Solving a logistic equation with varying growth rate

Consider a logistic growth function with a carrying capacity and a varying growth rate, where $p, \alpha, \beta \in [0, 1]$.

$$\dot{p} = p(1-p)\frac{\beta - \alpha}{(1-p)\alpha + p\beta} \tag{1}$$

The parameters α and β determine the sign of the growth rate. If $\beta > \alpha$ then the growth rate is positive, if $\beta < \alpha$ then the growth rate is negative.

Assuming for now that $\beta > \alpha$, we know that (1) exhibits a growth rate between the following two differential equations.

$$\dot{p}_\alpha = p(1-p)\frac{\beta - \alpha}{\alpha} \tag{2}$$

$$\dot{p}_\beta = p(1-p)\frac{\beta - \alpha}{\beta} \tag{3}$$

In fact, we can express the original equation as a combination of these other two, where $\sigma = \frac{p\beta}{(1-p)\alpha + p\beta}$.

$$\dot{p} = (1-\sigma)\dot{p}_\alpha + \sigma\dot{p}_\beta \tag{4}$$

Starting from an initial state with small p, the growth rate of (1) will initially look like $\dot{p}_\alpha$, but then become more like $\dot{p}_\beta$. That is, the growth rate will slow down over time.

We can visualize these as in Figure 1. The growth rate of the original differential equation is always between the other two. Intuitively, this means that any solution to $\dot{p}$ will be between the solutions to $\dot{p}_\alpha$ and $\dot{p}_\beta$.

Turning now to finding the general solution of the original differential equation, we can integrate and take antilogarithms, resulting in the following, where K comes from a constant of integration.

$$\frac{p^\alpha}{(1-p)^\beta} = Ke^{(\beta-\alpha)t} \tag{5}$$

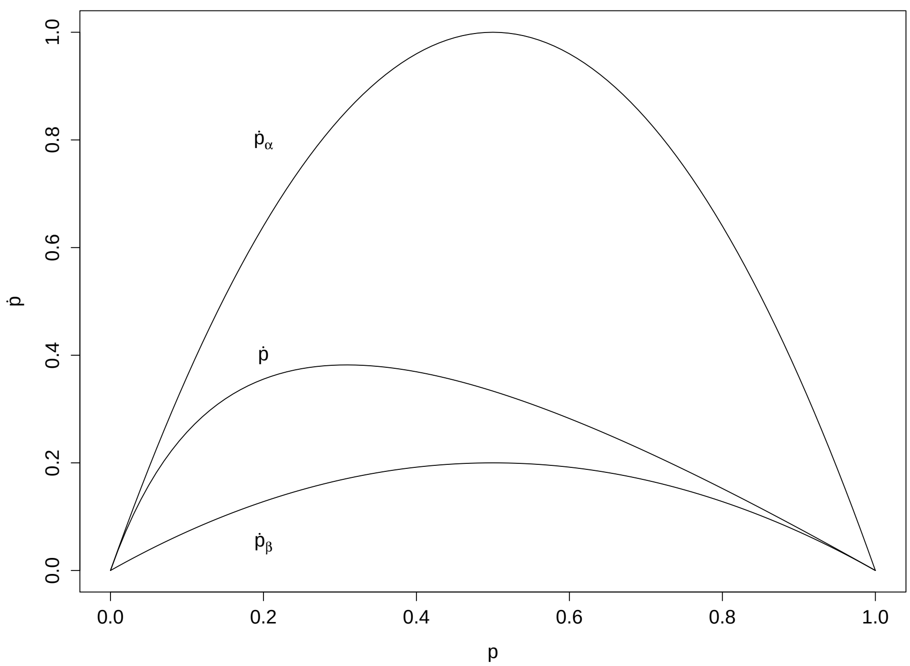

Figure 1: $\dot{p}_\alpha$ and $\dot{p}_\beta$ for $\alpha = .1, \beta = .5$

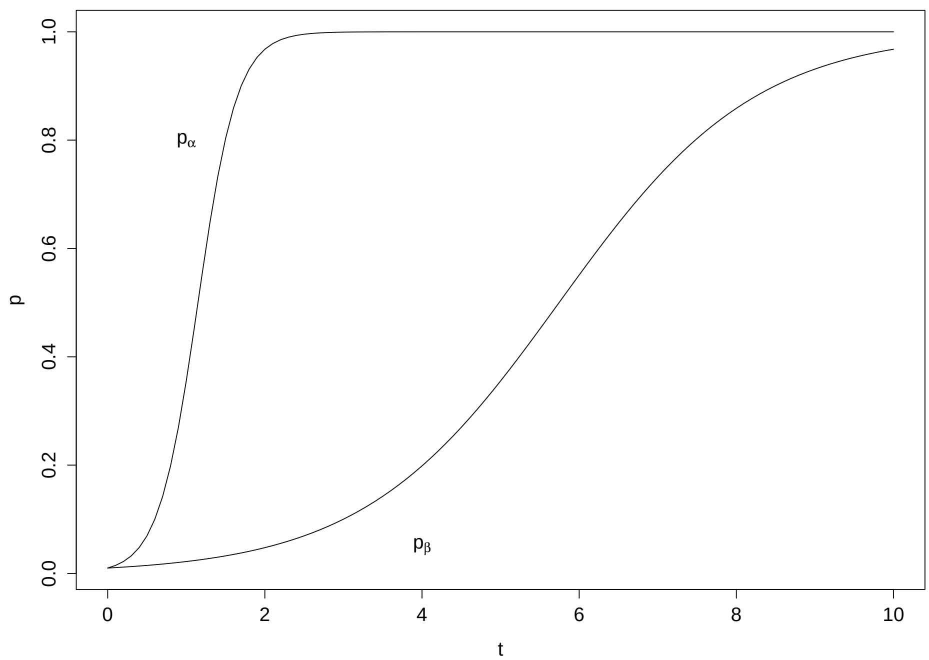

Figure 2: p_α and p_β for $p_0 = .01, \alpha = .1, \beta = .5$

Is there a means of solving for p? If not a closed-form solution, are there approximations that retain at least some of the information about the α and β?

For a constant growth rate, γ, the solution to the logistic is the following, where K results from a constant of integration.

$$p_\gamma = \frac{Ke^{\gamma t}}{1 + Ke^{\gamma t}} \tag{6}$$

Where $p(t_0) = p_0$, $K = \left(\frac{p_0}{1-p_o}\right)$, we have the following form for a particular solution.

$$p_\gamma = \frac{\left(\frac{p_0}{1-p_o}\right) e^{\gamma t}}{1 + \left(\frac{p_0}{1-p_o}\right) e^{\gamma t}} \tag{7}$$

We can visualize these two solutions, as in Figure 2.

Ideally, we'd be able to interpolate these solutions in a reasonable manner.