Contents MATHEMATICS-I

MODULE

1

CALCULUS-I

ROLLE'S THEOREM, MEAN VALUE THEOREMS

Rolle's Theorem –

Statement – If f(x) be a function such that

(i) f(x) is continuous in the interval $a \le x \le b$.

(ii) f'(x) exists for every point in the interval $a < x < b$, and

(iii) f(a) = f(b),

then there exists at least a point c at which f'(c) = 0,where $a < c < b$.

Proof. Since f(a) = f(b), therefore when x takes values greater than a (i.e. if we start from the point A in the fig. 1.1) then since f(x) again takes a value f(b) = f(a) it must cease to increase at some point C where x = c such that $a < c < b$ and begin to decrease thereafter.

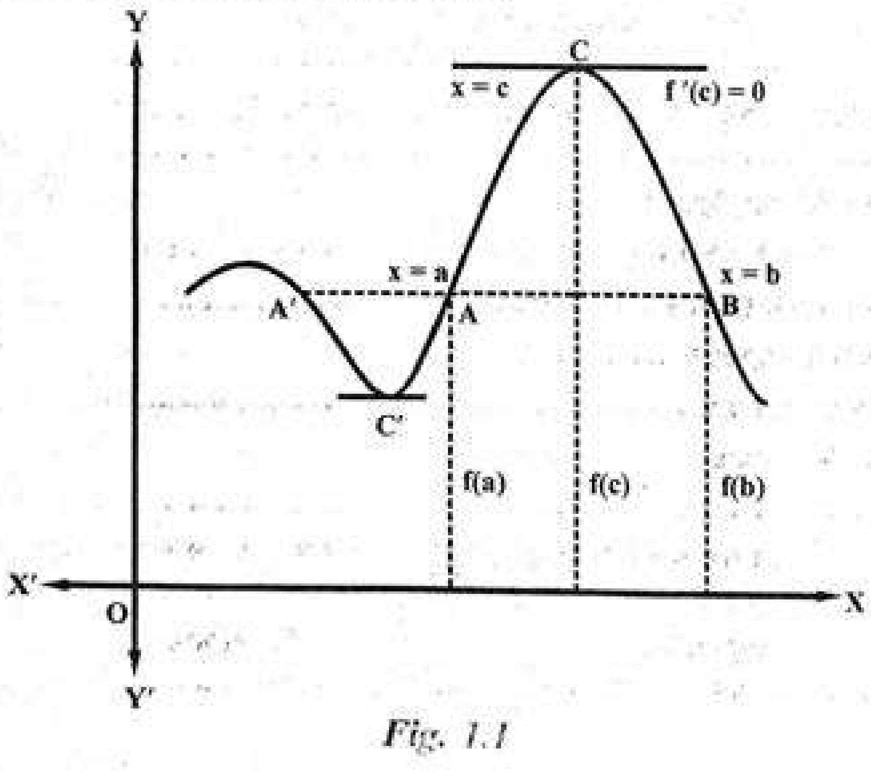


Fig. 1.1

At this point $x = c$, there is a maximum value of the function and so $f(c+h) - f(c)$ and $f(c-h) - f(c)$ are both negative, h being small and positive.

$$\therefore \qquad \frac{f(c+h)-f(c)}{h} < 0 \text{ and } \frac{f(c-h)-f(c)}{-h} > 0$$

So as $h \to 0$, the above expressions tend to negative and positive respectively unless each of them is zero.

If they have different limits, then $Rf'(c) \neq Lf'(c)$. i.e., $f'(c)$ does not exist which is against the hypothesis.

Hence each of the above limits must be zero,

i.e., $$f'(c) = 0, \text{ where } a < c < b.$$

Hence the theorem is proved.

Note – There may be more than one point like c at which $f'(x)$ vanishes i.e. if we start from the point A' instead of A in fig. 1.1, i.e. we take $x = a$ at A', we find that $f'(x)$ vanishes at C' and C i.e. at two points between A and B i.e. between $x = a$ and $x = b$, hence the expression there exists at least a point c in the statement of the Rolle's theorem.

Geometrical Significance of Rolle's Theorem – If the graph of the function $f(x)$ be drawn between $x = a$ and $x = b$, then it is a continuous curve between $x = a$ and $x = b$ having a unique tangent at all points in the above interval and $f(a) = f(b)$. Rolle's theorem asserts that there exists at least one point $x = c$ on the curve between $x = a$ and $x = b$ at which the tangent to the curve is parallel to the axis of x (see fig. 1.1).

Algebraic Interpretation of Rolle's Theorem – If $f(x)$ be a polynomial in x and $x = a$, $x = b$ be the two roots of the equation $f(x) = 0$, then from Rolle's theorem we find that at least one root of the equation $f'(x)= 0$ lies between a and b.

Lagrange's Mean Value Theorem –

Statement – If $f(x)$ be a function such that –

(i) $f(x)$ is continuous in the closed interval $a \le x \le b$, and

(ii) $f'(x)$ exists in the open interval $a < x < b$, then there exists at least one point c such that

$$f(b) - f(a) = (b-a)\, f'(c), \text{ where } a \le c \le b.$$

Proof. Let $\phi(x) = f(x) - Ax$ ...(i)

be an auxiliary function, where A is a constant, so chosen that

$$\phi(b) = \phi(a)$$

From equation (i), we have

$$f(b) - Ab = f(a) - Aa$$

$$f(b) - f(a) = Ab - Aa$$

or $$A = \frac{f(b)-f(a)}{b-a} \qquad ...(ii)$$

From equations (i) and (ii), we have

$$\phi(x) = f(x) - \left\{\frac{f(b)-f(a)}{b-a}\right\}x \qquad \text{...(iii)}$$

with the help of equation (iii) and the given conditions we find that $\phi(x)$ is continuous in $a \le x \le b$, differentiable in $a < x < b$ and $\phi(a) = \phi(b)$, hence all the conditions of Rolle's theorem are satisfied

Hence $\qquad \phi'(c) = 0$, where $a < c < b \qquad$ ...(iv)

From equation (i)

$$\phi'(x) = f'(x) - A$$

$\therefore$ From equation (iv)

$$\phi'(c) = f'(c) - A = 0$$

or $\qquad f'(c) = A = \dfrac{f(b)-f(a)}{b-a} \qquad$ [From equation (ii)]

or $\qquad (b-a)f'(c) = f(b) - f(a) \qquad$ ...(v)

or $\qquad f(b) = f(a) + (b-a)\ f'(c) \qquad$ ...(vi)

The equation (v) is called 'first mean value theorem of differential calculus' Also the equation (vi) is known as ***Lagrange's form of the mean value theorem.***

Since c lies in (a, b) so we can write

$$c = a + (b-a)\theta, \text{ where } 0 < \theta < 1.$$

From equation (vi), we get

$$f(b) = f(a) + (b-a)f'[a + (b-a)\theta] \qquad \text{...(vii)}$$

where $0 < \theta < 1$

Putting $b = a + h$ or $b - a = h$ in equation (vii), we have

$$f(a+h) = f(a) + hf'(a + \theta h), \text{ where } 0 < \theta < 1 \qquad \text{...(viii)}$$

This equation is an alternative form of the mean value theorem.

Geometrical Significance of Mean Value Theorem –

Let A and B be two points on the graph of $y = f(x)$ such that $x = a$ at A and $x = b$ at B.

Then the coordinates of A and B are $[a, f(a)]$ and $[b, f(b)]$ respectively. From A and B draw AM and BN perpendiculars to x-axis and also draw AK perpendicular to BN.

If θ be the inclination of AB to x-axis, then we have

$$\tan\theta = \tan\angle BAK = \frac{BK}{AK} = \frac{f(b)-f(a)}{b-a}$$

From mean value theorem equation (v) or (vi) we have $\dfrac{f(b)-f(a)}{b-a} = f'(c)$

So we have

$$\tan\theta = f'(c)$$

i.e., the slope of the tangent to the curve at $x = c$, from the definition of $f'(x)$ or $\frac{dy}{dx}$.

Hence from mean value theorem we conclude that there exists some point P on the arc AB of the curve, given by the fig. 1.2 of $y = f(x)$, the tangent at which is parallel to the chord AB.

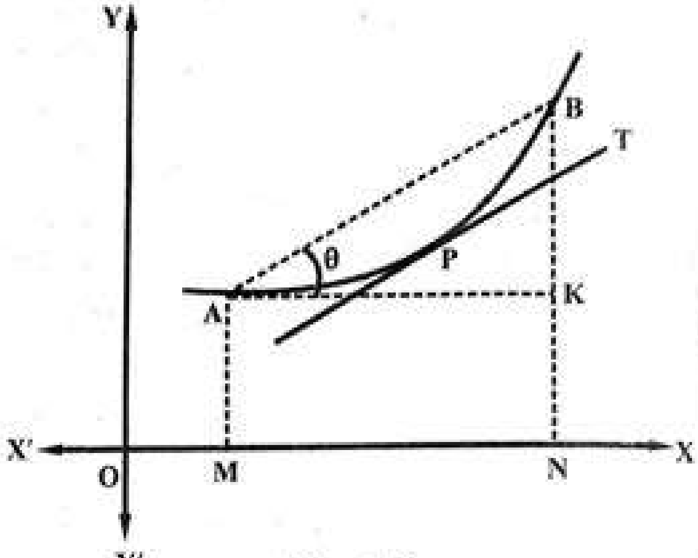


Fig. 1.2

Cor. If f(x) be a function such that $f'(x) = 0$ for all x in (a, b) then f(x) reduces to a constant in (a, b).

If x_1, x_2 be any two values of x in (a, b), then from mean value theorem, we have

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c), \quad a \le c \le b = 0$$

$\because$ $$f'(x) = 0 \vee x \text{ in } (a, b).$$

Thus $f(x_1) = f(x_2)$ i.e. f(x) has the same value for every value of x in (a, b).

Q.1. Write statement of Rolle's and Lagrange's theorem and explain their geometrical meaning. ***(R.G.P.V., Dec. 2017)***

Ans. Refer to the matter given on page 3, 4 and 5.

NUMERICAL PROBLEMS

Prob.1. Verify Rolle's theorem for the function $f(x) = x^2$ in (–1, 1). ***(R.G.P.V., Nov. 2018)***

Sol. Given that,

$$f(x) = x^2$$

Also $$f(-1) = (-1)^2 = 1$$

and $$f(1) = 1^2 = 1$$

$\therefore$ $$f(-1) = f(1) = 1$$

$$f'(x) = 2x \quad \text{...(i)}$$

Now $$Rf'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}\frac{h^2 + 2hx}{h} = \lim_{h\to 0}(h + 2x)$$

or $$Rf'(x) = 2x$$

and $$Lf'(x) = \lim_{h\to 0}\frac{f(x-h)-f(x)}{-h} = \lim_{h\to 0}\frac{(x-h)^2 - x^2}{-h}$$

$$= \lim_{h\to 0}\frac{h^2 - 2hx}{-h} = \lim_{h\to 0}(-h+2x)$$

$$Lf'(x) = 2x$$

Thus $$Rf'(x) = Lf'(x)$$

$\therefore$ f'(x) exists for all values of x in (–1, 1)

Also f(x) is continuous for all values of x in (–1, 1) since f(x) is differentiable for all values of x in (–1, 1)

Hence all the three conditions of Rolle's theorem are satisfied.

$\therefore$ f'(x) = 0 for at least one value of x in (–1, 1).

From equation (i), equating f'(x) to zero, we get x = 0.

Evidently x = 0 lies in (–1, 1)

Prob.2. Verify Rolle's theorem for the function $f(x) = x^2 + 2x - 8$ in the interval (–4, 2). ***(R.G.P.V., May 2018)***

Sol. Given that,

$$f(x) = x^2 + 2x - 8$$

Also $$f(-4) = (-4)^2 + 2(-4) - 8 = 16 - 8 - 8 = 0$$

and $$f(2) = (2)^2 + 2(2) - 8 = 4 + 4 - 8 = 0$$

$\therefore$ $$f(-4) = f(2) = 0$$

$$f'(x) = 2x + 2 \qquad \text{...(i)}$$

Now $$Rf'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{\{(x+h)^2 + 2(x+h) - 8\} - (x^2 + 2x - 8)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\{(x+h)^2 - x^2\} + 2\{(x+h) - x\}}{h}\right]$$

$$= \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} + 2\lim_{h\to 0}\frac{(x+h) - x}{h}$$

$$= \frac{d}{dx}(x^2) + 2\frac{d}{dx}(x) \qquad \text{(by definition)}$$

$$Rf'(x) = 2x + 2$$

$$= \lim_{h\to0}\frac{(x+h)^3-x^3}{h} - 12\lim_{h\to0}\frac{(x+h)-x}{h}$$

$$= \frac{d}{dx}(x^3) - 12\frac{d}{dx}(x) \qquad \text{(by definition)}$$

$$= 3x^2 - 12$$

and
$$Lf'(x) = \lim_{h\to0}\frac{f(x-h)-f(x)}{-h}$$

$$= \lim_{h\to0}\left[\frac{\{(x-h)^3-12(x-h)\}-(x^3-12x)}{-h}\right]$$

$$= \lim_{h\to0}\left[\frac{\{(x-h)^3-x^3\}-12\{(x-h)-x\}}{-h}\right]$$

$$= \lim_{h\to0}\frac{(x-h)^3-x^3}{-h} - 12\lim_{h\to0}\frac{(x-h)-x}{-h}$$

$$= 3x^2 - 12, \text{ as above}$$

Thus $\qquad Rf'(x) = Lf'(x) = 3x^2 - 12$

$\therefore\quad$ $f'(x)$ exists for all values of x in $(0, 2\sqrt{3})$.

Also $f(x)$ is continuous for all values of x in $(0, 2\sqrt{3})$ since $f(x)$ is differentiable for all values of x in $(0, 2\sqrt{3})$.

Hence all the three condition of Rolle's theorem are satisfied.

$\therefore\quad$ $f'(x) = 0$ for at least one value of x in $(0, 2\sqrt{3})$.

From equation (i), equating $f'(x)$ to zero, we get $x = \pm 2$.

Evidently $x = 2$ lies in $(0, 2\sqrt{3})$

Prob.5. The function f is defined in (0, 1) as follows –

$$f(x) = 1 \text{ for } 0 < x < \frac{1}{2} = 2 \text{ for } \frac{1}{2} \le x \le 1$$

Show that f(x) satisfies none of the conditions of Rolle's theorem yet f'(x) = 0 for many points in (0, 1).

Sol. Here we find that

$$f\left(\frac{1}{2}-0\right) = 1 \text{ whereas } f\left(\frac{1}{2}+0\right) = 2$$

i.e. $f\left(\frac{1}{2}-0\right) \ne f\left(\frac{1}{2}+0\right)$, though $f\left(\frac{1}{2}+0\right) = 2 = f\left(\frac{1}{2}\right)$ so the function $f(x)$ is discontinuous at $x = \frac{1}{2}$.

Also the continuity being a necessary condition for the existence of a finite derivative, the function f'(x) does not exist for every point in $0 \le x \le 1$.

Also f(0) = 1 and f(1) = 2 (Given)

So $f(0) \neq f(1)$

Hence all the conditions of Rolle's theorem are not satisfied by f(x) in (0, 1)

Here f(x) being a function free from x (i.e. constant) in (0, 1), there is possibility of the values of f'(x) to be zero at many points in (0, 1).

Prob.6. Verify Rolle's theorem, where $f(x) = 2x^3 + x^2 - 4x - 2$.

(R.G.P.V., Dec. 2015)

Sol. Given that,

$$f(x) = 2x^3 + x^2 - 4x - 2 \quad \text{...(i)}$$

Here f(x) = 0 gives $2x^3 + x^2 - 4x - 2 = 0$

or $\quad x^2(2x + 1) - 2(2x + 1) = 0$

or $\quad (2x + 1)(x^2 - 2) = 0$

or $$x = -\frac{1}{2}, \sqrt{2}, -\sqrt{2}$$

Now $f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^2 - 4\left(-\frac{1}{2}\right) - 2 = -\frac{1}{4} + \frac{1}{4} + 2 - 2 = 0$

$$f(\sqrt{2}) = 2(\sqrt{2})^3 + (\sqrt{2})^2 - 4\sqrt{2} - 2 = 4\sqrt{2} + 2 - 4\sqrt{2} - 2 = 0$$

and $$f(-\sqrt{2}) = 2(-\sqrt{2})^3 + (-\sqrt{2})^2 - 4(-\sqrt{2}) - 2 = -4\sqrt{2} + 2 + 4\sqrt{2} - 2 = 0$$

Thus, we find that $f(-\sqrt{2}) = 0 = f(\sqrt{2})$

Differentiating equation (i), we get

$$f'(x) = 6x^2 + 2x - 4 \quad \text{...(ii)}$$

Also $$Rf'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{\{2(x+h)^3 + (x+h)^2 - 4(x+h) - 2\} - (2x^3 + x^2 - 4x - 2)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{2\{(x+h)^3 - x^3\} + \{(x+h)^2 - x^2\} - 4\{(x+h) - x\}}{h} \right]$$

$$= 2 \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} + \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} - 4 \lim_{h \to 0} \frac{(x+h) - x}{h}$$

$$= 2\frac{d}{dx}(x^3) + \frac{d}{dx}(x^2) - 4\frac{d}{dx}(x) \quad \text{(by definition)}$$

or $$Rf'(x) = 6x^2 + 2x - 4$$

and $Lf'(x) = \lim_{h \to 0} \frac{f(x-h)-f(x)}{-h}$

$$= \lim_{h \to 0} \left[\frac{\left\{2(x-h)^3 + (x-h)^2 - 4(x-h) - 2\right\} - (2x^3 + x^2 - 4x - 2)}{-h} \right]$$

$$= \lim_{h \to 0} \left[\frac{2\{(x-h)^3 - x^3\} + \{(x-h)^2 - x^2\} - 4\{(x-h) - x\}}{-h} \right]$$

$$= 2 \lim_{h \to 0} \frac{(x-h)^3 - x^3}{-h} + \lim_{h \to 0} \frac{(x-h)^2 - x^2}{-h} - 4 \lim_{h \to 0} \frac{(x-h) - x}{-h}$$

$$= 2\frac{d}{dx}(x^3) + \frac{d}{dx}(x^2) - 4\frac{d}{dx}(x) \qquad \text{(by definition)}$$

or $\qquad Lf'(x) = 6x^2 + 2x - 4$

Thus $\qquad Rf'(x) = Lf'(x) = 6x^2 + 2x - 4$

$\therefore \quad f'(x)$ exists for all values of x in $(-\sqrt{2}, \sqrt{2})$

Also f(x) is continuous for all values of x in $(-\sqrt{2}, \sqrt{2})$ as it is differentiable for all values of x in $(-\sqrt{2}, \sqrt{2})$.

Hence all three conditions of Rolle's theorem are satisfied, and so $f'(x) = 0$ for at least one value of x, where $-\sqrt{2} < x < \sqrt{2}$.

From equation (ii) equating $f'(x)$ to zero, we get

$$6x^2 + 2x - 4 = 0$$

or $\qquad 3x^2 + x - 2 = 0$

or $\qquad 3x^2 + 3x - 2x - 2 = 0$

or $\qquad 3x(x+1) - 2(x+1) = 0$

or $\qquad (x+1)(3x-2) = 0$

or $\qquad x = -1, \frac{2}{3}$

Evidently $x = -1$ and $x = \frac{2}{3}$ lie in $(-\sqrt{2}, \sqrt{2})$.

Prob.7. Verify Rolle's theorem for the function $f(x) = x^3 - 6x^2 + 11x - 6$

[R.G.P.V., Nov. 2019 (O)]

Sol. Given that,

$$f(x) = x^3 - 6x^2 + 11x - 6 \qquad \text{...(i)}$$

Here $f(x) = 0$ gives $x^3 - 6x^2 + 11x - 6 = 0$

$(x-1)(x^2 - 5x + 6) = 0$

or $\quad (x-1)(x-2)(x-3) = 0$

or $\qquad x = 1, 2, 3$

$$f(1) = f(2) = f(3) = 0$$

Also $f'(x) = 3x^2 - 12x + 11$ ...(ii)

Now $$Rf'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{\{(x+h)^3 - 6(x+h)^2 + 11(x+h) - 6\} - (x^3 - 6x^2 + 11x - 6)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\{(x+h)^3 - x^3\} - 6\{(x+h)^2 - x^2\} + 11\{(x+h) - x\}}{h} \right]$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} - 6 \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} + 11 \lim_{h \to 0} \frac{(x+h) - x}{h} \quad \text{...(iii)}$$

$$= \frac{d}{dx}(x^3) - 6\frac{d}{dx}(x^2) + 11\frac{d}{dx}(x) \qquad \text{(by definition)}$$

$$= 3x^2 - 12x + 11$$

and $$Lf'(x) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h}$$

$$= \lim_{h \to 0} \frac{(x-h)^3 - x^3}{-h} - 6 \lim_{h \to 0} \frac{(x-h)^2 - x^2}{-h} + 11 \lim_{h \to 0} \frac{(x-h) - x}{-h}$$

[Replacing h by – h in equation (iii)]

$= 3x^2 - 12x + 11$, as above

Thus $$Rf'(x) = Lf'(x) = 3x^2 - 12x + 11$$

$\therefore$ $f'(x)$ exists for all values of x in (1, 3).

Also f(x) is continuous for all values of x in (1, 3) as it is differentiable for all values of x in (1, 3).

Hence all the three conditions of Rolle's theorem are satisfied, and so $f'(x) = 0$ for at least one value of x in (1, 3).

From equation (ii), equating f'(x) to zero, we get

$$3x^2 - 12x + 11 = 0$$

or $$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4.3.11}}{2.3}$$

$$= \frac{12 \pm \sqrt{144 - 132}}{6} = \frac{12 \pm \sqrt{12}}{6}$$

$$= 2 \pm \frac{\sqrt{3}}{3} = 2 \pm \frac{1.732}{3} = 2 \pm 0.577 = 2.577,\ 1.423$$

Both these values of x lies in (1, 3).

Prob.8. Verify the mean value theorem in interval (0, 4) for the function $f(x) = (x-1)(x-2)(x-3)$.

Sol. Given that,

$$f(x) = (x-1)(x-2)(x-3)$$
$$= (x-1)(x^2-5x+6)$$
$$= x^3-6x^2+11x-6$$

$\therefore$ $$f'(x) = 3x^2 - 12x + 11$$

So $$f'(c) = 3c^2 - 12c + 11 \qquad \text{...(i)}$$

Let $a = 0$, $b = 4$, then $f(a) = f(0)$, $f(b) = f(4)$

where $f(0) = (0-1)(0-2)(0-3) = -6$

and $f(4) = (4-1)(4-2)(4-3) = 6$

We know that

$$(b-a) f'(c) = f(b) - f(a)$$

$\therefore$ $$(4-0) f'(c) = 6 - (-6) = 12$$

or $$f'(c) = 3$$

From equation (i), we have

$$3 = 3c^2 - 12c + 11$$

or $$3c^2 - 12c + 8 = 0$$

$$c = \frac{-(-12) \pm \sqrt{(-12)^2 - 4.3.8}}{2.3} = \frac{12 \pm \sqrt{144-96}}{6}$$
$$= \frac{12 \pm \sqrt{48}}{6} = \frac{12 \pm 4\sqrt{3}}{6} = 2 \pm \frac{2}{\sqrt{3}}$$
$$= 2 \pm 1.155 = 3.155,\ 0.845$$

Both these values of c lies in (0, 4) and hence mean value theorem is verified for the given function f(x) in (0, 4).

Prob.9. Find the value of c in the Lagrange's mean value theorem $f(b) - f(a) = (b-a) f'(c)$, if $f(x) = Ax^2 + Bx + C$, where A, B, C are constants and $A \neq 0$.

Sol. Given that,

$$f(x) = Ax^2 + Bx + C \qquad \text{...(i)}$$

$\therefore$ $$f'(x) = 2Ax + B$$

So $$f'(c) = 2Ac + B$$

From equation (i), we have

$$f(b) = Ab^2 + Bb + C,\ f(a) = Aa^2 + Ba + C$$

We know that

$$(b-a) f'(c) = f(b) - f(a)$$

$\therefore$ $$(b-a)(2Ac + B) = (Ab^2 + Bb + C) - (Aa^2 + Ba + C)$$

or $$2Ac(b-a) + B(b-a) = A(b^2 - a^2) + B(b-a)$$

or $2Ac(b-a) = A(b^2 - a^2)$

or $A(b-a)[2c - (b+a)] = 0$

or $c = \dfrac{(a+b)}{2} \quad \because A \neq 0$ and $a \neq b$ Ans.

Prob.10. Verify the Lagrange's mean value theorem for the function $f(x) = x^2 - 2x + 4$ in the interval [1, 5]. ***(R.G.P.V., Dec. 2016)***

Sol. Given, $f(x) = x^2 - 2x + 4$ ...(i)

$\therefore \quad f'(x) = 2x - 2$

and so $\quad f'(c) = 2c - 2$ ...(ii)

Let a = 1 and b = 5, then from equation (i), we have

$f(a) = f(1) = (1)^2 - 2.1 + 4 = 1 - 2 + 4 = 3$

$f(b) = f(5) = (5)^2 - 2.5 + 4 = 25 - 10 + 4 = 19$

From Lagrange's mean value theorem, we have

$$f(b) = f(a) + (b-a) f'(c)$$

or $f(5) = f(1) + (5-1) f'(c)$

or $19 = 3 + 4f'(c)$

or $f'(c) = 4$

From equation (ii), we have

$$4 = 2c - 2$$

or $2c = 6$

or $c = 3$

c = 3 which lies in the interval (1, 5) and hence Lagrange's mean value theorem is verified in the interval (1, 5) for the given function.

Prob.11. Verify Lagrange's mean value theorem for the function f(x) $2x^2 - 7x + 10$ in the interval [2, 5]. ***(R.G.P.V., May 2019, June 2020)***

Sol. Given $\quad f(x) = 2x^2 - 7x + 10$ ...(i)

$\therefore \quad f'(x) = 4x - 7$

and so $\quad f'(c) = 4c - 7$ ...(ii)

Let a = 2 and b = 5, then from equation (i), we have

$f(a) = f(2) = 2(2)^2 - 7.2 + 10 = 8 - 14 + 10 = 4$

$f(b) = f(5) = 2(5)^2 - 7.5 + 10 = 50 - 35 + 10 = 25$

From Lagrange's mean value theorem, we have

$$\boxed{f(b) = f(a) + (b-a) f'(c)}$$

or $f(5) = f(2) + (5-2) f'(c)$

or $25 = 4 + 3f'(c)$

or $f'(c) = 7$

From equation (ii), we have

$$7 = 4c - 7$$

or $$4c = 14$$

or $$c = \frac{14}{4} = \frac{7}{2} = 3.5$$

c = 3.5 which lies in the interval (2, 5) and hence Lagrange's mean value theorem is verified in the interval (2, 5) for the given function.

Prob.12. Write the statement of Lagrange's mean value theorem and verify it for the function $f(x) = x^2 - 4x - 3$ in the interval [1,4].

(R.G.P.V., June 2017)

Sol. **Statement** – Refer to the matter given on page 4.

Proof – Given, $$f(x) = x^2 - 4x - 3 \qquad ...(i)$$

$\therefore$ $$f'(x) = 2x - 4$$

and so $$f'(c) = 2c - 4 \qquad ...(ii)$$

Let a = 1 and b = 4, then from equation (i), we have

$$f(a) = f(1) = (1)^2 - 4.1 - 3 = 1 - 4 - 3 = -6$$

$$f(b) = f(4) = (4)^2 - 4.4 - 3 = 16 - 16 - 3 = -3$$

From Lagrange's mean value theorem, we have

$$f(b) = f(a) + (b - a)f'(c)$$

or $$f(4) = f(1) + (4 - 1)f'(c)$$

or $$-3 = -6 + 3f'(c)$$

or $$f'(c) = 1$$

From equation (ii), we have

$$1 = 2c - 4$$

or $$2c = 5$$

or $$c = \frac{5}{2} = 2.5$$

c = 2.5 which lies in the interval (1, 4) and hence Lagrange's mean value theorem is verified in the interval (1, 4) for the given function.

EXPANSION OF FUNCTIONS BY MACLAURIN'S AND TAYLOR'S FOR ONE VARIABLE, TAYLOR'S THEOREM FOR FUNCTION OF TWO VARIABLES

Maclaurin's Theorem –

Statement – If f(x) be a function of x can be expanded in ascending powers of x, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^n}{n!}f^n(0) + ...$$

Proof. Suppose, $f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots$ ...(i)

where, A_0, A_1, A_2, ..., etc., are constants to be determined.

By successive differentiation, we get

$$f'(x) = A_1 + 2A_2x + 3A_3x^2 + \ldots \quad \ldots(ii)$$

$$f''(x) = 2A_2 + 3.2.A_3x + \ldots \quad \ldots(iii)$$

and $$f'''(x) = 3.2A_3 + 4.3.2\,A_4x + \ldots \quad \ldots(iv)$$

Substituting $x = 0$, in equations (i), (ii), (iii) and (iv), we get

$$f(0) = A_0,\ f'(0) = A_1,\ f''(0) = 2A_2$$

$$f'''(0) = 3.2A_3$$

$$\Rightarrow \quad A_0 = f(0),\ A_1 = f'(0),\ A_2 = \frac{f''(0)}{2!},\ A_3 = \frac{f'''(0)}{3!} \text{ and so on.}$$

Putting the values of $A_0, A_1, A_2, A_3, \ldots$, in equation (i), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots$$

This result is known as ***Maclaurin's theorem.***

If we take $f(x) = y$; $f(0) = (y)_0$; $f'(0) = (y_1)_0$;

$f''(0) = (y_2)_0$; $f'''(0)$

$= (y_3)_0$, then the Maclaurin's series (or theorem) takes the form as

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \ldots + \frac{x^n}{n!}(y_n)_0 + \ldots$$

Proved

Taylor's Theorem for Function of One Variables –

Statement – If f(x + h) be a function of x or h which can be expanded in powers of x or h then

$$f(x + h) = f(x) + h\,f'(x) + \frac{h^2}{2!}f''(x) + \ldots + \frac{h^n}{n!}f^n(x) + \ldots$$

or $$f(x + h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \ldots + \frac{x^n}{n!}f^n(h) + \ldots$$

or $$f(x) = f[a + (x - a)] = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^n}{n!}f^n(a) + \ldots$$

Proof. Suppose,

$$f(x + h) = A_0 + A_1h + A_2h^2 + A_3h^3 + A_4h^4 + \ldots + A_nh^n + \ldots \quad \ldots(i)$$

By successive differentiation of equation (i), with respect to h, we have

$$f'(x+h) = A_1 + 2A_2h + 3A_3h^2 + 4A_4h^3 + \dots \quad \text{...(ii)}$$

$$f''(x+h) = 2A_2 + 3.2\,A_3h + 4.3\,A_4h^2 + \dots \quad \text{...(iii)}$$

$$f'''(x+h) = 3.2.1.\,A_3 + 4.3.2.A_4h + \dots \quad \text{...(iv)}$$

....

....

Substituting h = 0 in equations (i), (ii), (iii) and (iv), we get

$$f(x) = A_0;\ f'(x) = A_1;\ f''(x) = 2A_2;\ f'''(x) = 3.2.1.A_3$$

$$\Rightarrow \quad A_0 = f(x);\ A_1 = f'(x);\ A_2 = \frac{1}{2!}f''(x)$$

$$A_3 = \frac{1}{3!}f'''(x) \text{ and so on.}$$

Putting these values in equation (i), we obtain the Taylor's theorem as

$$f(x+h) = \mathbf{f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots + \frac{h^n}{n!}f^n(x) + \dots} \quad \text{...(v)}$$

Proved

Note 1. Substituting x = a in equation (v), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^n(a) + \dots$$

Note 2. Substituting a = 0 and h = x in Note 1, we get the Maclaurin's theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^n(0) + \dots$$

Note 3. Substituting x = h and h = a in equation (v), we obtain

$$f(a+h) = f(h) + af'(h) + \frac{a^2}{2!}f''(h) + \dots + \frac{a^n}{n!}f^n(h) + \dots$$

Note 4. Substituting h = x − a in Note 1, we obtain

$$f(x) = f[a + (x-a)]$$
$$= f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots + \frac{(x-a)^n}{n!}f^n(a) + \dots$$

Taylor's Theorem for Function of Two Variables –

Statement – If f(x, y) and all its partial derivatives upto the n^{th} order are finite and continuous for all points (x, y) where $a \le x \le a + h$ and $b \le y \le b + k$

$$\textbf{Then } f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(a,b)$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(a,b) + \dots$$

Proof. Considering f(x + h, y + k) as a function of a single variable x where y is assumed as constant.

For one variable expanding by Taylor's theorem, we get

$$f(x + \delta x, y + \delta y) = f(x, y + \delta y) + \frac{\delta x}{1!}\frac{\partial}{\partial x}f(x, y+\delta y) + \frac{(\delta x)^2}{2!}\frac{\partial^2}{\partial x^2}f(x, y+\delta y) + ...$$

Now expanding for y, we have

$$= \left\{f(x, y) + \delta y\frac{\partial}{\partial y}f(x, y) + \frac{(\delta y)^2}{2!}.\frac{\partial^2}{\partial y^2}f(x, y) + ...\right\} + \delta x\frac{\partial}{\partial x}\left\{f(x, y) + \delta y\frac{\partial}{\partial y}f(x, y) + ...\right\}$$

$$+ \frac{(\delta x)^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x, y) + \delta y\frac{\partial}{\partial y}f(x, y) + ...\right\} + ...$$

$$= \left\{f(x, y) + \delta y\frac{\partial}{\partial y}f(x, y) + \frac{(\delta y)^2}{2!}.\frac{\partial^2}{\partial y^2}f(x, y) + ...\right\}$$

$$+ \delta x\left\{\frac{\partial f(x, y)}{\partial x} + \delta y\frac{\partial^2}{\partial x\,\partial y}f(x, y)\right\} + \frac{(\delta x)^2}{2!}\left\{\frac{\partial^2}{\partial x^2}f(x, y) + ...\right\} + ...$$

$$= f(x, y) + \left\{\delta x\frac{\partial f(x, y)}{\partial x} + \delta y\frac{\partial f(x, y)}{\delta y}\right\} + \frac{1}{2!}\left\{(\delta x)^2\frac{\partial^2 f(x, y)}{\partial x^2} + 2\delta x.\delta y\frac{\partial^2 f(x, y)}{\partial x\,\partial y} + (\delta y)^2\frac{\partial^2 f(x, y)}{\partial y^2}\right\} + ...$$

or $$f(a + h, b + k) = f(a, b) + \left\{h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right\} + \frac{1}{2!}\left\{h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right\} + ...$$

or $$\mathbf{f(a + h, b + k) = f(a, b) + \left\{h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right\}f(a, b) + \frac{1}{2!}\left\{h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right\}^2 f(a, b) + ...}$$

$$\text{Another form } f(x, y) = f(a, b) + \left\{(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right\} f(a, b)$$

$$+ \frac{1}{2!}\left\{(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right\}^2 f(a, b) + \ldots$$

or
$$f(x, y) = \mathbf{f(a, b) + \{(x-a)\, f_x(a, b) + (y-b)\, f_y(a, b)\}}$$

$$\mathbf{+ \frac{1}{2!}\{(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b)}$$

$$\mathbf{+ (y-b)^2 f_{yy}(a, b)\} + \ldots}$$

NUMERICAL PROBLEMS

Prob.13. Expand $f(x) = e^x$ in Maclaurin series. [R.G.P.V., Nov. 2018(O)]

Sol. Given that,

$$f(x) = e^x \qquad \ldots(i)$$

By Maclaurin's theorem, we know that

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots + \frac{x^n}{n!}f^n(0) \qquad \ldots(ii)$$

Here
$$f(x) = e^x \Rightarrow f(0) = 1$$
$$f'(x) = e^x \Rightarrow f'(0) = 1$$
$$f''(x) = e^x \Rightarrow f''(0) = 1$$
$$f'''(x) = e^x \Rightarrow f'''(0) = 1$$

$$f^n(x) = e^x \Rightarrow f^n(0) = 1$$

Substituting these values in equation (ii), we get

$$f(x) = e^x = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \ldots\ldots + \frac{x^n}{n!}(1)$$

$$e^x = \mathbf{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^n}{n!}} \qquad \textbf{Ans.}$$

Prob.14. Expand the function $f(x) = \cos x$ in Maclaurin series and hence find approximate value of cos 18°. (R.G.P.V., May 2018)

Sol. Given that,

$$f(x) = \cos x \qquad \ldots(i)$$

By Maclaurin's theorem, we know that

$$f(x) = f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+.....+\frac{x^n}{n!}f^n(0)+..... \quad ...(ii)$$

Here
$$f(x) = \cos x \Rightarrow f(0) = 1$$
$$f'(x) = -\sin x \Rightarrow f'(0) = 0$$
$$f''(x) = -\cos x \Rightarrow f''(0) = -1$$
$$f'''(x) = \sin x \Rightarrow f'''(0) = 0$$
$$f^{iv}(x) = \cos x \Rightarrow f^{iv}(0) = 1$$

$$f^n(x) = \cos\left(x+\frac{n\pi}{2}\right) \Rightarrow f^n(0) = \cos\frac{n\pi}{2}$$

Substituting these values in equation (ii), we get

$$f(x) = \cos x = 1+\frac{x}{1!}(0)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(0)+\frac{x^4}{4!}(1)+.....+\frac{x^n}{n!}\cos\frac{n\pi}{2}+.....$$

$$\cos x = \mathbf{1-\frac{x^2}{2!}+\frac{x^4}{4!}+.....+(-1)^m\frac{x^{2m}}{(2m)!}+.....\infty} \quad \textbf{Ans.}$$

Now putting x = 18°, we get

$$\cos 18° = 1-\frac{(18°)^2}{2!}+\frac{(18°)^4}{4!}-\frac{(18°)^6}{6!}+.....$$
$$= 1-\frac{1}{2}\left(18\times\frac{\pi}{180}\right)^2+\frac{1}{24}\left(18\times\frac{\pi}{180}\right)^4-\frac{1}{720}\left(18\times\frac{\pi}{180}\right)^6+.....$$
$$= 1-0.049348+0.00040587-0.00000133526+.....$$
$$= \mathbf{0.9510565347\ Approx.} \quad \textbf{Ans.}$$

Prob.15. If $y = \sin(m\sin^{-1}x)$,

prove that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0$. ***(R.G.P.V., Dec. 2015)***

Sol. Here, $y = \sin(m\sin^{-1}x)$ ...(i)

Differentiating equation (i) with respect to x, we get

$$\frac{dy}{dx} = \cos(m\sin^{-1}x).\left(m.\frac{1}{\sqrt{1-x^2}}\right)$$

or
$$\sqrt{1-x^2}\frac{dy}{dx} = m\cos(m\sin^{-1}x)$$

Squaring both sides, we have

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2 \cos^2(m \sin^{-1}x)$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2[1 - \sin^2(m \sin^{-1}x)]$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2(1 - y^2) \qquad \text{[From equation (i)]}$$

Again differentiating with respect to x, we have

$$(1-x^2)2\frac{dy}{dx}.\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2(-2x) = -2m^2y\frac{dy}{dx}$$

Divided by $2\frac{dy}{dx}$ both side, we have

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx} = -m^2y$$

or $$\mathbf{(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0} \qquad \textbf{Proved}$$

Prob.16. Find the Maclaurin's expansion of log (1 + x).

(R.G.P.V., June 2017)

Sol. Given that, f(x) = log (1 + x) ...(i)

By Maclaurin's theorem, we know that

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+.....+\frac{x^n}{n!}f^n(0)+..... \qquad ...(ii)$$

Here,

$$f(x) = \log(1+x) \Rightarrow f(0) = 0$$
$$f'(x) = (1+x)^{-1} \Rightarrow f'(0) = 1$$
$$f''(x) = -(1+x)^{-2} \Rightarrow f''(0) = -1$$
$$f'''(x) = 2(1+x)^{-3} \Rightarrow f'''(0) = 2$$
$$f^{iv}(x) = -6(1+x)^{-4} \Rightarrow f^{iv}(0) = -6$$

$$f^n(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$$

$$f^n(0) = (-1)^{n-1}(n-1)!$$

Substituting these values in equation (ii), we get

$$\log(1+x) = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \ldots\ldots$$

$$\ldots\ldots + \frac{x^n}{n!}(-1)^{n-1}(n-1)! + \ldots\ldots$$

$$\log(1+x) = \mathbf{x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots\ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots\ldots\ldots\infty} \quad \textbf{Ans.}$$

Prob.17. *Expand $e^{\sin x}$ by Maclaurin's series upto the terms containing x^4.*

(R.G.P.V., June 2007)

Or

Expand $e^{\sin x}$ in powers of x upto the terms containing x^4.

(R.G.P.V., June 2009, Feb. 2010)

Sol. Let, $y = e^{\sin x}$

Then
$$y_1 = e^{\sin x}.\cos x = y.\cos x$$
$$y_2 = y_1 \cos x - y \sin x$$
$$y_3 = y_2 \cos x - 2y_1 \sin x - y \cos x$$
$$y_4 = y_3 \cos x - 3y_2 \sin x - 3y_1 \cos x + y \sin x;\ \text{etc.}$$

Substituting x = 0, we get

$$(y)_0 = e^{\sin 0} = e^0 = 1$$
$$(y_1)_0 = (y)_0 \cos 0 = 1,\ (y_2)_0 = (y_1)_0 \cos 0 - (y)_0 \sin 0 = 1$$
$$(y_3)_0 = (y_2)_0 \cos 0 - 2(y_1)_0 \sin 0 - (y)_0 \cos 0 = 0$$
$$(y_4)_0 = (y_3)_0 \cos 0 - 3(y_2)_0 \sin 0 - 3(y_1)_0 \cos 0 + (y)_0 \sin 0 = -3$$

By Maclaurin's theorem, we get

$$e^{\sin x} = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \ldots\ldots$$

or
$$e^{\sin x} = \mathbf{1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \ldots\ldots} \quad \textbf{Ans.}$$

Prob.18. *Expand by Maclaurin's theorem $e^{x \cos x}$ as far as the term x^3.*

(R.G.P.V., Dec. 2015)

Sol. Let, $y = e^{x \cos x}$

Then
$$y_1 = e^{x\cos x}[\cos x - x \sin x] = y(\cos x - x \sin x)$$
$$y_2 = y_1(\cos x - x\sin x) + y[-\sin x - (\sin x + x\cos x)]$$
$$= y_1(\cos x - x \sin x) - y(2\sin x + x\cos x)$$
$$y_3 = y_2(\cos x - x\sin x) + y_1[-\sin x - (\sin x + x\cos x)]$$
$$- y_1(2\sin x + x\cos x) - y[2\cos x + (\cos x - x\sin x)]$$
$$= y_2(\cos x - x\sin x) - y_1(4\sin x + 2x\cos x) - y(3\cos x - x\sin x)$$

Putting x = 0, we get

$$(y)_0 = e^0 = 1$$
$$(y_1)_0 = (y)_0 (\cos 0 - 0.\sin 0) = 1(1-0) = 1$$
$$(y_2)_0 = (y_1)_0 (\cos 0 - 0.\sin 0) - (y)_0 (2\sin 0 + 0.\cos 0)$$
$$= 1(1-0) - 1(0+0) = 1$$
$$(y_3)_0 = (y_2)_0 (\cos 0 - 0.\sin 0) - (y_1)_0 (4\sin 0 + 2.0\cos 0) - (y)_0 (3\cos 0 - 0.\sin 0)$$
$$= 1(1-0) - 1(0+0) - 1(3-0) = 1 - 3 = -2$$

By Maclaurin's theorem, we get

$$e^{x\cos x} = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \ldots..$$
$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{3}\ldots.. \quad \textbf{Ans.}$$

Prob.19. Expand $\log\frac{1+x}{1-x}$ ***in powers of x using Maclaurin's theorem.***

(R.G.P.V., June 2014)

Sol. Let, $y = \log\frac{1+x}{1-x}$, $(y)_0 = \log 1 = 0$

$$y_1 = \frac{1-x}{1+x}\cdot\left(\frac{(1-x)1-(1+x).(-1)}{(1-x)^2}\right) = \frac{1-x}{1+x}\cdot\frac{2}{(1-x)^2} = \frac{2}{1-x^2}$$

$$(y_1)_0 = 2$$
$$(1-x^2)y_1 = 2 \quad \ldots(i)$$

Differentiating w.r. to x

$$(1-x^2)y_2 - 2xy_1 = 0$$
$$(y_2)_0 = 0 \quad \ldots(ii)$$

Differentiating equation (ii) n times by Leibnitz's theorem, we get

$$\left[(1-x^2)y_{n+2} + nc_1 y_{n+1}(-2x) + nc_2 y_n(-2.1)\right] - 2\left[xy_{n+1} + nc_1 y_n.1\right] = 0$$
$$(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - 2xy_{n+1} - 2ny_n = 0$$
$$(1-x^2)y_{n+2} - 2(n+1)xy_{n+1} - (n^2 - n + 2n)y_n = 0$$
$$(1-x^2)y_{n+2} - 2(n+1)x\, y_{n+1} - n(n+1)y_n = 0$$

Putting x = 0

$$(y_{n+2})_0 = n(n+1)(y_n)_0 \quad \ldots(iii)$$

Putting n = 1, 2, 3, ... in equation (iii), we get

$$(y_3)_0 = 1.2(y_1)_0 = 2 \times 2 = 4$$
$$(y_4)_0 = 2.3(y_2)_0 = 0$$
$$(y_5)_0 = 3.4(y_3)_0 = 12 \times 4 = 48 \text{ etc.}$$

By Maclaurin's theorem, we get

$$\log\left(\frac{1+x}{1-x}\right) = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \ldots..$$

$$= 0 + x.2 + \frac{x^2}{2!}.0 + \frac{x^3}{3!}4 + \frac{x^4}{4!}.0 + \frac{x^5}{5!}48 + \ldots..$$

$$= \mathbf{2x + \frac{4}{3!}x^3 + \frac{48}{5!}x^5 + \ldots..}$$ **Ans.**

Prob.20. Using Maclaurin's theorem prove that –

$$e^{ax} \cos bx = 1 + ax + (a^2 - b^2)\frac{x^2}{2!} + a(a^2 - 3b^2)\frac{x^3}{3!} + \ldots.$$

(R.G.P.V., Dec. 2008)

Sol. Here, $y = e^{ax} \cos bx$ ....(i)

Put x = 0, we get $(y)_0 = e^{a\times 0} \cos b \times 0 = 1.1 = 1$

Now differentiating equation (i), with respect to x successively, we get

$$y_1 = e^{ax}(-\sin bx.\ b) + \cos bx.\ e^{ax}.a$$

$$y_1 = -be^{ax} \sin bx + ay$$

or $$y_1 - ay = -be^{ax} \sin bx \quad \ldots(ii)$$

$$y_2 - ay_1 = -b[e^{ax} \cos bx.b + \sin bx.e^{ax}.a]$$

$$y_2 - ay_1 = -b^2 e^{ax} \cos bx - ab\, e^{ax} \sin bx$$

$$y_2 - ay_1 = -b^2 y + a(y_1 - ay)$$

$$y_2 - ay_1 = (-b^2 - a^2)\, y + ay_1$$

or $$y_2 - 2ay_1 + (a^2 + b^2)\, y = 0$$

$$y_3 - 2ay_2 + (a^2 + b^2)\, y_1 = 0$$

and so on.

Putting x = 0, we get

$$(y_1)_0 = a$$

$$(y_2)_0 = a^2 - b^2$$

$(y_3)_0 = a(a^2 - 3b^2)$, and so on.

Substituting the values of $(y)_0, (y_1)_0, (y_2)_0, (y_3)_0$.... etc. in the Maclaurin's series, we get

$$e^{ax}\cos bx = \mathbf{1 + ax + (a^2 - b^2)\frac{x^2}{2!} + a(a^2 - 3b^2)\frac{x^3}{3!} + \ldots..}$$ **Proved**

Prob.21. If f(x) = log (1 + x), x > 0, using Maclaurin's theorem, show that for 0 < θ < 1 –

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$$

Deduce that $\log(1 + x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ ***for x > 0. (R.G.P.V., June 2010)***

Sol. Given that, $f(x) = \log(1 + x)$ ...(i)

By Maclaurin's theorem, we know that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(\theta x) \quad ...(ii)$$

Here $f(x) = \log(1 + x)$, $f(0) = 0$

$$f'(x) = \frac{1}{1+x}, \quad f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2}, \quad f''(0) = -1$$

and

$$f'''(x) = \frac{2}{(1+x)^3}, \quad f'''(\theta x) = \frac{2}{(1+\theta x)^3}$$

Substituting these values in equation (ii), we get

$$\log(1 + x) = \mathbf{x} - \frac{\mathbf{x}^2}{\mathbf{2}} + \frac{\mathbf{x}^3}{\mathbf{3(1+\theta x)^3}} \qquad \textbf{Proved} \quad ...(iii)$$

Since $x > 0$ and $\theta > 0$, $\theta x > 0$

$$(1 + \theta x)^3 > 1 \text{ i.e., } \frac{1}{(1+\theta x)^3} < 1$$

$$x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} < x - \frac{x^2}{2} + \frac{x^3}{3}$$

Hence, $\log(1 + x) < x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$ [from equation (iii)]

Prob.22. Use Maclaurin's series to prove –

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + ...$$

(R.G.P.V., Jan./Feb. 2008, Dec. 2013)

Or

Find the Maclaurin's expansion of log (1 + e^x).[R.G.P.V., Nov. 2019 (O)

Sol. Suppose, $y = \log(1 + e^x)$ ...(i)

Putting $x = 0$ in equation (i), we get

and $(y)_0 = \log(1 + e^0) = \log 2$

By successive differentiation of equation (i), we obtain

$$y_1 = \frac{e^x}{1+e^x} = 1 - \frac{1}{(1+e^x)}$$

$$y_2 = -\left[\frac{-1}{(1+e^x)^2}.e^x\right] = \frac{e^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)}.\frac{1}{(1+e^x)} = y_1(1-y_1)$$

$$y_3 = y_2(1 - y_1) + y_1(-y_2) = y_2 - 2y_1y_2$$

and $y_4 = y_3 - 2y_1y_3 - 2y_2^2$ etc.

By Maclaurin's theorem, we have

$$\log(1+e^x)=(y)_0+x(y_1)_0+\frac{x^2}{2!}(y_2)_0+\frac{x^3}{3!}(y_3)_0+\frac{x^4}{4!}(y_4)_0+\ldots\ldots \quad \ldots\text{(ii)}$$

Putting x = 0, in y_1, y_2, y_3 and y_4, we get

$$(y_1)_0=\frac{1}{2}$$

$$(y_2)_0=(y_1)_0\left[1-(y_1)_0\right]=\frac{1}{2}\left[1-\frac{1}{2}\right]=\frac{1}{4}$$

$$(y_3)_0=(y_2)_0-2(y_1)_0(y_2)_0=\frac{1}{4}-2\times\frac{1}{2}\times\frac{1}{4}=0$$

$$(y_4)_0=(y_3)_0-2(y_3)_0(y_1)_0-2(y_2)_0^{\ 2}=-2\left(\frac{1}{4}\right)^2=-\frac{1}{8}$$

Substituting the values of $(y)_0$, $(y_1)_0$, $(y_2)_0$, $(y_3)_0$, $(y_4)_0$, ... in equation (ii), we get

$$\log(1+e^x)=\log 2+x.\frac{1}{2}+\frac{x^2}{2!}.\frac{1}{4}+\frac{x^3}{3!}.0+\frac{x^4}{4!}.\left(-\frac{1}{8}\right)+\ldots\ldots$$

or $\log(1+e^x)=\mathbf{\log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\ldots\ldots}$ **Ans.**

① ***Prob.23. Apply Maclaurin's theorem to prove that***

$$\log\sec x=\frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{45}x^6+\ldots\ldots$$ ***(R.G.P.V., June 2015)***

Sol. Given that, f(x) = log sec x ...(i)

By Maclaurin's theorem, we know that

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots+\frac{x^n}{n!}f^n(0)+\ldots\ldots \quad \ldots\text{(ii)}$$

Here, f(x) = log sec x

$\Rightarrow$ f(0) = 0

$\therefore$ $f'(x)=\frac{1}{\sec x}.\sec x\tan x=\tan x\Rightarrow f'(0)=0$

$f''(x)=\sec^2 x\Rightarrow f''(0)=1$

$f'''(x)=2\sec^2 x\tan x\Rightarrow f'''(0)=0$

$f^{iv}(x)=2\sec^4 x+4\sec^2 x\tan^2 x\Rightarrow f^{iv}(0)=2$

$f^{v}(x)=16\sec^4 x\tan x+8\sec^2 x\tan^3 x\Rightarrow f^{v}(0)=0$

$f^{vi}(x)=8\sec^4 x\tan^2 x+16\sec^6 x+16\sec^2 x\tan^4 x\Rightarrow f^{vi}(0)=16$

and so on.

Substituting these values in equation (ii), we get

$$\log \sec x = 0\frac{x}{1!}(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(2) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(16) + \ldots$$

$$\mathbf{\log \sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \ldots\ldots}$$ **Prove**

Prob.24. Expand $e^{a \sin^{-1} x}$ ***in ascending powers of x.***

[R.G.P.V., June 2008(N), April 2009, Dec. 201

Sol. We have, $$y = e^{a \sin^{-1} x} \quad \ldots(\text{i})$$

Differentiating equation (i), with respect to x, we get

$$y_1 = e^{a \sin^{-1} x} \frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{(1-x^2)}}$$

or $$(1-x^2)\, y_1^2 = a^2 y^2 \quad \ldots(\text{ii})$$

Again differentiating

$$(1 - x^2)\, 2y_1y_2 + (-2x)\, y_1^2 = 2a^2 y y_1$$

Dividing by $2y_1$, we get

$$(1 - x^2)\, y_2 - xy_1 - a^2 y = 0 \quad \ldots(\text{iii})$$

Differentiate it n times by Leibnitz's theorem, we get

$$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + a^2)\, y_n = 0$$

Putting x = 0,

$$(y_{n+2})_0 = (n^2 + a^2)\,(y_n)_0 \quad \ldots(\text{iv})$$

From equations (i), (ii) and (iii), we get

$$(y)_0 = 1,\ (y_1)_0 = a,\ (y_2)_0 = a^2$$

Putting n = 1, 2, 3... in equation (iv), we get

$$(y_3)_0 = (1^2 + a^2)\,(y_1)_0 = a(1^2 + a^2)$$

$$(y_4)_0 = (2^2 + a^2)\,(y_2)_0 = a^2(2^2 + a^2), \text{ and so on}$$

Substituting these values in the Maclaurin's series,

$$y = (y)_0 + \frac{(y_1)_0}{1!}x + \frac{(y_2)_0}{2!}x^2 + \frac{(y_3)_0}{3!}x^3 + \frac{(y_4)_0}{4!}x^4 + \ldots$$

We get,

$$e^{a \sin^{-1} x} = \mathbf{1 + ax + \frac{a^2}{2!}x^2 + \frac{a(1^2 + a^2)}{3!}x^3 + \frac{a^2(2^2 + a^2)}{4!}x^4 + \ldots}$$

Ans

Prob.25. Expand $\sin^{-1}x$ is power of x by Maclaurin's theorem.

(R.G.P.V., Dec. 2017)

Sol. Suppose, $y = \sin^{-1}x$ ...(i)

Putting x = 0 in equation (i), we get

$$(y)_0 = \sin^{-1}0 = 0$$

Differentiating equation (i), with respect to x, we get

$$y_1 = \frac{1}{\sqrt{1-x^2}} \qquad ...(ii)$$

Here we apply different approach for differentiating complicated form of function.

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

$$y_1^2 = \frac{1}{1-x^2}$$

$$y_1^2(1-x^2) = 1$$

Again differentiating

$$(1-x^2)2y_1y_2 + (-2x)y_1^2 = 0$$

Dividing by $2y_1$, we get

$$(1-x^2)y_2 - xy_1 = 0 \qquad ...(iii)$$

Differentiating in n times by Leibnitz's theorem, we get

$$(1-x^2)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n - (xy_{n+1} + {}^nC_1.1.y_n) = 0$$

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - (n^2-n)y_n - xy_{n+1} - ny_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0 \qquad ...(iv)$$

Putting x = 0,

$$(y_{n+2})_0 = n^2(y_n)_0 \qquad ...(v)$$

From equations (i), (ii) and (iii), we get

$$(y)_0 = 0, (y_1)_0 = 1, (y_2)_0 = 0$$

Putting n = 1, 2, 3, ... in equation (v), we get

$$(y_3)_0 = 1^2(y_1)_0 = 1.1 = 1$$

$$(y_4)_0 = 2^2(y_2)_0 = 4.0 = 0$$

$(y_5)_0 = 3^2(y_3)_0 = 3^2.1^2$, and so on.

By Maclaurin's theorem, we know that

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + ... \qquad ...(vi)$$

Substituting the values of $(y)_0$, $(y_1)_0$, $(y_2)_0$, $(y_3)_0$, $(y_4)_0$, ... in equatio (vi), we get

$$\sin^{-1}x = 0+\frac{x}{1!}.1+\frac{x^2}{2!}.0+\frac{x^3}{3!}.1+\frac{x^4}{4!}.0+\frac{x^5}{5!}.3^2.1^2+..... \quad \text{An}$$

Prob.26. Prove that

$$(\sin^{-1}x)^2 = \frac{2}{2!}x^2+\frac{2.2^2}{4!}x^4+\frac{2.2^2.4^2}{6!}x^6+....$$

and hence deduce

$$\theta^2 = 2\frac{\sin^2\theta}{2!}+2^2\frac{2\sin^4\theta}{4!}+2^2 4^2\frac{2\sin^6\theta}{6!}+....$$

(R.G.P.V., June 20

Sol. Let, $y=(\sin^{-1}x)^2$ and $(y_0)_0=(\sin^{-1}0)^2=0$

$\therefore$ Differentiating y w.r.t. x, we get

$$y_1 = 2(\sin^{-1}x).\frac{1}{\sqrt{1-x^2}} \text{ and } (y_1)_0 = 0$$

$$(1-x^2)y_1^2 = 4y$$

Again differentiating

$$(1-x^2)2y_1y_2 - 2xy_1^2 = 4y_1$$

$$\Rightarrow \quad (1-x^2)y_2 - xy_1 = 2 \text{ and } (y_2)_0 = 2$$

By Leibnitz's theorem, we get

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n = 0 \quad ...($$

Putting x = 0 in equation (i), we get

$$(y_{n+2})_0 = n^2(y_n)_0 \quad ...($$

Putting n = 1, 3, 5 ......, we get

$$(y_3)_0 = (y_5)_0 = (y_7)_0 = ... = 0$$

Putting n = 2, 4, 6 ...., we get

$$(y_4)_0 = 2^2(y_2)_0 = 2^2.2$$

$$(y_6)_0 = 4^2(y_4)_0 = 4^2.2^2.2 \text{ and so on.}$$

Hence substituting these values in Maclaurin's theorem, we get

$$(\sin^{-1}x)^2 = 0+x.0+\frac{x^2}{2!}.2+\frac{x^3}{3!}.0+\frac{x^4}{4!}.2^2.2+\frac{x^5}{5!}.0+\frac{x^6}{6!}4^2.2^2.2+..$$

$$= \frac{2x^2}{2!}+\frac{2.2^2}{4!}x^4+\frac{2.2^2.4^2}{6!}x^6+...... \quad ...(i$$

Now, putting x = sin θ in equation (iii), we get

$$\theta^2 = \frac{2\sin^2\theta}{2!} + 2^2.\frac{2\sin^4\theta}{4!} + 2^2.4^2.\frac{2\sin^6\theta}{6!} + \ldots\ldots \qquad \textbf{Proved}$$

Prob.27. Using Taylor's series, show that –

$$\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \ldots\ldots\ldots \quad \textit{(R.G.P.V., Jan./Feb. 2007)}$$

Or

Show that $\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} + \ldots\ldots$

(R.G.P.V., June 2015)

Sol. Suppose,

$$f(x+h) = \log(x+h) \qquad \ldots(i)$$

Substituting x = 0, in equation (i), we get

$$f(h) = \log h \qquad \ldots(ii)$$

By Taylor's series, we have

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \ldots\ldots \qquad \ldots(iii)$$

By successive differentiation of equation (ii), we get

$$f'(h) = \frac{1}{h},\ f''(h) = -\frac{1}{h^2},\ f'''(h) = \frac{2}{h^3}, \ldots\ldots$$

Putting the values of f(x + h), f(h), f′(h), f″(h), f‴(h), ... in equation (iii), we get

$$\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \ldots\ldots\ldots\ldots \qquad \textbf{Proved}$$

Prob.28. Use Taylor's theorem to prove that –

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin\theta.\frac{\sin\theta}{1} - (h\sin\theta)^2.\frac{\sin 2\theta}{2} + (h\sin\theta)^3.\frac{\sin 3\theta}{3} - \ldots\ldots$$

$$+ (-1)^{n-1}(h\sin\theta)^n.\frac{\sin n\theta}{n} + \ldots\ldots$$

where, $\theta = \cot^{-1}x.$ ***(R.G.P.V., Sept. 2009)***

Or

Prove that –

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin z.\frac{\sin z}{1} - \frac{(h\sin z)^2}{2}.\sin 2z + \ldots\ldots$$

where, $z = \cot^{-1}x.$ ***(R.G.P.V., June 2012)***

Sol. Suppose,

$$f(x+h) = \tan^{-1}(x+h) \quad \text{...(i)}$$

Substituting h = 0 in equation (i), we get

$$f(x) = \tan^{-1} x \quad \text{...(ii)}$$

Differentiating equation (ii), n times with respect to x, we get

$$f^n(x) = D^n(\tan^{-1}x) = (-1)^{n-1}(n-1)!\sin^n\theta \sin n\theta$$

where, $\theta = \cot^{-1}x$

For n = 1, $f'(x) = (-1)^0\, 0!\sin\theta\sin\theta = \sin\theta\sin\theta$

For n = 2, $f''(x) = (-1)^1\, 1!\sin^2\theta\sin 2\theta$

For n = 3, $f'''(x) = (-1)^2\, 2!\sin^3\theta\sin 3\theta$

....

....

By Taylor's theorem, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots + \frac{h^n}{n!}f^n(x) + \ldots\ldots \quad \text{...(iii)}$$

Putting the values of f(x + h), f(x), f′(x) .... in equation (iii), we get

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin\theta.\sin\theta - \frac{h^2}{2!}.\sin^2\theta.\sin 2\theta$$

$$+\frac{h^3}{3!}2\sin^3\theta\sin 3\theta + \ldots + \frac{h^n}{n!}(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta + \ldots$$

or $$\tan^{-1}(x+h) = \tan^{-1}x + h\sin\theta.\frac{\sin\theta}{1} - (h\sin\theta)^2.\frac{\sin 2\theta}{2}$$

$$+(h\sin\theta)^3\frac{\sin 3\theta}{3} - \ldots + (-1)^{n-1}(h\sin\theta)^n.\frac{\sin n\theta}{n} + \ldots$$

Proved

Prob.29. Calculate the approximate value of $\log_e 1.1$ correct to four decimal places using Taylor's expansion. ***(R.G.P.V., Nov./Dec. 200***

Or

Expand $\log_e x$ in powers of $(x-1)$ and hence evaluate $\log_e(1.1)$ correct to four decimal places. ***(R.G.P.V., Dec. 2002, March/April 20***

Or

Using Taylor series find value of $\log_e(1.1)$ correct upto three decimal place. ***(R.G.P.V., Nov. 20***

Or

Expand log x in power of (x – 1) by Taylor's theorem and hence find the value of log 1.1. ***(R.G.P.V., Nov. 20***

Sol. Let, $f(x) = \log_e x$, $\quad f(1) = 0$

$$f'(x) = \frac{1}{x} \qquad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \qquad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \qquad f'''(1) = 2$$

$$f^{(iv)}(x) = -\frac{6}{x^4} \qquad f^{(iv)}(1) = -6$$

etc. etc.

Substituting these values in the Taylor's series

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1)\frac{(x-1)^3}{3!}f'''(1) + ...$$

We get, $\log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + ...$

Now putting $x = 1.1$ so that $x - 1 = 0.1$, we have

$$\log_e 1.1 = 0.1 - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4 + ...$$

$$= 0.1 - 0.005 + 0.00033 - 0.000025 + ... = \mathbf{0.0953}$$ **Ans.**

Prob.30. Write one main condition when Maclaurin's theorem fails. Expand f(x) = sin x, in ascending powers of (x – π/2) using Taylor's theorem. ***(R.G.P.V., Dec. 2005)***

Sol. Suppose, f(x) is any function. Maclaurin's expansion of f(x) fails if any of the functions f(x), f '(x), f "(x), ... becomes infinite or discontinuous at any point of the interval [0, x] or if R_n does not tend to zero as $n \to \infty$.

Here, f(x) = sin x, we want to expand f(x) in powers of $x - \frac{1}{2}\pi$. We can write

$$f(x) = f\left[\frac{1}{2}\pi + \left(x - \frac{1}{2}\pi\right)\right]$$

Now expanding $f\left[\frac{1}{2}\pi + \left(x - \frac{1}{2}\pi\right)\right]$ by Taylor's theorem in power of $\left(x - \frac{1}{2}\pi\right)$, we get

$$f(x) = f\left[\frac{1}{2}\pi + \left(x - \frac{1}{2}\pi\right)\right]$$

$$= f\left(\frac{\pi}{2}\right) + \left(x - \frac{1}{2}\pi\right)f'\left(\frac{\pi}{2}\right) + \frac{1}{2!}\left(x - \frac{1}{2}\pi\right)^2 f''\left(\frac{\pi}{2}\right)$$

$$+ \frac{1}{3!}\left(x - \frac{1}{2}\pi\right)^3 f'''\left(\frac{\pi}{2}\right) + ... \quad ...(i)$$

Now, $f(x) = \sin x$. Therefore

$$f(\pi/2) = \sin \pi/2 = 1$$

$$f'(x) = \cos x,\ f'(\pi/2) = \cos \pi/2 = 0$$

$$f''(x) = -\sin x. \text{ so that } f''(\pi/2) = -\sin \pi/2 = -1$$

$$f'''(x) = -\cos x, \text{ so that } f'''(\pi/2) = -\cos \pi/2 = 0 \text{ etc.}$$

Substituting these values in series (i), we get

$$\sin x = 1 + \left(x - \frac{1}{2}\pi\right).0 + \frac{1}{2!}\left(x - \frac{1}{2}\pi\right)^2 (-1).....$$ A

Prob.31. Find Taylor's expansion of $y = \sin x$ about point $x = \frac{\pi}{2}$.

[R.G.P.V., Nov. 2019 (

Sol. Refer to Prob.30.

Prob.32. Expand sin x in powers of $(x - \pi/2)$. Hence, find the valu sin 91° correct to 4 decimal places. ***(R.G.P.V., Dec. 20***

Sol. For expansion of sin x in powers of $\left(x - \frac{\pi}{2}\right)$ refer to Prob.30.

To find value of sin 91° put x = 91° in Prob.30, we get

$$\sin 91° = 1 + (91° - 90°)(0) + \frac{1}{2!}(91° - 90°)^2(-1) + \frac{1}{3!}(91° - 90°)^3(0)$$

$$= 1 + 0 + \frac{1}{2}\left(1 \times \frac{\pi}{180}\right)^2(-1) + 0 + ...$$

$$= 1 - \frac{1}{2}(0.0175)^2$$

$$= 1 - 0.00015 = \mathbf{0.9998}$$ A

Prob.33. Find the Taylor series expansion of log cos x about the p $x = 0$. ***(R.G.P.V., Dec. 20***

Sol. Given, $f(x) = \log \cos x$

By the Taylor's series, we have

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + ...$$

$$+ \frac{(x-a)^n}{n!} f^n(a) + ...$$

where, $a = 0$.

By successive differentiation of equation (i), we get

$$f'(x) = -\frac{\sin x}{\cos x} = -\tan x$$

$$f''(x) = -\sec^2 x = -1 - \tan^2 x$$

$$f'''(x) = -2\tan x \sec^2 x$$

$$= -2\tan x(1+\tan^2 x) = -2\tan x - 2\tan^3 x$$

and so on.

Substituting x = 0 in above, we get

$$f(0) = \log\cos 0 = 0$$

$$f'(0) = -\tan 0 = 0$$

$$f''(0) = -1 - \tan^2(0) = -1$$

$$f'''(0) = -2\tan(0) - 2\tan^3(0) = 0$$

....

....

$$\Rightarrow \quad f(x) = f(0) + (x-0)f'(0) + \frac{(x-0)^2}{2!}f''(0) + \frac{(x-0)^3}{3!}f'''(0) + \ldots..$$

or $\log\cos x = -\frac{\mathbf{x^2}}{\mathbf{2!}} + \ldots..$ **Ans.**

Prob.34. Find the Taylor's series expansion of the function about the point π/3 –

f(x) = log cos x ***(R.G.P.V., Dec. 2003, June 2011)***

Sol. Given, $f(x) = \log\cos x$ ...(i)

By the Taylor's series, we have

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots...$$

$$+\frac{(x-a)^n}{n!}f^n(a) + \ldots... \quad \text{...(ii)}$$

where, $a = \pi/3$

By successive differentiation of equation (i), we get

$$f'(x) = -\frac{\sin x}{\cos x} = -\tan x$$

$$f''(x) = -\sec^2 x = -1 - \tan^2 x$$

$$f'''(x) = -2\tan x\sec^2 x = -2\tan x(1+\tan^2 x)$$

$$= -2\tan x - 2\tan^3 x$$

and so on

Substituting $x = \pi/3$ in above, we get

$$f\left(\frac{\pi}{3}\right) = \log\cos\frac{\pi}{3} = \log\frac{1}{2}$$

$$f'\left(\frac{\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}$$

$$f''\left(\frac{\pi}{3}\right) = -1-\tan^2\frac{\pi}{3} = -4$$

$$f'''\left(\frac{\pi}{3}\right) = -2\tan\frac{\pi}{3} - 2\tan^3\frac{\pi}{3} = -8\sqrt{3}$$

....

....

$$\Rightarrow \quad f(x) = f\left[\frac{\pi}{3}+\left(x-\frac{\pi}{3}\right)\right] = f\left(\frac{\pi}{3}\right)+\left(x-\frac{\pi}{3}\right)f'\left(\frac{\pi}{3}\right)+\frac{\left(x-\frac{\pi}{3}\right)^2}{2!}f''\left(\frac{\pi}{3}\right)$$

$$+\frac{\left(x-\frac{\pi}{3}\right)^3}{3!}f'''\left(\frac{\pi}{3}\right)+\ldots$$

or $$\log\cos x = \log\frac{1}{2} - \sqrt{3}\left(x-\frac{\pi}{3}\right) - 4\frac{\left(x-\frac{\pi}{3}\right)^2}{2!} - 8\sqrt{3}\frac{\left(x-\frac{\pi}{3}\right)^3}{3!}+\ldots$$

A

Prob.35. Compute the approximate value of $\sqrt{11}$ to four decimal pl by taking the first five terms of an approximate Taylor's expansion.

(R.G.P.V., Dec. 20

Sol. Let, $f(x) = x^{1/2}$ ...

Differentiating equation (i) successively w.r. to x, we get

$$f'(x) = \frac{1}{2}x^{-1/2}$$

$$f''(x) = -\frac{1}{4}x^{-3/2}$$

$$f'''(x) = \frac{3}{8}x^{-5/2}$$

$$f^{iv}(x) = -\frac{15}{16}x^{-7/2} \text{ and so on.}$$

By Taylor's series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) + \cdots$$

$$\therefore \quad (x+h)^{1/2} = x^{1/2} + h\left(\frac{1}{2}x^{-1/2}\right) + \frac{h^2}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^{-3/2} + \frac{h^3}{3!}\left(\frac{1}{2}\right)$$
$$\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2} + \frac{h^4}{4!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2} + \ldots\ldots$$
$$\sqrt{x+h} = \sqrt{x} + \frac{h}{2}.\frac{1}{\sqrt{x}} - \frac{h^2}{8}.\frac{1}{x\sqrt{x}} + \frac{h^3}{16}.\frac{1}{x^2\sqrt{x}} - \frac{5h^4}{128}.\frac{1}{x^3\sqrt{x}} + \ldots\ldots$$

Substituting x = 10 and h = 1 in above equation, we get

$$\sqrt{10+1} = \sqrt{10} + \frac{1}{2}.\frac{1}{\sqrt{10}} - \frac{1}{8}.\frac{1}{10\sqrt{10}} + \frac{1}{16}.\frac{1}{100\sqrt{10}} - \frac{5}{128}.\frac{1}{1000\sqrt{10}} + \ldots\ldots$$
$$\sqrt{11} = 3.16227 + 0.15811 - 0.00395 + 0.0001976 - 0.0000123 + \ldots\ldots$$
$$= 3.3166153 = \mathbf{3.3166\ (Approx.)} \qquad \textbf{Ans.}$$

Prob.36. Expand $f(x, y) = x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ by Taylor's theorem.

Sol. Here $a = 1, b = -2$

$f(x, y) = x^2y + 3y - 2$

$f(1, -2) = (1)^2(-2) + 3(-2) - 2 = -10$

$f_x(x, y) = 2xy \qquad \therefore f_x(1, -2) = 2(1)(-2) = -4$

$f_y(x, y) = x^2 + 3 \qquad f_y(1, -2) = (1)^2 + 3 = 4$

$f_{xx}(x, y) = 2y \qquad f_{xx}(1, -2) = 2(-2) = -4$

$f_{xy}(x, y) = 2x \qquad f_{xy}(1, -2) = 2(1) = 2$

$f_{yy}(x, y) = 0 \qquad f_{yy}(1, -2) = 0$

$f_{xxx}(x, y) = 0 \qquad f_{xxx}(1, -2) = 0$

$f_{xxy}(x, y) = 2 \qquad f_{xxy}(1, -2) = 2$

$f_{xyy}(x, y) = 0 \qquad f_{xyy}(1, -2) = 0$

$f_{yyy}(x, y) = 0 \qquad f_{yyy}(1, -2) = 0$

By Taylor's theorem, we have

$$f(x, y) = f(a, b) + \{(x-a)f_x(a, b) + (y-b)f_y(a, b)\} + \frac{1}{2!}\{(x-a)^2 f_{xx}(a, b)$$
$$+ 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)\} + \ldots\ldots \quad \ldots(i)$$

Putting a = 1, b = –2 and above values in equation (i), we get

$$x^2y + 3y - 2 = -10 + \{(x-1)(-4) + (y+2)(4)\} + \frac{1}{2!}\{(x-1)^2(-4)$$
$$+ 2(x-1)(y+2)(2) + (y+2)^2(0)\} + \frac{1}{3!}\{(x-1)^3(0)$$
$$+ 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^3(0)\}$$
$$x^2y + 3y - 2 = \mathbf{-10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2)}$$
$$\mathbf{+ (x-1)^2(y+2)} \qquad \textbf{Ans.}$$

PARTIAL DIFFERENTIATION

Function of Two Variables – Suppose z is a symbol which has a definite value of every x and y then z is said to be a function of two independent variables x and y and we write $z = f(x, y)$ or $\phi(x, y)$.

If z is a function of three or more variables x, y, t, ... are denoted by the relation $z = f(x, y, t, ...)$.

Limits of a Function of Two Variables – The function f(x, y) is said to tend to the limit l as $x \to a$ and $y \to b$ if the limit l is independent of the path followed by the point (x, y) as $x \to a$ and $y \to b$. Then

$$\underset{\substack{x \to a \\ y \to a}}{\text{limit}} f(x, y) = l$$

If corresponding a positive a number $\varepsilon \in (a, b)$, there exists another positive number δ, such that

$$|f(x, y) - l| < \varepsilon, \text{ for } 0 < (x - a)^2 + (y - b)^2 < \delta^2 \text{ for every point } (x, y) \text{ in R.}$$

Continuity of a Function of Two Variables – A function f(x, y) is called continuous at the point (a, b), if

$$\underset{\substack{x \to a \\ y \to b}}{\text{limit}} f(x, y) \text{ exists and } = f(a, b)$$

If a function is continuous at all points of a region, then it is called continuous in that region. A function which is not continuous at a point is called discontinuous at that point.

Partial Derivatives – Suppose $z = f(x, y)$ is a function of two variables. Now let us vary x while y is kept constant. Then z is a function of x only. The derivative of z with respect to x treating y as constant is said to be the partial derivative of z with respect to x, and is represented by one of the symbol

$$\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), D_x f.$$

Similarly, the derivative of z, with respect to y treating x as constant is said to be the partial derivative of z with respect to y and is represented by one of the symbol,

$$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y), D_y f.$$

We also represent $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ by f_{xx}; f_{yy}; f_{xy}; f_{yx}.

For the function of more than two variables on the same lines, these definitions can be applied.

It can easily be verified that, in all ordinary cases

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

and $$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t$$

Which Variable to be Treated as Constant –

Suppose, $x = r\cos\theta,\ y = r\sin\theta$ ....(i)

To find $\frac{\partial r}{\partial x}$, we need a relation between r and x. Such a relation will contain one more variable θ or y, for we can eliminate only one variable out of four from the equation (i). Thus the two possible relations are –

$$r = x\sec\theta \qquad \text{...(ii)}$$

and $$r^2 = x^2 + y^2 \qquad \text{...(iii)}$$

Differentiating equation (ii) partially with respect to x keeping θ as constant

$$\frac{\partial r}{\partial x} = \sec\theta \qquad \text{...(iv)}$$

Differentiating equation (iii) partially with respect to x keeping y as constant

$$2r\frac{\partial r}{\partial x} = 2x \text{ or } \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \qquad \text{...(v)}$$

From equation (iv), we have $\frac{\partial r}{\partial x} = \sec\theta$ and from equation (v), we have $\frac{\partial r}{\partial x} = \cos\theta$. These two values of $\frac{\partial r}{\partial x}$ make confusion.

To avoid the confusion we use the following notations –

Notation – (i) $\left(\frac{\partial r}{\partial x}\right)_\theta$ represent the partial derivative of r w.r.t. x, keeping θ as constant, from equation (iv), we have $\left(\frac{\partial r}{\partial x}\right)_\theta = \sec\theta$.

(ii) $\left(\frac{\partial r}{\partial x}\right)_y$ represent the partial derivative of r w.r.t. x keeping y as constant.

From equation (iv), we have

$$\left(\frac{\partial r}{\partial x}\right)_y = \cos\theta.$$

(iii) When no indication is given regarding the variables to be treated as constant.

$$\left(\frac{\partial}{\partial x}\right) \text{represent} \left(\frac{\partial}{\partial x}\right)_y, \quad \left(\frac{\partial}{\partial y}\right) \text{represent} \left(\frac{\partial}{\partial y}\right)_x$$

$$\left(\frac{\partial}{\partial r}\right) \text{represent} \left(\frac{\partial}{\partial r}\right)_\theta \text{ and } \left(\frac{\partial}{\partial \theta}\right) \text{represent} \left(\frac{\partial}{\partial \theta}\right)_r.$$

Homogeneous Function –

The function in which the degree of each term is same is said to be a ***homogeneous function,*** for example if we consider the function

$$z = f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n,$$

it is a homogeneous function of degree n.

$$z = f(x, y) = x^n\left\{a_0 + a_1\frac{y}{x} + a_2\left(\frac{y}{x}\right)^2 + \ldots + a^n\left(\frac{y}{x}\right)^n\right\} = x^n\phi\left(\frac{y}{x}\right) \text{ say.}$$

Therefore f(x, y) is a homogeneous function of nth degree and $\phi(y/x)$ is a homogeneous function of zeroth degree.

Note – If z is a homogeneous function of x and y of degree n, then function $\sin^{-1}\left(\frac{y}{x}\right)$ is a homogeneous function of degree 0. Then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are homogeneous functions of degree (n – 1) each.

Euler's Theorem on Homogeneous Functions –

Statement – Suppose u = f(x, y) is a homogeneous function of x and y of degree n, then we have

$$x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = nu$$

Proof. Since u = f(x, y) is a homogeneous function of x and y of degree n, so it can be written as

$$u = f(x, y) = x^n\phi\left(\frac{y}{x}\right) \qquad \ldots(i)$$

where, $\phi(y/x)$ is a homogeneous function of zero degree.

Differentiating equation (i) partially with respect to x and y respectively, we get

$$\frac{\partial u}{\partial x} = nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n\phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \qquad \ldots(ii)$$

and

$$\frac{\partial u}{\partial y} = x^n\phi'\left(\frac{y}{x}\right)\frac{1}{x} = x^{n-1}\phi'\left(\frac{y}{x}\right) \qquad \ldots(iii)$$

Multiplying equation (ii) by x and equation (iii) by y, and add we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nx^n\phi\left(\frac{y}{x}\right) \text{ or } x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu \qquad \textbf{Proved}$$

Note – Let $u = u(x_1, x_2 \ldots x_n)$, where u is a homogeneous functions of $x_1, x_2, x_3, \ldots, x_n$ of degree n, then

$$x_1\frac{\partial u}{\partial x_1}+x_2\frac{\partial u}{\partial x_2}+\ldots+x_n\frac{\partial u}{\partial x_n} = nu$$

Cor. 1. Let u be a homogeneous function of degree n, then

(i) $x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y} = (n-1)\frac{\partial u}{\partial x}$

(ii) $x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$

(iii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$

Important Deductions from Euler's Theorem –

(i) If z is a homogeneous function of x, y of degree n and $z = f(u)$, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

(ii) If it is a homogeneous function of x, y of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = g(u)\left[g'(u)-1\right]$$

where, $g(u) = n\frac{f(u)}{f'(u)}$.

Q.2. Define homogeneous functions and establish the Euler's theorem on homogeneous function. ***(R.G.P.V., June 2014)***

Ans. Refer to the matter given on page 40.

NUMERICAL PROBLEMS

Prob.37. If $u = f(y-z, z-x, x-y)$, prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = 0$.

(R.G.P.V, May 2019, June 2020)

Sol. Let $X = y-z$, $Y = z-x$ and $Z = x-y$ ...(i)

Then u = f(X, Y, Z), where each one of X, Y, Z is a function of x, y, z.
From (i), differentiate partially, we get

$$\frac{\partial X}{\partial x}=0, \frac{\partial X}{\partial y}=1, \frac{\partial X}{\partial z}=-1$$

$$\frac{\partial Y}{\partial x}=-1, \frac{\partial Y}{\partial y}=0, \frac{\partial Y}{\partial z}=1 \text{ and } \frac{\partial Z}{\partial x}=1, \frac{\partial Z}{\partial y}=-1, \frac{\partial Z}{\partial z}=0$$

Now, since

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}.\frac{\partial X}{\partial x}+\frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial x}+\frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial x}$$

$$= \frac{\partial u}{\partial X}(0)+\frac{\partial u}{\partial Y}(-1)+\frac{\partial u}{\partial Z}(1) = -\frac{\partial u}{\partial Y}+\frac{\partial u}{\partial Z} \quad \text{...(ii)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}.\frac{\partial X}{\partial y}+\frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial y}+\frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial y}$$

$$= \frac{\partial u}{\partial X}(1)+\frac{\partial u}{\partial Y}(0)+\frac{\partial u}{\partial Z}(-1) = \frac{\partial u}{\partial X}-\frac{\partial u}{\partial Z} \quad \text{...(iii)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}.\frac{\partial X}{\partial z}+\frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial z}+\frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial z}$$

$$= \frac{\partial u}{\partial X}(-1)+\frac{\partial u}{\partial Y}(1)+\frac{\partial u}{\partial Z}(0) = -\frac{\partial u}{\partial X}+\frac{\partial u}{\partial Y} \quad \text{...(iv)}$$

Now adding equations (ii), (iii) and (iv), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial Y}+\frac{\partial u}{\partial Z}+\frac{\partial u}{\partial X}-\frac{\partial u}{\partial Z}-\frac{\partial u}{\partial X}+\frac{\partial u}{\partial Y}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = 0$$

Proved

Prob.38. If $u = \log_e (x^3 + y^3 + z^3 - 3xyz)$, then prove that –

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{x+y+z}.$$

(R.G.P.V., June 2007)

Sol. Here, $u = \log_e (x^3 + y^3 + z^3 - 3xyz)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz} \quad \text{...(i)}$$

Similarly, $$\frac{\partial u}{\partial y}=\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz} \quad \text{...(ii)}$$

and $$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz} \quad \text{...(iii)}$$

Now adding equations (i), (ii) and (iii), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

or $$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \mathbf{\frac{3}{(x+y+z)}}$$ **Proved**

Prob.39. If $u = \log(x^3+y^3+z^3-3xyz)$, show that –

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$

(R.G.P.V., June 2005, Jan./Feb. 2008, Feb. 2010)

Sol. From Prob.38, we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3}{(x+y+z)}$$

Now, $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= 3\left[\frac{\partial}{\partial x}\frac{1}{(x+y+z)}+\frac{\partial}{\partial y}\frac{1}{(x+y+z)}+\frac{\partial}{\partial z}\frac{1}{(x+y+z)}\right]$$

$$= 3\left[-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}\right] = \frac{-9}{(x+y+z)^2}$$

i.e., $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\mathbf{\frac{9}{(x+y+z)^2}}$$ **Proved**

Prob.40. If $u = f\left(\frac{y}{x}\right)$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 0$.

