Ahern-Dissertation-AppendixB

November 22, 2015

1 Dynamics

```
In [1]: from scipy.special import beta as beta_func
        from scipy.misc import comb
        def beta_binomial(n, alpha, beta):
            return np.matrix([comb(n-1,k) * beta_func(k+alpha, n-1-k+beta) / beta_func(alpha,beta) for k
        def beta_binomial_expectation(prior):
            return sum([prior[i]*i/len(prior) for i in range(len(prior))])

In [2]: import numpy as np

In [3]: from scipy.integrate import odeint

In [4]: import time

In [5]: import matplotlib.pyplot as plt
        %matplotlib inline

In [6]: from scipy.integrate import odeint
        from scipy.optimize import minimize

In [7]: def discrete_time_replicator_dynamics(n_steps, X, Y, A, B, P):
            """Calculate the discrete-time replicator dynamics for"""
            # Get the number of states, signals, and actions
            X_nrow = X.shape[0]
            X_ncol = X.shape[1]
            Y_nrow = Y.shape[0] # Same as X_ncol
            Y_ncol = Y.shape[1] # Often, but not necessarily, the same as X_nrow
            # Create empty arrays to hold the population states over time
            X_t = np.empty(shape=(n_steps, X_nrow*X_ncol), dtype=float)
            Y_t = np.empty(shape=(n_steps, X_nrow*X_ncol), dtype=float)
            # Set the initial state
            X_t[0,:] = X.ravel()
            Y_t[0,:] = Y.ravel()
            # Iterate forward over (n-1) steps
            for i in range(1,n_steps):
                # Get the previous state
                X_prev = X_t[i-1,:].reshape(X_nrow, X_ncol)
                Y_prev = Y_t[i-1,:].reshape(Y_nrow, Y_ncol)
                # Calculate the scaling factors
                E_X = A * Y_prev.T
                X_bar = (((A * Y_prev.T) * X_prev.T).diagonal()).T
                X_hat = E_X / X_bar
                #
```

```
            C = np.divide(np.multiply(P.T, X_prev), (P * X_prev)[0])
            E_Y = (B.T * C).T
            Y_bar = ((E_Y*Y_prev.T).diagonal()).T
            Y_hat = np.divide(E_Y, Y_bar)
            # Calculate next states
            X_t[i,:] = np.multiply(X_prev, X_hat).ravel()
            Y_t[i,:] = np.multiply(Y_prev, Y_hat).ravel()
        return X_t, Y_t
```

In [8]:
```
def U_S(state, action, b):
    return 1 - (action - state - (1-state)*b)**2
    #return 1 - abs(action - state - (1-state)*b)
def U_R(state, action):
    return 1 - (action - state)**2
# Define functions to map integers to interval [0,1]
def t(i, n):
    return i/float(n)
def a(i, n):
    return i/float(n)
```

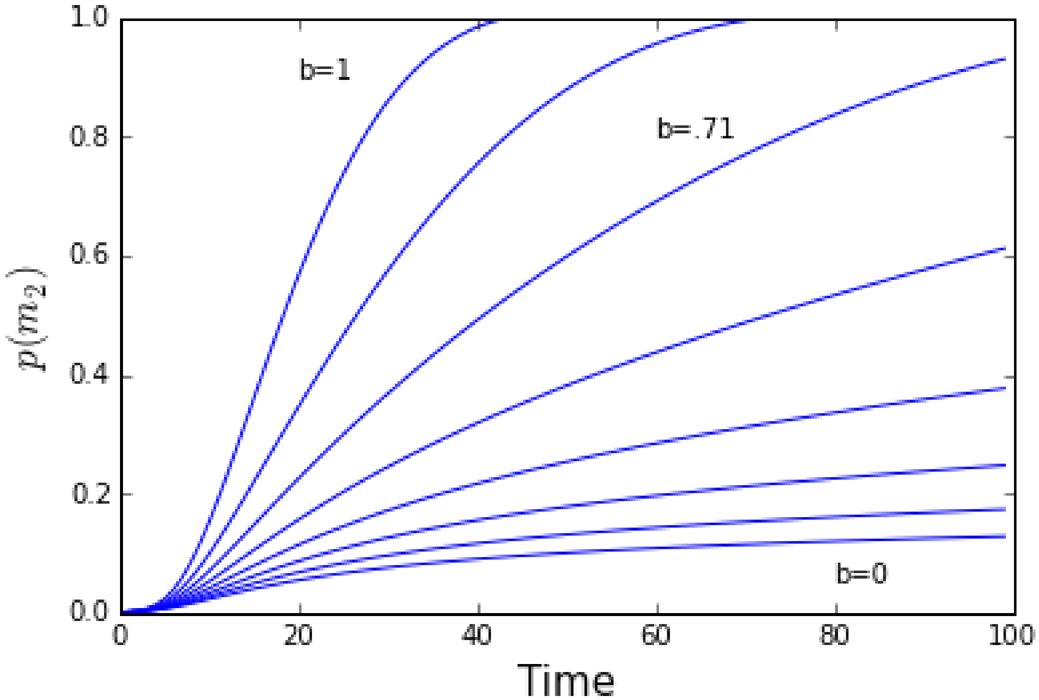
In [9]:
```
number=200
prior = beta_binomial(number, 1, 2)
```

