Inductive proof examples

CS161

July 16, 2016

1 Sum from 1 to n

We want to prove that $1 + 2 + ... + n$ can be calculated with $\frac{n(n+1)}{2}$

1.1 Definitions

We will define a function to let us talk about the sum of numbers from 1 to n. Let:

$$F(n) = 1 + 2 + ... + n \tag{1}$$

We will define a predicate to let us talk about the relationship between $F(n)$ and the shortcut calculation. Let:

$$P(n) : F(n) = \frac{n(n+1)}{2} \tag{2}$$

Note that $P(n)$ evaluates to a boolean. It can be true or false for any particular n. It is true for a particular value of n if $F(n)$ does in fact equal $\frac{n(n+1)}{2}$ and it is false if these two things are not equal.

1.2 Goal

Our goal is to prove that $P(n)$ holds (is true) for all values of n greater than 0. Prove:

$$\forall n \in N : P(n) \tag{3}$$

1.3 Proof by induction

1.3.1 Base case

To show our base case $P(1)$ is true, we will state the base case, then show that the left side does in fact equal the right side. Prove:

$$P(1) : F(1) = \frac{1(1+1)}{2} \tag{4}$$

$$F(1) = 1$$

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1$$

1.3.2 Inductive step

We will prove that **if** $P(k)$ holds (is true) for some $k \in N$, **then** $P(k+1)$ is also true. Prove:

$$P(k) \implies P(k+1), \forall k \in N \tag{5}$$

We start with the *inductive hypothesis*, we assume for the time that $P(k)$ holds. Assume:

$$P(k) : F(k) = \frac{k(k+1)}{2} \tag{6}$$

Now, assuming that $P(k)$ is true, prove:

$$P(k+1) : F(k+1) = \frac{(k+1)((k+1)+1)}{2} \tag{7}$$

By definition:

$$F(k+1) = 1 + 2 + ... + k + (k+1)$$

which is by definition:

$$F(k+1) = F(k) + (k+1)$$

which by our inductive hypothesis is:

$$F(k+1) = \frac{k(k+1)}{2} + (k+1)$$

simplifying is:

$$F(k+1) = (k+1)(\frac{k}{2} + 1)$$

which is equivalent to:

$$F(k+1) = (k+1)(\frac{k}{2} + \frac{2}{2})$$

which simplifies to:

$$F(k+1) = \frac{(k+1)(k+2)}{2}$$

which is clearly:

$$F(k+1) = \frac{(k+1)((k+1)+1)}{2}$$

And so we have proved $P(k+1)$ (7) by showing that the left side is equal to the right side (assuming that $P(k)$ is true).

1.4 Conclusion

We have proved that $P(n)$ holds for a base case of $P(1)$ and that for all $k \in N$, $P(k)$ being true implies that $P(k+1)$ is also true. Therefore $P(n)$ holds for all $n > 0$ (all natural numbers).

$$P(1) : F(1) = \frac{1(1+1)}{2}$$

$$P(k) \implies P(k+1), \forall k \in N$$

$$\therefore P(n), \forall n \in N$$

2 Making postage with 3 and 5 cent stamps

We want to prove that all postage amounts greater than 7 cents can be made with combinations of 3 and 5 cent stamps

2.1 Definitions

We will define a predicate to talk about whether a particular number can be represented as a summation of a non-negative multiple of 3 and a non-negative multiple of 5.

$$P(n) : n = 3a + 5b \mid a, b \in \mathbb{Z}_{\geq 0} \tag{8}$$

Note that $P(n)$ may be true or false for any given number n. For example, $P(2)$ is false, as 2 cents of postage cannot be made with 3 and 5 cent stamps. However, $P(11)$ is true, because 11 cents of postage can be made with a 5 cent stamp and two 3 cent stamps.