[R.G.P.V., Dec. 2014, Nob. 2019 (O)]

Sol. Given, $u = f\left(\frac{y}{x}\right)$ ...(i)

Differentiating equation (i) partially with respect to x, we get

$$\frac{\partial u}{\partial x} = f'\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right)$$

$$x\frac{\partial u}{\partial x} = -\frac{y}{x}.f'\left(\frac{y}{x}\right) \qquad ...(ii)$$

Differentiating equation (i) partially with respect to y, we get

$$\frac{\partial u}{\partial y} = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$y\frac{\partial u}{\partial y} = \frac{y}{x}f'\left(\frac{y}{x}\right) \qquad \text{...(iii)}$$

Now adding equations (ii) and (iii), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{y}{x}f'\left(\frac{y}{x}\right) + \frac{y}{x}f'\left(\frac{y}{x}\right)$$

Hence $\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0}$ **Proved**

Prob.41. If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$***, then evaluate –***

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}.$$ ***(R.G.P.V., Dec. 2010)***

Sol. Given,

$$u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

or $$u = x^2\{y - z\} - y^2\{x - z\} + z^2\{x - y\} \qquad \text{...(i)}$$

Differentiating equation (i) w.r.t. x, y and z respectively

$$\frac{\partial u}{\partial x} = 2x(y - z) - y^2 + z^2 \qquad \text{...(ii)}$$

$$\frac{\partial u}{\partial y} = x^2 - 2y(x - z) - z^2 \qquad \text{...(iii)}$$

$$\frac{\partial u}{\partial z} = -x^2 + y^2 + 2z(x - y) \qquad \text{...(iv)}$$

On adding equations (ii), (iii) and (iv), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 2x(y - z) - y^2 + z^2 + x^2 - 2y(x - z) - z^2$$
$$- x^2 + y^2 + 2z(x - y)$$
$$= 2x(y - z) - 2y(x - z) + 2z(x - y)$$
$$= 2xy - 2xz - 2xy + 2yz + 2xz - 2yz$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \mathbf{0}$$ **Ans.**

Prob.42. If u(x, y, z) = log (tan x + tan y + tan z), prove that –

$$\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}=2$$

(R.G.P.V., June 2013)

Sol. Given, $u = \log(\tan x + \tan y + \tan z)$ ...(i)

Partially differentiating equation (i) with respect to x, y and z respectively, we get

$$\frac{\partial u}{\partial x}=\frac{\sec^2 x}{(\tan x+\tan y+\tan z)} \quad ...(ii)$$

$$\frac{\partial u}{\partial y}=\frac{\sec^2 y}{(\tan x+\tan y+\tan z)} \quad ...(iii)$$

and $$\frac{\partial u}{\partial z}=\frac{\sec^2 z}{(\tan x+\tan y+\tan z)} \quad ...(iv)$$

Multiplying equation (ii) by sin 2x, equation (iii) by sin 2y and equation (iv) by sin 2z, and adding, we get

$$\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}$$

$$=\frac{\sin 2x.\frac{1}{\cos^2 x}}{(\tan x+\tan y+\tan z)}+\frac{\sin 2y.\frac{1}{\cos^2 y}}{(\tan x+\tan y+\tan z)}+\frac{\sin 2z.\frac{1}{\cos^2 z}}{(\tan x+\tan y+\tan z)}$$

$$=\frac{2\tan x}{(\tan x+\tan y+\tan z)}+\frac{2\tan y}{(\tan x+\tan y+\tan z)}+\frac{2\tan z}{(\tan x+\tan y+\tan z)}$$

$$=2.\frac{(\tan x+\tan y+\tan z)}{(\tan x+\tan y+\tan z)}=\mathbf{2}$$ **Proved**

✓***Prob.43. If z(x + y) = (x² + y²), show that –***

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$$ ***(R.G.P.V., June 2011)***

Sol. Given, $z(x+y)=(x^2+y^2)$

or $$z=\frac{x^2+y^2}{x+y} \quad ...(i)$$

Differentiating equation (i) partially with respect to x and y, we get

$$\frac{\partial z}{\partial x}=\frac{(x+y)(2x)-(x^2+y^2).1}{(x+y)^2}$$

$$=\frac{2x^2+2xy-x^2-y^2}{(x+y)^2}=\frac{x^2-y^2+2xy}{(x+y)^2} \quad ...(ii)$$

and $$\frac{\partial z}{\partial y} = \frac{(x+y)(2y)-(x^2+y^2).1}{(x+y)^2}$$

$$= \frac{2xy+2y^2-x^2-y^2}{(x+y)^2} = \frac{y^2-x^2+2xy}{(x+y)^2} \quad ...(ii$$

Taking $$\text{R.H.S.} = 4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$$

$$= 4\left\{1-\frac{(x^2-y^2+2xy)}{(x+y)^2}-\frac{(y^2-x^2+2xy)}{(x+y)^2}\right\}$$

$$= 4\left\{\frac{x^2+y^2+2xy-x^2+y^2-2xy-y^2+x^2-2xy}{(x+y)^2}\right\}$$

$$= 4\left\{\frac{x^2+y^2-2xy}{(x+y)^2}\right\} = 4\frac{(x-y)^2}{(x+y)^2}$$

Now

$$\text{L.H.S.} = \left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2 = \left\{\frac{x^2-y^2+2xy}{(x+y)^2}-\frac{y^2-x^2+2xy}{(x+y)^2}\right\}^2$$

$$= \left\{\frac{x^2-y^2+2xy-y^2+x^2-2xy}{(x+y)^2}\right\}^2 = \frac{(2x^2-2y^2)^2}{(x+y)^4}$$

$$= \frac{4(x-y)^2(x+y)^2}{(x+y)^4} = \frac{4(x-y)^2}{(x+y)^2} = \textbf{R.H.S.}$$ **Prove**

Prob.44. If $x^x y^y z^z = C$, then show that at $x = y = z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -\{x(\log ex)\}^{-1}$$

[R.G.P.V., Dec. 2004, 2006, June 2008 (O), May 201

Or

If $x^x y^y z^z = C$, then show that –

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$$ ***(R.G.P.V., Dec. 2013, 201***

Or

If $x^x y^y z^z = C$, then show that

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1} \text{ for } x = y = z.$$ ***[R.G.P.V., Nov. 2018 (O***

Sol. Here, $x^x y^y z^z = C$ ...(i)

Taking logarithm on both sides of equation (i), we get

$$x \log x + y \log y + z \log z = \log C \quad ...(ii)$$

Differentiating equation (ii), partially with respect to x, (z is a function of x and y here), we get

$$\left(x.\frac{1}{x} + 1.\log x\right) + \left(z.\frac{1}{z} + 1.\log z\right)\frac{\partial z}{\partial x} = 0$$

or $$(1 + \log x) + (1 + \log z)\frac{\partial z}{\partial x} = 0$$

or $$\frac{\partial z}{\partial x} = -\frac{(1 + \log x)}{(1 + \log z)} \quad ...(iii)$$

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{(1 + \log y)}{(1 + \log z)} \quad ...(iv)$$

Now differentiating equation (iv), partially with respect to x, we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{1 + \log y}{1 + \log z}\right) = -(1 + \log y).\frac{\partial}{\partial x}\left(\frac{1}{(1 + \log z)}\right)$$

$$= -(1 + \log y)\frac{-1}{(1 + \log z)^2}.\frac{1}{z}.\frac{\partial z}{\partial x} = \frac{(1 + \log y)}{(1 + \log z)^2}.\frac{1}{z}.\frac{\partial z}{\partial x}$$

$$= \frac{1 + \log y}{(1 + \log z)^2}\left\{-\frac{1}{z}\left(\frac{1 + \log x}{1 + \log z}\right)\right\} \quad \text{[by equation (iii)]}$$

$$= -\frac{(1 + \log y)(1 + \log x)}{z(1 + \log z)^3} \quad ...(v)$$

Since x = y = z, so that equation (v) become

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1 + \log x)(1 + \log x)}{x(1 + \log x)^3} = -\frac{1}{x(1 + \log x)}$$

or $$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(\log_e e + \log x)} = -\frac{1}{x \log(ex)}$$

Hence, $$\frac{\partial^2 z}{\partial x \partial y} = -[\mathbf{x \log(ex)}]^{-1}$$ **Proved**

Prob.45. If $u = \sin^{-1}\frac{x + y}{\sqrt{x} + \sqrt{y}}$***, prove that –***

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

(R.G.P.V., Sept. 2009, Dec. 2011)

Sol. Suppose $u = \sin^{-1}z$, where $z = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function of x and y of degree 1/2.

Then, $$z = \sin u$$

Now applying Euler's theorem on z, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z \qquad \text{...(i)}$$

Putting the value of z in equation (i), we get

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}(\sin u)$$

or $$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\sin u}{\cos u}$$

Hence, $$\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u}$$ **Proved**

Prob.46. If $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, ***prove that –***

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u\cos 2u}{4\cos^3 u}$$

(R.G.P.V., Dec. 2011)

Sol. We know

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\{g'(u) - 1\}$$

$$= \frac{1}{2}\tan u\left\{\frac{1}{2}\sec^2 u - 1\right\} = \frac{1}{2}\frac{\sin u}{\cos u}\left\{\frac{1}{2\cos^2 u} - 1\right\}$$

$$= \frac{1}{2}\frac{\sin u}{\cos u}\left\{\frac{1-2\cos^2 u}{2\cos^2 u}\right\} = -\frac{\sin u\cos 2u}{4\cos^3 u}$$ **Proved**

Prob.47. If $u = \tan^{-1}\left(\dfrac{x^2+y^2}{x-y}\right)$, ***show that*** $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\sin 2u$.

(R.G.P.V., Dec. 2017)

Sol. Suppose, $u = \tan^{-1}z$, where $z = \dfrac{x^2+y^2}{x-y}$ is a homogeneous function of x and y of degree 1.

Then $$z = \tan u$$

Now applying Euler's theorem on z, we get

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = z \qquad ...(i)$$

Putting the value of z in equation (i), we get

$$x\frac{\partial \tan u}{\partial x}+y\frac{\partial \tan u}{\partial y} = \tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y} = \tan u \quad \text{or} \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{\tan u}{\sec^2 u}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{1}{2}.2\sin u\cos u$$

$$\mathbf{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u}$$ **Proved**

[redacted]***rob.48. If*** $u = \sin^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$, ***then show that*** $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y} = \tan u$.

(R.G.P.V., Nov./Dec. 2007, June 2009, Nov. 2018, 2019, June 2020)

[redacted]*l.* Suppose, $u = \sin^{-1}z$, where $z = \dfrac{x^2+y^2}{x+y}$ is a homogeneous function [redacted]d y of degree 1.

[redacted]en $z = \sin u$

[redacted]w applying Euler's theorem on z, we get

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = z \qquad ...(i)$$

[redacted]ting the value of z in equation (i), we get

$$x\frac{\partial \sin u}{\partial x}+y\frac{\partial \sin u}{\partial y} = \sin u$$

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y} = \sin u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u}$$

[redacted], $$\mathbf{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \tan u}$$ **Proved**

[redacted]***9. Verify Euler's theorem for the function***

[redacted] $\sin^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$. ***(R.G.P.V., May 2018)***

Sol. We have $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

Here, u is not a homogeneous function but if

$$v = \sin u = \frac{x^2+y^2}{x+y} \quad \text{...(i)}$$

then v is a homogeneous function in x, y of degree 1.

Hence we have to prove

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v$$

Now $$v = \frac{x^2+y^2}{x+y}$$

$$\therefore \quad \frac{\partial v}{\partial x} = \frac{(x+y).2x-(x^2+y^2).1}{(x+y)^2}$$

$$= \frac{2x^2+2xy-x^2-y^2}{(x+y)^2} = \frac{x^2-y^2+2xy}{(x+y)^2}$$

or $$x\frac{\partial v}{\partial x} = \frac{x^3-xy^2+2x^2y}{(x+y)^2} \quad \text{...(ii)}$$

Again $$\frac{\partial v}{\partial y} = \frac{(x+y).2y-(x^2+y^2).1}{(x+y)^2}$$

$$= \frac{2xy+2y^2-x^2-y^2}{(x+y)^2} = \frac{-x^2+y^2+2xy}{(x+y)^2}$$

or $$y\frac{\partial v}{\partial y} = \frac{-x^2y+y^3+2xy^2}{(x+y)^2} \quad \text{...(iii)}$$

Now adding equations (ii) and (iii), we get

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{x^3-xy^2+2x^2y-x^2y+y^3+2xy^2}{(x+y)^2}$$

$$= \frac{x^3+xy^2+x^2y+y^3}{(x+y)^2}$$

$$= \frac{(x+y)(x^2+y^2)}{(x+y)^2} = \frac{x^2+y^2}{x+y} = v$$

i.e. $$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v$$

Hence theorem is verified **Proved**

Prob.50. ***If*** $u = sec^{-1}\left(\frac{x^3 - y^3}{x + y}\right)$, ***then prove that***

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u.$$

(R.G.P.V., Dec. 2015)

Sol. Given, $$u = \sec^{-1}\left(\frac{x^3 - y^3}{x + y}\right)$$

or $$\sec u = \frac{x^3 - y^3}{x + y} \quad ...(i)$$

Equation (i) is a homogeneous function of degree 2 in x and y.

Suppose, $z = \sec u$

Now applying Euler's theorem on z, we get

$$x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = 2z \quad ...(ii)$$

Putting z = sec u in equation (ii), we get

$$x\frac{\partial}{\partial x}(\sec u) + y\frac{\partial}{\partial y}(\sec u) = 2\sec u$$

$$x.\sec u.\tan u\frac{\partial u}{\partial x} + y\sec u.\tan u\frac{\partial u}{\partial y} = 2\sec u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2}{\tan u} \text{ or } \mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u} \quad \textbf{Ans.}$$

Prob.51. ***If*** $u = log\left(\frac{x^4 + y^4}{x + y}\right)$, ***show that*** $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$.

[R.G.P.V., Dec. 2016, Nov. 2018 (O), 2019 (O)]

Sol. Let, $\frac{x^4 + y^4}{x + y} = z$

Then $$z = e^u = \frac{x^4 + y^4}{x + y} = x^3.\frac{1 + (y/x)^4}{1 + (y/x)}$$

∴ z is a homogeneous function of degree 3 in x and y.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z \quad ...(i)$$

But, $$\frac{\partial z}{\partial x} = e^u\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = e^u\frac{\partial u}{\partial y}$$

Hence equation (i) becomes

$$x.e^u \frac{\partial u}{\partial x} + y.e^u \frac{\partial u}{\partial y} = 3e^u$$

or $$\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3}$$ **Proved**

Prob.52. If $u = \log\left(\frac{x^3+y^3}{x^2-y^2}\right)$, ***then find*** $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

(R.G.P.V., June 2017)

Sol. Let, $\frac{x^3+y^3}{x^2-y^2} = z$

Then $z = e^u = \frac{x^3+y^3}{x^2-y^2} = x.\frac{1+(y/x)^3}{1-(y/x)^2}$

$\therefore$ z is a homogeneous function of degree 1 in x and y.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \quad ...(i)$$

But $\frac{\partial z}{\partial x} = e^u \frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = e^u \frac{\partial u}{\partial y}$

Hence equation (i) becomes

$$x.e^u \frac{\partial u}{\partial x} + y.e^u \frac{\partial u}{\partial y} = e^u \text{ or } \mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1}$$ **Ans**

✓***Prob.53. If*** $u = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$, ***evaluate –***

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$ ***(R.G.P.V., March/April 2010)***

Sol. Given,

$$u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right) = u_1 - u_2 \quad ...(i)$$

$$\therefore \quad u_1 = x^2 \tan^{-1}\left(\frac{y}{x}\right) \text{ and } u_2 = y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

Clearly u_1 is a homogeneous function of degree 2. Therefore

By Euler's theorem,

$$x^2\frac{\partial^2 u_1}{\partial x^2} + 2xy\frac{\partial^2 u_1}{\partial x \partial y} + y^2\frac{\partial^2 u_1}{\partial y^2} = 2(2-1)u_1$$

or $$x^2\frac{\partial^2 u_1}{\partial x^2} + 2xy\frac{\partial^2 u_1}{\partial x \partial y} + y^2\frac{\partial^2 u_1}{\partial y^2} = 2u_1 \quad ...(ii)$$

Also u_2 is a homogeneous function of degree 2. Therefore

By Euler's theorem,

$$x^2\frac{\partial^2 u_2}{\partial x^2}+2xy\frac{\partial^2 u_2}{\partial x\partial y}+y^2\frac{\partial^2 u_2}{\partial y^2} = 2(2-1)u_2$$

or
$$x^2\frac{\partial^2 u_2}{\partial x^2}+2xy\frac{\partial^2 u_2}{\partial x\partial y}+y^2\frac{\partial^2 u_2}{\partial y^2} = 2u_2 \qquad ...(iii)$$

On subtracting equation (iii) from equation (ii), we get

$$x^2\frac{\partial^2}{\partial x^2}(u_1-u_2)+2xy\frac{\partial^2}{\partial x\partial y}(u_1-u_2)+y^2\frac{\partial^2}{\partial y^2}(u_1-u_2) = 2u_1-2u_2$$

or $\mathbf{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = 2u}$ [(from equation (i)] **Ans.**

Prob.54. If $u = x\phi(y/x) + \psi(y/x)$, then prove that –

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0$$

(R.G.P.V., June 2012, Dec. 2012)

Sol. Here $u = x\,\phi(y/x)+\psi(y/x)$ ...(i)

Differentiating equation (i), partially w.r. to x, we get

$$\frac{\partial u}{\partial x} = -\frac{y}{x}\phi'\left(\frac{y}{x}\right)+\phi\left(\frac{y}{x}\right)-\frac{y}{x^2}\psi'\left(\frac{y}{x}\right)$$

Again

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2}{x^3}\phi''\left(\frac{y}{x}\right)+\frac{y^2}{x^4}\psi''\left(\frac{y}{x}\right)+\frac{2y}{x^3}\psi'\left(\frac{y}{x}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} = \frac{y^2}{x}\phi''\left(\frac{y}{x}\right)+\frac{y^2}{x^2}\psi''\left(\frac{y}{x}\right)+\frac{2y}{x}\psi'\left(\frac{y}{x}\right) \qquad ...(ii)$$

Again differentiating equation (i) partially w.r. to y, we get

$$\frac{\partial u}{\partial y} = \phi'\left(\frac{y}{x}\right)+\frac{1}{x}\psi'\left(\frac{y}{x}\right) \qquad ...(iii)$$

Again

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{x}\phi''\left(\frac{y}{x}\right)+\frac{1}{x^2}\psi''\left(\frac{y}{x}\right)$$

$$y^2\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{x}\phi''\left(\frac{y}{x}\right)+\frac{y^2}{x^2}\psi''\left(\frac{y}{x}\right) \qquad ...(iv)$$

Now differentiating equation (iii), w.r.t. "x", we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{-y}{x^2}\phi''\left(\frac{y}{x}\right) - \frac{y}{x^3}\psi''\left(\frac{y}{x}\right) - \frac{1}{x^2}\psi'\left(\frac{y}{x}\right)$$

$$2xy\frac{\partial^2 u}{\partial x \partial y} = -\frac{2y^2}{x}\phi''\left(\frac{y}{x}\right) - \frac{2y^2}{x^2}\psi''\left(\frac{y}{x}\right) - \frac{2y}{x}\psi'\left(\frac{y}{x}\right) \quad ...($$

Adding equations (ii), (iv) and (v), we have

$$\mathbf{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0}$$ **Prov**

Prob.55. If $u = \tan^{-1}\dfrac{x^3 + y^3}{x - y}$***, prove that –***

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$

(R.G.P.V., Dec. 2002, 2003, 2005, April 20

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u.$

(R.G.P.V., Jan./Feb. 2008, April 20

Sol. (i) Given,

$$u = \tan^{-1}\left[\frac{x^3 + y^3}{x - y}\right] \text{ or } \tan u = \frac{x^3 + y^3}{x - y} \quad ..($$

Equation (i) is a homogeneous function of degree 2 in x and y.

Suppose, $z = \tan u$

Now applying Euler's theorem on z, we get

$$x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = 2z \quad ..($$

Putting $z = \tan u$ in equation (ii), we get

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

or $$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} \text{ or } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u \cos u$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \mathbf{\sin 2u} \quad ...(iii)$$ **Prov**

(ii) Differentiating equation (iii) partially with respect to x, we get

$$\left\{x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\right\}+y\frac{\partial^2 u}{\partial x \partial y}=2\cos 2u\ \frac{\partial u}{\partial x} \qquad \text{...(iv)}$$

Again differentiating equation (iii) partially with respect to y, we get

$$x\frac{\partial^2 u}{\partial y \partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=2\cos 2u\ \frac{\partial u}{\partial y} \qquad \text{...(v)}$$

Multiplying equation (iv) by x and equation (v) by y, then adding we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=2\cos 2u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\sin 2u=2\cos 2u \sin 2u$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2\cos 2u \sin 2u-\sin 2u$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 4u-\sin 2u$$

Hence $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\mathbf{2\cos 3u \sin u}$$ **Proved**

MAXIMA AND MINIMA (TWO AND THREE VARIABLES), METHOD OF LAGRANGES MULTIPLIERS

Maxima and Minima –

Definition – A function f(x) is called to have a maximum value at x = a, here exists a small number h, however small, such that

$$f(a) > \text{both } f(a-h) \text{ and } f(a+h)$$

A function f(x) is called to have a minimum value at x = a, if there exists nall number h, however small, such that

$$f(a) < \text{both } f(a-h) \text{ and } f(a+h)$$

Procedure for Finding Maxima and Minima –

(i) Firstly, put the given function = f(x)

(ii) Find derivative of function f(x) [i.e., f '(x)] and equate it to zero. e this equation and let its roots be a, b, c,..........

(iii) Find f"(x) and putting in it by turn x = a, b, c, ...

If f"(a) is negative then f(x) is maxima at x = a.

If f"(a) is positive then f(x) is minima at x = a.

(iv) Sometimes f"(x) may be difficult to find out or f "(x) may zero at x = a.

In such case, we see that, if f'(x) changes sign from positive to negati as x passes through a, then f(x) is maximum at x = 0.

If f'(x) changes sign from negative to positive as x passes through f(x) is minimum at x = a.

If f'(x) does not change sign while passing through x = a, then f(x) neither maximum nor minimum at x = a.

Two Variables Connected by a Relation – Let the function whose maxi and minima are to be obtained is expressed as a function of two variable connected by a given relation, one of the variables should first be eliminated

Maxima and Minima of Function of Two Variables –

Definition – A function f(x, y) is said to have a maximum or minimum x = a, y = b according as –

$$f(a + h, b + k) < \text{or} > f(a, b)$$

whatever be the relative values of h and k, positive or negative, provided h a k are finite and sufficiently small.

Considering, z = f(x, y) as a surface, maximum value of z occurs at top of an elevation (e.g. a dome) from which the surface descends in eve direction and a minimum value occurs at the bottom of a depression (example, a bowl) from which the surface ascends in every direction. Besid this, we have such a point of the surface, where the tangent plane is horizon and the surface falls for displacement in certain directions and rises displacements in other directions. Such a point is said to be a saddle point

Working Rule – (i) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate each to zero. Sol these as simultaneous equations in x and y. Suppose (a, b), (c, d) .... the pairs of values.

(ii) Calculate the values of, $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$ $t = \frac{\partial^2 f}{\partial y^2}$ for ea pair of values.

(a) If $rt - s^2 > 0$ and $r < 0$ at (a, b) then f(a, b) is a maximum val

(b) If $rt - s^2 > 0$ and $r > 0$ at (a, b), f(a, b) is a minimum val

(c) If $rt - s^2 < 0$ at (a, b), f(a, b) is not an extreme value i (a, b) is a saddle point.

(d) If $rt - s^2 = 0$ at (a, b), the case is doubtful and needs furt investigation.

Similarly, examine the other pairs of values one by one.

Note – Stationary Value – f(a, b) is called a stationary value of f(x, y), if $f_x(a, b) = 0$ and $f_y(a, b) = 0$ i.e., the function is stationary at (a, b).

Maxima and Minima of Function of Three Variables – Assume f(x, y, z) is a given function of three independent variables x, y and z. Evaluate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ and solve the simultaneous equation $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$. All the triads (a, b, c) of the values of x, y and z determined on solving these equations will give the stationary values of f(x, y, z) i.e., will give the points at which the function f(x, y, z) may be a maxima or a minima.

To discuss the maximum or minimum of f(x, y, z) at any point (a, b, c) determined on solving the equations $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$, we find the values at this point of the six partial derivatives of second order of f(x, y, z) symbolically denoted as given below –

$$P = \frac{\partial^2 f}{\partial x^2}, Q = \frac{\partial^2 f}{\partial y^2}, R = \frac{\partial^2 f}{\partial z^2}, S = \frac{\partial^2 f}{\partial y \partial z}, T = \frac{\partial^2 f}{\partial z\, \partial x} \text{ and } U = \frac{\partial^2 f}{\partial x \partial y}.$$

If the expressions

$$P, \begin{vmatrix} P & U \\ U & Q \end{vmatrix}, \begin{vmatrix} P & U & T \\ U & Q & S \\ T & S & R \end{vmatrix}$$

all positive, we shall have a minimum of f(x, y, z) at (a, b, c) and if these pressions be alternately negative and positive, we shall have a maximum f(x, y, z) at (a, b, c), whilst if these conditions are not satisfied, we shall general have neither a maximum nor a minimum of f(x, y, z) at (a, b, c).

Lagrange's Method of Undetermined Multipliers –

(i) Write $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$

(ii) Find the equations $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$

(iii) Solve the above equations together with $\phi(x, y, z) = 0$.

The values of x, y, z so found will give the stationary value of f(x, y, z).

NUMERICAL PROBLEMS

Prob.56. Find the maximum and minimum value of the function = $x^2 - 4x + 5$ in the interval $1 \le x \le 4$. ***(R.G.P.V., Dec. 2005)***

Sol. Here, $f(x) = x^2 - 4x + 5$

$\therefore$ $f'(x) = 2x - 4$

For maximum and minimum

$$f'(x) = 0 \Rightarrow 2x - 4 = 0$$

$$\Rightarrow \quad x = 2$$

$\because$ 2 lies in the interval $1 \le x \le 4$.

Now $\quad f''(x) = 2 > 0$

f(x) is minimum at $x = 2$

Then minimum value of f(x), at x = 2 is

$$f(2) = (2)^2 - 4 \times 2 + 5 = 4 - 8 + 5 = 1$$ An

Prob.57. Show that sin x (1 + cos x) is a maximum when $x = \frac{\pi}{3}$.

[R.G.P.V., June 2008(N), 200

Sol. Let, $\quad f(x) = \sin x\,(1 + \cos x)$

Then
$$\begin{aligned} f'(x) &= \cos x\,(1 + \cos x) + \sin x\,(-\sin x) \\ &= \cos x\,(1 + \cos x) - (1 - \cos^2 x) \\ &= (1 + \cos x)\,(2\cos x - 1) \end{aligned}$$

$\therefore \quad f'(x) = 0$, when $\cos x = \frac{1}{2}$ or -1 i.e., $x = \pi/3$ or

Now,
$$\begin{aligned} f''(x) &= -\sin x\,(2\cos x - 1) + (1 + \cos x)\,(-2\sin \\ &= -\sin x\,(4\cos x + 1) \end{aligned}$$

So that $\quad f''(\pi/3) = -\frac{3\sqrt{3}}{2}$ and $f''(\pi) = 0$

Hence, f(x) has a maximum at $x = \pi/3$.

Since $f''(\pi)$ is 0, let us see whether $f'(x)$ changes sign or not.

When x is slightly $< \pi$, $f'(x)$ is –ve, when x is slightly $> \pi$, $f'(x)$ is ag –ve i.e., $f'(x)$ does not change sign as x passes through π. So f(x) is neit maximum nor minimum at $x = \pi$. Prov

Prob.58. Find the largest and smallest values of $x^3 - 18x^2 + 96x$ in t interval (0, 9). ***(R.G.P.V., Jan./Feb. 20***

Sol. Let, $f(x) = x^3 - 18x^2 + 96x$

On differentiation, we have

$$f'(x) = 3x^2 - 36x + 96$$

and $$f''(x) = 6x - 36$$

Equating $f'(x)$ to zero, we get

$$x^2 - 36x + 96 = 0$$

or $$x^2 - 12x + 32 = 0$$

or $$(x - 8)(x - 4) = 0$$

or $$x = 4, 8, \text{ both of which lie in } (0, 9)$$ a

When x = 4, $f''(4) = 6(4) - 36 = -12 = -ve$

$\therefore$ f(x) is largest, when x = 4 and its maximum value

$$= (4)^3 - 18(4)^2 + 96(4) = 64 - 288 + 384$$
$$= 448 - 288 = \mathbf{160}$$ **Ans.**

When x = 8, $f''(8) = 6(8) - 36 = 12 = +ve$

$\therefore$ f(x) is minimum when x = 8, and its minimum value

$$= (8)^3 - 18(8)^2 + 96(8) = 512 - 1152 + 768 = \mathbf{128}$$ **Ans.**

Prob.59. Show that the diameter of the right circular cylinder of greatest urved surface which can be inscribed in a given cone is equal to the radius f the cone. ***(R.G.P.V., March/April 2010)***

Sol. Let r be the radius OA of the base and α the semi-vertical angle of e given cone. Inscribe a cylinder in it with base-radius OL = x.

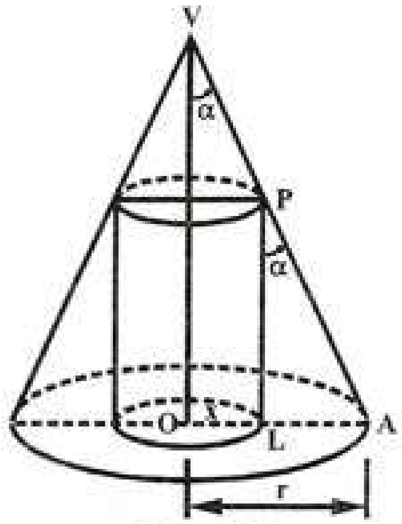


Fig. 1.3

Then the height of the cylinder LP

$$= LA \cot\alpha = (r - x)\cot\alpha$$

The curved surface S of the cylinder

$$= 2\pi x.LP = 2\pi x\,(r - x)\cot\alpha$$
$$= 2\pi\cot\alpha\,(rx - x^2)$$

$$\therefore \quad \frac{dS}{dx} = 2\pi\cot\alpha\,(r - 2x) = 0 \quad \text{for } x = \frac{r}{2}$$

$$\frac{d^2S}{dx^2} = -4\pi\cot\alpha$$

Hence S is maximum when $x = \frac{r}{2}$

rob.60. Find the maxima and minima of the following function –

$$\boldsymbol{\sin x + \sin y + \sin(x + y) \text{ in } \left[0 \le x \le \frac{\pi}{2}, 0 \le y \le \frac{\pi}{2}\right].}$$

(R.G.P.V., June 2012)

l. Here given function

t, $f(x, y) = \sin x + \sin y + \sin(x + y)$

$$p = \frac{\partial f(x,y)}{\partial x} = \cos x + \cos(x + y)$$

$$q = \frac{\partial f(x,y)}{\partial y} = \cos y + \cos(x + y)$$

maxima and minima

$$\frac{\partial f(x,y)}{\partial x} = 0, \quad \frac{\partial f(x,y)}{\partial y} = 0$$

$$\cos x + \cos(x + y) = 0 \Rightarrow \cos(x + y) = -\cos x \qquad \text{...(i)}$$
$$\cos y + \cos(x + y) = 0 \Rightarrow \cos(x + y) = -\cos y \qquad \text{....(ii)}$$

From (i) and (ii), we have

$$\cos x = \cos y$$

or $$x = y$$

Now $$r = \frac{\partial^2 f(x,y)}{\partial x^2} = -\sin x - \sin(x+y)$$

$$t = \frac{\partial^2 f(x,y)}{\partial y^2} = -\sin y - \sin(x+y)$$

$$s = \frac{\partial^2 f(x,y)}{\partial x \partial y} = -\sin(x+y)$$

At (0, 0) $\quad r = 0, s = 0, t = 0$ ...(ii

From (iii) $rt - s^2 = 0$ and therefore further investigation is required.

At $[\pi/2, \pi/2]$ $\quad r = -1, s = 0, t = -1$ ...(i

$$rt - s^2 = (-1) \times (-1) - (0)^2 = 1 > 0$$

Hence f(x, y) has a minimum at $(\pi/2, \pi/2)$ and minimum value at $(\pi/2, \pi/2$

$$f(\pi/2, \pi/2) = 1.$$

For points along the time y = x.

$$f(x, y) = 2 \sin x + \sin 2x$$

which is positive.

Thus in the neighbourhood of (0, 0), there are points where f(x, y) < f(0, and there are points where f(x, y) > f(0, 0).

Hence f(0, 0) is not a extreme value.

Prob.61. Discuss the maxima and minima of –

$$f(x, y) = x^3y^2(1 - x - y) \qquad \textit{(R.G.P.V., June 201}$$

Or

Discuss the maxima and minima of the function $u = x^3y^2(1 - x - y$

[R.G.P.V., Nov. 2018 (0

Or

Discuss the maximum and minimum value of $u = x^3y^2(1 - x - y)$.

(R.G.P.V., May 201

Sol. For maxima and minima of f(x, y), we find

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$$\Rightarrow x^2y^2(3 - 4x - 3y) = 0 \qquad ...($$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 2x^3y - 2x^4y - 3x^3y^2 = 0$$

$$\Rightarrow x^3y(2 - 2x - 3y) = 0 \qquad ...($$

On solving equations (i) and (ii), we get

$$x = \frac{1}{2}, y = \frac{1}{3}$$

Now, we find $$r = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$$

$= \frac{1}{2}$ and $y = \frac{1}{3}$ $\qquad r = -1/9,\ s = -\frac{1}{12}$ and $t = -\frac{1}{8}$

Now for maxima and minima, we calculate

$$rt - s^2 = \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2 = \frac{1}{72} - \frac{1}{144} = \frac{1}{144} = +\text{ve}$$

Since $r < 0$ and $rt - s^2 > 0$ therefore f(x, y) has a maximum at (1/2, 1/3)

Maximum value of

$$f(x, y) = \left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(1 - \frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{1}{72}\left(\frac{6-3-2}{6}\right) = \frac{1}{72} \times \frac{1}{6} = \mathbf{\frac{1}{432}} \qquad \textbf{Ans.}$$

Prob.62. Prove that if the perimeter of a triangle is constant, its area is mum when the triangle is equilateral. ***(R.G.P.V., June 2013)***

Sol. Let a, b, c be the sides of a triangle whose perimeter 2s is constant.

$\therefore\ 2s = a + b + c \Rightarrow c = -a - b + 2s$ ...(i)

We know

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{s(s-a)(s-b)(a+b-s)}$$

equation (i)]

et, $z = \Delta^2 = s(s-a)(s-b)(a+b-s) = f(a,b)$ (say)

$$\therefore\ \frac{\partial f}{\partial a} = f_a = s(s-b)\frac{\partial}{\partial a}[(s-a)(a+b-s)]$$

$$= s(s-b)[-(a+b-s)+(s-a)] = s(s-b)(2s-2a-b)$$

$$\frac{\partial f}{\partial b} = f_b = s(s-a)\frac{\partial}{\partial b}[(s-b)(a+b-s)]$$

$$= s(s-a)[-(a+b-s)+(s-b)] = s(s-a)(2s-a-2b)$$

$$\frac{\partial^2 f}{\partial a^2} = r = f_{aa} = -2s(s-b)$$

$$\frac{\partial^2 f}{\partial a \partial b} = s = f_{ab} = s[-(2s-a-2b)-(s-a)] = s(2a+2b-3s)$$

$$\frac{\partial^2 f}{\partial b^2} = t = f_{bb} = -2s(s-a)$$

Now for maxima and minima, we have

$$f_a = 0 \text{ and } f_b = 0$$

$$\Rightarrow \quad s(s-b)(2s-2a-b) = 0 \text{ and } s(s-a)(2s-a-2b) = 0$$

$$\Rightarrow \quad (s-b)(2s-2a-b) = 0 \quad \text{...(ii)}$$

$$\Rightarrow \quad (s-a)(2s-a-2b) = 0 \quad \text{...(iii)}$$

From equation (ii), $s = b$ or $2s = 2a + b$

When $s = b$, from equation (iii), we get

$$(b-a)(-a) = 0 \text{ or } b = a$$

When $2s = 2a + b$, from equation (iii), we get

$$\frac{b}{2}(a-b) = 0 \text{ or } a = b$$

Similarly we get $b = c$

$$\therefore \quad a = b = c = \frac{2s}{3}$$

Now $r = -2s\left(\frac{s}{3}\right) = -\frac{2s^2}{3} < 0$

$$s = s\left(\frac{4s}{3} + \frac{4s}{3} - 3s\right) = s\left(-\frac{s}{3}\right) = -\frac{s^2}{3}$$

$$t = -2s\left(\frac{s}{3}\right) = -\frac{2s^2}{3}$$

Now $rt - s^2 = \frac{4s^4}{9} - \frac{s^4}{9} = \frac{s^4}{3} > 0$ and $r < 0$

$\therefore$ Δ is maximum when $a = b = c = \frac{2s}{3}$, when the triangle is equilateral.

Proved

Prob.63. Discuss maxima and minima of the function
$f(x, y) = x^3 - 4xy + 2y^2$. ***(R.G.P.V., Dec. 2016)***

Or

Discuss the maximum or minima of the function
$f(x, y) = x^3 - 4xy + 2y^2$.

(R.G.P.V., Nov 2019, June 2020)

Sol. Given, $f(x, y) = x^3 - 4xy + 2y^2$ ...(i)

Differentiating equation (i), we have

$$f_x = \frac{\partial f}{\partial x} = 3x^2 - 4y$$

$$f_y = \frac{\partial f}{\partial y} = -4x + 4y$$

and

$$r = f_{xx} = 6x$$

$$s = f_{xy} = -4, \quad t = f_{yy} = 4$$

Now for maxima or minima, we must have $f_x = 0$, $f_y = 0$, we have

$$3x^2 - 4y = 0 \quad \text{...(ii)}$$

$$-4x + 4y = 0 \quad \text{...(iii)}$$

Solving equations (ii) and (iii), we get

$$3x^2 - 4x = 0 \text{ or } x(3x-4) = 0 \text{ or } x = 0, \frac{4}{3}$$

Now from equation (iii), we have

$$x = 0 \Rightarrow y = 0 \text{ and when } x = \frac{4}{3} \Rightarrow y = \frac{4}{3}$$

Thus the required stationary points are (0, 0) and $\left(\frac{4}{3}, \frac{4}{3}\right)$

At point $\left(\frac{4}{3}, \frac{4}{3}\right)$

$$r = 6 \times \frac{4}{3} = 8, s = -4, t = 4$$

$$rt - s^2 = 8 \times 4 - (-4)^2 = 32 - 16 = 16 > 0 \text{ and } r > 0$$

Hence, f(x, y) has a minima at $\left(\frac{4}{3}, \frac{4}{3}\right)$.

At point (0, 0) $\quad r = 6 \times 0 = 0, s = -4, t = 4$

$$rt - s^2 = 0 - (-4)^2 = -16 = -ve$$

Hence there is neither maxima nor minima at (0, 0).

Minimum value of $f(x, y) = \left[x^3 - 4xy + 2y^2\right]_{x=\frac{4}{3}, y=\frac{4}{3}}$

$$= \left(\frac{4}{3}\right)^3 - 4.\frac{4}{3}.\frac{4}{3} + 2.\left(\frac{4}{3}\right)^2$$

$$= \frac{64}{27} - \frac{64}{9} + \frac{32}{9} = \frac{64 - 192 + 96}{27} = -\frac{32}{27}$$ **Ans.**

Prob.64. Discuss the maxima and minima of the function f(x, y) = x³ + y³ – 3xy. ***(R.G.P.V., June 2017)***

Or

Discuss the maximum and minimum of $x^3 + y^3 - 3xy$.

(R.G.P.V., June 2015, Dec. 2017)

Ans. Given, $f(x, y) = x^3 + y^3 - 3xy$ ...(i)

Differentiating equation (i), we have

$$f_x = \frac{\partial f}{\partial x} = 3x^2 - 3y$$

$$f_y = \frac{\partial f}{\partial y} = 3y^2 - 3x$$

$$r = f_{xx} = 6x$$

$$s = f_{xy} = -3$$

$$t = f_{yy} = 6y$$

Now for maxima or minima, we must have $f_x = 0$, $f_y = 0$, we have

$$x^2 - y = 0 \quad \text{...(ii)}$$

and

$$y^2 - x = 0 \quad \text{...(iii)}$$

Solving equations (ii) and (iii), we get

$$(y^2)^2 - y = 0$$

or

$$y(y^3 - 1) = 0$$

or

$$y = 0, 1$$

Now from equation (ii), we have

when $y = 0 \Rightarrow x = 0$ and when $y = 1 \Rightarrow x = \pm 1$

But $x = -1$, $y = 1$, do not satisfy equation (iii) hence are not solutions.

Hence, the solutions are $x = 0$, $y = 0$ and $x = 1$, $y = 1$

At $x = 0$, $y = 0$, we have

$$r = 0, s = -3, t = 0$$

$$\therefore \quad rt - s^2 = 0 - (-3)^2 = -\text{ ve}$$

and hence there is neither maximum nor minimum at (0, 0).

At $x = 1$, $y = 1$, we have

$$r = 6, s = -3, t = 6$$

$$\therefore \quad rt - s^2 = 6.6 - (-3)^2 = 36 - 9 = 27 > 0$$

Also, $r = 6x = 6 \times 1 = 6 > 0$

Hence f(x, y) has a minimum at (1, 1)

Substituting (1, 1) in the equation (i), we get minimum value of f(x, y) = $(1)^3 + (1)^3 - 3.1.1$

$$= 1 + 1 - 3 = -1$$ **Ans.**

Prob.65. Discuss the maxima and minima of the function

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20$$ ***[R.G.P.V., Nov. 2019 (O)]***

Sol. Given, $f(x, y) = x^3 + y^3 - 3x - 12y + 20$ ...(i)

Differentiating equation (i), we have

$$f_x = \frac{\partial f}{\partial x} = 3x^2 - 3$$

$$f_y = \frac{\partial f}{\partial y} = 3y^2 - 12$$

and

$$r = f_{xx} = 6x$$

$$s = f_{xy} = 0$$

$$t = f_{yy} = 6y$$

Now for maxima or minima, we must have $f_x = 0$, $f_y = 0$, we have

$$x^2 - 1 = 0 \quad \text{...(ii)}$$

$$y^2 - 4 = 0 \quad \text{...(iii)}$$

Solving equations (ii) and (iii), we get

$$x = \pm 1 \text{ and } y = \pm 2$$

= 1, y = 2, we have

$$r = 6x = 6 \times 1 = 6$$
$$s = 0$$
$$t = 6y = 6 \times 2 = 12$$
$$rt - s^2 = 6 \times 12 - 0 = 72 > 0 \text{ and } r > 0$$

e f(x, y) has a minimum at (1, 2).

= – 1, y = – 2, we have

$$r = 6x = 6 \times (-1) = -6$$
$$= 0$$
$$= 6y = 6 \times (-2) = -12$$
$$= (-6) \times (-12) - 0 = 72 > 0 \text{ and } r < 0$$

f(x, y) has a maximum at (– 1, – 2). **Ans.**

66. Discuss the maximum and minimum value of $x^3 + y^3 - 3axy$.
(R.G.P.V., Dec. 2013)

Or

ss the maxima and minima of the function $x^3 + y^3 - 3axy$.
(R.G.P.V., Dec. 2014, Nov. 2018)

Or

he extreme values of the function $x^3 + y^3 - 3axy$.
(R.G.P.V., June 2014)

Or

he maximum and minimum value of the function $x^3 + y^3 - 3axy$.
(R.G.P.V., May 2018)

iven, $u = x^3 + y^3 - 3axy$ ...(i)

ntiating equation (i), we have

$$= \frac{\partial u}{\partial x} = 3x^2 - 3ay, \; q = \frac{\partial u}{\partial y} = 3y^2 - 3ax$$

$$\frac{\partial^2 u}{\partial x^2} = 6x, \; s = \frac{\partial^2 u}{\partial x \partial y} = -3a \quad \text{and} \quad t = \frac{\partial^2 u}{\partial y^2} = 6y$$

r maximum or minimum of u, we must have

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0$$

n $\frac{\partial u}{\partial x} = 0$, we get $x^2 - ay = 0$ ...(ii)

$\frac{\partial u}{\partial y} = 0$, we get $y^2 - ax = 0$ ...(iii)

equations (ii) and (iii), we get

$/a)^2 - ay = 0$ or $y^4 - a^3y = 0$ or $y(y^3 - a^3) = 0$ or $y = 0, a$

m equation (ii), we have

$\Rightarrow x = 0$, and when $y = a \Rightarrow x = \pm a$

But $x = -a$, $y = a$, do not satisfy equation (iii) hence are not solutions.

Hence, the solutions are $x = 0, y = 0$ and $x = a, y = a$

At $x = 0, y = 0$, we have

$$r = 0, s = -3a, t = 0$$

$$\therefore \quad rt - s^2 = 0 - (-3a)^2 = -ve$$

and hence there is neither maximum nor minimum at $x = 0, y = 0$.

At $x = a, y = a$, we have

$$r = 6a, s = -3a, t = 6a$$

$$\therefore \quad rt - s^2 = (6a)(6a) - (-3a)^2 = 36a^2 - 9a^2 > 0$$

Also, $r = 6a > 0$, if $a > 0$ and $r < 0$, if $a < 0$.

Hence, there is maximum or minimum according to $a < 0$ or $a > 0$. Ans.

Prob.67. Locate the stationary points of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and determine their nature. ***(R.G.P.V., Dec. 2015)***

Sol. Here, $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ ...(i)

Differentiating equation (i) partially w.r. to x and y respectively, we have

$$f_x = \frac{\partial f}{\partial x} = 4x^3 - 4x + 4y$$

$$f_y = \frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$$

and $$r = f_{xx} = 12x^2 - 4$$

$$s = f_{xy} = 4, \quad t = f_{yy} = 12y^2 - 4$$

When $f_x = 0, f_y = 0$, we have

$$4x^3 - 4x + 4y = 0$$

or $$x^3 - x + y = 0 \quad \text{...(ii)}$$

and $$4y^3 + 4x - 4y = 0$$

or $$y^3 + x - y = 0 \quad \text{...(iii)}$$

Adding equations (ii) and (iii), we get

$$x^3 + y^3 = 0$$

$$(x + y)(x^2 - xy + y^2) = 0$$

$$x = -y$$

Putting in equation (ii), we get

$$x^3 - 2x = 0$$

$$\Rightarrow \quad x = 0, x = \sqrt{2}$$

$$\therefore \quad y = 0, y = -\sqrt{2}$$

Thus the required stationary points are (0, 0) and $(\sqrt{2}, -\sqrt{2})$

Case-I – At point $(\sqrt{2}, -\sqrt{2})$

$$r = 12(\sqrt{2})^2 - 4 = 12 \times 2 - 4 = 24 - 4 = 20 > 0$$

$$s = 4 \text{ and } t = 12(-\sqrt{2})^2 - 4 = 24 - 4 = 20$$

$$\therefore \quad rt - s^2 = 20 \times 20 - (4)^2 = 400 - 16 = 384 > 0$$

Therefore f(x, y) is minimum at $(\sqrt{2}, -\sqrt{2})$

Case-II – At point (0, 0)

$$r = 12(0)^2 - 4 = -4 < 0$$

$$s = 4 \text{ and } t = 12(0)^2 - 4 = -4$$

$$\therefore \quad rt - s^2 = (-4)(-4) - (4)^2 = 16 - 16 = 0$$

The condition is doubt full and further investigation is needed.

Prob.68. If $p = x\cos\alpha + y\sin\alpha$, touches the curve

$$\left(\frac{x}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}} = 1, \textit{ prove that} -$$

$$p^n = (a\cos\alpha)^n + (b\sin\alpha)^n \qquad \textit{(R.G.P.V., Dec. 2011)}$$

Sol. Let, $f(x, y) = \left(\frac{x}{a}\right)^{n/(n-1)} + \left(\frac{y}{b}\right)^{n/(n-1)} - 1 = 0 \qquad ...(i)$

Differentiating equation (i), we get

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{\frac{1}{a}\left(\frac{x}{a}\right)^{1/(n-1)}}{\frac{1}{b}\left(\frac{y}{b}\right)^{1/(n-1)}}$$

Now, the equation of the tangent at (x, y) is

$$Y - y = -\frac{\frac{1}{a}\left(\frac{x}{a}\right)^{1/(n-1)}}{\frac{1}{b}\left(\frac{y}{b}\right)^{1/(n-1)}}(X - x)$$

$$\Rightarrow \frac{1}{b}Y\left(\frac{y}{b}\right)^{1/(n-1)} + \frac{1}{a}X\left(\frac{x}{a}\right)^{1/(n-1)} = \left(\frac{x}{a}\right)^{n/(n-1)} + \left(\frac{y}{b}\right)^{n/n-1}$$

$$\Rightarrow \frac{1}{a}\left(\frac{x}{a}\right)^{1/(n-1)}X + \frac{1}{b}\left(\frac{y}{b}\right)^{1/(n-1)}Y = 1 \qquad ...(ii)$$

$$\left(\because \left(\frac{x}{a}\right)^{n/(n-1)} + \left(\frac{y}{b}\right)^{n/(n-1)}\right) = 1$$

Let, the line $X\cos\alpha + Y\sin\alpha = p$ touches the curve then it should be identical with equation (ii).

Hence $\quad \frac{p}{1} = \frac{a\cos\alpha}{\left(\frac{x}{a}\right)^{1/(n-1)}} = \frac{b\sin\alpha}{\left(\frac{y}{b}\right)^{1/(n-1)}}$

$$\Rightarrow \quad \frac{p^n}{1} = \frac{(a\cos\alpha)^n}{\left(\frac{x}{a}\right)^{n/(n-1)}} = \frac{(b\sin\alpha)^n}{\left(\frac{y}{b}\right)^{n/(n-1)}}$$

$$= \frac{(a\cos\alpha)^n + (b\sin\alpha)^n}{\left(\frac{x}{a}\right)^{n/(n-1)} + \left(\frac{y}{b}\right)^{n/(n-1)}} = \frac{(a\cos\alpha)^n + (b\sin\alpha)^n}{1}$$

$$\Rightarrow \quad \mathbf{p^n = (a\cos\alpha)^n + (b\sin\alpha)^n}$$ **Proved**

Prob.69. Discuss the maximum or minimum values of u where $u = x^2 + y^2 + z^2 + x - 2z - xy$

Sol. For a maximum or a minimum of u, we must have

$$\frac{\partial u}{\partial x} = 2x - y + 1 = 0, \quad \frac{\partial u}{\partial y} = -x + 2y = 0 \text{ and } \frac{\partial u}{\partial z} = 2z - 2 = 0$$

Solving these equations, we get

$$x = -\frac{2}{3}, y = -\frac{1}{3}, z = 1$$

$\therefore \left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$ is the only point at which u is stationary i.e., at which u may have a maximum or a minimum.

Now $\frac{\partial^2 u}{\partial x^2} = 2, \frac{\partial^2 u}{\partial y^2} = 2, \frac{\partial^2 u}{\partial z^2} = 2, \frac{\partial^2 u}{\partial y \partial z} = 0, \frac{\partial^2 u}{\partial z \partial x} = 0, \frac{\partial^2 u}{\partial x \partial y} = -1$

If P, Q, R, S, T and U denote the respective values of these six partial derivatives of second order at the point $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$, then P = 2, Q = 2, R = 2, S = 0, T = 0, U = –1.

Now we have

$$P = 2$$

$$\begin{vmatrix} P & U \\ U & Q \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 2 \times 2 - (-1) \times (-1) = 4 - 1 = 3$$

and

$$\begin{vmatrix} P & U & T \\ U & Q & S \\ T & S & R \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2(4-0) + 1(-2-0) + 0 = 8 - 2 = 6$$

Since these three expressions are all positive, we have a minimum of u when

$$x = -\frac{2}{3}, y = -\frac{1}{3}, z = 1$$

Ans.

MODULE 2

CALCULUS-II

DEFINITE INTEGRAL AS A LIMIT OF A SUM AND ITS APPLICATION IN SUMMATION OF SERIES

Definite Integral –

The integral $\int_a^b f(x)\,dx$ is said to be the ***definite integral of f(x)*** with respect to x between the limits a and b or from a to b.
where, a and b are called its lower and upper limits. The interval (a, b) is said to be the range of integration.

Note – To distinguish it from a definite integral, the function F(x) i.e., $\int f(x)\,dx$ is some times said to be the indefinite integral of f(x). It would be noticed that an indefinite integral can be written, if necessary, as a definite integral. F

$$\int_a^x f(x)\,dx$$

is equal to F(x) – F(a), and so, the identical with the indefinite integral of), viz. F(x) + c, if c = – F(a).

Property I. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(It means that, the value of a definite integral depends on the limi: d not on the variables of integrations)

Let, $\int f(x)\,dx = \phi(x)$;

$\therefore$ $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$ i)

Then $\int f(t)\,dt = \phi(t)$

$\therefore$ $\int_a^b f(t)\,dt = \phi(b)$ ... i)

... and (ii), hence the

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+\ldots+f[a+(\overline{n-1})h]\right]$$

when, $\quad n \to \infty, h \to 0$ and $nh \to b-a$

Thus $\displaystyle\int_a^b f(x)\,dx = \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh)$, where $n \to \infty$ as $h \to 0$ and nh remains equal to $b-a$. We call $\int_a^b f(x)\,dx$ as the definite integral of f(x) with respect to x between the limits 'a' and 'b'.

Summation of Series – The summation definition of a definite integral enables us to express the limits of sums of certain types of series as definite integrals which can be easily evaluated. We have defined –

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h.\sum_{r=0}^{n-1} f(a+rh), \text{ where } nh = b-a.$$

Write down the general term [say rth or (r – 1)th term etc., as convenient] of the series, put in the form $\frac{1}{n}f\left(\frac{r}{n}\right)$. Then the series can be written as $\lim_{n\to\infty}\sum \frac{1}{n}f\left(\frac{r}{n}\right)$. Now to calculate $\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$ replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx and $\lim_{n\to\infty}\Sigma$ by the sign of integration i.e., by $\int$.

The lower limit of the definite integral will be the value of r/n for the first term as $n \to \infty$ and the upper limit will be the value of r/n for the last term as $n \to \infty$.

$$\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n}f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$$

To Find Limit of a Product by Integration – Suppose,

$$A = \lim_{n\to\infty} \text{(given product)}$$

Taking logarithm on both sides of above equation, so that

$$\log A = \lim_{n\to\infty} \text{(a series)} = k \text{ (say)}$$

Then $\qquad A = e^k.$

NUMERICAL PROBLEMS

Prob.1. Prove that $\int_0^1 \frac{\log(1+x)}{1+x^2}dx = \frac{\pi}{8}\log 2.$ ***(R.G.P.V., June 2015)***

Sol. Let, $I = \int_0^1 \frac{\log(1+x)}{1+x^2}dx$

Putting $x = \tan\theta$ so $dx = \sec^2\theta\, d\theta$

and limit $\theta \to 0$ to $\frac{\pi}{4}$

$$\therefore \quad I = \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\sec^2\theta\, d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\, d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\, d\theta \qquad \text{...(i)}$$

$$I = \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^{\frac{\pi}{4}} \log\left[1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right]d\theta \quad \left[\because \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}\right]$$

$$I = \int_0^{\frac{\pi}{4}} \log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$I = \int_0^{\frac{\pi}{4}} [\log 2 - \log(1+\tan\theta)]\,d\theta$$

or $$I = \log 2\int_0^{\frac{\pi}{4}} d\theta - I \qquad \text{[By equation (i)]}$$

or $$2I = \log 2[\theta]_0^{\pi/4} = \frac{\pi}{4}\log 2$$

$$I = \frac{\pi}{8}\log 2$$

Proved

Prob.2. Evaluate $\int_a^b x\,dx$ ***from the definition of integral as a limit of sum.***

(R.G.P.V., Dec. 2017)

Or

Evaluate $\int_a^b x\,dx$ ***directly from the definition as the limit of sum.***

[R.G.P.V., Nov. 2019 (O)]

Sol. We know that

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + \ldots.. + f\{a+(n-1)h\}]$$

Here, $f(x) = x$, so that $f(a) = a$, $f(a + h) = a + h$, $f(a + 2h) = a + 2h$, etc.

$$\therefore \quad \int_a^b x\,dx = \lim_{h\to 0} h[a + (a+h) + (a+2h) + \ldots.. + \{a + (n-1)h\}]$$

$$= \lim_{h\to 0} h\left[\frac{n}{2}\{2a + (n-1)h\}\right] \qquad \text{(Sum of A.P.)}$$

$$= \lim_{h\to 0} \frac{b-a}{2}[2a + (b-a) - h] \qquad (\because nh = b-a)$$

$$= \frac{b-a}{2}[2a + (b-a)] = \frac{(b-a)}{2}.(b+a) = \frac{1}{2}(b^2 - a^2) \qquad \text{Ans.}$$

Prob.3. Evaluate $\int_a^b x^2 dx$ ***on limit of sums.*** ***(R.G.P.V., May 2019)***

Sol. We know that

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + \ldots.. + f\{a+(n-1)h\}]$$

Here $f(x) = x^2$, so that $f(a) = a^2$, $f(a + h) = (a + h)^2$, $f(a + 2h) = (a + 2h)^2$, etc.

$$\therefore \quad \int_a^b x^2 dx = \lim_{h\to 0} h[a^2 + (a+h)^2 + (a+2h)^2 \ldots.. + \{a+(n-1)h\}^2]$$

$$= \lim_{h\to 0} h[a^2(1+1+1+\ldots..+1) + h^2(1^2 + 2^2 + 3^2 + \ldots.. + (n-1)^2)$$
$$+ 2ah\{1 + 2 + 3 + \ldots.. + (n-1)\}]$$

$$= \lim_{h\to 0} h\left[na^2 + h^2\frac{n(n-1)(2n-1)}{6} + 2ah\frac{n(n-1)}{2}\right]$$

$$\left[\because \sum_{i=1}^{n-1} i^2 = \frac{n(n-1)(2n-1)}{6}, \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}\right]$$

$$= \lim_{h\to 0}\left[(nh)a^2 + \frac{1}{6}(nh)(nh-h)(2nh-h) + a(nh)(nh-h)\right]$$

$$= \lim_{h\to 0}\left[(b-a)a^2 + \frac{1}{6}(b-a)(b-a-h)\{2(b-a)-h\} + a(b-a)\{(b-a)-h\}\right]$$

$$(\because nh = b-a)$$

$$= \left[(b-a)a^2 + \frac{1}{6}(b-a)(b-a)2(b-a) + a(b-a)(b-a)\right]$$

$$= (b-a)a^2 + \frac{1}{3}(b-a)^3 + a(b-a)^2 = \frac{1}{3}(b-a)[3a^2 + (b-a)^2 + 3a(b-a)]$$

$$= \frac{(b-a)}{3}(3a^2 + b^2 - 2ab + a^2 + 3ab - 3a^2)$$

$$= \frac{1}{3}(b-a)(a^2 + ab + b^2) = \frac{1}{3}(b^3 - a^3)$$ **Ans.**

Prob.4. Evaluate $\int_a^b e^x\,dx$ as limit of sum. *(R.G.P.V., Dec. 2015)*

Or

Evaluate $\int_a^b e^x\,dx$ from the definition of integral as limit of sum. *(R.G.P.V., Dec. 2016)*

Or

Using definition of integral as limit of sum, evaluate $\int_a^b e^x dx$. *(R.G.P.V., June 2017)*

Sol. We know that

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + \ldots\ldots + f\{a+(n-1)h\}]$$

Here, $f(x) = e^x$, so that $f(a) = e^a$, $f(a+h) = e^{a+h}$, etc.

$$\therefore \quad \int_a^b e^x dx = \lim_{h\to 0} h[e^a + e^{a+h} + e^{a+2h} + \ldots\ldots + e^{a+(n-1)h}]$$

$$= \lim_{h\to 0} he^a[1 + e^h + e^{2h} + \ldots\ldots + e^{(n-1)h}]$$

$$= \lim_{h\to 0} he^a\left[\frac{e^{nh}-1}{e^h-1}\right] = \lim_{h\to 0}\left(\frac{h}{e^h-1}\right)e^a(e^{nh}-1)$$

$$= \lim_{h\to 0}\left(\frac{h}{e^h-1}\right)e^a[e^{(b-a)}-1] \qquad (\because nh = b-a)$$

$$= \lim_{h\to 0}\left[e^{1/h}\right]e^a\left[e^{(b-a)}-1\right] \quad \left[\because \lim_{h\to 0}\left(\frac{h}{e^h-1}\right) = \lim_{h\to 0}\left(e^{\frac{1}{h}}\right)\right]$$

$$= 1.e^a[e^{(b-a)} - 1] = e^b - e^a$$ **Ans.**

Prob.5. Evaluate $\int_a^b \cos x\,dx$ as limit of sums. *(R.G.P.V., June 2015)*

Sol. We know that

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + \ldots\ldots + f\{a+(n-1)h\}\right]$$

Here, $f(x) = \cos x$, so that $f(a) = \cos a$, $f(a+h) = \cos(a+h)$, etc.

$$\therefore \int_a^b \cos x\,dx = \lim_{h\to 0} h[\cos(a) + \cos(a+h) + \cos(a+2h) + \ldots.. + \cos\{a+(n-1)h\}]$$

$$= \lim_{h\to 0} h\left[\frac{\sin\frac{nh}{2}\cos\left\{a+(n-1)\frac{h}{2}\right\}}{\sin\frac{h}{2}}\right] = \lim_{h\to 0} h\left[\frac{\sin\frac{nh}{2}\cos\left(a+\frac{nh}{2}-\frac{h}{2}\right)}{\sin\frac{h}{2}}\right]$$

$$= \lim_{h\to 0}\left(\frac{\frac{h}{2}}{\sin\frac{h}{2}}\right)\left[2\sin\left(\frac{b-a}{2}\right)\cos\left\{a+\left(\frac{b-a}{2}-\frac{h}{2}\right)\right\}\right] \quad (\because nh = b-a)$$

$$= \lim_{h\to 0}\left[2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}-\frac{h}{2}\right)\right] \quad \left[\because \lim_{h\to 0}\frac{\theta}{\sin\theta} = 1\right]$$

$$= 2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right)$$

$$= \sin\left(\frac{b-a}{2}+\frac{a+b}{2}\right) + \sin\left(\frac{b-a}{2}-\frac{a+b}{2}\right)$$

$$[\because 2\sin A\cos B = \sin(A+B) + \sin(A-B)]$$

$= \sin(b) + \sin(-a) = \mathbf{\sin b - \sin a}$ **Ans.**

Prob.6. Evaluate by definition of definite integral as the limit of a sum $\int_a^b \sin x\,dx$. ***(R.G.P.V., June 2014)***

Sol. By definition, we get

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots.. + f\{a+(n-1)h\}]$$

where, $h = \dfrac{b-a}{n}$.

$$\therefore \int_a^b \sin x\,dx = \lim_{h\to 0} h[\sin a + \sin(a+h) + \sin(a+2h) + \ldots.. + \sin\{a+(n-1)h\}]$$

$$= \lim_{h\to 0} h\left[\frac{\sin\left(a+(n-1)\frac{h}{2}\right)\sin\frac{nh}{2}}{\sin\frac{h}{2}}\right]$$

$$= \lim_{h\to 0} h\left[\frac{\sin\left(a+\frac{nh}{2}-\frac{h}{2}\right)\sin\frac{nh}{2}}{\sin\frac{h}{2}}\right]$$

$$= \lim_{h\to 0} h\left[\frac{\sin\left(a+\frac{b-a}{2}-\frac{h}{2}\right)\sin\frac{(b-a)}{2}}{\sin\frac{h}{2}}\right] \qquad (\because nh = b-a)$$

$$= \lim_{h\to 0}\left[\frac{\frac{h}{2}}{\sin\frac{h}{2}} 2\sin\left(\frac{a+b}{2}-\frac{h}{2}\right)\sin\left(\frac{b-a}{2}\right)\right]$$

$$= 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right) \qquad \left[\because \lim_{h\to 0}\frac{\theta}{\sin\theta}=1\right]$$

$$= \cos\left(\frac{a+b}{2}-\frac{b-a}{2}\right)-\cos\left(\frac{a+b}{2}+\frac{b-a}{2}\right)$$

$$[\because 2\sin A\sin B = \cos(A-B)-\cos(A+B)]$$

$= \mathbf{\cos a - \cos b}$ **Ans.**

Prob.7. Evaluate –

$$\lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right).....\left(1+\frac{n}{n}\right)\right\}^{\frac{1}{n}}.$$

(R.G.P.V., Jan./Feb. 2008, June 2017)

Sol. Suppose, the required limit is A, then

$$A = \lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right).....\left(1+\frac{n}{n}\right)\right\}^{\frac{1}{n}} \qquad ....(i)$$

Taking logarithms on both sides of equation (i), we get

$$\log A = \lim_{n\to\infty}\frac{1}{n}\left\{\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\log\left(1+\frac{3}{n}\right)+.....+\log\left(1+\frac{n}{n}\right)\right\}$$

$$= \lim_{n\to\infty}\Sigma\log\left(1+\frac{r}{n}\right)\frac{1}{n}$$

$$\log A = \int_0^1 \log(1+x)\,dx$$

$$= \left[x\log(1+x)\right]_0^1 - \int_0^1 \frac{x}{1+x}\,dx \qquad \text{(on integrating by parts)}$$

$$= \left[x\log(1+x)\right]_0^1 - \left[x - \log(1+x)\right]_0^1$$

$$= \log 2 - [1 - \log 2] = \log 2 - 1 + \log 2$$

$$= \log 4 - 1 = \log 4 - \log_e e$$

$\log A = \log(4/e)$ or $A = 4/e$ Ans.

Prob.8. Find the limit as $n \to \infty$ of the series –

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n}.$$

[R.G.P.V., June 2008(N), 2009, Dec. 2011]

Or

Evaluate – $\lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right]$

[R.G.P.V., Dec. 2016, Nov. 2018, Nov. 2018(O), June 2020]

Or

Evaluate by expressing the following limit of a sum in the form of a definite integral.

$$\lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right] \qquad \text{(R.G.P.V., Nov. 2019)}$$

Sol. Here, the general term

$$T_{r+1} = \frac{1}{n+r} = \frac{1}{n}\cdot\frac{1}{1+(r/n)}$$

$$= \frac{1}{1+x}\,dx, \qquad \left(\text{Putting, } \frac{r}{n} = x \text{ and } \frac{1}{n} = dx\right)$$

Now for the first term $r = 0$ and for the last term $r = n$.

$\therefore$ The lower limit of integration

$$= \lim_{n\to\infty}\left(\frac{0}{n}\right) = 0$$

and the upper limit of integration

$$= \lim_{n\to\infty}\left(\frac{n}{n}\right) = 1$$

Hence, the required limit

$$= \int_0^1 \frac{dx}{1+x} = \left|\log(1+x)\right|_0^1$$

$$= \log 2 - \log 1 = \mathbf{\log 2}$$ Ans.

Prob.9. Find the limit, when $n \to \infty$, of the series –

$$\frac{n}{n^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+(n-1)^2}.$$

(R.G.P.V., June 2004, Dec. 2008, April 2009)

Sol. Here, the given series

$$\frac{n}{n^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+(n-1)^2}$$

The general term = $(T_{r+1}) = \dfrac{1}{1+\left(\dfrac{r}{n}\right)^2}\cdot\dfrac{1}{n} = \dfrac{1}{1+x^2}dx$

$$\left[\text{Put } \frac{r}{n} = x \text{ and } \frac{1}{n} = dx\right]$$

Again for the first term r = 0 and for the last term r = n – 1.

Therefore, the lower limit of integration = $\lim\limits_{n\to\infty}\left(\dfrac{0}{n}\right) = 0$.

and the upper limit of integration = $\lim\limits_{n\to\infty}\left(\dfrac{n-1}{n}\right) = \lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right) = 1$

Hence, the required limit = $\displaystyle\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$

$= \pi/4$ **Ans.**

Prob.10. Evaluate –

$$\lim_{n\to\infty}\frac{1+2^{10}+3^{10}+\ldots+n^{10}}{n^{11}}.$$

Sol. The given limit can be written as

$$\lim_{n\to\infty}\left[\frac{1}{n}\left\{\left(\frac{1}{n}\right)^{10}+\left(\frac{2}{n}\right)^{10}+\left(\frac{3}{n}\right)^{10}+\ldots+\left(\frac{n}{n}\right)^{10}\right\}\right]$$

Now the r^{th} term = $\dfrac{1}{n}\cdot\left(\dfrac{r}{n}\right)^{10}$ and r varies from 1 to n

Therefore the given limit

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\cdot\left(\frac{r}{n}\right)^{10} = \int_0^1 x^{10}dx = \left[\frac{x^{11}}{11}\right]_0^1 = \frac{1}{11} \quad \textbf{Ans.}$$

Prob.11. Evaluate – $\lim_{n\to\infty}\left(\frac{1}{1+n^3}+\frac{4}{8+n^3}+\frac{9}{27+n^3}+.....+\frac{1}{2n}\right)$

(R.G.P.V., Dec. 2015)

Sol. The given series can be written as

$$\lim_{n\to\infty}\left(\frac{1^2}{1^3+n^3}+\frac{2^2}{2^3+n^3}+\frac{3^2}{3^3+n^3}+.....+\frac{n^2}{n^3+n^3}\right)$$

Now the r^{th} term $= \frac{r^2}{r^3+n^3}$, where r varies from 1 to n.

Therefore the given series

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{r^2}{r^3+n^3}\right) = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n^3}\left\{\frac{r^2}{\left(\frac{r^3}{n^3}\right)+1}\right\}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\left\{\frac{(r/n)^2}{(r/n)^3+1}\right\}$$

$$= \int_0^1\frac{x^2}{(x^3+1)}dx = \left[\frac{1}{3}\log(x^3+1)\right]_0^1 = \frac{1}{3}\log 2$$ Ans.

Prob.12. Evaluate the limit $\lim_{n\to\infty}\frac{(n!)^{1/n}}{n}$. ***(R.G.P.V., Dec. 2017)***

Or

Evaluate –

$$\lim_{n\to\infty}\left\{\frac{n!}{n^n}\right\}^{1/n}$$ ***(R.G.P.V., June 2012, Dec. 2012)***

Or

Evaluate the limit

$$\underset{n\to\infty}{Lt}\left\{\frac{n!}{n^n}\right\}^{1/n}$$ ***(R.G.P.V., May 2018)***

Sol. Let, $$P = \lim_{n\to\infty}\frac{(n!)^{1/n}}{n} = \lim_{n\to\infty}\left[\frac{n!}{n^n}\right]^{1/n} = \lim_{n\to\infty}\left\{\frac{1.2.3.4.5...n}{n.n.n.n.n...n}\right\}^{1/n}$$

$$= \lim_{n\to\infty}\left\{\left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\left(\frac{3}{n}\right)\left(\frac{4}{n}\right)\left(\frac{5}{n}\right)...\frac{n}{n}\right\}^{1/n}$$

Taking logarithms on both sides, we get

$$\log P = \lim_{n\to\infty} \frac{1}{n}\left[\log\left(\frac{1}{n}\right) + \log\left(\frac{2}{n}\right) + \log\left(\frac{3}{n}\right) + ... + \log\left(\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n}\log\left(\frac{r}{n}\right) = \int_0^1 \log x\, dx$$

$$= \left[(\log x).x\right]_0^1 - \int_0^1 \left(\frac{1}{x}\right) x\, dx \qquad \text{(by integrating by parts)}$$

$$= 0 - \int_0^1 dx = -[x]_0^1 = -1$$

$$P = e^{-1} = \mathbf{1/e}$$ **Ans.**

Prob.13. Evaluate –

$$\lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{1}{n}\sqrt{\frac{n+r}{n-r}}.$$ ***(R.G.P.V., June 2005)***

Sol. Let,

$$A = \lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{1}{n}\sqrt{\frac{n+r}{n-r}} = \lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{1}{n}\sqrt{\frac{1+r/n}{1-r/n}}$$

$$= \int_0^1 \sqrt{\left(\frac{1+x}{1-x}\right)} dx = \int_0^1 \frac{(1+x)}{\sqrt{(1-x^2)}} dx = \int_0^1 \frac{dx}{\sqrt{1-x^2}} + \int_0^1 \frac{x}{\sqrt{1-x^2}} dx$$

$$= \left[\sin^{-1} x\right]_0^1 + \left[-\sqrt{(1-x^2)}\right]_0^1 = \frac{\pi}{2} + [0+1] = \mathbf{\frac{\pi+2}{2}}$$ **Ans.**

Prob.14. Evaluate –

$$\lim_{n\to\infty}\left[\frac{1}{n} + \frac{1}{\sqrt{n^2-1^2}} + \frac{1}{\sqrt{n^2-2^2}} + ... + \frac{1}{\sqrt{n^2-(n-1)^2}}\right].$$

(R.G.P.V., Dec. 2004, 2013)

Or

Find the limit as n → ∞ of the series –

$$\sum_{r=0}^{n-1} \frac{1}{\sqrt{n^2-r^2}}.$$ ***(R.G.P.V., Feb. 2005, 2010)***

Sol. Here, $(r+1)^{th}$ term $= \dfrac{1}{\sqrt{n^2-r^2}} = \dfrac{1}{n}\dfrac{1}{\sqrt{1-\left(\frac{r}{n}\right)^2}}$

$\therefore$ The given series

$$= \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{\sqrt{1-\left(\frac{r}{n}\right)^2}} \cdot \frac{1}{n} = \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1} x\right]_0^1$$

$$= \sin^{-1}(1) - \sin^{-1}(0) = \pi/2$$

Ans.

Prob.15. Find the limit as $n \to \infty$ of the series –

$$\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \ldots + \frac{1}{8n}.$$ ***(R.G.P.V., Dec. 2005, 2006)***

Sol. The series can be written as

$$\lim_{n\to\infty}\left[\frac{n^2}{(n+0)^3} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \ldots + \frac{n^2}{(n+n)^3}\right]$$

Here, $(r+1)^{th}$ term $\dfrac{n^2}{(n+r)^3}$.

$\therefore$ The given series $= \lim\limits_{n\to\infty} \sum\limits_{r=0}^{\infty} \dfrac{n^2}{(n+r)^3} = \lim\limits_{n\to\infty} \sum\limits_{r=0}^{\infty} \dfrac{1}{n}\left[\dfrac{1}{\left(1+\frac{r}{n}\right)^3}\right]$

$$= \int_0^1 \frac{1}{(1+x)^3}\,dx = \left[-\frac{1}{2(1+x)^2}\right]_0^1 = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$$

Ans.

Prob.16. Evaluate –

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left[1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}}\right].$$ ***(R.G.P.V., June/July 2006)***

Sol.

Let, $A = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}}\left[1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \ldots + \dfrac{1}{\sqrt{n}}\right]$

Here, the general term $= T_{r+1} = \dfrac{1}{\sqrt{n}} \cdot \dfrac{1}{\sqrt{r}} = \dfrac{1}{n} \cdot \sqrt{\dfrac{n}{r}} = \dfrac{1}{n} \cdot \dfrac{1}{\sqrt{r/n}}$

$\therefore$ $$1 = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{1}{\sqrt{r/n}} = \int_0^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \left[2\sqrt{x}\right]_0^1 = 2\left(\sqrt{1} - \sqrt{0}\right) = 2$$

Ans.

Prob.17. Find ab-initio the value of the integral –

$$\int_0^{\pi/2} \sin x \, dx$$ ***(R.G.P.V., June 2012)***

Sol. From the definition

$$\int_a^b \sin x \, dx = \lim_{h\to 0} h \sum_{r=0}^{n-1} \sin(a + rh), \text{ where } rh = b - a$$

$$= \lim_{h\to 0} h[\sin a + \sin(a+h) + \sin(a+2h) + ... + n \text{ terms}]$$

$$= \lim_{h\to 0} h \sin\left(a + (n-1)\frac{h}{2}\right) \frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}}$$

$$= \lim_{h\to 0} \frac{\frac{h}{2}}{\sin\frac{h}{2}} 2\sin\frac{nh}{2} \sin\left\{a + (n-1)\frac{h}{2}\right\}$$

$$= \lim_{h\to 0} \frac{\frac{h}{2}}{\sin\frac{nh}{2}} \left[\cos\left(a - \frac{h}{2}\right) - \cos\left\{a + (2n-1)\frac{h}{2}\right\}\right]$$

$$= \lim_{h\to 0}\left[\cos\left(a - \frac{h}{2}\right) - \cos\left(a + nh - \frac{h}{2}\right)\right] \qquad \left[\because \lim_{h\to 0} \frac{\theta}{\sin\theta} = 1\right]$$

Put a = 0, and nh = π/2, we get

$$\lim_{h\to 0}\left[\cos\left(0 - \frac{h}{2}\right) - \cos\left(0 + \frac{\pi}{2} - \frac{h}{2}\right)\right] = \lim_{h\to 0}\left[\cos\frac{h}{2} - \cos\left(\frac{\pi}{2} - \frac{h}{2}\right)\right]$$

$$= \lim_{h\to 0}\left[\cos\frac{h}{2} - \sin\frac{h}{2}\right]$$

Taking limit $\quad 1 - 0 = \mathbf{1}$ **Ans.**

Prob.18. Evaluate –

$$\lim_{n\to\infty}\left[\left(1 + \frac{1}{n^2}\right)\left(1 + \frac{2^2}{n^2}\right)\left(1 + \frac{3^2}{n^2}\right)...\left(1 + \frac{n^2}{n^2}\right)\right]^{1/n}.$$

(R.G.P.V., Sept. 2009, June 2011, Dec. 2014)

Or

Evaluate by expressing the limit of a sum in the form of a definite integral –

$$\lim_{n\to\infty}\left[\left(1 + \frac{1}{n^2}\right)\left(1 + \frac{2^2}{n^2}\right)\left(1 + \frac{3^2}{n^2}\right)...\left(1 + \frac{n^2}{n^2}\right)\right]^{1/n}.$$

(R.G.P.V., June 2013)

Or

Prove that –

$$\lim_{n\to\infty}\left[\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n} = 2e^{\left(\frac{\pi-4}{2}\right)}$$

(R.G.P.V., June 2014)

Sol. Let, $A = \lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$

Taking logarithm on both sides of above equation, we have

$$\log A = \lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+\log\left(1+\frac{3^2}{n^2}\right)+\ldots+\log\left(1+\frac{n^2}{n^2}\right)\right]$$

Its general terms $= \frac{1}{n}\log\left[1+\left(\frac{r}{n}\right)^2\right] = \log(1+x^2)\,dx \quad \left(\text{put } \frac{r}{n} = x \text{ and } \frac{1}{n} = dx\right)$

Also for the first term, r = 1 and for the last term r = n

Therefore, the lower limit of integration $= \lim_{n\to\infty}\left(\frac{1}{n^2}\right) = 0$

and upper limit of integration $= \lim_{n\to\infty}\left(\frac{n^2}{n^2}\right) = 1$

$$\begin{aligned}\text{Hence, } \log A &= \int_0^1 \log(1+x^2).1\,dx \\ &= \left[\log(1+x^2).x\right]_0^1 - \int_0^1 \frac{1}{1+x^2}2x.x\,dx \\ &= [\log 2] - 2\int_0^1 \frac{x^2}{1+x^2}dx = \log 2 - 2\int_0^1\left(1-\frac{1}{1+x^2}\right)dx \\ &= \log 2 - 2\left[x-\tan^{-1}x\right]_0^1 = \log 2 - 2\left[1-\frac{\pi}{4}\right]\end{aligned}$$

$$\text{or} \quad \log A - \log 2 = \frac{1}{2}(\pi-4)$$

$$\text{or} \quad \log\frac{1}{2}A = \frac{1}{2}(\pi-4) \text{ or } \frac{1}{2}A = e^{(\pi-4)/2}$$

$$\text{or} \quad A = 2e^{(\pi-4)/2}$$

Ans.

Prob.19. Evaluate as limit of sums –

$$\int_1^3 (x^2+x)\,dx.$$

(R.G.P.V., Dec. 2010)

Sol. Let, $f(x) = x^2 + x$

Here, $a = 1,\ b = 3,\ nh = b - a = 3 - 1 = 2$

$$\begin{aligned}
f(a) &= f(1) = 2 \\
f(a+h) &= f(1+h) = (1+h)^2 + (1+h) \\
&= 1 + h^2 + 2h + 1 + h = h^2 + 3h + 2 \\
f(a+2h) &= f(1+2h) = (1+2h)^2 + (1+2h) \\
&= 1 + 4h + 4h^2 + 1 + 2h = 4h^2 + 6h + 2 \\
f(a+3h) &= f(1+3h) \\
&= (1+3h)^2 + (1+3h) = 9h^2 + 9h + 2 \\
f\{a+(n-1)h\} &= f\{1+(n-1)h\} = \{1+(n-1)h\}^2 + \{1+(n-1)h\} \\
&= (n-1)^2h^2 + 3(n-1)h + 2
\end{aligned}$$

We know that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h[f(a) + f(a+h) + f(a+2h) + \ldots + f\{a+(n-1)h\}$$

$$\int_1^3 (x^2+x)\,dx = \lim_{n\to\infty} h[2 + (h^2+3h+2) + (4h^2+6h+2) + (9h^2+9h+2) + \ldots + \{n-1)^2h^2 + 3(n-1)h + 2\}\ ]$$

$\{\text{where } h \to 0, \text{ as } n \to \infty \text{ and } nh = 3 - 1 = 2\}$

$$= \lim_{n\to\infty} h[h^2\{1+4+9+\ldots+(n-1)^2\} + h\{3+6+9+\ldots+3(n-1)\} + 2n]$$

$$= \lim_{n\to\infty} [h^3\{1^2+2^2+\ldots+(n-1)^2\} + 3h^2\{1+2+\ldots+(n-1)\} + 2nh]$$

$$= \lim_{n\to\infty}\left[h^3.\frac{1}{6}(n-1)n(2n-1) + 3h^2.\frac{1}{2}(n-1).n + 2nh\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{6}(nh-h)(nh)(2nh-h) + \frac{3}{2}(nh-h)nh + 2nh\right]$$

$$= \left[\frac{1}{6}(2-0).2(2\times 2-0) + \frac{3}{2}(2-0).2 + 2\times 2\right]$$

$$= \left[\frac{1}{6}\times 16 + \frac{3}{2}\times 4 + 4\right] = \left[\frac{8}{3} + 6 + 4\right] = \frac{38}{3}$$ **Ans.**

Prob.20. Prove that –

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ when } f(x) \text{ is even}$$
$$= 0, \quad \text{when } f(x) \text{ is odd.}$$

[R.G.P.V., June 2008 (O)]

Sol. We know that

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \quad \ldots(i)$$

[Property I]

In $\int_{-a}^{0} f(x)\,dx$, put $x = -t$, so that $dx = -dt$

$$\therefore \quad \int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-t)\,dt = \int_{0}^{a} f(-t)\,dt = \int_{0}^{a} f(-x)\,dx$$

[Property II]

Substituting in equation (i), we get

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \quad \text{...(ii)}$$

(i) If f(x) is an even function, $f(-x) = f(x)$.

$\therefore$ From equation (ii),

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

(ii) If f(x) is an odd function, $f(-x) = -f(x)$.

$\therefore$ From equation (ii),

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

Proved

BETA AND GAMMA FUNCTIONS AND THEIR PROPERTIES

Beta and Gamma Functions – Swiss mathematician Leonhard Euler (1707-1783). He studied under John Bernoulli and became a professor of mathematics in St. Petersburg Russia. He gave two definite integrals which after his name are styled as Eulerian integral of first kind and Eulerian integral of second kind. Above mentioned integrals are of great importance in the theory of definite integrals.

Beta Function and its Property – We defined the beta function $\beta(m, n)$ for $m > 0$, $n > 0$ by the relation.

$$\beta(m, n) = \int_{0}^{1} x^{m-1}(1-x)^{n-1}\,dx \quad \text{...(i)}$$

Beta function is also said to be the Eulerian integral of the first kind.

Substituting $x = 1 - y$ in equation (i), we get

$$\beta(m, n) = -\int_{1}^{0} (1-y)^{m-1} y^{n-1}\,dy = \int_{0}^{1} y^{n-1}(1-y)^{m-1}\,dy = \beta(n, m) \quad \text{...(ii)}$$

Therefore, symmetry of beta function $\beta(m, n) = \beta(n, m)$

Now substituting $x = \sin^2\theta$, so that $dx = 2\sin\theta\cos\theta\,d\theta$ in equation (i) we get

$$\beta(m, n) = \int_{0}^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}.2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_{0}^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

which is another form of $\beta(m, n)$.

Gamma Function – The define integral

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx \ (n > 0) \qquad \ldots(i)$$

is also known as Eulerian integral of the second kind. Gamma function defines a function of n for positive values of n.

Fundamental Property of Gamma Function – To prove that –

(i) $\Gamma(n+1) = n\Gamma n$, where $n > 0$

(ii) $\Gamma n = (n-1)!$ where n is a positive integer.

Proof. By the definition of gamma function, we have

$$\Gamma(n+1) = \int_0^{\infty} e^{-x} x^{(n+1)-1} dx = \int_0^{\infty} x^n e^{-x} dx$$

or $$\Gamma(n+1) = \left[-e^{-x} x^n\right]_0^{\infty} + \int_0^{\infty} e^{-x}\, n x^{n-1} dx \qquad \ldots(i)$$

Now $$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{x^n}{1 + x + \left(\frac{x^2}{2!}\right) + \ldots + \left(\frac{x^n}{n!}\right) + \ldots}$$

$$= \lim_{x\to\infty} \frac{1}{\frac{1}{x^n} + \frac{1}{x^{n-1}} + \ldots + \frac{1}{n!} + \frac{x}{(n+1)!} + \ldots} = \frac{1}{\infty} = 0$$

From equation (i), we get

$$\Gamma(n+1) = 0 + n\int_0^{\infty} e^{-x} x^{n-1} dx = n\Gamma(n)$$

or $$\mathbf{\Gamma(n+1) = n\Gamma(n)}$$

which proves the result (i)

(ii) We have

$$\Gamma n = \Gamma[(n-1)+1] = (n-1)\,\Gamma(n-1)$$

Similarly, $\Gamma(n-1) = (n-2)\Gamma(n-2)$ $\qquad [\because \Gamma(n+1) = n\Gamma n]$

Hence, if n is a positive integer, then proceedings, above, we get $\Gamma(n) = (n-1)(n-2)\ldots 2.1\ \Gamma(1)$

But, $$\Gamma(1) = \int_0^{\infty} e^{-x} x^{1-1} dx = \int_0^{\infty} e^{-x} x^0 dx = \int_0^{\infty} e^{-x}.1\, dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^{\infty} = -\left[\lim_{x\to\infty} \frac{1}{e^x} - e^0\right] = -[0-1] = 1$$

Hence, $\Gamma(n) = (n-1)(n-2)\ldots 2.1.1 = (n-1)!$

(if n is positive integer)

Remember – $\Gamma n = (n-1)\,\Gamma(n-1)$, where $n > 1$ and $\Gamma(1) = 1$. Also it may be remarked that $\Gamma(0) = \infty$ and $\Gamma(-n) = \infty$, where n is a positive integer.

The value of $\Gamma\frac{1}{2}$.