In [13]:
```
np.linspace(0,1, num=8)
```

Out[13]:
```
array([ 0.        ,  0.14285714,  0.28571429,  0.42857143,  0.57142857,
        0.71428571,  0.85714286,  1.        ])
```

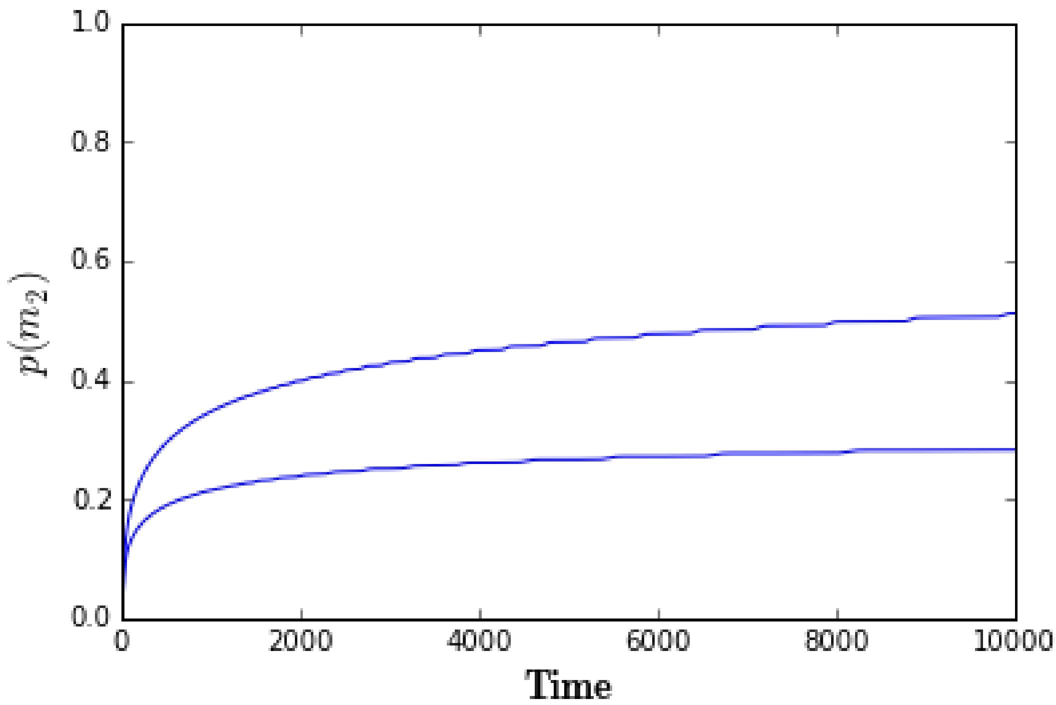
In [17]:
```
hfont = {'fontname':'Helvetica'}
for k in np.linspace(0, 1, num=8): # 10, 2, 2, .2
    #print k
    # Define number of states
    number = 200
    # Define prior probability
    prior = beta_binomial(number, 1, 2)
    P = np.repeat(prior, 2, axis=0)
    # Define payoff matrices
    b = k
    A =  np.matrix([[U_S(t(i, number-1), a(j,number-1), b) for j in range(number)] for i in ran
    B = np.matrix([[U_R(t(i, number-1), a(j,number-1)) for j in range(number)] for i in range(r
    # Define sender population
    X0_m2 = beta_binomial(number, 10, 1)
    X0_m1 = 1 - X0_m2
    X0 = np.vstack((X0_m1, X0_m2)).T
    # Define receiver population
    Y0 = np.vstack((beta_binomial(number, 1, 2), beta_binomial(number, 28, 2)))
    # Solve and plot
    X_sol, Y_sol = discrete_time_replicator_dynamics(100, X0, Y0, A, B, P)
    m2_sol = [prior.dot(line)[0,0] for line in X_sol[:,1::2]]
    plt.plot(m2_sol, 'b')
plt.ylim(0,1)
plt.xlabel('Time', fontsize=15, **hfont)
plt.ylabel(r'$p(m_2)$', fontsize=15, **hfont)
plt.text(20, .9, r'b=1')
plt.text(60, .8, r'b=.71')
```

```
plt.text(80, .05, r'b=0')
plt.savefig("replicator-multiple-b.eps", format='eps', dpi=1000)
plt.show()
```




```
In [14]: hfont = {'fontname':'Helvetica'}
         for k in [0, .16]: # 10, 2, 2, .2
             #print k
             # Define number of states
             number = 200
             # Define prior probability
             prior = beta_binomial(number, 1, 2)
             P = np.repeat(prior, 2, axis=0)
             # Define payoff matrices
             b = k
             A =  np.matrix([[U_S(t(i, number-1), a(j,number-1), b) for j in range(number)] for i in ran
             B = np.matrix([[U_R(t(i, number-1), a(j,number-1)) for j in range(number)] for i in range(
             # Define sender population
             X0_m2 = beta_binomial(number, 10, 1)
             X0_m1 = 1 - X0_m2
             X0 = np.vstack((X0_m1, X0_m2)).T
             # Define receiver population
             Y0 = np.vstack((beta_binomial(number, 1, 2), beta_binomial(number, 28, 2)))
             # Solve and plot
             X_sol, Y_sol = discrete_time_replicator_dynamics(10000, X0, Y0, A, B, P)
             m2_sol = [prior.dot(line)[0,0] for line in X_sol[:,1::2]]
             plt.plot(m2_sol, 'b')
         plt.ylim(0,1)
         plt.xlabel('Time', fontsize=15, **hfont)
```

```
plt.ylabel(r'$p(m_2)$', fontsize=15, **hfont)
#plt.savefig("replicator-multiple-b.eps", format='eps', dpi=1000)
plt.show()
```



