Pre-test practice recitation

CS161

July 16, 2016

1 General Instructions

Refer to the PDF in the class code (july-11/resources):
Inductive Proof Examples

2 Counting

1. How many non-negative integers less than 1000 (written with no leading zeros, but 0 is written as 0):
   (a) have exactly three digits? $1000 - 100$
   (b) have an odd number of digits? $10 + (1000 - 100)$
   (c) have at least one digit equal to 9? $1000 - 9^3$
   (d) have no odd digits? 5^3
   (e) have two consecutive fives? $10 + 10 - 1$
   (f) are palindromes? $90 + 9 + 10$

2. How many ways are there to choose a dozen donuts from 20 varieties (this will **not** be on the test, it's weird [and cool!!])
   (a) if there are no two donuts of the same variety? $\binom{20}{12}$
   (b) if all donuts are of the same variety? 20
   (c) if there are no restrictions? $\binom{20+12-1}{12}$
   (d) if there are at least two varieties? $\binom{20+12-1}{12} - 20$
   (e) if there must be at least six blueberry donuts? $\binom{20+6-1}{6}$

3. In how many different ways can five elements be selected in order from a set with three elements when repetition is allowed? 3^5

4. How many strings of six letters are there?

   (a) if only lowercase is allowed? 26^6

   (b) if any case is allowed? 52^6

5. In how many ways can you stack 7 different books, so that a specific book B is on the third place? ${}^6P_6$

6. In how many ways can you take 3 marbles out of a box with 15 different marbles? $\binom{15}{3}$

7. In how many ways can you take 5 cards, with at least 2 aces, out of a deck of 52 cards? All hands minus hands with no aces or one ace:

$$\binom{52}{5} - \left(\binom{48}{5} + \binom{4}{1} * \binom{48}{4}\right)$$

8. Find n if

   (a) ${}^nP_2 = 110$, then $n = 11$

   (b) ${}^nP_n = 5040$, then $n = 7$

   (c) ${}^nP_4 = 12 * {}^nP_2$, then $n = 6$

9. Find n if

   (a) $\binom{n}{2} = 45$, then $n = 10$

   (b) $\binom{n}{3} = {}^nP_2$, then $n = 8$

   (c) $\binom{n}{5} = \binom{n}{2}$, then $n = 7$

10. If the numbers from 1 to 1000 are written out on a piece of paper, how many 9's are on that paper?

    Numbers with one nine multiplied by one, numbers with two nines multiplied by two, and numbers with three nines multiplied by three. The first expression in parentheses reads as "choose a place to put the 9 then multiply by the ways you can fill in the other two spaces with digits that aren't 9's.

$$1 * \left(\binom{3}{1} * 9^2\right) + 2 * \left(\binom{3}{2} * 9\right) + 3 * \binom{3}{3}$$

3 Proofs

1. Prove that for all positive integers, n:

$$\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + ... + \frac{2}{3^n} = 1 - \frac{1}{3^n}$$

Let:

$$F(n) = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + ... + \frac{2}{3^n}$$

$$P(n) : F(n) = 1 - \frac{1}{3^n}$$

We will prove that $P(n)$ holds for n greater than 0.

$$\forall n \in N : P(n)$$

Base case:

$$P(1) : \frac{2}{3^1} = 1 - \frac{1}{3^1}$$

$$\frac{2}{3} = \frac{2}{3}$$

Inductive step. We want to prove $P(k) \implies P(k+1)$ for $k \in N$.

Inductive hypothesis, assume:

$$P(k) : F(k) = 1 - \frac{1}{3^k}$$

Now prove:

$$P(k+1) : F(k+1) = 1 - \frac{1}{3^{k+1}}$$

Starting with the left side:

$$\begin{aligned} F(k+1) &= \frac{2}{3} + \frac{2}{9} + ... + \frac{2}{3^k} + \frac{2}{3^{k+1}} \\ &= F(k) + \frac{2}{3^{k+1}} \\ &= 1 - \frac{1}{3^k} + \frac{2}{3^{k+1}} \\ &= 1 - \frac{3}{3^{k+1}} + \frac{2}{3^{k+1}} \\ &= 1 - \frac{1}{3^{k+1}} \end{aligned}$$

We have shown that a base case of $P(1)$ holds and that $P(k) \implies P(k+1)$ for $k \in N$. Therefore $P(n)$ holds for all $n \in N$.