We know that $\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$ ...(i)

Putting $n = \frac{1}{2}$, we have

$$\Gamma\frac{1}{2} = \int_0^\infty e^{-x}x^{-1/2}dx \qquad [\text{Put } x = y^2, \text{so that}, dx = 2y\,dy]$$

$$= 2\int_0^\infty e^{-y^2}dy \text{ which is also} = 2\int_0^\infty e^{-x^2}dx \qquad [\text{by property II}]$$

Therefore, $$\left[\Gamma\frac{1}{2}\right]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}r\,dr\,d\theta$$

$$= 4.\frac{\pi}{2}\int_0^\infty e^{-r^2}r\,dr = 2\pi\left[\left(-\frac{1}{2}\right)e^{-r^2}\right]_0^\infty = \pi$$

or $$\left[\Gamma\frac{1}{2}\right]^2 = \pi \text{ or } \left[\Gamma\frac{1}{2}\right] = \sqrt{\pi}$$ **Proved**

Relation between Beta and Gamma Functions –

Prove that –

$$\boldsymbol{\beta(m, n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}}, \textbf{ where } \boldsymbol{m > 0, n > 0}$$

Proof. We know that

$$\beta(m, n) = \frac{(m-1)!(n-1)!}{(n+m-1)!} = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} \qquad ...(i)$$

We have $$\Gamma m = \int_0^\infty e^{-t}t^{m-1}dt \qquad [\text{put } t = x^2 \text{ so that}, dt = 2x\,dx]$$

or $$\Gamma m = 2\int_0^\infty e^{-x^2}x^{2m-1}dx \qquad ...(ii)$$

Similarly, $$\Gamma(n) = 2\int_0^\infty e^{-y^2}y^{2n-1}dy \qquad ...(iii)$$

Now $$\Gamma m.\Gamma n = 4\int_0^\infty e^{-x^2}x^{2m-1}dx\int_0^\infty e^{-y^2}y^{2n-1}dy$$

or $$\Gamma m.\Gamma n = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}x^{2m-1}y^{2n-1}\,dx\,dy \qquad ...(iv)$$

[since the limit of integration are constant]

Now replace to polar co-ordinates by writing $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,d\theta\,dr$. To cover the region in equation (iv) which is the entire first quadrant, r varies from 0 to ∞ and θ from 0 to $\pi/2$. Therefore equation (iv), becomes

$$\Gamma m.\Gamma n = 4\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta\, dr$$

or $$\Gamma m.\Gamma n = \left[2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta\right] \times \left[2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr\right] \quad ...(v)$$

But by equation (ii), we get

$$2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr = \Gamma(m+n)$$

and by equation (iv), we get

$$2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta = \beta(m,n)$$

Therefore from equation (v), we get

$$\Gamma m.\Gamma n = \beta(m,n).\Gamma(m+n)$$

or $$\boldsymbol{\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}}$$ **Proved**

Cor.1. Rule to evaluate $\int_0^{\pi/2} \sin^p x \cos^q x\, dx$,

We have

$$\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

or $$\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{\Gamma\frac{p+1}{2}\Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}} \quad ...(vi)$$

In particular, when $q = 0$, and $p = n$, we have

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}.\frac{\sqrt{\pi}}{2}$$

$$...(vii)$$

Similarly, $$\int_0^{\pi/2} \cos^n x\, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}.\frac{\sqrt{\pi}}{2}$$

Transformation of Gamma Function –

Prove that

(i) $\int_0^\infty e^{-ky} y^{n-1} dy = \frac{\Gamma n}{k^n}$ (ii) $\int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy = \Gamma n$.

Proof. (i) We know that

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x}\, dx \qquad \text{....(i)}$$

Replace x by ky, so that dx = k dy, then equation (i), becomes,

$$\Gamma n = \int_0^\infty (ky)^{n-1} e^{-ky} k\, dy = k^n \int_0^\infty e^{-ky} y^{n-1}\, dy$$

or $$\int_0^\infty e^{-ky} y^{n-1}\, dy = \frac{\Gamma n}{k^n}$$ **Proved**

(ii) Replace e^{-x} by y, so that $-e^{-x} dx = dy$ and $-x = \log_e y$

$\Rightarrow x = \log_e \frac{1}{y}$, then equation (i) becomes.

$$\Gamma n = -\int_1^0 \left(\log \frac{1}{y}\right)^{n-1} y.\frac{dy}{e^{-x}} = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} .y.\frac{dy}{y}$$

or $$\Gamma n = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$$ **Proved**

Transformation of Beta Function –

Beta function can be transformed into many other a few of them are given below –

(i) We know that

$$\beta(l, m) = \int_0^1 x^{l-1} (1-x)^{m-1} dx \qquad \text{....(i)}$$

Substituting $x = \frac{1}{1+y}$ so that $dx = -\frac{1}{(1+y)^2} dy$ and $1 - x = \frac{y}{1+y}$, then equation (i) becomes,

$$\beta(l, m) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{l-1} \left(\frac{y}{1+y}\right)^{m-1} \left[\frac{-1}{(1+y)^2}\right] dy$$

$$= \int_0^\infty \left(\frac{1}{1+y}\right)^{l-1} \left(\frac{y}{1+y}\right)^{m-1} .\frac{dy}{(1+y)^2} = \int_0^\infty \frac{y^{m-1}}{(1+y)^{l+m}} dy$$

$\because$ l, m can be replaced in $\beta(l, m)$, then

$$\beta(m, l) = \int_0^\infty \frac{y^{l-1}}{(1+y)^{m+l}} dy = \beta(l, m) \qquad [\because \beta(m,l) = \beta(l,m)]$$

$$\beta(l, m) = \int_0^\infty \frac{x^{l-1}}{(1+x)^{m+l}} dx \qquad \text{[by property II]}$$

(ii) We know that $\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy = \beta(m, n)$

Now, $\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy = \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}} dy + \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$.

Replacing $y = \frac{1}{x}$ in the last integral, we obtain

$$\int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Therefore,

$$\beta(m, n) = \int_0^1 \frac{y^{m-1} dy}{(1+y)^{m+n}} + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx \quad \text{[by property II]}$$

$$= \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

Hence, $\boldsymbol{\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = \beta(m, n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}}$

Duplication Formula –

Prove that –

$$\Gamma(m)\, \Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m)$$

Proof. We know that

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad \text{...(i)}$$

$$2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad \text{...(ii)}$$

Putting $2n - 1 = 0$ or $n = \frac{1}{2}$, we get

$$2\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta = \frac{\Gamma(m)\,\Gamma(1/2)}{\Gamma(m+1/2)}$$

$$2\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta = \frac{\Gamma(m)\,\sqrt{\pi}}{\Gamma(m+1/2)} \quad \text{...(iii)}$$

Again substituting $n = m$ in equation (ii), we get

$$2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta\, d\theta = \frac{\Gamma(m)\,\Gamma(m)}{\Gamma(m+m)}$$

$$\frac{2}{2^{2m-1}}\int_0^{\pi/2} 2^{2m-1} \sin^{2m-1}\theta \cos^{2m-1}\theta\, d\theta = \frac{[\Gamma(m)]^2}{\Gamma(2m)}$$

$$\frac{2}{2^{2m-1}}\int_0^{\pi/2} (\sin 2\theta)^{2m-1}\, d\theta = \frac{[\Gamma(m)]^2}{\Gamma(2m)}$$

Put $2\theta = \phi$, so that $2d\theta = d\phi$

$$\frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi\, d\phi = \frac{[\Gamma(m)]^2}{\Gamma(2m)}$$

or
$$\frac{2}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1}\phi\, d\phi = \frac{[\Gamma(m)]^2}{\Gamma(2m)}$$

$$2\int_0^{\pi/2} \sin^{2m-1}\phi\, d\phi = 2^{2m-1}.\frac{[\Gamma(m)]^2}{\Gamma(2m)}$$

or
$$2\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta = 2^{2m-1}.\frac{[\Gamma(m)]^2}{\Gamma(2m)} \quad \text{(By property}$$

From equation (iii),

$$\frac{\Gamma(m)\,\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)} = 2^{2m-1}.\frac{[\Gamma(m)]^2}{\Gamma(2m)}$$

or
$$\boldsymbol{\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)}$$

Prov

Q.1. Define β(m, n). ***(R.G.P.V., June 201***

Ans. Refer to the matter giver ... page 86, under the heading beta functi and its property.

Q.2. Define gamma function and beta function and also establish the symmetry of beta function. ***(R.G.P.V., June 2014)***

Ans. Refer to the matter given on page 87 and 86.

Q.3. Define gamma function and prove that $\Gamma\frac{1}{2} = \sqrt{\pi}$.

(R.G.P.V., Dec. 2017)

Ans. Refer to the matter given on page 87 and 88.

NUMERICAL PROBLEMS

Prob.21. Define beta function and using its definition, evaluate $\int_0^1 x^4 (1-x)^3 \, dx$. ***(R.G.P.V., June 2017)***

Sol. **Beta Function** – Refer to the matter given on page 86.

$$\int_0^1 x^4 (1-x)^3 dx = \beta(5, 4)$$

$$= \frac{\Gamma 5 . \Gamma 4}{\Gamma 9} \qquad \left[\because \beta(m,n) = \frac{\Gamma m . \Gamma n}{\Gamma(m+n)}\right]$$

$$= \frac{4!.3!}{8!} = \frac{3!}{8.7.6.5} = \frac{1}{280} \qquad \textbf{Ans.}$$

Prob.22. Prove that $\Gamma(n+1) = n\Gamma(n)$. ***(R.G.P.V., May 2018)***

Or

Prove that $n\Gamma n = \Gamma(n+1), n > 0$. ***[R.G.P.V., Nov. 2018 (O)]***

Sol. Refer to the matter given on page 87, under heading 'Fundamental Property of Gamma Function'.

Prob.23. Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. ***(R.G.P.V., June 2017)***

Sol. Refer to the matter given on page 88.

Prob.24. Prove that $2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \beta(m,n)$.

(R.G.P.V., May 2018)

Or

Prove that $\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$.

[R.G.P.V., Nov. 2019 (O)]

Sol. We know that

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \qquad ...(i)$$

Let $\quad x = \sin^2\theta$

So that $\quad dx = 2\sin\theta\cos\theta\, d\theta$

When $\quad x = 0, \theta = 0$

$\quad x = 1, \theta = \dfrac{\pi}{2}$

$$\therefore \quad \beta(m, n) = \int_0^{\pi/2} \sin^{2(m-1)}\theta\,(1-\sin^2\theta)^{n-1}.2\sin\theta\cos\theta\, d\theta$$

$$\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,(\cos^2\theta)^{n-1}\cos\theta\, d\theta$$

$$\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

or
$$\mathbf{2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta = \beta(m, n)}$$
Proved

Prob.25. Express $\int_0^{\pi/2}\sqrt{\tan\theta}\, d\theta$ in terms of gamma functions.

Sol. Here,

$$\int_0^{\pi/2}\sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2}\sqrt{\frac{\sin\theta}{\cos\theta}}\, d\theta = \int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta\, d\theta$$

$$= \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\,\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

$$\left(\because \beta(p, q) = \int_0^{\pi/2}\sin^p\theta\cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}\right)$$

Hence, $\int_0^{\pi/2}\sqrt{\tan\theta}\, d\theta = \dfrac{1}{2}\Gamma\left(\dfrac{1}{4}\right)\Gamma\left(\dfrac{3}{4}\right)$ **Ans.**

Prob.26. Show that –

$$\beta(p, q) = \int_0^{\infty}\frac{y^{q-1}}{(1+y)^{p+q}}\, dy$$

(R.G.P.V., June 201

Sol. We know that

$$\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx \quad ...(i)$$

Substituting $x = \frac{1}{1+y}$ so that

$$dx = -\frac{1}{(1+y)^2}dy \text{ and } 1-x = \frac{y}{1+y}$$

Then equation (i) becomes,

$$\beta(p, q) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{p-1}\left(\frac{y}{1+y}\right)^{q-1}\left[\frac{-1}{(1+y)^2}\right]dy$$

$$= \int_0^\infty \left(\frac{1}{1+y}\right)^{p-1}\left(\frac{y}{1+y}\right)^{q-1}.\frac{dy}{(1+y)^2}$$

$$\beta(p, q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}dy$$ **Proved**

Prob.27. Evaluate –

$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx$$ ***(R.G.P.V., June 2012)***

Sol. Let,

$$I = \int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx = \int_0^\infty \frac{x^8}{(1+x)^{24}}dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}dx$$

$$= \beta(9, 15) - \beta(15, 9) = \mathbf{0} \qquad [\because \beta(m, l) = \beta(l, m)]$$ **Ans.**

Prob.28. Prove that –

$$\int_0^1 (1-x^n)^{1/n}dx = \frac{1}{n}.\frac{\left\{\Gamma\frac{1}{n}\right\}^2}{2.\Gamma\frac{2}{n}}$$ ***(R.G.P.V., Dec. 2010)***

Sol. Put $x^n = t$ or $x = t^{1/n}$

So that $dx = \frac{1}{n}t^{\left(\frac{1}{n}-1\right)}dt$, we have

$$\int_0^1(1-x^n)^{\frac{1}{n}}dx = \int_0^1(1-t)^{\frac{1}{n}}.\frac{1}{n}t^{\left(\frac{1}{n}-1\right)}dt = \frac{1}{n}\int_0^1(1-t)^{\frac{1}{n}}.t^{\left(\frac{1}{n}\right)-1}dt$$

$$= \frac{1}{n}\int_0^1(1-t)^{\left(1+\frac{1}{n}\right)-1}t^{\left(\frac{1}{n}\right)-1}dt$$

$$= \frac{1}{n}\beta\left(1+\frac{1}{n}, \frac{1}{n}\right) \text{ (By definition of Beta function)}$$

$$= \frac{1}{n}\cdot\frac{\Gamma\left(1+\frac{1}{n}\right)\cdot\Gamma\frac{1}{n}}{\Gamma\left(1+\frac{1}{n}+\frac{1}{n}\right)} \text{ (By relationship between Beta and Gamma function)}$$

$$= \frac{1}{n}\cdot\frac{\frac{1}{n}\Gamma\frac{1}{n}\Gamma\frac{1}{n}}{\Gamma\left(1+\frac{2}{n}\right)} \qquad [\because \Gamma(n+1) = n\Gamma n]$$

$$= \frac{1}{n}\cdot\frac{\frac{1}{n}\left(\Gamma\frac{1}{n}\right)^2}{\frac{2}{n}\Gamma\frac{2}{n}} = \frac{1}{n}\cdot\frac{\left\{\Gamma\frac{1}{n}\right\}^2}{2.\Gamma\frac{2}{n}}$$ **Proved**

Prob.29. Show that –

$$\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \frac{\Gamma p}{q^p},$$

where p, q > 0. ***(R.G.P.V., March/April 2010, June 2015)***

Sol. Here,

$$\text{L.H.S.} = \int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy$$

Put $\log\left(\frac{1}{y}\right) = t$ or $y = e^{-t}$ and $dy = -e^{-t}\,dt$

$$\text{L.H.S.} = \int_\infty^0 (e^{-t})^{q-1} t^{p-1}.(-e^{-t})\,dt = \int_0^\infty t^{p-1}(e^{-t})^{q-1+1}dt$$

$$= \int_0^\infty t^{p-1}(e^{-t})^q\,dt = \int_0^\infty t^{p-1}e^{-qt}\,dt \qquad \text{...(i)}$$

We know that $\Gamma p = \int_0^\infty x^{p-1}e^{-x}dx$ ...(ii)

Replace x by qt, so that dx = q dt, then equation (ii) becomes

$$\Gamma p = \int_0^\infty (qt)^{p-1}e^{-qt}q\,dt = q^p\int_0^\infty t^{p-1}e^{-qt}\,dt$$

or
$$\int_0^\infty t^{p-1}e^{-qt}\,dt = \frac{\Gamma p}{q^p} \qquad \text{...(iii)}$$

Substituting this value in equation (i)

$$\text{L.H.S.} = \frac{\Gamma p}{q^p} = \text{R.H.S.}$$ **Proved**

Prob.30. Prove that $\int_0^\infty \frac{x^c}{c^x}dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$.

(R.G.P.V., June 2007, Nov./Dec. 2007, Nov. 2019)

Sol. Here, $\int_0^\infty \frac{x^c}{c^x}dx = \int_0^\infty \frac{x^c}{e^{x\log c}}dx \qquad (\because c^x = e^{x\log c})$

or $$\int_0^\infty \frac{x^c}{c^x}dx = \int_0^\infty e^{-x\log c}.x^c dx \qquad ...(i)$$

Substituting x log c = t, so that $dx = \frac{dt}{\log c}$ in equation (i), we get

$$\int_0^\infty \frac{x^c}{c^x}dx = \int_0^\infty e^{-t}\left(\frac{t}{\log c}\right)^c \frac{dt}{\log c} = \frac{1}{(\log c)^{c+1}}\int_0^\infty t^c e^{-t}dt$$

Hence, $\int_0^\infty \frac{x^c}{c^x}dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$ **Proved**

Prob.31. Prove that $\Gamma n\Gamma(1-n) = \frac{\pi}{\sin n\pi}$, ***where*** $0 < n < 1$. ***From this result deduce the value of*** $\Gamma\left(\frac{1}{2}\right)$. ***(R.G.P.V., Dec. 2016)***

Sol. We know that

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)};\ m, n > 0$$

or $$\Gamma m\,\Gamma n = \Gamma(m+n).\beta(m,n)$$

or $$\Gamma m\,\Gamma n = \Gamma(m+n)\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx \qquad ....(i)$$

Substituting m + n = 1, i.e., m = 1 – n, in equation (i), we get

$$\Gamma n\,\Gamma(1-n) = \Gamma 1.\int_0^\infty \frac{x^{n-1}}{(1+x)}dx \text{ or } \Gamma n\,\Gamma(1-n) = \int_0^\infty \frac{x^{n-1}}{(1+x)}dx$$

or $$\mathbf{\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}}, \text{ where } 0 < n < 1$$

$$\left(\because \int_0^\infty \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi} \text{ for } 0 < n < 1\right)$$

Proved

Put $n = \frac{1}{2}$

$$\Gamma\frac{1}{2}\Gamma\left(1-\frac{1}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi}{2}\right)}$$

$$\Gamma\frac{1}{2}\Gamma\frac{1}{2}=\pi$$

$$\left(\Gamma\frac{1}{2}\right)^2=\pi \text{ or } \Gamma\frac{1}{2}=\sqrt{\pi}$$ Ans.

Prob.32. Prove that –

$$\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}},$$

where n is a positive integer and m > – 1. ***(R.G.P.V., Dec. 2008)***

Sol. Here,

$$\text{L.H.S.} = \int_0^1 x^m (\log x)^n . dx$$

Put $\log x = t$ or $x = e^t$, $dx = e^t\, dt$

$$= \int_{-\infty}^0 e^{mt}.t^n.e^t dt = \int_{-\infty}^0 e^{(m+1)t}.t^n dt$$

Put $(m+1)\,t = -y$, $dt = -\dfrac{1}{(m+1)}dy$

$$= \int_{\infty}^0 e^{-y}.\frac{(-1)^n}{(m+1)^n}.y^n.\left(\frac{-1}{m+1}\right)dy$$

$$= \int_0^\infty \frac{(-1)^n}{(m+1)^{n+1}}.e^{-y}.y^{n+1-1}dy$$

$$= \frac{(-1)^n}{(m+1)^{n+1}}\Gamma(n+1) = \frac{(-1)^n.n!}{(m+1)^{n+1}}$$ Proved

Prob.33. Prove that $\int_{-\infty}^{\infty} e^{-a^2x^2}dx = \dfrac{\sqrt{\pi}}{a}, a > 0.$ ***(R.G.P.V., Dec. 2015)***

Sol. Since $e^{-a^2x^2}$ is an even function of x and $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ if f(x) is an even function of x.

$$\therefore \quad \int_{-\infty}^{\infty} e^{-a^2x^2}dx = 2\int_0^\infty e^{-a^2x^2}dx$$

Put $a^2x^2 = t$,

$$\therefore \quad x = \frac{\sqrt{t}}{a}$$

So that $2a^2x\, dx = dt$ or $dx = \dfrac{dt}{2a^2x} = \dfrac{dt}{2a\sqrt{t}} = \dfrac{t^{-\frac{1}{2}}}{2a}dt$

When $x = 0, t = 0$

When $x \to \infty, t \to \infty$

$$\therefore \quad \int_{-\infty}^{\infty} e^{-a^2x^2} dx = 2\int_0^{\infty} e^{-t}.\frac{t^{-\frac{1}{2}}}{2a} dt = \frac{1}{a}\int_0^{\infty} e^{-t}.t^{\frac{1}{2}-1} dt$$

$$= \frac{1}{a}\Gamma\frac{1}{2} \qquad \left[\because \Gamma n = \int_0^{\infty} e^{-x}x^{n-1}dx \ (n > 0)\right]$$

$$= \frac{\sqrt{\pi}}{a}$$ **Proved**

Prob.34. Prove that $\int_0^{\infty} e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$. ***(R.G.P.V., May 2019)***

Sol. Let $I = \int_0^{\infty} e^{-x^2} dx$ ...(i)

Putting $x^2 = t$

So that $2x\, dx = dt$ or $dx = \frac{dt}{2x} = \frac{1}{2\sqrt{t}} dt = \frac{1}{2} t^{-1/2} dt$

Equation (i), becomes

$$I = \int_0^{\infty} e^{-t}.\frac{1}{2} t^{-1/2} dt = \frac{1}{2}\int_0^{\infty} e^{-t}.t^{\left(\frac{1}{2}-1\right)} dt$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \qquad \left[\because \Gamma n = \int_0^{\infty} e^{-x}x^{n-1}dx \ (n > 0)\right]$$

$$= \frac{1}{2}\sqrt{\pi}$$ **Proved**

Prob.35. Express in terms of the gamma function –

$$\int_0^{\infty} x^n e^{-a^2x^2} dx.$$

(R.G.P.V., April 2009, Dec. 2015)

Sol. Let, $I = \int_0^{\infty} x^n e^{-a^2x^2} dx$ ...(i)

Putting $t = a^2x^2$ or $x = \frac{\sqrt{t}}{a}$

$$dx = \frac{dt}{2a\sqrt{t}}$$

Equation (i), becomes

$$I = \int_0^{\infty} \left(\frac{\sqrt{t}}{a}\right)^n .e^{-t} \times \frac{1}{2a\sqrt{t}} dt = \frac{1}{2a^{n+1}}\int_0^{\infty} t^{\frac{n-1}{2}}.e^{-t} dt$$

$$= \frac{1}{2a^{n+1}}\int_0^{\infty} t^{\frac{n+1}{2}-1}.e^{-t} dt = \frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right)$$ **Ans.**

Prob.36. Using gamma function, evaluate $\int_0^{\infty} \sqrt{x}\, e^{-3\sqrt{x}}\, dx.$

(R.G.P.V., Dec. 2014)

Sol. Let, $I = \int_0^{\infty} \sqrt{x}\, e^{-3\sqrt{x}} dx$ ...(i)

Putting $t = 3\sqrt{x}$ or $x = \dfrac{t^2}{9}$

$$dx = \frac{2t}{9} dt$$

Equation (i), becomes

$$I = \int_0^{\infty} \frac{t}{3}.e^{-t}.\frac{2t}{9} dt$$

$$I = \frac{2}{27}\int_0^{\infty} e^{-t}.t^2 dt = \frac{2}{27}\int_0^{\infty} e^{-t}.t^{(3-1)} dt$$

We know that

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$\therefore$ $$I = \frac{2}{27}\Gamma 3 = \frac{2}{27} 2! = \mathbf{\frac{4}{27}}$$ Ans.

Prob.37. Prove that –

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1} dx = (b-a)^{m+n-1}\beta(m,n)$$

(R.G.P.V., Jan./Feb. 2007, June 2011)

Sol. The given integral is

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1} dx$$

Substituting $x = a + (b-a)y$, so that $dx = (b-a)\, dy$. Also when $x = a$, $y = 0$ and when $x = b$, $y = 1$.

$\therefore$ $\int_a^b (x-a)^{m-1}(b-x)^{n-1} dx$

$$= \int_0^1 [(b-a)y]^{m-1}[b-a-(b-a)y]^{n-1}(b-a)\, dy$$

$$= \int_0^1 (b-a)^{m-1} y^{m-1}(b-a)^{n-1}(1-y)^{n-1}(b-a)\, dy$$

$= (b-a)$ ... $^{n-1} dy$

$= (b$

Prob.38. Express $\int_0^1 x^m(1-x^n)^p\,dx$ in terms of beta function and hence evaluate $\int_0^1 x^5(1-x^3)^{10}\,dx$. ***(R.G.P.V., Dec. 2015, 2017)***

Sol. Let,
$$I = \int_0^1 x^m(1-x^n)^p\,dx \qquad ...(i)$$

Putting $x^n = z$ so that $n\,x^{n-1}\,dx = dz$

or $\quad dx = \frac{1}{n}z^{\frac{1}{n}-1}dz$, when $x = 0 \Rightarrow z = 0$ and $x = 1 \Rightarrow z = 1$ in equation (i), we get

$$I = \int_0^1 z^{m/n}.(1-z)^p.\frac{1}{n}z^{\frac{1}{n}-1}dz$$

or
$$I = \frac{1}{n}\int_0^1 z^{\frac{m+1}{n}-1}(1-z)^p\,dz$$

or
$$I = \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right) \qquad ...(ii)$$

Put m = 5, n = 3 and p = 10, we get

$$\int_0^1 x^5(1-x^3)^{10}dx = \frac{1}{3}\beta\left(\frac{5+1}{3}, 10+1\right) = \frac{1}{3}\beta(2,11) = \frac{1}{3}\frac{\Gamma 2\,\Gamma 11}{\Gamma(2+11)}$$

$$= \frac{1}{3}\frac{\Gamma 2\,\Gamma 11}{\Gamma 13} = \frac{1!(10)!}{3(12)!} = \frac{1}{3.12.11} = \mathbf{\frac{1}{396}} \qquad \textbf{Ans.}$$

Prob.39. Prove that $\beta(m, n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$, $(m, n > 0)$.

[R.G.P.V., Nov. 2018]

Or

Prove that $\beta(m, n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$. (R.G.P.V., June 2003, 2009, Feb. 20 , Dec. 2012, 2017, Nov. 2018, June 2)

Sol. Refer to the matter given on page 88.

Prob.40. Prove the duplication formula –

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m). \quad \textit{(R.G.P.V., Dec. 2011, 20)}$$

Prove the Legendre's duplication formula –

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m).$$ ***(R.G.P.V., Dec. 2014)***

Sol. Refer to the matter given on page 91.

Prob.41. Prove that –

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\,\Gamma(n+1)}.$$ ***(R.G.P.V., June 2005)***

Sol. From Duplication formula, we have

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m) \qquad ...(i)$$

Putting $m = n+\frac{1}{2}$ in equation (i), we get

$$\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2n}}\Gamma 2\left(n+\frac{1}{2}\right)$$

or $$\Gamma\left(n+\frac{1}{2}\right)\Gamma(n+1)=\frac{\sqrt{\pi}}{2^{2n}}\Gamma(2n+1)$$

or $$\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2n}}\cdot\frac{\Gamma(2n+1)}{\Gamma(n+1)}$$ **Proved**

Prob.42. If m, n are positive, then prove that –

(i) $$\frac{\beta(m,n+1)}{n}=\frac{\beta(m+1,n)}{m}=\frac{\beta(m,n)}{m+n}.$$

[R.G.P.V., Dec. 2006, June 2008(N), Dec. 2011]

(ii) $$\beta(m,n)=\beta(m+1,n)+\beta(m,n+1).$$

[R.G.P.V., June 2004, 2008 (O), 2013]

Sol. (i) Here,

$$\frac{\beta(m,n+1)}{n}=\frac{1}{n}\int_0^1 x^{m-1}(1-x)^n.dx=\frac{1}{n}\int_0^1(1-x)^n.x^{m-1}.dx$$

Integrating by part, we have

$$\frac{\beta(m,n+1)}{n}=\frac{1}{n}\left\{\left|\frac{(1-x)^n.x^m}{m}\right|_0^1-\int_0^1 n(1-x)^{n-1}.(-1).\frac{x^m}{m}.dx\right\}$$

or $$\frac{\beta(m,n+1)}{n}=\frac{1}{m}\int_0^1 x^m(1-x)^{n-1}dx=\frac{\beta(m+1,n)}{m} \qquad ...(i)$$

Now, $$\frac{\beta(m+1,n)}{m} = \frac{1}{m}\int_0^1 x^m(1-x)^{n-1}dx$$

or $$\frac{\beta(m+1,n)}{m} = \frac{1}{m}\int_0^1 x^{m-1}.x.(1-x)^{n-1}dx$$

$$\frac{\beta(m+1,n)}{m} = \frac{1}{m}\int_0^1 x^{m-1}[1-(1-x)](1-x)^{n-1}dx$$

$$\frac{\beta(m+1,n)}{m} = \frac{1}{m}\int_0^1 x^{m-1}.(1-x)^{n-1}dx - \frac{1}{m}\int_0^1 x^{m-1}(1-x)^n dx$$

$$\frac{\beta(m+1,n)}{m} = \frac{1}{m}\beta(m,n) - \frac{1}{m}\beta(m,n+1)$$

$$\frac{\beta(m+1,n)}{m} + \frac{1}{m}\beta(m,n+1) = \frac{1}{m}\beta(m,n)$$

$$\frac{\beta(m,n+1)}{n} + \frac{\beta(m,n+1)}{m} = \frac{\beta(m,n)}{m} \quad \left[\because \frac{\beta(m+1,n)}{m} = \frac{\beta(m,n+1)}{n}\right]$$

$$\left(\frac{m+n}{mn}\right).\beta(m,n+1) = \frac{\beta(m,n)}{m}$$

$$\frac{\beta(m,n+1)}{n} = \frac{\beta(m,n)}{m+n} \qquad \text{...(ii)}$$

From equations (i) and (ii), we get

$$\frac{\beta(m,n+1)}{n} = \frac{\beta(m+1,n)}{m} = \frac{\beta(m,n)}{m+n} \qquad \textbf{Proved}$$

(ii) Taking R.H.S. = $\beta(m+1,n)+\beta(m,n+1)$

$$= \int_0^1 x^m(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^n dx$$

$$= \int_0^1 \left[x^m(1-x)^{n-1} + x^{m-1}(1-x)^n\right]dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}[x+1-x]\,dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}dx = \beta(m,n) \qquad \textbf{Proved}$$

Prob.43. Express the integral $\int_0^1 x^3(1-x^2)^4 dx$ in terms of gamma function and hence evaluate. ***(R.G.P.V., Dec. 2016)***

Sol. Let, $I = \int_0^1 x^3(1-x^2)^4 dx$

Put $(1 - x^2) = t$ so that $-2x\,dx = dt$

$$\therefore \qquad I = \int_1^0 (1-t)t^4 \frac{dt}{-2} = \frac{1}{2}\int_0^1 (1-t)^{2-1} t^{5-1} dt$$

$$= \frac{1}{2}\beta(2,5) \qquad \text{(By definition of Beta function)}$$

$$= \frac{1}{2}\cdot\frac{\Gamma 2\,\Gamma 5}{\Gamma(2+5)} = \frac{1}{2}\cdot\frac{\Gamma 2\,\Gamma 5}{\Gamma 7} = \frac{1}{2}\cdot\frac{1!\,4!}{6!} = \frac{1}{2}\cdot\frac{1}{6.5} = \mathbf{\frac{1}{60}} \qquad \textbf{Ans.}$$

Prob.44. Express the integral $\int_0^1 x^m (1-x^n)^p dx$ in terms of gamma function and hence evaluate –

$$\int_0^1 x^2 (1-x^2)^4 dx. \qquad \textit{(R.G.P.V., Dec. 2004)}$$

Sol. Let, $\qquad I = \int_0^1 x^m (1-x^n)^p\, dx$

Put $(1 - x^n) = t$ so that $-nx^{n-1}\,dx = dt$

$$= -\int_1^0 (1-t)^{m/n}\, t^p . \frac{dt}{n(1-t)^{\frac{n-1}{n}}} = \frac{1}{n}\int_0^1 (1-t)^{\frac{m}{n} - \frac{(n-1)}{n}} t^p dt$$

$$= \frac{1}{n}\int_0^1 (1-t)^{\frac{m-n+1}{n}} t^p\, dt = \frac{1}{n}\int_0^1 (1-t)^{\left(\frac{m-n+1}{n}\right)} t^{p+1-1} dt$$

$$= \frac{1}{n}\int_0^1 (1-t)^{\frac{m+1}{n}-1} t^{(p+1)-1} dt = \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right)$$

or
$$I = \mathbf{\frac{1}{n} \frac{\Gamma\frac{m+1}{n}.\Gamma p+1}{\Gamma\left(\frac{m+1}{n}+p+1\right)}} \qquad \textbf{Ans.}$$

Given, integral

$$I = \int_0^1 x^2 (1-x^2)^4 dx$$

Here, $n = 2$, $m = 2$, $p = 4$.

Then
$$I = \frac{1}{2}\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma(4+1)}{\Gamma\left(\frac{2+1}{2}+4+1\right)} = \mathbf{\frac{1}{2}\cdot\frac{\Gamma\frac{3}{2}\Gamma(5)}{\Gamma\left(\frac{13}{2}\right)}} \qquad \textbf{Ans.}$$

Prob.45. Express the integral $\int_0^1 x^m (1-x^n)^p\, dx$ in terms of gamma function. *[R.G.P.V., Nov. 2018(O)]*

Sol. Refer to Prob.44.

APPLICATIONS OF DEFINITE INTEGRALS TO EVALUATE SURFACE AREAS AND VOLUMES OF REVOLUTIONS

Solids of Revolution –

Definition – If a plane area is revolved about a fixed line in its own plane, then the body so generated is known as solid of revolution.

Axis of Revolution – The fixed straight line about which the area revolves is said to be the axis of revolution or axis of rotation.

Surface of Revolution – If a plane curve is revolved about a fixed line lying in its own plane then surface generated by the perimeter of the curve is known as surface of revolution.

Surface Area of Revolution –

(i) Revolution About x-axis – The surface area of the solid generated by the revolution about x-axis, of the arc of the curve y = f(x) from x = a to x = b, is

$$\int_{x=a}^{x=b} 2\pi\, y\, ds$$

where s is the length of the arc measured from x = a to any point (x, y).

where, $\dfrac{ds}{dx} = \sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}$

(ii) Revolution About y-axis – Similarly the curved surface of the solid generated by the revolution about the y-axis of the area bounded by the curve x = f(x), the y-axis and the lines y = a, y = b is

$$\int_{y=a}^{y=b} 2\pi x\, ds$$

Volumes of Solids of Revolution –

(i) Revolution About x-axis – The volume of the solid generated by the revolution of the area bounded by the curve y = f(x), x-axis and the ordinates x = a, x = b about the x-axis is

$$\int_a^b \pi y^2\, dx$$

(ii) Revolution About the y-axis – Interchanging x and y in above formula, we obtain that the volume of the solid generated by the revolution, about y-axis of the area, between by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, $y = b$ is

$$\int_a^b \pi x^2 dy$$

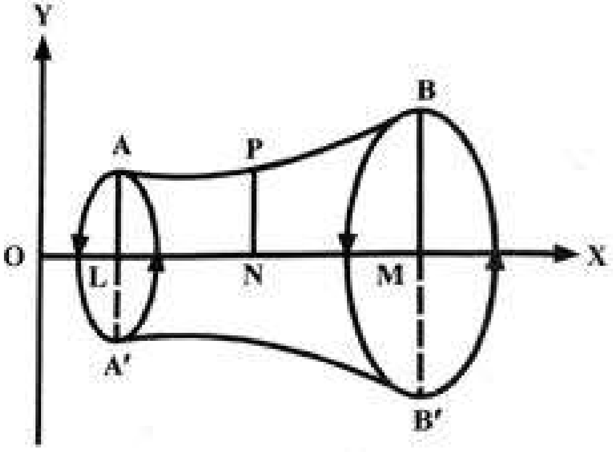


Fig. 2.1

(iii) Revolution About any Axis – The volume of the solid generated by the revolution about any axis LM of the area bounded by the curve AB, the axis LM and the perpendiculars AL, BM on the axis is

$$\int_{OL}^{OM} \pi(PN)^2 d(ON)$$

where O is a fixed point in LM and PN is perpendicular from any point P of the curve AB on LM.

(i) Volumes of Revolution (Polar Curves) – The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and radii vectors $\theta = \alpha$ and $\theta = \beta$.

(a) About the initial line (OX) $(\theta = 0)$ is given by,

$$V = \int_\alpha^\beta \frac{2\pi}{3} r^3 \sin\theta \, d\theta$$

(b) About the line OY $(\theta = \pi/2)$ is given by

$$V = \int_\alpha^\beta \frac{2\pi}{3} r^3 \cos\theta \, d\theta$$

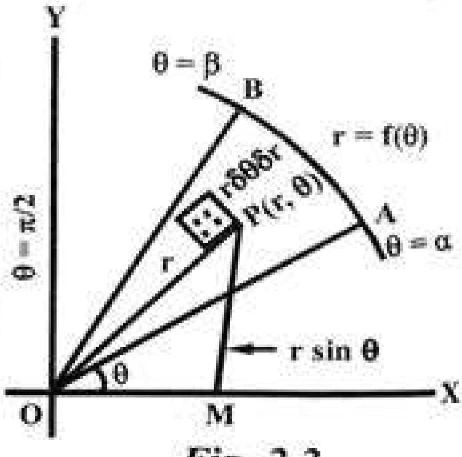


Fig. 2.2

(ii) Volume of a solid of revolution when the equation of the generating curve are given in parametric form.

(a) Let the curve be given by the parametric equations, say $x = \phi(t)$, $y = \psi(t)$, then the volume of the solid generated by the revolution about x-axis of the area bounded by the curve and the axis of x and the ordinates at the points where $t = a$ and $t = b$ is

$$= \int_a^b \pi y^2 \frac{dx}{dt} . dt = \pi \int_a^b \{\psi(t)\}^2 \phi'(t) \, dt$$

(b) The volume of the solid generated by the revolution about y-axis of the area between the curve $x = \phi(t)$, $y = \psi(t)$, the y-axis and the abscissae at the points where $t = a$, $t = b$ is

$$= \int_a^b \pi y^2 \frac{dy}{dt}.dt = \pi \int_a^b \{\phi(t)\}^2 \psi'(t)\, dt$$

(iii) Volume of solid of revolution when the equation of the generating curve is given in polar co-ordinates.

If the equation of the generating curve be given in the polar form and the curve is revolved about the initial line, then substituting $x = r\cos\theta$, $y = r\sin\theta$

the formula corresponding to $\int_a^b \pi y^2\, dx$ becomes $\int_\alpha^\beta \pi(r\sin\theta)^2 \frac{d}{d\theta}(r\cos\theta)\, d\theta$,

where α, β are the values of θ corresponding to $x = a$ and $x = b$.

NUMERICAL PROBLEMS

Prob.46. Find the area of the surface formed by the revolution of $x^2 + 4y^2 = 16$ about its major axis.

Sol. Here, the given equation of the curve is

$$x^2 + 4y^2 = 16 \qquad ....(i)$$

Differentiating equation (i) with respect to x, we obtain

$$2x + 8y\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = -\frac{x}{4y}$$

We know that by differential calculus

$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1 + \left(-\frac{x}{4y}\right)^2\right\}}$$

or $$\frac{ds}{dx} = \sqrt{1 + \frac{x^2}{16y^2}} = \frac{\sqrt{16y^2 + x^2}}{4y}$$

or $$\frac{ds}{dx} = \frac{\sqrt{\{x^2 + 4(16 - x^2)\}}}{4y}$$

or $$4y\, ds = \sqrt{64 - 3x^2}\, dx$$

Hence, the required surface $= 2.2\pi\int_{x=0}^{x=4} y\,ds$

or $$S = \pi\int_0^4 \sqrt{64-3x^2}\,dx$$

Put, $\sqrt{3}x = t$ so that $\sqrt{3}\,dx = dt$

$\therefore$ $$S = \frac{\pi}{\sqrt{3}}\int_0^{4\sqrt{3}} \sqrt{64-t^2}\,dt$$

or $$S = \frac{\pi}{\sqrt{3}}\left[\frac{1}{2}t\sqrt{64-t^2}+\frac{64}{2}\sin^{-1}\left(\frac{t}{8}\right)\right]_0^{4\sqrt{3}}$$

or $$S = \frac{\pi}{\sqrt{3}}\left[2\sqrt{3}\sqrt{(64-48)}+32\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

or $$S = \frac{\pi}{\sqrt{3}}\left[8\sqrt{3}+32.\frac{\pi}{3}\right] = 8\pi\left[1+\frac{4\pi}{3\sqrt{3}}\right] \textbf{ sq. units Ans.}$$

Prob.47. Find the area enclosed by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$. ***(R.G.P.V., Dec. 2015)***

Sol. We have to find the area OAMBO as shown in fig. 2.3.

Solving the given two equations simultaneously,

We have

$$x^4 = 16a^2y^2$$

or $$x^4 = 16a^2(4ax)$$

or $$x^4 = 64\,a^3x$$

or $$x^4 - 64a^3x = 0$$

or $$x(x^3 - 64a^3) = 0$$

$$x = 0,\ x^3 = 64a^3 = (4a)^3 \Rightarrow x = 4a$$

$\therefore$ $x = 0$ at O

and $x = 4a$ at B

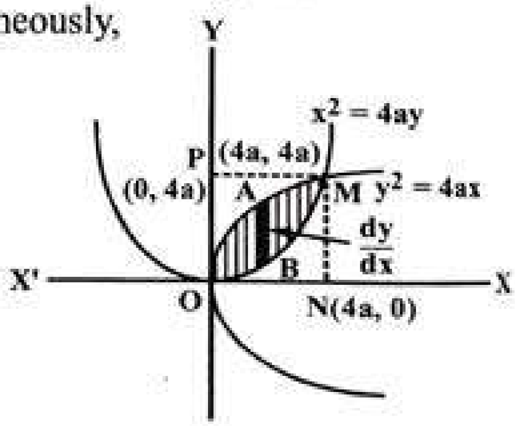


Fig. 2.3

Now area OAMBO = Area OAMNO − Area OBMNO

$$= \int_0^{4a} y_1dx - \int_0^{4a} y_2dx = \int_0^{4a} 2\,a^{1/2}x^{1/2}dx - \int_0^{4a}\frac{x^2}{4a}dx$$

$$= 2a^{\frac{1}{2}}\int_0^{4a} x^{\frac{1}{2}}dx - \frac{1}{4a}\int_0^{4a} x^2dx = 2a^{\frac{1}{2}}\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= 2.\frac{2}{3}a^{\frac{1}{2}}\left[(4a)^{3/2}-0\right] - \frac{1}{4a}.\frac{1}{3}\left[(4a)^3-0\right]$$

$$= \frac{4}{3}a^{1/2}.8a^{3/2} - \frac{1}{12a}64a^3 = \frac{32}{3}a^2 - \frac{16}{3}a^2$$

$$= \frac{32a^2 - 16a^2}{3} = \frac{16}{3}a^2 \text{ sq. units} \qquad \textbf{Ans.}$$

Prob.48. Find the whole area of astroid $x^{2/3} + y^{2/3} = a^{2/3}$.

(R.G.P.V., Dec. 2012)

Sol. The parametric equations of given curve are

$$x = a\cos^3 t$$
$$y = a\sin^3 t$$

$\therefore$
$$\frac{dx}{dt} = -3a\cos^2 t \sin t$$
$$\frac{dy}{dt} = 3a\sin^2 t \cos t$$

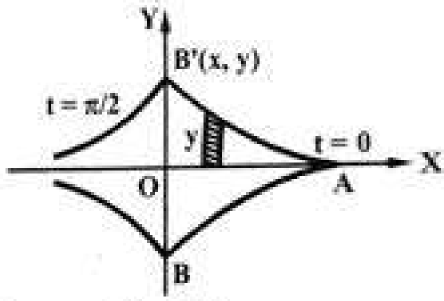


Fig. 2.4

Hence
$$\frac{dS}{dt} = \sqrt{\left\{\frac{dx}{dt}\right\}^2 + \left\{\frac{dy}{dt}\right\}^2}$$
$$= \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t} = 3a\sin t\cos t$$

For the arc AB, t varies from 0 to $\pi/2$

The required area is

= 2 × Area of surface generated by the revolution of AB

$$= 2\int_0^{\pi/2} 2\pi y dS = 4\pi\int_0^{\pi/2} y.\frac{dS}{dt}dt$$

$$= 4\pi\int_0^{\pi/2} a\sin^3 t\, 3a\sin t\cos t\, dt = 12\pi a^2\int_0^{\pi/2}\sin^4 t\cos t\, dt$$

$$= 12\pi a^2\frac{\Gamma\frac{5}{2}\Gamma 1}{2\Gamma\frac{7}{2}} = 12\pi a^2\frac{\Gamma\frac{5}{2}}{2.\frac{5}{2}\Gamma\frac{5}{2}} = \frac{\mathbf{12\pi a^2}}{\mathbf{5}} \textbf{ sq. units} \qquad \textbf{Ans.}$$

Prob.49. Find the volume of a spherical cap of height h cut-off from a sphere of radius a.

Sol. Equation of generating circle is

$$x^2 + y^2 = a^2$$

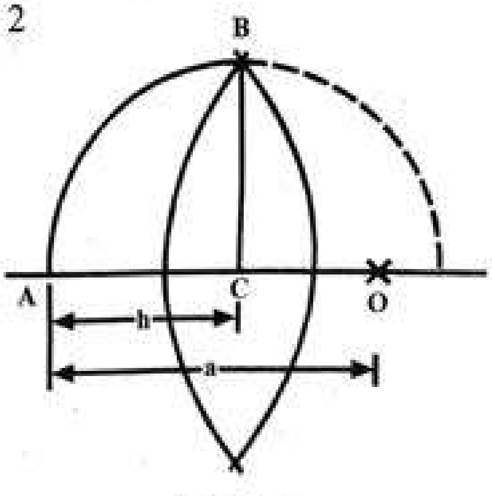


Fig. 2.5

Suppose,

AO = Radius of the circle = a

AC = Height of the spherical cap = h

When the portion ABC will be revolved about axis of symmetry, the spherical cap is formed.

Therefore, volume of the spherical cap $= \pi \int_{CO}^{AO} y^2 dx$

or $$V = \pi \int_{a-h}^{a} (a^2 - x^2)\, dx = \pi \left[a^2 x - \frac{x^3}{3} \right]_{a-h}^{a}$$

or $$V = \pi \left[a^3 - \frac{a^3}{3} - a^2 (a-h) + \frac{(a-h)^3}{3} \right]$$

$$= \pi h^2 \left(a - \frac{h}{3} \right) \textbf{ cubic units}$$ Ans.

Prob.50. Find the volume of the spindle shaped solid generated by revolving the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis.

Sol. Here, the given equation of the curve is

$x^{2/3} + y^{2/3} = a^{2/3}$ ....(i)

The parametric equations of astroid are

$x = a \cos^3 t,\ y = a \sin^3 t$ ....(ii)

The curve is symmetrical about both the axes and for the portion of the curve in the first quadrant, t varies 0 to $\frac{\pi}{2}$.

The required volume = 2 × Volume generated by revolving the area ABOA

or $$V = 2\int_0^{\pi/2} \pi y^2 . \frac{dx}{dt} . dt$$

or $$V = 2\pi \int_0^{\pi/2} a^2 \sin^6 t$$

$$(-3a \cos^2 t \sin t)\, dt$$

or $$V = -6\pi a^3 \int_0^{\pi/2} \sin^7 t \cos^2 dt$$

or $$V = -6\pi a^3 . \frac{\Gamma 4\, \Gamma \frac{3}{2}}{2 . \Gamma \frac{11}{2}}$$

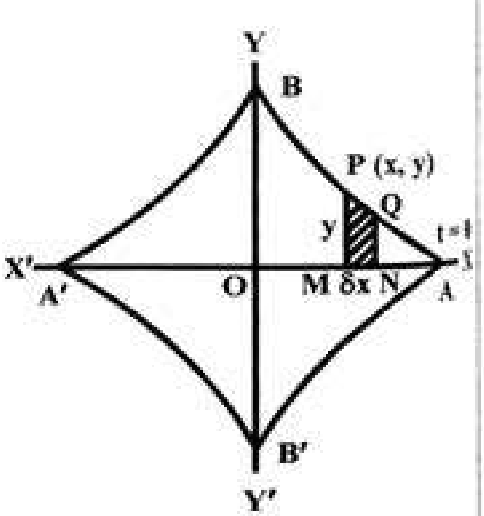


Fig. 2.6

or $$V = -6\pi a^3 . \frac{\frac{1}{2}.\sqrt{\pi}.3.2.1}{2.\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}} = -\frac{\mathbf{32\pi a^3}}{\mathbf{105}}$$ Ans.

MULTIPLE INTEGRAL

Double Integrals – Let a function f(x, y) of the independent variables be continuous inside some region R and on its boundary. Let us divide the domain R into n subdomain $R_1, R_2, \ldots, R_n$ of areas $\delta R_1, \delta R_2, \ldots, \delta R_n$. Suppose (x_r, y_r) is any point inside the r^{th} elementary area δR_r. Now consider the sum

$$S_n = f(x_1, y_1)\,\delta R_1 + f(x_2, y_2)\,\delta R_2 + \ldots + f(x_r, y_r)\,\delta R_r + \ldots$$

or $$S_n = \sum_{r=1}^{n} f(x_r, y_r)\,\delta R_r \quad \ldots(i)$$

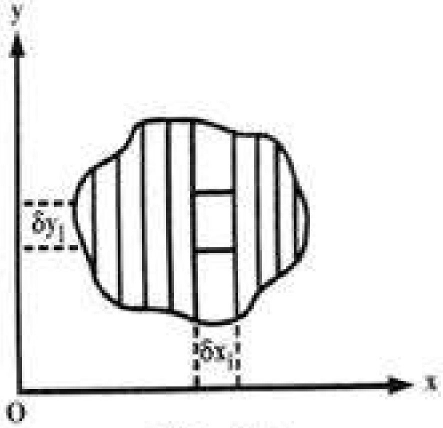


Fig. 2.7

The limit of this sum (i), if it exists as the number of sub-divisions increases indefinitely and area of each sub-division decreases to zero, is defined as the ***double integral*** of the function f(x, y) over the region R. It is denoted by

$\iint_R f(x, y)\, dR$ or $\iint_R f(x, y)\, dx\, dy$, and is read as "the double integral of f(x, y) over R."

To find the double integral we subdivided R by lines parallel to the co-ordinate axes. Creating a rectangular grid. Here the area of the typical rectangle is $\delta x_i . \delta x_j$, we have

$$\iint_R f(x,y)\, dx\, dy = \lim_{n\to\infty} \sum_{i,j=1}^{n} f(x_i, x_j)\,\delta x_i\, \delta x_j$$

Evaluation of Double Integrals –

Let, the region R be described by the inequalities,

$$\left.\begin{array}{r} c \le y \le d \\ \text{and} \quad g_1(y) \le x \le g_2(y) \end{array}\right\} \ldots(i)$$

as shown in fig. 2.8, then

$$\iint_R \mathbf{f(x, y)\, dx\, dy}$$

$$= \int_{y=c}^{y=d} \left\{ \int_{x=g_1(y)}^{x=g_2(y)} f(x, y)\, dx \right\} dy \quad \ldots(ii)$$

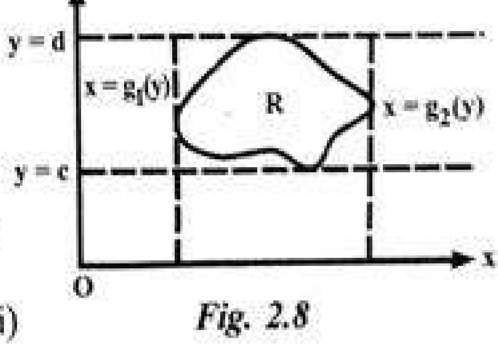


Fig. 2.8

Similarly, if the region R described by the inequalities

$$\left.\begin{array}{ll} & a \le x \le b \\ \text{and} & h_1(x) \le y \le h_2(x) \end{array}\right\} \quad \text{...(iii)}$$

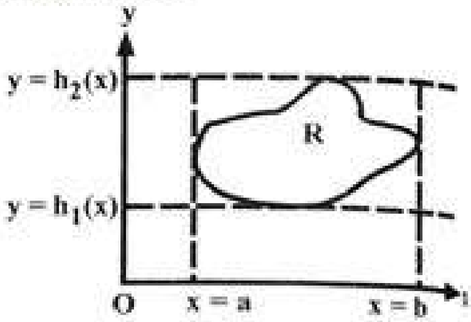


Fig. 2.9

as shown in fig. 2.9, then we write

$$\iint_R f(x, y)\, dx\, dy = \iint_R f(x, y)\, dy\, dx$$

or $$\iint_R \mathbf{f(x, y)\, dy\, dx} = \int_{x=a}^{x=b} \left\{ \int_{y=h_1(x)}^{y=h_2(x)} \mathbf{f(x, y)\, dy} \right\} \mathbf{dx} \quad \text{...(iv)}$$

Suppose the region R is bounded by the lines, x = a, x = b, y = c and y = d, as shown in fig. 2.10. Then

$$\iint_R f(x, y)\, dx\, dy = \int_{x=a}^{b} \left\{ \int_{y=c}^{d} f(x, y)\, dy \right\} dx$$

$$= \int_{y=c}^{d} \left\{ \int_{x=a}^{b} f(x, y)\, dx \right\} dy$$

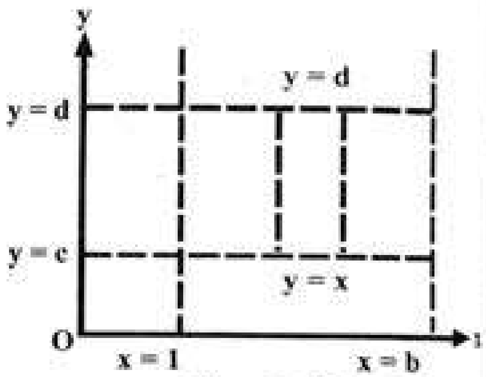


Fig. 2.10

i.e., in this case the order of integration is immaterial provided the limits of integration are changed accordingly.

Change of Variables –

Statement – Suppose the variables x, y in the double integral $\iint_R f(x, y)\, dx\, dy$ are changed to u, v by means of the relations, $x = \phi(u, v)$, $y = \psi(u, v)$, then the double integral is transformed into

$$\iint_{R'} f[\phi(u, v), \psi(u, v)]\, |J|\, du\, dv$$

where, $$J(\text{Jacobian}) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

and R' is the region in the uv-plane which corresponding to the region R in the xy-plane.

Working Rule – Change x, y by their equivalents in terms of u and v, the elementary area dx dy by |J| du dv, and also change the region R of integration in xy-plane by the region R' in the uv-plane.

Change to Polar Co-ordinates – Suppose δθ is the angle between two consecutive lines and δr is the distance between two consecutive circles

There is thus a network of elementary areas (say n in numbers) of which a typical one is PQRS. If P is the point (r, θ) the area of the element PQRS, situated at the point P is $1/2(r+\delta r)^2\delta\theta-(1/2)r^2\delta\theta = r\,\delta\theta\,\delta r$, by neglecting the term $1/2(\delta r)^2\delta\theta$ being an infinitesimal of higher order.

By the definition of the double integral of $f(r, \theta)$ over the region R, we have

$$\iint_R f(r,\theta)\,dR = \lim_{\substack{n\to\infty \\ \delta r\to 0 \\ \delta\theta\to 0}} \sum_{k=1}^{n} f(r_k,\theta_k)\,r_k\,\delta\theta\,\delta r.$$

where $r_k\delta\theta\delta r$ is the area of the element situated at the point (r_k, θ_k)

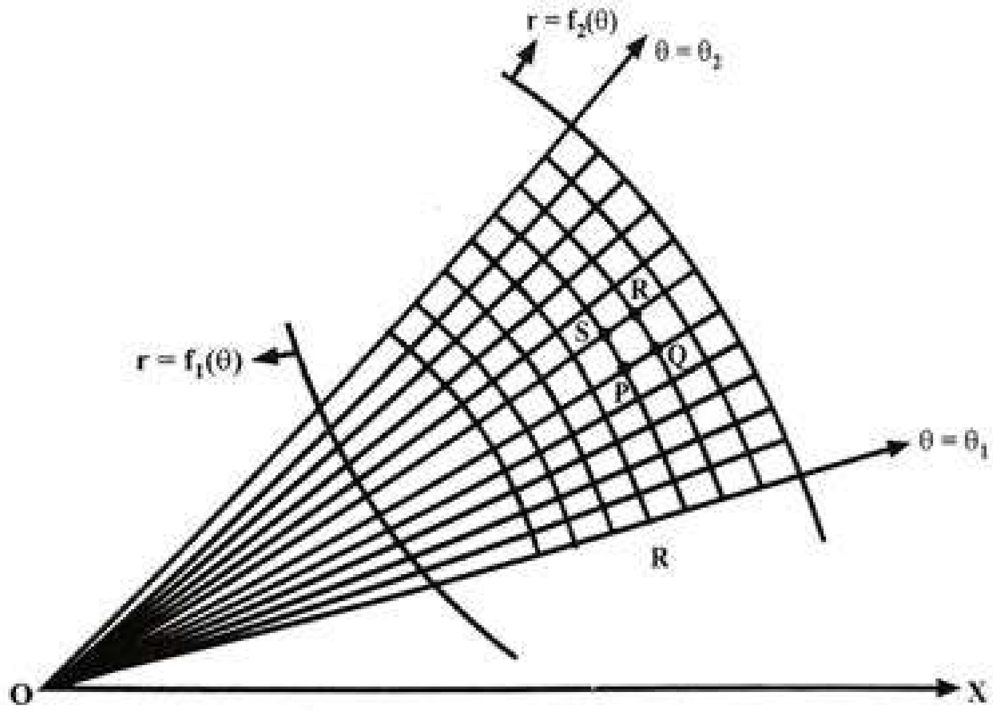


Fig. 2.11

Using the area of integration, this double integral is generally written as
$\int_{\theta_1}^{\theta_2}\int_{f_1(\theta)}^{f_2(\theta)} f(r,\theta)\,d\theta\,dr$ or $\int_{\theta_1}^{\theta_2} d\theta\int_{f_1(\theta)}^{f_2(\theta)} f(r,\theta)\,dr.$

The first integration is performed w.r.t.; r keeping θ as a constant. After putting the limits for r, the second integration w.r.t. θ is performed.

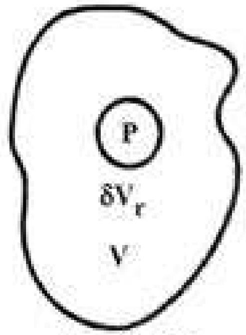


Fig. 2.12

Triple Integral – Let, $u = f(x, y, z)$ be single valued function of the independent variables x, y, z defined at every point of the three-dimensional finite region V. Let us divide the region V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_r$.

Let $P(x_r, y_r, z_r)$ be any point within the rth subregion δV_r. Take a point in each part and form the sum

$$S_n = f(x_1, y_1, z_1)\,\delta V_1 + f(x_2, y_2, z_2)\,\delta V_2 + \ldots + f(x_n, y_n, z_n)\,\delta V_n$$

$$= \sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r$$

Then the limit of this sum, if it exists, as $n \to \infty$ and $\delta V_r \to 0$ is said to be the triple integral of $f(x, y, z)$ over the region V and is denoted by

$$\iiint_V f(x, y, z)\,dV \text{ or } \iiint_V f(x, y, z)\,dx\,dy\,dz \qquad \ldots(i)$$

The triple integral can be used in evaluating a number of physical quantities like volume V etc.

$$\text{Volume} = \iiint_V dV, \text{ putting } f(x, y, z) = 1, \text{ also}$$

$$\text{Mass} = \iiint_V \rho\,dV, \text{ putting } f(x, y, z) = \rho.$$

Evaluation of Triple Integrals –

(i) Let, the region V be specified by the inequalities

$$a \le x \le b,\ \ c \le y \le d,\ \ e \le z \le f$$

then the triple integral

$$\iiint_V f(x, y, z)\,dx\,dy\,dz = \int_a^b \int_c^d \int_e^f f(x, y, z)\,dx\,dy\,dz$$

$$= \int_a^b dx \int_c^d dy \int_e^f f(x, y, z)\,dz$$

Here the order of integration is immaterial and the integration with respect to any of x, y and z can be performed first.

(ii) If the limits of z are given as functions of x and y and the limits of y as functions of x while x takes the constant values from $x = a$ to $x = b$ then

$$\iiint_V f(x, y, z)\,dx\,dy\,dz = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\,dz$$

i.e. first $f(x, y, z)$ is integrated with respect to z between the limits z_1 and z_2 regarding x and y as constant. Then the result just found integrated with respect to y between the limits y_1 and y_2 regarding x as a constant and in the last we perform the integration with respect to x.

NUMERICAL PROBLEMS

Prob.51. Evaluate $\int_0^2\int_0^1 (x^2+y^2)\,dx\,dy.$

[R.G.P.V., Dec. 2014, May 2018, Nov. 2018(O), May 2019, June 2020]

Sol. $\int_0^2\int_0^1(x^2+y^2)dx\,dy = \int_0^2\left[\int_0^1(x^2+y^2)dx\right]dy$

$$= \int_0^2\left[\frac{x^3}{3}+xy^2\right]_0^1 dy = \int_0^2\left[\frac{1}{3}+y^2\right]dy = \left[\frac{1}{3}y+\frac{y^3}{3}\right]_0^2 = \frac{2}{3}+\frac{8}{3} = \mathbf{\frac{10}{3}}\ \textbf{Ans.}$$

Prob.52. Evaluate – $\int_0^2\int_0^3 (x^2+y^2)\,dx\,dy.$ ***(R.G.P.V., Dec. 2017)***

Sol. $\int_0^2\int_0^3(x^2+y^2)dx\,dy = \int_0^2\left[\int_0^3(x^2+y^2)dx\right]dy$

$$= \int_0^2\left[\frac{x^3}{3}+xy^2\right]_0^3 dy = \int_0^2\left[9+3y^2\right]dy = \left[9y+y^3\right]_0^2$$

$$= 18+8 = \mathbf{26}$$ **Ans.**

Prob.53. Evaluate – $\int_1^2\int_1^3 xy^2\,dx\,dy$ ***(R.G.P.V., Dec. 2016)***

Sol. $\int_1^2\int_1^3 xy^2dx\,dy = \int_1^2\left[\int_1^3 xy^2dx\right]dy$

$$= \int_1^2\left[\frac{x^2y^2}{2}\right]_1^3 dy = \int_1^2\left[\frac{9}{2}y^2-\frac{y^2}{2}\right]dy = 4\int_1^2 y^2dy$$

$$= 4\left[\frac{y^3}{3}\right]_1^2 = 4\left[\frac{8}{3}-\frac{1}{3}\right] = 4\times\frac{7}{3} = \mathbf{\frac{28}{3}}$$ **Ans.**

Prob.54. Evaluate – $\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy.$ ***(R.G.P.V., June 2017)***

Sol. $\int_0^1\int_x^{\sqrt{x}}(x^2+y^2)\,dx\,dy = \int_0^1\left[\int_x^{\sqrt{x}}(x^2+y^2)dy\right]dx$

$$= \int_0^1\left[x^2y+\frac{y^3}{3}\right]_x^{\sqrt{x}} dx = \int_0^1\left[x^2\sqrt{x}+\frac{(\sqrt{x})^3}{3}-x^3-\frac{x^3}{3}\right]dx$$

$$= \left[x^{5/2} + \frac{1}{3}x^{3/2} - x^3 - \frac{1}{3}x^3\right]dx$$

$$= \left[\frac{2}{7}x^{7/2} + \frac{2}{15}x^{5/2} - \frac{x^4}{4} - \frac{x^4}{12}\right]_0^1$$

$$= \left[\frac{2}{7} + \frac{2}{15} - \frac{1}{4} - \frac{1}{12}\right] = \mathbf{\frac{3}{35}}$$ **Ans.**

Prob.55. Evaluate –

$$\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx.$$

[R.G.P.V., Nov./Dec. 2007, Nov. 2019 (O)]

Sol. Here,

$$\int_{x=0}^1 dx \int_{y=0}^{x^2} e^{y/x} dy = \int_{x=0}^1 dx \left[xe^{y/x}\right]_{y=0}^{x^2} = \int_0^1 (xe^x - x)\, dx$$

$$= \left[xe^x - e^x - \frac{x^2}{2}\right]_0^1 = e - e - \frac{1}{2} + 1 = \mathbf{\frac{1}{2}}$$ **Ans.**

Prob.56. Evaluate –

$$\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{\{(1-x^2)(1-y^2)\}}}.$$

(R.G.P.V., Dec. 2005)

Sol. Let,

$$I = \int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{\{(1-x^2)(1-y^2)\}}} = \int_{y=0}^1 \left[\int_{x=0}^1 \frac{dx}{\{(1-x^2)(1-y^2)\}^{1/2}}\right] dy$$

$$= \int_{y=0}^1 \frac{1}{\sqrt{1-y^2}} \left[\int_{x=0}^1 \frac{dx}{\sqrt{1-x^2}}\right] dy = \int_{y=0}^1 \frac{1}{\sqrt{1-y^2}} \left[\sin^{-1} x\right]_{x=0}^1 dy$$

$$= \int_{y=0}^1 \frac{1}{\sqrt{1-y^2}} \left[\frac{\pi}{2} - 0\right] dy = \frac{\pi}{2} \int_{y=0}^1 \frac{1}{\sqrt{1-y^2}} dy$$

$$= \frac{\pi}{2}\left[\sin^{-1} y\right]_{y=0}^1 = \frac{\pi}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{4}$$ **Ans.**

Prob.57. Evaluate –

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}.$$ *(R.G.P.V., Dec. 2010)*

Sol. Let,

$$I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2} = \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{(1+x^2)+y^2}\right] dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} - \tan^{-1} 0\right] dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} [\tan^{-1}(1) - \tan^{-1}(0)]\, dx = \frac{\pi}{4}\int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx$$

$$= \frac{\pi}{4}\left[\log\left\{x+\sqrt{1+x^2}\right\}\right]_0^1$$

$$= \frac{\pi}{4}\left[\log\left\{1+\sqrt{1+1}\right\} - \log\left\{0+\sqrt{1+0}\right\}\right] = \frac{\pi}{4}\left[\log\left(1+\sqrt{2}\right)\right]$$ **Ans.**

Prob.58. Compute the value of $\iint_R dx\,dy$ ***when R is the ellipse –***

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

of positive quadrant. *(R.G.P.V., April 2009)*

Sol. Dividing the area into vertical strips of width δx, y varies from K(y = 0) to $L[y = b\sqrt{(1-x^2/a^2)}]$ and then x varies from 0 to a as shown in fig. 2.13.

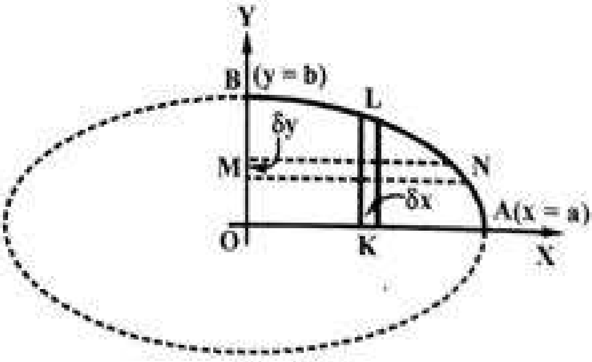


Fig. 2.13

$$I = \iint_R dx\,dy$$

$$= \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} dy = \int_0^a dx[y]_0^{b\sqrt{(1-x^2/a^2)}}$$

$$= \frac{b}{a}\int_0^a \sqrt{(a^2-x^2)}\, dx = \frac{\pi ab}{4}$$ **Ans.**

Prob.59. Evaluate $\iint y\,dx\,dy$ over the part of the plane bounded by the line $y = x$ and the parabola $y = 4x - x^2$. ***(R.G.P.V., Dec. 2015)***

Sol. The region R of integration is as shown in fig. 2.14.

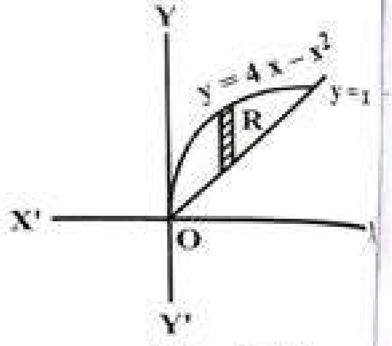


Fig. 2.14

Solving $y = 4x - x^2$ and $y = x$, we have

$$x = 4x - x^2$$

or $$3x - x^2 = 0$$

$$x(3 - x) = 0$$

$$x = 0 \text{ or } 3$$

$$y = 0 \text{ or } 3$$

Coordinates of intersections are O(0, 0) and A(3, 3).

Thus the region R is expressed as $0 \le x \le 3$, $x \le y \le 4x - x^2$.

Hence $I = \iint y\,dx\,dy = \int_{x=0}^{3}\int_{y=x}^{4x-x^2} y\,dx\,dy$

$$= \int_{x=0}^{3}\left[\frac{y^2}{2}\right]_{y=x}^{4x-x^2} dx = \frac{1}{2}\int_0^3\left[(4x-x^2)^2 - x^2\right]dx$$

$$= \frac{1}{2}\int_0^3 (16x^2 - 8x^3 + x^4 - x^2)dx = \frac{1}{2}\left[16\frac{x^3}{3} - 8\frac{x^4}{4} + \frac{x^5}{5} - \frac{x^3}{3}\right]$$

$$= \frac{1}{2}\left[\frac{16}{3}(3)^3 - 2(3)^4 + \frac{(3)^5}{5} - \frac{(3)^3}{3}\right] = \frac{1}{2}\left[144 - 162 + \frac{243}{5} - 9\right]$$

$$= \frac{1}{2}\left[-27 + \frac{243}{5}\right] = \frac{1}{2} \times \frac{108}{5} = \mathbf{\frac{54}{5}}$$

Ans.

Prob.60. Evaluate double integral $\iint_R xy\,dx\,dy$ over the region bounded by $x = 0$, $y = 0$ and $x + y = 1$. ***(R.G.P.V., Nov. 20…)***

Sol. Let $I = \iint_R xy\,dx\,dy$

$\because x + y = 1 \Rightarrow y = 1 - x$ and limits are $x \to 0$ to 1, $y \to 0$ to $1 - x$

$$\therefore \quad I = \int_{x=0}^{1}\int_{y=0}^{1-x} xy\,dx\,dy$$

$$= \int_{x=0}^{1} x\left[\int_{y=0}^{1-x} y\,dy\right]dx$$

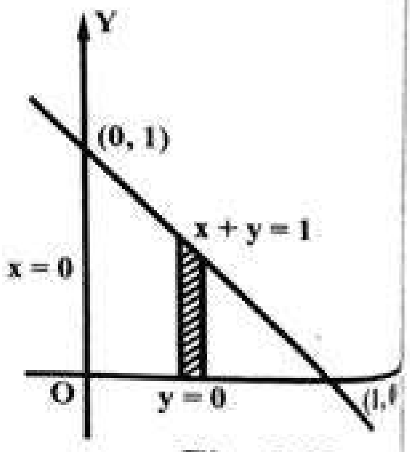


Fig. 2.15

$$= \int_{x=0}^{1} x\left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 x(1-x)^2 dx$$

$$= \frac{1}{2}\int_0^1 (x-2x^2+x^3)\, dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right] = \frac{1}{2}\times\frac{1}{12} = \frac{1}{24} \qquad \textbf{Ans.}$$

Prob.61. Evaluate $\iint_R xy\,(x+y)\,dx\,dy$, ***where R is the region bounded by the line*** $y = x$ ***and the curve*** $y = x^2$. ***(R.G.P.V., Dec. 2003, 2004)***

Sol. Let $I = \iint_R xy\,(x+y)\,dx\,dy$ ...(i)

The lines $y = x$ and the parabola $y = x^2$ intersect at P(1, 1) and O(0, 0), as shown in fig. 2.16.

Integrating first over a horizontal strip i.e. with respect to x from $x = y$ to $x = \sqrt{y}$ and then with respect to y from, $y = 0$ to $y = 1$, we have

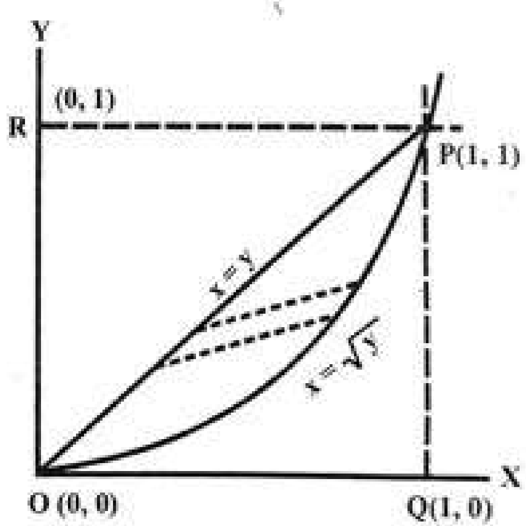


Fig. 2.16

$$I = \int_{y=0}^{1}\int_{x=y}^{\sqrt{y}} xy(x+y)\,dx\,dy$$

$$= \int_{y=0}^{1}\left\{\int_{x=y}^{\sqrt{y}} (x^2y+xy^2)\,dx\right\}dy = \int_0^1\left[\frac{x^3y}{3}+\frac{x^2y^2}{2}\right]_{x=y}^{\sqrt{y}} dy$$

$$= \int_{y=0}^{1}\left[\frac{y^2\sqrt{y}}{3}+\frac{y^3}{2}-\frac{y^4}{3}-\frac{y^4}{2}\right]dy = \int_{y=0}^{1}\left[\frac{y^2\sqrt{y}}{3}+\frac{y^3}{2}-\frac{5}{6}y^4\right]dy$$

$$= \left[\frac{2y^{7/2}}{21}+\frac{y^4}{8}-\frac{1}{6}y^5\right]_0^1 = \left[\frac{2}{21}+\frac{1}{8}-\frac{1}{6}\right] = \frac{3}{56} \qquad \textbf{Ans.}$$

Prob.62. Evaluate $\iint_R e^{2x+3y}\,dx\,dy$, ***where R is a triangle bounded by*** $x = 0, y = 0$ ***and*** $x + y = 1$. ***(R.G.P.V., Sept. 2009, Dec. 2017)***

Or

Evaluate $\iint_R e^{2x+3y}\,dx\,dy$, ***where R is the region bounded by*** $x = 0, y = 0$ ***and*** $x + y = 1$. ***[R.G.P.V., Nov. 2018(O)]***

Sol. Let,

$$I = \iint_R e^{2x+3y}\,dx\,dy$$

where R is the region shown in the fig. 2.17.

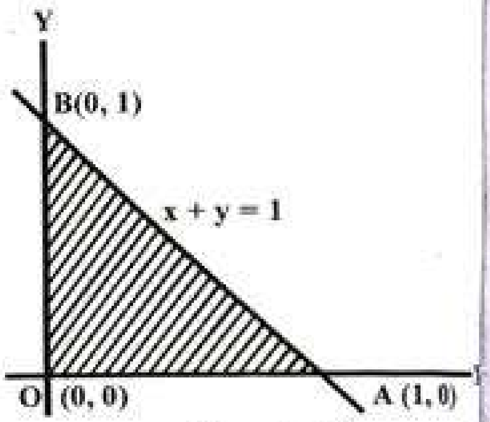


Fig. 2.17

We have,

$$I = \int_{y=0}^{1} e^{3y}dy\int_{x=0}^{x=1-y} e^{2x}dx$$

$$= \int_{y=0}^{1} e^{3y}\left[\frac{1}{2}e^{2x}\right]_{x=0}^{1-y} dy$$

$$= \frac{1}{2}\int_{y=0}^{1} e^{3y}\left[e^{2(1-y)} - e^{0}\right]dy$$

$$= \frac{1}{2}\int_{y=0}^{1}[e^{3y}.e^{2-2y} - 1.e^{3y}]\,dy$$

$$= \frac{1}{2}\int_{y=0}^{1}[e^{y+2} - e^{3y}]\,dy = \frac{1}{2}\int_{y=0}^{1}[e^{2}.e^{y} - e^{3y}]\,dy$$

$$= \frac{1}{2}\left[e^{2}.e^{y} - \frac{e^{3y}}{3}\right]_{y=0}^{1} = \frac{1}{2}\left[\left(e^{2}e^{1} - \frac{e^{3}}{3}\right) - \left(e^{2} - \frac{1}{3}\right)\right]$$

$$= \frac{1}{2}\left[e^{3} - \frac{e^{3}}{3} - e^{2} + \frac{1}{3}\right] = \frac{1}{2}\left[\frac{2e^{3}}{3} - e^{2} + \frac{1}{3}\right] = \mathbf{\frac{e^{3}}{3} - \frac{e^{2}}{2} + \frac{1}{6}}$$ **Ans.**

Prob.63. Evaluate $\iint_D x^2y^2\,dx\,dy$ ***where D is the region bounded by*** $x = 0$, $y = 0$ ***and*** $x^2 + y^2 = 1, x \geq 0, y \geq 0$. ***(R.G.P.V., Jan./Feb. 2007, Nov. 20…)***

Sol. Let, the region of integration be the first quardrant of the circle OAB.

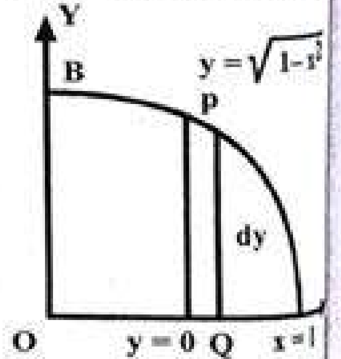


Fig. 2.18

$$\iint_D x^2y^2dx\,dy, \left(x^2 + y^2 = 1, y = \sqrt{1-x^2}\right)$$

First, we integrate with respect to y and then with respect to x.

The limits for y are 0 and $\sqrt{1-x^2}$ and for x, 0 to 1.

$$= \int_0^1 x^2 dx \int_0^{\sqrt{1-x^2}} y^2 dy$$

$$= \int_0^1 x^2 dx \left[\frac{y^3}{3}\right]_0^{\sqrt{1-x^2}} = \int_0^1 \frac{x^2(1-x^2)^{3/2}}{3} dx$$

$$= \frac{1}{3}\int_0^1 t\,(1-t)^{3/2} \frac{dt}{2\sqrt{t}} \qquad \left(\text{put, } x = \sqrt{t}, dx = \frac{1}{2\sqrt{t}} dt\right)$$

$$= \frac{1}{6}\int_0^1 t^{1/2}(1-t)^{3/2} dt = \frac{1}{6}\int_0^1 t^{3/2-1}(1-t)^{5/2-1} dt$$

$$= \frac{1}{6}\beta\left(\frac{3}{2},\frac{5}{2}\right) = \frac{1}{6}\frac{\Gamma\frac{3}{2}\Gamma\frac{5}{2}}{\Gamma\left(\frac{3}{2}+\frac{5}{2}\right)}$$

$$= \frac{1}{6}\frac{\frac{1}{2}\sqrt{\pi}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}{3!} = \frac{\pi}{96}$$ **Ans.**

Prob.64. Evaluate the triple integral $\int_{y=0}^{1}\int_{x=y^2}^{1}\int_{z=0}^{1-x} x\,dz\,dx\,dy.$

(R.G.P.V., May 2018, Nov. 2018)

Sol. Let,

$$I = \int_{y=0}^{1}\int_{x=y^2}^{1}\int_{z=0}^{1-x} x\,dz\,dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=y^2}^{1}[xz]_{z=0}^{1-x} dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=y^2}^{1}(x-x^2)\,dx\,dy$$

$$= \int_{y=0}^{1}\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{x=y^2}^{1} dy = \int_0^1\left[\frac{1}{2}-\frac{1}{3}-\frac{y^4}{2}+\frac{y^6}{3}\right]dy$$

$$= \int_0^1\left[\frac{1}{6}-\frac{1}{2}y^4+\frac{1}{3}y^6\right]dy = \left[\frac{1}{6}y-\frac{1}{2}\cdot\frac{y^5}{5}+\frac{1}{3}\frac{y^7}{7}\right]_0^1$$

$$= \left[\frac{1}{6}-\frac{1}{10}+\frac{1}{21}\right] = \frac{35-21+10}{210} = \frac{24}{210} = \frac{4}{35}$$ **Ans.**

Prob.65. Evaluate – $\int_0^3\int_0^2\int_0^1 (x+y+z)\,dx\,dy\,dz$. ***(R.G.P.V., June 2017)***

Sol. Let, $I = \int_0^3\int_0^2\int_0^1 (x+y+z)\,dx\,dy\,dz = \int_0^3\int_0^2\left[xz+yz+\frac{z^2}{2}\right]_0^1 dxdy$

$$=\int_0^3\left[\int_0^2\left(x+y+\frac{1}{2}\right)dx\right]dy$$

$$=\int_0^3\left[\frac{x^2}{2}+xy+\frac{x}{2}\right]_0^2 dy = \int_0^3 (2+2y+1)\,dy$$

$$=\int_0^3 (3+2y)\,dy = \left[3y+2\frac{y^2}{2}\right]_0^3 = 9+9=\mathbf{18}$$ Ans.

Prob.66. Evaluate – $\int_{-1}^{1}\int_0^z\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$.

(R.G.P.V., Dec. 2014, 201

Sol. Let,

$$I = \int_{-1}^{1}\int_0^z\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$$

$$= \int_{-1}^{1}\int_0^z\left[\int_{x-z}^{x+z}(x+y+z)dy\right]dx\,dz = \int_{-1}^{1}\int_0^z\left[xy+\frac{y^2}{2}+yz\right]_{x-z}^{x+z} dx\,dz$$

$$=\int_{-1}^{1}\int_0^z\left[x(x+z)+\frac{(x+z)^2}{2}+z(x+z)-x(x-z)-\frac{(x-z)^2}{2}-z(x-z)\right]dx$$

$$=\int_{-1}^{1}\int_0^z\left[x^2+xz+\frac{x^2}{2}+\frac{z^2}{2}+xz+xz+z^2-x^2+xz-\frac{x^2}{2}-\frac{z^2}{2}+xz-xz+z^2\right]dx$$

$$=\int_{-1}^{1}\left[\int_0^z(4xz+2z^2)dx\right]dz = \int_{-1}^{1}\left[2x^2z+2z^2x\right]_0^z dz = \int_{-1}^{1}(2z^3+2z^3)\,d$$

$$=4\int_{-1}^{1}z^3dz = 4\left[\frac{z^4}{4}\right]_{-1}^{1} = [1-1] = \mathbf{0}$$ A

Prob.67. Evaluate –

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2-z^2)}}.$$

(R.G.P.V., Dec.