2 Modeling

```
In [18]: import pandas as pd
         import numpy as np
         df1 = pd.read_csv("./first_curve_data.csv")
```

```
In [19]: def loss_first(params):
             # Unpack the paramters
             a_x = params[0]
             b_y1 = params[1]
             b_y2 = params[2]
             b = params[3]
             k=1 # Change this to alter scaling parameter
             # Construct initial states
             # Number of states and actions
             number = 200
             # Define prior probability
             prior = beta_binomial(number, 1, 2)
             P = np.repeat(prior, 2, axis=0)
             # Define payoff matrices
             A =  np.matrix([[U_S(t(i, number-1), a(j,number-1), b) for j in range(number)] for i in ran
             B = np.matrix([[U_R(t(i, number-1), a(j,number-1)) for j in range(number)] for i in range(r
             # Define sender population
```

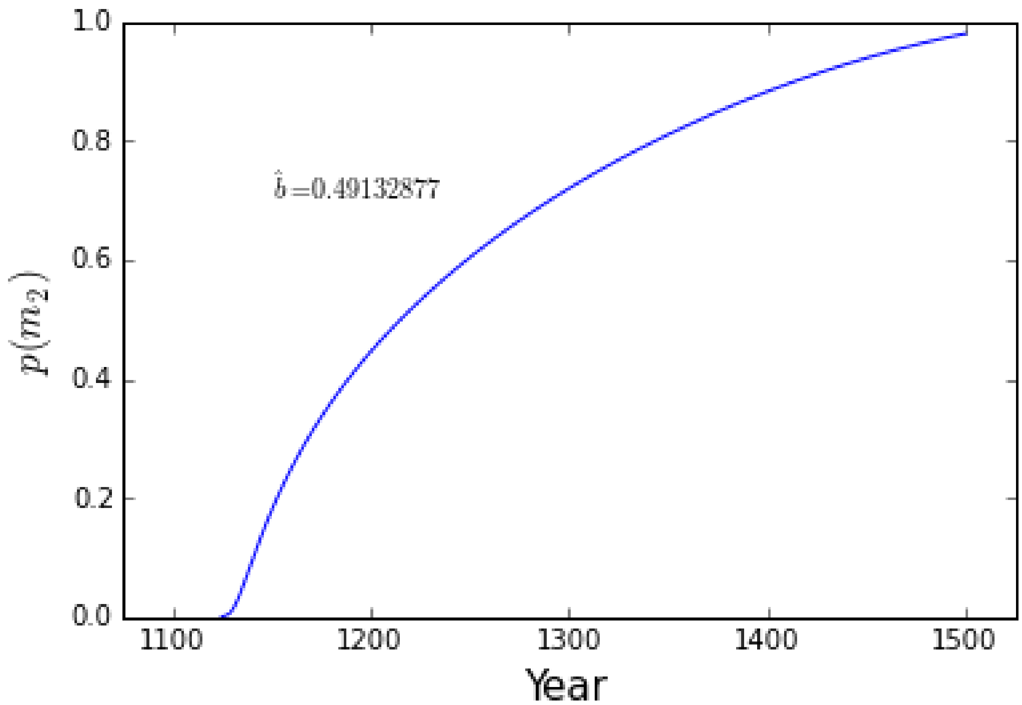
```
        X0_m2 = beta_binomial(number, a_x, 1)#b_x)
        X0_m1 = 1 - X0_m2
        X0 = np.vstack((X0_m1, X0_m2)).T
        # Calculate expected state given m2
        p_ti_m2 = np.multiply(X0[:,1], prior.T)
        p_m2 = prior * X0[:,1]
        p_t_m2 = p_ti_m2 / p_m2
        E_t_m2 = (np.array(range(200)) * p_t_m2) / number
        scale = E_t_m2[0,0]/(1 - E_t_m2[0,0])
        # Define receiver population
        Y0 = np.vstack((beta_binomial(number, .5*b_y1, b_y1), beta_binomial(number, scale*b_y2, b_y
        # Iterate through the discrete-time replicator dynamics
        X_sol, Y_sol = discrete_time_replicator_dynamics(376, X0, Y0, A, B, P)
        # Get p(m_2) over time
        m2_sol = [prior.dot(line)[0,0] for line in X_sol[:,1::2]]
        # Get data indices
        indices = np.subtract(df1.year.unique(), min(df1.year))
        # Get count of tokens for indices
        counts = df1.groupby('year').size() #.loc[index + 1125]
        # Get count of value==1 for indices
        values = df1.groupby('year').aggregate(np.sum)
        # Initialize RSS value
        RSS = 0
        # Loop over indices
        for index in indices:
            zero_count = counts.loc[index + min(df1.year)] - values.loc[index + min(df1.year), 'val
            one_count = values.loc[index + min(df1.year), 'value']
            #
            RSS += zero_count*(0 - m2_sol[index])**2 + one_count*(1 - m2_sol[index])**2
        return RSS
```

