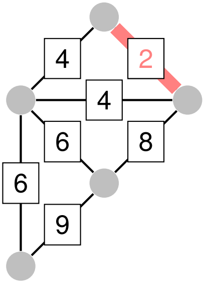

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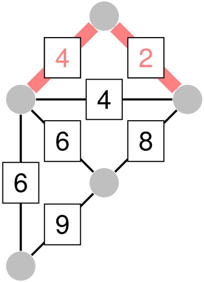

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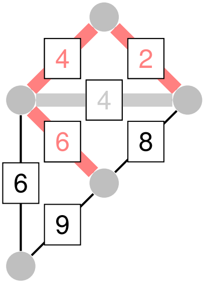

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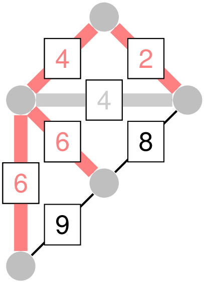

Kruskal's Algorithm – Implementation

```
vector<pair<int, pair<int,int>> Edgelist;
sort(Edgelist.begin(),Edgelist.end());
int mst_cost = 0;
UnionFind UF(V);
  // note 1: Pair object has built-in comparison;
  // note 2: Need to implement UnionSet class;

for (int i = 0; i < Edgelist.size(); i++) {
   pair <int, pair <int,int>> front = Edgelist[i];
   if (!UF.isSameSet(front.second.first,
                     front.second.second)) {
      mst_cost += front.first;
      UF.unionSet(front.second.first,front.second.second)
   }}

cout << "MST Cost: " << mst_cost << "\n"
```

Prim's Algorithm

Outline

Prim's algorith adds nodes to the MST one at a time, and keeps the edges connected to those nodes in a priority queue. It then tests each edge in the priority queue to add more nodes to the MST, avoiding cycles.

1. Add node 0 to MST;
2. Add all edges from new node to Priority Queue;
3. Visit smallest edge in Queue;
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5. Add new edges to Queue;
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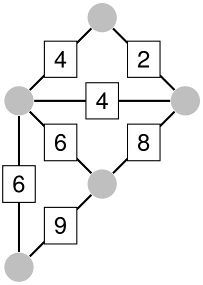

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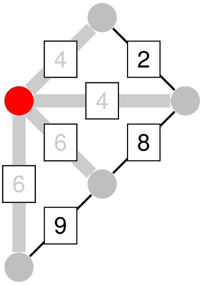

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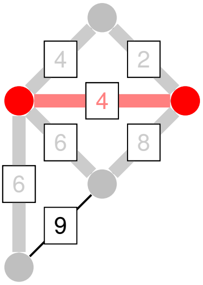

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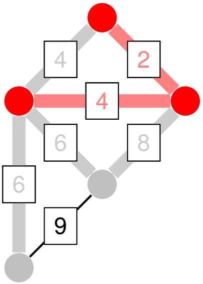

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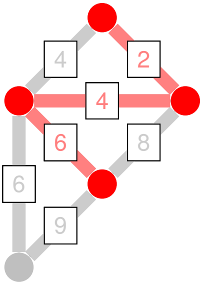

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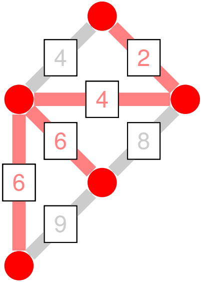


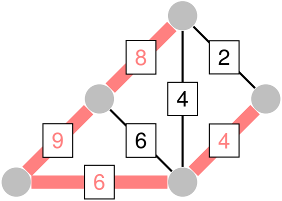
Prim's Algorithm – Implementation

```
vector <int> taken;
priority_queue <pair <int,int>> pq;

void process (int v) {
   taken[v] = 1;
   for (int j = 0; j < (int)AdjList[v].size(); j++) {
      pair <int,int> ve = AdjList[v][j];
      if (!taken[ve.first])
         pq.push(pair <int,int> (-ve.second,-ve.second)
}}
taken.assign(V,0);
process(0);
mst_cost = 0;

while (!pq.empty()) {
  vector <int,int> pq.top(); pq.pop();
  u = -front.secont, w = -front.first;
  if (!taken[u]) mst_cost += w, process(u);
}
```

MST variant 1 – Maximum Spanning tree

The Maximum Spanning Tree variant requires the spanning tree to have maximum possible weight.

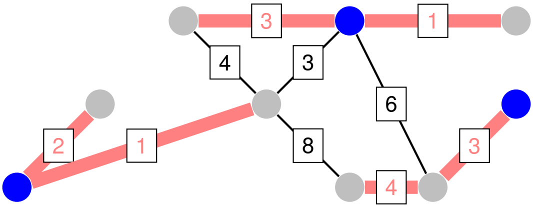
It is very easy to implement the Maximum MST by reversing the sort order of the edges (Kruskal), or the weighting of the priority Queue (Prim).