Sol. Let,

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2-z^2)}}$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\frac{z}{\sqrt{(1-x^2-y^2)}}\right]_0^{\sqrt{1-x^2-y^2}} dx\,dy$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} [\sin^{-1}(1)-\sin^{-1}(0)]\,dx\,dy = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2} dy\,dx$$

$$= \frac{\pi}{2}\int_0^1 \left[\int_0^{\sqrt{1-x^2}} dy\right] dx = \frac{\pi}{2}\int_0^1 [y]_0^{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\,dx$$

$$= \frac{\pi}{2}\left[\frac{x\sqrt{1-x^2}}{2}+\frac{1}{2}\sin^{-1}x\right]_0^1 = \frac{\pi}{4}(\sin^{-1}1) = \frac{\pi}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8} \quad \textbf{Ans.}$$

Prob.68. Evaluate – $\int_0^1\int_0^1\int_0^1 e^{x+y+z}dx\,dy\,dz$. ***(R.G.P.V., Dec. 2017)***

Sol. Let,

$$I = \int_0^1\int_0^1\int_0^1 e^{x+y+z}dx\,dy\,dz = \int_0^1\int_0^1\left[\int_0^1 e^{x+y+z}dx\right]dy\,dz$$

$$= \int_0^1\int_0^1\left[e^{x+y+z}\right]_0^1 dy\,dz = \int_0^1\left[\int_0^1\left\{e^{1+y+z}-e^{y+z}\right\}dy\right]dz$$

$$= \int_0^1\left[e^{1+y+z}-e^{y+z}\right]_0^1 dz = \int_0^1\left[e^{2+z}-e^{1+z}-e^{1+z}+e^z\right]dz$$

$$= \int_0^1\left[e^{2+z}-2e^{1+z}+e^z\right]dz = \int_0^1(e^2-2e+1)e^z dz$$

$$= \left[(e^2-2e+1)e^z\right]_0^1 = (e-1)^2(e-1) = \mathbf{(e-1)^3} \quad \textbf{Ans.}$$

Prob.69. Evaluate –

$$\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}dz\,dy\,dx.$$

(R.G.P.V., March/April 2010, May 2019, June 2020)

Sol. Let,

$$I = \int_{x=0}^a\int_{y=0}^x\int_{z=0}^{x+y} e^{x+y+z}\,dz\,dy\,dx = \int_{x=0}^a\int_{y=0}^x\int_{z=0}^{x+y} e^{x+y}.e^z\,dz\,dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{x}\left[\int_{z=0}^{x+y} e^z dz\right] e^{x+y} dy\,dx = \int_{x=0}^{a}\int_{y=0}^{x} e^{x+y}\left[e^z\right]_0^{x+y} dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{x} e^{x+y}[e^{x+y} - e^0]\,dy\,dx = \int_{x=0}^{a}\int_{y=0}^{x} [e^{2(x+y)} - e^{x+y}]\,dy\,dx$$

$$= \int_{x=0}^{a}\left[\int_{y=0}^{x} e^{2(x+y)} dy - \int_{y=0}^{x} e^{x+y}\,dy\right]dx$$

$$= \int_{x=0}^{a}\left[\int_{y=0}^{x} e^{2x}.e^{2y} dy - \int_{y=0}^{x} e^{x}.e^{y}\,dy\right]dx$$

$$= \int_{x=0}^{a}\left[e^{2x}\left(\frac{e^{2y}}{2}\right)_{y=0}^{x} - e^x\left(e^y\right)_{y=0}^{x}\right]dx$$

$$= \int_{x=0}^{a}\left[\frac{e^{2x}}{2}(e^{2x} - e^0) - e^x(e^x - e^0)\right]dx$$

$$= \int_{x=0}^{a}\left[\frac{e^{4x}}{2} - \frac{e^{2x}}{2} - e^{2x} + e^x\right]dx = \int_{x=0}^{a}\left[\frac{e^{4x}}{2} - \frac{3e^{2x}}{2} + e^x\right]dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^x\right]_0^a = \left[\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \frac{1}{8} + \frac{3}{4} - 1\right]$$

$$= \mathbf{\frac{1}{8}(e^{4a} - 6e^{2a} + 8e^a - 3)}$$ **An**

Prob.70. Evaluate – $\int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx$. ***(R.G.P.V., Dec. 2***

Sol. Let, $I = \int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx$

Integrating w.r.t. z regarding x and y as constants.

$$I = \int_0^1\int_0^{1-x} xy\left[\int_0^{1-x-y} z\,dz\right]dy\,dx$$

$$I = \int_0^1\int_0^{1-x} xy\left[\frac{z^2}{2}\right]_0^{1-x-y} dy\,dx$$

$$I = \frac{1}{2}\int_0^1\int_0^{1-x} xy\{(1-x) - y\}^2\,dy\,dx$$

$I = \frac{1}{2}\int_0^1\int_0^{1}$ [illegible] x [illegible] $-2(1-x)y$ [illegible] y] dy dx

Integrating w.r.t. y regarding x as constant.

$$I = \frac{1}{2}\int_0^1 x\left[\frac{(1-x)^2y^2}{2} - \frac{2(1-x)y^3}{3} + \frac{y^4}{4}\right]_0^{1-x} dx$$

$$I = \frac{1}{24}\int_0^1 x\left[6(1-x)^4 - 8(1-x)^4 + 3(1-x)^4\right]dx$$

$$I = \frac{1}{24}\int_0^1 x(1-x)^4(6-8+3)dx = \frac{1}{24}\int_0^1 x(1-x)^4 dx$$

Putting $x = \sin^2\theta$ so that, $dx = 2\sin\theta\cos\theta\, d\theta$, then

$$I = \frac{1}{24}\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^8\theta . 2\sin\theta\cos\theta\, d\theta = \frac{1}{12}\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^9\theta\, d\theta$$

$$= \frac{1}{12}.\frac{\Gamma\frac{4}{2}\Gamma\frac{10}{2}}{2\Gamma\frac{14}{2}} \qquad \left[\because \int_0^{\frac{\pi}{2}} \sin^p x\cos^q x\, dx = \frac{\Gamma\frac{p+1}{2}\Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}}\right]$$

$$= \frac{1}{12}.\frac{\Gamma 2\Gamma 5}{2\Gamma 7} = \frac{1}{12}.\frac{1!\,4!}{2\,6!} = \frac{1}{12}.\frac{1}{2.6.5} = \mathbf{\frac{1}{720}}$$ **Ans.**

Prob.71. Evaluate $\int_0^2\int_0^x\int_0^{x+y} e^x(y+2z)\,dx\,dy\,dz.$

[R.G.P.V., Nov. 2018(O), Nov. 2019 (O)]

Sol. Let, $I = \int^2\int^x\int^{x+y} e^x(y+2z)\,dx\,dy\,dz$

Integrating w.r.t. z regarding x and y as constants.

$$I = \int_0^2\int_0^x e^x[yz+z^2]_0^{x+y}\,dx\,dy$$

$$I = \int_0^2\int_0^x e^x[y(x+y)+(x+y)^2]\,dx\,dy$$

$$I = \int_0^2\int_0^x e^x(x^2+2y^2+3xy)\,dx\,dy$$

Integrating w.r.t. y regarding x as constant.

$$I = \int_0^2 e^x\left[x^2y+\frac{2}{3}y^3+\frac{3}{2}xy^2\right]_0^x dx$$

[illegible]

$$I = \frac{19}{6}\int_0^2 x^3.e^x dx$$

$$I = \frac{19}{6}\left[x^3.e^x - \int 3x^2.e^x dx\right]_0^2$$

$$I = \frac{19}{6}\left[x^3.e^x - 3\left\{x^2e^x - \int 2x.e^x dx\right\}\right]_0^2$$

$$I = \frac{19}{6}\left[x^3.e^x - 3x^2e^x + 6\left\{x.e^x - \int 1.e^x dx\right\}\right]_0^2$$

$$I = \frac{19}{6}\left[x^3e^x - 3x^2e^x + 6xe^x - 6e^x\right]_0^2$$

$$I = \frac{19}{6}\left[e^x(x^3 - 3x^2 + 6x - 6\right]_0^2$$

$$I = \frac{19}{6}\left[e^2\left\{(2)^3 - 3.(2)^2 + 6(2) - 6\right\} - e^0\{-6\}\right]$$

$$I = \frac{19}{6}(2e^2 + 6) = \mathbf{\frac{19}{3}e^2 + 19}$$ **Ans.**

CHANGE THE ORDER OF THE INTEGRATION

Change of Order of Integration – To evaluating a double integral we integrate first with respect to one variable and considering the other variable as constant, and then integrate with respect to the remaining variable. In the former case, limits of integration are determined in the given region by drawing strips parallel to y-axis while in the second case by drawing strips parallel to x-axis.

However, if the limits are constant, the order of integration is immaterial, and in such a case we have

$$\int_a^b \int_c^d f(x, y)\, dx\, dy = \int_c^d \int_a^b f(x, y)\, dy\, dx$$

i.e.,
$$\int_a^b dy \int_c^d f(x, y)\, dx = \int_c^d dx \int_a^b f(x, y)\, dy$$

But if the limits are variables and the integral f(x, y) in the double integral is either difficult or even impossible integrate in the given order then we change the order of integration and corresponding change is made in the limits of integration. By geometrical considerations therefore a clear sketch of the curve is to be drawn, the new limits are obtained.

NUMERICAL PROBLEMS

Prob.72. Evaluate –

$$\int_0^\infty \int_0^x xe^{-x^2/y}\, dy\, dx$$

by changing the order of integration. *(R.G.P.V., Dec. 2008, 2011)*

Sol. The region is given by $x = 0$, $x = \infty$, $y = 0$, $y = x$. Hence changing the order of integration the given integral reduced as,

$$= \int_0^\infty \int_y^\infty xe^{-x^2/y} dy\, dx$$

$$= \int_0^\infty \left(\frac{-y}{2}\right)\left[e^{-x^2/y}\right]_y^\infty dy$$

$$= \frac{1}{2}\int_0^\infty ye^{-y} dy$$

$$= -\frac{1}{2}\left[(y+1)e^{-y}\right]_0^\infty = \frac{1}{2}$$ **Ans.**

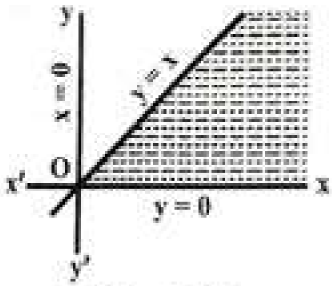


Fig. 2.19

Prob.73. Change the order of integration and evaluate it

$$\int_0^\infty \int_0^x e^{-xy} y\, dy\, dx$$ *(R.G.P.V., Nov. 2019)*

Sol. Here, $\int_0^\infty \int_0^x e^{-xy} y\, dy\, dx$

Here, the region OPQ of integration is bounded by $y = 0$ (x-axis), $y = x$ (a straight line), $x = 0$, i.e., y axis. A strip is drawn parallel to y-axis, y varies 0 to x and x varies 0 to ∞.

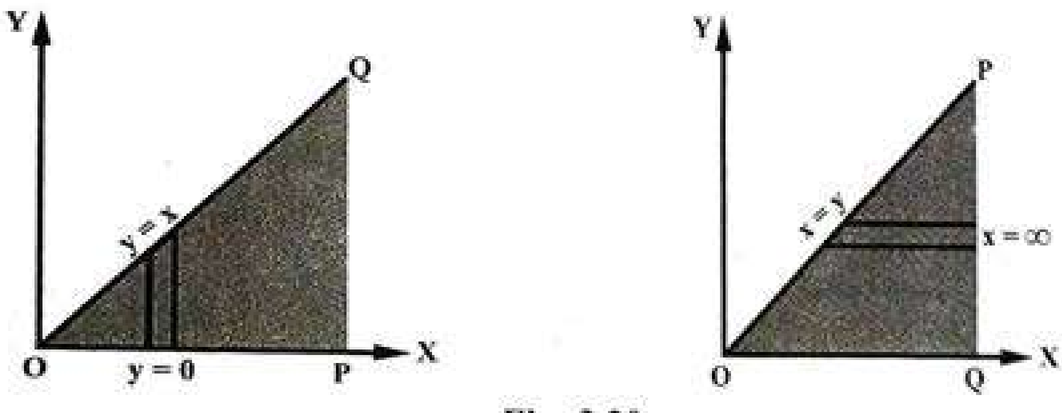


Fig. 2.20

On changing the order of integration, we first integrate, w.r.t. x and then w.r.t. y. A strip is drawn parallel to x-axis. On this strip x varies from y to ∞ and y varies from 0 to ∞.

Thus,

$$\int_0^\infty \int_0^x e^{-xy}\,y\,dy\,dx = \int_0^\infty y\,dy \int_y^\infty e^{-xy}dx = \int_0^\infty y\,dy\left[\frac{e^{-xy}}{-y}\right]_y^\infty$$

$$= \int_0^\infty \frac{y\,dy}{-y}\left[0 - e^{-y^2}\right] = \int_0^\infty e^{-y^2}dy = \frac{1}{2}\sqrt{\pi} \qquad \textbf{Ans.}$$

Prob.74. Evaluate –

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$$

by changing the order of integration. ***(R.G.P.V., June 2009)***

Or

By changing the order of integration, evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$.

(R.G.P.V., June 2017)

Sol. Here, $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$ ...(i)

Here, the elementary strip PQ extends from $y = x$ to $y = \infty$ and this vertical strip slides from $x = 0$ to $x = \infty$. The shaded portion of the figure is, therefore, the region of integration. On changing the order of integration, we first integrate, w.r.t. x along a horizontal strip RS which extends from $x = 0$ to $x = y$. To cover the given region, we then integrate w.r.t. y from $y = 0$ to $y = \infty$.

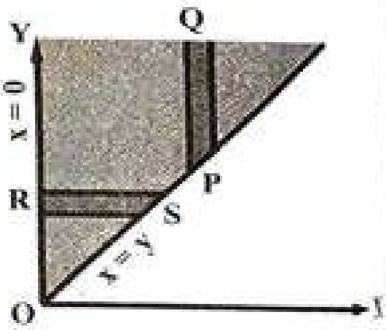


Fig. 2.21

Thus, $$\int_0^\infty dx \int_x^\infty \frac{e^{-y}}{y}dy = \int_0^\infty \frac{e^{-y}}{y}dy\int_0^y dx$$

$$= \int_0^\infty \frac{e^{-y}}{y}dy\,[x]_0^y = \int_0^\infty \frac{y.e^{-y}}{y}dy = \int_0^\infty e^{-y}dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^\infty = \left[-\frac{1}{e^y}\right]_0^\infty = -\left[\frac{1}{\infty} - 1\right] = \mathbf{1} \qquad \textbf{Ans.}$$

Prob.75. Change the order of integration $\int_{y=0}^a \int_{x=y}^a \frac{x\,dx\,dy}{x^2+y^2}$ ***and hence evaluate.*** ***[R.G.P.V., June 2003, 2004, 2008 (O***

Sol. Let, $$I = \int_{y=0}^a \int_{x=y}^a \frac{x\,dx\,dy}{x^2+y^2}$$ ...(i)

In the given integral, as it stands, it is to be integrated first with respect to x and then with respect to y, which is complicated. From the limits of integration, it is clear that the region of integration is bounded by $x = y$, $x = a$, $y = 0$ and $y = a$. Thus the region of integration is the Δ OPQ and is divided into horizontal strips. We divide the region of integration into vertical strips, for changing the order of integration. Thus, the new limits of integration becomes, y varies 0 to x and x varies 0 to a.

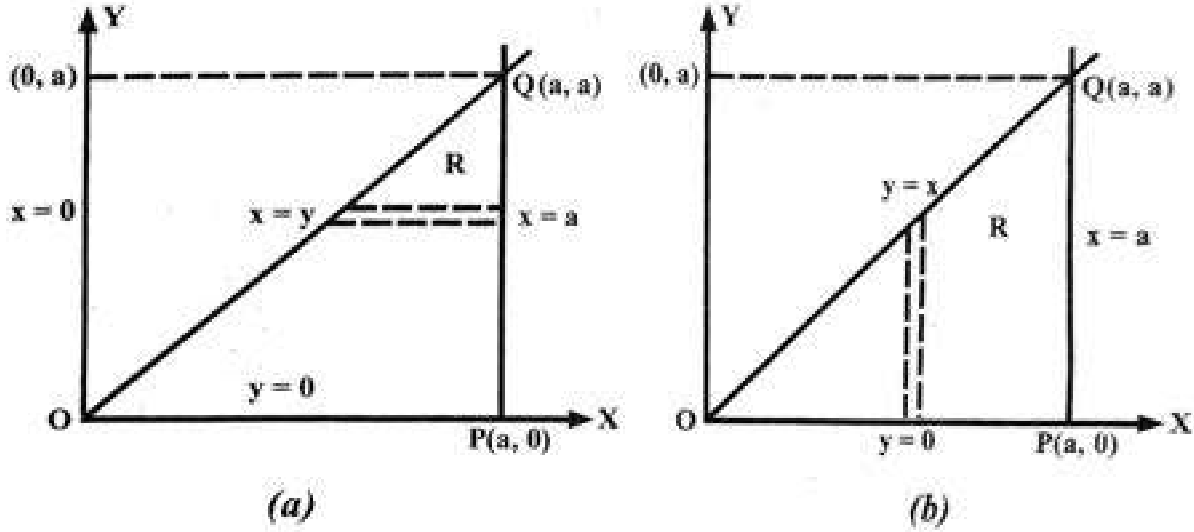


Fig. 2.22

Hence,

$$I = \int_{y=0}^{a}\int_{x=y}^{a}\frac{x}{x^2+y^2}\,dx\,dy = \int_{x=0}^{a}\int_{y=0}^{x}\frac{x}{x^2+y^2}\,dy\,dx$$

$$= \int_{x=0}^{a}\left[\int_{y=0}^{x}\frac{x\,dy}{x^2+y^2}\right]dx = \int_{x=0}^{a}\left[\frac{1}{x}.x\tan^{-1}\frac{y}{x}\right]_{y=0}^{x}dx$$

$$= \int_{x=0}^{a}\left[\frac{\pi}{4}-0\right]dx = \frac{\pi}{4}\int_{x=0}^{a}dx = \frac{\pi}{4}[x]_{x=0}^{a} = \frac{a\pi}{4} \qquad \textbf{Ans.}$$

Prob.76. Evaluate the following integral by changing the order of integration –

$$\int_0^1\int_x^{\sqrt{2-x^2}}\frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$$

[R.G.P.V., June 2008(N), Feb. 2010, June 2013]

Sol. The region to the integration is given by $x = 0$, $x = 1$, $y = x$, $x^2 + y^2 = 2$ and in the fig. 2.23 (a) OAC is the required area.

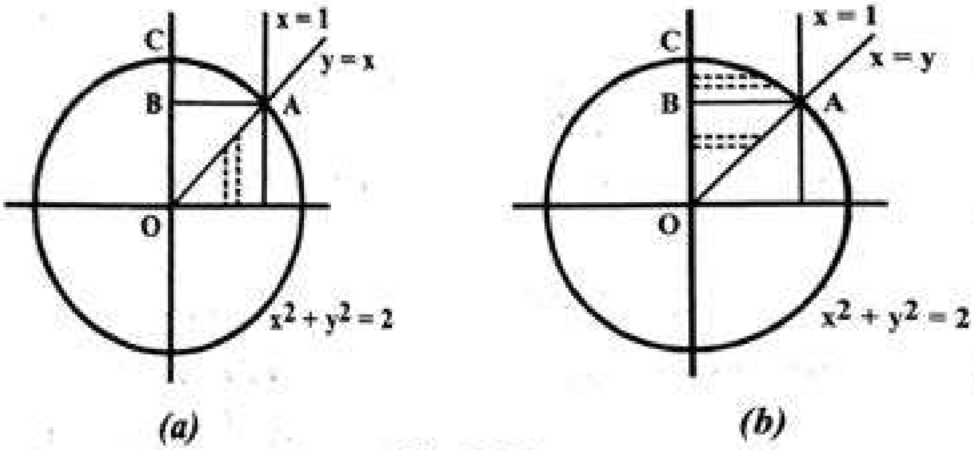


Fig. 2.23

We divide this area into two parts as shown in fig. 2.23 (b). The first part is OAB and second ABC. Then by changing the order of integration the given integral becomes

$$\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x\,dydx}{\sqrt{x^2+y^2}} = \int_0^1\int_0^y \frac{x}{\sqrt{x^2+y^2}}\,dydx + \int_1^{\sqrt{2}}\int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\,dydx$$

$$= \int_0^1 \left[\sqrt{x^2+y^2}\right]_0^y dy + \int_1^{\sqrt{2}}\left[\sqrt{x^2+y^2}\right]_0^{\sqrt{2-y^2}} dy$$

$$= (\sqrt{2}-1)\int_0^1 y\,dy + \int_1^{\sqrt{2}}\left[\sqrt{2}-y\right]dy$$

$$= (\sqrt{2}-1)\left[\frac{y^2}{2}\right]_0^1 + \left[\sqrt{2}y-\frac{y^2}{2}\right]_1^{\sqrt{2}}$$

$$= \frac{1}{2}\sqrt{2}-\frac{1}{2}+2-1-\sqrt{2}+\frac{1}{2} = 1-\frac{1}{2}\sqrt{2} = \mathbf{1-\frac{1}{\sqrt{2}}} \qquad \textbf{Ans.}$$

Prob.77. Change the order of integration of $\int_0^a\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\,dy$ ***and hence evaluate it.*** ***(R.G.P.V., June 2015)***

Sol. The curve given by the limit of first integral is $x = a-\sqrt{a^2-y^2}$ and $x = a+\sqrt{a^2-y^2}$.

$\Rightarrow$ $(x-a)^2+y^2+a^2=0$ i.e., $x^2+y^2-2ax=0$ which is a circle with the centre (a, 0) and radius = a.

Here the strip AB parallel to X-axis satisfy the limits of first integral.

If such strips are moved in Y-axis satisfy the limits of first integral.

If such strips are moved in Y-direction from y = 0 to y = a the region of integration is shaded semicircular area.

For changing the order of integration, we take strip EF parallel to Y-axis which extends from

$$F(y=0) \text{ to } E\left[y=\sqrt{(2ax-x^2)}\right]$$

Here x varies from x = 0 to x = 2a

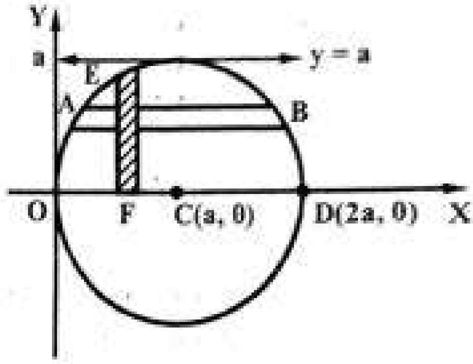


Fig. 2.24

$$\text{Hence } \quad I = \int_0^a \left(\int_{x=a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx \right) dy$$

$$= \int_{x=0}^{2a} \left(\int_{y=0}^{\sqrt{2ax-x^2}} dy \right) dx$$

$$= \int_0^{2a} [y]_0^{\sqrt{2ax-x^2}} dx = \int_0^{2a} \sqrt{2ax-x^2}\, dx$$

Putting $x = 2a\sin^2\theta$ so $dx = 4a\sin\theta\cos\theta\, d\theta$

and $\quad \theta \to 0 \text{ to } \frac{\pi}{2}$

$$\text{Hence } \quad I = \int_0^{\frac{\pi}{2}} \sqrt{4a^2\sin^2\theta - 4a^2\sin^4\theta}.4a\sin\theta\cos\theta\, d\theta$$

$$= 8a^2 \int_0^{\frac{\pi}{2}} \sin\theta\sqrt{1-\sin^2\theta}.\sin\theta\cos\theta\, d\theta = 8a^2 \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\, d\theta$$

$$= 8a^2 \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma 3} \left[\because \int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}\right]$$

$$= 8a^2 \frac{\frac{1}{2}\sqrt{\pi}.\frac{1}{2}\sqrt{\pi}}{2.2} = \frac{\pi}{2}a^2$$

Ans.

Prob.78. Change the order of integration in –

$$I = \int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} V\, dx\, dy \qquad \textit{(R.G.P.V., June 2011)}$$

Sol. Here the region of integration is bounded by $\sqrt{2ax-x^2} = y$ i.e, $x^2 + y^2 = 2ax$ i.e., $(x-a)^2 + y^2 = a^2$, the parabola $y^2 = 2ax$, y-axis i.e., $x = 0$, and the line $x = 2a$.

Clearly the area of integration is given by OABCEO. At the highest point A of the circle, the strips parallel to X-axis change their character. Through A, draw a straight line EAD parallel to X-axis. The region of integration is divided into three parts OEA, ADB and ECD.

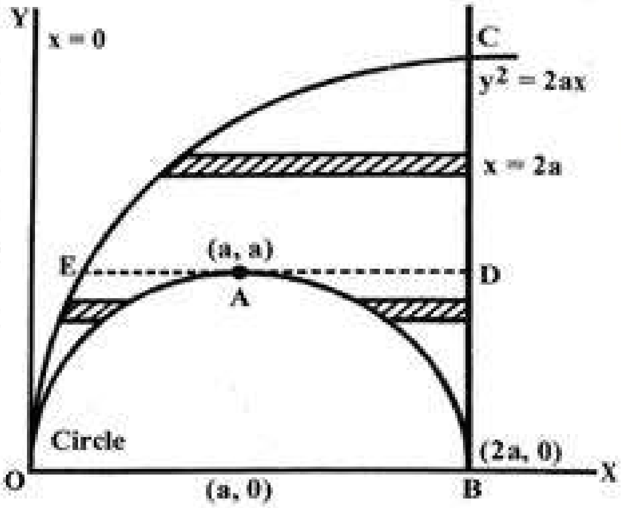


Fig. 2.25

Now from the equation of the circle

$$x^2 + y^2 = 2ax, \text{ we have}$$

$$x = a \pm \sqrt{a^2 - y^2}$$

For the region OEA, the limits for x are from $y^2/2a$ to $a - \sqrt{a^2 - y^2}$ and the limits for y are from $y = 0$ to $y = a$.

Clearly for the region ABD, the limits for x are from $a + \sqrt{a^2 - y^2}$ to 2a and that for y are from $y = 0$, to $y = a$.

Also for the region ECD, the limits for x are from $y^2/2a$ (from parabola) to 2a and that for y are from $y = a$ to $y = 2a$. Hence we have

$$\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} V\,dx\,dy = \int_0^{a}\int_{y^2/2a}^{a-\sqrt{a^2-y^2}} V\,dy\,dx + \int_0^{a}\int_{a+\sqrt{a^2-y^2}}^{2a} V\,dy\,dx + \int_a^{2a}\int_{y^2/2a}^{2a} V\,dy\,dx$$

Ans.

Prob.79. Change the order of integration –

$$\int_0^4\int_{x^2/4}^{2\sqrt{x}} dx\,dy$$

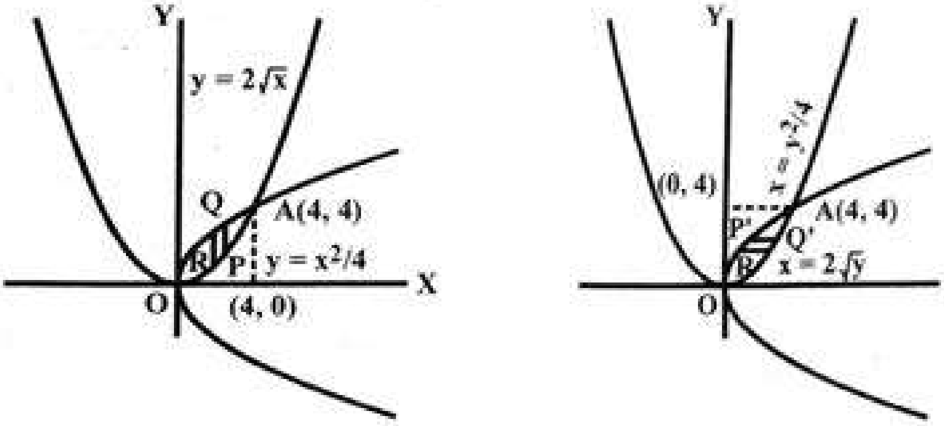
Hence evaluate it. ***(R.G.P.V., June 2012)***

Sol. See fig. 2.26.

$$R: 0 \le x \le 4, \frac{x^2}{4} \le y \le 2\sqrt{x}$$

$$R: 0 \le y \le 4, \frac{y^2}{4} \le x \le 2\sqrt{y}$$

From the limits of integration, it is clear that we have to integrate first w.r.t. y which varies from $y = \frac{x^2}{4}$ to $y = 2\sqrt{x}$ and then w.r.t. x which varies from $x = 0$ to $x = 4$. Thus integration is first performed along the vertical strip PQ which extends from a point P on the parabola $y = \frac{x^2}{4}$ (i.e., $x^2 = 4y$) to the point Q on the parabola $y = 2\sqrt{x}$ i.e. $y^2 = 4x$. Then this strip slides from O to A (4, 4) the point of intersection of the two parabolas.



(a) *(b)*

Fig. 2.26

For changing the order of integration, we divide the region of integration into horizontal strips P'Q' which extend from P' on the parabola $y^2 = 4x$ i.e., $x = \frac{y^2}{4}$ to Q' on the parabola $x^2 = 4y$ i.e., $x = 2\sqrt{y}$. Then this strip slides from O to A (4, 4) i.e., varies from 0 to 4.

$$\int_{x=0}^{4} \int_{y=\frac{x^2}{4}}^{2\sqrt{x}} dy\,dx = \int_{y=0}^{4} \int_{x=\frac{y^2}{4}}^{2\sqrt{y}} dx\,dy$$

$$= \int_{y=0}^{4} [x]_{\frac{y^2}{4}}^{2\sqrt{y}} dy = \int_{y=0}^{4} \left[2\sqrt{y} - \frac{y^2}{4}\right] dy$$

$$= \left[\frac{2y^{3/2}}{3/2} - \frac{y^3}{12}\right]_0^4 = \frac{4}{3} \times (4)^{3/2} - \frac{(4)^3}{12}$$

$$= \frac{4}{3} \times 8 - \frac{64}{12} = \frac{32}{3} - \frac{64}{12} = \frac{64}{12} = \mathbf{\frac{16}{3}}$$

Prob.80. Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$ and h... ce evaluate the same. ***(R.G.P.V., Jan./Feb. 2008, De... 2013, 2... 5)***

Sol. Let, $$I = \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx \qquad \text{...(i)}$$

In the given integral the limits of integration are given b... e., y, (which is a parabola passing through the origin) ... x, x = 0 and x = 1. Drawing those curves in one figure, we observe that the region of integration is the area OABMO.

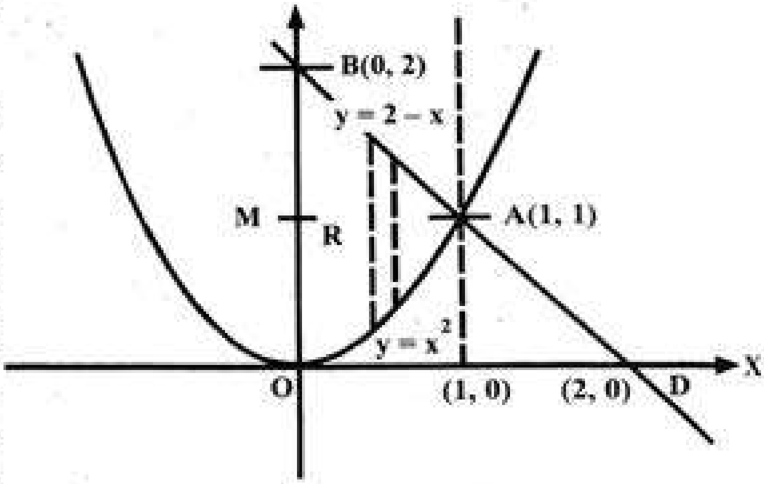


Fig. 2.27 $R: 0 \le x \le 1, x^2 \le y \le 2-x$

To change the order of integration first we divide the region of integration into two portions OAM and MAB, by drawing the line AM parallel to the X-axis. Now to reverse the order of integration, cover the whole region OABMO by strips parallel to the X-axis starting from the line x = 0. Some of these strips end on the arc OA while others end on the line AB.

For the point A, we have x = 1, substituting x = 1 in the equation of the line y = 2 − x, we get y =1.

For the region OAM, x varies from 0 to $\sqrt{y}$ and y varies from 0 to 1. Again for the region MAB. x varies from 0 to 2 − y and y varies from 1 to 2. Hence, the transformed integral is given by

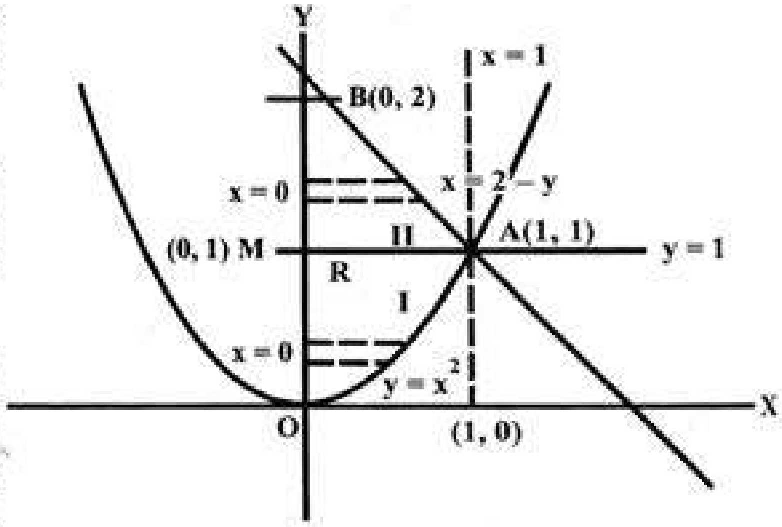


Fig. 2.28 $R: \begin{Bmatrix} I & 0 \le y \le 1, 0 \le x \le \sqrt{y} \\ II & 1 \le y \le 2, 0 \le x \le 2-y \end{Bmatrix}$

$$I = \int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dy\,dx + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dy\,dx$$

$$= \int_0^1 \left[\int_0^{\sqrt{y}} xy\,dx\right] dy + \int_{y=1}^{2}\left[\int_0^{2-y} xy\,dx\right] dy$$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]_{x=0}^{\sqrt{y}} dy + \int_{y=1}^{2} y\left[\frac{x^2}{2}\right]_{x=0}^{2-y} dy$$

$$= \int_0^1 y\left[\frac{y}{2} - 0\right] dy + \int_{y=1}^{2} y\left[\frac{(2-y)^2}{2} - 0\right] dy$$

$$= \int_0^1 \frac{y^2}{2} dy + \int_{y=1}^2 \frac{y(y^2-4y+4)}{2} dy$$

$$= \frac{1}{2}\int_0^1 y^2\, dy + \frac{1}{2}\int_{y=1}^2 \left(y^3 - 4y^2 + 4y\right) dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\left[\frac{y^4}{4} - \frac{4y^3}{3} + 2y^2\right]_{y=1}^2$$

$$= \frac{1}{2}\left[\frac{1}{3} - 0\right] + \frac{1}{2}\left[\left(4 - \frac{32}{3} + 8\right) - \left(\frac{1}{4} - \frac{4}{3} + 2\right)\right] = \frac{1}{6} + \frac{5}{24} = \frac{3}{8} \quad \textbf{Ans.}$$

Prob.81. Change the order of integration in –

$$I = \int_0^1 \int_0^{2-x} xy\, dx\, dy$$

and hence evaluate. *(R.G.P.V., March/April 2010)*

Sol. Let, $I = \int_0^1 \int_0^{2-x} xy\, dx\, dy$

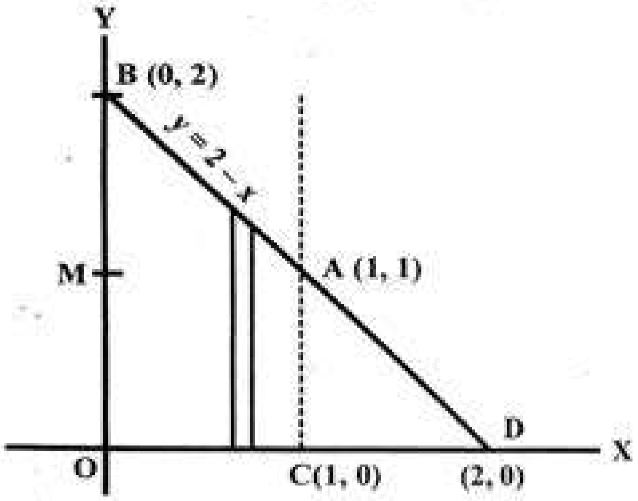


Fig. 2.29

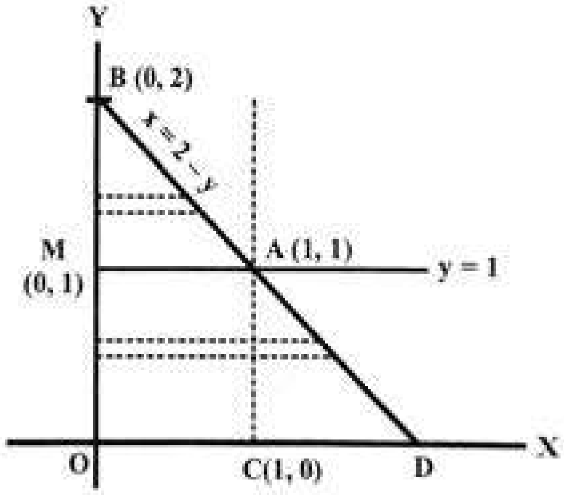


Fig. 2.30 $R : \begin{cases} I & 0 \le y \le 1,\ 0 \le x \le 1 \\ II & 1 \le y \le 2,\ 0 \le x \le 2 - y \end{cases}$

In the given integral the limits of integration are given by $y = 0$, the lines $y = 2 - x$, $x = 0$ and $x = 1$. Drawing these curves in one figure, we observe that the region of integration is OCABMO.

To change the order of integration first we divide the region of integration into two portions OCAMO and MABM, by drawing the line AM parallel to X-axis. Now to reverse the order of integration, cover the whole region OCABMO by strips parallel to the X-axis starting from the line $x = 0$. These strips end on the line AB.

For the point A, we have $x = 1$, substituting $x = 1$ in the equation of line $y = 2 - x$, we get $y = 1$.

For the region OCAMO, x varies from 0 to 1 and y varies from 0 to 1. Again for the region MABM, x varies from 0 to 2 – y and y varies from 1 to 2. Hence the transformed integral is given by

$$I = \int_0^1\int_0^1 xy\,dy\,dx + \int_1^2\int_0^{2-y} xy\,dy\,dx$$

$$= \int_0^1\left[\int_0^1 xy\,dx\right]dy + \int_1^2\left[\int_0^{2-y} xy\,dx\right]dy$$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]_0^1 dy + \int_1^2 y\left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \int_0^1 y\left[\frac{1}{2}-0\right]dy + \int_1^2 y\left[\frac{(2-y)^2}{2}-0\right]dy$$

$$= \int_0^1 \frac{y}{2}dy + \int_1^2 \frac{y}{2}(4+y^2-4y)\,dy$$

$$= \frac{1}{2}\int_0^1 y dy + \frac{1}{2}\int_1^2 (4y+y^3-4y^2)\,dy$$

$$= \frac{1}{2}\int_0^1 y\,dy + \frac{1}{2}\int_1^2 (y^3-4y^2+4y)\,dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2}\right]_0^1 + \frac{1}{2}\left[\frac{y^4}{4}-\frac{4y^3}{3}+\frac{4y^2}{2}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{1}{2}-0\right] + \frac{1}{2}\left[\frac{2^4}{4}-\frac{4.2^3}{3}+\frac{4.2^2}{2}-\frac{1}{4}+\frac{4}{3}-\frac{4}{2}\right]$$

$$= \frac{1}{4}+\frac{1}{2}\left[\frac{16}{4}-\frac{32}{3}+\frac{16}{2}-\frac{1}{4}+\frac{4}{3}-\frac{4}{2}\right] = \frac{1}{4}+\frac{5}{24} = \mathbf{\frac{11}{24}} \text{ Ans.}$$

Prob.82. Change the order of integration in $\int_0^2\int_0^{2-x} xy\,dx\,dy$ ***and hence evaluate it.*** ***(R.G.P.V., Dec. 2016)***

Sol. This problem can be solved in a similar way as Prob.81.

APPLICATIONS OF MULTIPLE INTEGRAL FOR CALCULATING AREA AND VOLUMES OF THE CURVES

Area by Double Integration –

(i) Cartesian Coordinates – Let us find the area enclosed between the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = a$, $x = b$.

$$\text{Area} = \int_{x=a}^{x=b}\int_{y=f_1(x)}^{y=f_2(x)} dx\,dy$$

Similarly, the region for area is bounded by the curves $y = a$, $y = b$, $x = f_1(y)$, $x = f_2(y)$ then

$$\text{Area} = \int_{y=a}^{y=b}\int_{x=f_1(y)}^{x=f_2(y)} dx\,dy$$

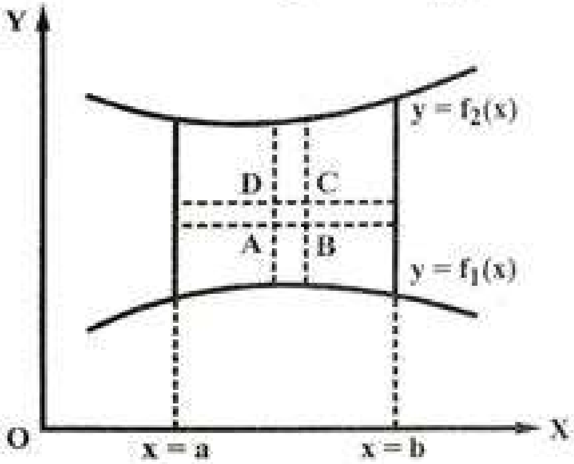


Fig. 2.31

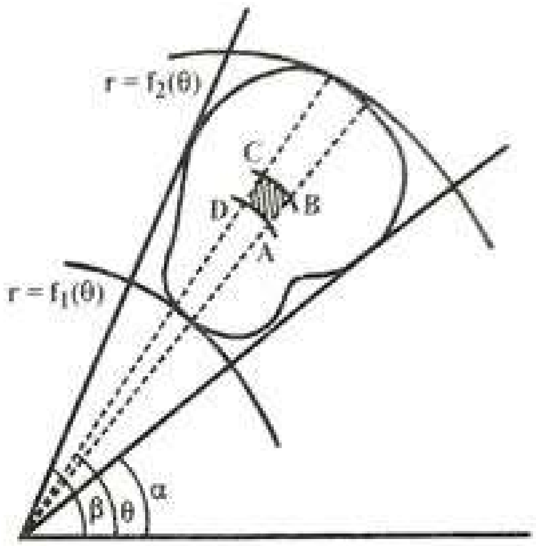


Fig. 2.32

(ii) Polar Coordinates –

Let, the region for area is bounded by the curves $r = f_1(\theta)$, $r = f_2(\theta)$ and the angles $\theta = \alpha$ to $\theta = \beta$ is given by

$$\text{Area} = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=f_1(\theta)}^{r=f_2(\theta)} r\,dr\,d\theta$$

Volume as a Double Integral –

(i) Cartesian Coordinates –

Consider a surface $z = f(x, y)$ then

$$\text{Volume} = \iint_R z\,dx\,dy$$

(ii) Cylindrical Coordinates –

Let the equation of the surface be $z = f(r, \phi)$ then

$$\text{Volume} = \iint_R z\,r\,dr\,d\theta$$

Volume as a Triple Integral –

In cartesian system,

$$\text{Volume} \qquad V = \iiint_V dx\,dy\,dz$$

Suppose the given region is bounded by the curves

$x = f_1(y, z)$, $x = f_2(y, z)$; $y = \phi_1(z)$

$y = \phi_2(z)$ and $z = a$, $z = b$ then

$$\text{Volume} = \int_a^b \int_{\phi(z)}^{\phi_2(z)} \int_{f_1(y,z)}^{f_2(y,z)} dx\, dy\, dz$$

In cylindrical coordinates,

$$\text{Volume, } V = \iiint\limits_V r\, dr\, d\phi\, dz$$

In spherical polar coordinates,

$$\text{Volume, } V = \iiint\limits_V r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Volume of Solid of Revolution about X-axis or Y-axis –

Let us consider the area A bounded by the curves $y = 0$, $y = f(x)$, $x = a$ and $x = b$ and the X-axis as shown in fig. 2.33. When we revolve the area ABCD, about the X-axis, then a solid is generated whose volume is to be determined by the formula

$$V = 2\pi \int_a^b \int_0^{f(x)} y\, dy\, dx$$

Similarly, when we revolve this area ABCD, about the Y-axis, then an another solid is generated whose volume will be

$$V = 2\pi \iint x\, dx\, dy$$

with appropriate limits.

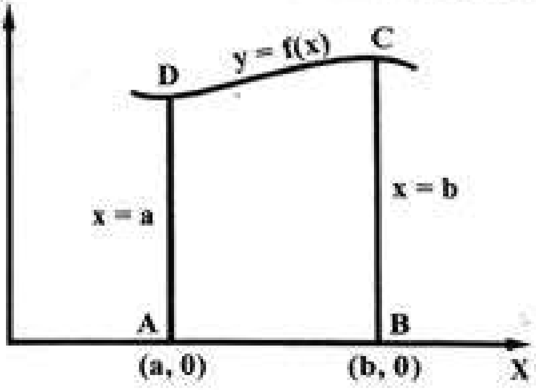


Fig. 2.33

In polar coordinates the corresponding formula for volume of solid of revolution of area A about X-axis, will be

$$V = 2\pi \iint r^2 \sin\theta\, d\theta\, dr$$

and the volume of solid of revolution of area A about the Y-axis, will be

$$V = 2\pi \iint r^2 \cos\theta\, d\theta\, dr$$

NUMERICAL PROBLEMS

Prob.83. Find the area lying between the parabola –

$$y = 4x - x^2$$

and the line $y = x$. ***(R.G.P.V., Dec. 2010)***

Sol. The parabola and line are bounded by the following curves

$$y = 4x - x^2 \quad \text{...(i)}$$

$$y = x \quad \text{...(ii)}$$

The points of intersection of (i) and (ii) are

$$x = 4x - x^2$$

$$\Rightarrow \quad 3x - x^2 = 0$$

$$x(3 - x) = 0$$

$$x = 0 \text{ or } 3$$

$$y = 0 \text{ or } 3$$

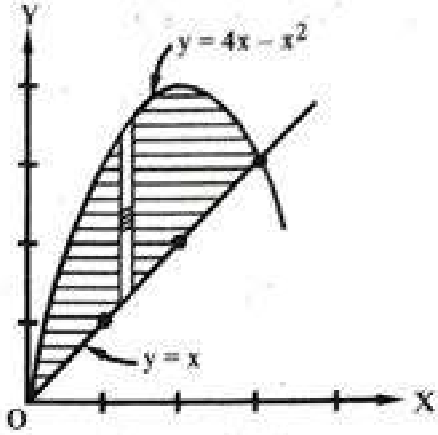


Fig. 2.34

Thus, the required area is the shaded area and is expressed by

$$0 \le x \le 3,\ x \le y \le 4x - x^2$$

$$\text{Hence, required area} = \int_0^3 \int_x^{4x-x^2} dx\, dy$$

$$= \int_0^3 [y]_x^{4x-x^2} dx$$

$$= \int_0^3 [4x - x^2 - x]\, dx = \int_0^3 (3x - x^2)\, dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \left[\frac{3\times 9}{2} - \frac{27}{3} - 0\right]$$

$$= \left[\frac{27}{2} - 9\right] = \mathbf{\frac{9}{2}} \qquad \textbf{Ans.}$$

Prob.84. Express the area between the curves $x^2 + y^2 = a^2$, and $x + y = a$ as double integral and evaluate it. ***(R.G.P.V., June 2003, 2015)***

Sol. The two curves intersect at points whose abscissae are given by

$$x^2 + (a - x)^2 = a^2$$

$$x^2 + a^2 + x^2 - 2ax = a^2$$

or $\quad 2x^2 - 2ax = 0 \Rightarrow x = 0, x = a$

The area can be considered as lying between the curves $x^2 + y^2 = a^2$ and $x + y = a$, $x = 0$ and $x = a$, so integrating along a vertical strip first we see that the required area

$$= \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} dy\, dx$$

$$= \int_0^a \left[\int_{a-x}^{\sqrt{a^2-x^2}} dy\right] dx$$

$$= \int_0^a [y]_{a-x}^{\sqrt{a^2-x^2}}\, dx$$

$$= \int_0^a \left[\sqrt{a^2 - x^2} - (a - x)\right] dx$$

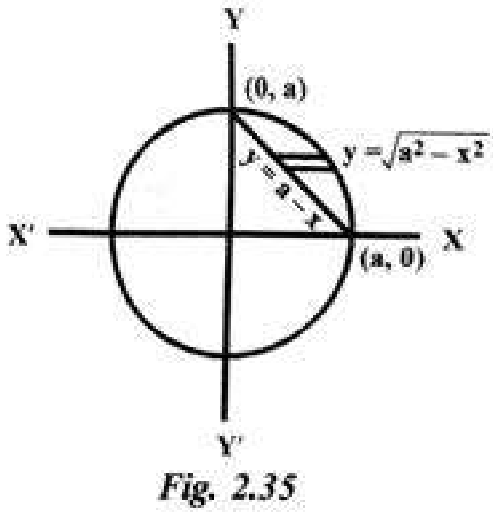


Fig. 2.35

$$= \int_0^a \sqrt{a^2 - x^2}\,dx - \int_0^a (a - x)\,dx$$

$$= \left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a - \left[ax - \frac{x^2}{2}\right]_0^a$$

$$= \left[0 + \frac{a^2}{2}\cdot\frac{\pi}{2}\right] - \left[a^2 - \frac{a^2}{2}\right] = \frac{a^2\pi}{4} - \frac{a^2}{2} = \mathbf{\frac{a^2(\pi - 2)}{4}} \qquad \textbf{Ans.}$$

Prob.85. Calculate the volume under the plane $z = 4 - x - y$ over the region $R : 0 \le x \le z,\ 0 \le y \le 1$ in the xy-plane. ***(R.G.P.V., June 2017)***

Sol. Equation of the plane $z = 4 - x - y$

$\therefore$ The required volume V is given by

$$V = \iint z\,dx\,dy = \int_0^2\int_0^1 (4 - x - y)\,dy\,dx$$

$$= \int_0^2\left[\int_0^1 (4 - x - y)\,dy\right]dx$$

$$= \int_0^2\left[4y - xy - \frac{y^2}{2}\right]_0^1 dx = \int_0^2\left(4 - x - \frac{1}{2}\right)dx$$

$$= \int_0^2\left(\frac{7}{2} - x\right)dx = \left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^2 = [7 - 2] = \mathbf{5} \qquad \textbf{Ans.}$$

Prob.86. Find the volume common to the cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$. ***[R.G.P.V., June 2008(N), Dec. 2011]***

Sol. Here, given cylinders

$$x^2 + y^2 = a^2,\ x^2 + z^2 = a^2$$

(i) z varies from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$

(ii) y varies from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$

(iii) x varies from $-a$ to $+a$

$$\text{Required volume} = \int_{-a}^{+a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz$$

$$= \int_{-a}^{+a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,[z]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}$$

$$= \int_{-a}^{+a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2 - x^2}\,dy = 2\int_{-a}^{+a} dx\sqrt{a^2 - x^2}\,[y]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}$$

$$= 2\int_{-a}^{+a} dx\sqrt{a^2-x^2}\left[\sqrt{a^2-x^2}+\sqrt{a^2-x^2}\right] = 4\int_{-a}^{+a}(a^2-x^2)\,dx$$

$$= 4\left[a^2x-\frac{x^3}{3}\right]_{-a}^{+a} = 4\left[a^3-\frac{a^3}{3}+a^3-\frac{a^3}{3}\right] = \frac{16a^3}{3} \qquad \textbf{Ans.}$$

Prob.87. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

(R.G.P.V., Nov./Dec. 2007, April 2009, Feb. 2010, June 2014)

Sol. Here, $x^2 + y^2 = 4$ or $y = \pm\sqrt{4-x^2}$

$y + z = 4$ or $z = 4 - y$ and $z = 0$

x varies from -2 to $+2$

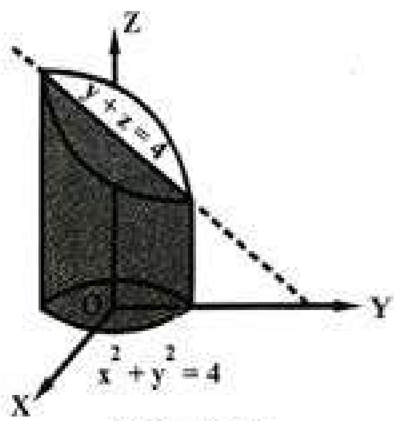


Fig. 2.36

$$V = \iiint dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-y} dz\,dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [z]_0^{4-y}\,dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\,dx\,dy = \int_{-2}^{2}\left[4y-\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2}\left[4\sqrt{4-x^2}-\frac{1}{2}(4-x^2)+4\sqrt{4-x^2}+\frac{1}{2}(4-x^2)\right]dx$$

$$= 8\int_{-2}^{2}\sqrt{4-x^2}\,dx = 8\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{-2}^{2} = 16\pi \qquad \textbf{Ans.}$$

Prob.88. Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder $x^2 + y^2 = ax$. ***(R.G.P.V., June 2013)***

Sol. Given that

$$x^2 + y^2 = ax \text{ i.e. } \left(x-\frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$$

i.e. it is the equation of cylinder with radius a/2 and centre at a distance of a/2 on X-axis. Clearly radius of cylinder is half the radius of the sphere.

$$\therefore \qquad \text{Volume} = 4\iiint_V dV = 4\int_0^a\int_0^{\sqrt{ax-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dx\,dy\,dz$$

$$= 4\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2-y^2}\,dx\,dy \qquad \text{...(i)}$$

Now changing to polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$

$\therefore \quad dx\,dy = r\,dr\,d\theta$

Also from

$$x^2 + y^2 = ax, \text{ we get}$$

$$r^2 = a\,r\cos\theta$$

i.e. $\quad r = 0, r = a\cos\theta$

i.e. the limits of r are from 0 to $a\cos\theta$ and clearly of θ are from 0 to $\pi/2$.

Now from equation (i)

$$\text{Volume} = 4\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta = 4\int_0^{\pi/2}\left[-\frac{(a^2-r^2)^{3/2}}{2\times\frac{3}{2}}\right]_0^{a\cos\theta} d\theta$$

$$= -\frac{4}{3}\int_0^{\pi/2}\left[(a^2-a^2\cos^2\theta)^{3/2}-(a^2)^{3/2}\right]d\theta = -\frac{4}{3}\int_0^{\pi/2} a^3\left[(\sin^2\theta)^{3/2}-1\right]d\theta$$

$$= \frac{4}{3}a^3\int_0^{\pi/2}(1-\sin^3\theta)d\theta = \frac{4}{3}a^3[\theta]_0^{\pi/2} - \frac{4}{3}a^3\int_0^{\pi/2}\sin^3\theta\,d\theta$$

$$= \frac{4}{3}a^3\frac{\pi}{2} - \frac{4}{3}a^3.\frac{\Gamma 2\Gamma\frac{1}{2}}{2\Gamma\frac{5}{2}} = \frac{2}{3}\pi a^3 - \frac{4}{3}a^3.\frac{1.\sqrt{\pi}}{2\frac{3}{2}\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{2}{3}\pi a^3 - \frac{8}{9}a^3 = \frac{2}{3}a^3\left(\pi - \frac{4}{3}\right)$$

Ans.

Prob.89. Find by triple integration, the volume of the sphere $x^2 + y^2 + z^2 = a^2$. ***(R.G.P.V., June 2009, 2010, Dec. 2012, May 2018)***

Or

Using triple integral, find volume of sphere $x^2 + y^2 + z^2 = a^2$. ***(R.G.P.V., Dec. 2016)***

Sol. Changing to polar spherical coordinates by putting

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

We have, $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$.

Also the volume of the sphere is 8 times the volume of its portion in the positive octant for which r varies from 0 to a, θ varies from 0 to $\pi/2$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ Volume of the sphere

$$= 8\int_0^a \int_0^{\pi/2}\int_0^{\pi/2} r^2 \sin\theta \, dr \, d\theta \, d\phi = 8\int_0^a r^2 dr\int_0^{\pi/2} \sin\theta \, d\theta\int_0^{\pi/2} d\phi$$

$$= 8\int_0^a r^2 dr\int_0^{\pi/2} \sin\theta \, d\theta[\phi]_0^{\pi/2} = 8\int_0^a r^2 dr\int_0^{\pi/2} \sin\theta \, d\theta\left[\frac{\pi}{2}-0\right]$$

$$= \frac{8\pi}{2}\int_0^a r^2 dr[-\cos\theta]_0^{\pi/2} = \frac{8\pi}{2}\int_0^a r^2 dr\left[-\cos\frac{\pi}{2}+\cos 0\right]$$

$$= \frac{8\pi}{2}\int_0^a r^2 dr = \frac{8\pi}{2}\left[\frac{r^3}{3}\right]_0^a = \frac{4\pi}{3}\left[a^3-0\right] = \frac{4\pi a^3}{3}$$ **Ans.**

Prob.90. By triple integration determine the volume of a hemisphere of radius 'a'. ***(R.G.P.V., Dec. 2013, 2017)***

Sol. Referred to centre as origin the equation of a sphere of radius 'a' is

$$x^2 + y^2 + z^2 = a^2 \qquad ...(i)$$

The hemisphere (i) is symmetrical in the four octants.

Therefore, volume of hemisphere

$= 4$ (the volume of the hemisphere lying in the positive octant)

Now for the region consisting of the volume of the hemisphere (i) lying in positive octant, we have

$$0 \le x \le a, \; 0 \le y \le \sqrt{a^2-x^2}, \; 0 \le z \le \sqrt{a^2-x^2-y^2}$$

Here, the required volume

$$= 4\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} dx \, dy \, dz$$

$$= 4\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} [z]_{z=0}^{\sqrt{a^2-x^2-y^2}} dx \, dy = 4\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2} \, dy \, dx$$

$$= 4\int_{x=0}^{a}\left[\frac{y}{2}\sqrt{(a^2-x^2)-y^2}+\frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= 4\int_0^a\left[0+\frac{a^2-x^2}{2}.\frac{\pi}{2}-0-0\right]dx = 4.\frac{\pi}{4}\int_0^a (a^2-x^2)\, dx$$

$$= \pi\left[a^2x-\frac{x^3}{3}\right]_0^a = \pi\left[a^3-\frac{a^3}{3}\right] = \frac{2}{3}\pi a^3$$ **Ans.**

Prob.91. Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$. ***(R.G.P.V., June 2007, 2011)***

Sol. The required volume is found by integrating $z = \frac{(x^2 + y^2)}{a}$ over the circle $x^2 + y^2 = 2ay$.

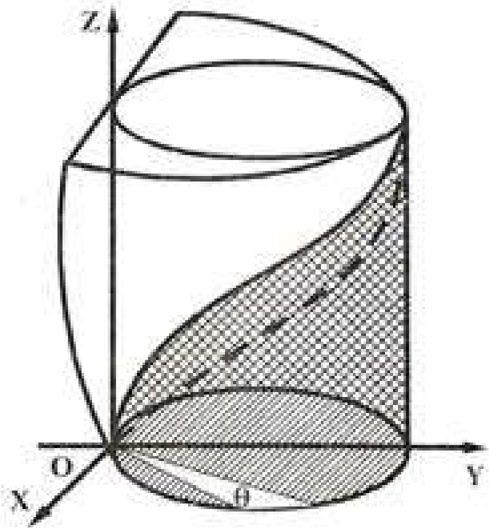


Fig. 2.37

Changing to polar coordinates in the xy-plane, we have $x = r\cos\theta$, $y = r\sin\theta$, so that $z = r^2/a$ and the polar equation of the circle is $r = 2a\sin\theta$.

To cover this circle, r varies from 0 to 2a sin θ and θ varies from 0 to π as shown in fig. 2.37.

Hence, the required volume

$$= \int_0^{\pi}\int_0^{2a\sin\theta} z.\, r\, d\theta\, dr = \frac{1}{a}\int_0^{\pi} d\theta \int_0^{2a\sin\theta} r^3\, dr$$

$$= \frac{1}{a}\int_0^{\pi} d\theta \left[\frac{r^4}{4}\right]_0^{2a\sin\theta} = 4a^3\int_0^{\pi} \sin^4\theta\, d\theta = \frac{3\pi a^3}{2} \quad \textbf{Ans.}$$

MODULE

3

SEQUENCES AND SERIES

CONVERGENCE OF SEQUENCE AND SERIES, TESTS FOR CONVERGENCE, POWER SERIES

Sequence – A sequence is defined as a function of positive integral variable denoted by f(n) or u_n, where n = 1, 2, 3, ...

From the above definition we find that a sequence is a set of numbers u_1, u_2, u_3, ... in a definite order of arrangement and these numbers are formed according to some definite rule.

The sequence u_1, u_2, u_3, ... is also denoted by (u_n).

A sequence having unlimited number of terms is known as an infinite sequence.

For example 1, 3, 5, ..., (2n – 1), ... are infinite sequence.

Limit of Sequence – If a sequence tends to a limit l, then we write

$$\lim_{n\to\infty} u_n = l$$

Convergence of Sequence – If the limit of a sequence is finite, the sequence is ***convergent*** sequence. If the limit of a sequence does not tend to a finite number, the sequence is known as *divergent* sequence.

Monotonic Sequence – The sequence is either increasing or decreasing, such sequences are known as monotonic sequence.

e.g. $u_{n+1} \geq u_n$ is monotonic increasing

$u_{n+1} \leq u_n$ is monotonic decreasing.

Series – The sum of a sequence is known as series. If there is infinite number of terms then the series is called infinite series.

e.g. $s_n = u_1 + u_2 + u_3 + ... + u_n + ... \infty$

It is denoted by $\sum_{n=1}^{\infty} u_n$ and also written as Σu_n.

Convergence, Divergence and Oscillation of a Series – Consider the infinite series $\Sigma u_n = u_1 + u_2 + ... + u_n + ... \infty$

and $\quad s_n = u_1 + u_2 + u_3 + ... + u_n$

Clearly, s_n is a function of n and as n increases indefinitely three possibilities arise –

(i) If s_n tends to a finite number as $n \to \infty$, the series Σu_n is said to be convergent.

(ii) If s_n tends to infinity as $n \to \infty$, the series Σu_n is said to be divergent.

(iii) If s_n does not tend to a unique limit, finite or infinite, the series Σu_n is known as oscillatory or non-convergent.

General Properties of Series – The truth of the following properties is self-evident and these may be regarded as axioms –

(i) If a series in which all the terms are +ve is convergent, the series remains convergent even when some or all of its terms are –ve; for the sum is clearly the greatest when all the terms are positive.

(ii) The convergence or divergence of an infinite series remains unaffected by multiplying each term by a finite number.

(iii) The convergence or divergence of an infinite series remains unaffected by the addition or removal of a finite number of its terms; for the sum of these terms being the finite quantity does not on addition or removal alter the nature of its sum.

Series of Positive Terms –

(i) An infinite series in which all the terms after some particular term are positive, is a positive term series.

e.g., $-7 - 5 - 2 + 3 + 8 + 11 + 21 + ...$ is a positive term series as all its terms after the third are positive.

(ii) A series of positive terms either converges or diverges to $+\infty$; for the sum of its first n terms, omitting the negative terms, tends to either a finite limit or $+\infty$.

Tests for Convergence – In mathematics, convergence tests are methods of testing for the convergence, conditional convergence, absolute convergence, interval of convergence or divergence of an infinite series $\sum_{n=1}^{\infty} u_n$.

There are following tests for convergence –

(i) Limit of the Summand – If the limit of the summand is undefined or non-zero, that is $\lim_{n\to\infty} u_n \neq 0$, then the series must diverge. In this sense, the partial sums are Cauchy only if this limit exists and is equal to 0. The test is inconclusive if the limit of the summand is 0.

(ii) Direct Comparison Test – If the series $\sum_{n=1}^{\infty} v_n$ is an absolutely convergent series and $|u_n| \le |v_n|$ for sufficiently large n, then the series $\sum_{n=1}^{\infty} u_n$ converges absolutely.

(iii) Limit Comparison Test – If $(u_n), (v_n) > 0$, and the limit $\lim_{n\to\infty} \frac{u_n}{v_n}$ exists, is constant positive and finite and is not zero, then $\sum_{n=1}^{\infty} u_n$ converges if and only if $\sum_{n=1}^{\infty} v_n$ converges.

(iv) Cauchy Condensation Test – Let (u_n) be a positive non increasing sequence. Then the sum $A = \sum_{n=1}^{\infty} u_n$ converges if and only if the sum $A^* = \sum_{n=0}^{\infty} 2^n a_{2n}$ converges. Moreover, if they converge, then $A \le A^* \le 2A$ holds.

(v) Ratio Test – This is also known as D'Alembert's criterion.

Suppose that there exists r such that

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = r$$

If $r < 1$, then the series absolutely convergent. If $r > 1$, then the series diverges. If $r = 1$, the ratio test is inconclusive, and the series may converge.

(vi) Root Test – This is also called the n^{th} root test or Cauchy's criterion.

Let $r = \lim_{n\to\infty} \sup \sqrt[n]{|u_n|}$

where lim sup denotes the limit superior. If $r < 1$, then the series converges. If $r > 1$, then the series diverges. If $r = 1$, the root test is inconclusive, and the series may converge or diverge.

The root test is more powerful then the ratio test.

(vii) Integral Test – A positive term series $f(1) + f(2) + f(3) + \ldots.. + f(n) + \ldots..$, where f(n) decreases as n increases, converges or diverges according as the integral

$$\int_1^{\infty} f(x)\,dx \qquad \ldots(i)$$

is finite or infinite.

The area under the curve $y = f(x)$, between any two ordinates lies between the set of inscribed and escribed rectangles formed by ordinates at $x = 1, 2, 3, \ldots..$ as in fig. 3.1. Then

$$f(1) + f(2) + f(3) + \ldots.. + f(n) \ge \int_1^{n+1} f(x)\,dx \ge f(2) + f(3) + \ldots.. + f(n+1)$$

or $$s_n \geq \int_1^{n+1} f(x)dx \geq s_{n+1} - f(1)$$

Taking limits as $n \to \infty$, we find from the second inequality that Lim $s_{n+1} \leq \int_1^{\infty} f(x)\,dx + f(1)$.

Hence if integral (i) is finite, so is Lim s_{n+1}. Similarly, from the first inequality, we see that if the integral (i) is infinite, so is Lim s_n. But the given series either converges or diverges to $+\infty$, i.e. Lim s_n is either finite or infinite as $n \to \infty$.

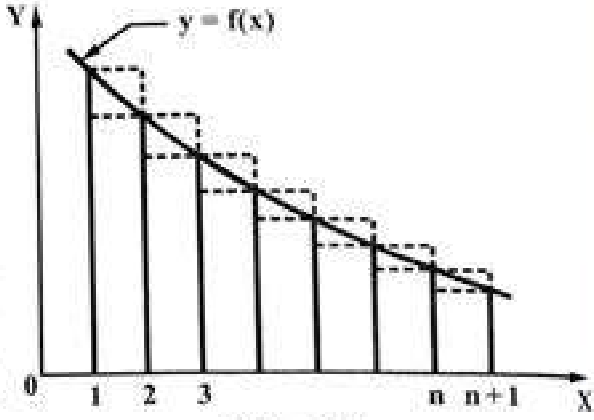


Fig. 3.1

(viii) Abel's Test –

Suppose the following statements are true –

(a) Σu_n is a convergent series

(b) (v_n) is a monotonic sequence

(c) (v_n) is bounded.

Then $\Sigma u_n v_n$ is also convergent.

(ix) Alternating Series Test –

(a) A series in which the terms are alternately positive or negative is known as an alternating series.

(b) An alternating series $u_1 - u_2 + u_3 - u_4 + \ldots\ldots$ converges if (1) each term is numerically less than its preceding term, and (2) $\lim\limits_{n\to\infty} u_n = 0$.

If $\lim\limits_{n\to\infty} u_n \neq 0$, the given series is oscillatory.

Given series $u_1 - u_2 + u_3 - u_4 + \ldots\ldots$

Let $u_1 > u_2 > u_3 > u_4 \ldots\ldots > u_{n+1} \ldots\ldots$ ...(i)

and $$\lim_{n\to\infty} u_n = 0 \quad \text{...(ii)}$$

Consider the sum of 2n terms. It can be written as

$$s_{2n} = (u_1 - u_2) + (u_3 - u_4) + \ldots\ldots + (u_{2n-1} - u_{2n}) \quad \text{...(iii)}$$

or as $$s_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \ldots\ldots - u_{2n} \quad \text{...(iv)}$$

By virtue of (i), the expressions within the brackets in (iii) and (iv) are all positive.

$\therefore$ It follows from (iii) that s_{2n} is positive and increases with n.

Also from (iv), we note that s_{2n} always remains less than u_1.

Hence s_{2n} must tend to a finite limit.

Moreover $\lim\limits_{n\to\infty} s_{2n+1} = \lim\limits_{n\to\infty}(s_{2n} + u_{2n+1}) = \lim\limits_{n\to\infty} s_{2n} + 0$ [By (ii)]

Thus $\lim_{n\to\infty} s_n$ tends to the same finite limit whether n is even or odd.

Hence given series is convergent.

When $\lim_{n\to\infty} u_n \neq 0, \lim_{n\to\infty} s_{2n} \neq \lim_{n\to\infty} s_{2n+1}$.

$\therefore$ The given series is oscillatory.

Alternating series test is also known as the Leibnitz criterion.

(x) Dirichlet's Test – If (u_n) is a sequence of real numbers and (v_n) is a sequence of complex numbers satisfying.

(a) $u_n \geq u_{n+1}$

(b) $\lim_{n\to\infty} u_n = 0$

(c) $\left|\sum_{n=1}^{N} v_n\right| \leq M$ for every positive integer N

where M is some constant, then the series

$$\sum_{n=1}^{\infty} u_n v_n$$

(xi) Raabe-Duhamel's Test – If Σu_n is a positive term series such that $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k$, then the series converges for $k > 1$ and diverges for $k < 1$, but the test fails for $k = 1$.

Let p be such that $k > p > 1$ and compare the given series Σu_n with the series $\sum \frac{1}{n^p}$ which is convergent as $p > 1$.

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} \quad \text{or} \quad \frac{u_n}{u_{n+1}} > \left(1 + \frac{1}{n}\right)^p$$

or $$\frac{u_n}{u_{n+1}} > 1 + \frac{p}{n} + \frac{p(p-1)}{2n^2} + \ldots\ldots \quad \text{or} \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{2n} + \ldots\ldots$$

or $$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > \lim_{n\to\infty}\left(p + \frac{p(p-1)}{2n} + \ldots\ldots\right)$$

i.e., if $k > p$ which is true. Hence Σu_n is convergent.

The other case when $k < 1$ can be proved similarly.

(xii) Logarithmic Test – If Σu_n is a positive term series such that $\lim_{n\to\infty}\left(n \log \frac{u_n}{u_{n+1}}\right) = k$, then the series converges for $k > 1$ and diverges for $k < 1$, but the test fails for $k = 1$.

Compare Σu_n with $\sum \frac{1}{n^p}$, if $k > p > 1$, then Σu_n converges

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p}$$

or $$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^p \qquad ...(i)$$

Taking logarithm of both sides of (i), we get

$$\log\frac{u_n}{u_{n+1}} > p\log\left(1+\frac{1}{n}\right)$$

We know that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

Hence

$$\log\frac{u_n}{u_{n+1}} > p\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + ...\right)$$

or $$n\log\frac{u_n}{u_{n+1}} > p\left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + ...\right)$$

or $$\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right) > \lim_{n\to\infty}\left[p\left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + ...\right)\right]$$

or $$\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right) > p$$

i.e. $k > p$ which is true as $k > p > 1$.

Hence Σu_n is convergent.

The other case when $k < 1$ can be proved similarly.

Power Series –

(i) Definition – A series of the form

$$a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$$

is a power series in x, here a's are independent of x.

(ii) Interval of Convergence –

$$u_n = a_nx^n$$

and $$u_{n+1} = a_{n+1}x^{n+1}$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{a_{n+1}x^{n+1}}{a_nx^n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}x$$

If $\frac{a_{n+1}}{a_n} = l$,

If $|lx| < 1$

or $\qquad |x| < \frac{1}{l}$, then the series is convergent.

Thus, the power series is convergent if $-\frac{1}{l} < x < \frac{1}{l}$.

Thus, the interval of the power series is $-\frac{1}{l}$ to $\frac{1}{l}$ for convergence outside this interval the series is divergent.

NUMERICAL PROBLEMS

Prob.1. Test for convergence of the following series

$$\sum u_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots$$ *(R.G.P.V., May 2019)*

Sol.
$$\sum u_n = s_n = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r}$$

Here, $r = \frac{1}{2}$ $\quad \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1}{1-r} - \frac{r^n}{1-r}\right) = \frac{1}{1-r} - 0 = \frac{1}{1-r}$

$$\therefore \qquad \lim_{n\to\infty} s_n = \frac{1}{1-\frac{1}{2}} = 2$$

Hence the series is **convergent**. **Ans.**

Prob.2. Examine for convergence the series

$1 + 2 + 3 + \ldots\ldots + n + \ldots\ldots \infty$

Sol.
$$s_n = \frac{n}{2}[2a + (n-1)d], \text{ where } a = 1, d = 1$$

$$= \frac{n}{2}[2 + (n-1)] = \frac{n(n+1)}{2}$$

$$\therefore \qquad \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left[\frac{n(n+1)}{2}\right] = \infty$$

Hence the series is **divergent**. **Ans.**

Prob.3. Examine for convergence the series

$$6 - 10 + 4 + 6 - 10 + 4 + 6 - 10 + 4 + \ldots\ldots \infty$$

Sol.

$$s_n = 6 - 10 + 4 + 6 - 10 + 4 + 6 - 10 + 4 + \ldots\ldots \infty$$
$$= 6 - 10n + 4n + 6(n-1) = 0$$

Hence the series is **oscillatory**. **Ans.**

Prob.4. Examine for convergence the series

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots\ldots \infty$$

Sol. Let $s = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots\ldots \infty$

$$T_n = \frac{1}{n(n+1)}$$

$$s_n = \sum\left(\frac{1}{n(n+1)}\right) = \sum\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(1 + \frac{1}{2} + \ldots\ldots \infty\right) - \left(\frac{1}{2} + \frac{1}{3} + \ldots\ldots \infty\right)$$

Hence finite series is **convergent.** **Ans.**

Prob.5. Test the convergence of the series

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \ldots\ldots$$

Sol. Here $u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}\left(1 + \frac{\sqrt{n+1}}{\sqrt{n}}\right)} = \frac{1}{\sqrt{n}\left(1 + \sqrt{\left(1 + \frac{1}{n}\right)}\right)}$

Let us compare Σu_n with Σv_n, where $v_n = \frac{1}{\sqrt{n}}$

$$\therefore \qquad \frac{u_n}{v_n} = \frac{1}{\sqrt{n}\left[1 + \sqrt{\left(1 + \frac{1}{n}\right)}\right]} \times \frac{\sqrt{n}}{1} = \frac{1}{1 + \sqrt{\left(1 + \frac{1}{n}\right)}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left[\frac{1}{1 + \sqrt{\left(1 + \frac{1}{n}\right)}}\right] = \frac{1}{1+1} = \frac{1}{2}$$

which is finite and non-zero.

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \sum \frac{1}{n^{1/2}}$ is of the form $\sum \frac{1}{n^p}$ with $p = \frac{1}{2} < 1$.

$\therefore$ Σv_n is divergent $= \Sigma u_n$ is divergent. **Ans.**

Prob.6. Test for convergence the series whose n^{th} term is $n^{\log x}$.

Sol. Given series is

$$1^{\log x} + 2^{\log x} + 3^{\log x} + \ldots\ldots \infty$$

$$= \frac{1}{1^{-\log x}} + \frac{1}{2^{-\log x}} + \frac{1}{3^{-\log x}} + \ldots\ldots \infty$$

$$= \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots \infty \quad [\text{Putting } (-\log x) = p]$$

(i) Which is convergent if $p > 1$,

i.e. $-\log x > 1$

or $\log_e \frac{1}{x} > \log_e e$

or $\frac{1}{x} > e$

or $x < \frac{1}{e}$ **Ans.**

(ii) Which is divergent if $p \le 1$

i.e. $-\log x \le 1$

or $\log \frac{1}{x} \le \log_e e$

or $\frac{1}{x} \le e$

or $x \ge \frac{1}{e}$ **Ans.**

Prob.7. Test for convergence the series

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots\ldots \infty$$

Sol. We have

$$u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{n\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{2-\frac{1}{n}}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Take $\quad v_n = \frac{1}{n^2}$

Then $$\frac{u_n}{v_n} = \frac{2-\frac{1}{n}}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} \times \frac{n^2}{1} = \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$= \frac{2}{(1)(1)} = 2, \text{ which is finite and non-zero.}$$

But $\Sigma v_n = \Sigma \frac{1}{n^2}$ is known to be convergent.

Hence Σu_n is also **convergent.** Ans.

Prob.8. Test the convergence of the series

$$\sum_{2}^{\infty} \frac{1}{\log n}$$

Sol. We have

$$u_n = \frac{1}{\log n}$$

$\because$ $\log n < n$, we have

$$\frac{1}{\log n} > \frac{1}{n}$$

Take $v_n = \frac{1}{n}$

As such $u_n > v_n$ and $\Sigma v_n = \Sigma \frac{1}{n}$ is divergent

Hence by comparison test Σu_n is also **divergent.** Ans.

Prob.9. Test the convergence or divergence of the series

$$s = \frac{1}{1.2} + \frac{2}{3.4} + \frac{3}{5.6} + \ldots.. \infty$$

Sol. We have

$$u_n = \frac{n}{(2n-1).2n} = \frac{1}{2(2n-1)}$$

$$s_n = \frac{1}{2}\left[1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \ldots..\right]$$

$$= \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \ldots.. = \frac{n}{n} = 1$$

Hence, the series is **convergent**. Ans.

Prob.10. Test the convergence or divergence of the series

$$s = \frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + ...$$

Sol. We have

$$u_n = \frac{n}{(2n-1)(2n+1)}$$

Take $$v_n = \frac{1}{2n-1}$$

$$\therefore \quad \frac{u_n}{v_n} = \frac{n}{2n+1}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{2n+1}$$

$$= \lim_{n\to\infty} \frac{1}{\left(2+\frac{1}{n}\right)} = \frac{1}{2}, \text{ which is finite.}$$

Hence $$v_n = \frac{1}{2n-1} \text{ is infinite.}$$

Thus the series is **divergent.** **Ans.**

Prob.11. Test the convergence or divergence of the series

$$s_n = \sum_{n=0}^{\infty} \frac{2n^3+5}{4n^5+1}$$

Sol. Here

$$u_n = \frac{2n^3+5}{4n^5+1} = \frac{n^3\left[2+\frac{5}{n^3}\right]}{n^5\left[4+\frac{1}{n^5}\right]} = \frac{1}{n^2}\left[\frac{2+\frac{5}{n^3}}{4+\frac{1}{n^5}}\right]$$

Take $$v_n = \frac{1}{n^2}$$

$$\frac{u_n}{v_n} = \frac{1}{n^2}\left[\frac{2+\frac{5}{n^3}}{4+\frac{1}{n^5}}\right] \times \frac{n^2}{1} = \frac{2+\frac{5}{n^3}}{4+\frac{1}{n^5}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left[\frac{2+\frac{5}{n^3}}{4+\frac{1}{n^5}}\right] = \frac{1}{2}$$

$\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent, hence Σu_n is also **convergent.** **Ans.**

Prob.12. Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

Sol. Here

$$u_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})}$$

$$= \sqrt{n+1}-\sqrt{n}$$

$$= \sqrt{n}\left[\sqrt{1+\frac{1}{n}}-1\right] = \sqrt{n}\left[\left(1+\frac{1}{n}\right)^{1/2}-1\right]$$

$$= \sqrt{n}\left[\left(1+\frac{1}{2n}-\frac{1}{8n^2}+.....\right)-1\right]$$

$$= \sqrt{n}\left(\frac{1}{2n}-\frac{1}{8n^2}+.....\right) = \frac{1}{\sqrt{n}}\left(\frac{1}{2}-\frac{1}{8n}+.....\right)$$

Taking $$v_n = \frac{1}{\sqrt{n}}$$

$$\frac{u_n}{v_n} = \frac{1}{2}-\frac{1}{8n}+.....$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left[\frac{1}{2}-\frac{1}{8n}+.....\right]$$

$$= \frac{1}{2}, \text{ which is finite and } \neq 0$$

$\therefore$ Σu_n and Σv_n convergene or divergence together.

But $\Sigma v_n = \Sigma \frac{1}{\sqrt{n}}$ is known to be divergent.

Hence Σu_n is also **divergent.** Ans

Prob.13. Test the convergence of the series

$$\sqrt{\frac{1}{2^3}}+\sqrt{\frac{2}{3^3}}+\sqrt{\frac{3}{4^3}}+\sqrt{\frac{4}{5^3}}+.........$$

(R.G.P.V., Nov. 201

Sol. We have

$$u_n = \sqrt{\frac{n}{(n+1)^3}} = \sqrt{\frac{n}{n^3\left(1+\frac{1}{n}\right)^3}} = \sqrt{\frac{1}{n^2\left(1+\frac{1}{n}\right)^3}} = \frac{1}{n\left(1+\frac{1}{n}\right)^{3/2}}$$

Taking $v_n = \frac{1}{n}$

$$\frac{u_n}{v_n} = \frac{1}{\left(1+\frac{1}{n}\right)^{3/2}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left[\frac{1}{\left(1+\frac{1}{n}\right)^{3/2}}\right]$$

= 1, which is finite and non-zero

$\therefore$ Σu_n and Σv_n convergence or divergence together.

Since $\Sigma v_n = \Sigma \frac{1}{n}$ is of the form $\Sigma \frac{1}{n^p}$ with p = 1

$\therefore$ Σv_n is divergent.

Hence Σu_n is also divergent. **Ans.**

Prob.14. Test for convergence the series

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots\ldots \infty$$

Sol. We have

$$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

and

$$u_{n+1} \quad \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$\therefore$

$$\frac{u_{n+1}}{u_n} = \frac{x^{2n}}{(n+2)\sqrt{n+1}} \cdot \frac{(n+1)\sqrt{n}}{x^{2n-2}}$$

$$= \frac{(n+1)\sqrt{n}}{(n+2)\sqrt{n+1}} . x^2 = \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)} \cdot \frac{1}{\sqrt{1+\frac{1}{n}}} . x^2$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left[\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)} \cdot \frac{1}{\sqrt{1+\frac{1}{n}}} . x^2\right]$$

$$= x^2$$

By D'Alembert's ratio test, Σu_n convergence if $x^2 < 1$ and divergence if $x^2 > 1$.

If $x^2 = 1$, $u_n = \dfrac{1}{(n+1)\sqrt{n}} = \dfrac{1}{n^{3/2}\left(1+\dfrac{1}{n}\right)}$

Take $\quad v_n = \dfrac{1}{n^{3/2}}$, we get

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right) = 1, \text{ which is finite}$$

$\therefore$ Both Σu_n and Σv_n convergence or divergence together.

But $\Sigma v_n = \Sigma\dfrac{1}{n^{3/2}}$ is a convergent series.

$\therefore$ Σu_n is also convergent

Hence, the given series converges if $x^2 \le 1$ and diverges if $x^2 > 1$. **Ans.**

Prob.15. Examine the convergence of $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\log n}$

Sol. We have

$$f(x) = \frac{1}{x\log x}$$

$$\int_2^{\infty}\frac{1}{x\log x}dx = \lim_{m\to\infty}[\log\log x]_2^m$$

$$= \lim_{m\to\infty}[\log\log m - \log\log 2] \to \infty$$

By Cauchy's integral test the series is **divergent.** **Ans.**

Prob.16. Test the convergence of the series

$$\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\frac{x^4}{4.5}+\ldots\ldots\infty$$

Sol. $\quad s_n = \sum\dfrac{x^n}{n(n+1)}$

$$u_n = \frac{x^n}{n(n+1)} \text{ and } u_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{x^n}{n(n+1)}\cdot\frac{(n+1)(n+2)}{x^{n+1}} = \left(1+\frac{2}{n}\right)\cdot\frac{1}{x}$$

or $$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)\cdot\frac{1}{x} = \frac{1}{x}$$

Hence the series is convergent for $x < 1$ and divergent for $x > 1$. But it fail at $x = 1$.

Let us try Raabe's test

$$\frac{u_n}{u_{n+1}} = 1+\frac{2}{n}$$

or $$n\left(\frac{u_n}{u_{n+1}}-1\right) = 2, \text{ for } 2 > k, \text{ k is the ratio.}$$

Hence the series is convergent for $x \le 1$ and divergent for $x > 1$. **Ans.**

Prob.17. Test the convergence of the series

$$x+\frac{2^2.x^2}{2!}+\frac{3^3.x^3}{3!}+\frac{4^4.x^4}{4!}+.....\infty$$

Sol. Here series is

$$x+\frac{2^2.x^2}{2!}+\frac{3^3.x^3}{3!}+\frac{4^4.x^4}{4!}+.....\infty$$

$$u_n = \frac{n^n.x^n}{n!}$$

and $$u_{n+1} = \frac{(n+1)^{(n+1)}.x^{(n+1)}}{(n+1)!}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{n^n.x^n}{n!}.\frac{(n+1)!}{(n+1)^{n+1}.x^{n+1}} = \frac{n^n}{(n+1)^n}.\frac{1}{x} = \frac{1}{\left(1+\frac{1}{n}\right)^n}.\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}.\frac{1}{x} = \frac{1}{ex}$$

If $\frac{1}{ex} > 1$ or $x < \frac{1}{e}$, the series is convergent.

If $\frac{1}{ex} < 1$ or $x > \frac{1}{e}$, the series is divergent.

If $\frac{1}{ex} = 1$ or $x = \frac{1}{e}$, ratio test fails.

Now $$\frac{u_n}{u_{n+1}} = \frac{e}{\left(1+\frac{1}{n}\right)^n}$$

$$\log\frac{u_n}{u_{n+1}} = \log_e e - \log\left(1+\frac{1}{n}\right)^n$$

$$= 1 - n\log\left(1+\frac{1}{n}\right) = 1 - n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right]$$

$$= 1 - 1 + \frac{1}{2n} - \frac{1}{3n^2} + \ldots = \frac{1}{2n} - \frac{1}{3n^2} + \ldots$$

So $$\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left[\frac{1}{2} - \frac{1}{3n} + \ldots\right] = \frac{1}{2} < 1$$

i.e. the series is divergent (by logarithmic test). **Ans.**

Prob.18. Find the values of x for which the series $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots\infty$ ***converges.***

Sol. Here

$$u_n = (-1)^{n-1}\frac{x^n}{n^2}$$

and $$u_{n+1} = (-1)^n\frac{x^{n+1}}{(n+1)^2}$$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{(-1)^n x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(-1)^{n-1}x^n} = -\frac{n^2}{(n+1)^2}x$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = -\lim_{n\to\infty}\frac{n^2}{(n+1)^2}x = -\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^2}x = -x$$

By D'Alembert's test the given series is convergent for $|x| < 1$ and divergent if $|x| > 1$.

At $x = 1$, the series becomes $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$

This is an alternately convergent series.

At $x = -1$, the series becomes $-1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \ldots$ which is divergent.

Hence the series is convergent for $-1 < x \le 1$. **Ans.**

Prob.19. If p > 1, prove that the p-series

$$\sum\frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \ldots + \frac{1}{n^p} + \ldots$$

converges. ***(R.G.P.V., Nov. 201***

Sol. The given series can be grouped as

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\left(\frac{1}{8^p}+\frac{1}{9^p}+\frac{1}{10^p}+\frac{1}{11^p}+\frac{1}{12^p}+\frac{1}{13^p}+\frac{1}{14^p}+\frac{1}{15^p}\right)+\ldots..$$

Now $$\frac{1}{1^p}=1 \qquad \ldots(i)$$

$$\frac{1}{2^p}+\frac{1}{3^p}<\frac{1}{2^p}+\frac{1}{2^p}=\frac{2}{2^p} \qquad \ldots(ii)$$

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}=\frac{4}{4^p} \qquad \ldots(iii)$$

$$\frac{1}{8^p}+\frac{1}{9^p}+\ldots..+\frac{1}{15^p}<\frac{1}{8^p}+\frac{1}{8^p}+\ldots..+\frac{1}{8^p}=\frac{8}{8^p} \qquad \ldots(iv)$$

On adding equations (i), (ii), (iii) and (iv), we get

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\left(\frac{1}{8^p}+\frac{1}{9^p}+\ldots..+\frac{1}{15^p}\right)+\ldots..$$

$$<\frac{1}{1^p}+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\ldots..$$

$$<1+\frac{1}{2^{p-1}}+\frac{1}{4^{p-1}}+\frac{1}{8^{p-1}}+\ldots..$$

$$<1+\left(\frac{1}{2}\right)^{p-1}+\left(\frac{1}{2}\right)^{2p-2}+\left(\frac{1}{2}\right)^{3p-3}+\ldots..$$

$$<\frac{1}{1-\left(\frac{1}{2}\right)^{p-1}} \qquad \left[\text{G.P., } r=\left(\frac{1}{2}\right)^{p-1}, s=\frac{1}{1-r}\right]$$

$<$ Finite number if $p>1$

Hence, the given series is convergent when $p>1$ **Proved**

TAYLOR'S SERIES, SERIES FOR EXPONENTIAL, TRIGONOMETRIC AND LOGARITHM FUNCTIONS

Taylor's Series

Statement – If a function f(z) be analytic at all points within a circle C with centre at 'a' and radius r, then at each point z inside C the series

$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + ... + \frac{(z-a)^n}{n!} f^n(a) + ...$ converges to f(z).

Proof. Let z be a point inside C. Draw a circle C' with at a enclosing z (fig. 3.2). The function f(z) is analytic inside and on C'. Hence by Cauchy's integral formula. Let w be a point on circle C'.