In [20]:
```
# Start with initial guesses and bounds
guess = np.array([10, 2, 2, .2])
bnds = ((1, 100), (1, 100), (1, 100), (0,1))
# Start timer
start = time.time()
# Solve for various numbers of iterations
res = minimize(loss_first, x0=guess, method="COBYLA", bounds=bnds, options={"maxiter" : 2000,
end = time.time()
elapsed = end - start
m, s = divmod(elapsed, 60)
print res.message + " " + "%02d:%02d" % (m, s)
print res
```

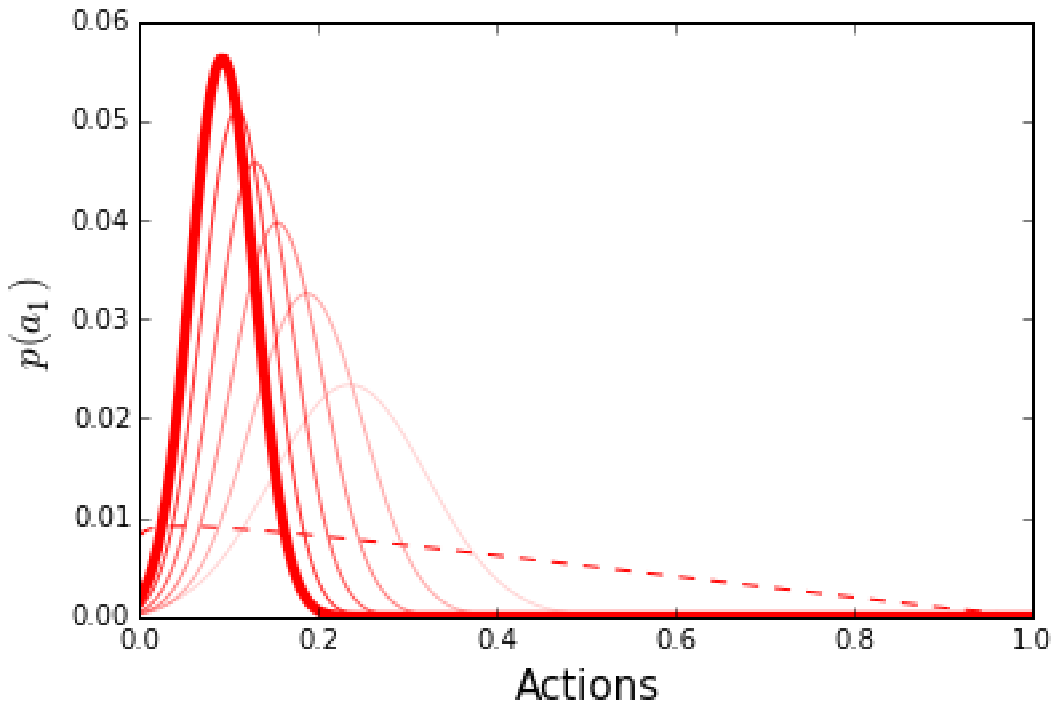
```
Optimization terminated successfully. 12:52
  status: 1
    nfev: 1509
   maxcv: 0.0
 success: True
     fun: 566.75226355693087
       x: array([ 9.57057411,  2.11689308,  2.96493063,  0.49132877])
 message: 'Optimization terminated successfully.'

//anaconda/lib/python2.7/site-packages/scipy/optimize/_minimize.py:397: RuntimeWarning: Method COBYLA ca
  RuntimeWarning)
```