MST variant 2 – Minimum Spanning Subgraph, Forest

In one importante variant of the MST, a subset of edges or vertices are pre-selected.

- In the case of pre-selected vertices, add them to the "taken" list in Kruskal's algorithm before starting;
- In the case of edges, add the end vertices to the "taken" list;
- What if you are given a number of Connected Components?



MST Variant 3 – *n*th Best MST

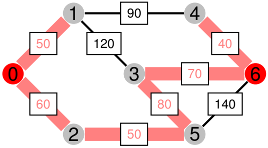
Problem Definition

Consider that you can order MST by their costs: $G_1, G_2, \ldots, G_n$. This variant asks you to calculate the n^{th} best spanning tree.

Basic Idea:

- Calculate the MST (using Kruskal or Prim);
- For every edge in the MST, remove that edge from the graph and calculate a new MST;
- The new MST with minimum weight is the 2^{nd} best MST;

MST Variant 4 – Min-max (or Max-min)



Problem Definition

Given two vertices i, j, find a path $i \to j$ so that the cost of the most expensive edge is minimized;

Another way to write this problem is: Find the cheapest path where the cost of the path is the cost of the most expensive edge.

How to solve

The MST finds the path that connects all nodes while keeping the cost of individual edges minimal.

To solve the minimax problem, we calculate the MST for G, and then find the path from i to j in the MST.

Summary

- Graphs come in a wide variety of types;
- Graph problems also have many different types;
- Most problems involve small modifications of DFS and BFS;

Next Week

More Graphs!

- Shortest Paths (Single Source and All Pairs);
- Network Flow (and related problems);
- Graph Matching (bipartite matching, etc) (and related problems);

This Week's Problems

- Dominator;
- Knight in a War grid;
- Wetlands in Florida;
- Battleships;
- Pick up Sticks;
- Place the Guards;
- Street Directions;
- Dominos;
- Freckles;
- Artic Network;

Problem Hints (0)

- All the problems this week (and next week!) include graphs, and probably need BFS and/or DFS;
- Prepare a “template” of an Adjacency list and DFS/BFS, and put it in the code before starting;
- Try to draw the problem on paper before coding;
- Remember to test “tricky” cases: Graphs with 1 node, disconnected graphs, self-edges, multi-edges;

Problem Hints (1)

Dominator

- Remember: A node is not dominated by anyone if it is not connected to the root (node 0);
- Basic algorithm discussed in class: Calculate all nodes reachable from root. Then remove one node at a time, and node which ones are not reachable anymore;
- If removing node *i* makes node *j* not reachable, then *i* dominates *j*.
- To “remove” a node, modify the DFS(root,i) so that it returns if *i* is reached;

Problem Hints (2)

Knight in a War Grid

- The problem only wants to know which squares are reachable, it is not worried about minimum distance;
- Be careful, M or N can be zero!
- Be careful, if $M == N$, the graph becomes multigraph!
- This graph is implicit, the connections are given by the knight step, the board size, and the impossible squares;

Problem Hints (3)

Wetlands of Florida

- Make a graph with 0 and 1 indicating water or no water;
- Flood-fill the graph at the requested location;
- Multiple-case input is a bit hard to read, make sure to test that;

Battleships

- Scan the graph (double fors).
- For each unvisited 'x' or '@', flood fill the ship (mark visited) and add the ship;
- A ship with only @'s should not be counted.

Problem Hints (4)

Pick up sticks

- The input gives you directed nodes.
- Try to build a topological order (follow the class code)
- Any order is fine. If you find a cycle, print “impossible”

Palace Guards

- Each junction is a node, each street is an edge.
- We have junctions with guards and without guards. (No guard can be near each other)
- There is a solution if the graph is bipartite!
- How do you calculate the smallest number of guards?

Problem Hints (5)

Street Directions

- We have to convert two way streets to one way streets
- Undirected graph to directd graph.
- When is a 2-way street necessary?
- How can you generate 1 way streets?
- Hint: you need to draw the graph on paper

Dominos

- The dominos falling is a directed graph.
- Each domino that falls, we visit one node.
- How many nodes do we need to start, to visit all nodes?

Problem Hints (6)

Freckles

- The problem requires the minimum ink (cost) among all freckles;
- This is straight up MST code;
- Be careful when rounding up values;

Arctic Network

- Also wants to calculate the MST (minimum radio power necessary);
- However, we can use *S* “satellite” links, which cost 0;
- Remember that two stations need a satellite link to talk;