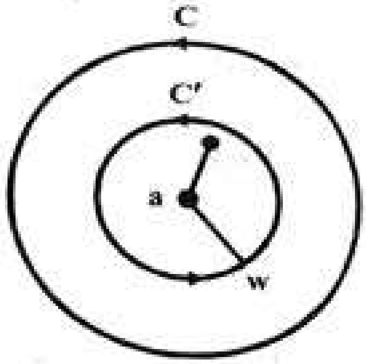


Fig. 3.2

$$f(z) = \frac{1}{2\pi i}\int_{C'} \frac{f(w)dw}{w-z} \quad ...(i)$$

Now, $\frac{1}{w-z} = \frac{1}{w-a-(z-a)}$

$$= \frac{1}{(w-a)\left\{1-\left(\frac{z-a}{w-a}\right)\right\}} = \frac{1}{(w-a)}\left[1-\left(\frac{z-a}{w-a}\right)\right]^{-1}$$

$$= \frac{1}{(w-a)}\left[1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+......\left(\frac{z-a}{w-a}\right)^{n-1}+\frac{\left(\frac{z-a}{w-a}\right)^n}{1-\frac{(z-a)}{(w-a)}}\right]$$

$$= \frac{1}{w-a}+\frac{(z-a)}{(w-a)^2}+\frac{(z-a)^2}{(w-a)^3}+......+\frac{(z-a)^{n-1}}{(w-a)^n}+\frac{(z-a)^n}{(w-a)^n(w-z)} \quad ...(ii)$$

Since for any complex number α,

$$1+\alpha+\alpha^2+........+\alpha^{n-1} = \frac{1-\alpha^n}{1-\alpha}$$

$$\therefore \quad \frac{1}{1-\alpha} = 1+\alpha+\alpha^2+....+\alpha^{n-1}+\frac{\alpha^n}{1-\alpha}$$

As $|z-a| < |w-a|$ or $\frac{|z-a|}{|w-a|} < 1$

so the series coverage uniformly. Hence the series is integrable.

Multiplying both sides by $\frac{f(w)}{2\pi i}$ of equation (ii) and integrating around C', we get

$$\frac{1}{2\pi i}\int_{C'} \frac{f(w)dw}{w-z}$$

$$= \frac{1}{2\pi i}\int_{C'} \frac{f(w)\,dw}{w-a} + (z-a)\frac{1}{2\pi i}\int_{C'} \frac{f(w)\,dw}{(w-a)^2} + ... + \frac{(z-a)^n}{2\pi i}\int_{C'} \frac{f(w)\,dw}{(w-a)^n(w-z)}$$

Using Cauchy's integral formula and formulae for derivative,

$$f(z) = f(a) + (z-a)\, f'(a) + \frac{(z-a)^2}{2!} f''(a) + ... + \frac{(z-a)^{n-1}}{(n-1)!} f^{n-1}(a) + R_n$$

where, $R_n = \dfrac{(z-a)^n}{2\pi i} \displaystyle\int_{C'} \frac{f(w)dw}{(w-a)^n (w-z)}$

It can be shown that $|R_n| \to 0$ as $n \to \infty$. Therefore, in the limit,

$$f(z) = f(a) + \sum_{r=1}^{\infty} \frac{(z-a)^n}{r!} f^{(r)}(a) \qquad ...(iii)$$

The series on the R.H.S. of equation (iii) is known as the Taylor's series of f(z) about z = a.

The series on the right hand side of equation (iii) represents f(z) at all points of z interior to C'. Since for any z inside C, corresponding C' can be found the above representation is valid for any z inside C.

Taylor Series for the Exponential Function, e^z – Consider the function $f(z) = e^z$. All the derivatives of this function are equal to e^z. In particular

$$f^{(n)}(z) = e^z \qquad ...(i)$$

By Taylor series, we have

$$f(z) = f(a) + (z-a)\, f'(a) + \frac{(z-a)^2}{2!} f''(a) + ... + \frac{(z-a)^n}{n!} f^{(n)}(a) + ...$$

Putting a = 0, we have

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2!} f''(0) + ... + \frac{z^n}{n!} f^{(n)}(0) + ... \qquad ...(ii)$$

Also from equation (i), we have

$$f(0) = e^0 = 1,\ f^{(n)}(0) = e^0 = 1,\ \forall\, n$$

i.e., $f'(0) = 1 = f''(0) = f'''(0) = ...$

Putting these values in equation (ii), we have

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ... + \frac{z^n}{n!} + ...$$

Taylor Series for Trigonometric Functions – In given below example we determine the Taylor series for the sine function. The function and its derivatives are

$$\text{so} \quad \left.\begin{aligned} f(z) &= \sin z \\ f^{(n)}(z) &= \sin\left(\frac{1}{2}n\pi + z\right) \end{aligned}\right\} \qquad ...(i)$$

By Taylor's series, we have

$$f(z)=f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+.....+\frac{(z-a)^n}{n!}f^{(n)}(a)+.....$$

Putting a = 0, we have

$$f(z)=f(0)+zf'(0)+\frac{z^2}{2!}f''(0)+.....+\frac{z^n}{n!}f^{(n)}(0)+..... \quad ...(ii)$$

From equation (i) we have

$$f(0)=\sin 0=0,\ f^{(n)}(0)=\sin\left(\frac{1}{2}n\pi\right)$$

i.e. $f^{(n)}(0) = (-1)^{(n-1)/2}$ or 0 according as n is odd or even.

Putting n = 1, 2, 3, 4, ... we get

$$f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -1,\ f^{iv}(0) = 0,\ f^{v}(0) = 1,\ .....$$

Substituting these values in equation (ii), we get

$$\sin z=0+z(1)+\frac{z^2}{2!}(0)+\frac{z^3}{3!}(-1)+\frac{z^4}{4!}(0)+\frac{z^5}{5!}(1)+..... + \frac{z^{2m+1}}{(2m+1)!}(-1)^m+.....$$

or $$\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}+.....+(-1)^m\frac{z^{2m+1}}{(2m+1)!}+.....$$

Taylor Series for Logarithm Function –

Let $f(z) = ln(1 + z) = \log_e(1 + z)$

so $f(0) = 0$ ...(i)

By Taylor's series, we know that

$$f(z)=f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+.....+\frac{(z-a)^n}{n!}f^{(n)}(a)+..... \quad ...(ii)$$

From equation (i), we get

$$f'(z)=\frac{1}{1+z}=(1+z)^{-1} \quad \Rightarrow f'(0)=1$$

$$f''(z)=-(1+z)^{-2} \quad \Rightarrow f''(0)=-1$$

$$f'''(z) \quad \Rightarrow f'''(0)=2=2!$$

$$f^{iv}(z) \quad \Rightarrow f^{iv}(0)=-6=-3!$$

$$f^{v}(z) \quad \Rightarrow f^{v}(0)=24=$$

....

....

Putting a = 0 in equation (ii), we get

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \frac{z^3}{3!}f'''(0) + ...$$

$$ln(1+z) = 0 + z(1) + \frac{z^2}{2!}(-1) + \frac{z^3}{3!}(2!) + \frac{z^4}{4!}(-3!) + \frac{z^5}{5!}(4!) + ...$$

or $$ln\,(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - .....$$

NUMERICAL PROBLEMS

Prob.20. Expand $f(z) = z^3 - 3z^2 + 4z - 2$ in a Taylor series about $z = 2$.

Sol. Here a = 2 and $f(z) = z^3 - 3z^2 + 4z - 2$

$\therefore$ $f'(z) = 3z^2 - 6z + 4$

$f''(z) = 6z - 6$

$f'''(z) = 6$ and all higher derivatives are 0.

Putting a = 2 in these, we get

$f(a) = f(2) = (2)^3 - 3(2)^2 + 4(2) - 2 = 8 - 12 + 8 - 2 = 2$

$f'(a) = f'(2) = 3(2)^2 - 6(2) + 4 = 12 - 12 + 4 = 4$

$f''(a) = f''(2) = 6(2) - 6 = 12 - 6 = 6$

$f'''(a) = f'''(2) = 6$

$f^{iv}(a)$ etc. are all 0.

By Taylor's series, we have

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + ...$$

Substituting these values in above series, we get

$$z^3 - 3z^2 + 4z - 2 = 2 + (z-2)(4) + \frac{(z-2)^2}{2!}(6) + \frac{(z-2)^3}{3!}(6)$$

or $$z^3 - 3z^2 + 4z - 2 = 2 + 4(z-2) + 3(z-2)^2 + (z-2)^3$$

Prob.21. Expand $f(z) = \dfrac{1}{(1-z)}$ in a Taylor's series about z = i.

Sol. By Taylor's series, we have

Here a = i, so reduce the above equation

$$f(z) = f(i) + (z-i)\,f'(i) + \frac{(z-i)^2}{2!}f''(i) + \ldots.. \qquad \ldots(i)$$

Now $f(z) = \dfrac{1}{(1-z)} = (1-z)^{-1} \Rightarrow f(i) = \dfrac{1}{(1-i)} = \dfrac{(1+i)}{1-i^2} = \dfrac{1+i}{2}$

$$f'(z) = (1-z)^{-2} \Rightarrow f'(i) = \frac{1}{(1-i)^2} = \frac{(1+i)^2}{(1-i^2)^2} = \frac{(1+i)^2}{4}$$

$$f''(z) = 2(1-z)^{-3} \Rightarrow f''(i) = \frac{2}{(1-i)^3} = \frac{2(1+i)^3}{(1-i^2)^3} = \frac{(1+i)^3}{4}$$

....

....

Substituting these values in equation (i), we get

$$\frac{1}{(1-z)} = \left(\frac{1+i}{2}\right) + (z-i)\frac{(1+i)^2}{4} + \frac{(z-i)^2}{2!}\frac{(1+i)^3}{4} + \ldots..$$

$$= \left(\frac{1+i}{2}\right)\left[1 + \left(\frac{1+i}{2}\right)(z-i) + \left(\frac{1+i}{2}\right)^2 (z-i)^2 + \ldots..\right] \quad \textbf{Ans.}$$

Prob.22. Expand cosh z with the help of Taylor's series.

Sol. Here a = 0 and f(z) = cosh z

$f'(z) = \sinh z$, $f''(z) = \cosh z$, $f'''(z) = \sinh z$, $f^{iv}(z) = \cosh z$, ...

$\therefore$ $f(a) = f(0) = \cosh(0) = 1$

$f'(a) = f'(0) = \sinh(0) = 0$

$f''(a) = f''(0) = \cosh(0) = 1$

$f'''(a) = f'''(0) = \sinh(0) = 0$

$f^{iv}(a) = f^{iv}(0) = \cosh(0) = 1$

....

....

By Taylor's series, we have

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \ldots..$$

Substituting these values in above equation, we get

$$\cosh z = 1 + z(0) + \frac{z^2}{2!}(1) + \frac{z^3}{3!}(0) + \frac{z^4}{4!}(1) + \ldots..$$

$$= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \ldots.. + \frac{z^{2n}}{2n!} + \ldots.. \qquad \textbf{Ans.}$$

Prob.23. Find the Taylor series for $f(z) = e^{-z}$ about $z = 0$.

Sol. By Taylor's series, we have

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + ... \quad ...(i)$$

Here a = 0 and $f(z) = e^{-z}$, which gives

$f'(z) = -e^{-z}$, $f''(z) = e^{-z}$, $f'''(z) = -e^{-z}$, ...

$f^n(z) = (-1)^n e^{-z}$

$\therefore \quad f(a) = f(0) = e^{-(0)} = 1$

$f'(a) = f'(0) = -e^{-0} = -1$

$f''(a) = f''(0) = e^{-0} = 1$

$f'''(a) = f'''(0) = -e^{-0} = -1$

....

....

$f^n(a) = f^n(0) = (-1)^n e^{-0} = (-1)^n$

Substituting these values in equation (i), we get

$$e^{-z} = 1 + z(-1) + \frac{z^2}{2!}(1) + \frac{z^3}{3!}(-1) + ... + \frac{z^n}{n!}(-1)^n + ...$$

$$= \mathbf{1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + ... + \frac{z^n}{n!}(-1)^n + ...} \qquad \textbf{Ans.}$$

FOURIER SERIES – HALF RANGE SINE AND COSINE SERIES, PARSEVAL'S THEOREM

Periodic Functions – If the value of each ordinate f(x) repeats itself at equal intervals of abscissa, then f(x) is said to be a ***periodic function.***

If f(x) = f(x + T) = f(x + 2T) = ... then T is said to be the period of the function f(x).

Example – Consider the function y = f(x) = cos x. We have cos x = cos(x + 2π) = cos(x + 4π) = ... Thus cos x is a periodic function with period 2π. This function is known as sinusoidal periodic function. Behaviour of this periodic function is shown in fig. 3.3.

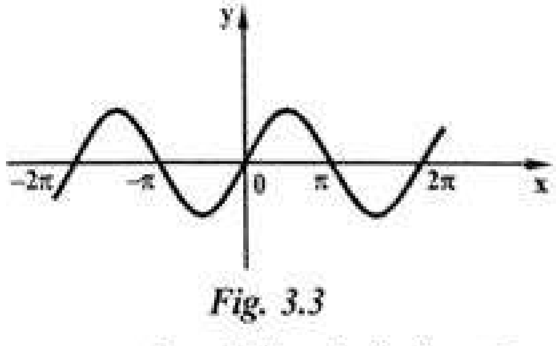


Fig. 3.3

Fourier Series – Here we will express a non-sinusoidal periodic function into a fundamental and its harmonics. A series of sines and cosines of an angle and its multiples of the form

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + ... + a_n \cos nx + ...$$
$$+ b_1 \sin x + b_2 \sin 2x + ... + b_n \sin nx$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

is said to be the Fourier series, where $a_0, a_1, a_2, \ldots, a_n, \ldots, b_1, b_2, b_3, \ldots, b_n, \ldots$ are constants.

Dirichlet's Conditions for a Fourier Series –

If the function f(x) for the interval $(-\pi, \pi)$

(i) is single-valued.

(ii) is bounded.

(iii) has at most a finite number of maxima and minima.

(iv) has only a finite number of discontinuity.

(v) is $f(x + 2\pi) = f(x)$ for value of x outside $[-\pi, \pi]$, then

$$S_P(x) = \frac{a_0}{2} + \sum_{n=1}^{P} a_n \cos nx + \sum_{n=1}^{P} b_n \sin nx$$

converges to f(x) as $P \to \infty$ at values of x for which f(x) is continuous and to

$\frac{1}{2}\left[f(x+0) + f(x-0)\right]$ at points of discontinuity.

Fourier Series for Discontinuous Function – At a point of discontinuity, Fourier series gives the value of f(x) as the arithmetic mean of left and right limits at the point of discontinuity, $x = c$. At $x = c, f(x) = \frac{1}{2}\left[f(c+0) + f(c-0)\right]$.

Determination of Fourier Coefficients – To determine the coefficients a_0, a_n, b_n, we suppose that the series (i) is uniformly convergents so that the term by term integration is permissible from $x = \alpha$ to $x = (\alpha + 2\pi)$.

To Find a_0 – Suppose f(x) is represented in the interval $(\alpha, \alpha + 2\pi)$ by the Fourier series –

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{....(i)}$$

Integrating on both sides of series (i) with respect to x between the limits α to $(\alpha + 2\pi)$, we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} dx + \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2\pi} \cos nx\,dx + \sum_{n=1}^{\infty} b_n \int_{\alpha}^{\alpha+2\pi} \sin nx\,dx$$

$$= \frac{1}{2}a_0.2\pi + a_n.0 + b_n.0 \qquad \text{[by definite integral]}$$

Hence, $\mathbf{a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx}$

To Find a_n – Multiplying each sides of series (i) by cos nx and integrate with respect to x between the limits α to $\alpha + 2\pi$, we obtain

$$\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi}\cos nx\,dx + \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2\pi}\cos^2 nx\,dx + \sum_{n=1}^{\infty} b_n \int_{\alpha}^{\alpha+2\pi}\cos nx \sin nx\,dx$$

$$= \frac{1}{2}a_0.0 + a_n.\pi + b_n.0 = \pi a_n \qquad \text{[by definite integral]}$$

Hence, $\mathbf{a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx}$

To Find b_n – Multiplying each sides of series (i) by sin nx and integrate with respect to x between the limits α to $\alpha + 2\pi$, we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi}\sin nx\,dx + \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2\pi}\cos nx \sin nx\,dx + \sum_{n=1}^{\infty} b_n \int_{\alpha}^{\alpha+2\pi}\sin^2 nx\,dx$$

$$= \frac{1}{2}a_0.0 + a_n.0 + b_n.\pi = \pi b_n \qquad \text{[by definite integral]}$$

Hence $\qquad \mathbf{b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx}$

Particular Cases –

(i) If $\alpha = 0$, the interval becomes $0 < x < 2\pi$ and the coefficients reduce to the form –

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,\quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \text{ and } b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

(ii) If $\alpha = -\pi$, the interval becomes $-\pi < x < \pi$, and the coefficients reduce to the form –

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

(iii) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(t)\,dt + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{+\pi} f(t)(\cos nt \cos nx + \sin nt \sin nx)\,dt$$

or $$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(t).\cos n(x-t)\,dt,\ \text{where } -\pi \le x \le \pi$$

Even and Odd Functions – A function f(x) is said to be ***even (or symmetric)*** if

$$f(-x) = f(x) \text{ e.g., } x^2, \cos x.$$

A function f(x) is said to be ***odd (or skew-symmetric)*** if

$$f(-x) = -f(x), \text{ e.g., } x, \sin x$$

Expansion of Even or Odd Periodic Function – When an even function f(x) is expanded in a Fourier series over the interval from $-\pi$ to π, the coefficients of the series will be given by,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \quad \text{and} \quad b_n = 0$$

When an odd function is developed in this interval, we have

$$a_0 = 0, \; a_n = 0$$

and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Half-range Series – Many times, it is desired to find a Fourier expansion of a function f(x) for the range (0, c), which is half the period of the Fourier series and hence the named half-range. As it is immaterial whatever the function may be outside the interval (0, c), we extend the function to cover the range $-c < x < c$ so that the new function may be odd or even.

Sine Series – To expand f(x) as a sine series we extend the function in the interval $(-c, c)$ as an odd function.

$$\left.\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c} \\ \text{where} \quad b_n &= \frac{2}{c}\int_0^{c} f(x)\sin\frac{n\pi x}{c}\,dx, \text{ and } a_n = 0 \end{aligned}\right\} \quad ....(i)$$

Cosine Series – To expand f(x) as a cosine series we extend the function in the interval $(-c, c)$ as an even function.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c}, \text{ where } a_0 = \frac{2}{c}\int_0^{c} f(x)\,dx$$

and

$$a_n = \frac{2}{c}\int_0^{c} f(x)\cos\frac{n\pi x}{c}\,dx \text{ and } b_n = 0$$

Change of Interval – If we have a function f(x) defined in $(\alpha, \alpha + 2c)$ and we have to change the period to 2π. For this, put

$$z = \frac{\pi x}{c} \text{ or } x = \frac{zc}{\pi} \quad ....(i)$$

$\therefore$ When $x = \alpha$, $z = \dfrac{\alpha\pi}{c} = \beta$ (say),

when $x = \alpha + 2c$, $z = (\alpha + 2c)\dfrac{\pi}{c} = \beta + 2\pi$

Hence the function f(x) of period 2c in $(\alpha, \alpha + 2c)$ is transformed to the function $f\left(\dfrac{cz}{\pi}\right) = F(z)$ [say] of period 2π in $(\beta, \beta + 2\pi)$. Hence $f\left(\dfrac{cz}{\pi}\right)$ can be expressed as the Fourier series.

$$f\left(\frac{cz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \qquad \text{....(ii)}$$

where,

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) dz \\ a_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) \cos nz\, dz \\ \text{and} \quad b_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) \sin nz\, dz \end{aligned}\right\} \qquad \text{...(iii)}$$

Making the inverse substitutions $z = \dfrac{\pi x}{c}, dz = \left(\dfrac{\pi}{c}\right) dx$ in equations (i) and (ii), the Fourier expansion of f(x) in the interval $(\alpha, \alpha + 2c)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$$

where $\quad a_0 = \dfrac{1}{c}\displaystyle\int_{\alpha}^{\alpha+2c} f(x)\, dx$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x) \cos \frac{n\pi x}{c} dx \text{ and } b_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x) \sin \frac{n\pi x}{c} dx$$

Complex Form of Fourier Series – The Fourier series of a periodic function f(x) of period $2l$, is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right) \qquad \text{...(i)}$$

We know that $\cos\theta = \dfrac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \dfrac{1}{2i}(e^{i\theta} - e^{-i\theta})$

On putting the values of cos x and sin x in series (i), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\left(\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right) + b_n\left(\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}\right)\right\}$$

$$= c_0 + \sum_{n=1}^{\infty}\left\{c_n e^{in\pi x/l} + c_{-n} e^{-in\pi x/l}\right\} \qquad \text{...(ii)}$$

where $\quad c_0 = \frac{1}{2}a_0,\ c_n = \frac{1}{2}(a_n - ib_n),\ c_{-n} = \frac{1}{2}(a_n + ib_n)$

Now $\quad c_n = \frac{1}{2l}\left\{\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx - i\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx\right\}$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}dx$$

Similarly, $c_{-n} = \frac{1}{2l}\int_{-l}^{l} f(x)e^{in\pi x/l}dx$, where $n = 0, \pm 1, \pm 2, \pm 3......$

Then the series (ii) can be compactly written as –

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} \qquad ...(iii)$$

which is the complex form of Fourier series and its coefficient are given by Fourier series given in equation (iii).

Parseval's Theorem –

Statement – Suppose f(x) is a periodic function with period 2c defined in the interval (– c, c).

Then $\int_{-c}^{c}[f(x)]^2 dx = c\left[\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$, where a_0, a_n, b_n are the Fourier coefficients of f(x).

Proof. We know that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{c} + b_n\sin\frac{n\pi x}{c}\right) \qquad ...(i)$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{c}\int_{-c}^{c} f(x)\,dx \\ a_n &= \frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{n\pi x}{c}\right)dx \\ \text{and} \quad b_n &= \frac{1}{c}\int_{-c}^{c} f(x)\sin\left(\frac{n\pi x}{c}\right)dx \end{aligned}\right\} \qquad ....(ii)$$

On multiplying both sides of equation (i) by f(x) and then integrating term by term between the limits – c to c, we get

$$\int_{-c}^{c}[f(x)]^2 dx = \frac{a_0}{2}\int_{-c}^{c} f(x)\,dx + \sum_{n=1}^{\infty}\left[a_n\int_{-c}^{c} f(x)\cos\left(\frac{n\pi x}{c}\right)dx\right]$$
$$+ \sum_{n=1}^{\infty}\left[b_n\int_{-c}^{c} f(x)\sin\left(\frac{n\pi x}{c}\right)dx\right]$$

$$\text{or} \quad \int_{-c}^{c} [f(x)]^2\, dx = \frac{a_0}{2}.(ca_0) + \sum_{n=1}^{\infty} a_n.(ca_n) + \sum_{n=1}^{\infty} b_n.(cb_n)$$

$$\text{or} \quad \int_{-c}^{c} [f(x)]^2\, dx = c\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right] \quad \text{....(iii)}$$

which is the Parseval's formula. **Proved**

NUMERICAL PROBLEMS

Prob.24. Find the Fourier series representing

$$f(x) = x,\ 0 < x < 2\pi$$

and sketch its graph from $x = -\pi$ to $x = 4\pi$.

Sol. Suppose

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{...(i)}$$

$$\text{Here} \quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{2\pi} x\, dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$$

$$\text{Also} \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{\cos 2\pi n}{n^2} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}(1-1) = 0$$

$$\text{and} \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{-2\pi\cos 2n\pi}{n}\right] = -\frac{2}{n}$$

Substituting these values of a_0, a_n and b_n in equation (i), we get

$$x = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right] \quad \textbf{Ans.}$$

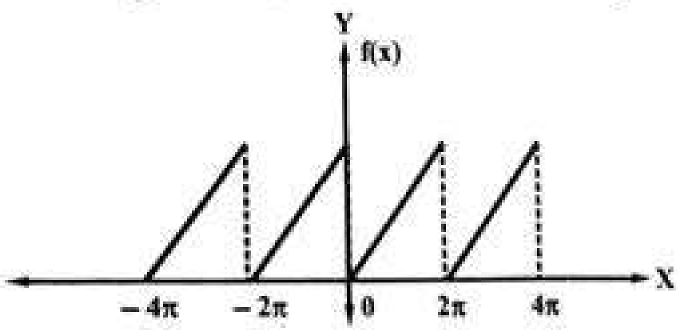


Fig. 3.4

Prob.25. Expand $f(x) = x \sin x$, $0 < x < 2\pi$ in a Fourier series.

[R.G.P.V., Dec. 2008 (III-Sem), 2012 (III-Sem), June 2015 (III-Sem)]

Sol. Let $f(x)=\dfrac{a_0}{2}+\sum\limits_{n=1}^{\infty} a_n \cos nx+\sum\limits_{n=1}^{\infty} b_n \sin nx$ ....(i)

where $$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$$

$$=\frac{1}{\pi}\left[x(-\cos x)-1(-\sin x)\right]_0^{2\pi}=\frac{1}{\pi}[-2\pi]=-2$$

and $$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x(2\cos nx.\sin x)\,dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x-\sin(n-1)x]\,dx$$

$$=\frac{1}{2\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right\}-1\left\{\frac{-\sin(n+1)x}{(n+1)^2}+\frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$=\frac{1}{2\pi}\left[2\pi\left\{\frac{-\cos(2n\pi+2\pi)}{n+1}+\frac{\cos(2n\pi-2\pi)}{n-1}\right\}\right]$$

$$=-\frac{1}{n+1}+\frac{1}{n-1}=\frac{-n+1+n+1}{(n-1)(n+1)}=\frac{2}{n^2-1},n\neq 1$$

Now for $n = 1$, we have,

$$a_1=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx=\frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx=\frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right)-1\left(\frac{-\sin 2x}{4}\right)\right]_0^{2\pi}=\frac{1}{2\pi}\left[-\frac{2\pi}{2}\right]=-\frac{1}{2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx=\frac{1}{2\pi}\int_0^{2\pi} x(2\sin nx.\sin x)\,dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x\{\cos(n-1)x-\cos(n+1)x\}\,dx$$

$$=\frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1}-\frac{\sin(n+1)x}{n+1}\right\}-1\left\{\frac{-\cos(n-1)x}{(n-1)^2}+\frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos(2n\pi - 2\pi)}{(n-1)^2} - \frac{\cos(2n\pi + 2\pi)}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] = 0, \quad n \neq 1$$

When n = 1, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \frac{1}{2\pi}(2\pi^2) = \pi$$

$\therefore$ From equation (i)

$$f(x) = \frac{a_0}{2} + a_1\cos x + b_1\sin x + \sum_{n=2}^{\infty} a_n\cos nx + \sum_{n=2}^{\infty} b_n\sin nx$$

$$= -1 - \frac{1}{2}\cos x + \pi\sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx$$

$$\therefore \quad x\sin x = -1 - \frac{1}{2}\cos x + \pi\sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx \qquad \textbf{Ans.}$$

Prob.26. Find the Fourier series expansion of f(x), when

$$f(x) = -\pi, \quad -\pi < x < 0$$
$$x, \quad 0 < x < \pi$$

hence deduce $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$. ***[R.G.P.V., Dec. 2004 (III-Sem), Jan./Feb. 2007 (III-Sem), Dec. 2008 (III-Sem), 2014 (III-Sem)]***

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$ ...(i)

Then $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx + \int_0^{\pi} x\,dx\right] = \frac{1}{\pi}\left[-\pi|x|_{-\pi}^{0} + \left|\frac{x^2}{2}\right|_0^{\pi}\right] = \frac{-\pi}{2}$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \left|\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\pi - 1)$$

Therefore, $a_1 = \frac{-2}{\pi.1^2}$, $a_2 = 0$, $a_3 = -\frac{2}{\pi.3^2}$, $a_4 = 0$, $a_5 = \frac{-2}{\pi.5^2}$ etc.

and $$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{\pi\cos nx}{n}\right|_{-\pi}^{0} + \left|\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1 - 2\cos n\pi)$$

Therefore, $b_1 = 3$, $b_2 = -\frac{1}{2}$, $b_3 = 1$, $b_4 = -\frac{1}{4}$ etc.

Thus putting the values of a's and b's in series (i), we get

$$f(x) = \frac{-\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$$

$$+ 3\sin x - \frac{\sin 2x}{2} + \sin 3x - \ldots \qquad \textbf{Ans.}$$

Putting x = 0, in above series we obtain

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\infty\right) \qquad \ldots\text{(ii)}$$

Now f(x) is discontinuous at x = 0. As a matter of fact

$f(0-0) = -\pi$ and $f(0+0) = 0$

$$\therefore \quad f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -\frac{\pi}{2}$$

Hence equation (ii) takes the form

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8} \qquad \textbf{Proved}$$

Prob.27. Find the Fourier series to represent the function f(x) given by –

$$f(x) = \begin{cases} 0 & \text{for } -\pi \le x \le 0 \\ \sin x & \text{for } 0 \le x \le \pi \end{cases}$$

Deduce that $\dfrac{1}{1.3} - \dfrac{1}{3.5} + \dfrac{1}{5.7} - \ldots\ldots = \dfrac{\pi - 2}{4}$.

[R.G.P.V., Dec. 2015 (III-Sem)]

Sol. Given $f(x) = \begin{cases} 0; & -\pi \le x \le 0 \\ \sin x; & 0 \le x \le \pi \end{cases}$ ...(i)

Here $2c = 2\pi - 0$ i.e. $c = \pi$

Suppose the Fourier series of f(x) with period 2c is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{c}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{c}\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad (\because c = \pi)$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx \qquad \text{...(ii)}$$

Then, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin x\,dx = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = -\frac{1}{\pi}\left[\cos\pi - \cos 0\right] = \frac{2}{\pi}$$

and $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos nx\,dx = \frac{1}{2\pi}\int_{0}^{\pi} 2\sin x\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx \qquad \text{...(iii)}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{(n+1)} + \frac{\cos(n-1)\pi}{(n-1)} + \frac{1}{(n+1)} - \frac{1}{(n-1)}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\cos n\pi}{(n+1)} - \frac{\cos n\pi}{(n-1)} + \frac{1}{(n+1)} - \frac{1}{(n-1)}\right]$$

$$= \frac{1}{2\pi}\left[(-1)^n\left\{\frac{1}{(n+1)} - \frac{1}{(n-1)}\right\} + \frac{1}{(n+1)} - \frac{1}{(n-1)}\right]$$

$$[\because \cos n\pi = (-1)^n]$$

$$= \frac{1}{2\pi}\left[(-1)^n\left\{\frac{-2}{(n^2-1)}\right\} + \left\{\frac{-2}{(n^2-1)}\right\}\right]$$

$$= -\frac{1}{\pi}\cdot\frac{1}{(n^2-1)}[(-1)^n + 1] \ ; n \neq 1$$

For n = 1, from equation (iii), we get

$$a_1 = \frac{1}{2\pi}\int_0^\pi \sin 2x\, dx = \frac{1}{2\pi}\left[\frac{-\cos 2x}{2}\right]_0^\pi = -\frac{1}{4\pi}[1-1] = 0$$

and
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\, dx + \int_0^\pi f(x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\int_0^\pi \sin x \sin nx\, dx \qquad \text{...(iv)}$$

$$= 0,\ n \neq 1 \qquad \left[\because \int_0^\pi \sin mx \sin nx\, dx = 0,\ m \neq n\right]$$

For n = 1, from equation (iv), we get

$$b_1 = \frac{1}{\pi}\int_0^\pi \sin^2 x\, dx = \frac{1}{2\pi}\int_0^\pi (1-\cos 2x)dx$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{1}{2\pi}[(\pi - 0) - (0 - 0)] = \frac{1}{2}$$

From equation (ii), we have

$$f(x) = \frac{1}{\pi} + 0 - \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{[(-1)^n + 1]}{(n-1)(n+1)}\cos nx + \frac{1}{2}\sin x + 0$$

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} + \ldots\right) + \frac{1}{2}\sin x \qquad \text{...(v)}$$

$\because x = \frac{\pi}{2}$, is point of continuity, then

$$f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1 \qquad \text{[By equation (i)]}$$

Putting $x = \frac{\pi}{2}$ in equation (v), we get

$$1 = \frac{1}{\pi} - \frac{2}{\pi}\left[-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + ...\right] + \frac{1}{2}.1$$

$$1 - \frac{1}{2} - \frac{1}{\pi} = \frac{2}{\pi}\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - .....\right]$$

$$\frac{2\pi - \pi - 2}{2\pi} = \frac{2}{\pi}\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - .....\right]$$

or $$\mathbf{\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - ..... = \frac{\pi - 2}{4}}$$ **Proved**

Prob.28. Expand in Fourier series $f(x) = x + x^2, -\pi < x < \pi$.
[R.G.P.V., Dec. 2016 (III-Sem)]

Sol. Let $f(x) = x + x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ...(i)

We have, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right] = \frac{1}{\pi}\left[\frac{2\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

Also $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\left|(x + x^2)\left(\frac{\sin nx}{n}\right)\right|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(1 + 2x)\left(\frac{\sin nx}{n}\right)dx\right]$$

$$= \frac{1}{\pi}\left[0 - \left\{\left|(1 + 2x)\left(\frac{-\cos nx}{n^2}\right)\right|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2\left(\frac{-\cos nx}{n^2}\right)dx\right\}\right]$$

$$= \frac{1}{\pi}\left[\left|(1 + 2x)\left(\frac{\cos nx}{n^2}\right)\right|_{-\pi}^{\pi} - 2\left|\frac{\sin nx}{n^3}\right|_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(1 + 2\pi)\frac{\cos n\pi}{n^2} - (1 - 2\pi)\frac{\cos(-n\pi)}{n^2} - 0\right]$$

$$= \frac{1}{\pi}\left[4\pi\frac{\cos n\pi}{n^2}\right] = \frac{4(-1)^n}{n^2} \qquad [\because \cos n\pi = (-1)^n]$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\, dx$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{-\cos nx}{n}\right)-(1+2x)\left(\frac{-\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{\cos n\pi}{n}+\frac{2\cos n\pi}{n^3}+(-\pi+\pi^2)\frac{\cos(-n\pi)}{n}-\frac{2\cos(-n\pi)}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{\cos n\pi}{n}+\frac{2\cos n\pi}{n^3}+(-\pi+\pi^2)\frac{\cos n\pi}{n}-\frac{2\cos n\pi}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n}(-\pi-\pi^2-\pi+\pi^2)\right]$$

$$= \frac{1}{\pi}\left[\frac{-2\pi}{n}\cos n\pi\right] = \frac{-2}{n}(-1)^n \qquad [\because \cos n\pi = (-1)^n]$$

From equation (i), we have

$$x+x^2 = \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}-2\sum_{n=1}^{\infty}\frac{(-1)^n\sin nx}{n}$$

$$= \frac{\pi^2}{3}+4\left[-\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+\ldots\right]-2\left[-\frac{\sin x}{1}+\frac{\sin 2x}{2}-\frac{\sin 3x}{3}+\ldots\right]$$

$$= \frac{\pi^2}{3}-4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\ldots\right]+2\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\ldots\right]$$

Ans.

Prob.29. Find a Fourier series to represent $f(x) = x - x^2$ from $x = -\pi$ to $x = \pi$. Also deduce that $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots\ldots\ldots = \frac{\pi^2}{12}$.

[R.G.P.V., June/July 2006 (III-Sem), Dec. 2013 (III-Sem)]

Sol. Let, $f(x) = x - x^2 = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$ ...(i)

We have, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)dx = \frac{1}{\pi}\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{-\pi}^{\pi}$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2}-\frac{\pi^3}{3}-\frac{\pi^2}{2}-\frac{\pi^3}{3}\right] = \frac{1}{\pi}\left[\frac{-2\pi^3}{3}\right] = \frac{-2\pi^2}{3}$$

Also,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{\sin nx}{n}\right)-(1-2x)\left(\frac{-\cos nx}{n^2}\right)+(-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(1-2\pi)\cos n\pi}{n^2}-\frac{(1+2\pi)\cos n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{-4\pi\cos n\pi}{n^2}\right] = \frac{-4(-1)^n}{n^2} \qquad [\because \cos n\pi = (-1)^n]$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(-\frac{\cos nx}{n}\right)-(1-2x)\left(-\frac{\sin nx}{n^2}\right)+(-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi-\pi^2)\left(-\frac{\cos n\pi}{n}\right)-2\frac{\cos n\pi}{n^3}+(-\pi-\pi^2)\frac{\cos n\pi}{n}+\frac{2\cos n\pi}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-2\pi\frac{(-1)^n}{n}\right] = \frac{-2(-1)^n}{n}$$

$\therefore$ From equation (i), we have

$$x-x^2 = \frac{-\pi^2}{3}-4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}-2\sum_{n=1}^{\infty}\frac{(-1)^n\sin nx}{n}$$

$$= \frac{-\pi^2}{3}-4\left[\frac{-\cos x}{1^2}+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+\ldots..\right]+2\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\ldots..\right]$$

$$= \frac{-\pi^2}{3}+4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\ldots..\right]+2\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\ldots..\right]$$

Ans.

Putting $x = 0$, we get

$$0 = \frac{-\pi^2}{3}+4\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\ldots..\right]$$

or $$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots\ldots\ldots = \frac{\pi^2}{12}$$ **Proved**

Prob.30. Find the coefficient a_0 in the Fourier expansion of the even function $f(x) = |\cos x|$ in the interval $(-\pi, \pi)$. ***[R.G.P.V., June 2016 (III-Sem)]***

Sol. Here $f(x) = |\cos x|$

As $f(-x) = |\cos(-x)| = |\cos x| = f(x)$, $|\cos x|$ is an even function and so Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Then,
$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} |\cos x|dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi}(-\cos x)dx\right]$$

$$\left[\because \cos x \text{ is } -\text{ve, when} \frac{\pi}{2} < x < \pi\right]$$

$$= \frac{2}{\pi}\left\{[\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\{(1-0)-(0-1)\} = \frac{2}{\pi}(1+1) = \frac{4}{\pi}$$ Ans.

Prob.31. Find the Fourier series expansion of the function $f(x) = |x|$ for $-\pi \le x \le \pi$. Hence deduce that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$ ***[R.G.P.V., June 2017 (III-Sem)]***

Sol. Since, $f(-x) = |-x| = |x| = f(x)$

$\therefore$ $f(x)$ is an even function and so $b_n = 0$.

$\therefore$ Fourier series expansion is given by

$$f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \ldots(i)$$

Then, $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}|x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\frac{\cos n\pi - 1}{n^2}\right] = \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$\therefore$ $$a_1 = \frac{-4}{\pi.1^2},\ a_2 = 0,\ a_3 = \frac{-4}{\pi.3^2},\ a_4 = 0$$

$$a_5 = \frac{-4}{\pi.5^2},\ \ a_6 = 0 \text{ and so on.}$$

From equation (i)

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right)$$ **Ans.**

Putting x = 0 in above series, we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right)$$

or $$\frac{\pi}{2} = \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right)$$

or $$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...$$ **Proved**

Prob.32. Find the Fourier series for the even function

$$f(x) = \begin{cases} 1 + \dfrac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi}, & 0 < x < \pi \end{cases}$$

Hence deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}.$$ *[R.G.P.V., Dec. 2016 (III-Sem)]*

Sol. Since $f(-x) = 1 - \dfrac{2x}{\pi}$ in $(-\pi, 0) = f(x)$ in $(0, \pi)$

and $f(-x) = 1 + \dfrac{2x}{\pi}$ in $(0, \pi) = f(x)$ in $(-\pi, 0)$

Therefore f(x) is an even function in $(-\pi, \pi)$. This is also clear from its graph P'QP (see fig. 3.5) which is symmetrical about the y-axis.

$$\therefore\ \ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad ...(i)$$

Then $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi} = 0$$

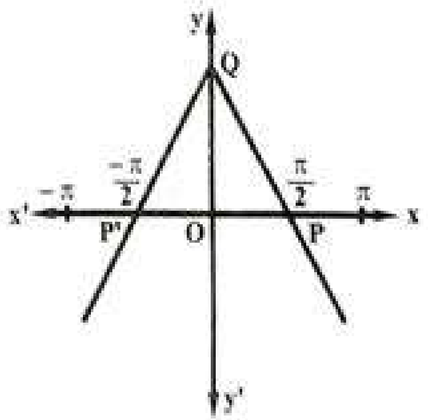


Fig. 3.5

and $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\left\{\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)\right\}_0^{\pi} - \int_0^{\pi}\left(-\frac{2}{\pi}\right)\left(\frac{\sin nx}{n}\right)dx\right]$$

$$= \frac{2}{\pi}\left[0+\frac{2}{\pi}\left\{\frac{-\cos nx}{n^2}\right\}_0^{\pi}\right] = \frac{4}{\pi^2}\left[-\frac{\cos n\pi}{n^2}+\frac{1}{n^2}\right]$$

$$= \frac{4}{n^2\pi^2}(-\cos n\pi + 1) = \frac{4}{n^2\pi^2}[1-(-1)^n] \quad [\because \cos n\pi = (-1)^n]$$

$$\therefore \quad a_1 = \frac{8}{1^2\pi^2},\ a_2 = 0,\ a_3 = \frac{8}{3^2\pi^2},\ a_4 = 0,\ a_5 = \frac{8}{5^2\pi^2},$$

$a_6 = 0$ and so on.

Substituting these values in equation (i), we get

$$f(x) = 0+\frac{8}{1^2\pi^2}\cos x+\frac{8}{3^2\pi^2}\cos 3x+\frac{8}{5^2\pi^2}\cos 5x+.....$$

or $$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+.....\right] \quad \textbf{Ans.}$$

Putting x = 0, we get

$$1-\frac{2}{\pi}.0 = \frac{8}{\pi^2}\left[\frac{\cos 0}{1^2}+\frac{\cos 0}{3^2}+\frac{\cos 0}{5^2}+.....\right]$$

$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+......\right]$$

or $$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+.......=\frac{\pi^2}{8} \quad \textbf{Proved}$$

Prob.33. Obtain the Fourier series for the function $f(x) = x$ in the interval $(-\pi, \pi)$. ***[R.G.P.V., June 2016 (III-Sem)]***

Sol. $f(x) = x$ is an odd function.

$$\therefore \quad a_0 = 0 \text{ and } a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$[\because \int uv = uv_1 - u'v_2 + u''v_3 - .....]$$

$$= \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] = -\frac{2}{n}(-1)^n$$

$\therefore$
$$x = \sum_{n=1}^{\infty}\left[-\frac{2}{n}(-1)^n \sin nx\right]$$

$$= 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + ...\right]$$ **Ans.**

Prob.34. Find the Fourier series to represent the function
$f(x) = x^2, -l < x < l.$ [R.G.P.V., June 2005 (III-Sem), Feb. 2010 (III-Sem), June 2012 (III-Sem)]

Sol. Here $f(x) = x^2$

Since $f(-x) = (-x)^2 = x^2 = f(x)$,

So given function is even in $(-l, l)$, so Fourier expansion is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \qquad ...(i)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l x^2\,dx = \frac{2}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{2l^2}{3}$

and $a_n = \frac{2}{l}\int_0^l x^2\cos\frac{n\pi x}{l}dx$

$$= \frac{2}{l}\left[x^2\left(\frac{\sin n\pi x/l}{n\pi/l}\right) - 2x\left(-\frac{\cos n\pi x/l}{n^2\pi^2/l^2}\right) + 2\left(-\frac{\sin n\pi x/l}{n^3\pi^3/l^3}\right)\right]_0^l$$

or $a_n = \frac{4l^2(-1)^n}{n^2\pi^2}$

$\therefore$ $a_1 = -\frac{4l^2}{1^2.\pi^2}$, $a_2 = \frac{4l^2}{2^2.\pi^2}$, $a_3 = -\frac{4l^3}{3^2.\pi^2}$, $a_4 = \frac{4l^2}{4^2.\pi^2}$ etc.

Putting these values in equation (i), we get

$$x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{\cos\pi x/l}{1^2} - \frac{\cos 2\pi x/l}{2^2} + \frac{\cos 3\pi x/l}{3^2} - ....\right]$$ **Ans.**

Prob.35. Obtain Fourier series for the function –

$$f(x)=\begin{cases}\pi x, & 0\le x\le 1\\ \pi(2-x), & 1\le x\le 2\end{cases}$$

[R.G.P.V., Dec. 2004 (III-Sem), June 2008 (III-Sem), 2011 (III-Sem)]

Sol. Here the length of the interval is 2 and therefore c = 1.

∴ The required series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{1}+b_n\sin\frac{n\pi x}{1}\right] \quad ...(i)$$

where $a_0=\frac{1}{1}\int_0^2 f(x)\,dx=\int_0^1 f(x)\,dx+\int_1^2 f(x)\,dx$

$$=\int_0^1 \pi x\,dx+\int_1^2 \pi(2-x)\,dx=\left[\frac{\pi x^2}{2}\right]_0^1+\left[\pi\left(2x-\frac{x^2}{2}\right)\right]_1^2=\pi$$

$$a_n=\frac{1}{1}\int_0^2 f(x)\cos n\pi x\,dx=\int_0^1 \pi x\cos n\pi x\,dx+\int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$=\left[\frac{\pi x\sin n\pi x}{n\pi}+\frac{\pi\cos n\pi x}{n^2\pi^2}\right]_0^1+\left[\frac{\pi(2-x)\sin n\pi x}{n\pi}-\frac{\pi\cos n\pi x}{n^2\pi^2}\right]_1^2$$

$$=\frac{2}{n^2\pi}\left[(-1)^n-1\right]=0,\text{ when n is even.}=\frac{-4}{n^2\pi},\text{ when n is odd.}$$

and $b_n=\frac{1}{1}\int_0^2 f(x)\sin n\pi x\,dx=\int_0^1 \pi x\sin n\pi x\,dx+\int_1^2 \pi(2-x)\sin n\pi x\,dx$

$$=\left[\frac{-\pi x\cos n\pi x}{n\pi}+\frac{\pi\sin n\pi x}{n^2\pi^2}\right]_0^1+\left[\frac{-\pi(2-x)\cos n\pi x}{n\pi}-\frac{\pi\sin n\pi x}{n^2\pi^2}\right]_1^2=0$$

Hence the required Fourier series is

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left[\cos\pi x+\frac{1}{3^2}\cos 3\pi x+\frac{1}{5^2}\cos 5\pi x+\ldots\right] \quad \text{Ans.}$$

Prob.36. Expand the function $f(x)=\begin{cases}0, & -\pi<x<0\\ \frac{1}{4}\pi x, & 0<x<\pi\end{cases}$ ***in a Fourier series in interval (– π, π).***

[R.G.P.V., June 2013 (III-Sem)]

Sol. Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \quad ...(i)$

Here the given interval being $(-\pi, \pi)$, the values of a_0, a_n and b_n are as follows –

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\,dx + \int_0^{\pi}\frac{\pi x}{4}\,dx\right] = \frac{1}{\pi}\cdot\frac{\pi}{4}\int_0^{\pi} x\,dx = \frac{1}{4}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi^2}{8}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_0^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\cos nx\,dx + \int_0^{\pi}\frac{\pi x}{4}\cos nx\,dx\right] = \frac{1}{4}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{1}{4}\left[\left\{x.\left(\frac{\sin nx}{n}\right)\right\}_0^{\pi} - \int_0^{\pi} 1.\left(\frac{\sin nx}{n}\right)dx\right]$$

$$= \frac{1}{4}\left[(0) - \frac{1}{n}\left(\frac{-\cos nx}{n}\right)_0^{\pi}\right] = \frac{1}{4n^2}(\cos n\pi - \cos 0) = \frac{[(-1)^n - 1]}{4n^2}$$

$= 0$, when n is even.

$= -\dfrac{1}{2n^2}$, when n is odd.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_0^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\sin nx\,dx + \int_0^{\pi}\frac{\pi x}{4}\sin nx\,dx\right] = \frac{1}{4}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{1}{4}\left[\left\{x.\left(\frac{-\cos nx}{n}\right)\right\}_0^{\pi} - \int_0^{\pi} 1.\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= \frac{1}{4}\left[\left(\frac{-\pi}{n}\cos n\pi\right) + \frac{1}{n}\left(\frac{\sin nx}{n}\right)_0^{\pi}\right] = \frac{1}{4n}[-\pi\cos n\pi + 0] = -\frac{(-1)^n\pi}{4n}$$

From equation (i), we get

$$f(x) = \frac{\pi^2}{16} + \sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{4n^2}\cos nx + \sum_{n=1}^{\infty}\frac{-(-1)^n\pi}{4n}\sin nx$$

$$= \frac{\pi^2}{16} + \left(-\frac{\cos x}{2} - \frac{\cos 3x}{2.3^2} - \frac{\cos 5x}{2.5^2} - \ldots\ldots\right)$$

$$+ \left(\frac{\pi}{4}\sin x - \frac{\pi}{4.2}\sin 2x + \frac{\pi}{4.3}\sin 3x - \ldots\ldots\right)$$

Ans.

Prob.37. Expand –

$$f(x) = \begin{cases} 1/4 - x, & \text{if } 0 < x < 1/2 \\ x - 3/4, & \text{if } 1/2 < x < 1 \end{cases}$$

as the Fourier series of sine terms. ***[R.G.P.V., Sept. 2009 (III-Sem)]***

Sol. Let f(x) represent an odd function in (– 1, 1) so that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

where $\quad b_n = \frac{2}{1}\int_0^1 f(x) \sin n\pi x \, dx$

$$= 2\left[\int_0^{1/2}\left(\frac{1}{4} - x\right)\sin n\pi x \, dx + \int_{1/2}^1 \left(x - \frac{3}{4}\right)\sin n\pi x \, dx\right]$$

$$= 2\left[-\left(\frac{1}{4} - x\right)\frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right]_0^{1/2} + 2\left[-\left(x - \frac{3}{4}\right)\frac{\cos n\pi x}{n\pi} + \sin\frac{n\pi x}{n^2\pi^2}\right]_{1/2}^1$$

$$= 2\left[\frac{1}{4n\pi}\cos\frac{n\pi}{2} + \frac{1}{4n\pi} - \frac{\sin n\pi/2}{n^2\pi^2}\right] + 2\left[\frac{-1}{4n\pi}\cos n\pi - \frac{1}{4n\pi}\cos\frac{n\pi}{2} - \sin\frac{n\pi/2}{n^2\pi^2}\right]$$

$$= \frac{1}{2n\pi}[1 - (-1)^n] - \frac{4\sin n\pi/2}{n^2\pi^2}$$

Hence $b_1 = \frac{1}{\pi} - \frac{4}{\pi^2}; b_2 = 0;\ b_3 = \frac{1}{3\pi} + \frac{4}{3^2\pi^2}$

$$b_4 = 0;\ b_5 = \frac{1}{5\pi} - \frac{4}{5^2\pi^2}, b_6 = 0, \text{etc.}$$

Hence,

$$f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} + \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 5\pi x + \ldots$$

Ans.

Prob.38. Expand $\pi x - x^2$ in a half-range sine series in the interval (0, π) upto the first three terms. ***[R.G.P.V., June 2005 (III-Sem), 2012 (III-Sem), Dec. 2015 (III-Sem)]***

Sol. Let the required series be of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c} \quad \ldots(i)$$

Here ... nec... half-

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}dx = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\pi x \sin nx\, dx - \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\, dx$$

$$= 2\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} - \frac{2}{\pi}\left[-\frac{x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_0^{\pi}$$

$$= 2\left[-\frac{\pi\cos n\pi}{n}\right] - \frac{2}{\pi}\left[-\frac{\pi^2\cos n\pi}{n} + \frac{2\cos n\pi}{n^3} - \frac{2}{n^3}\right]$$

$$= -\frac{2\pi\cos n\pi}{n} + \frac{2\pi\cos n\pi}{n} - \frac{4\cos n\pi}{\pi n^3} + \frac{4}{\pi n^3}$$

$$= \frac{4}{\pi n^3} - \frac{4\cos n\pi}{\pi n^3} = \frac{4}{\pi n^3}[1-(-1)^n]$$

Putting $n = 1, 2, \ldots..$

$$b_1 = \frac{4}{\pi.1^3}\times 2,\ \ b_2 = 0,\ \ b_3 = \frac{4}{\pi.3^3}\times 2,\ \ b_4 = 0 \ldots.$$

Hence

$$\pi x - x^2 = \frac{8}{\pi}\left\{\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots..\right\}$$ **Ans.**

Prob.39. Determine half-range sine series for the function f defined by $f(x) = x^2 + x,\ 0 \le x \le \pi$ ***[R.G.P.V., Dec. 2012 (III-Sem)]***

Sol. Let required series be of the form

$$f(x) = \sum_{n=1} b_n \sin\frac{n\pi x}{c} \qquad \ldots(i)$$

Here $c = \pi$, because it is half-range series.

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}dx = \frac{2}{\pi}\int_0^{\pi}(x^2+x)\sin nx\, dx$$

$$= \frac{2}{\pi}\left[(x^2+x)\left(-\frac{\cos nx}{n}\right) - (2x+1)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(\pi^2+\pi)\left(-\frac{\cos n\pi}{n}\right) + 2\left(\frac{\cos n\pi}{n^3}\right) - 2\frac{1}{n^3}\right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}(\pi^2+\pi)(-1)^n + \frac{2}{n^3}(-1)^n - \frac{2}{n^3}\right]$$

Hence $x^2 + x = \sum_{n=1}^{\infty}\left[-\frac{2}{n}(\pi+1)(-1)^n + \frac{4}{n^3\pi}\{(-1)^n - 1\}\right]\sin nx$ **Ans.**

Prob.40. Obtain a half-range cosine series for –

$$f(x) = \begin{cases} kx, & \text{for } 0 \le x \le \frac{l}{2} \\ k(l-x), & \text{for } \frac{l}{2} \le x \le l \end{cases}$$

Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$

[R.G.P.V., Dec. 2006 (III-Sem), 2013 (III-Sem), June 2014 (III-Sem), Dec. 2014 (III-Sem)]

Sol. Let the required series be of the form –

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} \qquad \ldots(i)$$

Here, $c = l$, because it is half-range series.

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^{l/2} kx\,dx + \frac{2}{l}\int_{l/2}^{l} k(l-x)\,dx$$

$$= \frac{2}{l}\left|\frac{kx^2}{2}\right|_0^{l/2} + \frac{2}{l}\left|k\left(lx - \frac{x^2}{2}\right)\right|_{l/2}^{l} = \frac{kl}{2}$$

and $$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$$

$$= \frac{2}{l}\int_0^{l/2} kx\cos\frac{n\pi x}{l}\,dx + \frac{2}{l}\int_{l/2}^{l} k(l-x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[kx.\frac{l}{n\pi}\sin\frac{n\pi x}{l} + \frac{kl^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right]_0^{l/2}$$

$$+\frac{2}{l}\left[k(l-x)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - \frac{kl^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right]_{l/2}^{l}$$

$$= \frac{2}{l}\left[\frac{2kl^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{kl^2}{n^2\pi^2}\cos n\pi - \frac{kl^2}{n^2\pi^2}\right]$$

or $$a_n = \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - (-1)^n - 1\right]$$

Here,

$a_1 = 0$, $a_2 = -\frac{8kl}{2^2\pi^2}$, $a_3 = 0$, $a_4 = 0$, $a_5 = 0$, $a_6 = -\frac{8kl}{6^2\pi^2}$ and so on.

Substituting above values in series (i), we get

$$f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \frac{1}{10^2}\cos\frac{10\pi x}{l} + \ldots\right] \quad \textbf{Ans.}$$

Substituting x = 0 in above series, we set

$$0 = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos 0 + \frac{1}{6^2}\cos 0 + \frac{1}{10^2}\cos 0 + \ldots\ldots\right]$$

or $$\frac{kl}{4} = \frac{8kl}{\pi^2}\cdot\frac{1}{2^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right]$$

or $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8} \quad \textbf{Ans.}$$

Prob.41. Develop sin $\left(\frac{\pi x}{l}\right)$ in half-range cosine series in the range $0 < x < l$. *[R.G.P.V., Dec. 2005 (III-Sem), 2015 (III-Sem), June 2017 (III-Sem)]*

Sol. Let the required series be of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} \quad \ldots(i)$$

Here c = l, because it is half-range series.

Then

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx = \frac{2}{l}\int_0^l \sin\left(\frac{\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[-\frac{l}{\pi}\cos\frac{\pi x}{l}\right]_0^l = -\frac{2}{\pi}(\cos\pi - \cos 0) = \frac{4}{\pi}$$

and $$a_n = \frac{2}{l}\int_0^l \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_0^l \left[\sin\left(\frac{\pi x}{l} + \frac{n\pi x}{l}\right) - \sin\left(\frac{n\pi x}{l} - \frac{\pi x}{l}\right)\right]dx$$

$$= \frac{1}{l}\int_0^l \sin(n+1)\frac{\pi x}{l}dx - \frac{1}{l}\int_0^l \sin(n-1)\frac{\pi x}{l}dx$$

$$= \frac{1}{l}\left[-\frac{l}{(n+1)\pi}\cos(n+1)\frac{\pi x}{l}\right]_0^l - \frac{1}{l}\left[-\frac{l}{(n-1)\pi}\cos(n-1)\frac{\pi x}{l}\right]_0^l$$

$$= -\frac{1}{(n+1)\pi}[\cos(n+1)\pi - \cos 0] + \frac{1}{(n-1)\pi}[\cos(n-1)\pi - \cos 0]$$

$$= -\frac{1}{(n+1)\pi}[(-1)^{n+1}-1]+\frac{1}{(n-1)\pi}[(-1)^{n+1}-1]$$

$$= (-1)^{n+1}\left[-\frac{1}{(n+1)\pi}+\frac{1}{(n-1)\pi}\right]+\left[\frac{1}{(n+1)\pi}-\frac{1}{(n-1)\pi}\right]$$

$$= \frac{2(-1)^{n+1}}{(n^2-1)\pi}-\frac{2}{(n^2-1)\pi} = \frac{2}{(n^2-1)\pi}[(-1)^{n+1}-1]$$

$$= \frac{-4}{(n^2-1)\pi}, \text{ when n is even} \qquad = 0, \text{ when n is odd.}$$

The above formula for obtaining the value of a_1 is not applicable.

$$a_1 = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{\pi x}{l}dx = \frac{1}{l}\int_0^l \sin\frac{2\pi x}{l}dx$$

$$= \frac{1}{l}\left(-\frac{l}{2\pi}\cos\frac{2\pi x}{l}\right)_0^l = -\frac{1}{2\pi}(\cos 2\pi - \cos 0) = 0$$

Hence

$$f(x) = \frac{a_0}{2}+a_1\cos\frac{\pi x}{l}+a_2\cos\frac{2\pi x}{l}+a_3\cos\frac{3\pi x}{l}+a_4\cos\frac{4\pi x}{l}+\ldots..$$

or $$f(x) = \frac{2}{\pi}-\frac{4}{\pi}\left[\frac{1}{3}\cos\frac{2\pi x}{l}+\frac{1}{15}\cos\frac{4\pi x}{l}+\frac{1}{35}\cos\frac{6\pi x}{l}+\ldots.\right]$$ Ans.

Prob.42. Express $f(x) = x$, as a half-range cosine series in $0 < x < 2$.
[R.G.P.V., June/July 2006 (III-Sem), Dec. 2011 (III-Sem), 2016 (III-Sem), May 2019]

Sol. The graph of $f(x) = x$ in $(0, 2)$ is the line OA. Let us extend the function $f(x)$ in the interval $(-2, 0)$ shown by the line OB' so that the new function is symmetrical about the Y-axis and, therefore, represents an even function in $(-2, 2)$. (Fig. 3.6)

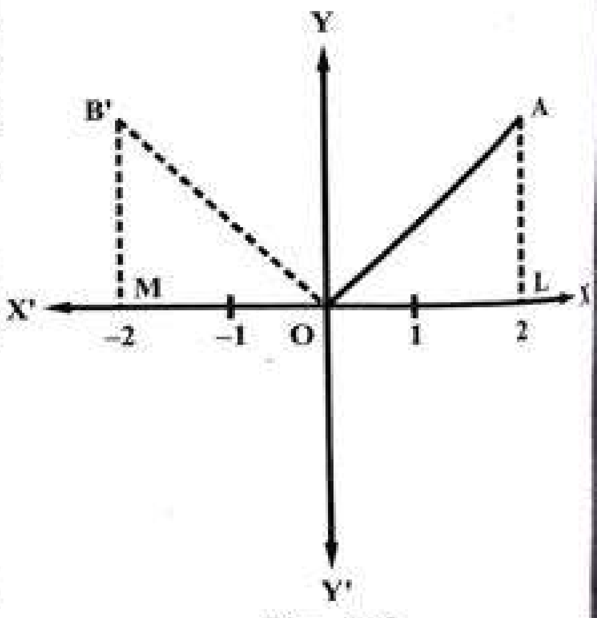


Fig. 3.6

Hence the Fourier series for $f(x)$ over the full period $(-2, 2)$ will contain only cosine terms given by

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{2}$$

where $a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 x\,dx = 2$

and $a_n = \dfrac{2}{2}\displaystyle\int_0^2 f(x)\cos\frac{n\pi x}{2}dx = \int_0^2 x\cos\frac{n\pi x}{2}dx$

$$= \left[\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_0^2 = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

Hence $a_1 = -8/\pi^2$, $a_2 = 0$, $a_3 = \dfrac{-8}{3^2\pi^2}$, $a_4 = 0$, $a_5 = \dfrac{-8}{5^2\pi^2}$ etc.

Hence the desired Fourier series for f(x) over the half-range (0, 2) is

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos \pi x/2}{1^2} + \frac{\cos 3\pi x/2}{3^2} + \frac{\cos 5\pi x/2}{5^2} + ...\right] \quad \textbf{Ans.}$$

Prob.43. Express f(x) = x as a half-range sine series in 0 < x < 2.

[R.G.P.V., Jan./Feb. 2007 (III-Sem), June 2009 (III-Sem)]

Sol. The graph of f(x) = x in 0 < x < 2 is the line OA. Let us extend the function f(x) in the interval – 2 < x < 0 (shown by the line BO) so that the new function is symmetrical about the origin and, therefore, represents an odd function in (– 2, 2) (Fig. 3.7).

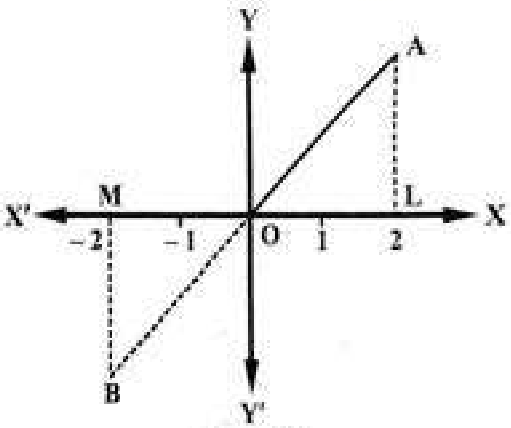


Fig. 3.7

Hence, the Fourier series for f(x) over the full period (– 2, 2) will contain only sine terms given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

where $b_n = \dfrac{2}{2}\displaystyle\int_0^2 f(x)\sin\frac{n\pi x}{2}dx = \int_0^2 x\sin\frac{n\pi x}{2}dx$

$$= \left[\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2 = -\frac{4(-1)^n}{n\pi}$$

Hence, $b_1 = 4/\pi$, $b_2 = -4/2\pi$, $b_3 = 4/3\pi$, $b_4 = -4/4\pi$ etc.

Hence the Fourier sine series for f(x) over half-range (0, 2) is

$$f(x) = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \frac{1}{4}\sin\frac{4\pi x}{2} + ...\right) \quad \textbf{Ans.}$$

Prob.44. Find the complex form of the Fourier series of $f(x) = e^{-x}$ in $-1 \le x \le 1$.

Sol. Here $f(x) = e^{-x}$, $-1 \le x \le 1$.

We have $$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x} \quad \text{....(i)}$$

where $$c_n = \frac{1}{2}\int_{-1}^{1} e^{-x}.e^{-in\pi x}dx = \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}dx$$

$$= \frac{1}{2}\left|\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right|_{-1}^{1} = \frac{e^{1+in\pi} - e^{-(1+in\pi)}}{2(1+in\pi)}$$

$$= \frac{e.(\cos n\pi + i\sin n\pi) - e^{-1}(\cos n\pi - i\sin n\pi)}{2(1+in\pi)}$$

$$= \frac{e-e^{-1}}{2}.(-1)^n.\frac{(1-in\pi)}{1+n^2\pi^2} = \frac{(-1)^n(1-in\pi)\sinh 1}{1+n^2\pi^2}$$

Putting the value of c_n in series (i), we get

$$e^{-x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1-in\pi)}{1+n^2\pi^2}\sinh 1.e^{in\pi x} \quad \textbf{Ans.}$$

Prob.45. From the Fourier series expansion of $f(x) = x^2$ in the interval $-\pi < x < \pi$, prove that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$. [R.G.P.V., June 2009 (III-Sem), Dec. 2010 (III-Sem)]

Sol. Since f(x) is even function, so Fourier series expansion is given by

$$f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{....(i)}$$

where $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right] = \frac{2\pi^2}{3}$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}.\frac{2\pi}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$$

Substituting these values in equation (i), we get

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \qquad \text{....(ii)}$$

which is Fourier expansion of $f(x) = x^2$ in the interval $(-\pi, \pi)$.

By Parseval's theorem, we have

$$\int_{-\pi}^{\pi} [f(x)]^2 \, dx = \pi\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right] \qquad \text{....(iii)}$$

where $f(x) = x^2$, $a_0 = \frac{2\pi^2}{3}$, $a_n = \frac{4}{n^2}(-1)^n$ and $b_n = 0$.

Hence

$$\int_{-\pi}^{\pi} x^4 dx = \pi\left[\frac{4\pi^4}{2.9} + \sum_{n=1}^{\infty}\left(\frac{16}{n^4}\right)\right]$$

or $$\left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{2\pi^5}{9} + \pi\sum_{n=1}^{\infty}\left(\frac{16}{n^4}\right)$$

or $$\frac{2\pi^5}{5} - \frac{2\pi^5}{9} = 16\pi\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right)$$

or $$\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right) = \frac{\pi^4}{90} \quad \textbf{Proved}$$

Prob.46. Find the Fourier series for the function $f(x) = x^2, -\pi \le x \le \pi$.

(R.G.P.V., Nov. 2019)

Sol. Refer to Prob.45.

Prob.47. Prove that –

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}, -\pi < x < \pi$$

and hence show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

[R.G.P.V., June 2010 (III-Sem)]

Sol. For I[st] part, refer to Prob.45.

Put x = π in equation (ii) of Prob.45, we get

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.(-1)^n \qquad \{\because \cos n\pi = (-1)^n\}$$

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$\frac{2\pi^2}{3\times 4} = \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \{\because (-1)^{2n} = 1\}$$

or $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$ **Proved**

MODULE 4

VECTOR SPACES

VECTOR SPACE, VECTOR SUBSPACE, LINEAR COMBINATION OF VECTORS

Vector Space –

Let (F, +, .) be a field. The elements of F will be called ***scalars.*** Let V be a non-empty set whose elements will be called ***vectors***. Then V is a vector space over the field F, if

(i) There is defined an internal composition in V called ***addition of vectors*** and denoted by '+'. Also for this composition V is an abelian group i.e.,

(a) $\alpha + \beta \in V$ for all $\alpha, \beta \in V$

(b) $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in V$

(c) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ for all $\alpha, \beta, \gamma \in V$

(d) $\exists$ an element $\mathbf{0} \in V$ such that $\alpha + \mathbf{0} = \alpha$ for all $\alpha \in V$.

This element $\mathbf{0} \in V$ will be called the ***zero vector***.

(e) To every vector $\alpha \in V$, there exists a vector $-\alpha \in V$ such that

$$\alpha + (-\alpha) = \mathbf{0}$$

(ii) There is an external composition in V over F called ***scalar multiplication*** and denoted multiplicatively i.e., $a\alpha \in V$ for all $a \in F$ and for all $\alpha \in V$. In other words V is called closed with respect to scalar multiplication.

(iii) The two compositions i.e., scalar multiplication and addition of vectors satisfy the following postulates

(a) $a(\alpha + \beta) = a\alpha + a\beta \ \forall\, a \in F$ and $\forall\, \alpha, \beta \in V$.

(b) $(a + b)\alpha = a\alpha + b\alpha \ \forall\, a, b \in F$ and $\forall\, \alpha \in V$.

(c) $(ab).\alpha = a(b\alpha) \ \forall\, a, b \in F$ and $\forall\, \alpha \in V$.

(d) $1.\alpha = \alpha \ \forall\, \alpha \in V$ and 1 is unit element of the field F.

General Properties of Vector Spaces –

Let V(F) be a vector space over the field (F, +, .) and $\mathbf{0}$ be the additive identity of V. Also let $\alpha, \beta \in V$, $a \in F$, 0 be the additive identity of F, then

(i) $a\mathbf{0} = \mathbf{0}$ (ii) $0\alpha = \mathbf{0}$

(iii) $a(-\alpha) = -(a\alpha)$ (iv) $(-a)\alpha = -(a\alpha)$

(v) $a(\alpha - \beta) = a\alpha - a\beta$ (vi) $a\alpha = \mathbf{0} \Rightarrow a = \mathbf{0}$ or $\alpha = \mathbf{0}$; $\forall a \in F,\ \alpha, \beta \in V$

Proof. (i) We have $a\mathbf{0} = a(\mathbf{0} + \mathbf{0}) = a\mathbf{0} + a\mathbf{0}$ $(\because \mathbf{0} + \mathbf{0} = \mathbf{0})$ (by distributive law)

Also $a\mathbf{0} = a\mathbf{0} + \mathbf{0}$ ($\because \mathbf{0}$ being additive identity in V)

$\therefore$ $a\mathbf{0} + \mathbf{0} = a\mathbf{0} + a\mathbf{0}$

($\because$ V is abelian group, therefore using cancellation law)

$\Rightarrow$ $a\mathbf{0} = \mathbf{0}$ **Proved**

(ii) We have $0 + 0 = 0, \forall 0 \in F$

Let $\alpha \in V$, then $(0 + 0)\alpha = 0.\alpha$

or $0\alpha + 0\alpha = 0\alpha + \mathbf{0}$ $(\because \alpha \in V \text{ and } \mathbf{0} + 0\alpha = 0\alpha)$

Since V is an abelian group for addition, therefore by using cancellation law, we get

$$0\alpha = \mathbf{0}$$ **Proved**

(iii) Let $a \in F, \alpha \in V$

$a \in V \Rightarrow -a \in V$, then we have

$a[\alpha + (-\alpha)] = a\alpha + a(-\alpha)$ (by distributive law)

$a\mathbf{0} = a\alpha + a(-\alpha)$ $(\because \alpha + (-\alpha) = 0 \in V)$

$\mathbf{0} = a\alpha + a(-\alpha)$ $(\because a\mathbf{0} = \mathbf{0})$

i.e. $a\alpha + a(-\alpha) = \mathbf{0}$

$\Rightarrow$ $-(a\alpha) = a(-\alpha)$

which shows that $a.\alpha$ and $a(-\alpha)$ are inverse of each other with respect to addition in V. **Proved**

(iv) We know that, if $a \in F \Rightarrow -a \in F$

and also $a + (-a) = 0$.

Now $[a + (-a)]\alpha = 0\alpha$

$a\alpha + (-a)\alpha = 0\alpha,$ (by distributive law)

$a\alpha + (-a)\alpha = \mathbf{0},$ $(\because 0\alpha = \mathbf{0} \in V)$

$\therefore$ $a\alpha$ and $(-a)\alpha$ are inverse of each other respect to addition in V. **Proved**

(v) Let $a \in F$, $\alpha, \beta \in V$, then we have

$a(\alpha - \beta) = a[\alpha + (-\beta)] = a\alpha + a(-\beta)$ (by distributive law)

$= a\alpha - a\beta$ $(\because a(-\beta) = -a\beta)$ **Proved**

(vi) Let $a\alpha = \mathbf{0}, a \neq 0$. Then there exists $a^{-1} \in F$ to each non-zero $a \in F$.

$\therefore$ $a\alpha = \mathbf{0}$

$\Rightarrow$ $a^{-1}.(a\alpha) = a^{-1}\mathbf{0}$

$\Rightarrow$ $(a^{-1}a)\alpha = \mathbf{0}$ (by associative law and $a^{-1}\mathbf{0} = \mathbf{0}$)

$\Rightarrow$ $1.\alpha = \mathbf{0}$ $(\because a^{-1}a = 1 \in F)$

$\Rightarrow$ $\alpha = \mathbf{0}$.

Again let $a\alpha = \mathbf{0}$ and $\alpha \neq \mathbf{0}$. Then we are required to prove that $a = 0$.

Let $\quad a \neq 0 \in F \Rightarrow a^{-1} \in F.$

Consider $\quad a.\alpha = \mathbf{0}$

$\Rightarrow \quad a^{-1}(a\alpha) = a^{-1}\mathbf{0}$

$\Rightarrow \quad (a^{-1} \cdot a)\ \alpha = \mathbf{0}$ (by associative law and $a^{-1}.\mathbf{0} = \mathbf{0}$)

$\Rightarrow \quad 1\alpha = \mathbf{0}$ $(\because a^{-1}a = 1)$

$\Rightarrow \quad \alpha = \mathbf{0}$

which is contradiction because $\alpha \neq 0$.

Therefore a must be equal to zero. Hence

$$\alpha \neq \mathbf{0} \text{ and } a\alpha = \mathbf{0} \Rightarrow a = 0$$ **Proved**

Vector Subspace – Let V be a vector space over the field F and let $W \subseteq V$. Then W is called ***a subspace of V if W itself is a vector space over F*** with respect to the operations of vector addition and scalar multipilication in V.

Theorem 1. A subset W of a vector space V(F) is a subspace of V, iff $\forall\, \alpha, \beta \in W$ and a, $b \in F \Rightarrow a\alpha + b\beta \in W$. *[R.G.P.V., May/June 2006 (III-Sem)]*

***Proof.* The Condition is Necessary** – Let V(F) be a vector space over F and W be a subspace of V, hence it must be closed for vector addition and scalar multiplication; i.e.,

$\forall\, a \in F, \alpha \in W \Rightarrow a\alpha \in W$ (closed for scalar multiplication)

$\forall\, b \in F, \beta \in W \Rightarrow b\beta \in W$ (closed for scalar multiplication)

Now $\quad a\alpha \in W, b\beta \in W \Rightarrow (a\alpha + b\beta) \in W.$

Hence the condition is necessary.

The Condition is Sufficient – Let us assume that W is a non-empty subset of V, subjected to the condition

$$a\alpha + b\beta \in W, \forall \alpha, \beta \in W \text{ and } a, b \in F$$

i.e., W is closed under vector addition and scalar multiplication. Then we are required to prove that W is a vector subspace of V(F); W satisfies all the postulates of vector space.

Since, $\forall$ a, $b \in F$, so let us take a = 1, b = 1, then from the given condition, we have

$1.\alpha + 1.\beta = \alpha + \beta \in W, \forall \alpha, \beta \in W \quad (\because \alpha \in W \Rightarrow \alpha \in V, 1.\alpha = \alpha \in V)$

Hence, W is closed under vector addition.

Now taking $a = -1$, $b = 0$, we see that if $\alpha \in W$ then $(-1)\alpha + 0\alpha \in W$ [In place of β we have taken α]

$\Rightarrow -(1\alpha) + 0 \in W \Rightarrow -\alpha \in W$

Hence the additive inverse of each element of W is also in W.

Now, taking a = 0, b = 0, then from the given condition, we have

$0\alpha + 0.\beta \in W \Rightarrow \mathbf{0} + \mathbf{0} \in W \Rightarrow \mathbf{0} \in W.$

Thus the zero vector of V belongs to W. It will also be the zero vector of W.

As $W \subseteq V$, therefore, vector addition will be, associative as well as commutative in W.

Again, taking $\beta = \mathbf{0}$, then from the given condition, we have

$a.\alpha + b.\mathbf{0} \in W \Rightarrow a\alpha + \mathbf{0} \in W \Rightarrow a\alpha \in W, \forall a \in F, \alpha \in W$

and hence W is closed under scalar multiplication, so the remaining postulates of a vector space hold in W as $W \subseteq V$.

Hence, W(F) is vector subspace of V(F). **Proved**

Theorem 2. The non-empty subset W of a vector space if and only if

(i) $u \in W, v \in W \Rightarrow u - v \in W$

(ii) $a \in F, u \in W \Rightarrow au \in W$

[R.G.P.V., Dec. 2003 (III-Sem)]

Proof. **The Conditions are Necessary** – If W is a subspace of V, then W is an abelian group with respect to vector addition. Therefore $u \in W, v \in W \Rightarrow u - v \in W$. Also W must be closed under scalar multiplication. Therefore $a \in F, u \in W \Rightarrow a\,u \in W$. Hence the conditions are necessary.

The Conditions are Sufficient – Now let W be a non-empty subset of V satisfying the two given conditions. From condition (i), we have

$$u \in W, u \in W \Rightarrow u - u \in W \Rightarrow \mathbf{0} \in W$$

Hence the zero vector of V belongs to W and it will also be the zero vector of W.

Now $\mathbf{0} \in W, u \in W \Rightarrow \mathbf{0} - u \in W \Rightarrow -u \in W$

Hence the additive inverse of each element of W is also in W.

Again $u \in W, v \in W \Rightarrow u \in W, -v \in W$

$\Rightarrow \quad u - (-v) \in W \Rightarrow u + v \in W$

Hence W is closed w.r.t. vector addition. Because the elements of W are also the elements of V, so vector addition will be commutative as well as associative in W. Thus W is an abelian group under vector addition. Also from condition (ii), W is closed under scalar multiplication. The remaining postulates of a vector space will hold in W since they hold in V of which W is a subset. Hence W is a subspace of V. **Proved**

Theorem 3. The necessary and sufficient condition for a non-empty subset W of a vector space V(F) to be a subspace of V is that W is closed under vector addition and scalar multiplication.

Proof. **The Condition is Necessary** – Let V(F) be a vector space over the field F and W be a subspace of V;

i.e. W itself is a vector space for the given two compositions for which V is a vector space and consequently W is closed under vector addition and scalar multiplication.

The Condition is Sufficient – Now if $W \subseteq V$ and W is closed with respect to vector addition and scalar multiplication than we shall show that W is a vector space.

Let $\alpha \in W$ and 1 be the unit element of F.

$$1 \in F \Rightarrow -1 \in F$$

$\because \quad -1 \in F, \alpha \in W \Rightarrow (-1).\ \alpha \in W,$

($\because$ W is closed for scalar multiplication)

$\Rightarrow \quad -(1.\alpha) \in W \Rightarrow -\alpha \in W$

i.e., the additive inverse of each $\alpha \in W$ is $-\alpha \in W$.

Also, as W is closed for vector addition

$\therefore \quad \alpha \in W, -\alpha \in W \Rightarrow \alpha + (-\alpha) \in W \Rightarrow \mathbf{0} \in W$

where $\mathbf{0}$ is the zero vector of V, i.e., zero vector of V is also a zero vector of W.

Also, associative and commutative law holds good in W, since they hold in V and $W \subseteq V$.

Thus, (W, +) is an abelian group.

Also, it is given that W is closed for scalar multiplication. The remaining postulates of vector space also holds good in W, since they hold in V of which W is a subset.

Hence, W is a vector space of V(F). **Proved**

Algebra of Subspaces –

Theorem 4. The intersection of any two subspaces of a vector space V(F) is also a subspace of V(F).

Proof. Let W_1 and W_2 be any two subspaces of vector space V(F). Then we have to prove $W_1 \cap W_2$ is also a subspace of V(F). Since W_1 and W_2 are subspace so, at least $0 \in W_1$ and W_2

$\therefore \quad 0 \in W_1 \cap W_2$

Hence $\quad W_1 \cap W_2 \neq 0$

i.e., $\quad W_1 \cap W_2$ is a non-empty set.

Let $\quad \alpha, \beta \in W_1 \cap W_2$ and $a, b \in F$.

Now $\quad \alpha \in W_1 \cap W_2 \Rightarrow \alpha \in W_1$ and $\alpha \in W_2$

Also $\quad \beta \in W_1 \cap W_2 \Rightarrow \beta \in W_1$ and $\beta \in W_2$.

Since W_1 and W_2 are subspaces of V, therefore $\forall\, a, b \in F;\ \alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1$

$$a, b \in F,\ \alpha, \beta \in W_2 \Rightarrow a\alpha + b\beta \in W_2.$$

Now $\quad a\alpha + b\beta \in W_1,\ a\alpha + b\beta \in W_2 \Rightarrow a\alpha + b\beta \in W_1 \cap W_2$.

Hence, $\quad W_1 \cap W_2$ is subspace of V(F). **Proved**

Theorem 5. The union of two subspaces is a subspace if one is contained in the other.

Proof. Let W_1 and W_2 be two subspaces of a vector space V.

Suppose $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. Then $W_1 \cup W_2 = W_2$ or W_1. But W_1, W_2 are subspaces and therefore, $W_1 \cup W_2$ is also a subspace.

Conversely, suppose $W_1 \cup W_2$ is a subspace

To prove that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Let us assume that W_1 is not a subset of W_2 and W_2 is also not a subset of W_1.

Now W_1 is not a subset of $W_2 \Rightarrow \exists \alpha \in W_1$ and $\alpha \notin W_2$ ...(i)

and W_2 is not a subset of $W_1 \Rightarrow \exists \beta \in W_2$ and $\beta \notin W_1$ ...(ii)

From relations (i) and (ii), we have

$\alpha \in W_1 \cup W_2$ and $\beta \in W_1 \cup W_2$

Since $W_1 \cup W_2$ is a subspace, therefore

$\alpha + \beta$ is also in $W_1 \cup W_2$

But $\alpha + \beta \in W_1 \cup W_2 \Rightarrow \alpha + \beta \in W_1$ or W_2

Suppose $\alpha + \beta \in W_1$. Since $\alpha \in W_1$ and W_1 is a subspace, therefore $(\alpha + \beta) - \alpha = \beta$ is in W_1. But from relation (ii), we have $\beta \notin W_1$. Hence we get a contradiction. Again suppose that $\alpha + \beta \in W_2$. Since $\beta \in W_2$ and W_2 is a subspace, therefore $(\alpha + \beta) - \beta = \alpha$ is in W_2. But from relation (i), we have $\alpha \notin W_2$. Hence here also we get a contradiction. Hence either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Quotient Spaces – Let V(F) be a vector space over F and W be a subspace of V. Then W is a subgroup of the abelian group V with respect to addition of vectors. If $\alpha \in V$, then $W + \alpha$ is a right coset of W in V and $\alpha + W$ is a left coset of W in V.

Also, since W is an abelian group for the addition composition, therefore each left coset is equal to the right coset; i.e.

$$W + \alpha = \alpha + W$$

i.e., there is no distinction between left coset and a right coset. Let

$$\frac{V}{W} = \{W + \alpha : \alpha \in V\}.$$

Theorem 6. If W is any subspace of a vector space V(F), then the set V/W of all cosets W + α, where α∈V is a vector space over F for the addition and scalar multiplication composition defined as follows –

$$\text{(i)} \quad (W + \alpha) + (W + \beta) = W + (\alpha + \beta)$$

and $$\text{(ii)} \quad a(W + \alpha) = W + a\alpha, \forall a \in F$$

Proof. Before going to the proof of the theorem, it is desirable to prove that the two compositions are well defined; i.e., to show that if,

$$W + \alpha = W + \alpha' \quad \text{and} \quad W + \beta = W + \beta'$$

where $\alpha, \alpha', \beta, \beta' \in V$, then

(i) $W+(\alpha+\beta) = W+(\alpha'+\beta')$

(ii) $W + a\alpha = W + a\alpha'$ for every $a \in F$, we have

$$W + \alpha = W + \alpha' \Rightarrow \alpha - \alpha' \in W.$$

and $$W + \beta = W + \beta' \Rightarrow \beta - \beta' \in W$$

Also, $(\alpha-\alpha') \in W$ and $(\beta-\beta') \in W$

$\Rightarrow$ $(\alpha-\alpha')+(\beta-\beta') \in W$ ($\because$ W is a subspace)

$\Rightarrow$ $(\alpha+\beta)-(\alpha'+\beta') \in W$

$\Rightarrow$ $W+(\alpha+\beta) = W+(\alpha'+\beta')$.

Therefore, the addition in set of all cosets V/W is well defined.

Again,

$a \in F, (\alpha-\alpha') \in W \Rightarrow a.(\alpha-\alpha') \in W \Rightarrow a\alpha - a\alpha' \in W$

$\Rightarrow$ $W + a\alpha = W + a\alpha'$

$\Rightarrow$ $a(W+\alpha) = a(W+\alpha')$

i.e., scalar multiplication in V/W is also well defined. $V_1 : \left(\frac{V}{W}, '+'\right)$ is an abelian group

(i) Closure Property – Let $\alpha, \beta \in V$ and $a \in F$, then we have

$\because$ $\alpha, \beta \in V \Rightarrow \alpha+\beta \in V$ also $a \in F, \alpha \in V \Rightarrow a\alpha \in V$,

Hence, $W+(\alpha+\beta) \in \frac{V}{W}$ and $W + a\alpha \in \frac{V}{W}$.

Thus, $\frac{V}{W}$ is closed for both the compositions.

(ii) Commutativity of Addition – Let $W + \alpha, W + \beta \in \frac{V}{W}$, then we have

$$(W+\alpha)+(W+\alpha) = W+(\alpha+\beta) = W+(\beta+\alpha) \quad (\because V \text{ is commutative})$$
$$= (W + \beta) + (W + \alpha)$$

i.e., addition of cosets is commutative.

(iii) Associativity of Addition – Let $W + \alpha, W + \beta, W + \gamma \in \frac{V}{W}$; $\forall \alpha, \beta, \gamma \in V$, then we have

$$(W+\alpha)+[(W+\beta)+(W+\gamma)] = (W+\alpha)+[W+(\beta+\gamma)]$$
$$= W+[\alpha+(\beta+\gamma)] = W+[(\alpha+\beta)+\gamma]$$
$$= [W+(\alpha+\beta)]+(W+\gamma)$$
$$= [(W+\alpha)+(W+\beta)]+(W+\gamma)$$

i.e., addition of cosets is associative.