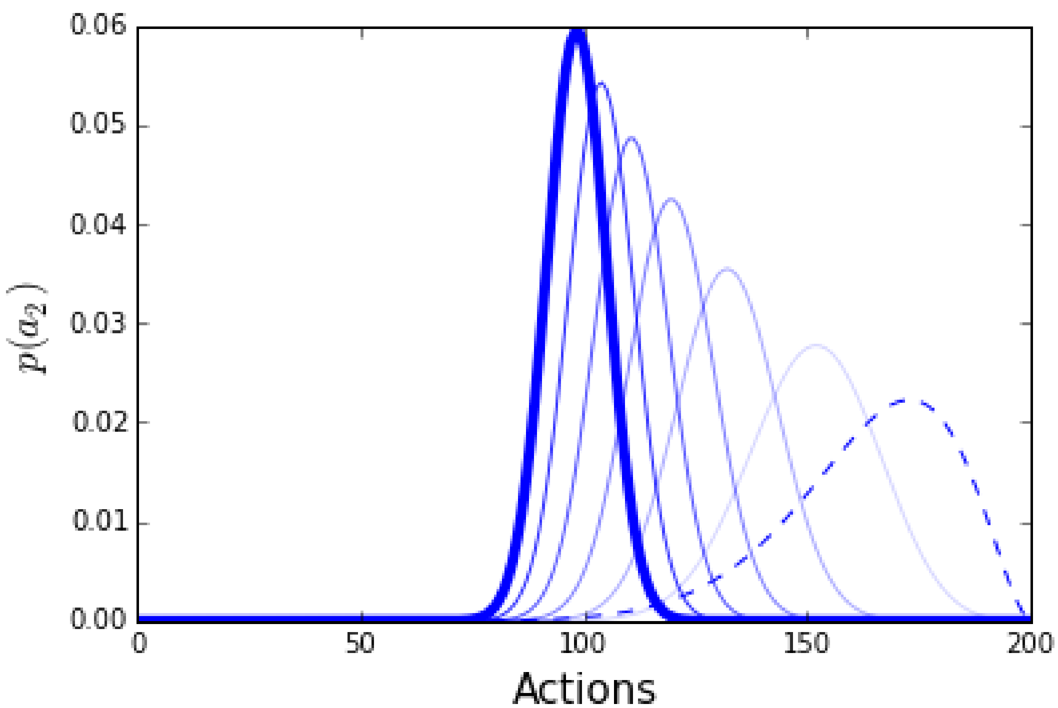
Use the following if you don't want to run the following optimization.

```
In [21]: # short_res = res.x # Use this to define the shortcut
         short_res = [ 9.57057411,  2.11689308,  2.96493063,  0.49132877]

In [43]: b = short_res[3]
         A =  np.matrix([[U_S(t(i, number-1), a(j,number-1), b) for j in range(number)] for i in range(n
         B = np.matrix([[U_R(t(i, number-1), a(j,number-1)) for j in range(number)] for i in range(numbe
         # Define sender population
         X0_m2 = beta_binomial(number,  res.x[0], 1)
         X0_m1 = 1 - X0_m2
         X0 = np.vstack((X0_m1, X0_m2)).T

         p_ti_m2 = np.multiply(X0[:,1], prior.T)
         p_m2 = prior * X0[:,1]
         p_t_m2 = p_ti_m2 / p_m2
         E_t_m2 = (np.array(range(200)) * p_t_m2) / number
         scale = E_t_m2[0,0]/(1 - E_t_m2[0,0])

         # Define receiver population
         Y0 = np.vstack((beta_binomial(number, .5*short_res[1], short_res[1]),\
                       beta_binomial(number, scale*short_res[2], short_res[2])))
         # Solve and plot
         X_sol, Y_sol = discrete_time_replicator_dynamics(376, X0, Y0, A, B, P)
         m2_sol = [prior.dot(line)[0,0] for line in X_sol[:,1::2]]
         years = [1125 + item for item in range(376)]
         plt.plot(years, m2_sol, 'b')
         plt.ylim(0,1)
         plt.xlim(1075, 1525)
         plt.xlabel('Year', fontsize=15, **hfont)
         plt.ylabel(r'$p(m_2)$', fontsize=15)
         plt.text(1150, .7, r'$\hat{b}=0.49132877$')
         plt.savefig("m2_sol.eps", format='eps', dpi=1000)
         plt.show()
```