2.2 Goal

Our goal is to prove that $P(n)$ holds for all values of n greater than 7.

$$\forall n \in \mathbb{Z}_{>7} : P(n) \tag{9}$$

2.3 Proof by induction

2.3.1 Base case

To show that the base case $P(8)$ is true, we will state the base case, then show that we can find suitable non-negative integers a and b. Prove:

$$P(8) : 8 = 3a + 5b \tag{10}$$

$$a, b = 1$$

2.3.2 Inductive step

We will prove that **if** $P(k)$ holds (is true) for some $k \in \mathbb{Z}_{>7}$, **then** $P(k+1)$ is also true. Prove:

$$P(k) \implies P(k+1), \forall k \in \mathbb{Z}_{>7} \tag{11}$$

We start with the *inductive hypothesis*, we assume for the time that $P(k)$ holds. Assume:

$$P(k) : k = 3a + 5b \mid a, b \in \mathbb{Z}_{\geq 0} \tag{12}$$

Now, assuming that $P(k)$ is true, prove:

$$P(k+1) : k + 1 = 3c + 5d \mid c, d \in \mathbb{Z}_{\geq 0} \tag{13}$$

If $b > 0$, then $d = b - 1$ and $c = a + 2$ results in:

$$3c + 5d = 3(a + 2) + 5(b - 1) = 3a + 5b + 1 = k + 1$$

By our inductive hypothesis, we assumed that a and b were non-negative integers, and so c and d will be too.

If $b = 0$, then k must be a multiple of 3. The smallest multiple of 3 in $\mathbb{Z}_{>7}$ is 9, which is $3 * 3$. All other multiples of 3 in $\mathbb{Z}_{>7}$ will have a greater or equal number of 3 in their all-3 representation ($b = 0$). Therefore if $b = 0$, then $a \geq 3$. So $d = b + 2$ and $c = a - 3$ results in:

$$3c + 5d = 3(a - 3) + 5(b + 2) = 3a + 5b + 1 = k + 1$$

By our inductive hypothesis and our reasoning that a must be greater than or equal to 3 in the case where $b = 0$, then c and d will be non-negative integers too.

And so we have proved $P(k + 1)$ (13) by assuming $P(k)$ (12) was true. If $P(k)$, then $P(k + 1$ (11).

2.4 Conclusion

We have proved that $P(n)$ holds for a base case of $P(8)$ and that for all $k \in \mathbb{Z}_{>7}$, $P(k)$ being true implies that $P(k+1)$ is also true. Therefore $P(n)$ holds for all $n > 7$.

$$P(8) : 8 = 3(1) + 5(1)$$

$$P(k) \implies P(k + 1), \forall k \in \mathbb{Z}_{>7}$$

$$\therefore P(n), \forall n \in \mathbb{Z}_{>7}$$

3 Proof with inequality

We want to prove that for any integer n greater than 6, $n!$ is greater than 3^n.

$$n! > 3^n \mid n > 6$$

3.1 Definitions

We will define a predicate to let us talk about an inequality relationship between $n!$ and 3^n. Let:

$$P(n) : n! > 3^n \tag{14}$$

Note that $P(n)$ may be true or false for any particular choice of n. For example, $P(2)$ is false because $2!$ is not greater than 3^2. But $P(7)$ is true because $7!$ (5040) **is** greater than 3^7 (2187).

3.2 Goal

Our goal is to prove that $P(n)$ holds for all values of n greater than 6.

$$\forall n \in \mathbb{Z}_{>6} : P(n) \tag{15}$$

3.3 Proof by induction

3.3.1 Base case

To show that the base case $P(7)$ is true, we will state the base case, then show that the inequality is true. Prove:

$$P(7) : 7! > 3^7 \tag{16}$$

$$7! = 5040 > 2187 = 3^7$$

3.3.2 Inductive step

We will prove that **if** $P(k)$ holds (is true) for some $k \in \mathbb{Z}_{>6}$, **then** $P(k+1)$ is also true. Prove:

$$P(k) \implies P(k+1), \forall k \in \mathbb{Z}_{>6} \tag{17}$$

We start with the *inductive hypothesis*, we assume for the time that $P(k)$ holds. Assume:

$$P(k) : k! > 3^k \mid k \in \mathbb{Z}_{>6} \tag{18}$$

Now, assuming that $P(k)$ is true, prove:

$$P(k+1) : (k+1)! > 3^{k+1} \tag{19}$$

Starting with the left side:

$$(k+1)! = (k+1)k!$$

Using our inductive hypothesis:

$$(k+1)! > (k+1)3^k$$
$$(k+1)! > \frac{1}{1}(k+1)3^k$$
$$(k+1)! > \frac{3}{3}(k+1)3^k$$
$$(k+1)! > \frac{k+1}{3}3 * 3^k$$
$$(k+1)! > \frac{k+1}{3}3^1 * 3^k$$
$$(k+1)! > \frac{k+1}{3}3^{k+1}$$

And if $\frac{k+1}{3}$ is greater than 1:

$$(k+1)! > \frac{k+1}{3}3^{k+1} > 3^{k+1}$$

therefore:

$$(k+1)! > 3^{k+1}$$

Since $k \in \mathbb{Z}_{>6}$, we know that $\frac{k+1}{3}$ will always be greater than $\frac{6}{3}$ which is greater than 1. And so we have proved $P(k+1)$ (19) by assuming $P(k)$ (18) was true. If $P(k)$, then $P(k+1$ (17).