(iv) Existence of Additive Identity – Let 0 be the zero vector of V. Then we have

$$W + \mathbf{0} = W \in \frac{V}{W} \text{ and also let } W + \alpha \in \frac{V}{W}, \forall \alpha \in V.$$

Now, $(W + \mathbf{0}) + (W + \alpha) = W + (\mathbf{0} + \alpha) = W + \alpha \quad (\because \mathbf{0} + \alpha = \alpha)$

Similarly, $(W+\alpha)+(W+\mathbf{0}) = W + \alpha$

i.e., $W + \mathbf{0}$ is the additive identity.

(v) Existence of Additive Inverse – Let $W + \alpha \in \frac{V}{W}, \forall \alpha \in V$, then we have

$$\alpha \in V \Rightarrow -\alpha \in V \quad \text{and} \quad W + \alpha \in \frac{V}{W} \Rightarrow W - \alpha \in \frac{V}{W}.$$

Now, $(W+\alpha)+(W-\alpha) = W+[(\alpha)+(-\alpha)] = W + \mathbf{0} = W,$

i.e., $W - \alpha$ is the additive inverse of $W + \alpha$, in $\frac{V}{W}$,

V_2 : Scalar Multiplication is Distributive Over Vector Addition –

Let $W + \alpha$, $W + \beta \in \frac{V}{W}$, $\forall \alpha, \beta \in V$ and also $a \in F$, then we have

$$\begin{aligned} a[(W + \alpha) + (W + \beta)] &= a.[W + (\alpha + \beta)] \\ &= W + a.(\alpha+\beta) = W + (a.\alpha + a.\beta) \\ &= (W+a.\alpha)+(W+a.\beta) = a(W+\alpha)+a(W+\beta) \end{aligned}$$

V_3 : Multiplication by Vector is Distributive Over Scalar Addition –

Let $W+\alpha \in \frac{V}{W}, \forall \alpha \in V$ and also $a, b \in F$.

$$\begin{aligned} \text{Now} \quad (a + b).(W + \alpha) &= W + (a + b).\alpha \\ &= W + a.\alpha + b.\alpha = (W + a.\alpha) + (W + b.\alpha) \\ &= a.(W + \alpha) + b.(W + \alpha). \end{aligned}$$

V_4 : Scalar Multiplication is Associative –

Let $W+\alpha \in \frac{V}{W}, \forall \alpha \in V$ and also $a, b \in F$.

$$\begin{aligned} \text{Now} \quad (ab).(W + \alpha) &= W + (ab)\alpha = W + a(b\alpha) \\ &= a.(W + b\alpha) = a.[b(W + \alpha)] \end{aligned}$$

V_5 : If 1 is the multiplicative identity in F, then

$$1.(W + \alpha) = W + 1.\alpha = W + \alpha$$

$\therefore$ V/W satisfies all the properties of being a vector space and also as its elements are cosets, hence it is called the quotient space over V relative to W.

Proved

Direct Sum of Two Subspaces – If W_1 and W_2 be two subspaces of a vector space V(F). Then V is said to be ***direct sum of W_1 and W_2*** denoted as $V = W_1 \oplus W_2$ if every element γ of V can be written uniquely as $\gamma = \alpha + \beta$ for unique elements $\alpha \in W_1$ and $\beta \in W_2$.

Also, the subspaces W_1 and W_2 are said to be ***complementary subspaces.***

A vector space V(F) is said to be direct sum of its subspaces W_1, W_2, ..., W_n, i.e.;

$$V = W_1 \oplus W_2 \oplus \ldots \oplus W_n$$

if and only if a vector $\gamma \in V$ is uniquely expressible as

$$\gamma = \alpha_1 + \alpha_2 + \ldots + \alpha_n$$

where $\qquad \alpha_1 \in W_1, \alpha_2 \in W_2, \ldots, \alpha_n \in W_n$.

Two subspaces W_1 and W_2 are said to be disjoint if and only if their intersection is a zero subspace, i.e.,

$$W_1 \cap W_2 = \{0\}.$$

Linear Combination of Vectors – Let V(F) be the vector space over the field F. If $\alpha_1, \alpha_2, \ldots, \alpha_n \in V$ then any element

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n \text{ of } V$$

where $\quad a_1, a_2, \ldots, a_n \in F$

is called the ***linear combination*** of the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Linear Span – Let V(F) be a vector space over field F and S be non-empty subset of vector space V(F). Then the linear span of S denoted as L(S) is the set of all linear combinations of finite set S. Thus we have

$$L(S) = \{a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n;\ \alpha_1.\alpha_2....\alpha_n$$

is any arbitrary finite subset of S and $a_1, a_2, \ldots, a_n$ is any arbitrary subset of F}

Linear Sum of Two Subspaces – Let W_1 and W_2 be two subspaces of a vector space V over the field F, then the linear sum of these subspaces is the set of sum $\alpha_1 + \alpha_2$, such that $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ and denoted by $W_1 + W_2$.

Hence $W_1 + W_2 = \{\alpha_1 + \alpha_2 : \alpha_1 \in W_1, \alpha_2 \in W_2\}$

Theorem 7. If W_1 and W_2 be two subspaces of V(F) then $W_1 + W_2$ is also a subspace of V(F).

Proof. Let $\alpha_1, \beta_1 \in W_1$ and $\alpha_2, \beta_2 \in W_2$

$$\alpha = \alpha_1 + \alpha_2 \in W_1 + W_2$$

and

$$\beta = \beta_1 + \beta_2 \in W_1 + W_2$$

In order to prove that, $W_1 + W_2$ is subspace of V, it is sufficient to prove that

$$a\alpha + b\beta \in W_1 + W_2;\ \forall a, b \in F \text{ and } \alpha, \beta \in W_1 + W_2.$$

Since W_1, W_2 are subspaces of V, therefore

$$a\alpha_1 + b\beta_1 \in W_1,\ a\alpha_2 + b\beta_2 \in W_2.$$

Hence, we have $(a\alpha_1 + b\beta_1) + (a\alpha_2 + b\beta_2) \in W_1 + W_2$.

i.e., $$a(\alpha_1 + \alpha_2) + b(\beta_1 + \beta_2) \in W_1 + W_2$$

i.e., $$a\alpha + b\beta \in W_1 + W_2$$

Hence $W_1 + W_2$ is a subspace of V(F). **Proved**

Theorem 8. The linear span L(S) of any subset S of a vector space V(F) is a subspace of V generated by S, i.e., L(S) = S.

Proof. Suppose α, β are any two elements of L(S). Then

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + ...+ a_m\alpha_m$$

and $$\beta = b_1\beta_1 + b_2\beta_2 + ...+ b_n\beta_n.$$

where the a's and b's are elements of F and the α's and β's are elements of S.

If a, b are any two elements of F, then

$$\begin{aligned} a\alpha + b\beta &= a\,(a_1\alpha_1 + a_2\alpha_2 + ...+ a_m\alpha_m) \\ &\quad + b\,(b_1\beta_1 + b_2\beta_2 + ...+ b_n\beta_n) \\ &= a\,(a_1\alpha_1) + a\,(a_2\alpha_2) + ...+ a\,(a_m\alpha_m) \\ &\quad + b\,(b_1\beta_1) + b\,(b_2\beta_2) + ...+ b\,(b_n\beta_n) \\ &= (aa_1)\,\alpha_1 + (aa_2)\,\alpha_2 + ...+ (aa_m)\alpha_m \\ &\quad + (bb_1)\,\beta_1 + (bb_2)\beta_2 + ...+ (bb_n)\beta_n \end{aligned}$$

Thus $a\alpha + b\beta$ has been expressed as a linear combination of a finite set $\alpha_1, \alpha_2, ..., \alpha_m, \beta_1, \beta_2, ..., \beta_n$ of the elements of S consequently $a\alpha + b\beta \in L(S)$

Thus $a, b \in F$ and $\alpha, \beta \in L(S) \Rightarrow a\,\alpha + b\,\beta \in L(S)$

Hence L(S) is a subspace of V(F).

Also each element of S belongs to L(S), because if $\alpha_r \in S$, then $\alpha_r = 1\alpha_r$ and this implies that $\alpha_r \in L$. Thus L(S) is a subspace of V and S is contained in L(S). Now if W is any subspace of V containing S, then each element of L(S) must be in W because W is to be closed under vector addition and scalar multiplication. Therefore L(S) will be contained in W.

Hence L(S) = S, i.e., L(S) is the smallest subspace of V containing S.

Proved

NUMERICAL PROBLEMS

Prob.1. Show that the set of all ordered n-tuples of the elements of any field f, is a vector space over the field F.

Sol. Let F be a field, then an ordered set of all n-tuple elements of F, i.e., $\alpha = (a_1, a_2, \ldots, a_n)$ is called an n-tuple over F, and the set of all these n-tuples over a field F will be denoted by V_n (F) or V_n.

Let us define the vector addition, scalar multiplication and equality of n-tuples as follow –

Addition Composition – Let α, β be any two arbitrary elements of $V_n(F)$; i.e.,

$$\alpha = (a_1, a_2, \ldots, a_n), \beta = (b_1, b_2, \ldots, b_n)$$

$$\therefore \quad \alpha + \beta = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

Obviously, $\alpha + \beta \in V_n(F)$.

Hence, the set $V_n(F)$ is closed w.r.t. addition of n-tuples.

Scalar Multiplication Composition – Let

$\alpha = (a_1, a_2, \ldots, a_n) \in V_n$ and $a \in F$, then we define scalar multiplication in V_n (F) as

$$a\alpha = (aa_1, aa_2, \ldots, aa_n), \forall a \in F$$

as $aa_1, aa_2, \ldots, aa_n$ all are elements of F.

Hence the set $V_n(F)$ is closed w.r.t. scalar multiplication.

Equality of Two n-tuples – Two n-tuples

$$\alpha = (a_1, a_2, \ldots, a_n); \beta = (b_1, b_2, \ldots, b_n)$$

are said to be equal if and only if

$$a_i = b_i, \text{ for each } i = 1, 2, \ldots, n.$$

Now let us verify that the set $V_n(F)$ is a vector space for the above defined composition.

V_1 : $V_n(F)$ is an Abelian Group for Addition –

(i) Associativity of Addition – Let $\alpha, \beta, \gamma \in V_n(F)$, where

$$\alpha = (a_1, a_2, \ldots, a_n), \beta = (b_1, b_2, \ldots, b_n)$$

and $\gamma = (c_1, c_2, \ldots c_n)$

Now
$$\begin{aligned}
\alpha + (\beta + \gamma) &= (a_1, a_2, \ldots, a_n) + \{(b_1, b_2, \ldots, b_n) + (c_1, c_2 \ldots, c_n)\} \\
&= (a_1, a_2, \ldots, a_n) + (b_1 + c_1, b_2 + c_2, \ldots, b_n + c_n) \\
&= \{a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), \ldots, a_n + (b_n + c_n)\} \\
&= \{(a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \ldots, (a_n + b_n) + c_n\} \\
&= (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) + (c_1, c_2, \ldots, c_n) \\
&= \{(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n)\} + (c_1, c_2, \\
&\quad \ldots, c_n) \\
&= (\alpha + \beta) + \gamma
\end{aligned}$$

$\therefore$ $V_n(F)$ is associativity for addition.

(ii) Commutativity for Addition – Let $\alpha, \beta \in V_n(F)$, where

$$\alpha = (a_1, a_2, \ldots, a_n),$$

$$\beta = (b_1, b_2, \ldots, b_n)$$

Now $\alpha + \beta = (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n)$
$= (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$
$= (b_1 + a_1, b_2 + a_2, \ldots, b_n + a_n)$
($\because$ Scalar addition is commutative)
$= (b_1, b_2, \ldots, b_n) + (a_1, a_2, \ldots, a_n) = \beta + \alpha$

$\therefore$ $V_n(F)$ is commutative for addition

(iii) Existence of Additive Identity – We have

$\mathbf{0} = (0, 0, \ldots, 0) \in V_n(F)$

and also $\alpha = (a_1, a_2, \ldots, a_n) \in V_n(F)$

Now $\mathbf{0} + \alpha = (0, 0, \ldots, 0) + (a_1, a_2, \ldots, a_n)$
$= (0 + a_1, 0 + a_2, \ldots, 0 + a_n) = (a_1, a_2, \ldots, a_n) = \alpha$

i.e., $\mathbf{0} = (0, 0, \ldots, 0)$ is the additive identity in $V_n(F)$.

(iv) Existence of Additive Inverse – Let $\alpha = (a_1, a_2, \ldots, a_n) \in V_n(F)$ then $-\alpha = (-a_1, -a_2, \ldots, -a_n) \in V_n(F)$

Also, $\alpha + (-\alpha) = (a_1, a_2, \ldots, a_n) + (-a_1, -a_2, \ldots, -a_n)$
$= (a_1 - a_1, a_2 - a_2, \ldots, a_n - a_n)$
$= (0, 0, \ldots, 0) = \mathbf{0}$

i.e., $\alpha + (-\alpha) = \mathbf{0}$

i.e. $-\alpha = (-a_1, -a_2, \ldots, -a_n)$ is the additive inverse of, $\alpha = (a_1, a_2, \ldots, a_n)$.

Thus, $V_n(F)$ is an abelian group for addition.

V_2 : Scalar Multiplication is Distributive Over Vector Addition –

Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n) \in V_n$ and $a \in F$, then

$$\begin{aligned} a.(\alpha + \beta) &= a.\{(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n)\} \\ &= a.(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \\ &= (aa_1 + ab_1, aa_2 + ab_2, \ldots, aa_n + ab_n) \\ &= (aa_1, aa_2, \ldots, aa_n) + (ab_1, ab_2, \ldots, ab_n) \\ &= a(a_1, a_2, \ldots, a_n) + a(b_1, b_2, \ldots, b_n) = a\alpha + a\beta. \end{aligned}$$

i.e., scalar multiplication is distributive over vector addition in $V_n(F)$.

V_3 : Multiplication by Vector is Distributive Over Scalar Addition –

Let $\alpha = (a_1, a_2, \ldots, a_n) \in V_n$ and $a, b \in F$. Consider

$$\begin{aligned} (a + b).\alpha &= (a + b).(a_1, a_2, \ldots, a_n) \\ &= \{(a + b).a_1, (a + b).a_2, (a + b).a_3, \ldots, (a + b).a_n\} \\ &= (aa_1 + ba_1, aa_2 + ba_2, \ldots, aa_n + ba_n) \\ &= (aa_1, aa_2, \ldots, aa_n) + (ba_1, ba_2, \ldots, ba_n) \\ &= a.(a_1, a_2, \ldots, a_n) + b.(a_1, a_2, \ldots, a_n) = a.\alpha + b.\alpha \end{aligned}$$

i.e. multiplication by vector is distributive over scalar addition in $V_n(F)$.

V_4 : Scalar Multiplication is Associative –

Let $\alpha = (a_1, a_2, \ldots, a_n) \in V_n$ and a, b $\in$F. Consider

$$(ab).\alpha = (ab).(a_1, a_2, \ldots, a_n) = (aba_1, aba_2, \ldots, aba_n)$$
$$= a.(ba_1, ba_2, \ldots, ba_n)$$
$$= a.\{(b(a_1, a_2, \ldots, a_n)\} = a.\{b(\alpha)\} = a.(b\alpha)$$

i.e. scalar multiplication is associative in $V_n(F)$.

V_5 : Let 1 be the unit element of F and $\alpha = (a_1, a_2, \ldots, a_n) \in V_n$ then

$$1.\alpha = 1.(a_1, a_2, \ldots, a_n) = (1.a_1, 1.a_2, \ldots, 1.a_n) = \alpha$$

Since all the postulates of vector space are satisfied and so the set $V_n(F)$ of all ordered n-tupled is a vector space over F. **Proved**

Prob.2. If V = {(a, b); a, b $\in$ R} and R is a field, show that V is not a vector space over R under the addition and scalar multiplication defined by –

$$(a, b) + (c, d) = (0, b + d)$$
$$\alpha (a, b) = (a\alpha, \alpha b)$$

[R.G.P.V., Dec. 2004 (III-Sem)]

Sol. V is not a vector space under the composition defined by

$$(a, b) + (c, d) = (0, b + d)$$
$$\alpha (a, b) = (\alpha a, \alpha b)$$

because $(\alpha_1 + \alpha_2)(a, b) = [(\alpha_1 + \alpha_2)a, (\alpha_1 + \alpha_2)b]$

$[\because \alpha (a, b) = (\alpha a, \alpha b)]$

or $(\alpha_1 + \alpha_2)(a, b) = (\alpha_1 a + \alpha_2 a, \alpha_1 b + \alpha_2 b)$ ...(i)

and $\alpha_1 (a, b) + \alpha_2 (a, b) = (\alpha_1 a, \alpha_1 b) + (\alpha_2 a, \alpha_2 b)$

$[\because \alpha (a, b) = (\alpha a, \alpha b)]$

$= (0, \alpha_1 b + \alpha_2 b)$ $[\because (a, b) + (c, d) = (0, b + d)]$

or $\alpha_1 (a, b) + \alpha_2 (a, b) = (0, \alpha_1 b + \alpha_2 b)$ ...(ii)

From relations (i) and (ii), we get

$$(\alpha_1 + \alpha_2)(a, b) \neq \alpha_1 (a, b) + \alpha_2 (a, b)$$

Hence, V is not satisfied. **Proved**

Prob.3. If V(F) be a vector space, then if

(i) a, b $\in$F and α is a non-zero vector of V, we have

$$a\alpha = b\alpha \Rightarrow a = b$$

(ii) $\alpha, \beta \in$V and a is a non-zero element of F, we have

$$a\alpha = a\beta \Rightarrow \alpha = \beta.$$

Sol. (i) We have $a\alpha = b\alpha$

$\Rightarrow \quad a\alpha - b\alpha = \mathbf{0}$

$\Rightarrow \quad (a - b)\alpha = \mathbf{0}$ (by distributive law in V)

$$\Rightarrow \quad \{a + (-b)\}\alpha = 0$$

$$\Rightarrow \quad a + (-b) = 0 \qquad [\because \alpha \neq 0 \text{ (given)}]$$

$$\Rightarrow \quad a = b$$

Proved

(ii) We have $a\alpha = a\beta$

$$\Rightarrow \quad a\alpha - a\beta = 0 \Rightarrow a(\alpha - \beta) = 0$$

$$\Rightarrow \quad \alpha - \beta = 0 \qquad [\because a \neq 0 \text{ (given)}]$$

$$\Rightarrow \quad \alpha = \beta.$$

Proved

Prob.4. If all the ordered vectors $(x, 2x, -3x, x)$ of a four dimensional real number space V_4 are in S, than show that S is a subspace of V_4.

Sol. In order to show that S is a subspace of V_4, it is enough to prove that

$\forall \alpha, \beta \in W$; $a, b \in F \Rightarrow a\alpha + b\beta \in W$.

Let $\alpha = (x_1, 2x_1, -3x_1, x_1)$

and $\beta = (x_2, 2x_2, -3x_2, x_2) \in S$ and $a, b \in F$

We have $a\alpha + b\beta = a(x_1, 2x_1, -3x_1, x_1) + b(x_2, 2x_2, -3x_2, x_2)$

$$= (ax_1, 2ax_1, -3ax_1, ax_1) + (bx_2, 2bx_2, -3bx_2, bx_2)$$

$$= \{(ax_1 + bx_2), 2(ax_1 + bx_2), -3(ax_1 + bx_2), (ax_1 + bx_2)\} \in S$$

(Because the elements are of the nature of elements of S)

$\therefore$ S is subspace of V_4. **Proved**

Prob.5. If V is a set of all $(n \times n)$ matrices over any field F, then show that a set W of all $(n \times n)$ symmetric matrices forms a vector subspace of V(F). [R.G.P.V., June 2003 (III-Sem), Dec. 2006 (III-Sem)]

Sol. In order to show that the set W of all $(n \times n)$ symmetric matrices forms a vector subspace of V(F), it is enough to show that W is closed for vector addition and scalar multiplication.

Any matrix $A = [a_{ij}]_{(n \times n)}$ will be called as ***symmetric matrix,*** if $A = A'$

Let $A = [a_{ij}]_{(n \times n)}$ and $B = [b_{ij}]_{(n \times n)}$ be two symmetric matrices, than we have

$$A + B = [a_{ij}]_{(n \times n)} + [b_{ij}]_{(n \times n)}$$

$$= [a_{ij} + b_{ij}]_{(n \times n)}, \qquad \text{(By addition of matrix)}$$

$$= [a_{ji} + b_{ji}]_{(n \times n)} \qquad (\because \text{A and B are symmetric})$$

$$= (A + B)'$$

i.e., the sum of any two symmetric matrices is again a symmetric matrix.

Hence, W is closed under matrix; i.e.,

for all $A, B \in W$, we have $A + B \in W$.

Also, if $a \in F$ and $A = [a_{ij}]_{(n \times n)} \in W$,

Now $a.A = a.[a_{ij}]_{(n \times n)} = [a \,.\, a_{ij}]_{(n \times n)} \qquad (\because A = A')$

$$= [a \,.\, a_{ji}]_{(n \times n)} = (a.\, A)'$$

i.e., a. A is a symmetric matrix. Hence W is closed under scalar multiplication. Hence W(F) is a vector subspace of V(F). **Proved**

Prob.6. Let V(R) be the vector space of complete real continuous functions. Then show that the solution set W of the differential equation

$$2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0, \text{ where } y = f(x), \text{ is the subspace of } V.$$

Sol. It is given that W is the solution set of the differential equation

$$2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0$$

i.e., $W = \{y : 2y'' - 9y' + 2y = 0\}$

where $y = f(x)$

Since, above is a linear differential equation of second degree, so it must have two solutions say y_1 and $y_2 \in W$, i.e.,

$$2y_1'' - 9y_1' + 2y_1 = 0 \text{ and } 2y_2'' - 9y_2' + 2y_2 = 0$$

Let a, $b \in R$.

If W is to be a subspace then we should show that $ay_1 + by_2$ also belong to W, i.e., it is a solution of the given differential equation. We have

$$2\frac{d^2}{dx^2}(ay_1 + by_2) - 9\frac{d}{dx}(ay_1 + by_2) + 2(ay_1 + by_2)$$

$$= 2(ay_1'' + by_2'') - 9(ay_1' + by_2') + 2(ay_1 + by_2)$$

$$= a(2y_1'' - 9y_1' + 2y_1) + b(2y_2'' - 9y_2' + 2y_2)$$

$$= a.0 + b.0 = 0$$

i.e. $\forall a, b \in R, y_1, y_2 \in W \Rightarrow ay_1 + by_2 \in W$

$\therefore$ W is a subspace of W. **Proved**

Prob.7. Is the vector (2, –5, 3) in the subspace of R^3 spanned by the vectors (1, –3, 2) (2, – 4, 1), (1, –5, 7) ? [R.G.P.V., June 2005 (III-Sem)]

Sol. Let $\alpha = (2, -5, 3)$, $\alpha_1 = (1, -3, 2)$, $\alpha_2 = (2, -4, 1)$, $\alpha_3 = (1, -5, 7)$. If α can be expressed as a linear combination of the vectors $\alpha_1, \alpha_2, \alpha_3$ then it will be in the subspace of R^3 spanned by these vectors otherwise it will not be.

Let $\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$, where $a_1, a_2, a_3 \in R$.

Then $(2, -5, 3) = a_1(1, -3, 2) + a_2(2, -4, 1) + a_3(1, -5, 7)$

or $(2, -5, 3) = (a_1 + 2a_2 + a_3, -3a_1 - 4a_2 - 5a_3, 2a_1 + a_2 + 7a_3)$.

$\therefore$

$$a_1 + 2a_2 + a_3 = 2 \qquad \ldots(i)$$

$$-3a_1 - 4a_2 - 5a_3 = -5 \qquad \ldots(ii)$$

$$2a_1 + a_2 + 7a_3 = 3 \qquad \ldots(iii)$$

On solving above equations, we get

$$a_1 = \frac{1}{4}, a_2 = \frac{3}{4}, a_3 = \frac{1}{4}$$

The values of a_1, a_2, a_3, show that the above equations are consistent. Hence the vector α can be expressed as a linear combination of the vectors, α_1, α_2, α_3. Therefore α is the subspace of R^3 generated by the vectors α_1, α_2, α_3. **Ans.**

Prob.8. Show that the set $w = \{(a, b, 0) : a, b \in R\}$ is a subspace of R^3. ***(R.G.P.V., Nov. 2018)***

Sol. Let $\alpha, \beta \in w$, $c \in R$.

Then $\alpha = (a_1, b_1, 0)$

and $\beta = (a_2, b_2, 0)$

for some $a_i, b_i, \in R$

$$\alpha + \beta = (a_1 + a_2, b_1 + b_2, 0) \in w$$

and $c\,\alpha = (ca_1, cb_1, 0) \in w$

Hence w is a subspace of R^3 **Proved**

Prob.9. Let $V = R^3$, show that w is not a subspace of V, where
$w = \{(a, b, c) : a \geq 0\}$. ***(R.G.P.V., June 2020)***

Sol. Let $w = \{\alpha \in R^3 : \alpha = (a, b, c) \text{ such that } a \geq 0\}$.

To determine whether w is not a subspace of V (= R^3).

Let $p \in w$ and $\alpha \in R$ be arbitrary, then $p = (a, b, c)$ such that $a \geq 0$.

Now, $\alpha p = (\alpha a, \alpha b, \alpha c)$.

$a \geq 0$ does not imply $\alpha a \geq 0$

(This can be verified if we take $\alpha = -3$, $a = 5$ $\Rightarrow a > 0, \alpha a < 0$)

Consequently $\alpha p \notin w$

Thus $\alpha \in R, p \in w \Rightarrow \alpha p \notin w$

So w is not a subspace of V. **Proved**

Prob.10. If $\alpha_1 = (1, 2, -1)$, $\alpha_2 = (2, -3, 2)$, $\alpha_3 = (4, 1, 3)$ and $\alpha_4 = (-3, 1, 2)$ be the vectors in $V_3(R)$ show that

$$L(\{\alpha_1, \alpha_2\}) \neq L(\{\alpha_3, \alpha_4\}).$$

Sol. Let $L(\{\alpha_1, \alpha_2\}) = L(\{\alpha_3, \alpha_4\})$.

Then there will exist scalars $a_1, a_2, a_3, a_4 \in R$, such that

$$a_1\alpha_1 + a_2\alpha_2 = a_3\alpha_3 + a_4\alpha_4$$

$$\Rightarrow \quad a_1(1, 2, -1) + a_2(2, -3, 2) = a_3(4, 1, 3) + a_4(-3, 1, 2)$$

$$\Rightarrow \quad a_1 + 2a_2 = 4a_3 - 3a_4 \quad \ldots(i)$$

$$2a_1 - 3a_2 = a_3 + a_4 \quad \ldots(ii)$$

$$-a_1 + 2a_2 = 3a_3 + 2a_4 \quad \ldots(iii)$$

Solving equations (i) and (iii), we obtain

$$a_1 = \frac{a_3 - 5a_4}{2}, \quad a_2 = \frac{7a_3 - a_4}{4}$$

Substituting these values of a_1 and a_2 in equation (ii), we get

$$2a_1 - 3a_2 = 2\left(\frac{a_3 - 5a_4}{2}\right) - 3\left(\frac{7a_3 - a_4}{4}\right) = \left(1 - \frac{21}{4}\right)a_3 - \left(5 - \frac{3}{4}\right)a_4$$

$$= -\frac{17}{4}a_3 - \frac{17}{4}a_4 \neq a_3 + a_4$$

Thus equation (ii) is not satisfied

$$L(\{\alpha_1, \alpha_2\}) \neq L(\{\alpha_3, \alpha_4\})$$ **Proved**

LINEARLY DEPENDENT, LINEARLY INDEPENDENT

Linearly Dependent – Let V(F) be a vector space. A finite set $\{\alpha_1, \alpha_2 .. \alpha_n\}$ of vectors of V is said to be ***linearly dependent*** if there exist scalar $a_1, a_2 ... a_n \in F$ not all of them 0 (some of them may be zero) such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + ... + a_n\alpha_n = 0$$

Linearly Independent – Let V(F) be a vector space. A finite set $\{\alpha_1, \alpha_2 .. \alpha_n\}$ of vectors of V is said to be ***linearly independent*** if every relation of the form

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + ... + a_n\alpha_n = 0$$

$$a_i \in F, 1 \leq i \leq n \Rightarrow a_i = 0 \text{ for each } 1 \leq i \leq n.$$

An ***infinite set*** of vectors of V is said to be linearly independent if its every finite subset is linearly independent, otherwise it is linearly dependent.

NUMERICAL PROBLEMS

Prob.11. Find whether the set of vectors $v_1 = (1, 2, 1)$, $v_2 = (3, 1, 5)$, $v_3 = (3, -4, 7)$ is linearly independent or dependent.

[R.G.P.V., June 2002 (III-Sem), Dec. 2006 (III-Sem)]

Sol. Let a_1, a_2, a_3 be three scalars such that

$$a_1v_1 + a_2v_2 + a_3v_3 = 0$$

$$\Rightarrow \quad a_1(1, 2, 1) + a_2(3, 1, 5) + a_3(3, -4, 7) = 0$$

$$\Rightarrow \quad (a_1 + 3a_2 + 3a_3, 2a_1 + a_2 - 4a_3, a_1 + 5a_2 + 7a_3) = 0$$

$$\Rightarrow \quad a_1 + 3a_2 + 3a_3 = 0$$

$$2a_1 + a_2 - 4a_3 = 0$$

$$a_1 + 5a_2 + 7a_3 = 0$$

The coefficients matrix of these equation is

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{bmatrix}, \qquad \therefore\ |A| = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{vmatrix}$$

$$= 1\,(7 + 20) - 3\,(14 + 4) + 3\,(10 - 1) = 27 - 54 + 27 = 0$$

and $\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 6 = -5 \neq 0,$

$\therefore\quad \rho(A) = 2.$

i.e., so the rank of matrix A < no. of unknown quantities.

The system of equations will have $3 - 2 = 1$ non-zero solutions and hence the set of vectors are linearly dependent. **Ans.**

Prob.12. Determine whether or not the vectors u = (1, 1, 2), v = (2, 3, 1), w = (4, 5, 5) in R^3 are linearly dependent.

(R.G.P.V., Nov. 2019, June 2020)

Sol. Let a_1, a_2, a_3 be three scalars such that

$$a_1u + a_2v + a_3w = 0$$

$\Rightarrow\quad a_1\,(1, 1, 2) + a_2\,(2, 3, 1) + a_3\,(4, 5, 5) = 0$

$\Rightarrow\quad (a_1 + 2a_2 + 4a_3,\ a_1 + 3a_2 + 5a_3,\ 2a_1 + a_2 + 5a_3) = 0$

$$\Rightarrow\qquad \begin{aligned} a_1 + 2a_2 + 4a_3 &= 0 \\ a_1 + 3a_2 + 5a_3 &= 0 \\ 2a_1 + a_2 + 5a_3 &= 0 \end{aligned}$$

The coefficient matrix of these equation is

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 2 & 1 & 5 \end{bmatrix}$$

$$\therefore\qquad |A| = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 2 & 1 & 5 \end{bmatrix}$$

$$= 1\,(15 - 5) - 2(5 - 10) + 4(1 - 6)$$

$$= 10 + 10 - 20 = 0$$

and $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = 3 - 2 = 1 \neq 0$

$\therefore\qquad \rho(A) = 2$

i.e., so the rank of matrix A < no. of unknown quantities.

The system of equations will have $3 - 2 = 1$, non-zero solutions and hence the set of vectors are linearly dependent. **Ans.**

Prob.13. Show that the set {1, x, 1 + x + x^2} is a linearly independent set of vectors in the vector space of all polynomial over the real number field. ***[R.G.P.V., June 2004 (III-Sem)]***

Sol. Let a, b, c be scalars (real numbers) such that

$$a(1) + bx + c(1 + x + x^2) = 0$$

We have $(a + c) + (b + c)x + cx^2 = 0$

$\Rightarrow$ $a + c = 0, b + c = 0, c = 0$

$\Rightarrow$ $a = 0, b = 0, c = 0$

Therefore the vectors 1, x, $1 + x + x^2$ are linearly independent over the field of real numbers. **Proved**

Prob.14. Show that the set S of vectors (1, 0, 0), (1, 1, 0) and (1, 1, 1) is linearly independent. ***(R.G.P.V., May 2019)***

Sol. Let a_1, a_2, a_3 be three scalars (real numbers) such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$$

Here $\alpha_1 = (1, 0, 0)$, $\alpha_2 = (1, 1, 0)$ and $\alpha_3 = (1, 1, 1)$

$\therefore$ $a_1(1, 0, 0) + a_2(1, 1, 0) + a_3(1, 1, 1) = 0$

$\Rightarrow$ $(a_1 + a_2 + a_3, a_2 + a_3, a_3) = (0, 0, 0)$

$\Rightarrow$ $a_1 + a_2 + a_3 = 0$

$a_2 + a_3 = 0$

$a_3 = 0$

$\Rightarrow$ $a_1 = 0, a_2 = 0, a_3 = 0$

Therefore the vectors $a_1 + a_2 + a_3$, $a_2 + a_3$, a_3 are linearly independent over the field of real numbers. **Proved**

Prob.15. Are the vectors (2, 2, 2, 4), (2, –2, –4, 0), (4, –2, –5, 2), (4, 2, 1, 6) linearly independent ?

[R.G.P.V., Dec. 2001 (III-Sem), June 2007 (III-Sem)]

Sol. Let a_1, a_2, a_3 and a_4 be four scalars such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0$$

Here $\alpha_1 = (2, 2, 2, 4)$, $\alpha_2 = (2, -2, -4, 0)$, $\alpha_3 = (4, -2, -5, 2)$

and $\alpha_4 = (4, 2, 1, 6)$

$\therefore$ $a_1(2, 2, 2, 4) + a_2(2, -2, -4, 0) + a_3(4, -2, -5, 2) + a_4(4, 2, 1, 6) = 0$

$\Rightarrow$ $(2a_1 + 2a_2 + 4a_3 + 4a_4, 2a_1 - 2a_2 - 2a_3 + 2a_4, 2a_1 - 4a_2 - 5a_3 + a_4$
$4a_1 + 2a_3 + 6a_4) = (0, 0, 0, 0)$

$\Rightarrow$ $2a_1 + 2a_2 + 4a_3 + 4a_4 = 0$

$2a_1 - 2a_2 - 2a_3 + 2a_4 = 0$

$2a_1 - 4a_2 - 5a_3 + a_4 = 0$

$4a_1 + 2a_3 + 6a_4 = 0$

The coefficients matrix of these equation is

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{bmatrix}, \qquad \therefore \ |A| = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{vmatrix}$$

$$= 1\,(7+20) - 3\,(14+4) + 3\,(10-1) = 27 - 54 + 27 = 0$$

and $\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 6 = -5 \neq 0,$

$\therefore \quad \rho(A) = 2.$

i.e., so the rank of matrix A < no. of unknown quantities.

The system of equations will have $3 - 2 = 1$ non-zero solutions and hence the set of vectors are linearly dependent. **Ans.**

Prob.12. Determine whether or not the vectors $u = (1, 1, 2)$, $v = (2, 3, 1)$, $w = (4, 5, 5)$ in R^3 are linearly dependent.

(R.G.P.V., Nov. 2019, June 2020)

Sol. Let a_1, a_2, a_3 be three scalars such that

$$a_1u + a_2v + a_3w = 0$$

$$\Rightarrow \quad a_1(1, 1, 2) + a_2(2, 3, 1) + a_3(4, 5, 5) = 0$$

$$\Rightarrow \quad (a_1 + 2a_2 + 4a_3,\ a_1 + 3a_2 + 5a_3,\ 2a_1 + a_2 + 5a_3) = 0$$

$$\Rightarrow \quad \begin{aligned} a_1 + 2a_2 + 4a_3 &= 0 \\ a_1 + 3a_2 + 5a_3 &= 0 \\ 2a_1 + a_2 + 5a_3 &= 0 \end{aligned}$$

The coefficient matrix of these equation is

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 2 & 1 & 5 \end{bmatrix}$$

$$\therefore \qquad |A| = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 2 & 1 & 5 \end{bmatrix}$$

$$= 1\,(15-5) - 2(5-10) + 4(1-6)$$

$$= 10 + 10 - 20 = 0$$

and $\qquad \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = 3 - 2 = 1 \neq 0$

$\therefore \qquad \rho(A) = 2$

i.e., so the rank of matrix A < no. of unknown quantities.

The system of equations will have $3 - 2 = 1$, non-zero solutions and hence the set of vectors are linearly dependent. **Ans.**

Prob.13. Show that the set {1, x, 1 + x + x²} is a linearly independent set of vectors in the vector space of all polynomial over the real number field. *[R.G.P.V., June 2004 (III-Sem)]*

Sol. Let a, b, c be scalars (real numbers) such that

$$a(1) + bx + c(1 + x + x^2) = 0$$

We have $\quad (a + c) + (b + c)x + cx^2 = 0$

$\Rightarrow \quad a + c = 0,\ b + c = 0,\ c = 0$

$\Rightarrow \quad a = 0,\ b = 0,\ c = 0$

Therefore the vectors $1, x, 1 + x + x^2$ are linearly independent over the field of real numbers. **Proved**

Prob.14. Show that the set S of vectors (1, 0, 0), (1, 1, 0) and (1, 1, 1) is linearly independent. *(R.G.P.V., May 2019)*

Sol. Let a_1, a_2, a_3 be three scalars (real numbers) such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$$

Here $\alpha_1 = (1, 0, 0)$, $\alpha_2 = (1, 1, 0)$ and $\alpha_3 = (1, 1, 1)$

$\therefore \quad a_1(1, 0, 0) + a_2(1, 1, 0) + a_3(1, 1, 1) = 0$

$\Rightarrow \quad (a_1 + a_2 + a_3, a_2 + a_3, a_3) = (0, 0, 0)$

$\Rightarrow \quad a_1 + a_2 + a_3 = 0$

$a_2 + a_3 = 0$

$a_3 = 0$

$\Rightarrow \quad a_1 = 0,\ a_2 = 0,\ a_3 = 0$

Therefore the vectors $a_1 + a_2 + a_3, a_2 + a_3, a_3$ are linearly independent over the field of real numbers. **Proved**

Prob.15. Are the vectors (2, 2, 2, 4), (2, –2, –4, 0), (4, –2, –5, 2), (4, 2, 1, 6) linearly independent ?

[R.G.P.V., Dec. 2001 (III-Sem), June 2007 (III-Sem)]

Sol. Let a_1, a_2, a_3 and a_4 be four scalars such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0$$

Here $\quad \alpha_1 = (2, 2, 2, 4)$, $\alpha_2 = (2, -2, -4, 0)$, $\alpha_3 = (4, -2, -5, 2)$

and $\quad \alpha_4 = (4, 2, 1, 6)$

$\therefore \quad a_1(2, 2, 2, 4) + a_2(2, -2, -4, 0) + a_3(4, -2, -5, 2) + a_4(4, 2, 1, 6) = 0$

$\Rightarrow \quad (2a_1 + 2a_2 + 4a_3 + 4a_4, 2a_1 - 2a_2 - 2a_3 + 2a_4, 2a_1 - 4a_2 - 5a_3 + a_4$
$4a_1 + 2a_3 + 6a_4) = (0, 0, 0, 0)$

$\Rightarrow \quad 2a_1 + 2a_2 + 4a_3 + 4a_4 = 0$

$2a_1 - 2a_2 - 2a_3 + 2a_4 = 0$

$2a_1 - 4a_2 - 5a_3 + a_4 = 0$

$4a_1 + 2a_3 + 6a_4 = 0$

The coefficient matrix of these equation is

$$A = \begin{bmatrix} 2 & 2 & 4 & 4 \\ 2 & -2 & -2 & 2 \\ 2 & -4 & -5 & 1 \\ 4 & 0 & 2 & 6 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ and $R_4 \to R_4 - 2R_1$, we get

$$A = \begin{bmatrix} 2 & 2 & 4 & 4 \\ 0 & -4 & -6 & -2 \\ 0 & -6 & -9 & -3 \\ 0 & -4 & -6 & -2 \end{bmatrix}$$

Applying $R_3 \to 2R_3 - 3R_2$ and $R_4 \to R_4 - R_2$, we get

$$A = \begin{bmatrix} 2 & 2 & 4 & 4 \\ 0 & -4 & -6 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \qquad \rho(A) = 2$$

i.e., so the rank of matrix A < no. of unknown quantities.

The system of equations will have $4 - 2 = 2$, non-zero solutions and hence the set of vectors are linearly dependent. Hence given vectors are not linearly independent. **Ans.**

Prob.16. Prove that, if two vectors are linearly dependent, then one of them is a scalar multiple of other.

Sol. Let V be a vector space over the field F and α, β be two linearly dependent vectors of V. Then there exists two scalars, $a, b \in F$ not all zero such that $a\alpha + b\beta = 0$

If possible let $a \neq 0,\ a \in F \Rightarrow a^{-1} \in F$

Now, we have $a\alpha = -b\beta$

$$\Rightarrow \quad a^{-1}(a\alpha) = a^{-1}(-b\beta)$$

$$\Rightarrow \quad (a.^{-1}a)\ \alpha = (-a.^{-1}b)\beta$$

$$\Rightarrow \quad 1.\alpha = [(-a^{-1})b]\beta \qquad (\because\ a^{-1}a = 1)$$

$$\Rightarrow \quad \alpha = -\frac{b}{a}\beta$$

$\Rightarrow$ α is a scalar multiple of β.

Similarly, by taking $b \neq 0$, we can express β as a scalar multiple of α.

Proved

Prob.17. Show that the set of polynomials

$$p(x) = 1 + x + 2x^2,\ q(x) = 2 - x + x^2,\ r(x) = -4 + 5x + x^2$$

over the vector space R[x] is linearly dependent.

Sol. Let a_1, a_2 and a_3 be three scalars such that

$$a_1.p(x) + a_2.q(x) + a_3.r(x) = 0(x) \qquad \text{(zero polynomial)}$$

$$\Rightarrow \quad a_1(1 + x + 2x^2) + a_2(2 - x + x^2) + a_3(-4 + 5x + x^2) = 0(x)$$

$$\Rightarrow \quad (2a_1 + a_2 + a_3)x^2 + (a_1 - a_2 + 5a_3)x + (a_1 + 2a_2 - 4a_3) = 0x^2 + 0x + 0$$

Then by definition of equality of two polynomials, we get

$$2a_1 + a_2 + a_3 = 0$$
$$a_1 - a_2 + 5a_3 = 0$$
$$a_1 + 2a_2 - 4a_3 = 0$$

The coefficient matrix of these equations is

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 5 \\ 1 & 2 & -4 \end{bmatrix}$$

Applying $R_2 \to 2R_2 - R_1$, $R_3 \to 2R_3 - R_1$, we get

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -3 & 9 \\ 0 & 3 & -9 \end{bmatrix}$$

Applying $R_3 \to R_3 + R_2$, we get

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -3 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \qquad \rho(A) = 2$$

The given equations may be written as

$$2a_1 + a_2 + a_3 = 0$$
$$-3a_2 + 9a_3 = 0$$

Now $$\frac{a_1}{9+3} = \frac{-a_2}{18} = \frac{a_3}{-6} \text{ or } \frac{a_1}{2} = \frac{a_2}{-3} = \frac{a_3}{-1}$$

$$\therefore \qquad a_1 = 2,\ a_2 = -3,\ a_3 = -1$$

Since $\rho(A) <$ no. of unknown quantities, therefore a_1, a_2 and a_3 are not all zero so the given set of vectors are linearly dependent. **Ans.**

BASIS OF A VECTOR SPACE

Basis of a Vector Space –

A subset S of a vector space V(F) is said to be ***a basis of V(F)*** if

(i) S consists of linearly independent vectors.

(ii) S generates V(F) i.e., L(S) = V, i.e., each vector in V is a linear combination of a finite number of elements of S.

Finite Dimensional Vector Spaces –

The vector space V(F) is said to be ***finite dimensional or finitely generated*** if there exists a finite subset S of V such that V = L(S). The vector space $V_n(F)$ of n-tuples is a finite dimensional vector space.

The vector space F[x] of all polynomials over a field F is not finite dimensional. There exists no finite subset S of F[x] which spans F[x].

A vector space which is not finitely generated may be referred to as an infinite dimensional space.

Coordinates and Coordinate Vectors – The n vectors $e_1 = (1, 0, 0, ... 0)$, $e_2 = (0, 1, 0, 0, ... 0)$ ..... $e_n = (0, 0, ... 0,1)$ are called unit (or elementary) vectors over field F. The unit vector e_m, whose mth components is 1 is known as the mth unit vector.

$\{e_1, e_2 ... e_n\}$ is an important (standard) basis for $V_n(F)$.

Every vector $v \in V_n(F)$ can be expressed uniquely as

$$v = a_1e_1 + a_2e_2 + ... + a_ne_n = \sum_{i=1}^{n} a_ie_i$$

The components $a_1, a_2, ... a_n$ are called the coordinates of v with respect,to the basis $\{e_1, e_2 ... e_n)$ and $(a_1, a_2 .... a_n)$ is called the coordinate vector of v with respect to (or relative to) the basis $\{e_1, e_2, ... e_n\}$ or $\{e_i\}$ and this vector is denoted by $[V]_e$ or $[v]$, i.e.,

$$[v] = (a_1, a_2 .... a_n)$$

Theorem 9. *Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a basis of a finite dimensional vector space V(F) of dimensions n. Then every element α of V can be uniquely expressed as*

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n \quad \text{where } a_1, a_2, ...a_n \in F$$

Proof. Since S is a basis of V

$\therefore \qquad L(S) = V.$

Any vector $\alpha \in V$ can be expressed as $\alpha = a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$.

If possible, let us assume that it has another representation, say

$$\alpha = b_1\alpha_1 + b_2\alpha_2 + ...+b_n\alpha_n.$$

Then we have to prove that $a_1 = b_1, a_2 = b_2, ..., a_n = b_n$

Also $\quad a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n = b_1\alpha_1 + b_2\alpha_2 + ... + b_n\alpha_n$

$\Rightarrow \quad (a_1 - b_1)\alpha_1 + (a_2 - b_2)\alpha_2 + ... + (a_n - b_n)\alpha_n = 0$

(as $\alpha_1, \alpha_2, ..., \alpha_n$ are linearly independent)

$\Rightarrow \quad a_1 - b_1 = 0, a_2 - b_2 = 0, ..., a_n - b_n = 0$

$\Rightarrow \quad a_1 = b_1, a_2 = b_2, ..., a_n = b_n$

and hence the result. **Proved**

Co-ordinate Representation of a Vector Space – Basis plays an important role in introducing the co-ordinates in a finite dimensional vector space let dimension of V is n. Corresponding to each $\alpha \in V$, there exists one and only one n-tuple $(a_1, a_2, ..., a_n)$ in V. Which may be called as co-ordinates of α in V.

Let V(F) be a finite dimensional vector space of dimensions n over a field F and $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$
be a basis of V. Then there exists a unique n-tuples $\{a_1, a_2, ..., a_n\}$ of scalar, such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + ...+ a_n\alpha_n.$$

The n-tuple $(a_1, a_2, ..., a_n)$ is called the n-tuple of co-ordinates of α relative to the ordered basis S. The scalar a_i is called the ith co-ordinate of α relative to the ordered basis S.

NUMERICAL PROBLEMS

Prob.18. Prove that the system S consisting on n vectors

$$e_1 = (1, 0, 0 .... 0), \ e_2 = (0, 1, 0, 0 .... 0)$$
$$e_n = (0, 0, ... 0, 1) \text{ is a basis of } V_n(F).$$

[R.G.P.V., Dec. 2004 (III-Sem)]

Sol. Let $a_1, a_2, ... a_n$ be scalars such that

$$a_1e_1 + a_2e_2 + ... + a_ne_n = 0$$
$$\Rightarrow a_1(1, 0, 0, ... 0) + a_2(0, 1, 0, 0, ... 0) + ... + a_n(0, 0, ... 0, 1) = (0, 0, ... 0)$$
$$\Rightarrow (a_1, a_2, ... a_n) = (0, 0, ... 0)$$
$$\Rightarrow a_1 = a_2 = ... = a_n = 0$$

Hence, S is linearly independent.

Now we should prove that $L(S) = V_n(F)$. We have always $L(S) \subseteq V_n(F)$. Therefore we should prove that $V_n(F) \subseteq L(S)$. i.e., each vector in $V_n(F)$ is a linear combination of elements of S.

Suppose, $\alpha = (a_1, a_2 .... a_n)$ be any vector in $V_n(F)$. We can write

$$(a_1, a_2 ...., a_n) = a_1 (1, 0, 0 .... 0) + a_2 (0, 1, 0 .... 0) + ... + a_n (0, 0 ..... 0, 1)$$

i.e. $\alpha = a_1e_1 + a_2e_2 + ... + a_ne_n$

Hence S is a basis of $V_n(F)$. We shall call this particular basis the standard basis of $V_n(F)$.

Proved

Prob.19. Show that the set of vectors S = {a, b, c} forms a basis for R^3, where, $\alpha_1 = (1, 2, 1)$, $\alpha_2 = (2, 9, 0)$, $\alpha_3 = (3, 3,4)$.

Sol. Let a_1, a_2 and a_3 be three scalars such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$$
$$\Rightarrow a_1 (1, 2, 1) + a_2 (2, 9, 0) + a_3 (3, 3, 4) = (0, 0, 0)$$
$$\Rightarrow (a_1 + 2a_2 + 3a_3, 2a_1 + 9a_2 + 3a_3, a_1 + 4a_3) = (0, 0, 0)$$
$$\Rightarrow a_1 + 2a_2 + 3a_3 = 0$$
$$2a_1 + 9a_2 + 3a_3 = 0$$
$$a_1 + 4a_3 = 0$$

The coefficient matrix of the above equations is

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 9 & 3\\ 1 & 0 & 4\end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$, we get

$$A=\begin{bmatrix}1 & 2 & 3\\ 0 & 5 & -3\\ 0 & -2 & 1\end{bmatrix}$$

Applying $R_3 \to 5R_3 + 2R_2$, we get

$$A=\begin{bmatrix}1 & 2 & 3\\ 0 & 5 & -3\\ 0 & 0 & -1\end{bmatrix}$$

$\therefore \qquad \rho(A) = 3$

$\Rightarrow \qquad a_1 = a_2 = a_3 = 0.$

Hence, S is linearly independent.

Now we will show that $L(S) = R^3$.

Let $(a, b, c) \in S$, and $(a, b, c) = a_1 (1, 2, 1) + a_2 (2, 9, 0) + a_3 (3, 3, 4)$

$\Rightarrow \qquad (a, b, c) = (a_1 + 2a_2 + 3a_3, 2a_1 + 9a_2 + 3a_3, a_1 + 4a_3)$

$$\Rightarrow \qquad \begin{aligned} a_1 + 2a_2 + 3a_3 &= a\\ 2a_1 + 9a_2 + 3a_3 &= b\\ a_1 + 4a_3 &= c\end{aligned}$$

Above can be written as

$$\begin{bmatrix}1 & 2 & 3\\ 2 & 9 & 3\\ 1 & 0 & 4\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix}=\begin{bmatrix}a\\ b\\ c\end{bmatrix}$$

Operate $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix}1 & 2 & 3\\ 0 & 5 & -3\\ 0 & -2 & 1\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix}=\begin{bmatrix}a\\ b-2a\\ c-a\end{bmatrix}$$

Operate $R_3 \to 5R_3 + 2R_2$, we get

$$\begin{bmatrix}1 & 2 & 3\\ 0 & 5 & -3\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix}=\begin{bmatrix}a\\ b-2a\\ 5c-9a+2b\end{bmatrix}$$

which show that for $a_1, a_2, a_3 \in R$ any element. (a, b, c) of S may be expressed as linear combination of elements of S; i.e.,

$$L(S) = R^3.$$

Since S consists of linearly independent vectors and also

$L(S) = R^3$, so S is a basis of $V_3(R)$. **Proved**

Prob.20. Show that the set –

$$S = \{(1, 2, 1), (2, 1, 0), (1, -1, 2)\}$$

forms a basis of $V_3(F)$.

[R.G.P.V., Dec. 2003 (III-Sem), June 2004 (III-Sem)]

Sol. Let

$$v_1 = (1, 2, 1), v_2 = (2, 1, 0), v_3 = (1, -1, 2)$$

Let $a_1v_1 + a_2v_2 + a_2v_3 = 0$, where $a_1, a_2, a_3 \in F$

$\Rightarrow$ $a_1 (1, 2, 1) + a_2 (2, 1, 0) + a_3 (1, -1, 2) = 0$

$\Rightarrow$ $(a_1 + 2a_2 + a_3, 2a_1 + a_2 - a_3, a_1 + 2a_3) = (0, 0, 0)$

$\Rightarrow$ $a_1 + 2a_2 + a_3 = 0, 2a_1 + a_2 - a_3 = 0, a_1 + 2a_3 = 0$

$\Rightarrow$ $a_1 = 0 = a_2 = a_3$

Therefore the given set of vectors is linearly independent.

Again let the unit vector $(1, 0, 0) = av_1 + bv_2 + cv_3$, where $a, b, c \in F$.

Then $(1, 0, 0) = a (1, 2, 1) + b (2, 1, 0) + c (1, -1, 2)$

or $(1, 0, 0) = (a + 2b + c, 2a + b - c, a + 2c)$

or $a + 2b + c = 1, 2a + b - c = 0, a + 2c = 0$

On solving above equation, we get

$$a = -\frac{2}{9}, b = \frac{5}{9}, c = \frac{1}{9}$$

$$\therefore \quad (1, 0, 0) = -\frac{2}{9}(1,2,1)+\frac{5}{9}(2,1,0)+\frac{1}{9}(1,-1,2)$$

Similary we can show that

$$(0, 1, 0) = \frac{4}{9}(1,2,1)-\frac{1}{9}(2,1,0)-\frac{2}{9}(1,-1,2)$$

Also $V_3(F)$ is generated by the unit vectors (1, 0, 0), (0, 1, 0), (0, 0, 1) and every element of $V_3(F)$ is a linear combination of the given set as the unit vectors are them selves linear combinations of the given set of vectors as shown above.

Hence the given set of vectors forms a basis of $V_3(F)$. **Proved**

Prob.21. Show that the vectors (1, 0, –1), (1, 2, 1), (0, –3, 2) form a basis for $X_3(R)$. *[R.G.P.V., Dec. 2001 (III-Sem)]*

Sol. Let $S = \{ \alpha_1, \alpha_2, \alpha_3\}$

where $\alpha_1 = (1, 0, -1), \alpha_2 = (1, 2, 1), \alpha_3 = (0, -3, 2)$

Consider three scalars $a_1, a_2, a_3 \in R$ such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$$

$\Rightarrow$ $a_1(1, 0, -1) + a_2(1, 2, 1) + a_3(0, -3, 2) = 0$

$\Rightarrow$ $(a_1 + a_2, 2a_2 - 3a_3, -a_1 + a_2 + 2a_3) = (0, 0, 0)$

$$\Rightarrow \quad a_1 + a_2 = 0$$
$$2a_2 - 3a_3 = 0$$
$$-a_1 + a_2 + 2a_3 = 0$$

The coefficients matrix of these equations is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{bmatrix}$$

Applying $R_3 \to R_3 + R_1$, we get

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 2 & 2 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\therefore \quad \rho(A) = 3$$

So, $\rho(A)$ = No. of unknown quantities

$$\therefore \quad a_1 = a_2 = a_3 = 0.$$

Hence, the system S consists of linearly independent vectors.

Now we shall show that S generates $X_3(R)$.

Let $(a, b, c) \in X_3(R)$. Then

$$(a, b, c) = a_1 (1, 0, -1) + a_2 (1, 2, 1) + a_3 (0, -3, 2)$$
$$\Rightarrow \quad (a, b, c) = (a_1 + a_2, 2a_2 - 3a_3, -a_1 + a_2 + 2a_3)$$
$$\Rightarrow \quad a_1 + a_2 = a$$
$$2a_3 - 3a_3 = b$$
$$-a_1 + a_2 + 2a_3 = c$$

The coefficient matrix of the above equations is given by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{bmatrix}, \quad |A| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{vmatrix}$$

$$\therefore \quad |A| = 1(4 + 3) - 1(0 - 3) = 7 + 3 = 10 \neq 0.$$

The above equations have unique solution.

$\therefore$ $(a, b, c) \in X_3(R)$ can be expressed as linear combination of the vectors of S. Hence S is a basis of $X_3(R)$. **Proved**

Prob.22. Show that the vectors (1, 0, 0), (1, 1, 0), (1, 1, 1) form a basis for R^3. ***[R.G.P.V., June 2007 (III-Sem)]***

Sol. Let the set

$$S = \{\alpha_1, \alpha_2, \alpha_3\}$$

where $\alpha_1 = (1, 0, 0)$, $\alpha_2 = (1, 1, 0)$ and $\alpha_3 = (1, 1, 1)$

Consider three scalars $a_1, a_2, a_3 \in R$, such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$$

$$\Rightarrow \quad a_1 (1, 0, 0) + a_2 (1, 1, 0) + a_3 (1, 1, 1) = 0$$

$$\Rightarrow \quad (a_1 + a_2 + a_3, a_2 + a_3, a_3) = (0, 0, 0)$$

$$\Rightarrow \quad a_1 + a_2 + a_3 = 0$$

$$\Rightarrow \quad a_2 + a_3 = 0$$

$$\Rightarrow \quad a_3 = 0$$

The coefficients matrix of these equations is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying operations $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$, we get

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ Rank of matrix, $\rho(A) = 3$

So, $\rho(A)$ = No. of unknown quantities

$$a_1 = a_2 = a_3 = 0$$

Hence, the systems consist S of linearly independent vectors

Now we shall show that S generates $V_3(R)$.

Let $(a, b, c) \in V_3(R)$, then

$$(a, b, c) = a_1 (1, 0, 0) + a_2 (1, 1, 0) + a_3 (1, 1, 1)$$

$$\Rightarrow \quad a_1 + a_2 + a_3 = a$$

$$a_2 + a_3 = b$$

$$a_3 = c$$

The coefficient matrix of the above equations is given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

or

$$|A| = 1 (1 - 0) - 1 (0 - 0) + 1 (0 - 0) = 1 \neq 0$$

The above equations has unique solution.

$\therefore$ $(a, b, c) \in V_3(R)$ can be expressed as linear combination of the vectors of S. Hence S is a basis of $V_3(R)$. **Proved**

Prob.23. Show that the vectors (2, 1, 4), (1, –1, 2) and (3, 1, –2) from a basis for R^3. ***(R.G.P.V., Nov. 2018)***

Sol. Let the set

$$S = \{\alpha_1, \alpha_2, \alpha_3\}$$

where $\alpha_1 = (2, 1, 4)$, α_2 $(1, -1, 2)$ and $\alpha_3 = (3, 1, -2)$

Consider three scalars $a_1, a_2, a_3 \in R$, such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$$

$$\Rightarrow \quad a_1(2, 1, 4) + a_2(1, -1, 2) + a_3(3, 1, -2) = 0$$

$$\Rightarrow \quad (2a_1 + a_2 + 3a_3, a_1 - a_2 + a_3, 4a_1 + 2a_2 - 2a_3) = (0, 0, 0)$$

$$\Rightarrow \quad 2a_1 + a_2 + 3a_3 = 0$$

$$\Rightarrow \quad a_1 - a_2 + a_3 = 0$$

$$\Rightarrow \quad 4a_1 + 2a_2 - 2a_3 = 0$$

The coefficients matrix of these equations is

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$$

Applying $R_1 \leftrightarrow R_2$, we have

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 4 & 2 & -2 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 4R_1$, we have

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & 6 & -6 \end{bmatrix}$$

Applying $C_2 \to C_2 + C_1$ and $C_3 \to C_3 - C_1$, we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 6 & -6 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$, we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & -8 \end{bmatrix}$$

Applying $C_2 \to \frac{1}{3}C_2$, we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -8 \end{bmatrix}$$

Applying $C_3 \to C_3 - C_2$, we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -8 \end{bmatrix}$$

Applying $C_3 \to -\frac{1}{8}C_3$, we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ Rank of matrix, $\rho(A) = 3$

So, $\rho(A)$ = No. of unknown quantities

$$a_1 = a_2 = a_3 = 0$$

Hence, the systems consist S of linearly independent vectors.

Now we shall show that S generates $V_3(R)$.

Let $(a, b, c) \in V_3(R)$, then

$(a, b, c) = a_1(2, 1, 4) + a_2(1, -1, 2) + a_3(3, 1, -2)$

$\Rightarrow \quad 2a_1 + a_2 + 3a_3 = a$

$a_1 - a_2 + a_3 = b$

$4a_1 + 2a_2 - 2a_3 = c$

The coefficient matrix of the above equations is given by

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$$

Then $$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{vmatrix}$$

or $$|A| = 2(2-2) - 1(-2-4) + 3(2+4)$$
$$= 0 + 6 + 18 = 24 \neq 0$$

The above equations has unique solution.

$\therefore$ $(a, b, c) \in V_3(R)$ can be expressed as linear combination of the vectors of S. Hence S is a basis of $V_3(R)$. **Proved**

Prob.24. If $V_3(R)$ is a vector space then show that

$S = \{(0, 1, -1), (1, 1, 0), (1, 0, 2)\}$ is basis of V_3 and hence find the coordinates of vector $(1, 0, -1)$ with respect to this basis.

[R.G.P.V., Dec. 2005 (III-Sem)]

Sol. Let $S = \{\alpha_1, \alpha_2, \alpha_3\}$

where $\alpha_1 = (0, 1, -1), \alpha_2 = (1, 1, 0)$ and $\alpha_3 = (1, 0, 2)$

Consider the three scalar $a_1, a_2, a_3 \in R$ such that

$$a_1 \alpha_1 + a_2 \alpha_2 + a_3 \alpha_3 = 0$$

$$\Rightarrow \quad a_1(0, 1, -1) + a_2 (1, 1, 0) + a_3 (1, 0, 2) = (0, 0, 0)$$

$$\Rightarrow \quad (a_2 + a_3, a_1 + a_2, -a_1 + 2a_3) = (0, 0, 0)$$

$$\Rightarrow \quad a_2 + a_3 = 0$$

$$a_1 + a_2 = 0 \text{ and } -a_1 + 2a_3 = 0$$

$$\Rightarrow \quad a_1 = 0 = a_2 = a_3 \qquad \text{(on solving)}$$

Therefore the given set of vectors is linearly independent.

Now we shall show that $L(S) = V_3(R)$

Let $a, b, c, \in S$ and

$$(a, b, c) = a_1 (0, 1, -1) + a_2 (1, 1, 0) + a_3 (1, 0, 2)$$

or

$$(a, b, c) = (a_2 + a_3, a_1 + a_2, -a_1 + 2a_3)$$

$$\therefore \quad a_2 + a_3 = a$$

$$a_1 + a_2 = b$$

$$-a_1 + 2a_3 = c$$

The coefficient matrix of the above equations is given by

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

$$\because \quad |A| = 0(2-0) - 1(2-0) + 1(0+1) = -1 \neq 0$$

Clearly $\rho(A) = 3$

$\Rightarrow$ The above equation have unique solutions

$$L(S) = V_3(R)$$

$\therefore$ S forms a basis of $V_3(R)$ **Proved**

Now we shall find out coordinate vector.

Let a'_1, a'_2, a'_3 be unknown scalars such that

$$a'_1\alpha_1 + a'_2\alpha_2 + a'_3\alpha_3 = \alpha$$

$$a'_1(0, 1, -1) + a'_2(1, 1, 0) + a'_3(1, 0, 2) = (1, 0, -1)$$

$$(a'_2 + a'_3, a'_1 + a'_2, -a'_1 + 2a'_3) = (1, 0, -1)$$

$$\Rightarrow \quad a'_2 + a'_3 = 1 \qquad .....(i)$$

$$a'_1 + a'_2 = 0 \qquad .....(ii)$$

$$-a'_1 + 2a'_3 = -1 \qquad ...(iii)$$

On solving above equations, we get

$$a'_1 = -3, \quad a'_2 = 3, \quad a'_3 = -2$$

Hence the coordinate vector is **$(-3, 3, -2)$** **Ans.**

Prob.25. Show that the following set is a basis of $V_4(R)$ – (1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1). *[R.G.P.V., June 2002 (III-Sem)]*

Sol. Let $\alpha_1 = (1, 0, 0, 0), \alpha_2 = (1, 1, 0, 0), \alpha_3 = (1, 1, 1, 0)$

and $\alpha_4 = (1, 1, 1, 1)$

$\therefore$ $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$

Let a_1, a_2, a_3 and a_4 be four scalars such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0$$

$$\Rightarrow a_1(1, 0, 0, 0) + a_2(1, 1, 0, 0) + a_3(1, 1, 1, 0) + a_4(1, 1, 1, 1) = 0$$

$$\Rightarrow (a_1 + a_2 + a_3 + a_4, a_2 + a_3 + a_4, a_3 + a_4, a_4) = (0, 0, 0, 0)$$

$$\Rightarrow a_1 + a_2 + a_3 + a_4 = 0$$

$$a_2 + a_3 + a_4 = 0$$

$$a_3 + a_4 = 0$$

$$a_4 = 0$$

The coefficients matrix of these equations is

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$ and $R_3 \to R_3 - R_4$, we get

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $\rho(A) = 4$

So $\rho(A) =$ No. of unknown quantities

$\therefore$ $a_1 = a_2 = a_3 = a_4 = 0.$

$\Rightarrow$ S is linearly independent.

Now we shall show that $L(S) = V_4(R)$

Let a, b, c, d$\in$S and $(a, b, c, d) = a_1(1, 0, 0, 0) + a_2(1, 1, 0, 0) + a_3(1, 1, 1, 0) + a_4(1, 1, 1, 1)$

or $(a, b, c, d) = (a_1 + a_2 + a_3 + a_4, a_2 + a_3 + a_4, a_3 + a_4, a_4)$

$$\Rightarrow a_1 + a_2 + a_3 + a_4 = a$$

$$a_2 + a_3 + a_4 = b$$

$$a_3 + a_4 = c$$

$$a_4 = d$$

The coefficient matrix of the above equations is given by

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Clearly $\rho(A) = 4$

$\Rightarrow$ The above equations have unique solutions, $L(S) = V_4(R)$

$\therefore$ S forms a basis of $V_4(R)$ **Proved**

Prob.26. Prove that the set $S = \{a + ib, c + id\}$ is a basis set of $C(R)$ if $ad - bc \neq 0$

Sol. Let $a_1, a_2 \in R$, such that

$$a_1\alpha_1 + a_2\alpha_2 = 0$$

Here $\alpha_1 = a + ib, \alpha^2 = c + id$

$\therefore$ $a_1 (a + ib) + a_2 (c + id) = (0, 0)$

$\Rightarrow$ $(a_1a + a_2c) + i (a_1b + a_2d) = (0, 0)$

$\Rightarrow$ $a_1a + a_2c = 0$ ...(i)

and $a_1b + a_2d = 0$ ...(ii)

Eliminating a_1 from above two equations, we get

$$a_2 (cb - ad) = 0$$

i.e., either $a_2 = 0$ or $bc - ad = 0$ or both

$\Rightarrow$ $a_2 = 0$ only if $bc - ad \neq 0$.

Therefore, S is linearly independent set of vectors, if $ad - bc \neq 0$, and hence will form a basis set. **Proved**

Prob.27. In the vector space R^3, let $\alpha_1 = (1, 2, 1)$, $\alpha_2 = (3, 1, 5)$ and $\alpha_3 = (3, -4, 7)$, show that there exists more than one basis for the subspace spanned by the set $T = \{\alpha_1, \alpha_2, \alpha_3\}$

Sol. Let $S = \{\alpha_1, \alpha_2\}$

then we will show that S is linearly independent and also

$$L(S) = L(T).$$

Consider,

$$a_1\alpha_1 + a_2\alpha_2 = 0$$

$\Rightarrow$ $a_1 (1, 2, 1) + a_2 (3, 1, 5) = 0$

$\Rightarrow$ $(a_1 + 3a_2, 2a_1 + a_2, a_1 + 5a_2) = (0, 0, 0)$

$\Rightarrow$ $a_1 + 3a_2 = 0$

$2a_1 + a_2 = 0$

$a_1 + 5a_2 = 0.$

The coefficient matrix of these equation is

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 5 \end{bmatrix}$$

Applying, $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$, we get

$$A = \begin{bmatrix} 1 & 3 \\ 0 & -5 \\ 0 & 2 \end{bmatrix}$$

Operate, $R_3 \to 5R_3 + 2R_2$

$$A = \begin{bmatrix} 1 & 3 \\ 0 & -5 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ $\rho(A) = 2$

So $\rho(A) =$ No. of unknown quantities

$\therefore$ $a_1 = a_2 = 0$

$\Rightarrow$ S is linearly independent

L(S) = L(T) – L(T) denotes the linear span of T and consists of those vectors which are linear combination of vectors in T.

$$\therefore \quad L(T) = \{a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 : a_1, a_2, a_3 \in R\}$$

Then, $a_1(1, 2, 1) + a_2(3, 1, 5) + a_3(3, -4, 7) \in L(T)$.

Let $(3, -4, 7) = x(1, 2, 1) + y(3, 1, 5)$

$\Rightarrow$ $(3, -4, 7) = (x + 3y, 2x + y, x + 5y)$

i.e.,

$$x + 3y = 3$$
$$2x + y = -4$$
$$x + 5y = 7$$

On solving, we get $x = -3, y = 2$

$\therefore$ $(3, -4, 7) = -3(1, 2, 1) + 2(3, 1, 5)$

Hence one element of

$$\begin{aligned} L(T) &= a_1(1, 2, 1) + a_2(3, 1, 5) - 3a_3(1, 2, 1) + 2a_3(3, 1, 5) \\ &= (a_1 - 3a_3)(1, 2, 1) + (a_2 + 2a_3)(3, 1, 5) \\ &= a'(1, 2, 1) + b'(3, 1, 5) \end{aligned}$$

where $a' = a_1 - 3a_3,\ b' = a_2 + 2a_3 \in R = a'\alpha_1 + b'\alpha_2 \in L(S)$

i.e. $L(T) = L(S)$

$\Rightarrow$ S spans L(T).

Since S spans L(T) and consists of linearly independent vectors so

$$S = \{\alpha_1, \alpha_2\}$$

forms basis set for L(T) and also since S consists of two elements so

$$\dim [L(T)] = 2$$

Proceeding in a similar fashion we can show that the set (α_1, α_3) and (α_2, α_3) also forms basis set for L(T). **Proved**

Prob.28. Let $V_3(R)$ be a finite dimensional vector space. Find the co-ordinate vector of $\alpha = (3, 1, -4)$ relative to the basis

$$\alpha_1 = (1, 1, 1),\ \alpha_2 = (0, 1, 1),\ \alpha_3 = (0, 0, 1)$$

Sol. Let a_1, a_2, a_3 be unknowns scalars, such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \alpha$$

or $a_1(1, 1, 1) + a_2(0, 1, 1) + a_3(0, 0, 1) = (3, 1, -4)$

$$\Rightarrow \quad (a_1, a_1 + a_2, a_1 + a_2 + a_3) = (3, 1, -4)$$

$$\Rightarrow \quad a_1 = 3 \quad \ldots(i)$$

$$a_1 + a_2 = 1 \quad \ldots(ii)$$

$$a_1 + a_2 + a_3 = -4 \quad \ldots(iii)$$

From equation (i), we get $\quad a_1 = 3$

From equation (ii), we get

$$a_2 = 1 - a_1 \text{ or, } \quad a_2 = -2$$

Putting the values of a_1 and a_2 in equation (iii), we obtains

$$3 - 2 + a_3 = -4$$

$$\Rightarrow \quad a_3 = -4 - 3 + 2$$

or

$$a_3 = -5$$

$$\therefore \quad a_1 = 3, a_2 = -2, a_3 = -5.$$

Hence, the co-ordinate is **(3, –2, –5)** relative to the given basis. **Ans.**

Prob.29. Find the co-ordinate vector $V = (3, 5, -2)$ relative to the basis of $e_1 = (1, 1, 1)$, $e_2 = (0, 2, 3)$, $e_3 = (0, 2, -1)$.

[R.G.P.V., June 2005 (III-Sem)]

Sol. Let $a_1, a_2, a_3 \in R$, such that

$$V = a_1e_1 + a_2e_2 + a_3e_3 \quad \ldots(i)$$

i.e.

$$(3, 5, -2) = a_1(1, 1, 1) + a_2(0, 2, 3) + a_3(0, 2, -1)$$

$$\therefore \quad 3 = a_1 \quad \ldots(ii)$$

$$5 = a_1 + 2a_2 + 2a_3 \quad \ldots(iii)$$

$$-2 = a_1 + 3a_2 - a_3 \quad \ldots(iv)$$

On solving above equations, we get

$$a_1 = 3, a_2 = -1, a_3 = 2$$

$\therefore$ The coordinate vector or V relative to basis is

$$[V] = \mathbf{(3, -1, 2)}$$ **Ans.**

LINEAR TRANSFORMATIONS

Vector Space Homomorphism or Linear Transformation – Let U(F) and V(F) be two vector spaces over the field F. Then a mapping $f : U \to V$ is called a ***linear transformation or homomorphism of U into V*** if

(i) $\quad f(\alpha + \beta) = f(\alpha) + f(\beta)$

and (ii) $\quad f(a\alpha) = af(\alpha), \forall a \in F, \alpha, \beta \in U$

or in other words $f(a\alpha + b\beta) = af(\alpha) + bf(\beta), \forall a, b, \in F; \alpha, \beta \in U$

V is called the **homomorphic image of U**.

Theorem 10. If f is homomorphism of U(F) into V(F), then

(i) $f(0) = 0'$; where 0 and 0' are the zero vectors of U and V respectively.

(ii) $f(-\alpha) = -f(\alpha), \forall \alpha \in U$.

Proof. (i) Let $\alpha \in U$

$\Rightarrow$ $f(\alpha) \in V$

Also $f(\alpha) + 0' = f(\alpha)$, (0' is the zero vector of V)

$= f(\alpha + 0)$ $(\alpha \in U, \alpha + 0 = \alpha)$

$= f(\alpha) + f(0)$ ($\therefore$ f preserves the composition)

Now, V is an abelian group with respect to addition of vectors, so by using left cancellation law, we have

$$f(\alpha) + 0' = f(\alpha) + f(0)$$

$$\Rightarrow \quad \mathbf{0' = f(0)}$$ **Proved**

(ii) Also, let

$$\alpha \in U \Rightarrow -\alpha \in U \text{ and } \alpha + (-\alpha) = 0$$

Consider $f[\alpha + (-\alpha)] = f(0)$

$\Rightarrow$ $f(\alpha) + f(-\alpha) = f(0)$

$\Rightarrow$ $f(\alpha) + f(-\alpha) = 0'$, $(\because f(0) = 0')$

$\Rightarrow$ $\mathbf{f(-\alpha) = -f(\alpha)}$, (definition of additive inverse)

Proved

Kernel of a Homomorphism – Let f be homomorphism of a vector space U(F) into a vector space V(F). Then the set K of all the elements of U which are mapped into zero element of V is called ***kernel of homomorphism*** and is defined as

$$K = \{\alpha \in U : f(\alpha) = 0', \text{ where } 0' \text{ is the zero vector of } V\}$$

Theorem 11. The kernel of a homomorphism is a subspace of U(F).

Proof. Let U(F) and V(F) be two vector spaces over the field F and f be a homomorphism of U into V. Let K be the kernel of f, i.e.,

$$K = \{\alpha \in U : f(\alpha) = 0'\}.$$

In order to prove that K is subspace of U it is sufficient to prove that $a\alpha + b\beta \in K, \forall a, b \in F : a, b \in K$

Let $\alpha, \beta \in K$. Then

$$f(\alpha) = 0' \text{ and } f(\beta) = 0'.$$

Now $f(a\alpha + b\beta) = f(a\alpha) + f(b\beta) = af(\alpha) + bf(\beta) = a.0' + b.0'$

$= 0' + 0' = 0'$ where $a, b \in F$

$\Rightarrow$ $a\alpha + b\beta \in K, \forall a, b \in F; \alpha \in \beta \in K.$

Hence, K is a subspace of U. **Proved**

Isomorphism of Vector Spaces – Let U(F) and V(F) be two vector spaces over the field F. Then a mapping $f : U \to V$ is called an ***isomorphism of U into V***, if

(i) f is one-one

(ii) f is onto

(iii) $f(\alpha + \beta) = f(\alpha) + f(\beta)$

nd (iv) $f(a\alpha) = af(\alpha), \forall a \in F; \alpha, \beta \in U$.

The vector space V(F) is called the ***isomorphic image of U(F)***.

Matrix of a Linear Transformation – Let U be an n-dimensional vector space over the field f and let V be an m-dimensional vector space over F. Suppose $B = \{\alpha_1, \alpha_2, ... \alpha_n\}$ and $B' = \{\beta_1, \beta_2, ... \beta_m\}$ are ordered based for U and V respectively. Suppose T is a linear transformation from U into V. We know that T is completely determined by its action on the vectors α_j belonging to a basis for U. Each of the n vectors $T(\alpha_j)$ is uniquely expressible as a linear combination of $\beta_1 .... \beta_m$ because $T(\alpha_j) \in V$ and these m vectors form a basis for V.

Let for j = 1, 2, ... n.

$$T(\alpha_j) = a_{1j}\beta_1 + a_{2j}\beta_2 + ... + a_{mj}\beta_m = \sum_{i=1}^{m} a_{ij}\, \beta_i$$

The scalars $a_{1j}, a_{2j}, ... a_{mj}$ are the coordinates of $T(\alpha_j)$ in the ordered basis B'. The $m \times n$ matrix whose jth column (j = 1, 2, ... n) consists of these coordinates is called the matrix of the linear transformation T relative to the pair of ordered bases B and B'.

We shall denote it by the symbol [T, B, B'] or simply by [T] if the bases are understood. Thus

$$\begin{aligned}[T] &= [T, B, B'] = \text{Matrix of T relative to order bases B and B'}. \\ &= [a_{ij}]_{m \times n}\end{aligned}$$

where $T(\alpha_j) = \sum_{i=1}^{m} a_{ij}\, \beta_i$, for each j = 1, 2, ... n. ...(i)

The coordinates of $T(\alpha_1)$ in ordered basis B' form the first column of this matrix, the coordinates of $T(\alpha_2)$ in ordered basis B' form the second column of this matrix and so on. The $m \times n$ matrix $[a_{ij}]_{m \times n}$ completely determine the linear transformation T through the formulae given in relation (i). Therefore the matrix $[a_{ij}]_{m \times n}$ represents the transformation T.

Range and Null Space of a Linear Transformation – Let U(F) and V(F) be two vector spaces and let T be a linear transformation from U into V. Then the image or range of T denoted as R(T) is the set of all vectors $\beta \in V$, such that

$$T(\alpha) = \beta, \text{ for some } \alpha \in U$$

i.e., $$R(T) = \{\beta \in V : T(\alpha) = \beta, \text{ for some } \alpha \in U\}$$

and null space of T denoted as N(T) is the set of all vectors $\alpha \in U$, such that

$$T(\alpha) = 0$$

i.e.,

$$N(T) = \{\alpha \in U : T(\alpha) = 0 \in V\}.$$

But, if we regard the linear transformation T from U into V as a vector space homomorphism of U into V, then the null space of T is also called the kernel of T.

Rank and Nullity of a Linear Transformation –

Let T be a linear transformation from a vector space U(F) into a vector space V(F), where U(F) is a finite dimensional vector space. Then the dimension of the range of T is called ***the rank of*** T and is denoted by $\rho(T)$ i.e.,

$$\rho(T) = \text{ rank of } T = \dim R(T)$$

and the ***nullity of T*** is defined as the dimension of the null space of T, denoted as $\nu(T)$ i.e.,

$$\nu(T) = \text{ nullity of } T = \dim N(T).$$

NUMERICAL PROBLEMS

Prob.30. Show that the mapping

$f : V_3(R) \to V_2(R)$, defined as below

$f(a, b, c) = (c, a + b)$ is linear.

[R.G.P.V., June 2003 (III-Sem)]

Sol. Let $\alpha = (a_1, b_1, c_1)$ and $\beta = (a_2, b_2, c_2)$ be any two elements of $V_3(R)$. Also, let $a \in F$

Now
$$\begin{aligned} f(a\alpha + b\beta) &= f(a(a_1, b_1, c_1) + b(a_2, b_2, c_2)) \\ &= f\{(aa_1, ab_1, ac_1) + (ba_2, bb_2, bc_2)\} \\ &= f(aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2) \\ &= (ac_1 + bc_2, aa_1 + ba_2 + ab_1 + bb_2) \\ &= (ac_1, aa_1 + ab_1) + (bc_2, ba_2 + bb_2) \\ &= a(c_1, a_1 + b_1) + b(c_2, a_2 + b_2) \\ &= af(a_1, b_1, c_1) + bf(a_2, b_2, c_2) \\ &= af(\alpha) + bf(\beta). \end{aligned}$$

$\Rightarrow$
$$f(a\alpha + b\beta) = af(\alpha) + bf(\beta), \forall \alpha, \beta \in V_3(R); a, b \in F$$

Hence f is linear.

Otherwise –
$$\begin{aligned} f(\alpha + \beta) &= f\{(a_1, b_1, c_1) + (a_2, b_2, c_2)\} \\ &= f(a_1 + a_2, b_1 + b_2, c_1 + c_2) \\ &= \{(c_1 + c_2), (a_1 + a_2) + (b_1 + b_2)\} \\ &= \{(c_1, a_1 + b_1) + (c_2, a_2 + b_2)\} \\ &= f(a_1, b_1, c_1) + f(a_2, b_2, c_2) \\ &= f(\alpha) + f(\beta) \end{aligned}$$

i.e.,
$$f(\alpha + \beta) = f(\alpha) + f(\beta).$$

Also, $f(a\alpha) = f\{a\,(a_1, b_1, c_1)\} = f(aa_1, ab_1, ac_1)$
$= (ac_1, aa_1 + ab_1) = a\,(c_1, a_1 + b_1)$
$= af(a_1, b_1, c_1) = af(\alpha)$

i.e., $f(a\alpha) = af(\alpha)$.

Hence f is linear. **Proved**

Prob.31. Show that the mapping $f : V_2(R) \to V_3(R)$ defined as below
$$f(a, b) = (a, b, 0)$$
is a linear transformation form $V_2(R)$ onto $V_3(R)$.

[R.G.P.V., Dec. 2001 (III-Sem)]

Sol. Let $\alpha = (a_1, b_1)$ and $\beta = (a_2, b_2)$ be any two elements of $V_2(R)$ and also let $a, b \in F$.

Now $f(\alpha + \beta) = f[(a_1, b_1) + (a_2, b_2)] = f(a_1 + a_2, b_1 + b_2)$
$= (a_1 + a_2, b_1 + b_2, 0)$
$= (a_1, b_1, 0) + (a_2, b_2, 0) = f(a_1, b_1) + f(a_2, b_2)$
$= f(\alpha) + f(\beta)$.

Also $f(a\alpha) = f(a\,(a_1, b_1)) = f(aa_1, ab_1) = (aa_1, ab_1, 0)$
$= a\,(a_1, b_1, 0) = af(a_1, b_1) = af(\alpha)$.

Hence f is linear. **Proved**

Prob.32. Show that the mapping $f : V_3(R) \to V_2(R)$ defined by $f(a, b, c) = (a - b, a + c)$ is linear. ***[R.G.P.V., Dec. 2006 (III-Sem)]***

Sol. Here $f : V_3(R) \to V_2(R)$ defined as above is said to be linear, if
$$f(\alpha+\beta) = f(\alpha)+f(\beta)$$
and
$$f(a\alpha) = af(\alpha), \forall a \in F, \alpha, \beta \in V_3(R)$$

Let $\alpha = (a_1, a_2, a_3), \beta = (b_1, b_2, b_3)$ and $a \in F$.

Now $f(\alpha+\beta) = f[(a_1, a_2, a_3) + (b_1, b_2, b_3)]$
$= f(a_1 + b_1, a_2 + b_2, a_3 + b_3)$
$= [(a_1 + b_1) - (a_2 + b_2), (a_1 + b_1) + (a_3 + b_3)]$
$= [(a_1 - a_2), (a_1 + a_3)] + [(b_1 - b_2), (b_1 + b_3)]$
$= f(a_1, a_2, a_3) + f(b_1, b_2, b_3) = f(\alpha)+f(\beta)$

$\therefore$ $f(\alpha+\beta) = f(\alpha)+f(\beta)$ ...(i)

Also $f(a\alpha) = f[a\,(a_1, a_2, a_3)] = f(aa_1, aa_2, aa_3)$
$= (aa_1 - aa_2, aa_1 + aa_3) = a\,(a_1 - a_2, a_1 + a_3)$
$= af(a_1, a_2, a_3) = af(\alpha)$

or $f(a\alpha) = af(\alpha)$ ...(ii)

From relations (i) and (ii), it is clear that the given mapping is linear.

Proved

Prob.33. Show that the transformation mapping $f: V_2(R) \to V_2(R)$ defined by $f(x, y) = (x + 2, y + 3)$ is not linear. *[R.G.P.V., Dec. 2002 (III-Sem)]*

Sol. Let $\alpha = (x_1, y_1)$ and $\beta = (x_2, y_2)$ be any two elements of $V_2(R)$ and $a \in F$.

Now,
$$\begin{aligned} f(\alpha + \beta) &= f[(x_1, y_1) + (x_2, y_2)] = f(x_1 + x_2, y_1 + y_2) \\ &= (x_1 + x_2 + 2,\ y_1 + y_2 + 3) \\ &\neq (x_1 + 2, y_1 + 3) + (x_2 + 2, y_2 + 3) \neq f(x_1, y_1) + f(x_2, y_2) \\ &\neq f(\alpha) + f(\beta) \end{aligned}$$

$\therefore$ $f(\alpha + \beta) \neq f(\alpha) + f(\beta)$

Hence f is not linear. **Proved**

Prob.34. Show that the mapping

$$f: V_3(R) \to V_2(R) \text{ defined as}$$

$$f(a_1, a_2, a_3) = (a_1 - a_2, a_1 - a_3) \text{ is linear transformation.}$$

Sol. $f: V_3(R) \to V_2(R)$ defined as above is said to be linear, if

$$f(\alpha + \beta) = f(\alpha) + f(\beta)$$

and
$$f(a\alpha) = af(\alpha),\ \forall a \in F,\ \alpha, \beta \in V_3(R)$$

Let $\alpha = (a_1, a_2, a_3)$,

$\beta = (b_1, b_2, b_3)$ and $a \in F$.