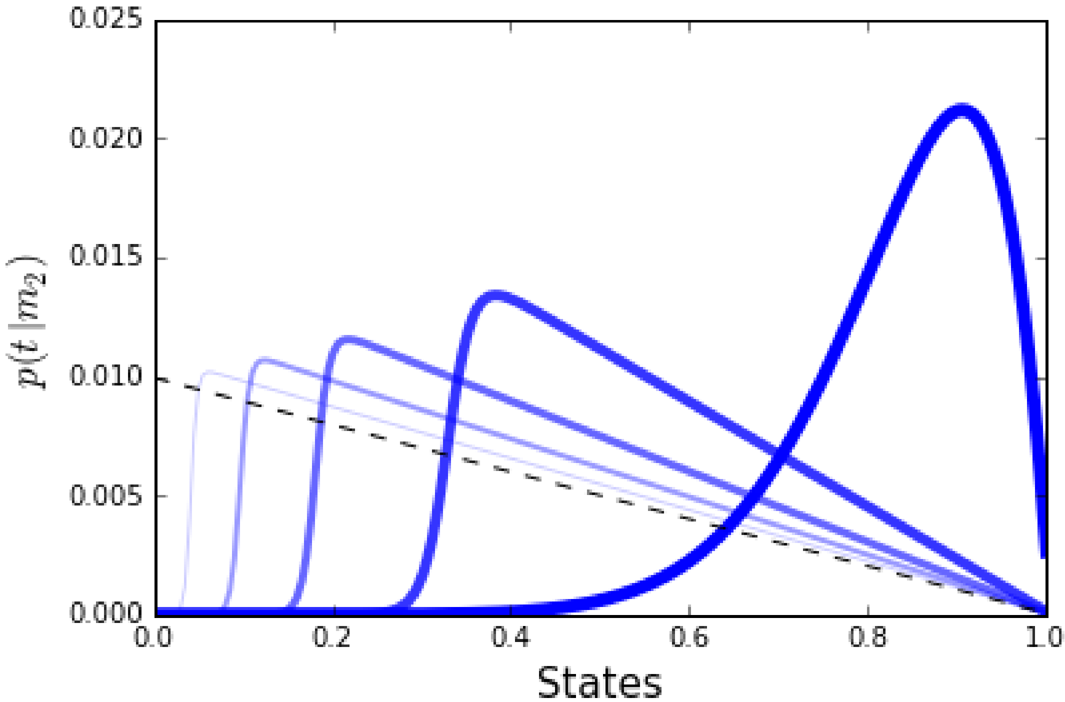
In [24]:
```
timesteps=376
states = np.linspace(0,1, num=200)
for i in range(1,timesteps, timesteps/6):
    plt.plot(states, Y_sol[i,:200], color='r', alpha=(i/float(timesteps)))
plt.plot(states, Y_sol[0,:200], 'r--')
plt.plot(states, Y_sol[-1,:200], 'r', linewidth=4)
plt.ylabel(r'$p(a_1)$', fontsize=15, **hfont)
plt.xlabel('Actions', fontsize=15, **hfont)
plt.savefig("a1_rd.eps", format='eps', dpi=1000)
plt.show()
#for line in Y_sol[:,:200]:
#    plt.plot(line)
#plt.show()
for i in range(1,timesteps, timesteps/6):
    plt.plot(Y_sol[i,200:], color='b', alpha=(i/float(timesteps)))
plt.plot(Y_sol[0,200:], 'b--')
plt.plot(Y_sol[-1,200:], 'b', linewidth=4)
plt.ylabel(r'$p(a_2)$', fontsize=15, **hfont)
plt.xlabel('Actions', fontsize=15, **hfont)
plt.savefig("a2_rd.eps", format='eps', dpi=1000)
plt.show()
```