3.4 Conclusion

We have proved that $P(n)$ holds for a base case of $P(7)$ and that for all $k \in \mathbb{Z}_{>6}$, $P(k)$ being true implies that $P(k+1)$ is also true. Therefore $P(n)$ holds for all $n > 6$.

$$P(7) : 7! > 3^7$$
$$P(k) \implies P(k+1), \forall k \in \mathbb{Z}_{>6}$$
$$\therefore P(n), \forall n \in \mathbb{Z}_{>6}$$

4 Proof of divisibility

This is related to our stamp problem. We want to prove that for any integer n greater than or equal to 0, $3^{2n} - 1$ is divisible by 8.

$$3^{2n} - 1 \equiv 0 \mod 8$$

4.1 Definitions

We will define a predicate to let us talk about the divisibility of our expression. Let:

$$P(n) : 3^{2n} - 1 \equiv 0 \mod 8 \tag{20}$$

Note that $P(n)$ may be true or false for any particular choice of n. For example, $P(0)$ is true because $3^0 - 1 = 0$ is divisible by 8. But $P(-1)$ is false because $3^{-2} - 1 = -\frac{8}{9}$ is not divisible by 8.

4.2 Goal

Our goal is to prove that $P(n)$ holds for all values of n greater than or equal to 0.

$$\forall n \in \mathbb{Z}_{\geq 0} : P(n) \tag{21}$$

4.3 Proof by induction

4.3.1 Base case

To show that the base case $P(0)$ is true, we will state the base case, then show it is true. Prove:

$$P(0) : 3^{2(0)} - 1 \equiv 0 \mod 8 \tag{22}$$
$$3^0 - 1 = 0$$

4.3.2 Inductive step

We will prove that **if** $P(k)$ holds (is true) for some $k \in \mathbb{Z}_{\geq 0}$, **then** $P(k+1)$ is also true. Prove:

$$P(k) \implies P(k+1), \forall k \in \mathbb{Z}_{\geq 0} \tag{23}$$

We start with the *inductive hypothesis*, we assume for the time that $P(k)$ holds. Assume:

$$P(k) : 3^{2k} - 1 \equiv 0 \mod 8 \mid k \in \mathbb{Z}_{\geq 0} \tag{24}$$

Now, assuming that $P(k)$ is true, prove:

$$P(k+1) : 3^{2(k+1)} - 1 \equiv 0 \mod 8 \tag{25}$$

Starting with the left side:

$$3^{2(k+1)} - 1 = 3^{2k+2} - 1$$

$$3^{2(k+1)} - 1 = 3^{2k} * 9 - 1$$

$$3^{2(k+1)} - 1 = 3^{2k} * (8+1) - 1$$

$$3^{2(k+1)} - 1 = 3^{2k} * 8 + 3^{2k} - 1$$

The left expression, $(3^{2k} * 8)$ is clearly divisible by 8 because any whole number multiplied by 8 is divisible by 8. (We know it is a whole number because k is non-negative). And the right expression $(3^{2k} - 1)$ is, by our *inductive hypothesis*, divisible by 8. So

$$3^{2(k+1)} - 1 = 8a + 8b \mid a, b \in \mathbb{Z}$$

$$3^{2(k+1)} - 1 = 8(a+b) \mid a, b \in \mathbb{Z}$$

Therefore $3^{2(k+1)} - 1$ is divisible by 8, because any whole number times 8 is divisible by 8. And so we have proved $P(k+1)$ (25) by assuming $P(k)$ (24) was true. If $P(k)$, then $P(k+1)$ (23).