Now
$$\begin{aligned} f(\alpha + \beta) &= f[(a_1, a_2, a_3) + (b_1, b_2, b_3)] \\ &= f(a_1 + b_1, a_2 + b_2, a_3 + b_3) \\ &= \{(a_1 + b_1) - (a_2 + b_2), (a_1 + b_1) - (a_3 + b_3)\} \\ &= (a_1 - a_2, a_1 - a_3) + (b_1 - b_2, b_1 - b_3) \\ &= f(a_1, a_2, a_3) + f(b_1, b_2, b_3) = f(\alpha) + f(\beta) \end{aligned}$$

$\therefore$ $f(\alpha + \beta) = f(\alpha) + f(\beta)$.

Also
$$\begin{aligned} f(a\alpha) &= f[a(a_1, a_2, a_3)] = f(aa_1, aa_2, aa_3) \\ &= (aa_1 - aa_2, aa_1 - aa_3) = a(a_1 - a_2, a_1 - a_3) \\ &= af(a_1, a_2, a_3) = af(\alpha) \end{aligned}$$

$\therefore$ $f(a\alpha) = af(\alpha)$.

Hence, f is linear. **Proved**

Prob.35. Show that the function $T: R^3 \to R^2$ defined by –

$$T(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3)$$

is a linear transformation. *[R.G.P.V., May/June 2006 (III-Sem)]*

Sol. Let $\alpha = (x_1, x_2, x_3), \beta = (y_1, y_2, y_3) \in V_3\ (R)$

If a, b $\in$ R, then

$$\begin{aligned}
T(a\alpha + b\beta) &= T[a\ (x_1, x_2, x_3) + b\ (y_1, y_2, y_3)] \\
&= T(ax_1 + by_1, ax_2 + by_2, ax_3 + by_3) \\
&= [(ax_1 + by_1) - (ax_2 + by_2), (ax_1 + by_1) + (ax_3 + by_3)] \\
&= [a\ (x_1 - x_2) + b\ (y_1 - y_2), a\ (x_1 + x_3) + b\ (y_1 + y_3)] \\
&= [a\ (x_1 - x_2), a\ (x_1 + x_3)] + [b\ (y_1 - y_2), b\ (y_1 + y_3)] \\
&= a\ [(x_1 - x_2), (x_1 + x_3)] + b\ [(y_1 - y_2), (y_1 + y_3)] \\
&= aT(x_1, x_2, x_3) + bT(y_1, y_2, y_3) = aT(\alpha) + bT(\beta)
\end{aligned}$$

$\therefore$ T is a linear transformation from R_3 into R_2. **Proved**

Prob.36. Show that the map $T : R^2 \to R^3$ given by

$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_2)$ is linear.

(R.G.P.V., May 2019)

Sol. Let $\alpha = (x_1, x_2), \beta = (y_1, y_2) \in V_2(R)$

If a, b $\in$ R, then

$$\begin{aligned}
T(a\alpha + b\beta) &= T[a(x_1, x_2) + b(y_1, y_2)] \\
&= T(ax_1 + by_1, ax_2 + by_2) \\
&= [(ax_1 + by_1) + (ax_2 + by_2), (ax_1 + by_1) \\
&\qquad - (ax_2 + by_2), (ax_2 + by_2)] \\
&= [a(x_1 + x_2) + b(y_1 + y_2), a(x_1 - x_2) + b(y_1 - y_2), \\
&\qquad (ax_2 + by_2)] \\
&= a[(x_1 + x_2), (x_1 - x_2), x_2] + b[(y_1 + y_2), \\
&\qquad (y_1 - y_2), y_2] \\
&= aT(x_1, x_2) + bT(y_1, y_2) \\
&= aT(\alpha) + bT(\beta)
\end{aligned}$$

$\therefore$ T is a linear transformation from R_2 into R_3. **Proved**

Prob.37. Let U and V be two vector subspaces over the same field F. Show that a function $T : U \to V$ is a linear transformation iff –

$T(au + v) = a\ T(u) + T(v)$, for all $u, v \in U$ and $a \in F$

[R.G.P.V., June 2007 (III-Sem)]

Sol. Consider any u, v $\in$ U and a $\in$ F.

Let T be a linear transformation.

So $\qquad T(au) = aT(u)$

Consequently

$$T(au + v) = T(au) + T(v) = aT(u) + T(v)$$

Conversely, let T satisfy the given condition

Then $\quad T(u + v) = T(1.u + v)$

$$= 1.T(u) + T(v) = T(u) + T(v)$$

We know that

If T is a linear transformation from a vector space U(F) into a vector space V(F).

Then

$$T(0) = 0$$

So that $\quad T(au) = T(au + 0)$

$$= T(au) + T(0) = aT(u) + 0 = aT(u)$$

Hence T is a linear transformation. **Proved**

Prob.38. Show that the mapping $f : V_3(R) \rightarrow V_2(R)$ defined by

$f(a_1, a_2, a_3) = (3a_1 - 2a_2 + a_3, a_1 - 3a_2 - 2a_3)$ is a linear mapping from $V_3(R)$ into $V_2(R)$.

Sol. $f : V_3(R) \rightarrow V_2(R)$ defiend as above is said to be linear, if

$$f(\alpha + \beta) = f(\alpha) + f(\beta)$$

and $\quad f(a\alpha) = af(\alpha), \forall a \in F; \alpha, \beta \in V_3(R)$

Let $\quad \alpha = (a_1, a_2, a_3)$ and $\beta = (b_1, b_2, b_3)$.

Consider

$$\begin{aligned} f(\alpha + \beta) &= f\{(a_1, a_2, a_3) + (b_1, b_2, b_3)\} \\ &= f(a_1 + b_1, a_2 + b_2, a_3 + b_3) \\ &= \{3(a_1 + b_1) - 2(a_2 + b_2) + (a_3 + b_3), (a_1 + b_1) \\ &\quad - 3(a_2 + b_2) - 2(a_3 + b_3)\} \\ &= (3a_1 + 3b_1 - 2a_2 - 2b_2 + a_3 + b_3, \\ &\quad a_1 + b_1 - 3a_2 - 3b_2 - 2a_3 - 2b_3) \\ &= (3a_1 - 2a_2 + a_3, a_1 - 3a_2 - 2a_3) + (3b_1 - 2b_2 \\ &\quad + b_3, b_1 - 3b_2 - 2b_3) \\ &= f(a_1, a_2, a_3) + f(b_1, b_2, b_3) = f(\alpha) + f(\beta) \end{aligned}$$

$\therefore \quad f(\alpha + \beta) = f(\alpha) + f(\beta),$

and

$$\begin{aligned} f(a\alpha) &= f\{a(a_1, a_2, a_3)\} = f(aa_1, aa_2, aa_3) \\ &= (3aa_1 - 2aa_2 + aa_3, aa_1 - 3aa_2 - 2aa_3) \\ &= a(3a_1 - 2a_2 + a_3, a_1 - 3a_2 - 2a_3) \\ &= af(a_1, a_2, a_3) = af(\alpha) \end{aligned}$$

$\therefore \quad f(a\alpha) = af(\alpha).$

Hence, f is linear. **Proved**

Prob.39. Find the matrix representation of the linear mappings relative to the usual basis for R^n.

(i) $F: R^3 \to R^3$ *given by*

$$F(x, y, z) = (x, y, 0)$$

(ii) $F: R^3 \to R^3$ *given by*

$$F(x, y, z) = (z, y + z, x + y + z).$$

Sol. (i) The mapping $F : R^3 \to R^3$ given by

$$F(x, y, z) = (x, y, 0)$$

and the usual basis of R^3 is $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$.

Now $\quad T(1, 0, 0) = (1, 0, 0) = 1.e_1 + 0.e_2 + 0.e_3$

$\quad T(0, 1, 0) = (0, 1, 0) = 0.e_1 + 1.e_2 + 0.e_3$

and $\quad T(0, 0, 1) = (0, 0, 0) = 0.e_1 + 0.e_2 + 0.e_3$

The coefficient matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and its transpose matrix is $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$

Ans.

(ii) The mapping $F : R^3 \to R^3$ given by

$$F(x, y, z) = (z, y + z, x + y + z)$$

Now $\quad T(e_1) = T(1, 0, 0) = (0, 0, 1) = 0.e_1 + 0.e_2 + 1.e_3$

$\quad T(e_2) = T(0, 1, 0) = (0, 1, 1) = 0.e_1 + 1.e_2 + 1.e_3$

$\quad T(e_3) = T(0, 0, 1) = (1, 1, 1) = 1.e_1 + 1.e_2 + 1.e_3$

The coefficient matrix is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Its transpose $= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}$ **Ans.**

Prob.40. Find the matrix representation of linear transformation T on $V_3(R)$ defines as –

$$T(a, b, c) = (2b + c, a - 4b, 3a)$$

corresponding to the basis $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

[R.G.P.V., Dec. 2005 (III-Sem)]

Sol. Here, $T(a, b, c) = (2b + c, a - 4b, 3a)$

We have

$T(1, 0, 0) = (0, 1, 3) = 0(1, 0, 0) + 1(0, 1, 0) + 3(0, 0, 1)$

$T(0, 1, 0) = (2, -4, 0) = 2(1, 0, 0) - 4(0, 1, 0) + 0(0, 0, 1)$

and $\quad T(0, 0, 1) = (1, 0, 0) = 1(1, 0, 0) + 0(0, 1, 0) + 0(0, 0, 1)$

Therefore by definition of matrix of T, with respect to B, we have

$$[T]_B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix} \qquad \textbf{Ans.}$$

Prob.41. Let T be the linear operator on R^2 defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$$

What is the matrix of T of the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (-1, 2, 1)$ and $\alpha_3 = (2, 1, 1)$.

Sol. We have

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$$

$$\therefore \quad T(\alpha_1) = T(1, 0, 1) = (4, -2, 3)$$

$$T(\alpha_2) = T(-1, 2, 1) = (-2, 4, 9)$$

and $$T(\alpha_3) = T(2, 1, 1) = (7, -3, 4).$$

Now, let $$(a, b, c) = x\alpha_1 + y\alpha_2 + z\alpha_3$$
$$= x(1, 0, 1) + y(-1, 2, 1) + z(2, 1, 1)$$
$$= (x - y + 2z, 2y + z, x + y + z)$$

$$\Rightarrow \quad x - y + 2z = a$$
$$2y + z = b$$
$$x + y + z = c$$

On solving, we get

and $$\left.\begin{aligned} x &= \frac{5c - 3b - a}{4} \\ y &= \frac{b + c - a}{4} \\ z &= \frac{b - c + a}{2} \end{aligned}\right\} \qquad \ldots(i)$$

Putting a = 4, b = –2, c = 3 in equation (i), we get

$$x = \frac{17}{4}, y = -3/4, z = -1/2$$

$$\therefore \quad T(\alpha_1) = \frac{17}{4}\alpha_1 - \frac{3}{4}\alpha_2 - \frac{1}{2}\alpha_3 \qquad \ldots(ii)$$

Putting a = –2, b = 4, c = 9 in equation (i), we obtain

$$x = \frac{35}{4}, y = 15/4, z = -7/2$$

$$\therefore \quad T(\alpha_2) = \frac{35}{4}\alpha_1 + \frac{15}{4}\alpha_2 - \frac{7}{2}\alpha_3 \qquad \ldots(iii)$$

Putting a = 7, b = –3, c = 4 in equation (i), we get

$$x = 11/2,\ y = -3/2,\ z = 0$$

$$\therefore \qquad T(\alpha_3) = \frac{11}{2}\alpha_1 - \frac{3}{2}\alpha_2 + 0\alpha_3 \qquad \ldots(\text{iv})$$

From equations (ii), (iii) and (iv), we obtain the coefficient matrix as

$$\begin{bmatrix} 17/4 & -3/4 & -1/2 \\ 35/4 & 15/4 & -7/2 \\ 11/2 & -3/2 & 0 \end{bmatrix} \text{ and its transpose is } \begin{bmatrix} 11/4 & 35/4 & 11/2 \\ -3/4 & 15/4 & -3/2 \\ -1/2 & -7/2 & 0 \end{bmatrix} \qquad \textbf{Ans.}$$

MODULE 5

MATRICES

RANK OF A MATRIX

Submatrix and Minor of a Matrix –

Let A be any matrix of the order $m \times n$. Then a matrix obtained by leaving some rows and columns from matrix A is said to be a ***submatrix of A***. In particular the matrix A itself is a submatrix of A because matrix A is obtained from matrix A by leaving no rows or columns.

Suppose A is a square or rectangular matrix, from it delete all rows leaving certain n rows and all column leaving certain n columns. If $n > 1$ then the elements that are left, constitutes a square matrix of order n and the determinant of this matrix is said to be a ***minor of A of order n***.

Rank of a Matrix – A matrix is said to be of rank r, when

(i) There is at least one minor of A of order r which does not vanish.

(ii) Every minor of A of order $(r + 1)$ or higher vanishes.

Rank = Number of non-zero row in upper triangular matrix.

Again each minor of order $(r + 1)$ can be expanded by its first rows as a sum of multiples of minors of order $(r + 1)$, therefore when all minors of order $(r + 1)$ are zero, the minors of order $(r + 2)$ will vanish.

Briefly, the rank of a matrix is the largest order of any non-vanishing minor of the matrix.

From the above definition of the rank of a matrix we have the following two useful results.

(i) If a matrix has a non-zero minor of order r, its rank is $\geq r$.

(ii) If all minors of a matrix of order $r + 1$ are zero, its rank is $\leq r$.

The rank of a matrix A shall be denoted by $\rho(A)$.

Elementary Transformation of a Matrix –

The following operations, three of which refer to rows and three to columns are called an elementary transformations –

(i) The interchange of any two columns (rows).

(ii) The multiplication of any column (row) by a non-zero number.

(iii) The addition of a constant multiple of the elements of any column (row) to the corresponding elements of any other column (row).

Notation – The following notation will be applied to denote the column elementary transformation.

(a) C_{ij} for the interchange of the ith and jth columns.

(b) kC_i for multiplication of the ith column by k.

(c) $C_i + pC_j$ for addition to the ith column, p times the jth column.

The corresponding row transformation will be denoted by R in place of C.

Equivalent Matrix –

Two matrices A and B are called equivalent if one can be obtained from the other by a sequence of elementary transformations. Two equivalent matrices have the same order and the same rank.

The symbol " ~ " is used for equivalence

The following three properties of the relation '~' in the set of all $m \times n$ matrices are quite obvious.

(i) Reflexivity – If A be any $m \times n$ matrix, then $A \sim A$. Obviously A can be obtained from A by the elementary transformation.

$$R_i \to kR_i, \quad \text{where } k = 1$$

(ii) Symmetry – If $A \sim B$, then $B \sim A$. If B can be obtained from A by a sequence of elementary transformations of A, then A can also be obtained from B by a sequence of elementary transformation of B.

(iii) Transitivity – If $A \sim B$, $B \sim C$, then $A \sim C$. If B can be obtained from A by a sequence of elementary transformations of A, C can be obtained from B by a sequence of elementary transformations of B, then C can also be obtained from A by a sequence of elementary transformations of A. So the relation '~' in the set of all $m \times n$ matrices is an equivalence relation.

Elementary Matrices –

A matrix, which is obtained from a unit matrix by subjecting it to any of the elementary transformations is called an ***elementary matrix.***

Examples of elementary matrices obtained from

$$I_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \text{ are } R_{23} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = C_{23}$$

$$kR_2 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{vmatrix}; R_1 + pR_2 = \begin{bmatrix} 1 & p & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem 1. Elementary row (column) transformations of a matrix A can be obtained by pre-multiplying (post-multiplying) A by the corresponding elementary matrices.

Proof. Let, $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Then, $R_{23} \times A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{bmatrix}$

Therefore, a pre-multiplication by R_{23} has interchanged the 2nd and 3rd rows of matrix A.

Similarly, pre-multiplication by kR_2 will multiply the 2nd row of matrix A by k and pre-multiplication by $R_1 + pR_2$ will result in the addition of p times the 2nd row of matrix A to its 1st row.

Thus, the pre-multiplication of matrix A by elementary matrices results in the corresponding elementary row transformation of matrix A. It can easily be seen that post-multiplication will perform the elementary column transformations.

Invariance of Rank through Elementary Transformations –

Theorem 2. Elementary transformations do not alter the rank of a matrix, i.e., equivalent matrices have the same rank.

Proof. Let A be a matrix of order $m \times n$, i.e.,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots\cdots & a_{3n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & a_{mn} \end{bmatrix}_{m \times n}$$

Suppose, a sub-matrix A_r of order r belong to the first r-rows of the matrix A. By the properties of the determinants following three conditions are always satisfied –

(i) Any determinant $|A_r|$ either remains unaltered or changes into $-|A_r|$, if two rows are interchanged.

(ii) Any determinant $|A_r|$ becomes $\lambda|A_r|$ or remains unaltered, if one row is multiplied by λ.

(iii) Any determinant $|A_r|$ either remains unaltered or change into a sum of two of the original determinants, if any row is changed by addition to it another row.

Similar statements are true for elementary transformations on columns.

Now suppose the matrix B is equivalent to A i.e., $B \sim A$. From above three conditions it is obvious that if all the determinant of order r in A are zero, then all the determinants of order 'r' in B will also be zero.

Hence, $\rho(B) \leq \rho(A)$ ...(i)

But since $A \sim B \Rightarrow \rho(A) \leq \rho(B)$ ...(ii)

From equations (i) and (ii), we get

$$\rho(A) = \rho(B)$$

Hence it follows that if $B \sim A$, then

$$\boldsymbol{\rho(B) = \rho(A)}$$ **Proved**

Echelon Form of a Matrix –

A matrix $A = (a_{ij})_{m \times n}$ is said to be in Echelon form, if

(i) Every row of A which has all its entries 0 occurs below every row which has a non-zero entry.

(ii) The number of zero preceding the first non-zero element in a row is less than that the number of such zeros in the succeeding (or next) row.

(iii) The first non-zero element in every row is unity.

When a matrix is converted in Echelon form, then the number of non-zero rows of the matrix is known as the rank of the matrix A.

For example, matrix $A = \begin{bmatrix} 1 & 3 & -2 & 6 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in the Echelon form.

Theorem 3. The rank of the transpose of a matrix is the same as that of the original matrix.

Proof. Suppose, A is any matrix and A' is the transpose of A. Suppose $\rho(A) = r$, and $\rho(A') = s$.

Then we have to prove, $r = s$

$\because \quad \rho(A) = r$

$\Rightarrow$ There exists at least one r-rowed square submatrix, say R of A such that

$$|R'| = |R| \neq 0$$

$\Rightarrow$ There exists at least one r-rowed square submatrix, R' of A' such that

$\Rightarrow \quad |R'| = |R| \neq 0$

$\Rightarrow \quad \rho(A) \geq r \Rightarrow s \geq r$ ...(i)

$\therefore$ Interchanging the roles of A and A' in the equation (i), we have

$$r \geq s$$ ...(ii)

From equations (i) and (ii), we get

$$\mathbf{r = s}$$

Hence the result. **Proved**

Normal Form of a Matrix –

Every non-zero matrix [say $A = (a_{ij})_{m \times n}$] of rank r, by a sequence of elementary transformations can be reduced to the form.

$$\begin{bmatrix} I_r & : & O \\ .. & & .. \\ O & : & O \end{bmatrix}, \begin{bmatrix} I_r \\ .. \\ O \end{bmatrix}, [I_r : O] \text{ or } [I_r]$$

where I_r is a $r \times r$ unit matrix of order r and O represented null matrix of any order.

These form are said to be the normal form or canonical form of the given matrix A.

Procedure to Obtain Normal Form –

On applying to A the elementary transformations in the following manner, the normal form or canonical form of the matrix A can easily be obtained.

(i) To obtain a non-zero element in the first row and the first column of the given matrix, interchange row (or columns).

(ii) Divide the first row by this element if it is not one.

(iii) By subtracting suitable multipliers of the first row from other rows, obtain zeros in the remainder of the first column.

(iv) By subtracting suitable multiples of the first column from the other columns, obtain zeros in the remainder of the first-row.

(v) Starting with element in the second row and second column, repeat the above four steps.

(vi) Continue the process down the main-diagonal either until end of the diagonal is reached or until all the remaining element of the matrix are zero.

Theorem 4. The rank of a product of two matrices cannot exceed the rank of either matrix. i.e., $\rho(AB) \le \rho(A)$ and $\rho(AB) \le \rho(B)$.

Proof. Suppose $\rho(AB) = r, \rho(A) = r_1$ and $\rho(B) = r_2$.

Then we have to show $r \le r_1$ and $r \le r_2$. Suppose there exists a non-singular matrix P, such that

$$\rho(PAB) = \rho(AB) = r \text{ and } PA = \begin{bmatrix} G \\ . \quad . \\ O \end{bmatrix} \qquad ...(i)$$

where, G is a matrix of rank r_1 with r_1 rows.

Post-multiplying equation (i) by B, we have

$$PAB = \begin{bmatrix} G \\ . \quad . \\ O \end{bmatrix} B \qquad ....(ii)$$

$$\therefore \qquad \rho(AB) = \rho(PAB) = \rho\left(\begin{bmatrix} G \\ .. \\ O \end{bmatrix} B\right) \qquad ....(iii)$$

Since G has r_1 non-zero rows $\begin{bmatrix} G \\ .. \\ O \end{bmatrix} B$ cannot have more than r_1 non-zero rows.

Consequently rank of $\begin{bmatrix} G \\ O \end{bmatrix} B \le r_1$

That is $\rho(AB) \le r_1$ or $r \le r_1$

Again $\rho(AB) = \rho(AB)' = \rho(B'A') \le \rho(B') = \rho(B)$

Therefore, $\rho(AB) \le \rho(B)$ that is $\mathbf{r \le r_2}$ **Proved**

Q.1. What are the elementary transformations of a matrix ?

(R.G.P.V., Jan./Feb. 2006, Nov./Dec. 2007)

Ans. Refer to the matter given on page 241.

NUMERICAL PROBLEMS

Prob.1. Find the rank of the matrix –

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}.$$

(R.G.P.V., Jan./Feb. 2006, Dec. 2006, Nov./Dec. 2007)

Sol. Let, $$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$ so that the given matrix.

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

Obviously, the 3rd order minor of A vanishes. Also its 2nd order minors formed by its 2nd and 3rd rows are all zero. But another 2nd order minor is

$$\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = -1 \ne 0$$

$\therefore$ $\rho(A) = 2$. Hence, the rank of the given matrix is **2**. **Ans.**

Prob.2. Determine the rank of the matrix –

$$\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$$

[R.G.P.V., June 2008(N)]

Sol. Let,
$$A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$$

Applying $R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 + R_1$ so that the given matrix

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

Obviously, the 3rd order minor of A vanishes. Also its 2nd order minors formed by its 2nd and 3rd rows are all zero. But another 2nd order minor is

$$\begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix} = 12 \neq 0$$

Hence, the rank of the given matrix is 2. **Ans.**

Prob.3. Find one non zero minor of highest order of the matrix $A =$ $\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}$ ***and hence find the rank of the matrix A.***

(R.G.P.V., Dec. 2014)

Sol. Here,
$$A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}_{3\times 3}$$

$$|A|_{3\times 3} = \begin{vmatrix} -1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{vmatrix}$$

$$= -1(28 + 2) + 2(-14 - 1) + 3(-4 + 4)$$

$$= -30 - 30 + 0$$

i.e.,
$$|A|_{3\times 3} = -60 \neq 0$$

Therefore A is non singular or non vanishing minor of A of order 3.

Hence $\rho(A) = 3$ **Ans.**

Prob.4. Determine the rank of the following matrix –

$$\begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix}$$

(R.G.P.V., June 2014)

Sol. If A denotes the matrix, we have $|A| = 0$, and each of the second order minor is also 0. Also the matrix is non-zero.

Hence the rank of the matrix is 1, i.e., $\rho(A) = 1$ **Ans.**

Prob.5. Find the rank of the matrix –

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -2 & -4 \end{bmatrix}$$

[R.G.P.V., June 2008 (O)]

Sol. Here, there is no minor of order 4. Now consider the following minor of order 3.

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & -2 & -2 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -2 \end{vmatrix} \quad \text{(By taking 2 common from } R_2\text{)}$$

$= 0$, since R_1 and R_2 are identical.

Similarly, we can prove that all other minors of order 3 are zero.

Now, let us consider a minor of order 2.

$$= \begin{vmatrix} 2 & 3 \\ -2 & -2 \end{vmatrix} \neq 0.$$

Hence the rank of matrix A is 2, i.e., $\rho(A) = 2$ **Ans.**

Prob.6. Find rank of the matrix A, where

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$ ***(R.G.P.V., Dec. 2013, Nov. 2018)***

Sol. Here, there is no minor of order 4. Now consider the following minor of order 3.

$$\begin{vmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ 1 & 3 & 4 \end{vmatrix} = 0, \text{ since } R_1 \text{ and } R_3 \text{ are identical.}$$

Similarly, we can prove that all other minors of order 3 are zero.

Now, let us consider a minor of order 2.

$$\begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 3\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \quad \text{(By taking 3 common from } R_2\text{)}$$

$$= 0$$

Similarly, we can prove that all other minors of order 2 are zero.

Hence we are left with minors of order unity, viz. the elements of the given matrix, which are not equal to zero.

Hence the rank of matrix A is 1, i.e. $\rho(A) = 1$. **Ans.**

Prob.7. Find the rank and nullity of the following matrix –

$$A = \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$$

(R.G.P.V., June 2015)

Sol. Here, $A = \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$

Applying $R_3 \to R_3 - R_1$ and $R_4 \to R_4 - R_2$, we get

$$A \sim \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the rank of given matrix A is 2

i.e., $\rho(A) = \mathbf{2}$ **Ans.**

$\because$ rank A + nullity A = n

$\therefore$ 2 + nullity A = 4

or nullity A = 4 – 2 = **2** **Ans.**

Prob.8. Find rank of the matrix –

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

(R.G.P.V., June/July 2006, May 2019, Nov. 2019, June 2020)

Sol. Here, $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$, we have

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

Now applying $R_3 \to R_3 + R_2$, we get

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Hence rank of the matrix A = **2** **Ans.**

Prob.9. Find the rank of the following matrix A –

$$A = \begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}.$$

(R.G.P.V., Jan./Feb. 2008, June 2011)

Sol. Here, $$|A| = \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$$

On simplification, we get

$$|A| = 0$$

$\therefore$ Rank of A is not 4

Then consider a minor of order 3

$$= \begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$$

By applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$, we get

$$= \begin{vmatrix} 1 & 3 & 5 \\ 4 & 5 & 7 \\ 9 & 7 & 9 \end{vmatrix}$$

By applying $C_3 \to C_3 - C_2$, we get

$$= \begin{vmatrix} 1 & 3 & 2 \\ 4 & 5 & 2 \\ 9 & 7 & 2 \end{vmatrix}$$

Now applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$= \begin{vmatrix} 1 & 3 & 2 \\ 3 & 2 & 0 \\ 8 & 4 & 0 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 \\ 8 & 4 \end{vmatrix} = 2(12 - 16) = -8 \neq 0$$

Hence the rank of given matrix A is 3, i.e., $\rho(A) = \mathbf{3}$ **Ans.**

Prob.10. Define rank of a matrix. Find the rank of matrix A, where

$$A = \begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}$$

(R.G.P.V., June 2013)

***Sol.* Rank of Matrix** – Refer to the matter given on page 241.

To find the rank of matrix A, refer Prob.9.

Prob.11. Find the rank of the matrix –

$$A = \begin{bmatrix} 1 & 4 & 3 & 6 & 1 \\ 0 & 2 & 3 & 1 & 4 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times 5}$$

by defining it in Echelon form. ***(R.G.P.V., June 2012)***

Sol. Here $A = \begin{bmatrix} 1 & 4 & 3 & 6 & 1 \\ 0 & 2 & 3 & 1 & 4 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Applying $\frac{1}{2} R_2$, we get

$$A \sim \begin{bmatrix} 1 & 4 & 3 & 6 & 1 \\ 0 & 1 & 3/2 & 1/2 & 2 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, $-1\,(R_4)$, we get

$$A \sim \begin{bmatrix} 1 & 4 & 3 & 6 & 1 \\ 0 & 1 & 3/2 & 1/2 & 2 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the Echelon form of matrix A. The number of non-zero is 4 and hence $\rho(A) = \mathbf{4}$

Ans.

Prob.12. Find the normal form of the matrix $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

and hence find its rank. ***(R.G.P.V., Dec. 2003, June 2005, 2007, Dec. 2008, 2012, 2014)***

Sol. Here, $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Operating $R_1 \leftrightarrow R_2$, we have

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Operating $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$ and $R_4 \to R_4 - 6R_1$, we have

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

Operating $C_2 \to C_2 + C_1, C_3 \to C_3 + 2C_1$ and $C_4 \to C_4 + 4C_1$, we have

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

Operating $R_4 \to R_4 - R_3 - R_2$, we have

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operating $R_2 \to R_2 - R_3$, we have

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operating $R_3 \to R_3 - 4R_2$, we have

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operating $C_3 \to C_3 + 6C_2$ and $C_4 \to C_4 + 3C_2$, we have

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operating $C_3 \to \frac{1}{33} C_3$, we have

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operating $C_4 \to C_4 - 22C_3$, we have

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, $\rho(A) = 3$ **Ans.**

Prob.13. Reduce the following matrix into its normal form and hence find its rank –

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

(R.G.P.V., June 2009, March/April 2010)

Sol. Here, $$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Applying operation $R_1 \leftrightarrow R_3$, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix}$$

Applying operations $R_2 - 3R_1$, $R_3 - 2R_1$, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

Applying operations $C_2 - C_1$, $C_3 - C_1$, $C_4 - 2C_1$, we get

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

Applying operations $R_2 \rightarrow -R_2$, $R_3 \rightarrow -R_3$, $R_2 \leftrightarrow R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 6 & 2 & 4 \end{bmatrix}$$

Applying operation $R_3 - 6R_2$, we get

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & -28 & -56 \end{bmatrix}$$

Applying operations $C_3 - 5C_2$, $C_4 - 10C_2$, we get

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -28 & -56 \end{bmatrix}$$

Applying operation $R_3 \rightarrow -\frac{1}{28} R_3$, we get

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Applying operation $C_4 - 2C_3$, we get

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

= $[I_3, 0]$ is the normal form of A, its rank is 3. **Ans.**

Prob.14. Reduce the matrix A to the normal form and find its rank –

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

(R.G.P.V., June 2003, April 2009, Dec. 2011)

Sol. Let,
$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

Operating $R_1 \to \frac{1}{2}R_1$, then we get

$$A \sim \begin{bmatrix} 1 & 3/2 & 2 & 5/2 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

Operating $R_2 - 3R_1$, $R_3 - 4R_1$, $R_4 - 9R_1$, then we get

$$A \sim \begin{bmatrix} 1 & \frac{3}{2} & 2 & \frac{5}{2} \\ 0 & -\frac{1}{2} & -1 & -\frac{3}{2} \\ 0 & -1 & -2 & -3 \\ 0 & -\frac{7}{2} & -7 & -\frac{21}{2} \end{bmatrix}$$

Operating $C_2 - \frac{3}{2}C_1$, $C_3 - 2C_1$ and $C_4 - \frac{5}{2}C_1$, then we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -1 & -\frac{3}{2} \\ 0 & -1 & -2 & -3 \\ 0 & -\frac{7}{2} & -7 & -\frac{21}{2} \end{bmatrix}$$

Operating $R_4 - 7R_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -1 & -\frac{3}{2} \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operating $R_3 - 2R_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -1 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operating $C_3 - 2C_2$, $C_4 - 3C_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operating $R_2 \to -2R_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ which is of the form } \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, the rank of matrix A is **2**. **Ans.**

Prob.15. For the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$. *Find non-singular matrices* ***P and Q such that PAQ is in the normal form.*** *Also find rank of A.*

(R.G.P.V., Dec. 2012)

Sol. We shall find elementary matrix by using row operations on a unit matrix I_3 to get pre-factor P. Also to get post-factor Q, we shall find elementary matrix by applying column operations on a unit matrix I_3 of order 3.

Hence we get PAQ, we write $A = I_3 A I_3$

i.e.
$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$, we get

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 + R_2$, we get

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - 2C_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $C_3 \to C_3 - C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

which is of the normal form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and $\rho(A) = 2$ **Ans.**

SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS BY ELEMENTARY TRANSFORMATION, CONSISTENCY OF EQUATION

Types of Linear Equations –

(i) Consistent – A system of equations is called consistent, if they have one or more solution.

i.e., $x + 2y = 4$ $\quad$ $x + 2y = 4$

$3x + 2y = 2$ $\quad$ $3x + 6y = 12$

Unique solution $\quad$ Infinite solutions

(ii) Inconsistent – A system of equation is said to be inconsistent, if they have no solution.

Example –

$$x + 5y = 9 \qquad ...(i)$$

$$2x + 10y = 14 \qquad ...(ii)$$

Divided equation (ii) by 2, we get

$$x + 5y = 7$$

and from equation (i)

$$x + 5y = 9$$

which is not possible. Hence given system of equations have no solution i.e., inconsistent.

Systems of Linear Non-homogeneous Equations – Consider the system of m linear equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n &= b_2 \\ \dots\dots \quad \dots\dots \quad \dots\dots \quad \dots\dots & \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \dots(i)$$

involving the n-unknown $x_1, x_2, \dots\dots, x_n$.

If $$A = \begin{bmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

then the system (i) can be written as

$$AX = B \quad \dots(ii)$$

To determine whether the system of equations (i) are consistent (i.e., possess a solution) or not, we consider the rank of the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots\dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots\dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_n \end{array}\right]$$

Here A is the coefficient matrix and [A|B] is called the augmented matrix of the system of equations (i).

Working Rule for Finding the Solution of the Equation AX = B –

Obtain the ranks of the coefficient matrix A and augmented matrix [A|B], by reducing A to the triangular form by suitable elementary row operations. Suppose the rank of coefficient matrix A is r and that of [A|B] is r'. Then the following cases may arise –

Case I. If $\rho(A) < \rho[A|B]$. In this case the equation AX = B are inconsistent, that is, they have no solution.

Case II. If $\rho(A) = \rho[A|B]$ **(i.e., r = r')**

In this case the equation $AX = B$ are consistent, that is, they have a solution.

(i) If $\mathbf{r = r' < m}$, then $\mathbf{(m - r)}$ equations will be eliminated in the process of reducing the matrix [A|B] to Echelon form. By an equivalent system of r equations, the given system of m-equations will then be replaced. From these r equations we shall be able to express the values of some r unknowns in terms of the remaining (n – r) unknowns which can be given any arbitrarily chosen values.

(ii) If $\mathbf{r = r' = n}$, then the equations are consistent and there is a unique solution.

(iii) If $\mathbf{r = r' < n}$, the equations are consistent and there are infinite number of solutions. Giving arbitrary values to n – r of the unknowns. We may express the other r unknowns in terms of these.

(iv) If $\mathbf{m < n}$, then $\mathbf{r \le m < n}$. Thus in this case $\mathbf{n - r > 0}$. So, when the number of equations is less than the number of unknowns, the equations will always have an infinite number of solution, if the number of equations are less than the number of unknowns, provided they are consistent.

System of Linear Homogeneous Equations –

Consider the homogeneous linear equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots\dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots\dots + a_{2n}x_n = 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots\dots\dots\dots + a_{3n}x_n = 0 \\ ------------------------ \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots\dots\dots + a_{mn}x_n = 0 \end{array}\right\} \quad \dots(i)$$

Containing the n-unknowns, $x_1, x_2 \dots x_n$.

The system (i) can be written as

$$AX = O \qquad \dots(ii)$$

$$\text{where, } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & ----- & a_{1n} \\ a_{21} & a_{22} & a_{23} & ----- & a_{2n} \\ a_{31} & a_{32} & a_{33} & ----- & a_{3n} \\ --- & --- & --- & --- & --- \\ a_{m1} & a_{m2} & a_{m3} & --- & a_{mn} \end{bmatrix}_{m \times n}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \quad \text{and } O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

Here, the matrix A is said to be the coefficient matrix of the system of equation (i).

Here the coefficient matrix A and augmented matrix [A|B], i.e., [A|O] are the same so that

$$\rho(A) = \rho\,[A|B] \text{ (i.e., rank of A = rank of [A|B])}$$

The system of homogeneous equations is always consistent $\Rightarrow x_1 = x_2 = x_3 = \ldots\ldots\ldots\ldots x_n = 0$ is always a solution if det $A \neq 0$. Which is called a trivial solution or zero solution. Again suppose X_1 and X_2 are two solutions of equation (ii), then their linear combination $k_1X_1 + k_2X_2$, where k_1 and k_2 are any arbitrary numbers is also a solution of equation (ii)

We have,

$$A(k_1X_1 + k_2X_2) = k_1(AX_1) + k_2\,(AX_2)$$

or $$A(k_1X_1 + k_2X_2) = k_1(O) + k_2(O) \qquad [\because AX_1 = O, AX_2 = O]$$

or $$A[k_1X_1 + k_2X_2] = 0$$

Hence, $k_1X_1 + k_2X_2$ is also a solution of equation (ii).

So the collection of all the solutions of the system of equations AX = O, form a subspace of the n-vector space V_n.

The Nature of Solutions of the Equation AX = O –

Let we have m-equations in n unknowns; therefore the coefficient matrix A will be of the type m × n. Suppose r is the rank of the matrix A. Obviously r cannot be greater than n (the number of column of the matrix A). So we have either r = n or r < n.

Case I. Let $\rho(A) = r = n\,(\leq m)$.

In this case the number of variables to be assigned arbitrary values in $n - r = 0$.

Hence the system of equations (i) have only a trivial zero solution i.e., $x_1 = x_2 = \ldots\ldots = x_n = 0$.

Let $\rho(A) = r < n\,(\leq m)$.

In this case, we shall have n – r linear independent solutions. Any linear combination of these n – r solutions will also be a solution of AX = O. Thus in this case the equation AX = O will have an infinite number of solution.

Case II. When m < n, that is, number of equations is less than the number of unknowns the solution is always other than $x_1 = x_2 = \ldots\ldots\ldots = x_n = 0$. The number of solution is infinite.

Case III. When m = n (i.e., the number of equations is equal to number of variables), the necessary and sufficient condition for solutions other than $x_1 = x_2 = \ldots\ldots\ldots\ldots = x_n = 0$, is that the determinant of the coefficient matrix is zero. In this case the equation are said to be consistent and a such solution is called wen-trivial solution. The determinant is called the eliminant of the equation.

NUMERICAL PROBLEMS

Prob.16. Solve the system of equations using matrix method.

$$x + 3y - 2z = 0$$
$$2x - y + 4z = 0$$
$$x - 11y + 14z = 0$$

(R.G.P.V., June 2014)

Sol. Given equations can be written in matrix form as

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix A of coefficients

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$, we get

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$, we get

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Applying $R_2 \to \left(-\frac{1}{7}\right)R_2$, we get

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -8/7 \\ 0 & 0 & 0 \end{bmatrix}$$

Above is Echelon form of A and its rank = 2 the number of non-zero rows. Rank A = r < n i.e., 3. Hence the system has non-trivial solution. We shall assign arbitrary values to n – r = 3 – 2 = 1 variable and remaining r = 2. Variable shall be found in terms of these. The equivalent system of equations is

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -8/7 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

or $$x + 3y - 2z = 0$$

$$y - \frac{8}{7}z = 0$$

Suppose $z = k$, then we have

$$y - \frac{8}{7}k = 0 \text{ or } y = \frac{8}{7}k$$

Also $$x = 2k - 3\left(\frac{8}{7}k\right) = \frac{14k - 24k}{7} = -\frac{10}{7}k$$

Hence $$x = -\frac{10}{7}k,\ y = \frac{8}{7}k,\ z = k$$ **Ans.**

Prob.17. Test for consistency and solve –

$$2x - 3y + 7z = 5$$
$$3x + y - 3z = 13$$
$$2x + 19y - 47z = 32$$

(R.G.P.V., Dec. 2008, June 2009)

Sol. We have

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

Applying operation $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 0 & 22 & -54 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 27 \end{bmatrix}$$

Applying operation $R_2 \to R_2 - \frac{3}{2}R_1$, we get

$$\begin{bmatrix} 2 & -3 & 7 \\ 0 & 11/2 & -27/2 \\ 0 & 22 & -54 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 11/2 \\ 27 \end{bmatrix}$$

Now applying operation $R_3 \to R_3 - 4R_2$, we get

$$\begin{bmatrix} 2 & -3 & 7 \\ 0 & 11/2 & -27/2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 11/2 \\ 5 \end{bmatrix}$$

The rank of coefficient matrix = 2 and rank of augmented matrix = 3

So, given system of equation is inconsistent. **Ans.**

Prob.18. Investigate the values of λ and μ so that the equations –

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$

have (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

(R.G.P.V., June 2007, Sept. 2009, June 2010)

NUMERICAL PROBLEMS

Prob.16. Solve the system of equations using matrix method.

$$x + 3y - 2z = 0$$
$$2x - y + 4z = 0$$
$$x - 11y + 14z = 0$$

(R.G.P.V., June 2014)

Sol. Given equations can be written in matrix form as

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

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Applying $R_2 \to \left(-\frac{1}{7}\right)R_2$, we get

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -8/7 \\ 0 & 0 & 0 \end{bmatrix}$$

Above is Echelon form of A and its rank = 2 the number of non-zero rows. Rank A = r < n i.e., 3. Hence the system has non-trivial solution. We shall assign arbitrary values to n – r = 3 – 2 = 1 variable and remaining r = 2. Variable shall be found in terms of these. The equivalent system of equations is

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -8/7 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

or $$x + 3y - 2z = 0$$

$$y - \frac{8}{7}z = 0$$

Suppose z = k, then we have

$$y - \frac{8}{7}k = 0 \text{ or } y = \frac{8}{7}k$$

Also $$x = 2k - 3\left(\frac{8}{7}k\right) = \frac{14k - 24k}{7} = -\frac{10}{7}k$$

Hence $$\mathbf{x = -\frac{10}{7}k\,,\ y = \frac{8}{7}k\,,\ z = k}$$ **Ans.**

Prob.17. Test for consistency and solve –

$$2x - 3y + 7z = 5$$
$$3x + y - 3z = 13$$
$$2x + 19y - 47z = 32$$

(R.G.P.V., Dec. 2008, June 2009)

Sol. We have

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

Applying operation $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 0 & 22 & -54 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 27 \end{bmatrix}$$

Applying operation $R_2 \to R_2 - \frac{3}{2} R_1$, we get

$$\begin{bmatrix} 2 & -3 & 7 \\ 0 & 11/2 & -27/2 \\ 0 & 22 & -54 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 11/2 \\ 27 \end{bmatrix}$$

Now applying operation $R_3 \to R_3 - 4R_2$, we get

$$\begin{bmatrix} 2 & -3 & 7 \\ 0 & 11/2 & -27/2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 11/2 \\ 5 \end{bmatrix}$$

The rank of coefficient matrix = 2 and rank of augmented matrix = 3
So, given system of equation is inconsistent. **Ans.**

Prob.18. Investigate the values of λ and μ so that the equations –

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$

have (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

(R.G.P.V., June 2007, Sept. 2009, June 2010)

Sol. Here, $A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$

and augmented matrix $[A|B] = \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right]$

Applying operation $R_2 \to R_2 - \frac{7}{2} R_1$ and $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 0 & -15/2 & -39/2 & -47/2 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right]$$

Now following cases arise –

Case (i). There exist a unique solution, if $\rho([A|B]) = 3$ and $\rho(A) = 3$, then we have $\lambda - 5 \neq 0$ and $\mu - 9 \neq 0$ i.e., $\boldsymbol{\lambda \neq 5, \mu \neq 9}$ **Ans.**

Case (ii). There exist no solution if $\rho(A) \neq \rho(A|B)$

$\Rightarrow \quad \lambda - 5 = 0$ and $\mu - 9 \neq 0$ [then $\rho(A) = 2$, $\rho(A|B) = 3$]

$\boldsymbol{\lambda = 5}$ **and** $\boldsymbol{\mu \neq 9}$ **Ans.**

Case (iii). There exist infinite number of solutions

$\Rightarrow \quad \rho(A|B) = 2 = \rho(A)$

$\therefore \quad \lambda - 5 = 0$ and $\mu - 9 = 0$

$\therefore \quad \boldsymbol{\lambda = 5}$ **and** $\boldsymbol{\mu = 9}$ **Ans.**

Prob.19. For what values of λ, the equations

$$x + y + z = 1$$
$$x + 2y + 4z = \lambda$$
$$x + 4y + 10z = \lambda^2$$

have a solution and solve completely in each case.

[R.G.P.V., Feb. 2005, June 2008 (O), Dec. 2014]

Sol. Here, given equations can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} \quad \text{...(i)}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda - 1 \\ \lambda^2 - 1 \end{bmatrix}$$

Now applying $R_3 \to R_3 - 3R_2$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda - 1 \\ \lambda^2 - 3\lambda + 2 \end{bmatrix}$$

For solution $\lambda^2 - 3\lambda + 2 = 0$ or $\lambda = 1$ and 2

Hence, the solution for $\lambda = 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

or $$x + y + z = 1 \quad \text{...(ii)}$$

and $$y + 3z = 0 \quad \text{...(iii)}$$

From equations (ii) and (iii), we get

$\mathbf{x = 1 + 2z, \; y = -3z}$ **Ans.**

For $\lambda = 2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

or $$x + y + z = 1 \quad \text{...(iv)}$$

$$y + 3z = 1 \quad \text{...(v)}$$

From equations (iv) and (v), we get

$\mathbf{x = 2z, \; y = 1 - 3z}$ **Ans.**

Prob.20. Find the values of k such that the system of equations –

$$x + ky + 3z = 0$$
$$4x + 3y + kz = 0$$
$$2x + y + 2z = 0$$

has non-trivial solution. *(R.G.P.V., June 2012)*

Sol. The set of equations is written in the form of matrices

$$\begin{bmatrix} 1 & k & 3 \\ 4 & 3 & k \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = B$$

$$C = (A|B)$$

$$C = \left[\begin{array}{ccc|c} 1 & k & 3 & 0 \\ 4 & 3 & k & 0 \\ 2 & 1 & 2 & 0 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 4 & 3 & k & 0 \\ 1 & k & 3 & 0 \end{array}\right]$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - \frac{1}{2}R_1$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & 1 & k-4 & 0 \\ 0 & k-1/2 & 2 & 0 \end{array}\right]$$

Applying $R_3 \to R_3 - (k - 1/2)R_2$

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & 1 & k-4 & 0 \\ 0 & 0 & 2-(k-1/2)(k-4) & 0 \end{array}\right]$$

For a non-trivial solution or for infinite solution

$$\rho(A) = \rho(C) = 2$$

$$\therefore \qquad 2 - (k - 1/2)(k - 4) = 0$$

or $$2 - k^2 + 4k + \frac{k}{2} - 2 = 0 \quad \text{or} \quad -k^2 + \frac{9}{2}k = 0$$

or $$k\left(-k + \frac{9}{2}\right) = 0 \text{ or } k = \frac{9}{2}, k = 0$$ **Ans.**

Prob.21. Show that the following system of equation is inconsistent –

$$x - 2y + z - w = -1$$
$$3x - 2z + 3w = -4$$
$$5x - 4y + w = -3$$

(R.G.P.V., June 2015)

Sol. The given system of equations are equivalent to

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 3 & 0 & -2 & 3 \\ 5 & -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ -3 \end{bmatrix}$$

or $$AX = B$$

The augmented matrix

$$[A \mid B] = \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & -1 \\ 3 & 0 & -2 & 3 & -4 \\ 5 & -4 & 0 & 1 & -3 \end{array}\right]$$

Operating $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 5R_1$, we get

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & -1 \\ 0 & 6 & -5 & 6 & -1 \\ 0 & 6 & -5 & 6 & 2 \end{array}\right]$$

Operating $R_3 \to R_3 - R_2$, we get

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & -1 \\ 0 & 6 & -5 & 6 & -1 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right]$$

The last equivalent matrix is in Echelon form and the new equivalent matrix equation is

$$\begin{bmatrix}1 & -2 & 1 & -1\\0 & 6 & -5 & 6\\0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}-1\\-1\\3\end{bmatrix}$$

which gives $x-2y+z-w=-1$

$$0x+6y-5z+6w=-1$$
$$0x+0y+0z+0w=3$$

The last equation shows that given system of equations is inconsistent. In this problem, we observe that the rank of coefficient matrix A is 2 as it has only two non-zero rows. Whereas rank of $[A|B]$ is 3. Since it has three non-zero rows. Thus $\rho(A) \neq \rho[A|B]$.

Hence we infer that if in the matrix equation $AX = B$, rank A and $[A|B]$ are unequal the given system of equation is **inconsistent**. **Proved**

Prob.22. Solve completely the system of equations $2w+3x-y-z=0$, $4w-6x-2y+2z=0$, $-6w+12x+3y-4z=0$. ***(R.G.P.V., June 2013)***

Sol. Given

$$2w+3x-y-z=0 \qquad ...(i)$$
$$4w-6x-2y+2z=0 \qquad ...(ii)$$
$$-6w+12x+3y-4z=0 \qquad ...(iii)$$

These equations can be written as

$$AX=0$$

i.e.,
$$\begin{bmatrix}2 & 3 & -1 & -1\\4 & -6 & -2 & 2\\-6 & 12 & 3 & -4\end{bmatrix}\begin{bmatrix}w\\x\\y\\z\end{bmatrix}=0 \qquad ...(iv)$$

Here
$$A=\begin{bmatrix}2 & 3 & -1 & -1\\4 & -6 & -2 & 2\\-6 & 12 & 3 & -4\end{bmatrix}$$

Operating $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + 3R_1$

$$A=\begin{bmatrix}2 & 3 & -1 & -1\\0 & -12 & 0 & 4\\0 & 21 & 0 & -7\end{bmatrix}$$

Operating $R_2 \to -\frac{1}{4}R_2$ and $R_3 \to \frac{1}{7}R_3$

$$A=\begin{bmatrix}2 & 3 & -1 & -1\\0 & 3 & 0 & -1\\0 & 3 & 0 & -1\end{bmatrix}$$

Operating $R_3 \to R_3 - R_2$

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\because$ $\rho(A) = 2 < n$ (n = 4, number of unknowns)

$\therefore$ The system of equations is consistent and have infinite number of solutions.

Here $n - \rho(A) = 4 - 2 = 2$

Hence arbitrary values will be given to two unknowns.

Now equation (iv) becomes

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = 0$$

$$2w + 3x - y - z = 0,\ 3x - z = 0$$

If $y = \mathbf{k_1}$, $z = \mathbf{k_2}$, then $x = \frac{1}{3}\mathbf{k_2}$, $w = \frac{1}{2}\mathbf{k_1}$ **Ans.**

Prob.23. Find for what values of λ and μ, the equations $x + y + z = 6$; $x + 2y + 3z = 10$; $x + 2y + \lambda z = \mu$.

Have (i) No solution (ii) A unique solution

(iii) Infinite number of solution.

[R.G.P.V., June 2008 (N), 2015]

Or

Investigate the values of λ and μ in the simultaneous equations –

$x + y + z = 6,\ x + 2y + 3z = 10,\ x + 2y + \lambda z = \mu$

(R.G.P.V., March/April 2010)

Sol. Given that –

$$x + y + z = 6 \quad \text{...(i)}$$

$$x + 2y + 3z = 10 \quad \text{...(ii)}$$

$$x + 2y + \lambda z = \mu \quad \text{...(iii)}$$

The matrix form of the system of equations is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

Operating $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$, we have

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right]$$

Operating $R_3 \to R_3 - R_2$, we have

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

If $\lambda \neq 3$, then $\rho\,[A|B] = \rho(A) = 3$

In this case, the system of equations are consistent. Because rank A = the number of unknowns, so the given system of equations possesses a unique solution. Thus, if $\lambda \neq 3$, the system of equations has a unique solution for any value of μ.

If $\lambda = 3$ and $\mu \neq 10$, then $\rho[A|B] = 3$ and $\rho(A) = 2$. In this case, $\rho\,[A|B] \neq \rho\,(A)$ and thus given system of equations are inconsistent, therefore given system of equations have no solution.

If $\lambda = 3$ and $\mu = 10$, then $\rho\,[A|B] = \rho(A)$. In this case given system of equations are consistence. Since rank of A < number of unknowns, therefore in this case the given system of equations have an infinite no. of solutions. **Ans.**

Prob.24. Test for consistency and solve –

$$5x + 3y + 7z = 4,\;\; 3x + 26y + 2z = 9,\;\; 7x + 2y + 10z = 5$$

[R.G.P.V., Nov./Dec. 2007, Jan./Feb. 2008, June 2008 (N), April 2009, Feb. 2010, June 2011, Dec. 2011]

Or

Test the consistency of the following system of equations and solve using matrix methods.

$$5x + 3y + 7z = 4,\; 3x + 26y + 2z = 9,\; 7x + 2y + 10z = 5$$

(R.G.P.V., Dec. 2012)

Sol. Given that – $5x + 3y + 7z = 4$ ...(i)

$$3x + 26y + 2z = 9 \quad ...(ii)$$

$$7x + 2y + 10z = 5 \quad ...(iii)$$

Equations (i), (ii) and (iii) can be written in matrix form as

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} \Rightarrow AX = B$$

Here, $$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

The augmented matrix –

$$[A|B] = \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$$

Operating $R_1 \to \frac{1}{5}R_1$, we get

$$\sim\left[\begin{array}{ccc|c} 1 & \frac{3}{5} & \frac{7}{5} & \frac{4}{5} \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$$

Operating $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 7R_1$, we get

$$\sim\left[\begin{array}{ccc|c} 1 & \frac{3}{5} & \frac{7}{5} & \frac{4}{5} \\ 0 & \frac{121}{5} & -\frac{11}{5} & \frac{33}{5} \\ 0 & -\frac{11}{5} & \frac{1}{5} & -\frac{3}{5} \end{array}\right]$$

Operating $R_3 \to R_3 + \left(\frac{1}{11}\right)R_2$, we get

or
$$[A|B] \sim \left[\begin{array}{ccc|c} 1 & \frac{3}{5} & \frac{7}{5} & \frac{4}{5} \\ 0 & \frac{121}{5} & -\frac{11}{5} & \frac{33}{5} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, rank of ([A|B]) = 2, and the rank of [A] = 2.

Therefore, the system is consistent but as the rank is less than the number of unknown. Therefore, there exist infinite number of solutions.

In matrix form, the given system reduces to the form.

$$\begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & \frac{121}{5} & -\frac{11}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{33}{5} \\ 0 \end{bmatrix}$$

$$x + \frac{3}{5}y + \frac{7}{5}z = \frac{4}{5} \quad \text{...(iv)}$$

$$\frac{121}{5}y - \frac{11}{5}z = \frac{33}{5} \quad \text{...(v)}$$

Suppose z = k, then we have

$$11y - k = 3$$

or
$$y = \frac{3}{11} + \frac{k}{11}$$

Also
$$x = -\frac{16k}{11} + \frac{7}{11}$$

Here k can take infinite number of values so x, y, z also take infinite number of values, thus there exist infinite number of solutions. **Ans.**

Prob.25. Show that the system of equations –

$$3x + 3y + 2z = 1, \quad x + 2y = 4$$
$$10y + 3z = -2, \quad 2x - 3y - z = 5$$

are consistent and hence solve it. ***(R.G.P.V., Dec. 2010)***

Or

Solve the system of equations –

$$3x + 3y + 2z = 1, x + 2y = 4$$
$$10y + 3z = -2 \text{ and } 2x - 3y - z = 5.$$

(R.G.P.V., May 2019, June 2020)

Sol. Given equations can be written in matrix form as

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

The augmented matrix

$$[A \mid B] = \left[\begin{array}{ccc|c} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$$

Applying $R_2 \to R_2 - 3R_1$ and $R_4 \to R_4 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{array}\right]$$

Applying $R_3 \to R_3 + 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 1 & 9 & -35 \\ 0 & -7 & -1 & -3 \end{array}\right]$$

Applying $R_2 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 9 & -35 \\ 0 & -3 & 2 & -11 \\ 0 & -7 & -1 & -3 \end{array}\right]$$

Applying $R_3 \to R_3 + 3R_2$ and $R_4 \to R_4 + 7R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & 29 & -116 \\ 0 & 0 & 62 & -248 \end{array}\right]$$

Applying $R_4 \to R_4 - \frac{62}{29}R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & 29 & -116 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since $\rho(A \vdots B) = \rho(A) =$ Number of unknowns.

$\therefore$ Given system of equations is consistent and must have unique solution. Therefore,

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 29 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -35 \\ -116 \\ 0 \end{bmatrix}$$

$$x + 2y = 4 \qquad \text{...(i)}$$

$$y + 9z = -35 \qquad \text{...(ii)}$$

$$29z = -116 \qquad \text{...(iii)}$$

On solving above equations, we get

$$x = 2, y = 1, z = -4$$ **Ans.**

✓Prob.26. Show that the following system of equation is inconsistent. $5x + 3y + 14z = 4, y + 2z = 1, x - y + 2z = 0, 2x + y + 6z = 2.$

(R.G.P.V., Nov. 2018)

Sol. Given equations can be written in matrix form as

$$\begin{bmatrix} 5 & 3 & 14 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \\ 2 & 1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

The augmented matrix

$$[A \mid B] = \left[\begin{array}{ccc|c} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \\ 2 & 1 & 6 & 2 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \\ 2 & 1 & 6 & 2 \end{array}\right]$$

Applying $R_3 \to R_3 - 5R_1$ and $R_4 \to R_4 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \\ 0 & 3 & 2 & 2 \end{array}\right]$$

Applying $R_3 \to R_3 - 8R_2$ and $R_4 \to R_4 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \\ 0 & 0 & -4 & -1 \end{array}\right]$$

Applying $R_4 \to 3R_4$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \\ 0 & 0 & -12 & -3 \end{array}\right]$$

Applying $R_4 \to R_4 - R_3$

$$[A\,|\,B] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

which is in Echelon form.

$\therefore \quad \rho(A\,|\,B) = 4$ and $\rho(A) = 3$

$\Rightarrow \quad \rho(A\,|\,B) \neq \rho(A)$

$\Rightarrow$ The system is **inconsistent.** **Proved**

Prob.27. Show that the following equations are consistent or not.
$5x + 3y + 14z = 4,\ y + 2z = 1,\ x - y + 2z = 0$

(R.G.P.V., Nov. 2019, June 2020)

Sol. Given equations can be written in matrix form as

$$\begin{bmatrix} 5 & 3 & 14 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

The augmented matrix

$$A\,|\,B = \left[\begin{array}{ccc|c} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{array}\right]$$

Applying $R_2 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & 1 & 9 & -35 \\ 0 & -3 & 2 & -11 \\ 0 & -7 & -1 & -3 \end{array}\right]$$

Applying $R_3 \to R_3 + 3R_2$ and $R_4 \to R_4 + 7R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & 29 & -116 \\ 0 & 0 & 62 & -248 \end{array}\right]$$

Applying $R_4 \to R_4 - \dfrac{62}{29} R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & 29 & -116 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since $\rho(A \vdots B) = \rho(A) =$ Number of unknowns.

$\therefore$ Given system of equations is consistent and must have unique solution. Therefore,

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 29 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -35 \\ -116 \\ 0 \end{bmatrix}$$

$$x + 2y = 4 \qquad \text{...(i)}$$
$$y + 9z = -35 \qquad \text{...(ii)}$$
$$29z = -116 \qquad \text{...(iii)}$$

On solving above equations, we get

$$\mathbf{x = 2, y = 1, z = -4}$$ **Ans.**

Prob.26. Show that the following system of equation is inconsistent. $5x + 3y + 14z = 4, y + 2z = 1, x - y + 2z = 0, 2x + y + 6z = 2.$

(R.G.P.V., Nov. 2018)

Sol. Given equations can be written in matrix form as

$$\begin{bmatrix} 5 & 3 & 14 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \\ 2 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

The augmented matrix

$$[A \mid B] = \left[\begin{array}{ccc:c} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \\ 2 & 1 & 6 & 2 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \\ 2 & 1 & 6 & 2 \end{array}\right]$$

Applying $R_3 \rightarrow R_3 - 5R_1$ and $R_4 \rightarrow R_4 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \\ 0 & 3 & 2 & 2 \end{array}\right]$$

Applying $R_3 \rightarrow R_3 - 8R_2$ and $R_4 \rightarrow R_4 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \\ 0 & 0 & -4 & -1 \end{array}\right]$$

Applying $R_4 \rightarrow 3R_4$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \\ 0 & 0 & -12 & -3 \end{array}\right]$$

Applying $R_4 \rightarrow R_4 - R_3$

$$[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

which is in Echelon form.

$\therefore \quad \rho(A \mid B) = 4$ and $\rho(A) = 3$

$\Rightarrow \quad \rho(A \mid B) \neq \rho(A)$

$\Rightarrow$ The system is **inconsistent.** **Proved**

Prob.27. Show that the following equations are consistent or not.

$$5x + 3y + 14z = 4,\ y + 2z = 1,\ x - y + 2z = 0$$

(R.G.P.V., Nov. 2019, June 2020)

Sol. Given equations can be written in matrix form as

$$\begin{bmatrix} 5 & 3 & 14 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

The augmented matrix

$$A \mid B = \left[\begin{array}{ccc|c} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{array}\right]$$

Applying $R_3 \to R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{array}\right]$$

Applying $R_3 \to R_3 - 8R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{array}\right]$$

Clearly $\rho(A|B) = 3$ and $\rho(A) = 3$

$$\rho(A) = \rho(A|B) = 3$$

Hence the given system is **consistent**. **Ans.**

Prob.28. Show that the system

$$x + 2y - 5z = 9$$
$$3x - y + 2z = 5$$
$$2x + 3y - z = 3$$

is consistent and solve it. ***(R.G.P.V., Dec. 2013)***

Sol. Given equation can be written in matrix form as

$$\begin{bmatrix} 1 & 2 & -5 \\ 3 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 3 \end{bmatrix}$$

The augmented matrix

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 2 & -5 & 9 \\ 3 & -1 & 2 & 5 \\ 2 & 3 & -1 & 3 \end{array}\right]$$

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -5 & 9 \\ 0 & -7 & 17 & -22 \\ 0 & -1 & 9 & -15 \end{array}\right]$$

Applying $R_3 \to R_3 - \frac{1}{7}R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -5 & 9 \\ 0 & -7 & 17 & -22 \\ 0 & 0 & \frac{46}{7} & -\frac{83}{7} \end{array}\right]$$

Clearly $\rho([A|B]) = 3$ and $\rho([A]) = 3$

$$\rho(A) = \rho([A|B]) = 3$$

Hence the given system is consistent and possesses a unique solution.
In matrix form, the above system reduces to

$$\begin{bmatrix} 1 & 2 & -5 \\ 0 & -7 & 17 \\ 0 & 0 & \frac{46}{7} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -22 \\ -\frac{83}{7} \end{bmatrix}$$

$x + 2y - 5z = 9,\ -7y + 17z = -22,\ \frac{46}{7}z = \frac{-83}{7}$

On solving these, we get

$$z = \frac{-83}{46},\ y = \frac{-57}{46},\ x = \frac{113}{46}$$ **Ans.**

EIGEN VALUES AND EIGEN VECTORS, DIAGONALIZATION OF MATRICES, CAYLEY-HAMILTON THEOREM AND ITS APPLICATIONS TO FIND INVERSE

Eigen Values – Let A be any square matrix of order 'n', we can form the matrix $(A - \lambda I)$, where I is the unit matrix of order n. The determinant of this matrix equated to zero i.e.,

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

is called the ***characteristic equation of A***. On solving the determinant, the characteristic equation takes the form –

$$\boxed{(-1)^n\lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \ldots + k_n = 0}$$

The root of this equation are called ***the characteristic roots*** or ***latent roots*** or ***eigen values*** of the matrix A.

Eigen Vectors –

If $$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

then the linear transformation $\boxed{Y = AX}$ ...(i)

Carries the column vector X into the column vector Y by means of the square matrix A. In practices, it is often required to find such vectors which transform into themselves or to a scalar multiple of themselves.

Let X be such a vector which transforms into λX by means of the transformation (i) then

$$\lambda X = AX$$

$$\Rightarrow \quad (A - \lambda I)X = 0 \quad \ldots(ii)$$

This matrix equation represents n homogeneous linear equations –

$$\left.\begin{array}{l} (a_{11}-\lambda)x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{22}x_1 + (a_{22}-\lambda)x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + (a_{nn}-\lambda)x_n = 0 \end{array}\right\} \quad \ldots(iii)$$

it will have a non-trivial solution only if the coefficient matrix is singular i.e., if

$$|A - \lambda I| = 0$$

The equation (ii) or (iii) will have a non-zero solution

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

which is known as the ***eigen vector*** or ***latent vector***.

Theorem 5. If A is non-singular matrix. Prove that the eigen values of A^{-1} are the reciprocal of eigen values of A.

(R.G.P.V., Dec. 2006, June 2014)

Proof. Let A be a non-singular matrix. Then we are required to show that eigen values of A^{-1} are the reciprocal of eigen values of A. Now, A is non-singular and let λ be the eigen value of A.

$$AX = \lambda X \quad \ldots(i)$$

Premultiplying both sides by A^{-1} of equation (i), we have

$$A^{-1}(AX) = A^{-1}(\lambda X)$$

$$(A^{-1}A)X = \lambda(A^{-1}X)$$

$$\Rightarrow \quad IX = \lambda(A^{-1}X) \quad (\because A^{-1}A = I)$$

$$\Rightarrow \quad X = \lambda(A^{-1}X)$$

$$\Rightarrow \quad \lambda^{-1}X = \lambda^{-1}\lambda(A^{-1}X)$$

$$\Rightarrow \quad \lambda^{-1}X = A^{-1}X$$

This being of the same form as equation (i) shows that λ^{-1} is an eigen value of A^{-1}. **Proved**

Theorem 6. Show that, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of a matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ (m being a positive integer).

Proof. Suppose λ_i is an eigen value of matrix A, and X_i is the corresponding eigen vector. Then,

$$AX_i = \lambda_i X_i \quad \ldots(i)$$

Premultiplying both sides of equation (i) by A, then we have

$$A.AX_i = A\lambda_i X_i$$

$$A^2X_i = \lambda_i(AX_i)$$

$$A^2X_i = \lambda_i\ (\lambda_i X_i) \qquad (\because AX_i = \lambda_i X_i)$$

$$A^2X_i = \lambda_i^2 X_i \qquad \ldots\text{(ii)}$$

Again premultiplying both sides of equation (ii) by A, we get

$$A.(A^2X_i) = A\ (\lambda_i^2\ X_i)$$

$$(A.A^2)\ X_i = \lambda_i^2(AX_i)$$

$$A^3X_i = \lambda_i^2.(\lambda_i X_i) \qquad (\because\ AX_i = \lambda_i X_i)$$

$$A^3X_i = \lambda_i^3 X_i$$

In general $\qquad A^m X_i = \lambda_i^m X_i$

which is of the same form of equation (i)

Hence, λ_i^m is an eigen value of A^m. The corresponding eigen vector is the same X_i.

Proved

Theorem 7. λ is a characteristic root of a matrix A if and only if there exist a non-zero vector X such that, AX = λX.

Proof. **The Condition is Necessary** – Suppose A is a square matrix of order n. If λ is a characteristic root of the matrix A, then $|A - \lambda I| = 0$, and so the characteristic matrix $(A - \lambda I)$ is singular.

That means the matrix equations, $(A - \lambda I)\ X = 0$, possesses a non-zero solution i.e., there exists a non-zero vector X such that

$$(A - \lambda I)\ X = 0 \Rightarrow AX - \lambda IX = 0 \Rightarrow AX = \lambda X$$

Hence, the condition is necessary.

The Condition is Sufficient – Suppose there exists a non-zero vector. X such that

$$AX = \lambda X$$

or $\qquad AX - \lambda XI = 0$

or $\qquad (A - \lambda I)X = 0$

Again since the matrix equation $(A - \lambda I)X = 0$ possesses a non-zero solution, therefore the coefficient matrix $(A - \lambda I)$ must be singular i.e., $|A - \lambda I| = 0$

Hence, the condition is sufficient λ is a characteristic root of the matrix A.

Proved

Theorem 8. If A is a square matrix, show that the characteristic roots of A and its transpose A' are identical.

Proof. Let us consider

$$(A - \lambda I)' = A' - \lambda I'$$

$$= A' - \lambda I \qquad (\because I' = I)$$

$$\therefore \quad |(A - \lambda I)'| = |A' - \lambda I|$$

$$\Rightarrow \quad |A - \lambda I| = |A' - \lambda I| \qquad (\because |A'| = |A|)$$

$$\therefore \quad |A - \lambda I| = 0, \text{ if and only if } |A' - \lambda I| = 0$$

$\Rightarrow$ λ is an eigen value of A if and only if λ is also an eigen value of A'.

Proved

Theorem 9. If A and P be square matrices of the same type and if P be invertible, show that the matrices A and $P^{-1}AP$ have the same characteristic roots.

Proof. Suppose $B = P^{-1}AP$, then we will show that characteristic equations for both A and B are the same, and hence they have the same characteristic roots.

Now $\quad B - \lambda I = P^{-1}AP - P^{-1}\lambda IP = P^{-1}(A - \lambda I)P$

Therefore, $\quad |B - \lambda I| = |P^{-1}(A - \lambda I)P| = |P^{-1}||A - \lambda I||P|$

$$= |A - \lambda I||P^{-1}||P|$$

$$= |A - \lambda I||P^{-1}P| = |A - \lambda I||I| \qquad [\because P^{-1}P = I]$$

or $\quad |B - \lambda I| = |A - \lambda I|$

Hence the matrices A and B have the same characteristic roots i.e., the same characteristic equations. **Proved**

Theorem 10. Show that '0' is a characteristic root of a matrix if and only if the matrix is singular.

Proof. **The Condition is Necessary** – Characteristic equation of matrix A is,

$$\det(A - \lambda I) = |A - \lambda I| = 0 \qquad \ldots(i)$$

When $\lambda = 0$

$$\therefore \quad \det(A - 0.I) = |A - 0.I| = 0 \Rightarrow \det A = |A| = 0$$

That is matrix A is singular.

Hence, the necessary condition.

The Condition is Sufficient – Let the matrix A be a singular, then $|A| = 0$

Now $\quad |A - \lambda I| = 0$

$$\Rightarrow \quad |A| - \lambda |I| = 0$$

$$\Rightarrow \quad 0 - \lambda |I| = 0 \qquad (\because |A| = 0)$$

$$\Rightarrow \quad 0 - \lambda = 0 \qquad (\because |I| = 1)$$

$$\Rightarrow \quad \lambda = 0$$

Proved

Hence, the sufficient condition.

Theorem 11. The modulus of each characteristic root of a unitary matrix is unity.

Proof. Suppose A is unitary matrix, then we have $A^\theta A = I$.

Further, if λ is a characteristic root of matrix A, then $AX = \lambda X$ ...(i)

Now
$$(AX)^\theta = (\lambda X)^\theta$$
or
$$X^\theta A^\theta = \bar{\lambda} X^\theta \qquad \text{...(ii)}$$

Postmultiplying both sides of equation (ii) by AX, we have
$$X^\theta A^\theta AX = \bar{\lambda} X^\theta AX$$
$$X^\theta A^\theta AX = \bar{\lambda} X^\theta . \lambda X$$
$$X^\theta . IX = \bar{\lambda}\lambda X^\theta X \qquad \text{[by equation (i)]}$$
$$X^\theta . X = \bar{\lambda}\lambda X^\theta X$$
$$(1-\bar{\lambda}\lambda) X^\theta X = 0$$
$$1-\bar{\lambda}\lambda = 0 \qquad (\because X \neq 0, \text{ so } X^\theta X \neq 0)$$
$$1 = \bar{\lambda}\lambda$$
$$|\lambda|^2 = 1$$

Hence, the characteristic roots are uni-modular. **Proved**

Theorem 12. Show that the product of the eigen values of a square matrix is equal to the determinant of the matrix.

Proof. Let A be a square matrix of order n,
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The characteristic values of A will be given by
$$|A - \lambda I| = 0$$
$$\Rightarrow \quad \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$
$$\Rightarrow \quad (-1)^n (\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_n) = 0$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the n eigen values of A

Let
$$\phi(\lambda) = (-1)^n (\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_n) \qquad \text{...(i)}$$

Putting $\lambda = 0$ in equation (i), we obtain
$$\phi(0) = (-1)^n \lambda_1 . \lambda_2 \ldots \lambda_n$$
or
$$\phi(0) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = (-1)^n . \lambda_1\lambda_2\lambda_3 \ldots \lambda_n$$
$$\Rightarrow \quad |A| = \lambda_1 . \lambda_2 . \lambda_3 \ldots \lambda_n$$

Hence, the product of eigen values of A is equal to determinant of A.

Proved

Theorem 13. If λ is the eigen value of an invertible matrix A, then show that $\frac{|A|}{\lambda}$ is the eigen value of adj A.

Proof. Since λ is the eigen value of an invertible matrix A, so $\lambda \neq 0$.

Also we know that

$$AA^{-1} = A^{-1}A = I \qquad \text{...(i)}$$

and

$$A^{-1} = \frac{\text{adj } A}{|A|} \qquad \text{...(ii)}$$

The characteristics equation of the matrix A will be

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad |A - \lambda AA^{-1}| = 0 \qquad \text{[by equation (i)]}$$

$$\Rightarrow \quad \left|AI_n - \lambda A.\frac{\text{adj } A}{|A|}\right| = 0 \qquad \text{[by equation (ii)]}$$

$$\Rightarrow \quad \left|A\left(I_n - \lambda\frac{\text{adj } A}{|A|}\right)\right| = 0$$

$$\Rightarrow \quad \left|-\frac{A\lambda}{|A|}\left(\text{adj } A - \frac{|A|}{\lambda}I_n\right)\right| = 0$$

$$\Rightarrow \quad -\lambda\left|\frac{A}{A}\right|\left|\text{adj } A - \frac{|A|}{\lambda}I_n\right| = 0 \Rightarrow \left|\text{adj } A - \frac{|A|}{\lambda}I_n\right| = 0 \text{ as } \lambda \neq 0$$

$\Rightarrow$ The given value of the matrix adj A will be $\frac{|A|}{\lambda}$. **Proved**

Similarity – Let A and B be $n \times n$ (square) matrices over the field F for which there exists an invertible matrix P such that, $B = P^{-1}AP$. Then B is said to be ***similar to A*** over F and it is expressed by –

$$BA$$

(i) Two diagonal matrices are similar if and only if their diagonal elements differ only in order.

(ii) If eigen values of two matrices are same, then it is not necessary that they are similar.

Diagonalization of Matrices – The object of this section to obtain the simplest form of the matrix which is similar to a given square matrix. One of the simplest forms of a matrix is a diagonal form.

diag $(a_1, a_2, \text{..........}, a_n)$, where $a, a_2, \text{.........}, a_n \in F$ (field).

Definition – A matrix A over field F is said to be diagonalizable if it is similar to a diagonal matrix over the field F. Thus a matrix A is diagonalizable if there exists an invertible matrix P such that, $P^{-1}AP = D$.

where D is a diagonal matrix.

Also the matrix P is then said to diagonalize A or transform A to a diagonal form. We know that –

(i) The eigen values of a diagonal matrix are its diagonal elements, and

(ii) The eigen values of similar matrices are the same. Hence, if

$$P^{-1}AP = D = \text{diag}(\lambda_1, \lambda_2 \ldots\ldots\ldots \lambda_n)$$

then the eigen values of A are $\lambda_1, \lambda_2, \ldots\ldots\ldots\ldots \lambda_n$.

Diagonalization of a Symmetric Matrix – Any real symmetric matrix can be diagonalized irrespective of they being distinct or repeated. This fact will be clear by the theorem being given below. But before giving the theorem 12 we would like to give the following definition.

Definition – If P and A be orthogonal matrices and

$$B = P^{-1}AP,$$

then B is called orthogonally similar to A.

The Cayley-Hamilton Theorem –

Statement – Every square matrix satisfies its characteristic equation i.e., if for a square matrix A of order n.

$$|A - \lambda I| = [(-1)^n \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n]$$

Then the matrix equation

$$(-1)^n X^n + a_1X^{n-1} + a_2X^{n-2} + a_3X^{n-3} + \ldots + a_nI = O$$

is satisfied by X = A, i.e.,

$$(-1)^n A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots + a_nI = O$$

Proof. Suppose, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

is a square matrix of order n, then

$$|a - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda \end{vmatrix}$$

and suppose that $|A - \lambda I| = (-1)^n \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n$ ...(i)

Since the elements of $A - \lambda I$ are atmost of the first degree in λ, therefore the elements of adj $(A - \lambda I)$ are ordinary polynomial in λ of degree $(n - 1)$ or less. So, adj $(a - \lambda I)$ can be written as a matrix polynomial in λ, given by

$$\text{adj}(A - \lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + B_2\lambda^{n-3} + \ldots + B_{n-2}\lambda + B_{n-1} \quad \ldots\text{(ii)}$$

where $B_0, B_1, B_2 \ldots, B_{n-1}$ are matrices of the type $n \times n$, whose elements are functions of a_{ij}'s.

Also $$A\,(\text{adj } A) = |A|.\, I = (\text{adj } A).\, A$$
where I, is an $n \times n$ identity matrix

$$(A - \lambda I)\text{ adj }(A - \lambda I) = |A - \lambda I|.I \qquad \ldots(iii)$$

Putting the values of adj $(A - \lambda I)$ and $|A - \lambda I|$ in equation (iii), we get

$$(A - \lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + B_2\lambda^{n-3} + \ldots + B_{n-2}\lambda + B_{n-1})$$
$$= [(-1)^n \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n].I$$

Comparing coefficients of like powers of λ on both sides, we get

$$-IB_0 = (-1)^n I$$
$$AB_0 - IB_1 = a_1 I$$
$$AB_1 - IB_2 = a_2 I$$

$$AB_{n-1} = a_n I$$

Premultiplying the above equations by $A^n, A^{n-1}, A^{n-2}, \ldots, I$ respectively and adding, we obtain

$$O = (-1)^n A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I$$

or $$(-1)^n A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I = O \qquad \ldots(iv)$$

where O is null matrix of order $n \times n$. **Proved**

Cor. 1. Inverse of a matrix A by Cayley-Hamilton theorem.

Multiplying equation (iv) by A^{-1}, we get

$$(-1)^n A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \ldots + a_{n-1} I + a_n A^{-1} = 0$$

Hence, $$A^{-1} = \frac{-1}{a_n}\left[(-1)^n A^{n-1} + a_1 A^{n-2} + \ldots + a_{n-1}\right]$$

Above result gives the inverse of a in terms of $\overline{n-1}$ powers of A and is considered as a practical method for the computation of the inverse of the large matrices. As a by product of the computation the characteristics equation and the determinant of the matrix are also obtained.

Q.2. Prove that product of all eigen values of a matrix A is equal to determinant of A. ***(R.G.P.V., Dec. 2013)***

Ans. Refer to Theorem 12 given on page 277.

Q.3. State and prove Cayley-Hamilton theorem. ***(R.G.P.V., Dec. 2010)***

Ans. Refer to the matter given on page 279.

NUMERICAL PROBLEMS

Prob.29. Find the eigen values and eigen vectors for the matrix A –

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

(R.G.P.V., Nov./Dec. 2007, Nov. 2019, June 2020)

Sol. The characteristic equation is

$$|A - \lambda I| = 0$$

i.e.,
$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

or
$$\lambda^2 - 7\lambda + 6 = 0$$

or
$$(\lambda - 6)(\lambda - 1) = 0$$

$$\therefore \qquad \lambda = 6, 1$$

Thus, the eigen values are **6** and **1**. **Ans.**

If x, y be the components of an eigen vector. Corresponding to the eigen value λ, then

$$[A - \lambda I]\,X = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Corresponding to $\lambda = 6$, we have

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

which gives only one independent equation

$$-x + 4y = 0$$

$\therefore \qquad \dfrac{x}{4} = \dfrac{y}{1}$ giving the eigen vector **(4, 1)**.

Corresponding to $\lambda = 1$, we have

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

which gives only one independent equation $x + y = 0$.

$\therefore \qquad \dfrac{x}{1} = \dfrac{y}{-1}$ giving the eigen vector **(1, – 1)** **Ans.**

Prob.30. Find the eigen values of the matrix –

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}.$$

[R.G.P.V., June 2008(N)]

Sol. Given that –

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Here, the characteristic equation of matrix A is given by –

$$|A - \lambda I| = 0$$

or
$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

or $(8-\lambda)\{(7-\lambda)(3-\lambda)-16\}+6\{(-6)(3-\lambda)+8)\}+2\{24-2(7-\lambda)\} = 0$

or $(8-\lambda)(21-7\lambda-3\lambda+\lambda^2-16)+6(-18+6\lambda+8)+2(24-14+2\lambda)=0$

or $(8-\lambda)(5-10\lambda+\lambda^2)+6(-10+6\lambda)+2(10+2\lambda)=0$

or $40-80\lambda+8\lambda^2-5\lambda+10\lambda^2-\lambda^3-60+36\lambda+20+4\lambda=0$

or $-\lambda^3+18\lambda^2-45\lambda=0$ or $-\lambda(\lambda^2-18\lambda+45)=0$

or $-\lambda(\lambda^2-15\lambda-3\lambda+45)=0$ or $-\lambda[\lambda(\lambda-15)-3(\lambda-15)]=0$

or $-\lambda(\lambda-3)(\lambda-15)=0$

$\Rightarrow$ $\lambda = 0, 3, 15$

Thus, the eigen value of matrix A are **0, 3, 15** **Ans.**

Prob.31. Determine the eigen values and the corresponding eigen vectors of the matrix –

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}.$$

(R.G.P.V., Dec. 2004, Jan./Feb. 2008, Dec. 2012)

Sol. **Eigen Values of the Matrix** – Refer to Prob.30.

Now the eigen vector of the matrix A, corresponding the eigen value λ is given by the non-zero solution of the equation.

$$(A - \lambda I)X = 0$$

or
$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{...(i)}$$

When $\lambda = 0$, we have

$$\begin{bmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_1 \to R_1 - 4R_3$ and $R_2 \to R_2 + 3R_3$, we get

$$\begin{bmatrix} 0 & 10 & -10 \\ 0 & -5 & 5 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_1 \leftrightarrow R_3$ and $R_2 \to -\left(\frac{1}{5}\right)R_2$, we get

$$\begin{bmatrix} 2 & -4 & 3 \\ 0 & 1 & -1 \\ 0 & 10 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_3 \to R_3 - 10R_2$, we get

$$\begin{bmatrix} 2 & -4 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 - 4x_2 + 3x_3 \\ x_2 - x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1 - 4x_2 + 3x_3 = 0 \qquad \text{...(ii)}$$

$$x_2 - x_3 = 0 \qquad \text{...(iii)}$$

From equation (iii), we have

$$\frac{x_2}{1} = \frac{x_3}{1} = k \Rightarrow x_2 = k, x_3 = k$$

Putting the values of x_2 and x_3 in equation (ii), we get

$$2x_1 - 4k + 3k = 0$$

$$x_1 = \frac{1}{2}k \Rightarrow \frac{x_1}{1/2} = \frac{x_2}{1} = \frac{x_3}{1} = k$$

Thus, we have

$$X_1 = \begin{bmatrix} \mathbf{1/2} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} \qquad (\because k = 1)$$

is an eigen vector of A, corresponding to eigen value $\lambda = 0$

When $\lambda = 3$, we have

$$(A - 3I)X = 0$$

$$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_1 \to R_1 + R_2$, we get

$$\begin{bmatrix} -1 & -2 & -2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \to R_2 - 6R_1$, $R_3 \to R_3 + 2R_1$, we get

$$\begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & -8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now operating $R_3 \to R_3 + \frac{1}{2}R_2$, we get

$$\begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, we have

$$-x_1 - 2x_2 - 2x_3 = 0$$
$$16x_2 + 8x_3 = 0$$

From last equation, we have

$$\frac{x_2}{1} = \frac{x_3}{-2} = k_1 \text{(say)}$$

or $\quad x_2 = k_1,\ x_3 = -2k_1$

Hence, we get $x_1 = -2x_2 - 2x_3 = -2k_1 + 4k_1 = 2k_1$

$$\Rightarrow \quad \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2} \Rightarrow X_2 = \begin{bmatrix} \mathbf{2} \\ \mathbf{1} \\ \mathbf{-2} \end{bmatrix}$$

is an eigen vector corresponding to eigen value $\lambda = 3$

Now we shall obtain eigen vector corresponding the eigen value $\lambda = 15$,

When $\lambda = 15$, we have

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying operations $R_1 + 4R_3$ and $R_2 + 3R_3$, we get

$$\begin{bmatrix} 1 & -22 & -46 \\ 0 & -20 & -40 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now applying operation $R_3 - 2R_1$, we get

$$\begin{bmatrix} 1 & -22 & -46 \\ 0 & -20 & -40 \\ 0 & 40 & 80 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Again applying operation $R_3 + 2R_2$, we get

$$\begin{bmatrix} 1 & -22 & -46 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 - 22x_2 - 46x_3 \\ -20x_2 - 40x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \qquad x_1 - 22x_2 - 46x_3 = 0 \qquad \text{...(iv)}$$

$$-20x_2 - 40x_3 = 0 \qquad \text{...(v)}$$

From equation (v), we get

$$\frac{x_2}{-2} = \frac{x_3}{1} = k' \text{ or } x_2 = -2k', x_3 = k'$$

Putting the values of x_2 and x_3 in equation (iv), we get

$$x_1 + 44k' - 46k' = 0$$

$$x_1 = 2k'$$

Therefore, we have

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1} = k' \Rightarrow X_3 = \begin{bmatrix} \mathbf{2} \\ \mathbf{-2} \\ \mathbf{1} \end{bmatrix}$$

is an eigen vector corresponding the eigen value $\lambda = \mathbf{15}$. **Ans.**

Prob.32. Find eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}. \quad \textit{(R.G.P.V., Dec. 2013, June 2015)}$$

Sol. Given that –

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Here, the characteristic equation of matrix A is given by –

$$|A - \lambda I| = 0$$

$$\Rightarrow \qquad \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)\{(3-\lambda)(2-\lambda)-2\}-2\{1(2-\lambda)-1\}+1\{2-(3-\lambda)\}=0$

$(2-\lambda)\{6-5\lambda+\lambda^2-2\}-2\{1-\lambda\}+1\{-1+\lambda\}=0$

$(2-\lambda)\{\lambda^2-5\lambda+4\}-2+2\lambda-1+\lambda=0$

$2\lambda^2-10\lambda+8-\lambda^3+5\lambda^2-4\lambda-2+2\lambda-1+\lambda=0$

$-\lambda^3+7\lambda^2-11\lambda+5=0$

Hence, the characteristic equation of matrix A is –

$\lambda^3-7\lambda^2+11\lambda-5=0$

$\therefore \qquad \lambda = 1, 1, 5$

Thus, the eigen values of matrix A are **1, 1, 5** **Ans.**

Let x_1, x_2, x_3 be the components of an eigen vectors corresponding to the eigen value λ, we have

$$(A-\lambda I)X=0$$

$$\Rightarrow \quad \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad ...(i)$$

When $\lambda = 5$, we have, $(A-5I)X=0$

$$\Rightarrow \quad \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now applying operations R_1+3R_2, R_2-R_3, we get

$$\begin{bmatrix} 0 & -4 & 4 \\ 0 & -4 & 4 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Again applying operations $R_1 \to \left(\frac{1}{4}\right)R_1$ and $R_2 \to \left(\frac{1}{4}\right)R_2$, we get

$$\begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Again applying operations R_2-R_1 and R_3+2R_1, we get

$$\begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -x_2+x_3=0 \qquad ...(ii)$$

$$x_1-x_3=0 \qquad ...(iii)$$

From equations (ii) and (iii), we get

$$x_1=x_2=x_3=k$$

Thus, we have $\qquad X_1=\begin{bmatrix} k \\ k \\ k \end{bmatrix}=[k, k, k]'$

It is clear that, we have infinite number of eigen vector corresponding to eigen value $\lambda = 5$.