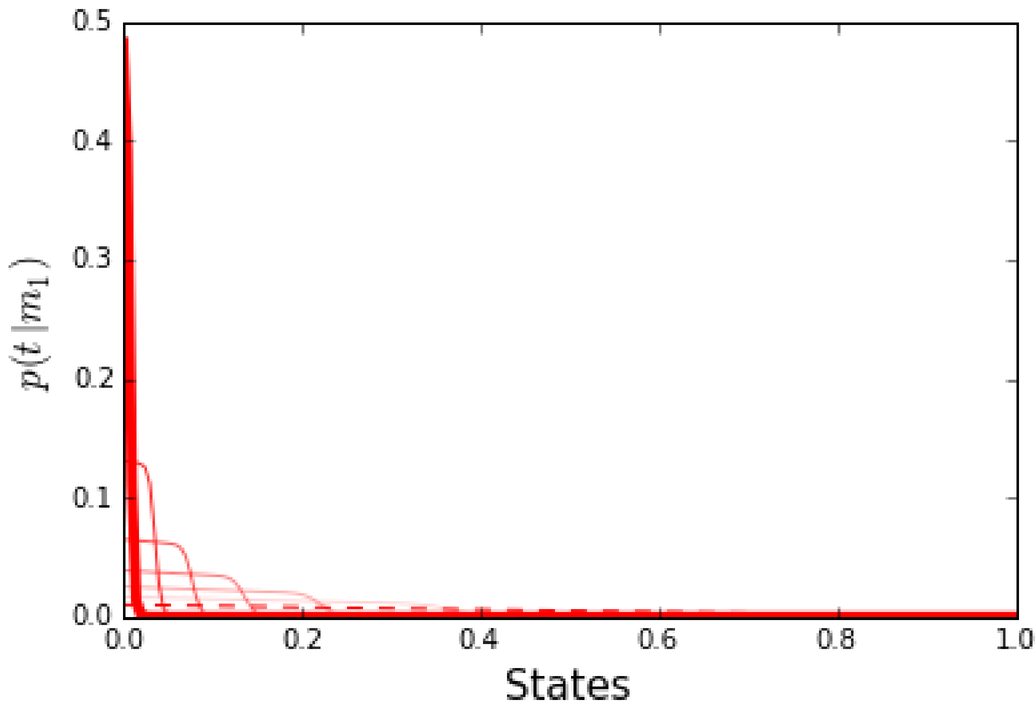

```
In [41]: for j, i in enumerate(range(1,timesteps, timesteps/5)):
             p_ti_m2 = np.multiply(X_sol[i,1::2], prior)
```

```
        p_m2 =  X_sol[i,1::2] * prior.T
        p_t_m2 = p_ti_m2 / p_m2
        plt.plot(states, p_t_m2.tolist()[0], color='b', linewidth=5-j, alpha=(1 - (i/float(timeste
    # First state
    #p_ti_m2 = np.multiply(X_sol[0,1::2], prior)
    #p_m2 =  X_sol[0,1::2] * prior.T
    #p_t_m2 = p_ti_m2 / p_m2
    #plt.plot(states, p_t_m2.tolist()[0], 'b--')
    # Last state
    #p_ti_m2 = np.multiply(X_sol[-1,1::2], prior)
    #p_m2 =  X_sol[-1,1::2] * prior.T
    #p_t_m2 = p_ti_m2 / p_m2
    #plt.plot(states, p_t_m2.tolist()[0], 'b', linewidth=4)
    #
    plt.plot(states, prior.tolist()[0], 'k--')
    plt.ylabel(r'$p(t \mid m_2)$', fontsize=15, **hfont)
    plt.xlabel('States', fontsize=15, **hfont)
    plt.savefig("p_t_m2_rd.png", format='png')
    plt.show()
```




```
In [44]: for i in range(1,timesteps, timesteps/6):
             p_ti_m1 = np.multiply(X_sol[i,0::2], prior)
             p_m1 =  X_sol[i,0::2] * prior.T
             p_t_m1 = p_ti_m1 / p_m1
             plt.plot(states, p_t_m1.tolist()[0], color='r', alpha=(i/float(timesteps)))
         # First state
         p_ti_m1 = np.multiply(X_sol[0,0::2], prior)
         p_m1 =  X_sol[0,0::2] * prior.T
```