4.4 Conclusion

We have proved that $P(n)$ holds for a base case of $P(0)$ and that for all $k \in \mathbb{Z}_{\geq 0}$, $P(k)$ being true implies that $P(k+1)$ is also true. Therefore $P(n)$ holds for all $n \geq 0$.

$$P(0) : 3^{2(0)} - 1 \equiv 0 \mod 8$$

$$P(k) \implies P(k+1), \forall k \in \mathbb{Z}_{\geq 0}$$

$$\therefore P(n), \forall n \in \mathbb{Z}_{\geq 0}$$

5 Another inequality proof

Prove that for all integers $n \geq 4$ that $n^2 \leq n!$

5.1 Definitions

Let:

$$P(n) : n^2 \leq n! \tag{26}$$

5.2 Goal

We will prove

$$\forall n \in \mathbb{Z}_{\geq 4} : P(n) \tag{27}$$

5.3 Proof by induction

5.3.1 Base case

$$\begin{aligned} P(4) : 4^2 &\leq 4! \\ 16 &\leq 24 \end{aligned} \tag{28}$$

5.3.2 Inductive step

We will prove:

$$P(k) \implies P(k+1), \forall k \in \mathbb{Z}_{\geq 4} \tag{29}$$

Inductive hypothesis, assume:

$$P(k) : k^2 \leq k! \tag{30}$$

Now prove:

$$P(k+1) : (k+1)^2 \leq (k+1)! \tag{31}$$

Starting with the left side:

$$(k+1)^2 = k^2 + 2k + 1$$

Using our inductive hypothesis:

$$(k+1)^2 \leq k! + 2k + 1$$

If we can prove that for $k \geq 4$:

$$P'(k) : k! + 2k + 1 \leq (k+1)!$$

then we will have proved $P(k+1)$.

We can rewrite $P'(k)$ to equivalent expressions:

$$P'(k) : 2k + 1 \leq (k+1)! - k!$$

$$P'(k) : 2k + 1 \leq (k+1)k! - k!$$
$$P'(k) : 2k + 1 \leq ((k+1) - 1)k!$$
$$P'(k) : 2k + 1 \leq k * k!$$

$$P'(k) : 1 + \frac{1}{2k} \leq \frac{k * k!}{2k}$$
$$P'(k) : 1 + \frac{1}{2k} \leq \frac{k!}{2}$$

If $k > 0$, then $1 + \frac{1}{2k} < 2$. And if $k > 2$, then $\frac{k!}{2} = k * (k-1) * ... * 3$ which is greater than 2. So for $k \geq 4$:

$$1 + \frac{1}{2k} < 2 < \frac{k!}{2}$$

Therefore $P'(k)$ and $P(k+1)$ are true.

5.4 Conclusion

We have proved that $P(n)$ holds for a base case of $P(4)$ and that for all $k \in \mathbb{Z}_{\geq 4}$, $P(k)$ implies $P(k+1)$. Therefore $P(n)$ holds for all $n \geq 4$.

6 Another divisibility proof

Prove that 6 evenly divides $n^3 - n$ for every non-negative integer n.

6.1 Definitions

Let:

$$P(n) : n^3 - n \equiv 0 \mod 6 \tag{32}$$

6.2 Goal

We will prove that $P(n)$ holds for all values of n greater than or equal to 0.

$$\forall n \in \mathbb{Z}_{\geq 0} : P(n) \tag{33}$$

6.3 Proof by induction

6.3.1 Base case

$$P(0) : 0^3 - 0 \equiv 0 \mod 6 \tag{34}$$
$$0^3 - 0 = 0$$

6.3.2 Inductive step

We will prove:

$$P(k) \implies P(k+1), \forall k \in \mathbb{Z}_{\geq 0} \tag{35}$$

Inductive hypothesis, assume:

$$P(k) : k^3 - k \equiv 0 \mod 6 \tag{36}$$

Now prove:

$$P(k+1) : (k+1)^3 - (k+1) \equiv 0 \mod 6 \tag{37}$$

Starting with the left side:

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1)$$

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 2k$$
$$(k+1)^3 - (k+1) = k^3 - k + 3k^2 + 3k$$
$$(k+1)^3 - (k+1) = (k^3 - k) + 3(k^2 + k)$$

By our *inductive hypothesis* the expression $k^3 - k$ is divisible by 6. If we can prove that $k^2 + k$ is a multiple of 2 (even), then the expression $3(k^2 + k)$ is also divisible by 6, proving $P(k+1)$.

If k is even, then $k^2 + k$ is even, and if k is odd then $k^2 + k$ is also even. Therefore $k^2 + k$ must be a multiple of 2.

6.4 Conclusion

We have proved that $P(n)$ holds for a base case of $P(0)$ and that for all $n \in \mathbb{Z}_{\geq 0}$, $P(n)$ being true implies that $P(n+1)$ is also true. Therefore $P(n)$ holds for all $n \geq 0$.