2. Which amounts of postage can you make using 5 and 9 cent stamps?

   We can make all postage 32 cents and greater. You can make 32 cents with three 9's and one 5. If you have a postage of k where $k \geq 32$, you can get a postage of $k+1$ in one of two ways:

   (a) If you have a 9 cent stamp in your k postage, trade it for two 5's. Now you have $k+1$ postage.

   $$k = 9a + 5b \mid a, b \in \mathbb{Z}$$

   $$9(a-1) + 5(b+2) = 9a + 5b + 1 = k + 1$$

   (b) Otherwise, trade seven 5's for four 9's. If we have no 9's, then we must have at least seven 5's because we are only working with postages of 32 cents and greater, so trading seven 5's will not result in a negative number of 5 cent stamps.

   $$k = 5b \mid b \in \mathbb{Z}$$

   $$9 * 4 + 5(b-7) = 5b + 1 = k + 1$$

3. Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$ if $f(n)$ is defined recursively by $f(0) = 1$ and for $n = 0, 1, 2, ...$

   (a) $f(n+1) = f(n) + 2$

   $$f(1) = 3, f(2) = 5, f(3) = 7, f(4) = 9$$

   (b) $f(n+1) = 3f(n)$

   $$f(1) = 3, f(2) = 9, f(3) = 27, f(4) = 81$$

   (c) $f(n+1) = 2^{f(n)}$

   $$f(1) = 2, f(2) = 4, f(3) = 16, f(4) = 65536$$

   (d) $f(n+1) = f(n)^2 + f(n) + 1$

   $$f(1) = 3, f(2) = 13, f(3) = 183, f(4) = 33673$$

4. Let $P(n)$ be the statement that

   $$1 + \frac{1}{4} + \frac{1}{9} + ... + \frac{1}{n^2} < 2 - \frac{1}{n}$$

   where n is an integer greater than 1.

   (a) What is the statement $P(2)$?

   $$P(2) : 1 + \frac{1}{2^2} < 2 - \frac{1}{2}$$

(b) Show that $P(2)$ is true, completing the basis step of the proof.

$$P(2) : \frac{5}{4} < \frac{3}{2}$$

(c) What is the *inductive hypothesis*? For $k > 1$,

$$P(k) : 1 + \frac{1}{4} + ... + \frac{1}{k^2} < 2 - \frac{1}{k}$$

(d) What do you need to prove in the inductive step? That $P(k)$ implies $P(k+1)$ for $k > 1$.

(e) Complete the inductive step. Prove:

$$P(k+1) : 1 + \frac{1}{4} + \frac{1}{9} + ... + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$$

Starting with the left side:

$$1 + \frac{1}{4} + ... + \frac{1}{k^2} + \frac{1}{(k+1)^2}$$

Using the inductive hypothesis:

$$1 + \frac{1}{4} + ... + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

If we could prove for $k > 1$ that

$$P'(k) : 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$$

then we would be done. This is equivalent to proving that

$$P'(k) : -\frac{1}{k} + \frac{1}{(k+1)^2} \leq -\frac{1}{k+1}$$

which is equivalent to proving that

$$P'(k) : \frac{1}{(k+1)^2} \leq \frac{1}{k} - \frac{1}{k+1}$$

And since

$$\frac{1}{k} - \frac{1}{k+1} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{1}{k(k+1)}$$

then $P'(k)$ is equivalent to

$$P'(k) : \frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)}$$

which is equivalent to proving

$$P'(k) : (k+1)^2 \geq k(k+1)$$

which, because we are only working with $k > 1$ is equivalent to

$$P'(k) : k+1 \geq k$$

which is clearly true. Therefore $P(k+1)$ is true if $P(k)$ is true.

(f) Explain why these steps show that this inequality is true for all n where n is an integer greater than 1.
Since we have shown a base case of $P(2)$ to be true and that for all $n > 1$, that $P(n)$ implies $P(n+1)$, we have proved that $P(n)$ holds for all $n > 1$.

5. Prove by induction that:

$$1 + 3 + 5 + ... + (2n-1) = n^2$$

Let:

$$F(n) = 1 + 3 + 5 + ... + (2n-1)$$
$$P(n) : F(n) = n^2$$

We will prove that $P(n)$ holds for n greater than 0.

$$\forall n \in N : P(n)$$

Base case:

$$P(1) : 1 = 1^2$$

Inductive step. We want to prove $P(k) \implies P(k+1)$ for $k \in N$.

Inductive hypothesis, assume:

$$P(k) : F(k) = k^2$$

Now prove:

$$P(k+1) : F(k+1) = (k+1)^2$$

Starting with the left side:

$$\begin{aligned}
F(k+1) &= 1 + 3 + 5 + ... + (2k-1) + (2(k+1)-1) \\
&= F(k) + (2(k+1)-1) \\
&= k^2 + (2(k+1)-1) \\
&= k^2 + 2k + 1 \\
&= (k+1)(k+1) \\
&= (k+1)^2
\end{aligned}$$

We have shown that a base case of $P(1)$ holds and that $P(k) \implies P(k+1)$ for $k \in N$. Therefore $P(n)$ holds for all $n \in N$.