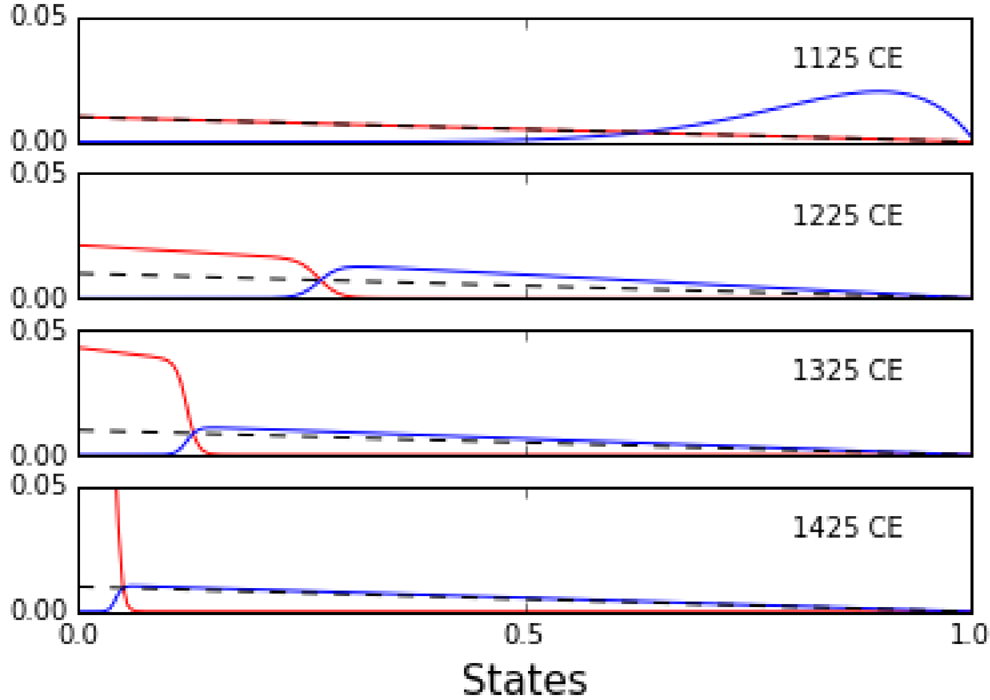
```
p_t_m1 = p_ti_m1 / p_m1
plt.plot(states, p_t_m1.tolist()[0], 'r--')
# Last state
p_ti_m1 = np.multiply(X_sol[-1,0::2], prior)
p_m1 =  X_sol[-1,0::2] * prior.T
p_t_m1 = p_ti_m1 / p_m1
plt.plot(states, p_t_m1.tolist()[0], 'r', linewidth=4)
#
plt.ylabel(r'$p(t \mid m_1)$', fontsize=15, **hfont)
plt.xlabel('States', fontsize=15, **hfont)
plt.savefig("p_t_m1_rd.png", format='png')
plt.show()
```




```
In [63]: # Push-chain
f, (ax1, ax2, ax3, ax4) = plt.subplots(4, sharex=True, sharey=True)
i=0
p_ti_m1 = np.multiply(X_sol[i,0::2], prior)
p_m1 =  X_sol[i,0::2] * prior.T
p_t_m1 = p_ti_m1 / p_m1
ax1.plot(states, p_t_m1.tolist()[0], 'r')
p_ti_m2 = np.multiply(X_sol[i,1::2], prior)
p_m2 =  X_sol[i,1::2] * prior.T
p_t_m2 = p_ti_m2 / p_m2
ax1.plot(states, p_t_m2.tolist()[0], 'b')
ax1.plot(states, prior.tolist()[0], 'k--')
ax1.text(.8, .03, str(1125+i)+' CE', **hfont)
#ax.annotate('Test', xy=(1, 0), xycoords='axes fraction', fontsize=16,
 #                horizontalalignment='right', verticalalignment='bottom')
```

```
#ax1.set_ylabel('common xlabel')
#ax1.ylim(0,.1)
#plt.show()
#
i=100
#plt.subplot(3,2,1)
p_ti_m1 = np.multiply(X_sol[i,0::2], prior)
p_m1 =  X_sol[i,0::2] * prior.T
p_t_m1 = p_ti_m1 / p_m1
ax2.plot(states, p_t_m1.tolist()[0], 'r')
p_ti_m2 = np.multiply(X_sol[i,1::2], prior)
p_m2 =  X_sol[i,1::2] * prior.T
p_t_m2 = p_ti_m2 / p_m2
ax2.plot(states, p_t_m2.tolist()[0], 'b')
ax2.plot(states, prior.tolist()[0], 'k--')
ax2.text(.8, .03, str(1125+i)+' CE', **hfont)
#ax2.ylim(0,.1)
#plt.show()
#
i=200
#plt.subplot(3,3,1)
p_ti_m1 = np.multiply(X_sol[i,0::2], prior)
p_m1 =  X_sol[i,0::2] * prior.T
p_t_m1 = p_ti_m1 / p_m1
ax3.plot(states, p_t_m1.tolist()[0], 'r')
p_ti_m2 = np.multiply(X_sol[i,1::2], prior)
p_m2 =  X_sol[i,1::2] * prior.T
p_t_m2 = p_ti_m2 / p_m2
ax3.plot(states, p_t_m2.tolist()[0], 'b')
ax3.plot(states, prior.tolist()[0], 'k--')
ax3.text(.8, .03, str(1125+i)+' CE', **hfont)
#
i=300
#plt.subplot(3,3,1)
p_ti_m1 = np.multiply(X_sol[i,0::2], prior)
p_m1 =  X_sol[i,0::2] * prior.T
p_t_m1 = p_ti_m1 / p_m1
ax4.plot(states, p_t_m1.tolist()[0], 'r')
p_ti_m2 = np.multiply(X_sol[i,1::2], prior)
p_m2 =  X_sol[i,1::2] * prior.T
p_t_m2 = p_ti_m2 / p_m2
ax4.plot(states, p_t_m2.tolist()[0], 'b')
ax4.plot(states, prior.tolist()[0], 'k--')
ax4.text(.8, .03, str(1125+i)+' CE', **hfont)
#
f.subplots_adjust(hspace=.25)
plt.locator_params(nbins=2)
plt.setp([a.get_xticklabels() for a in f.axes[:-1]], visible=False)
#
#plt.ylabel(r'£p(t \mid m)£', fontsize=15, **hfont)
plt.xlabel('States', fontsize=15, **hfont)
#
plt.ylim(0,.05)
plt.savefig("push_chain.eps", format='eps', dpi=1000)
```

```
plt.show()
```




```
In [1]: from scipy.stats import beta

In [35]: def f(params):
             return abs((1-.49) - beta.cdf(.5, params[0], params[1]))
         x0 = [1,2]
         b_fit = minimize(f, x0, method="Nelder-Mead")
         b_fit.x[0]/b_fit.x.sum()

Out[35]: 0.49339524133564144

In [18]: .28/(.5 + .28)

Out[18]: 0.358974358974359

In [ ]: def b_fit(params):
            # Unpack the paramters
            a = params[0]
            b = params[1]
            RSS = beta.cdf(0, .5, a, b)
            return RSS
```