Now, we shall obtain eigen vector corresponding to eigen value $\lambda = 1$.

When $\lambda = 1$, we have, $(A - 1.I)\,X = 0$

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying operations $R_2 - R_1$ and $R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which gives –

$$x_1 + 2x_2 + x_3 = 0$$

Let, $$x_1 = c_1,\ x_2 = c_2$$

$\therefore$ $$x_3 = -(c_1 + 2c_2)$$

Thus, we have

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ -(c_1 + 2c_2) \end{bmatrix} = \left[c_1, c_2, -(c_1 + 2c_2)\right]'$$

X_2 is a corresponding eigen vector to eigen value $\lambda = 1$. **Ans.**

Prob.33. Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

(R.G.P.V., Dec. 2014)

Sol. Refer to Prob.32.

Prob.34. Find the characteristic equation of the matrix –

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

(R.G.P.V., June 2011, Nov. 2018, June 2020)

Sol. Characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)\{(2-\lambda)^2 - 1\} + 1\{-(2-\lambda) + 1\} + 1\{1 - (2-\lambda)\} = 0$$

$$\Rightarrow \quad (2-\lambda)^3 - (2-\lambda) - (2-\lambda) + 1 + 1 - (2-\lambda) = 0$$

$$\Rightarrow \quad (2-\lambda)^3 - 3(2-\lambda) + 2 = 0$$

$$\Rightarrow \quad 8 - \lambda^3 - 12\lambda + 6\lambda^2 - 6 + 3\lambda + 2 = 0$$

$$\Rightarrow \quad -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0 \ \text{ or } \ \boldsymbol{\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0}$$ **Ans.**

Prob.35. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \textit{(R.G.P.V. June 2004, Dec. 2011)}$$

Sol. Characteristic equation of given matrix A is given by

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\{(2-\lambda)(1-\lambda)-0\} - 1\{1-\lambda-0\} + 1\{0-0\} = 0$$

$$\Rightarrow \quad (2-\lambda)^2(1-\lambda)-(1-\lambda) = 0$$

$$\Rightarrow \quad (\lambda-1)^2(\lambda-3) = 0$$

$$\therefore \quad \lambda = 1, 1, 3$$

Therefore the eigen values of matrix A are **1, 1, 3** **Ans.**

The eigen vector of the matrix A corresponding to the eigen value λ is given by the non-zero solution of the equation $(A - \lambda I) X = 0$

or
$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

When $\lambda = 1$, the corresponding eigen vector is given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad x_1 + x_2 + x_3 = 0$$

Now let $x_1 = k_1$ and $x_2 = k_2$, so that $x_3 = -(k_1 + k_2)$.

When $\lambda = 3$, the corresponding eigen vector is given by

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad -x_1 + x_2 + x_3 = 0 \quad \text{...(i)}$$

$$x_1 - x_2 + x_3 = 0 \quad \text{...(ii)}$$

$$-2x_3 = 0 \quad \text{...(iii)}$$

From equation (iii), we get $\quad x_3 = 0$

Putting above value, in equation (ii), we get

$$x_1 - x_2 = 0$$

or
$$\frac{x_1}{1} = \frac{x_2}{1} = k_1(\text{say})$$

$$\Rightarrow \quad x_1 = k_1, x_2 = k_1$$

Hence the eigen vectors are $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}k_1\\k_2\\-(k_1+k_2)\end{pmatrix}$ **Ans.**

Prob.36. Find eigen values of the matrix $\begin{bmatrix}2 & 1 & 1\\1 & 2 & 1\\0 & 0 & 1\end{bmatrix}$.

(R.G.P.V., Nov. 2018, June 2020)

Sol. Refer to Prob.35.

Prob.37. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix}-2 & 2 & -3\\2 & 1 & -6\\-1 & -2 & 0\end{bmatrix}.$$

(R.G.P.V., Dec. 2002, 2006, June 2013, 2014)

Or

Find the characteristic roots and the characteristic vectors of the matrix –

$$\begin{bmatrix}-2 & 2 & -3\\2 & 1 & -6\\-1 & -2 & 0\end{bmatrix}$$ ***(R.G.P.V., March/April 2010)***

Sol. Given that –

$$A = \begin{bmatrix}-2 & 2 & -3\\2 & 1 & -6\\-1 & -2 & 0\end{bmatrix}$$

Here, characteristic equation of matrix A is –

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix}-2-\lambda & 2 & -3\\2 & 1-\lambda & -6\\-1 & -2 & -\lambda\end{vmatrix} = 0$$

$$(-2-\lambda)\{(1-\lambda)(-\lambda)-12\}-2\{2(-\lambda)-6\}-3\{-4+1(1-\lambda)\}=0$$

or $\quad (-2-\lambda)(-\lambda+\lambda^2-12)-2(-2\lambda-6)-3(-4+1-\lambda) = 0$

or $\quad 2\lambda-2\lambda^2+24+\lambda^2-\lambda^3+12\lambda+4\lambda+12+12-3+3\lambda = 0$

or $\quad \lambda^3+\lambda^2-21\lambda-45=0$

or $\quad (\lambda-5)(\lambda+3)(\lambda+3)=0$

or $\quad \lambda = -3, -3, 5$

Thus the eigen values of the matrix A are **5, – 3, – 3.**

Again, the equation $(A-\lambda I)=0$, for the matrix A is –

$$\begin{bmatrix}-2-\lambda & 2 & -3\\2 & 1-\lambda & -6\\-1 & -2 & -\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad ...(i)$$

Prob.35. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$ *(R.G.P.V. June 2004, Dec. 2011)*

Sol. Characteristic equation of given matrix A is given by

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\{(2-\lambda)(1-\lambda)-0\} - 1\{1-\lambda-0\} + 1\{0-0\} = 0$$

$$\Rightarrow \quad (2-\lambda)^2(1-\lambda) - (1-\lambda) = 0$$

$$\Rightarrow \quad (\lambda-1)^2(\lambda-3) = 0$$

$$\therefore \quad \lambda = 1, 1, 3$$

Therefore the eigen values of matrix A are **1, 1, 3** **Ans.**

The eigen vector of the matrix A corresponding to the eigen value λ is given by the non-zero solution of the equation $(A - \lambda I)X = 0$

or $$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

When $\lambda = 1$, the corresponding eigen vector is given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad x_1 + x_2 + x_3 = 0$$

Now let $x_1 = k_1$ and $x_2 = k_2$, so that $x_3 = -(k_1 + k_2)$.

When $\lambda = 3$, the corresponding eigen vector is given by

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad -x_1 + x_2 + x_3 = 0 \quad \text{...(i)}$$

$$x_1 - x_2 + x_3 = 0 \quad \text{...(ii)}$$

$$-2x_3 = 0 \quad \text{...(iii)}$$

From equation (iii), we get $x_3 = 0$

Putting above value, in equation (ii), we get

$$x_1 - x_2 = 0$$

or $$\frac{x_1}{1} = \frac{x_2}{1} = k_1(\text{say})$$

$$\Rightarrow \quad x_1 = k_1, x_2 = k_1$$

Hence the eigen vectors are $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} k_1 \\ k_2 \\ -(k_1+k_2) \end{pmatrix}$ **Ans.**

Prob.36. Find eigen values of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(R.G.P.V., Nov. 2018, June 2020)

Sol. Refer to Prob.35.

Prob.37. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$

(R.G.P.V., Dec. 2002, 2006, June 2013, 2014)

Or

Find the characteristic roots and the characteristic vectors of the matrix –

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$ ***(R.G.P.V., March/April 2010)***

Sol. Given that –

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Here, characteristic equation of matrix A is –

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)\{(1-\lambda)(-\lambda)-12\} - 2\{2(-\lambda)-6\} - 3\{-4+1(1-\lambda)\} = 0$$

or $$(-2-\lambda)(-\lambda+\lambda^2-12) - 2(-2\lambda-6) - 3(-4+1-\lambda) = 0$$

or $$2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 4\lambda + 12 + 12 - 3 + 3\lambda = 0$$

or $$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

or $$(\lambda-5)(\lambda+3)(\lambda+3) = 0$$

or $$\lambda = -3, -3, 5$$

Thus the eigen values of the matrix A are **5, – 3, – 3.**

Again, the equation $(A - \lambda I) = 0$, for the matrix A is –

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ...(i)$$

When $\lambda = 5$, we have

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding eigen vector given by following equations –

$$-7x_1 + 2x_2 - 3x_3 = 0 \quad \text{...(ii)}$$

$$2x_1 - 4x_2 - 6x_3 = 0 \quad \text{...(iii)}$$

$$-x_1 - 2x_2 - 5x_3 = 0 \quad \text{...(iv)}$$

On solving equations (iii) and (iv), we get

$$\frac{x_1}{20-12} = \frac{-x_2}{-10-6} = \frac{x_3}{-4-4}$$

$$\frac{x_1}{8} = \frac{x_2}{16} = \frac{x_3}{-8}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1} = k$$

Hence, the eigen vector is $X_1 = \begin{bmatrix} k \\ 2k \\ -k \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

When $\lambda = -3$, we have

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 - 3x_3 = 0 \quad \text{...(v)}$$

$$2x_1 + 4x_2 - 6x_3 = 0 \quad \text{...(vi)}$$

$$-x_1 - 2x_2 + 3x_3 = 0 \quad \text{...(vii)}$$

Equations (v), (vi) and (vii) are the same. Let $x_1 = k_1$, $x_2 = k_2$ then $x_3 = \frac{1}{3}(k_1 + 2k_2)$

Hence, the eigen vector is $\begin{bmatrix} k_1 \\ k_2 \\ \frac{1}{3}(k_1 + 2k_2) \end{bmatrix}$

Let $k_1 = 0$, $k_2 = 3$, Hence $X_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

Since, the matrix is non-systematic the corresponding eigen vector X_2 and X_3 must be linearly independent. This can be done by choosing $k_2 = 0$, and $k_1 = 3$. Hence, $X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

$$\lambda_1 = 5 \rightarrow X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \lambda = -3 \rightarrow X_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \lambda = -3 \rightarrow X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \textbf{ Ans.}$$

Prob.38. Find the eigen values and eigen vectors of the matrix –

$$A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$$

(R.G.P.V., June 2012)

Sol. The characteristic equation is $|A - \lambda I| = \begin{vmatrix} 3-\lambda & -4 & 4 \\ 1 & -2-\lambda & 4 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

$$(3-\lambda)\{(-2-\lambda)(3-\lambda)+4\}+4\{(3-\lambda)-4\}+4\{-1-1(-2-\lambda)\}=0$$
$$(3-\lambda)(-6+2\lambda-3\lambda+\lambda^2+4)+4(-1-\lambda)+4(1+\lambda)=0$$
$$(3-\lambda)(\lambda^2-\lambda-2)-4-4\lambda+4+4\lambda=0$$
$$(3-\lambda)(\lambda^2-\lambda-2)=0$$
$$(3-\lambda)(\lambda+1)(\lambda-2)=0$$

Hence the eigen value of the matrix

– 1, 2 and **3.**

Now the eigen vector of the matrix A, corresponding to the eigen value λ is given by the non-zero solution of the equation.

$$(A - \lambda I)\ X = 0$$

$$\begin{bmatrix} 3-\lambda & -4 & 4 \\ 1 & -2-\lambda & 4 \\ 1 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

When $\lambda = -1$

$$\begin{bmatrix} 4 & -4 & 4 \\ 1 & -1 & 4 \\ 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - R_2$, we get

$$\begin{bmatrix} 4 & -4 & 4 \\ 1 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Again applying $R_1 \rightarrow R_1 - R_2$, we get

$$\begin{bmatrix} 3 & -3 & 0 \\ 1 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3x_1 - 3x_2 \\ x_1 - x_2 + 4x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x_1 - 3x_2 = 0 \quad \text{...(i)}$$

$$x_1 - x_2 + 4x_3 = 0 \quad \text{...(ii)}$$

From equation (i), we have

$$\frac{x_1}{1} = \frac{x_2}{1} = k$$

Putting the values of x_1 and x_2 in equation (ii), we get

$$k - k + 4x_3 = 0$$

$$x_3 = 0$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0} = k$$

Thus, we have $$X_1 = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}$$

When $\lambda = 2$

$$\begin{bmatrix} 1 & -4 & 4 \\ 1 & -4 & 4 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$, we get

$$\begin{bmatrix} 1 & -4 & 4 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \leftrightarrow R_3$, we get

$$\begin{bmatrix} 1 & -4 & 4 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$, we get

$$\begin{bmatrix} 1 & -4 & 4 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 - 4x_2 + 4x_3 \\ 3x_2 - 3x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 4x_2 + 4x_3 = 0 \quad \text{....(iii)}$$

$$3x_2 - 3x_3 = 0 \quad \text{....(iv)}$$

From equation (iv), we have

$$\frac{x_2}{1} = \frac{x_3}{1} = k$$

Putting the values of x_2 and x_3 in equation (iii), we get

$$x_1 - 4k + 4k = 0$$
$$x_1 = 0$$
$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1} = k$$

Thus, we have $X_2 = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}$

When $\lambda = 3$

$$\begin{bmatrix} 0 & -4 & 4 \\ 1 & -5 & 4 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4x_2 + 4x_3 \\ x_1 - 5x_2 + 4x_3 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-4x_2 + 4x_3 = 0 \qquad \text{...(v)}$$
$$x_1 - 5x_2 + 4x_3 = 0 \qquad \text{...(vi)}$$
$$x_1 - x_2 = 0 \qquad \text{...(vii)}$$

From equation (v), we have

$$-x_2 + x_3 = 0$$
$$\frac{x_2}{1} = \frac{x_3}{1} = k$$

From equation (vii), we get

$$\frac{x_1}{1} = \frac{x_2}{1} = k$$

Hence

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} = k$$

Thus, we have

$$X_3 = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}$$

Thus, the eigen vector A corresponding to the eigen value $\lambda = 3$. **Ans.**

Prob.39. Find the eigen values and eigen vectors of the matrix –

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$ ***(R.G.P.V., June 2011)***

Sol. Characteristic equation of given matrix A is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 0-\lambda & 1 & 2 \\ 1 & 0-\lambda & -1 \\ 2 & -1 & 0-\lambda \end{vmatrix} = 0$$

$$(-\lambda)\{(-\lambda)(-\lambda)-1\}-1\{-\lambda+2\}+2\{-1+2\lambda\}=0$$
$$-\lambda(\lambda^2-1)+\lambda-2-2+4\lambda=0$$
$$-\lambda^3+\lambda+\lambda-4+4\lambda=0$$
$$\lambda^3-6\lambda+4=0$$

Since $\lambda = 2$ satisfies it, we can write this equation as

$$\lambda^2(\lambda-2)+2\lambda(\lambda-2)-2(\lambda-2)=0$$
$$(\lambda-2)(\lambda^2+2\lambda-2)=0$$
$$\lambda=2,\ \lambda=-1\pm\sqrt{3}$$

Thus the eigen values of matrix A are $\mathbf{2, -1 \pm \sqrt{3}}$

Now the eigen vector of the matrix A corresponding the eigen value λ as given by the non-zero solution of the equation

$$(A-\lambda I)X=0$$

or
$$\begin{bmatrix} 0-\lambda & 1 & 2 \\ 1 & 0-\lambda & -1 \\ 2 & -1 & 0-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

When $\lambda = 2$, we have

$$\begin{bmatrix} -2 & 1 & 2 \\ 1 & -2 & -1 \\ 2 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1+x_2+2x_3=0$$
$$x_1-2x_2-x_3=0$$
$$2x_1-x_2-2x_3=0$$
$$\frac{x_1}{-1+4}=\frac{x_2}{2-2}=\frac{x_3}{4-1} \text{ or } \frac{x_1}{3}=\frac{x_2}{0}=\frac{x_3}{3}$$

Thus we have $X_1 = \begin{bmatrix} \mathbf{3} \\ \mathbf{0} \\ \mathbf{3} \end{bmatrix}$

When $\lambda = -1+\sqrt{3}$, we have

$$\begin{bmatrix} 1-\sqrt{3} & 1 & 2 \\ 1 & 1-\sqrt{3} & -1 \\ 2 & -1 & 1-\sqrt{3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(1-\sqrt{3})x_1+x_2+2x_3=0$$
$$x_1+(1-\sqrt{3})x_2-x_3=0$$
$$2x_1-x_2+(1-\sqrt{3})x_3=0$$
$$\frac{x_1}{-1-2(1-\sqrt{3})}=\frac{x_2}{2+1(1-\sqrt{3})}=\frac{x_3}{(1-\sqrt{3})^2-1}$$
$$\frac{x_1}{-3+2\sqrt{3}}=\frac{x_2}{3-\sqrt{3}}=\frac{x_3}{3-2\sqrt{3}}$$

Thus

$$X_2 = \begin{bmatrix} -3+2\sqrt{3} \\ 3-\sqrt{3} \\ 3-2\sqrt{3} \end{bmatrix} \text{ when } \lambda = -1-\sqrt{3}, \text{ we have}$$

$$\begin{bmatrix} 1+\sqrt{3} & 1 & 2 \\ 1 & 1+\sqrt{3} & -1 \\ 2 & -1 & 1+\sqrt{3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(1+\sqrt{3})\,x_1 + x_2 + 2x_3 = 0$$

$$x_1 + (1+\sqrt{3})x_2 - x_3 = 0$$

$$2x_1 - x_2 + (1+\sqrt{3})x_3 = 0$$

$$\frac{x_1}{-1-2(1+\sqrt{3})} = \frac{x_2}{2+1(1+\sqrt{3})} = \frac{x_3}{(1+\sqrt{3})^2-1}$$

$$\frac{x_1}{-3-2\sqrt{3}} = \frac{x_2}{3+\sqrt{3}} = \frac{x_3}{3+2\sqrt{3}}$$

Thus

$$X_3 = \begin{bmatrix} -3-2\sqrt{3} \\ 3+\sqrt{3} \\ 3+2\sqrt{3} \end{bmatrix}$$

Ans.

Prob.40. Determine a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix where –

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol. The characteristic equation of the matrix A is

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

or $(6-\lambda)\,[9+\lambda^2-6\lambda-1] + 2\,[-6+2\lambda+2] + 2\,[2-6+2\lambda] = 0$

or $(6-\lambda)\,(\lambda^2-6\lambda+8) - 8 + 4\lambda - 8 + 4\lambda = 0$

or $6\lambda^2 - 36\lambda + 48 - \lambda^3 + 6\lambda^2 - 8\lambda - 16 + 8\lambda = 0$

or $-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$

or $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

or $(\lambda-2)^2\,(\lambda-8) = 0$

or $\lambda = 2, 2, 8$

Therefore the eigen values of matrix A are **2, 2, 8**.

Eigen vector for $\lambda = 2$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \left(\text{by } R_1 \to \frac{1}{2}R_1\right)$$

or $$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{pmatrix}\text{by } R_2 \to R_2 + R_1 \\ \text{and } R_3 \to R_3 - R_1\end{pmatrix}$$

or $$2x_1 - x_2 + x_3 = 0$$

This equation is satisfied by $x_1 = 0, x_2 = 1, x_3 = 1$ and $x_1 = 1, x_2 = 3, x_3 = 1$.

Eigen vectors are $[0, 1, 1]'$, $[1, 3, 1]'$

Eigen vector for $\lambda = 8$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} 1 & 1 & -1 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \left(\text{by } R_1 \to -\frac{1}{2}R_1\right)$$

or $$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{pmatrix}\text{by } R_2 \to R_2 + 2R_1 \\ \text{and } R_3 \to R_3 - 2R_1\end{pmatrix}$$

or $$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{pmatrix}\text{by } R_2 \to -\frac{1}{3}R_2 \\ \text{and } R_3 \to R_3 - R_2\end{pmatrix}$$

$$x_1 + x_2 - x_3 = 0 \text{ and } x_2 + x_3 = 0$$

Suppose $x_3 = 1, x_2 = -1$, and so $x_1 = 2$

Eigen vector is $[2, -1, 1]'$

$$\therefore \qquad P = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix},\ P^{-1} = -\frac{1}{6}\begin{bmatrix} 4 & 1 & -7 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{bmatrix}$$

Now

$$P^{-1}AP = -\frac{1}{6}\begin{bmatrix} 4 & 1 & -7 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{8} \end{bmatrix}$$

Ans.

Prob.41. Find eigen values of the matrix

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

(R.G.P.V., May 2019)

Sol. Refer to Prob.40.

Prob.42. Show that the matrix A is not diagonalizable, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

Sol. The characteristic equation of the matrix A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (-\lambda)(-3\lambda + \lambda^2 + 3) - 1(-1) = 0$$

$$\Rightarrow \quad 3\lambda^2 - \lambda^3 - 3\lambda + 1 = 0$$

$$\Rightarrow \quad \lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$\therefore \quad (\lambda - 1)^3 = 0$$

$$\therefore \quad \lambda = 1, 1, 1$$

Now, we will find the eigen vectors corresponding to the eigen value $\lambda = 1$, which is a non-zero solution of the equation

$$(A - 1I)X = 0$$

$$\Rightarrow \quad \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad -x_1 + x_2 = 0$$

$$-x_2 + x_3 = 0$$

$$x_1 - 3x_2 + 2x_3 = 0$$

Solving the above equations, we get

$$x_1 = x_2 = x_3 = k$$

$\therefore$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the corresponding eigen vector and every scalar multiple $\begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix}$ is also an eigen vector.

$$E_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

dim of $E_1 = 1 =$ Dimension of null space of $(A - 1I)$.

But the multiplicity of the eigen value is 3.

Since dimension of $E_1 \neq$ the multiplicity of the eigen value.

Hence A is not diagonalizable. **Proved**

Prob.43. Verify Cayley-Hamilton's theorem for the matrix

$$\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

(R.G.P.V., May 2019)

Sol. Characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\{(1-\lambda)(-1-\lambda)-3\} - 2\{(-1-\lambda)-1\} - 2\{3-(1-\lambda)\} = 0$$

$$(1-\lambda)(-1-\lambda+\lambda+\lambda^2-3) - 2(-2-\lambda) - 2(2+\lambda) = 0$$

$$(1-\lambda)(\lambda^2-4) + 4 + 2\lambda - 4 - 2\lambda = 0$$

$$\lambda^2 - 4 - \lambda^3 + 4\lambda = 0$$

$$-\lambda^3 + \lambda^2 + 4\lambda - 4 = 0 \quad \text{or} \quad \lambda^3 - \lambda^2 - 4\lambda + 4 = 0$$

By Cayley-Hamilton theorem, we have

$$A^3 - A^2 - 4A + 4I = 0 \qquad \text{...(i)}$$

$$A^2 = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix}$$

Substituting these values of A, A^2, A^3 in equation (i), we have

$$A^3 - A^2 - 4A + 4I = \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1-4+4 & 6+2-8+0 & -6-2+8+0 \\ 7-3-4+0 & 6-6-4+4 & 2+2-4+0 \\ 7-3-4+0 & 14-2-12+0 & -6-2+4+4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}$$

Proved

Hence, given matrix satisfied Cayley-Hamilton theorem.

Prob.44. Verify the Cayley-Hamilton theorem for the matrix –

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$ *(R.G.P.V., June 2012)*

Sol. From Prob.34, characteristic equation is

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

By Cayley-Hamilton theorem, we have

$$A^3 - 6A^2 + 9A - 4I = 0 \qquad \text{...(i)}$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Substituting these values of A, A^2, A^3 in equation (i), we have

$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9-0 & 21-30+9-0 \\ -21+30-9-0 & 22-36+18-4 & -21+30-9-0 \\ 21-30+9-0 & -21+30-9-0 & 22-36+18-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}$$ **Proved**

Hence, given matrix satisfied Cayley-Hamilton theorem

Prob.45. Verify Cayley-Hamilton theorem for the matrix –

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and find its inverse.

(R.G.P.V., June 2002, Dec. 2008, Feb. 2010)

Sol. **Verification of Cayley-Hamilton Theorem** – Refer to Prob.44.

For computing A^{-1}, multiplying equation (i) in Prob.44 by A^{-1}, we get

$$4A^{-1} = A^2 - 6A + 9I$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

or $$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$ **Ans.**

Prob.46. Find the characteristic equation of the matrix A and hence find A^{-1}.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

(R.G.P.V., June 2010, Nov. 2019)

Sol. The characteristic equation of the given matrix is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)\{(1-\lambda)(2-\lambda) - 0\} - 1\{0-0\} + 1\{0 - 1(1-\lambda)\} = 0$$

$$\Rightarrow \quad (2-\lambda)\{2 + \lambda^2 - 3\lambda\} - 1 + \lambda = 0$$

$$\Rightarrow \quad 4 + 2\lambda^2 - 6\lambda - 2\lambda - \lambda^3 + 3\lambda^2 - 1 + \lambda = 0$$

$$\Rightarrow \quad -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$$

$$\Rightarrow \quad \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

Hence, the characteristic equation of A is given by

$$A^3 - 5A^2 + 7A - 3 = 0 \qquad \text{...(i)}$$

On multiplying A^{-1}, we get

$$A^2 - 5A + 7I - 3A^{-1} = 0 \text{ or } 3A^{-1} = A^2 - 5A + 7I$$

or $$3A^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 10 & 5 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$ **Ans.**

Prob.47. Find the eigen values of A and using Cayley-Hamilton theorem, find A^n (n is a positive integer); given that

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

(R.G.P.V., June 2014)

Sol. Given that

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

Characteristic equation of given matrix is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda)-8=0$$
$$3-\lambda-3\lambda+\lambda^2-8=0$$
$$\lambda^2-4\lambda-5=0 \qquad \text{...(i)}$$
$$\lambda^2-5\lambda+\lambda-5=0$$
$$\lambda(\lambda-5)+1(\lambda-5)=0$$
or $$(\lambda+1)(\lambda-5)=0$$
$$\lambda=-1,5$$

Thus, the eigen values of the matrix A are **–1, 5** **Ans.**

By Cayley-Hamilton theorem, the matrix A satisfy its characteristic equation (i).

We have, $A^2-4A-5I=0$

or $$A^2=4A+5I \qquad \text{...(ii)}$$

Multiplying both sides of equation (ii) by A^{n-2}, we have

$$A^n=4A^{n-1}+5A^{n-2}$$

Where n is a positive integer. **Ans.**

Prob.48. Verify Cayley-Hamilton theorem for the matrix –

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}$$

Hence find A^{-1}.

(R.G.P.V., June 2015)

Sol. The characteristic equation is given by

$$|A-\lambda I|=0$$

or $$\begin{vmatrix}1-\lambda & 2 & 3\\ 2 & 4-\lambda & 5\\ 3 & 5 & 6-\lambda\end{vmatrix}=0$$

$$(1-\lambda)\{(4-\lambda)(6-\lambda)-25\}-2\{2(6-\lambda)-15\}+3\{10-3(4-\lambda)\}=0$$
$$(1-\lambda)(24-4\lambda-6\lambda+\lambda^2-25)-2(12-2\lambda-15)+3(10-12+3\lambda)=0$$
$$(1-\lambda)(\lambda^2-10\lambda-1)-2(-3-2\lambda)+3(-2+3\lambda)=0$$
$$\lambda^2-10\lambda-1-\lambda^3+10\lambda^2+\lambda+6+4\lambda-6+9\lambda=0$$
$$-\lambda^3+11\lambda^2+4\lambda-1=0$$

Therefore, the characteristic equation of A is

$$\lambda^3-11\lambda^2-4\lambda+1=0$$

Verification – We have to show that

$$A^3-11A^2-4A+I=0 \qquad \text{...(i)}$$

Now $A^2=A.A$

$$=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}=\begin{bmatrix}14 & 25 & 31\\ 25 & 45 & 56\\ 31 & 56 & 70\end{bmatrix}$$

and $\quad A^3 = A^2.A$

$$= \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}$$

Hence,

$$A^3 - 11A^2 - 4A + I$$

$$= \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} - 11\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}$$ **Proved**

Hence, A satisfied the characteristic equation.

For computing A^{-1}, multiplying equation (i) by A^{-1}, we get

$$A^{-1} = -A^2 + 11A + 4I$$

$$A^{-1} = -\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + 11\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{-3} & \mathbf{2} \\ \mathbf{-3} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{0} \end{bmatrix}$$

Ans.

Prob.49. Show that Cayley-Hamilton theorem is satisfied by the matrix A.

where $A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$

Hence find A^{-1}.

(R.G.P.V., June 2013)

Sol. The characteristics equation is given by $|A - \lambda I| = 0$, i.e.

$$\begin{vmatrix} 0-\lambda & 0 & 1 \\ 3 & 1-\lambda & 0 \\ -2 & 1 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad -\lambda(1-\lambda)(4-\lambda) + 1\{3 + 2(1-\lambda)\} = 0$

$\Rightarrow \quad -\lambda(4 - \lambda - 4\lambda + \lambda^2) + (3 + 2 - 2\lambda) = 0$

$\Rightarrow \quad -\lambda(\lambda^2 - 5\lambda + 4) + 5 - 2\lambda = 0$

$\Rightarrow \quad -\lambda^3 + 5\lambda^2 - 4\lambda + 5 - 2\lambda = 0$

$\Rightarrow \quad -\lambda^3 + 5\lambda^2 - 6\lambda + 5 = 0$

or $\quad \lambda^3 - 5\lambda^2 + 6\lambda - 5 = 0 \quad$ ...(i)

Verification – We have to show that

$$A^3 - 5A^2 + 6A - 5I = 0 \quad \text{...(ii)}$$

Now $\quad A^2 = A \times A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix}$

and $\quad A^3 = A^2 \times A = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -5 & 5 & 14 \\ -3 & 4 & 15 \\ -13 & 19 & 51 \end{bmatrix}$

Hence from equation (ii)

$A^3 - 5A^2 + 6A - 5I$

$$= \begin{bmatrix} -5 & 5 & 14 \\ -3 & 4 & 15 \\ -13 & 19 & 51 \end{bmatrix} - 5\begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix} + 6\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}$$

Proved

Hence, A satisfied the characteristics equation.

Now we have $5I = A^3 - 5A^2 + 6A$

or $\quad 5A^{-1} = A^2 - 5A + 6I$

i.e. $\quad 5A^{-1} = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix} - 5\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{bmatrix}$

$\Rightarrow \quad A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{bmatrix}$ **Ans.**

Prob.50. Using Cayley-Hamilton's theorem, find A^{-2}, where –

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$ *[R.G.P.V., June 2008 (O)]*

Sol. Characteristic equation of matrix A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$(1 - \lambda)\,[(1 + \lambda)\,(1 + \lambda) - 0] - 2[2(-1 - \lambda) - 0]$

$(1 - \lambda)\,(1 + \lambda)^2 + 4 + 4\lambda = 0$

$(1 - \lambda)\,(1 + \lambda^2 + 2\lambda) + 4 + 4\lambda = 0$

$1 + \lambda^2 + 2\lambda - \lambda - \lambda^3 - 2\lambda^2 + 4 + 4\lambda = 0$

$-\lambda^3 - \lambda^2 + 5\lambda + 5 = 0$

$\lambda^3 + \lambda^2 - 5\lambda - 5 = 0$

By Cayley-Hamilton theorem, we have

$$A^3 + A^2 - 5A - 5I = 0$$

or $$A^2 + A - 5I - 5A^{-1} = 0$$

or $$5A^{-1} = A^2 + A - 5I \qquad \text{...(i)}$$

$$A^2 = A.A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From equation (i), we have

$$A^{-1} = \frac{1}{5}\left\{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} = \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

Now

$$A^{-2} = (A^{-1})^2$$

$$= \left\{\frac{1}{25}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}\right\} = \frac{1}{25}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

or $$A^{-2} = \begin{bmatrix} \mathbf{1/5} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1/5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$ **Ans.**

■■■

RGPV

B.E. (First/Second Semester) EXAMINATION, June 2012
(Grading System)
(Common for all Branches)
ENGINEERING MATHEMATICS-I
[BE – 102(GS)]

Note : Attempt all questions. All questions carry equal marks.

1. **(a)** Prove that –

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin z.\frac{\sin z}{1} - \frac{(h\sin z)^2}{2}.\sin 2z + \ldots..$$

where $z = \cot^{-1}x$. (See Unit-I, Page 31, Prob.28)

(b) If $u = x\phi(y/x) + \psi(y/x)$, then prove that –

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$ (See Unit-I, Page 53, Prob.54)

Or

(a) What error in the common logarithm of a number will be produced by an error of 1% in the number ? **

(b) Find the maxima and minima of the following function –

$$\sin x + \sin y + \sin(x+y) \text{ in } \left[0 \le x \le \frac{\pi}{2}, 0 \le y \le \frac{\pi}{2}\right]$$

(See Unit-I, Page 59, Prob.60)

2. **(a)** Find ab-initio the value of the integral –

$$\int_0^{\pi/2} \sin x \, dx$$ (See Unit-II, Page 83, Prob.17)

(b) Evaluate – $$\int_0^{\infty} \frac{x^8(1-x^6)}{(1+x)^{24}} dx$$ (See Unit-II, Page 95, Prob.27)

Or

(a) Evaluate – $$\lim_{n\to\infty}\left\{\frac{n!}{n^n}\right\}^{1/n}$$ (See Unit-II, Page 80, Prob.12)

(b) Change the order of integration –

$$\int_0^4 \int_{x^2/4}^{2\sqrt{x}} dx\, dy$$

Hence evaluate it. (See Unit-II, Page 132, Prob.79)

3. **(a)** Solve – **

$$y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$$

(b) Solve – **

$$\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x$$

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

Or

(a) Solve – $p(py) = x(x + y)$ **

where $p \equiv \frac{dy}{dx}$.

(b) Solve – $\frac{dx}{dt} + y = \sin t$ **

$\frac{dy}{dt} + x = \cos t$

Given that $x = 2$ and $y = 2$, when $t = 0$.

4. (a) Find the rank of the matrix –

$$A = \begin{bmatrix} 1 & 4 & 3 & 6 & 1 \\ 0 & 2 & 3 & 1 & 4 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times5}$$

by defining it in Echelon form. **(See Unit-V, Page 251, Prob.11)**

(b) Find the eigen values and eigen vectors of the matrix – 10

$$A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$$

(See Unit-V, Page 291, Prob.38)

Or

(a) Find the values of k such that the system of equations –

$x + ky + 3z = 0$

$4x + 3y + kz = 0$

$2x + y + 2z = 0$

has non-trivial solution. **(See Unit-V, Page 263, Prob.20)**

(b) Verify the Cayley-Hamilton theorem for the matrix –

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

(See Unit-V, Page 299, Prob.44)

5. (a) Define the following terms for a graph – **

(i) Subgraph (ii) Degree of a vertex

(iii) Composition and De-composition (iv) Rooted tree

(b) Define fuzzy logic and its applications in science and engineering. **

Or

(a) Prepare a truth table to get the negation of the statement "Sita is dull and careless." **

(b) Prove that – **

a.b + b.c + c.a = (a + b).(b + c).(c + a) $\forall$ a, b, c, $\in$ B

Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus

RGPV

B.E. (First/Second Semester) EXAMINATION, Dec. 2012

(Grading System)

(Common for all Branches)

ENGINEERING MATHEMATICS-I

(BE – 102)

Note : **(i)** Attempt all questions

(ii) All questions carry equal marks

(iii) Internal choices are also given.

Unit-I

1. (a) Expand sin x in powers of $(x - \pi/2)$. Hence, find the value of sin 91° correct to 4 decimal places. (See Unit-I, Page 34, Prob.32)

(b) Prove that if the perimeter of a triangle is constant its area is maximum when the triangle is equilateral.

Or

2. (a) If $u = x\phi(y/x) + \psi(y/x)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$ (See Unit-I, Page 53, Prob.54)

(b) Show that the radius of curvature at any point on the cardioid. **

$r = a(1 - \cos\theta)$ is $2/3\sqrt{2ar}$

Unit-II

3. (a) Evaluate $\lim\limits_{n \to \infty}\left\{\frac{n!}{n^n}\right\}^{1/n}$. (See Unit-II, Page 80, Prob.12)

(b) Find the whole area of astroid $x^{2/3} + y^{2/3} = a^{2/3}$.

(See Unit-II, Page 109, Prob.48)

Or

4. (a) Find, by triple integration, the volume of the sphere $x^2 + y^2 + z^2 = a^2$.

(See Unit-II, Page 142, Prob.89)

(b) Prove that $\beta(m, n) = \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)}$. (See Unit-II, Page 101, Prob.39)

Unit-III

5. (a) Solve the differential equation. **

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x$$

(b) Solve the following differential equation by method of variation of parameters

$(D^2 + a^2)y = \sec ax.$ **

Or

6. (a) Solve the differential equation **

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-12y = x^3 \log x$$

(b) Solve – $\frac{dx}{dt}-7x+y = 0$ **

$$\frac{dy}{dt}-2x-5y = 0$$

Unit-IV

7. (a) Find the normal form of the matrix A and hence find its rank, where

$$A=\begin{bmatrix}2 & 3 & -1 & -1\\ 1 & -1 & -2 & -4\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix}$$ **(See Unit-V, Page 251, Prob.12)**

(b) For the matrix $A=\begin{bmatrix}1 & 1 & 2\\ 1 & 2 & 3\\ 0 & -1 & -1\end{bmatrix}$. Find non-singular matrices P and Q such that PAQ is in the normal form. Also find rank of A.

(See Unit-V, Page 255, Prob.15)

Or

8. (a) Determine the eigen values and the corresponding eigen vectors of the matrix

$$A=\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$$ **(See Unit-V, Page 282, Prob.31)**

(b) Test the consistency of the following system of equations and solve using matrix methods.

$$5x + 3y + 7z = 4$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5$$

(See Unit-V, Page 267, Prob.24)

Unit-V

9. (a) Prove that the proposition **

$P \rightarrow (q \rightarrow r) \leftrightarrow (p \wedge q) \rightarrow r$ is a tautology.

(b) Define a tree and prove that a tree T with n vertices has exactly (n – 1) edges. **

Or

10.(a) Let (B, +, ..') be a Boolean algebra and a, b be any two elements of B. Then prove that **

(i) $(a + b)' = a'.b'$ (ii) $(a.b)' = a' + b'$

(b) Define the following terms : **

(i) Support of a fuzzy set (ii) Complement of a fuzzy set

(iii) Union of two fuzzy set (iv) Intersection of two fuzzy set.

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

RGPV

BE-102

B.E. (First/Second Semester) EXAMINATION, June 2013

ENGINEERING MATHEMATICS-I

Note : (i) Attempt *five* questions.

(ii) Select *one* question from each Unit.

(iii) All questions carry equal marks.

Unit-I

1. (a) Prove that

$$(\sin^{-1}x)^2 = \frac{2}{2!}x^2 + \frac{2.2^2}{4!}x^4 + \frac{2.2^2.4^2}{6!}x^6 + \ldots$$

and hence deduce

$$\theta^2 = 2\frac{\sin^2\theta}{2!} + 2^2\frac{2\sin^4\theta}{4!} + 2^2 4^2\frac{2\sin^6\theta}{6!} + \ldots$$

(See Unit-I, Page 30, Prob.26)

(b) If $u(x, y, z) = \log(\tan x + \tan y + \tan z)$, prove that

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2$$

(See Unit-I, Page 45, Prob.42)

Or

2. (a) Prove that if the perimeter of a triangle is constant, its area is maximum when the triangle is equilateral. (See Unit-I, Page 61, Prob.62)

(b) Determine the curvature of the parabola $y^2 = 2px$ at **

(i) an arbitrary point (x, y)

(ii) the point $\left(\frac{p}{2}, p\right)$ and (iii) the point $(0, 0)$.

Unit-II

3. (a) Evaluate by expressing the limit of a sum in the form of a definite integral :

$$\lim_{x\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$$

(See Unit-II, Page 83, Prob.18)

(b) Define $\beta(m, n)$. Prove that

$$\beta(m, n) = \beta(m+1, n) + \beta(m, n+1) \quad m, n > 0.$$

(See Unit-II, Page 92, 102, Q.1, Prob.42)

Or

4. (a) Evaluate the following integral by changing the order of integration :

$$\int_0^1\int_x^{\sqrt{2-x^2}}\frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$$

(See Unit-II, Page 129, Prob.76)

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

(b) Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder $x^2 + y^2 = ax$. **(See Unit-II, Page 141, Prob.88)**

Unit-III

5. (a) Solve $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$. **

(b) Solve $y - x = x\dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^2$. **

Or

6. (a) Solve $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 13y = 8e^{3x}\sin 4x + 2^x$. **

(b) Solve $\dfrac{dx}{dt} + 4x + 3y = t$ **

$$\frac{dy}{dt} + 2x + 5y = e^t$$

Unit-IV

7. (a) Define rank of a matrix. Find the rank of matrix A, where

$$A = \begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}$$

(See Unit-V, Page 250, Prob.10)

(b) Solve completely the system of equations $2w + 3x - y - z = 0$, $4w - 6x - 2y + 2z = 0$, $-6w + 12x + 3y - 4z = 0$. **(See Unit-V, Page 265, Prob.22)**

Or

8. (a) Determine the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

(See Unit-V, Page 289, Prob.37)

(b) Show that Cayley-Hamilton theorem is satisfied by the matrix A.

where $A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$

Hence find A^{-1}. **(See Unit-V, Page 302, Prob.49)**

Unit-V

9. (a) Write the following function into disjunctive normal form of 3 variables x, y, z : **

(i) $x' + y'$ (ii) $xy' + x'y$

(b) In a Boolean algebra B. Prove that the identity elements $0, 1 \in B$ are unique and prove $0' = 1$, $1' = 0$. **

Or

10.(a) Define the following terms giving examples : **

(i) Support of fuzzy set. (ii) Complement of a fuzzy set.

Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus

(iii) Union of two fuzzy sets. (iv) Intersection of two fuzzy sets.

(b) Prove that the number of vertices of odd degree in a graph is always even. **

RGPV

BE-102

B.E. (First/Second Semester)

EXAMINATION, Dec. 2013

ENGINEERING MATHEMATICS-I

Note : (i) All questions carry equal marks. **(ii)** Internal choices are given.

Unit-I

1. (a) Use Maclaurin's series to prove –

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \ldots\ldots$$ **(See Unit-I, Page 26, Prob.22)**

(b) If $x^x y^y z^z = C$, then show that

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$$ **(See Unit-I, Page 46, Prob.44)**

Or

2. (a) Prove that curvature of the circle $x^2 + y^2 = a^2$ is constant. **

(b) Discuss the maximum and minimum value of $x^3 + y^3 - 3axy$.

(See Unit-I, Page 65, Prob.66)

Unit-II

3. (a) Evaluate –

$$\lim_{n\to\infty}\left[\frac{1}{n} + \frac{1}{\sqrt{n^2-1^2}} + \frac{1}{\sqrt{n^2-2^2}} + \ldots + \frac{1}{\sqrt{n^2-(n-1)^2}}\right]$$

(See Unit-II, Page 81, Prob.14)

(b) Prove the duplication formula

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$ **(See Unit-II, Page 101, Prob.40)**

Or

4. (a) Using triple integration determine the volume of a hemisphere of radius 'a'. **(See Unit-II, Page 143, Prob.90)**

(b) Change the order of integration $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate the same. **(See Unit-II, Page 133, Prob.80)**

Unit-III

5. (a) Solve the differential equation **

$$(D^2 + a^2)\,y = \sin ax$$

(b) Solve by method of variation of parameters **

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x$$

Or

6. (a) Solve the differential equation **

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x$$

(b) Solve **

$$\frac{dx}{dt} + y = \sin t,\ \frac{dy}{dt} + x = \cos t.$$

Unit-IV

7. (a) Find rank of the matrix A, where

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

(See Unit-V, Page 248, Prob.6)

(b) Show that the system

$x + 2y - 5z = 9$
$3x - y + 2z = 5$
$2x + 3y - z = 3$

is consistent and solve it. (See Unit-V, Page 272, Prob.28)

Or

8. (a) Find eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

(See Unit-V, Page 285, Prob.32)

(b) Prove that product of all eigen values of a matrix A is equal to determinant of A. (See Unit-V, Page 280, Q.2)

Unit-V

9. (a) Prove that following propositions is tautologies. **

$\sim (p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$

(b) Draw the simplified switching circuit of the function **

$f(x, y, z) = (x + y).\ (x' + y'\ .\ z')$

Or

10. (a) Prove that if G is a connected graph with n vertices and $(n - 1)$ edges, then G is a tree. **

(b) Define the following terms – **

(i) Euler's graphs
(ii) Spanning tree
(iii) Fuzzy set
(iv) Union and intersection of two fuzzy sets.

Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus

BE-102

B.E. (First/Second Semester)

EXAMINATION, June 2014

ENGINEERING MATHEMATICS-I

Note : (i) Answer five questions. In each question part A, B, C is compulsory and D part has internal choice.

(ii) All parts of each question are to be attempted at one place.

(iii) All questions carry equal marks, out of which part A and B (Max. 50 words) carry 2 marks, part C (Max. 100 words) carry 3 marks, part D (Max. 400 words) carry 7 marks.

(iv) Except numericals, Derivation, Design and Drawing etc.

Unit-I

1. (a) Expand $\log \frac{1+x}{1-x}$ in powers of x using Maclaurin's theorem. 2

(See Unit-I, Page 24, Prob.19)

(b) Define homogeneous functions and composite function and establish the Euler's theorem on homogeneous function. 2

(See Unit-I, Page 41, Q.2)

(c) Find the extreme values of the function $x^3 + y^3 - 3axy$. 3

(See Unit-I, Page 65, Prob.66)

(d) If the sides and angles of a triangle ABC vary in such a way that its circum radius remains constant, prove that ** 7

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$

Or

Prove that the radius of curvature for the catenary $y = c \cosh (x/c)$ is equal to the portion of the normal intercepted between the curve and the x-axis and that it varies as the square of the ordinate. ** 7

Unit-II

2. (a) Define gamma function and beta function and also establish the symmetry of beta function. (See Unit-II, Page 93, Q.2) 2

(b) Evaluate the following integral by changing the order of integration

$$\int_0^1 \int_{e^x}^{e} \frac{dydx}{\log y}.$$ 2

(c) Evaluate by definition of definite integral as the limit of a sum

$$\int_a^b \sin x \, dx.$$ (See Unit-II, Page 76, Prob.6) 3

(d) Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$. (See Unit-II, Page 141, Prob.87) 7

Or

Prove that – 7

**Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus

$$\lim_{n\to\infty}\left[\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}=2e^{\left(\frac{\pi-4}{2}\right)}$$

(See Unit-II, Page 100, Prob.20)

Unit-III

3. (a) Define the order and degree of a differential equation with one example also explain that the elimination of n arbitrary constants from an equation leads us to which order derivative and hence a differential equation of which order. ** 2

(b) Solve – ** 2

$$-y\,dx + x\,dy = \sqrt{x^2+y^2}\,dx$$

(c) A bacterial population β is known to have a rate of growth α to β itself. If between noon and 2 pm the population triples, at what time, no controls being exerted should β become 100 times what it was at noon. ** 3

(d) Solve – ** 7

$$x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x+\log x.$$

Or

Solve the following differential equation by using the method of variation of parameters – ** 7

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=e^x\tan x$$

Unit-IV

4. (a) Determine the rank of the following matrix – 2

$$\begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix}.$$

(See Unit-V, Page 247, Prob.4)

(b) Solve the system of equations using matrix method – 2

$x + 3y - 2z = 0$
$2x - y + 4z = 0$
$x - 11y + 14z = 0$

(See Unit-V, Page 260, Prob.16)

(c) If A is a non-singular matrix, prove that the eigen values of A^{-1} are the reciprocals of the eigen values of A. (See Unit-V, Page 274, Theorem 5) 3

(d) Find the eigen values and eigen vectors of the matrix – 7

$$A=\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

(See Unit-V, Page 289, Prob.37)

Or

Find the eigen values of A and using Cayley-Hamilton theorem, find A^n (n is a positive integer); given that 7

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

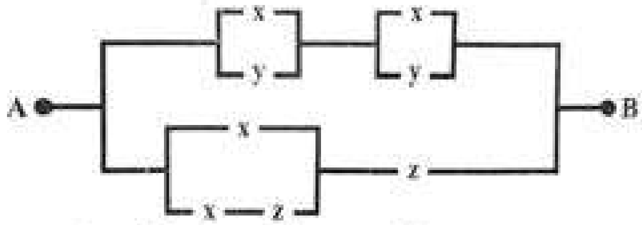
$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$ **(See Unit-V, Page 300, Prob.47)**

Unit-V

5. (a) What do you mean by logical equivalence and prove that the statement $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction ? ** 2

(b) For a simple graph of n vertices, the number of edges is $\frac{1}{2}n(n-1)$. ** 2

(c) Simplify the following circuit ** 3



(d) A simple graph with n vertices and k components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges. ** 7

Or

Express the following functions into disjunctive normal form $f(x, y, z) = x.y' + x.z + x.y$. ** 7

RGPV

BE-102

B.E. (First/Second Semester) EXAMINATION, Dec. 2014

ENGINEERING MATHEMATICS-I

Note : (i) Answer five questions. In each question part A, B, C is compulsory and D part has internal choice.

(ii) All parts of each questions are to be attempted at one place.

(iii) All questions carry equal marks, out of which part A and B (Max. 50 words) carry 2 marks, part C (Max. 100 words) carry 3 marks, part D (Max. 400 words) carry 7 marks.

(iv) Except numericals, Derivation, Design and Drawing etc.

1. (a) Define curvature of a curve at a point and find the radius of curvature at any point (s, ψ) of the curve $s = 4a \sin \psi$. **

(b) If $u = f\left(\frac{y}{x}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

(See Unit-I, Page 43, Prob.40)

(c) Discuss the maxima and minima of the function $x^3 + y^3 - 3axy$.

(See Unit-I, Page 65, Prob.66)

(d) Compute the approximate value of $\sqrt{11}$ to four decimal place by taking the first five terms of an approximate Taylor's expansion.

(See Unit-I, Page 36, Prob.35)

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

Or

If $x^x y^y z^z = C$, then show that

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$$ (See Unit-I, Page 46, Prob.44)

2. **(a)** Using gamma function, evaluate $\int_0^\infty \sqrt{x}\, e^{-3\sqrt{x}} dx$.

(See Unit-II, Page 100, Prob.36)

(b) Evaluate $\int_0^2 \int_0^1 (x^2 + y^2) dx\, dy$. (See Unit-II, Page 115, Prob.51)

(c) Evaluate $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x + y + z)\, dx\, dy\, dz$. (See Unit-II, Page 122, Prob.66)

(d) Evaluate –

$$\lim_{n \to \infty} \left[\left(1 + \frac{1}{n^2}\right)\left(1 + \frac{2^2}{n^2}\right)\left(1 + \frac{3^2}{n^2}\right) \cdots \left(1 + \frac{n^2}{n^2}\right) \right]^{1/n}$$

(See Unit-II, Page 83, Prob.18)

Or

Prove the Legendre's duplication formula

$$\Gamma(m)\,\Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m)$$ (See Unit-II, Page 101, Prob.40)

3. **(a)** State whether the differential equation $(e^y + 1) \cos x\, dx + e^y \sin x\, dy = 0$ is exact differential equation or not. **

(b) Solve the differential equation $p = \sin(y - xp)$. **

(c) Solve the differential equation – **

$$\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$$

(d) Solve $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = (1 + x)^2$. **

Or

Solve the simultaneous equations – **

$$\frac{dx}{dt} + 5x + y = e^t$$

$$\frac{dy}{dt} - x + 3y = e^{2t}$$

4. **(a)** Find one non zero minor of highest order of the matrix $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}$ and hence find the rank of the matrix A.

(See Unit-V, Page 247, Prob.3)

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

(b) Find the sum and product of eigen values of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & 1 \\ 2 & -1 & 3 \end{pmatrix}$ without actually computing them.

(c) Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

(See Unit-V, Page 287, Prob.33)

(d) Find the normal form of the matrix $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ and hence find its rank.

(See Unit-V, Page 251, Prob.12)

Or

For what values of λ, the equations

$$x + y + z = 1$$
$$x + 2y + 4z = \lambda$$
$$x + 4y + 10z = \lambda^2$$

have a solution and solve completely in each case.

(See Unit-V, Page 262, Prob.19)

5. (a) Let $p \equiv$ Raju is tall, $q \equiv$ Raju is handsome and $r \equiv$ People like Raju then write the following statements in language. **

(i) $(p \Rightarrow q) \vee (p \Rightarrow r)$ (ii) $p \Rightarrow (q \vee r)$

(iii) $\sim p \vee \sim q$ (iv) $\sim(\sim p \vee \sim q)$.

(b) In a Boolean algebra B, prove that $a + b = b \Rightarrow a.b = a, \forall\, a, b \in B$. **

(c) Draw the switching circuit for the following function and replace it by simpler one – **

$F(x, y, z) = x.y.z + (x + y).(x + z)$

(d) Prove that a tree with n vertices has $(n - 1)$ edges. **

Or

If p, q, r are three statements then show that $(p \Leftrightarrow q) \wedge (q \Leftrightarrow r) \Rightarrow (p \Leftrightarrow r)$ is a tautology. **

RGPV

BE-102

B.E. (First/Second Semester)

EXAMINATION, June 2015

ENGINEERING MATHEMATICS-II

Note : **(i)** Answer five questions. In each question part A, B, C is compulsory and D part has internal choice.

(ii) All parts of each questions are to be attempted at one place.

(iii) All questions carry equal marks, out of which part A and B

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

(Max. 50 words) carry 2 marks, part C (Max. 100 words) carry 3 marks, Part D (Max. 400 words) carry 7 marks.

(iv) Except numericals, derivation, Design and Drawing etc.

Unit-I

1. **(a)** Apply Maclaurin's theorem to prove that

$$\log \sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \ldots\ldots$$ **(See Unit-I, Page 27, Prob.23)**

(b) Show that $\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} + \ldots\ldots$

(See Unit-I, Page 31, Prob.27)

(c) Discuss the maximum and minimum of $x^3 + y^3 - 3xy$.

(See Unit-I, Page 63, Prob.64)

(d) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then show that **

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u$$

Or

If ρ_1 and ρ_2 be the radii of curvature at the xtremities of two conjugate diameters of an ellipse, prove that **

$$\left\{(\rho_1)^{2/3} + (\rho_2)^{2/3}\right\}(ab)^{2/3} = a^2 + b^2.$$

Unit-II

2. **(a)** Prove that $\int_0^1 \frac{\log(1+x)}{1+x^2}dx = \frac{\pi}{8}\log 2.$ **(See Unit-II, Page 73, Prob.1)**

(b) Evaluate $\int_a^b \cos x \, dx$ as limit of sums. **(See Unit-II, Page 75, Prob.5)**

(c) Show that $\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \frac{\Gamma(p)}{q^p}$, where $p, q > 0$.

(See Unit-II, Page 96, Prob.29)

(d) Change the order of integration of $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\, dy$ and hence evaluate it. **(See Unit-II, Page 130, Prob.77)**

Or

Express the area between the curves $x^2 + y^2 = a^2$, and $x + y = a$ as

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

double integral and evaluate it. (See Unit-II, Page 139, Prob.84)

Unit-III

3. **(a)** Solve $(1+e^{x/y})\,dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$ **

(b) Solve $y^2 \log y = x\,yp + p^2$. **

(c) Solve $(D^3 + 3D^2 + 2D)y = x^2$ **

(d) Solve the differential equation **

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x$$

Or

Solve by method of variation of parameters **

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x$$

Unit-IV

4. **(a)** Find the rank and nullity of the following matrix –

$$A = \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$$

(See Unit-V, Page 249, Prob.7)

(b) Show that the following system of equation is inconsistent –

$x - 2y + z - w = -1$
$3x - 2z + 3w = -4$
$5x - 4y + w = -3$

(See Unit-V, Page 264, Prob.21)

(c) Find eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

(See Unit-V, Page 285, Prob.32)

(d) Verify Cayley-Hamilton theorem for the matrix –

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$. Hence find A^{-1}. (See Unit-V, Page 301, Prob.48)

Or

Find for what value of λ and μ, the equations –
$x + y + z = 6$

Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus

$x + 2y + 3z = 10$
$x + 2y + \lambda z = \mu$
have

(i) No solution (ii) A unique solution (iii) Infinite many solutions.

(See Unit-V, Page 266, Prob.23)

Unit-V

5. (a) Define each of the following and give examples – **

(i) Graph (ii) Digraph
(iii) Pseudo graph (iv) Order of a graph

(b) Construct the truth table for the proposition – **

(i) $\sim p \wedge q$ (ii) $p \wedge (p \vee q)$

(c) For any Boolean algebra (B, +, ., '), prove that – **

(i) Additive identity is unique
(ii) Multiplicative identity is unique
(iii) For each $a \in B$, complement a' is unique.

(d) Prove that a simple graph with n vertices and k components can have at most $\frac{1}{2}(n - k)(n - k + 1)$ edges. **

Or

Prove that every non-trival tree has atleast 2-vertices of degree 1. **

RGPV

MA-110
B.E. (All Branches), 1-Year, I-Semester
Examination, December 2015
Choice Based Credit System (CBCS)
MATHEMATICS-I

Note : Attempt any five questions. All questions carry equal marks.

1. (a) If $y = \sin(m \sin^{-1}x)$. 3

prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$. (See Unit-I, Page 21, Prob.15)

(b) The equation of the tangent at the point (2, 3) of the curve $y^2 = ax^3 + b$ is $y = 4x - 5$. Find the values of a and b. ** 4

(c) Evaluate $\int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$. ** 5

2. (a) Expand by Maclaurin's theorem $e^{x \cos x}$ as far as the term x^3. 3

(See Unit-I, Page 23, Prob.18)

(b) Prove that the curvature at the point (x, y) of the catenary $y = c\cosh\left(\frac{x}{c}\right)$ is $\frac{y^2}{c}$. ** 4

(c) Locate the stationary points of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and determine their nature. (See Unit-I, Page 66, Prob.67) 5

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

3. **(a)** If $u = \sec^{-1}\left(\frac{x^3 - y^3}{x + y}\right)$, then prove that

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u.$ **(See Unit-I, Page 51, Prob.50)** 3

(b) The radius of a sphere is found to be 10 cm with a possible error of 0.02 cm. What is the relative error in computing the volume ? ** 4

(c) If $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$, then show that

$\frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} = r^2\sin\theta.$ ** 5

4. **(a)** Evaluate $\lim_{n\to\infty}\left(\frac{1}{1+n^3} + \frac{4}{8+n^3} + \frac{9}{27+n^3} + ... + \frac{1}{2n}\right)$ 3

(See Unit-II, Page 80, Prob.11)

(b) Prove that $\int_{-\infty}^{\infty} e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{a}, a > 0.$ 4

(See Unit-II, Page 98, Prob.33)

(c) Express $\int_0^1 x^m(1-x^n)^p\,dx$ in terms of Beta function and hence evaluate $\int_0^1 x^5(1-x^3)^{10}\,dx.$ **(See Unit-II, Page 101, Prob.38)** 5

5. **(a)** Evaluate $\iint y\,dx\,dy$ over the part of the plane bounded by the line $y = x$ and the parabola $y = 4x - x^2$. **(See Unit-II, Page 118, Prob.59)** 3

(b) Evaluate $\int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx.$ **(See Unit-II, Page 124, Prob.70)** 4

(c) Find the area enclosed by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$. 5

(See Unit-II, Page 108, Prob.47)

6. **(a)** Evaluate $\int_a^b e^x\,dx$ as limit of sum. **(See Unit-II, Page 75, Prob.4)** 3

(b) Express in terms of the Gamma function –

$\int_0^\infty x^n e^{-a^2x^2}\,dx.$ **(See Unit-II, Page 99, Prob.35)** 4

(c) Change the order of integration in $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate the same. **(See Unit-II, Page 133, Prob.80)** 5

7. **(a)** Verify Rolle's theorem, where $f(x) = 2x^3 + x^2 - 4x - 2$. 3

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

(See Unit-I, Page 11, Prob.6)

(b) If $u = f(y - z, z - x, x - y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$ ** 4

(c) Trace the curve $y^2(2a - x) = x^3$. ** 5

RGPV

MA-110
B.E. (All Branches), I-Semester
Examination, December 2016
Choice Based Credit System (CBCS)
MATHEMATICS-I

Note : (i) Attempt any five out of eight questions.
(ii) All questions carry equal marks.

1. (a) Verify the Lagrange's mean value theorem for the function $f(x) = x^2 - 2x + 4$ in the interval [1, 5]. (See Unit-I, Page 15, Prob.10)

(b) Using integration by parts, evaluate $\int x.\tan^{-1} x\, dx$. **

2. (a) Define tangent line of a curve. Find equation of the tangent line at point (3, 1) on the curve $4x^2 + 9y^2 = 45$. **

(b) Find the Taylor series expansion of log cos x about the point x = 0. (See Unit-I, Page 34, Prob.33)

3. (a) If $u = \log_e\left(\frac{x^4 + y^4}{x + y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3.$ (See Unit-I, Page 51, Prob.51)

(b) Discuss maxima and minima of the function
$$f(x, y) = x^3 - 4xy + 2y^2.$$
(See Unit-I, Page 62, Prob.63)

4. (a) Find radius of curvature at point 't' of the curve $x = at^2$, $y = 2at$. **

(b) Evaluate $\int_a^b e^x\, dx$ from the definition of integral as limit of sum. (See Unit-II, Page 75, Prob.4)

5. (a) Evaluate $\lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \ldots.. + \frac{1}{2n}\right]$ (See Unit-II, Page 78, Prob.8)

(b) Evaluate $\int_1^2\int_1^3 xy^2\, dx\, dy$ (See Unit-II, Page 115, Prob.53)

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

6. (a) Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$

(See Unit-II, Page 122, Prob.66)

(b) Change the order of integration in $\int_{0}^{2}\int_{0}^{2-x} xy\,dx\,dy$ and hence evaluate it. **(See Unit-II, Page 136, Prob.82)**

7. (a) Express the integral $\int_{0}^{1} x^3(1-x^2)^4\,dx$ in terms of gamma function and hence evaluate. **(See Unit-II, Page 103, Prob.43)**

(b) Prove that $\Gamma n\Gamma(1-n) = \frac{\pi}{\sin n\pi}$, where $0 < n < 1$. From this result deduce the value of $\Gamma\left(\frac{1}{2}\right)$. **(See Unit-II, Page 97, Prob.31)**

8. (a) Using triple integral, find volume of sphere $x^2 + y^2 + z^2 = a^2$.

(See Unit-II, Page 142, Prob.89)

(b) Trace the curve $ay^2 = x^3$. **

MA-110
B.E. (All Branches), I & II -Semester
Examination, June 2017
Choice Based Credit System (CBCS)
MATHEMATICS-I

RGPV

Note : (i) Attempt any five out of eight questions.
(ii) All questions carry equal marks.

1. (a) Write the statement of Lagrange's mean value theorem and verify it for the function $f(x) = x^2 - 4x - 3$ in the interval [1, 4].

(See Unit-I, Page 16, Prob.12)

(b) Find equation of tangent and normal to the curve at a point (1, 1) of the curve $2y = 3 - x^2$ **

2. (a) Find the Maclaurin's expansion of $\log(1 + x)$

(See Unit-I, Page 22, Prob.16)

(b) Find the radius of curvature at any point 't' of the circle $x = a\cos t$, $y = a\sin t$. **

3. (a) Discuss the maxima and minima of the function $f(x, y) = x^3 + y^3 - 3xy$ **(See Unit-I, Page 63, Prob.64)**

(b) If $u = f(y - z, z - x, x - y)$, then prove that **

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

4. (a) If $u = \log\left(\frac{x^3 + y^3}{x^2 - y^2}\right)$, then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

(See Unit-I, Page 52, Prob.52)

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

(b) Find the percentage error in the area of an ellipse if 1% error is made in measuring the major and minor axis. **

5. **(a)** Using definition of integral as limit of sum, evaluate $\int_a^b e^x dx$

(See Unit-II, Page 75, Prob.4)

(b) Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ **(See Unit-II, Page 93, Prob.24)**

6. **(a)** Evaluate $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\ldots\left(1+\frac{n}{n}\right)\right]^{1/n}$

(See Unit-II, Page 77, Prob.7)

(b) Define Beta function and using its definition, evaluate

$\int_0^1 x^4(1-x)^3 dx$ **(See Unit-II, Page 93, Prob.21)**

7. **(a)** Evaluate $\int_0^1\int_x^{\sqrt{x}}(x^2+y^2)dxdy$ **(See Unit-II, Page 115, Prob.54)**

(b) Evaluate $\int_0^3\int_0^2\int_0^1(x+y+z)\,dxdydz$ **(See Unit-II, Page 122, Prob.65)**

8. **(a)** Calculate the volume under the plane $z = 4 - x - y$ over the region $R{:}0 \le x \le 2, 0 \le y \le 1$ in the xy-plane. **(See Unit-II, Page 140, Prob.85)**

(b) By changing the order of integration, evaluate $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}dxdy$.

(See Unit-II, Page 128, Prob.74)

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BT-1002 (CBGS)

B.Tech., First Semester, Examination, Dec. 2017

Choice Based Grading System (CBGS)

MATHEMATICS - I

Note : **(i)** Attempt any five questions out of eight.

(ii) All questions carry equal marks.

1. **(a)** Expand $\sin^{-1}x$ is power of x by Maclaurin's theorem.

(See Unit-I, Page 29, Prob.25)

(b) Show that the curvature of a circle is constant and is equal to the reciprocal of its radius. **

2. **(a)** Write statement of Rolle's and Lagrange's theorem and explain their geometrical meaning. **(See Unit-I, Page 6, Q.1)**

(b) Discuss the maximum and minimum of $x^3 + y^3 - 3xy$.

(See Unit-I, Page 63, Prob.64)

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

3. **(a)** If $u = \tan^{-1}\left(\frac{x^2+y^2}{x-y}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u$$

(See Unit-I, Page 48, Prob.47)

(b) Find the percentage error in calculating the area of a rectangle when an error of 1% is made in measuring its length and breadth. **

4. **(a)** Evaluate $\int_a^b x\,dx$ from the definition of integral as a limit of sum.

(See Unit-II, Page 74, Prob.2)

(b) Evaluate the limit $\lim\limits_{n\to\infty} \frac{(n!)^{1/n}}{n}$. **(See Unit-II, Page 80, Prob.12)**

5. **(a)** Evaluate $\iint_R e^{2x+3y} dx\,dy$, where R is a triangle bounded by x = 0, y = 0 and x + y = 1. **(See Unit-II, Page 120, Prob.62)**

(b) Evaluate $\int_0^1\int_0^1\int_0^1 e^{x+y+z} dx\,dy\,dz$ **(See Unit-II, Page 123, Prob.68)**

6. **(a)** By triple integration determine the volume of a hemisphere of radius 'a'. **(See Unit-II, Page 143, Prob.90)**

(b) Evaluate $\int_0^2\int_0^3 (x^2+y^2)dx\,dy$. **(See Unit-II, Page 115, Prob.52)**

7. **(a)** Express $\int_0^1 x^m(1+x^n)^p dx$ is terms of beta functions and hence evaluate $\int_0^1 x^5(1+x^3)^{10}dx$. **(See Unit-II, Page 101, Prob.38)**

(b) Prove that $\beta(m, n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$ **(See Unit-II, Page 101, Prob.39)**

8. **(a)** Find the equation of the tangent and normal at the point $(at^2, 2at)$ on the parabola $y^2 = 4ax$. **

(b) Define gamma function and prove that $\Gamma\frac{1}{2} = \sqrt{\pi}$.

(See Unit-II, Page 93, Q.3)

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

RGPV

BT-1002 (CBGS)
B.Tech., I & II Semester
Examination, May 2018
Choice Based Grading System (CBGS)
Mathematics-I

Note : **(i)** Attempt any five questions out of eight.
(ii) All questions carry equal marks.

1. (a) Verify Rolle's theorem for the function $f(x)=x^2+2x-8$, in the interval $(-4, 2)$. **(See Unit-I, Page 7, Prob.2)**

(b) Find the slope and equation of the tangent to the curve $y=x^3-x$ at $x=2$. **

2. (a) Evaluate $\int e^x \sin x\, dx$ using integration by parts. **

(b) Find equations of the tangent and normal to the curve $y=x^2$ at the piont (0, 0). **

3. (a) Expand the function $f(x)=\cos x$ in Maclaurin series and hence find approximate value of $\cos 18°$. **(See Unit-I, Page 20, Prob.14)**

(b) Verify Euler's theorem for the function

$$u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$$

(See Unit-I, Page 49, Prob.49)

4. (a) Find the maximum and minimum value of the function $x^3+y^3-3a\,xy$.

(See Unit-I, Page 65, Prob.66)

(b) If $x^x y^y z^z = c$, then show that $x = y = z$, $\frac{\partial^2 z}{\partial x\, \partial y} = -(x \log ex)^{-1}$

(See Unit-I, Page 46, Prob.44)

5. (a) Evaluate the limit $\underset{n\to\infty}{\mathrm{Lt}}\left\{\frac{n!}{n^n}\right\}^{1/n}$. **(See Unit-II, Page 80, Prob.12)**

(b) Prove that $\Gamma(n+1)=n\Gamma(n)$. **(See Unit-II, Page 110, Prob.22)**

6. (a) Prove that $2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = B(m,n)$.

(See Unit-II, Page 93, Prob.24)

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

(b) Evaluate $\int_0^2 \int_0^1 (x^2 + y^2)\, dx\, dy$. (See Unit-II, Page 115, Prob.51)

7. (a) Evaluate the triple integral $\int_{y=0}^{1} \int_{x=y^2}^{1} \int_{z=0}^{1-x} x\, dz\, dx\, dy$.

(See Unit-II, Page 121, Prob.64)

(b) Find by triple integration, the volume of the sphere $x^2 + y^2 + x^2 = a^2$.

(See Unit-II, Page 142, Prob.89)

8. (a) Find radius of curvature at a point 't' of the curve x = a cos t, y = b sin t. **

(b) If u = x log xy where $x^3 + y^3 + 3xy = 1$, then find $\frac{du}{dx}$. **

RGPV

BT-102 (CBGS)
B.Tech., I Semester
Examination, November 2018
Choice Based Grading System (CBGS)
Mathematics-I

Note : (i) Attempt any five questions out of eight.
(ii) All questions carry equal marks.

1. (a) Verify Rolle's theorem for the function $f(x) = x^2$ in $[-1, 1]$.

(See Unit-I, Page 6, Prob.1)

(b) Using Taylor series find value of $\log_e(1.1)$ correct upto three dedimal place. (See Unit-I, Page 32, Prob.29)

2. (a) If $u = \sin^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$.

(See Unit-I, Page 49, Prob.48)

(b) Discuss the maxima and minima of the function $x^3 + y^3 - 3axy$.

(See Unit-I, Page 65, Prob.66)

3. (a) Evaluate $\lim_{n \to \infty} \left\{\frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{2n}\right\}$.

(See Unit-II, Page 78, Prob.8)

(b) Prove that $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$. (See Unit-II, Page 101, Prob.39)

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

4. **(a)** Evaluate double integral $\int\int_R xy\,dx\,dy$ over the region R bounded by $x = 0$, $y = 0$ and $x + y = 1$. **(See Unit-II, Page 118, Prob.60)**

(b) Evaluate triple integral $\int_{y=0}^{1}\int_{x=y^2}^{1}\int_{z=0}^{1-x} x\,dz\,dy\,dx$.

(See Unit-II, Page 121, Prob.64)

5. **(a)** Discuss the convergence of the geometric series $\sum_{n=0}^{\infty} x^n$.

(b) If $p > 1$, prove that the p-Series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + ... + \frac{1}{n^p} + ...$ converges. **(See Unit-III, Page 160, Prob.19)**

6. **(a)** Show that the vectors (2, 1, 4), (1, – 1, 2) and (3, 1, – 2) form a basis for R^3. **(See Unit-IV, Page 224, Prob.23)**

(b) Show that the set $w = \{(a,b,0) : a, b \in R\}$ is a subspace of R^3.

(See Unit-IV, Page 212, Prob.8)

7. **(a)** Find rank of the matrix $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 1 \end{bmatrix}$.

(See Unit-V, Page 248, Prob.6)

(b) Show that the following system of equation is inconsistent..
$5x + 3y + 14z = 4$; $y + 2z = 1$; $x - y + 2z = 0$; $2x + y + 6z = 2$

(See Unit-V, Page 270, Prob.26)

8. **(a)** Find Eigen values of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(See Unit-V, Page 289, Prob.36)

(b) Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

(See Unit-V, Page 287, Prob.34)

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

RGPV

BT-1002 (CBGS)
B.Tech., I & II Semester
Examination, November 2018
Choice Based Grading System (CBGS)
Mathematics-I

Note : (i) Attempt any five questions out of eight.
(ii) All questions carry equal marks.

1. **(a)** Verify Rolle's theorem for the function $f(x)=x^3-12x$ in the interval $\left[0,2\sqrt{3}\right]$. **(See Unit-I, Page 9, Prob.4)**

(b) Find the equation of tangent and normal at the point 't' on the curve $x=a\cos^3 t;\ y=a\sin^3 t$. **

2. **(a)** Evaluate the following – **

(i) $\int \frac{x^3}{1+x^8}dx$

(ii) $\int x\sin^{-1}x\,dx$.

3. **(a)** Expand $f(x) = e^x$ in Maclaurin series.
(See Unit-I, Page 20, Prob.13)

(b) Discuss the maxima and minima of the function $u=x^3y^2(1-x-y)$.
(See Unit-I, Page 60, Prob.61)

4. **(a)** Find radius of curvature for the curve $x^2 + y^2 = a^2$. **

(b) If $u=\log\left(\frac{x^4+y^4}{x+y}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$.
(See Unit-I, Page 51, Prob.51)

5. **(a)** If $x^xy^yz^z=c$ then show that $\frac{\partial^2 z}{\partial x\,\partial y}=-(x\log ex)^{-1}$ for $x=y=z$.
(See Unit-I, Page 46, Prob.44)

(b) Evaluate $\lim_{n\to\infty}\left\{\frac{1}{n+1}+\frac{1}{n+2}+......+\frac{1}{2n}\right\}$.
(See Unit-II, Page 78, Prob.8)

6. **(a)** Prove that $n\lceil n = \lceil(n+1), n > 0$. **(See Unit-II, Page 93, Prob.22)**

(b) Prove that $\beta(m, n) = \dfrac{\overline{|(m)}\,\overline{|(n)}}{\overline{|m+n}}, (m, n > 0)$.

(See Unit-II, Page 101, Prob.39)

7. (a) Express the integral $\int_0^1 x^m (1-x^n)^p \, dx$ in terms of Gamma function.

(See Unit-II, Page 104, Prob.45)

(b) Evaluate $\int_0^2 \int_0^1 (x^2 + y^2) \, dx \, dy$. (See Unit-II, Page 115, Prob.51)

8. (a) Evaluate $\int_0^2 \int_0^x \int_0^{x+y} e^x (y+2z) \, dx \, dy \, dz$.

(See Unit-II, Page 125, Prob.71)

(b) Evaluate $\iint_R e^{2x+3y} \, dx \, dy$ where R is the region bounded by $x = 0$, $y = 0$ and $x + y = 1$. (See Unit-II, Page 120, Prob.62)

RGPV

BT-102 (CBGS)
B.Tech., I & II Semester
Examination, May 2019
Choice Based Grading System (CBGS)
Mathematics-I

Note : (i) Attempt any five questions.
(ii) All questions carry equal marks.

1. (a) Discuss the maximum and minimum value of $u = x^3 y^2 (1 - x - y)$.

(See Unit-I, Page 60, Prob.61)

(b) Expand $\log_e x$ in powers of (xy) and hence evaluate $\log_e (1.1)$ correct to 4 decimal places.

2. (a) Verify Lagrange's mean value theorem for the function $f(x) = 2x^2 - 7x + 10$ in the interval [2, 5].

(See Unit-I, Page 15, Prob.11)

(b) If $u = f(y - z, z - x, x - y)$, prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

(See Unit-I, Page 41, Prob.37)

3. (a) Evaluate $\int_a^b x^2 \, dx$ on limit of sums. (See Unit-II, Page 74, Prob.5)

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

(b) Prove that $\int_0^{\infty} e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$. (See Unit-II, Page 99, Prob.34)

4. (a) Evaluate $\int_0^2 \int_0^1 \left(x^2 + y^2\right) dx\,dy$. (See Unit-II, Page 115, Prob.51)

(b) Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$. (See Unit-II, Page 123, Prob.69)

5. (a) Test for convergence of the following series –

$\sum u_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$ (See Unit-III, Page 151, Prob.1)

(b) Express f(x) = x as half range cosine series in $0 < x < 2$.
(See Unit-III, Page 192, Prob.42)

6. (a) Show that the map $T : R^2 \to R^3$ given by
$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_2)$ is linear.
(See Unit-IV, Page 236, Prob.36)

(b) Show that the set S of vectors (1, 0, 0), (1, 1, 0) and (1, 1, 1) is linearly independent. (See Unit-IV, Page 215, Prob.14)

7. (a) Find rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$. (See Unit-V, Page 249, Prob.8)

(b) Solve the system of equations $3x + 3y + 2z = 1$; $x + 2y = 4$; $10y + 3z = -2$; and $2x - 3y - z = 5$. (See Unit-V, Page 269, Prob.25)

8. (a) Find Eigen values of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

(See Unit-V, Page 297, Prob.41)

(b) Verify Cayley-Hamilton's theorem for the matrix $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$.

 (See Unit-V, Page 298, Prob.43)

RGPV

BT-102(CBGS)
B.Tech., I & II Semester
EXAMINATION, November 2019
Choice Based Grading System (CBGS)
MATHEMATICS-I

Note : (i) Attempt any five questions.
(ii) All questions carry equal marks.

1. (a) Verify Rolle's theorem for the function $f(x) = x^2 - x - 12$ in the interval $[-3, 4]$. **(See Unit-I, Page 8, Prob.3)**

(b) Expand log x in power of $(x - 1)$ by Taylor's theorem and hence find the value of log 1.1. **(See Unit-I, Page 32, Prob.29)**

2. (a) If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = \tan u$.

(See Unit-I, Page 49, Prob.48)

(b) Discuss the maximum or minima of the function $f(x, y) = x^3 - 4xy + 2y^2$. **(See Unit-I, Page 62, Prob.63)**

3. (a) Evaluate by expressing the following limit of a sum in the form of a definite intergral.

$$\lim_{n\to\infty}\left\{\frac{1}{n+1}+\frac{1}{n+2}+.....+\frac{1}{2n}\right\}.$$

(See Unit-II, Page 78, Prob.8)

(b) Prove that $\int_0 \frac{X^C}{C^X}dx = \frac{\sqrt{C+1}}{(\log C)^{c+1}}$

4. (a) Evaluate $\iint_D x^2y^2\,dxdy$, Where D is the region bounded by $x = 0$, $y = 0$ and $x^2 + y^2 = 1$, $x \geq 0, y \geq 0$. **(See Unit-II, Page 120, Prob.63)**

(b) Change the order of integration and evaluate it $\int_0^{\infty}\int_0^{x} e^{-xy}\,y\,dydx$.

(See Unit-II, Page 127, Prob.73)

5. (a) Find the Fourier series for the function $f(x) = x^2, -\pi \leq x \leq \pi$

(See Unit-III, Page 195, Prob.46)

(b) Test the convergence of the series

$$\sqrt{\frac{1}{2^3}}+\sqrt{\frac{2}{3^3}}+\sqrt{\frac{3}{4^3}}+\sqrt{\frac{4}{5^3}}+.....$$

(See Unit-III, Page 156, Prob.13)

6. **(a)** Determine whether or not the vectors u = (1, 1, 2), v = (2, 3, 1), w = (4, 5, 5) in R^3 are linearly dependent.
(See Unit-IV, Page 214, Prob.12)

(b) Let V = R^3, show that w is not a subspace of V, where w = {(a, b, c): a $\geq$ 0}.

7. **(a)** Find the rank of the matrix –

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

(See Unit-V, Page 249, Prob.8)

(b) Find the characteristic equation of the matrix A and hence find A^{-1}.
(See Unit-V, Page 300, Prob.46)

8. **(a)** Find the Eigen values and Eigen vectors for the matrix A –

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

(See Unit-V, Page 281, Prob.29)

(b) Show that the following equations are consistent or not.
5x + 3y + 14z = 4, y + 2z = 1, x – y + 2z = 0.
(See Unit-V, Page 271, Prob.27)

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BT-102(CBGS)
B.Tech., I & II Semester
EXAMINATION, November 2019
Choice Based Grading System (CBGS)
MATHEMATICS-I

Note : (i) Attempt any five questions.
(ii) All questions carry equal marks.

1. **(a)** Find the tangent at a point 't' on the curve
x = a cos ht, y = b sin ht. **

(b) Verify Rolle's theorem for the function $f(x) = x^3 - 6x^2 + 11x - 6$.
(See Unit-I, Page 12, Prob.7)

2. **(a)** Find equation of normal at point (x, y) for the curve $y^2 = 4ax$.
**

(b) Evaluate $\int xe^x dx$.

3. **(a)** Find the Maclaurin's expansion of log $(1 + e^x)$.
(See Unit-I, Page 26, Prob.22)

(b) Find radius of curvature for the curve x = a cos t, y = a sin t. **

4. **(a)** Find Taylor's expansion of y = sin x about point $x = \pi/2$.
(See Unit-I, Page 34, Prob.31)

(b) Discuss the maxima and minima of the function
$f(x, y) = x^3 + y^3 - 3x - 12y + 20.$ (See Unit-I, Page 64, Prob.65)

5. (a) If $u = \log\left(\frac{x^4+y^4}{x+y}\right)$, show that $x\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial y} = 3$.

(See Unit-I, Page 51, Prob.51)

(b) If $u = f(y/x)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = 0$.

(See Unit-I, Page 43, Prob.40)

6. (a) The radius of a sphere is found to be 20 cm with possible a error of 0.02 cm. Find the relative error in calculating the volume. **

(b) If $x^y + y^x = c$, then find $\frac{dy}{dx}$. **

7. (a) Eavaluate $\int_a^b x\,dx$ directly from the definition as the limit of sum.

(See Unit-II, Page 74, Prob.2)

(b) Prove that $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$.

(See Unit-II, Page 93, Prob.24)

8. (a) Evaluate $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$. (See Unit-II, Page 116, Prob.55)

(b) Evaluate $\int_0^2 \int_0^x \int_0^{x+y} e^x (y+2z)\, dx\, dy\, dz$.

(See Unit-II, Page 125, Prob.71)

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BT-102-(CBGS)
B.Tech., I & II Semester
EXAMINATION, June 2020
Choice Based Grading System (CBGS)
MATHEMATICS-I

Note : (i) Attempt any five questions.
(ii) All questions carry equal marks.
(iii) In case of any doubt or dispute the English version question should be treated as final.

1. (a) Verify Lagrange's mean value theorem for the function
$f(x) = 2x^2 - 7x + 10$ in the interval [2, 5].

(See Unit-I, Page 15, Prob.11)

(b) if u = f(y – z, z – x, x – y), prove that $\frac{\partial u}{\partial x}+\frac{du}{\partial y}+\frac{\partial u}{\partial z}=0.$

(See Unit-I, Page 41, Prob.37)

2. (a) Evaluate $\int_0^2\int_0^1\left(x^2+y^2\right)dxdy.$ (See Unit-II, Page 115, Prob.51)

(b) Evaluate $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}dz\,dy\,dx$ (See Unit-II, Page 123, Prob.69)

3. (a) Find rank of the matrix

$$\begin{bmatrix}1 & 2 & 3 & 2\\ 2 & 3 & 5 & 1\\ 1 & 3 & 4 & 5\end{bmatrix}$$

(See Unit-V, Page 249, Prob.8)

(b) Sovle the system of equations

3x + 3y + 2z = 1; x + 2y = 4; 10y + 3z = – 2 and 2x – 3y – z = 5

(See Unit-V, Page 269, Prob.25)

4. (a) If u = $\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$

(See Unit-I, Page 49, Prob.48)

(b) Discuss the maximum or minima of the function

$f(x, y) = x^3 - 4xy + 2y^2$ (See Unit-I, Page 62, Prob.63)

5. (a) Determine whether or not the vectors u (1, 1, 2), v(2, 3, 1), w(4, 5, 5) in R^3 are linearly dependent. (See Unit-IV, Page 214, Prob.12)

(b) Let V = R^3, show that w is not a subspace of V, where

$w=\{(a,b,c): a\geq 0\}$ (See Unit-IV, Page 212, Prob.9)

6. (a) Find the eigen values and eigen vectors for the matrix A –

$$A=\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}$$

(See Unit-V, Page 281, Prob.29)

(b) Show that the following equations are consistent or not.

$$5x + 3y + 14z = 4,\ y + 2z = 1,\ x - y + 2z = 0$$

(See Unit-V, Page 271, Prob.27)

7. **(a)** Evaluate $\lim_{n\to\infty}\left\{\frac{1}{n+1}+\frac{1}{n+2}+-----+\frac{1}{2n}\right\}$.

(See Unit-II, Page 78, Prob.8)

(b) Prove that –

$$\beta(m, n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

(See Unit-II, Page 101, Prob.39)

8. **(a)** Find Eigen values of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(See Unit-V, Page 289, Prob.36)

(b) Find the characteristics equation of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

(See Unit-V, Page 287, Prob.34)

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